Exponential Lag Synchronization of Cohen-Grossberg Neural Networks with Discrete and Distributed Delays on Time Scales

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Abstract: In this article, we investigate exponential lag synchronization results for the Cohen-Grossberg neural networks (C-GNNs) with discrete and distributed delays on an arbitrary time domain by applying feedback control. We formulate the problem by using the time scales theory so that the results can be applied to any uniform or non-uniform time domains. Also, we provide a comparison of results that shows that obtained results are unified and generalize the existing results. Mainly, we use the unified matrix-measure theory and Halanay inequality to establish these results. In the last section, we provide a simulated example for different time domains to show the effectiveness and generality of the obtained analytical results.

Novelty statement: This is the first attempt to discuss the exponential lag synchronization results for the generalized C-GNNs with mixed delays on time scales. The results are obtained by applying the novel unified matrix-measure theory and Halanay inequality. A comparison of results shows that these results unify and generalize the existing results. An example with simulation for different time domains is given to illustrate the analytical results.

1 Introduction

Since the 1980s, neural networks (NNs), including recurrent NNs, Hopfield NNs, cellular NNs, and bi-directional associative NNs, have been a subject of intense study because of their large number of potential applications in many fields, such as the classification of patterns, signal and image processing, optimization problems, associative memory, parallel computing, and so on. In 1983, Cohen-Grossberg [8] introduced the C-GNNs which are recognized as one of the most important and typical NNs because some other well-known NNs, for example, recurrent NNs, cellular NNs, and Hopfield NNs are special cases of C-GNNs. As a result, these types of networks have attracted considerable research attention and have been extensively studied regarding their dynamical properties such as state estimation [36], periodicity [5], stability [34, 43], boundedness [15], and synchronization [17, 41]. Furthermore, due to the importance of discrete-time C-GNNs as discussed in [40], the dynamics of discrete-time C-GNNs have become a popular research topic; see, for example, [9, 18, 30, 31].

Synchronization is one of the most important qualitative properties of dynamic systems and means that two or more dynamic systems lead to a common dynamical behaviour by using some coupling or external forces. The concept of synchronization of drive-response systems was first introduced by Pecora and Carrol [28]. Since then, the problem of synchronization has been capturing increasing attention both from a fundamental application-driven point of view. Potential applications can be found in many areas of applied sciences, such as harmonic oscillation generation, information science, human heartbeat regulation, chemical and biological systems, and secure communication [6,26,39]. In the last few years, various types of synchronization phenomena have been discovered and investigated such as complete synchronization [7], exponential synchronization [23, 41], finite-time synchronization [19, 29], lag synchronization [1], adaptive synchronization [10], and projective synchronization [2]. Among them, lag synchronization has been extensively studied [12,13,25,37] because in many connected electronic networks we have some constant time shifts between the drive and response systems, and due to that, the complete synchronization is hard to implement effectively.

Due to the importance of both discrete and continuous dynamical systems in many practical applications, the authors proved the results for the discrete and continuous dynamic systems but most of these results are investigated separately. Therefore, to avoid proving the results twice, in 1988, Hilger [11], introduced the so-called *time scales theory (or measure chain theory)* to unify the discrete and continuous analysis into a single comprehensive analysis, i.e., the study of dynamic equations on time scales turns out to be difference equations and differential equations if the time scale is chosen to be the set of integers, and real numbers, respectively.

Furthermore, apart from the discrete and continuous-time domains, there are many other time domains which can be very useful to study the dynamic behaviours of dynamic systems more accurately. For example, to model the growth process of some species like Magicicada Septendecim, Magicicada Cassini, and Pharaoh Cicada, we need a time domain of the form $T = \cup_{k=0}^{\infty}[k(a+b), k(a+b)+b],\ a, b \in (0, \infty)$, which is neither discrete nor continuous, and hence, neither the difference equation nor the differential equations can give the accurate behaviour of such types of models. But the time scales theory can overcome such difficulties as it gives the freedom to work on the general domain, i.e., the results obtained by using the time scales will also be valid for uniform and non-uniform time domains such as the non-overlapping closed intervals, a mixture of closed intervals and discrete points, and even a discrete non-uniform time domain. Thus, we can summarize the above and state that "Unification and Extension" are two main features of the time scales theory. In the last few years, the study of dynamic equations on time scales has drawn a tremendous amount of attention across the world and many researchers found its applications in many fields, such as epidemiology, economics, and control theory [3, 27]. Therefore, it is worth to investigate the dynamic equations on time scales. For more study on time scales, one can refer to the monograph [4].

Recently, many authors established different types of qualitative behaviours of dynamic systems on time scales, for example, the existence of solutions, stability analysis, stabilization, and synchronization [14,16,32,35,38]. Also, few authors established the existence of periodic, anti-periodic, almost-periodic solutions and their stability results of the C-GNNs [20–22, 24, 33, 42]. In [21], the authors studied the existence of anti-periodic solution and exponential stability for C-GNNs with time-varying delays on time scales. In [24], the authors established the existence and global exponential stability of almost periodic solutions for C–GNNs with distributed delays on time scales while in [22], the authors considered the impulsive C-GNNs with distributed delays on time scales and studied the existence and exponential stability of periodic solutions by using Lyapunov functions, M-matrix theory, and coincidence degree theory.

However, the synchronization problem of C-GNNs on time scales has not been studied so far to the best of our knowledge. Therefore, to fill this gap, in this work, we establish exponential lag synchronization results for C-GNNs with discrete and distributed time delays on time scales by using feedback control and a novel unified matrix-measure technique as well as the Halanay inequality. In short, the essential commitment and benefit of this manuscript can be summarized as follows:

- The C-GNNs with discrete and distributed delays on arbitrary time domains are considered to study exponential lag synchronization.
- The problem is formulated by using the time scales theory and the results are derived based on a novel unified matrix-measure theory and the Halanay inequality.
- The results for different special cases are given which shows that the obtained results unify and generalize the existing results.
- A simulated example for different time scales including continuous, discrete and non-overlapping closed intervals, is given to verify the obtained analytical outcomes.

The remaining part of the manuscript is organized as follows: In Section 2, we recall some basics from matrix theory and time scales which we need throughout this manuscript. In Section 3, we formulate our statement of the problem. In Section 4, the main results are discussed and in the last Section 5, a numerical example with simulation is given to verify the obtained results.

2 Preliminaries

Throughout this paper, the notations $\mathbb{R}, \mathbb{Z}$ and $\mathbb{N}$ denote the set of all real, integers. and natural numbers, respectively; $\mathbb{T}$ denotes the time scales; $\emptyset$ denotes the empty set; $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the n-dimensional Euclidean space and the set of all $n \times m$ matrices, respectively; diag$\{\ldots\}$ denotes the diagonal matrix; Superscript $*$ denotes the matrix transpose; Id and O denote the identity and zero matrices of appropriate dimensions, respectively; $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$, denotes the time scale interval. For any $a, b \in \mathbb{R}, C([a, b], \mathbb{R}^n)$ denotes the set of continuous functions from $[a, b]$ into $\mathbb{R}^n$; $\|\cdot\|_p$, $(p = 1, 2, \infty)$ is used to denote the p-norm for a vector or for a matrix.

Next, we recall some basic definitions and results about time scale calculus.

A *time scale* is an arbitrary non-empty closed subset of the real numbers. $\mathbb{R}, h\mathbb{Z}(h > 0)$, $\mathbb{P}_{a,b} = \cup_{k=0}^{\infty}[k(a + b), k(a + b) + a]$ and any discrete set are some examples of time scales. The *forward and backward jump operators* $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$, respectively with the substitution $\sup \mathbb{T} = \inf \emptyset$ and $\inf \mathbb{T} = \sup \emptyset$. Also the graininess functions $\mu : \mathbb{T} \to [0, \infty)$ is given by $\mu(t) = \sigma(t) - t$. A point $t \in \mathbb{T}$ is called *right-dense* if $t < \max\{\mathbb{T}\}$ and $\sigma(t) = t$, *left-dense* if $t > \min\{\mathbb{T}\}$ and $\rho(t) = t$, *right-scattered* if $\sigma(t) > t$, and *left-scattered* if $\rho(t) < t$. If $\mathbb{T}$ has a left-scattered maximum M, then we set $\mathbb{T}^k = \mathbb{T} \setminus \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

Definition 2.1 ([35], Def. 1)**.** *Let $t \in \mathbb{T}^k$ and $f : \mathbb{T} \to \mathbb{R}$ be a function. Then the* delta derivative *of f at a point t is defined as a number $f^{\Delta}(t)$ (provided it exists) whenever for each $\epsilon > 0$ there exists a neighborhood U of t such that*

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s| \text{ for all } s \in U.$$

Further, if the neighborhood U is replaced by the right-hand sided neighborhood U^+, then the delta derivative is called the upper right Dini-delta-derivative *and denoted by $D^+_{\Delta} f(t)$.*

Remark 2.1. *In the above Definition 2.1, if $\mu(t) = 0$, then we have*

$$D^+_{\Delta} f(t) = \lim_{h\to 0^+} \frac{f(t + h) - f(t)}{h}.$$

Remark 2.2. *Let $f : \mathbb{T} \to \mathbb{R}$ is differentiable at $t \in \mathbb{T}^k$, then the forward operator σ and the delta derivative of f are related by the formula $f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t)$.*

A function $f : \mathbb{T} \to \mathbb{R}$ is called *regressive (or positive regressive)* if $1 + \mu(t)f(t) \neq 0(or \; > 0)$ for all $t \in \mathbb{T}$. Also, f is called *regulated* provided its right-side limit exists (finite) at all right-dense points of $\mathbb{T}$ and its left-side limit exist (finite) at all left-dense points of $\mathbb{T}$. Furthermore, f is called a *rd-continuous function* if it is regulated and it is continuous at all right-dense points of $\mathbb{T}$. The collection of all regressive (or positive regressive) functions and rd-continuous functions from $\mathbb{T}$ to $\mathbb{R}$ are defined, respectively, by $\mathcal{R}(or\ \mathcal{R}^+)$ and $C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.2 ([32], Def. 3)**.** *For any $p \in \mathcal{R}$ and $t \in \mathbb{T}^\kappa$, we define $\ominus p$ by*

$$(\ominus p)(t) = -\frac{p(t)}{1 + \mu(t)p(t)}.$$

Remark 2.3. *If $p \in \mathcal{R}$, then $\ominus p \in \mathcal{R}$.*

Next, we define the time scales version of the exponential function.

Definition 2.3 ([4], Def. 2.30)**.** *Let $p \in \mathcal{R}$, then we define the exponential function on time scales by*

$$e_p(t, s) = \exp\left(\int_s^t \zeta_{\mu(z)}(p(z))\Delta z\right) \text{ for } t, s \in \mathbb{T}$$

with

$$\zeta_{\mu(s)}(p(s)) = \begin{cases} \dfrac{1}{\mu(s)} \log(1 + p(s)\mu(s)), & \text{if } \mu(s) \neq 0, \\ p(s), & \text{if } \mu(s) = 0. \end{cases}$$

Next, we define the delta-integral on time scales.

Definition 2.4 ([4], Def. 1.71)**.** *Let $f : \mathbb{T} \to \mathbb{R}$ be a regulated function, then a function $F : \mathbb{T} \to \mathbb{R}$ is called an* anti-derivative of f *if $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. Also, we define the Cauchy integral by*

$$\int_a^b f(t)\Delta(t) = F(b) - F(a) \text{ for all } \; a, b, \in \mathbb{T}.$$

Remark 2.4. *For any $a, b \in \mathbb{T}$ and $f \in C_{rd}(\mathbb{T}, \mathbb{R})$, if we set $\mathbb{T} = \mathbb{R}$, then we have*

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt.$$

Further, if $[a, b)_\mathbb{T}$ consists of only isolated points, then we have

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t\in[a,b)_\mathbb{T}} \mu(t)f(t) & \text{if } a < b, \\ 0 & \text{if } a = b, \\ -\sum_{t\in[a,b)_\mathbb{T}} \mu(t)f(t) & \text{if } a > b. \end{cases}$$

Next, we recall some basics from matrix-measure theory.

Definition 2.5 ([38], Def. 1)**.** *The* generalized matrix-measure *and* classical matrix-measure *of a real square matrix $W = (w_{kl})_{n\times n}$ with respect to the $p-$norm ($p = 1, 2$ or ∞) are defined by*

$$\omega_p(W, h) = \frac{\| \operatorname{Id} + hW\|_p - 1}{h} \text{ and } \Lambda_p(W) = \lim_{s\to 0^+} \frac{\| \operatorname{Id} + sW\|_p - 1}{s},$$

respectively, where $h > 0$. The matrix norms and corresponding classical matrix-measures are given in Table 1.

Matrix norm	Matrix-measure
$\|W\|_1 = \max_j \sum_{i=1}^n \|w_{ij}\|$	$\Lambda_1(W) = \max_j w_{jj} + \sum_{i=1,i\neq j}^n \|w_{ij}\|$
$\|W\|_2 = \sqrt{\lambda_{\max}(W^TW)}$	$\Lambda_2(W) = \frac{1}{2}\lambda_{\max}(W^T + W)$
$\|W\|_\infty = \max_i \sum_{j=1}^n \|w_{ij}\|$	$\Lambda_\infty(W) = \max_i w_{ii} + \sum_{j=1,\neq i}^n \|w_{ij}\|$

Table 1: Matrix norms and corresponding classical matrix-measures

Definition 2.6 ([38], Def. 2)**.** *Let $W \in \mathbb{R}^{n\times n}$ be a real matrix and let $\mathbb{T}$ be an arbitrary time scale. Then the* unified matrix-measure on $\mathbb{T}$ with respect to the $p-$norm ($p = 1, 2$ *or* ∞) *is defined as*

$$M_p(W,\mathbb{T}) = \begin{cases} \max\left\{\dfrac{\|\operatorname{Id} + \mu(t)W\|_p - 1}{\mu(t)} : t \in \mathbb{T}\right\}, \textit{if } \mu(t) > 0, \forall\ t \in \mathbb{T}, \\ \max\left\{\Lambda_p(W),\ \max\left\{\dfrac{\|\operatorname{Id} + \mu(t)W\|_p - 1}{\mu(t)} : t \in \mathbb{T}, \mu(t) > 0\right\}\right\},\ \textit{else.} \end{cases}$$

Note that for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = h\mathbb{Z}$, $h > 0$, Definition 2.6 reduces to Definition 2.5.

3 Statement of Problem

We consider a class of C-GNNs with discrete and distributed delays on time scales of the following form:

$$\begin{cases} x^\Delta(t) &= -\Gamma(x(t))[\Upsilon(x(t)) - P\mathcal{F}(x(t)) - Q\mathcal{F}(x(t-\eta_1)) - R\int_{t-\eta_2}^t \mathcal{F}(x(s))\Delta s - I],\ t \in [0,\infty)_\mathbb{T}, \\ x(s) &= \phi(s),\ s \in [-\eta, 0]_\mathbb{T}, \end{cases} \tag{1}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^* \in \mathbb{R}^n$ is the state vector; $P = (r_{ij})_{n\times n} \in \mathbb{R}^{n\times n}, Q = (s_{ij})_{n\times n} \in \mathbb{R}^{n\times n}, R = (t_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$ are the connection, discrete delay connection and distributed delay connections strength matrices, respectively; $\eta_1(> 0)$ and $\eta_2(> 0)$ are the discrete and distributed delay, respectively, such that $t-\eta_1 \in \mathbb{T}$ and $t-\eta_2 \in \mathbb{T}$; $\eta = \max\{\eta_1, \eta_2\}$; $\Gamma(x(t)) = \operatorname{diag}\{\Gamma_1(x(t)), \Gamma_2(x(t)), \ldots, \Gamma_n(x(t))\} \in \mathbb{R}^{n\times n}$ is the state-dependent amplification function; $\Upsilon(x(t)) = [\Upsilon_1(x(t)), \Upsilon_2(x(t)), \ldots, \Upsilon_n(x(t))]^* \in \mathbb{R}^n$ is the appropriate behaviour function; $\mathcal{F}(x(\cdot)) = [\mathcal{F}_1(x(\cdot)), \mathcal{F}_2(x(\cdot)), \ldots, \mathcal{F}_n(x(\cdot))]^* \in \mathbb{R}^n$ denotes the activation function; I is the external bias term; $\phi \in C_{rd}([-\eta, 0]_\mathbb{T}, \mathbb{R}^n)$.

In this paper, we shall establish synchronization results by using the drive-response technique. Therefore, we consider system (1) as the drive system and, correspondingly, we consider a copy of the C-GNNs as response system as follows

$$\begin{cases} y^\Delta(t) &= -\Gamma(y(t))[\Upsilon(y(t)) - P\mathcal{F}(y(t)) - Q\mathcal{F}(y(t-\eta_1)) - R\int_{t-\eta_2}^t \mathcal{F}(y(s))\Delta s - I] + u(t), \\ & \qquad t \in [0,\infty)_\mathbb{T}, \\ y(s) &= \psi(s),\ s \in [-\eta, 0]_\mathbb{T}, \end{cases} \tag{2}$$

where $y(t) \in \mathbb{R}^n$; $\psi \in C_{rd}([\eta, 0]_\mathbb{T}, \mathbb{R}^n)$; $u(t)$ is the control function defined as

$$u(t) = -K(y(t) - x(t-\beta)), \tag{3}$$

with K as the feedback gain matrix and β is the transmittal delay such that $t - \beta \in \mathbb{T}$.

Remark 3.1. *The considered class of C-GNNs is defined on the general time domain, and hence, it contains the usual continuous-time C-GNNs, discrete-time C-GNNs, and many more. For example, if we consider the* ***continuous-time domain****, i.e.,* $\mathbb{T}=\mathbb{R}$*, then the drive system* (1) *becomes*

$$x'(t) = -\Gamma(x(t))[\Upsilon(x(t)) - P\mathcal{F}(x(t)) - Q\mathcal{F}(x(t-\eta_1)) - R\int_{t-\eta_2}^{t}\mathcal{F}(x(s))ds - I] \tag{4}$$

and the response system (2) *becomes*

$$y'(t) = \Gamma(y(t))[\Upsilon(y(t)) - P\mathcal{F}(y(t)) - Q\mathcal{F}(y(t-\eta_1)) - R\int_{t-\eta_2}^{t}\mathcal{F}(y(s))ds - I] + u(t), \tag{5}$$

where $t\in[0,\infty)$ *and the rest of the parameters are the same as defined previously. Also, if we choose, the* $h-$***difference discrete-time domain****, i.e.,* $\mathbb{T}=h\mathbb{Z}$*,* $h>0$*, then drive system* (1) *is converted to*

$$x(t+h) = x(t) - h\Gamma(x(t))\left[\Upsilon(x(t)) - P\mathcal{F}(x(t)) - Q\mathcal{F}(x(t-\eta_1)) - R\sum_{k=\frac{t-\eta_2}{h}}^{\frac{t}{h}-1} h\mathcal{F}(x(kh)) - I\right] \tag{6}$$

and the response system (2) *is converted to*

$$y(t+h) = y(t) - h\Gamma(y(t))\left[\Upsilon(y(t)) - P\mathcal{F}(y(t)) - Q\mathcal{F}(y(t-\eta_1)) - R\sum_{k=\frac{t-\eta_2}{h}}^{\frac{t}{h}-1} h\mathcal{F}(y(kh)) - I\right] + hu(t), \tag{7}$$

where $t\in[0,\infty)_{h\mathbb{Z}}$*. Furthermore, for the* ***non-overlapping time domain*** $\mathbb{T}=\cup_{i=0}^{\infty}[i,i+h], 0<h<1$*, the concrete expression of drive system* (1) *is*

$$\begin{cases} x'(t) = -\Gamma(x(t))[\Upsilon(x(t)) - P\mathcal{F}(x(t)) - Q\mathcal{F}(x(t-\eta_1)) - R\int_{t-\eta_2}^{t}\mathcal{F}(x(s))ds - I], \\ \qquad t\in\cup_{i=0}^{\infty}[i,i+h), \\ x(t+1-h) = x(t) - (1-h)\Gamma(x(t))\Big[\Upsilon(x(t)) - P\mathcal{F}(x(t)) - Q\mathcal{F}(x(t-\eta_1)) \\ \qquad -R\sum_{k=\frac{t-\eta_2}{1-h}}^{\frac{t}{1-h}-1}(1-h)\mathcal{F}(x(k(1-h))) - I\Big],\ t=\cup_{i=0}^{\infty}\{i+h\} \end{cases} \tag{8}$$

and the response system (2) *is*

$$\begin{cases} y'(t) = -\Gamma(y(t))[\Upsilon(y(t)) - P\mathcal{F}(y(t)) - Q\mathcal{F}(y(t-\eta_1)) - R\int_{t-\eta_2}^{t}\mathcal{F}(y(s))ds - I] + u(t), \\ \qquad t\in\cup_{i=0}^{\infty}[i,i+h), \\ y(t+1-h) = y(t) - (1-h)\Gamma(y(t))\Big[\Upsilon(y(t)) - P\mathcal{F}(y(t)) - Q\mathcal{F}(y(t-\eta_1)) \\ \qquad -R\sum_{k=\frac{t-\eta_2}{1-h}}^{\frac{t}{1-h}-1}(1-h)\mathcal{F}(y(k(1-h))) - I\Big] + (1-h)u(t),\ t=\cup_{i=0}^{\infty}\{i+h\}. \end{cases} \tag{9}$$

The main idea of synchronization is that the response system (2) utilizes a feasible controller to synchronize itself with the drive system (1). Mathematically, we can define it in the following definition.

Definition 3.1. *The drive system* (1) *and the response system* (2) *are said to be* exponentially lag synchronized *in the timescale sense under the control protocol* (3) *if there exist two constants* $C>0$ *and* $\nu>0$*, such that the following inequality holds*

$$\|y(t)-x(t-\beta)\|_p \le Ce_{\ominus\nu}(t,t_0),\ t\ge t_0.$$

Remark 3.2. *In the above Definition 3.1, if $\beta = 0$, then the drive system (1) and the response system (2) are called exponentially synchronized.*

Now, to prove the synchronization results, we define the error between drive system (1) and response system (2) by $\zeta(t) = y(t) - x(t-\beta)$, then the error dynamics can be written as

$$\zeta^{\Delta}(t) = -K\zeta(t) - \tilde{\Gamma}(\zeta(t))[\tilde{\Upsilon}(\zeta(t)) - P\tilde{\mathcal{F}}(\zeta(t)) - Q\tilde{\mathcal{F}}(\zeta(t-\eta_1)) - R\int_{t-\eta_2}^{t}\tilde{\mathcal{F}}(\zeta(s))\Delta s - I], \qquad (10)$$

where $\zeta(t) \in \mathbb{R}^n$ and

$$\begin{aligned}
\tilde{\Gamma}(\zeta(t))\tilde{\Upsilon}(\zeta(t)) &= \Gamma(y(t))\Upsilon(y(t)) - \Gamma(x(t-\beta))\Upsilon(x(t-\beta)),\\
\tilde{\Gamma}(\zeta(t))P\tilde{\mathcal{F}}(\zeta(t)) &= \Gamma(y(t))P\mathcal{F}(y(t)) - \Gamma(x(t-\beta))P\mathcal{F}(x(t-\beta)),\\
\tilde{\Gamma}(\zeta(t))Q\tilde{\mathcal{F}}(\zeta(t-\eta_1)) &= \Gamma(y(t))Q\mathcal{F}(y(t-\eta_1)) - \Gamma(x(t-\beta))Q\mathcal{F}(x(t-\beta-\eta_1)),\\
\tilde{\Gamma}(\zeta(t))R\int_{t-\eta_2}^{t}\tilde{\mathcal{F}}(\zeta(s))\Delta s &= \Gamma(y(t))R\int_{t-\eta_2}^{t}\mathcal{F}(y(s))\Delta s - \Gamma(x(t-\beta))R\int_{t-\eta_2}^{t}\mathcal{F}(x(s-\beta))\Delta s,\\
\tilde{\Gamma}(\zeta(t))I &= \Gamma(y(t))I - \Gamma(x(t-\beta))I.
\end{aligned}$$

From the definition of $\zeta(t)$, it is clear that if the error system (10) is exponentially stable, then the drive system (1) and the response system (2) are exponentially lag synchronized. Therefore, our goal is to show the exponential stability of the error system (10).

To deal with the lag delay, we set $x(s) = \phi(-\eta)$ for all $s \in [-\eta-\beta, -\eta]_{\mathbb{T}}$ and

$$\Psi(s) = \begin{cases} \phi(s), & s \in [-\eta, 0]_{\mathbb{T}},\\ \phi(-\eta), & s \in [-\eta-\beta, -\eta]_{\mathbb{T}},\end{cases}$$

then, we can define the initial condition for the error system (10) as follows

$$\zeta(s) = \psi(s) - \Psi(s-\beta), \; s \in [-\eta, 0]_{\mathbb{T}}.$$

In order to prove the main results, we need the following assumption.

Assumption 1. For any $x, y \in \mathbb{R}^n$, there exist positive constants $L_\Gamma, L_\Upsilon, L_\mathcal{F}$ such that

$$\|\Gamma(x) - \Gamma(y)\|_p \le L_\Gamma\|x-y\|_p, \; \|\Upsilon(x) - \Upsilon(y)\|_p \le L_\Upsilon\|x-y\|_p, \; \|\mathcal{F}(x) - \mathcal{F}(y)\|_p \le L_\mathcal{F}\|x-y\|_p.$$

Also, there exist positive constants $M_\Gamma, M_\Upsilon, M_\mathcal{F}$ such that

$$\|\Gamma(x)\|_p \le M_\Gamma\|x\|_p, \; \|\Upsilon(x)\|_p \le M_\Upsilon\|x\|_p, \; \|\mathcal{F}(x)\|_p \le M_\mathcal{F}\|x\|_p.$$

4 Exponential Lag Synchronization Results

In this section, we provide the main results of this manuscript. Before that, we are giving an important lemma which is useful to establish these results.

Lemma 4.1 ([38], Lemma 2)**.** *For any real scalars a and b such that $a > b > 0$ and $-a \in \mathcal{R}^+$, let $x(t)$ be a non-negative right-dense continuous function satisfying*

$$D_{\Delta}^{+}x(t) \le -ax(t) + b\hat{x}, \; t \in [t_0, \infty)_{\mathbb{T}},$$

where $D_{\Delta}^{+}x(t)$ is the upper right Dini-Δ-derivative of x at t, $\hat{x} = \sup_{s\in[t-\eta,t]_{\mathbb{T}}} x(s)$. Then the inequality

$$x(t) \le \hat{x}(t_0)e_{\ominus\lambda}(t, t_0),$$

holds, where $\lambda > 0$ is a solution of the inequality $\lambda + b\exp(\lambda\eta) < a$.

Now, we are ready to give the first main result of this article in the following theorem.

Theorem 4.1. *Let Assumption 1 hold. Then the drive system (1) and response system (2) are exponentially lag synchronized if for some $p \in \{1, 2, \infty\}$, there exist a non-singular matrix Z, a control gain matrix K such that $\mathcal{M}_1^p - \mathcal{M}_2^p > 0$ and $-\mathcal{M}_1^p \in \mathcal{R}^+$, where*

$$\mathcal{M}_1^p = -(M_p(-ZKZ^{-1}, \mathbb{T}) + \|Z\|_p\|Z^{-1}\|_p((M_\Gamma L_\Upsilon + M_\Upsilon L_\Gamma) + (M_\Gamma L_\mathcal{F} + M_\mathcal{F} L_\Gamma)\|P\|_p + L_\Gamma\|I\|_p)),$$
$$\mathcal{M}_2^p = \|Z\|_p\|Z^{-1}\|_p(M_\Gamma L_\mathcal{F} + M_\mathcal{F} L_\Gamma)(\|Q\|_p + \eta\|R\|_p)$$

and $M_p(\cdot, \mathbb{T})$ denotes the unified matrix-measure as defined in Definition 2.6.

Proof. For any non-singular matrix Z, we define

$$V(\zeta(t)) = \|Z\zeta(t)\|_p.$$

Now, for any arbitrary point $t \in \mathbb{T}$, from the definition of $\mu(t)$, we have either $\mu(t) = 0$ or $\mu(t) > 0$. Therefore, we split the proof into the following two steps:
Step 1: When $\mu(t) > 0$, then for any $t \in \mathbb{T}$, we have

$$\begin{aligned}
\frac{\|Z\zeta(\sigma(t))\|_p - \|Z\zeta(t)\|_p}{\mu(t)} &= \frac{1}{\mu(t)}\Big\{\|Z\zeta(t) + \mu(t)Z\zeta^\Delta(t)\|_p - \|Z\zeta(t)\|_p\Big\} \\
&= \frac{1}{\mu(t)}\Bigg\{\Bigg\|Z\zeta(t) + \mu(t)Z(-K\zeta(t) - \tilde{\Gamma}(\zeta(t))[\tilde{\Upsilon}(\zeta(t)) - P\tilde{\mathcal{F}}(\zeta(t)) \\
&\quad - Q\tilde{\mathcal{F}}(\zeta(t-\eta_1)) - R\int_{t-\eta_2}^{t}\tilde{\mathcal{F}}(\zeta(s))\Delta s - I])\Bigg\|_p - \|Z\zeta(t)\|_p\Bigg\} \\
&\leq \frac{1}{\mu(t)}\{\|Z\zeta(t) + \mu(t)(-ZK)\zeta(t)\|_p - \|Z\zeta(t)\|_p\} \\
&\quad + \|Z\tilde{\Gamma}(\zeta(t))\tilde{\Upsilon}(\zeta(t))\|_p + \|Z\tilde{\Gamma}(\zeta(t))P\tilde{\mathcal{F}}(\zeta(t))\|_p + \|Z\tilde{\Gamma}(\zeta(t))I\|_p \\
&\quad + \|Z\tilde{\Gamma}(\zeta(t))Q\tilde{\mathcal{F}}(\zeta(t-\eta_1))\|_p + \left\|Z\tilde{\Gamma}(\zeta(t))R\int_{t-\eta_2}^{t}\tilde{\mathcal{F}}(\zeta(s))\Delta s\right\|_p. \qquad (11)
\end{aligned}$$

Now, from the definition of $\tilde{\Gamma}, \tilde{\Upsilon}, \tilde{\mathcal{F}}$ and the Assumption 1, we have

$$\begin{aligned}
\|\tilde{\Gamma}(\zeta(t))\tilde{\Upsilon}(\zeta(t))\|_p &= \|\Gamma(y(t))\Upsilon(y(t)) - \Gamma(x(t-\beta))\Upsilon(x(t-\beta))\|_p \\
&\leq \|\Gamma(y(t))\Upsilon(y(t)) - \Gamma(y(t))\Upsilon(x(t-\beta))\|_p \\
&\quad + \|\Gamma(y(t))\Upsilon(x(t-\beta)) - \Gamma(x(t-\beta))\Upsilon(x(t-\beta))\|_p \\
&\leq (M_\Gamma L_\Upsilon + M_\Upsilon L_\Gamma)\|\zeta(t)\|_p, \qquad (12)
\end{aligned}$$

Similarly, one can obtain

$$\begin{aligned}
\|\tilde{\Gamma}(\zeta(t))P\tilde{\mathcal{F}}(\zeta(t))\|_p &= \|\Gamma(y(t))P\mathcal{F}(y(t)) - \Gamma(x(t-\beta))P\mathcal{F}(x(t-\beta))\|_p \\
&\leq (M_\Gamma L_\mathcal{F} + M_\mathcal{F} L_\Gamma)\|P\|_p\|\zeta(t)\|_p, \qquad (13)
\end{aligned}$$

$$\|\tilde{\Gamma}(\zeta(t))Q\tilde{\mathcal{F}}(\zeta(t-\eta_1))\|_p \leq (M_\Gamma L_\mathcal{F} + M_\mathcal{F} L_\Gamma)\|Q\|_p \sup_{s\in[t-\eta_1,t]_\mathbb{T}}\|\zeta(s)\|_p, \qquad (14)$$

$$\left\|\tilde{\Gamma}(\zeta(t))R\int_{t-\eta_2}^{t}\tilde{\mathcal{F}}(\zeta(s))\Delta s\right\|_p \leq \eta(M_\Gamma L_\mathcal{F} + M_\mathcal{F} L_\Gamma)\|R\|_p \sup_{s\in[t-\eta_2,t]_\mathbb{T}}\|\zeta(s)\|_p \qquad (15)$$

and

$$\|\tilde{\Gamma}(\zeta(t))I\|_p \leq L_\Gamma\|I\|_p\|\zeta(t)\|_p. \qquad (16)$$

Now, from the inequalities (11), (12), (13), (14), (15) and (16), we get

$$\begin{aligned}\frac{\|Z\zeta(\sigma(t))\|_p - \|Z\zeta(t)\|_p}{\mu(t)} &\leq \frac{\|\operatorname{Id} + \mu(t)(-ZKZ^{-1})\|_p - 1}{\mu(t)} \|Z\zeta(t)\|_p + \|Z\|_p L_\Gamma \|I\|_p \|\zeta(t)\|_p \\ &\quad + \|Z\|_p (M_\Gamma L_\Upsilon + M_\Upsilon L_\Gamma)\|\zeta(t)\|_p \\ &\quad + \|Z\|_p (M_\Gamma L_{\mathcal{F}} + M_{\mathcal{F}} L_\Gamma)\|P\|_p \|\zeta(t)\|_p \\ &\quad + \|Z\|_p (M_\Gamma L_{\mathcal{F}} + M_{\mathcal{F}} L_\Gamma)\|Q\|_p \sup_{s\in[t-\eta_1,t]_{\mathbb{T}}} \|Z\zeta(s)\|_p \\ &\quad + \eta\|Z\|_p (M_\Gamma L_{\mathcal{F}} + M_{\mathcal{F}} L_\Gamma)\|R\|_p \sup_{s\in[t-\eta_2,t]_{\mathbb{T}}} \|\zeta(s)\|_p \\ &\leq (M_p(-ZKP^{-1}, \mathbb{T}) + \|Z\|_p L_\Gamma \|I\|_p \|Z^{-1}\|_p \|Z\zeta(t)\|_p \\ &\quad + \|Z\|_p (M_\Gamma L_\Upsilon + M_\Upsilon L_\Gamma)\|Z^{-1}\|_p \|Z\zeta(t)\|_p \\ &\quad + \|Z\|_p (M_\Gamma L_{\mathcal{F}} + M_{\mathcal{F}} L_\Gamma)\|P\|_p \|Z^{-1}\|_p \|Z\zeta(t)\|_p \\ &\quad + \|Z\|_p (M_\Gamma L_{\mathcal{F}} + M_{\mathcal{F}} L_\Gamma)\|Q\|_p \|Z^{-1}\|_p \sup_{s\in[t-\eta_1,t]_{\mathbb{T}}} \|Z\zeta(s)\|_p \\ &\quad + \eta\|Z\|_p (M_\Gamma L_{\mathcal{F}} + M_{\mathcal{F}} L_\Gamma)\|R\|_p \|Z^{-1}\|_p \sup_{s\in[t-\eta_2,t]_{\mathbb{T}}} \|Z\zeta(s)\|_p \\ &\leq -\mathcal{M}_1^p \|Z\zeta(t)\|_p + \mathcal{M}_2^p \sup_{s\in[t-\eta,t]_{\mathbb{T}}} \|Z\zeta(s)\|_p.\end{aligned}$$

Hence, using Definition 2.1, we get

$$D_\Delta^+ V(\zeta(t)) = -\mathcal{M}_1^p V(\zeta(t)) + \mathcal{M}_2^p \sup_{s\in[t-\eta,t]_{\mathbb{T}}} V(\zeta(s)). \tag{17}$$

Step 2: When $\mu(t) = 0$, the derivative is the classical derivative, therefore, by using the formula $x(t+h) = x(t) + x'(t)h + o(h)$ with $\lim_{h\to 0} \frac{\|o(h)\|_p}{h} = 0$, we can calculate

$$\begin{aligned}\lim_{h\to 0^+} \frac{\|Z\zeta(t+h)\|_p - \|Z\zeta(t)\|_p}{h} &= \lim_{h\to 0^+} \frac{1}{h}\Big\{\|Z\zeta(t) + hZ\zeta^\Delta(t) + o(h)\|_p - \|Z\zeta(t)\|_p\Big\} \\ &= \lim_{h\to 0^+} \frac{1}{h}\Big\{\|Z\zeta(t) + hZ(-K\zeta(t) - \tilde{\Gamma}(\zeta(t))[\tilde{\Upsilon}(\zeta(t)) - P\tilde{\mathcal{F}}(\zeta(t)) \\ &\quad - Q\tilde{\mathcal{F}}(\zeta(t-\eta_1)) - R\int_{t-\eta_2}^{t} \tilde{\mathcal{F}}(\zeta(s))\Delta s - I]) + o(h)\|_p - \|Z\zeta(t)\|_p\Big\} \\ &\leq (M_p(-ZKP^{-1}, \mathbb{T}) + \|Z\|_p L_\Gamma \|I\|_p \|Z^{-1}\|_p \|Z\zeta(t)\|_p \\ &\quad + \|Z\|_p (M_\Gamma L_\Upsilon + M_\Upsilon L_\Gamma)\|Z^{-1}\|_p \|Z\zeta(t)\|_p \\ &\quad + \|Z\|_p (M_\Gamma L_{\mathcal{F}} + M_{\mathcal{F}} L_\Gamma)\|P\|_p \|Z^{-1}\|_p \|Z\zeta(t)\|_p \\ &\quad + \|Z\|_p (M_\Gamma L_{\mathcal{F}} + M_{\mathcal{F}} L_\Gamma)\|Q\|_p \|Z^{-1}\|_p \sup_{s\in[t-\eta_1,t]_{\mathbb{T}}} \|Z\zeta(s)\|_p \\ &\quad + \eta\|Z\|_p (M_\Gamma L_{\mathcal{F}} + M_{\mathcal{F}} L_\Gamma)\|R\|_p \|Z^{-1}\|_p \sup_{s\in[t-\eta_2,t]_{\mathbb{T}}} \|Z\zeta(s)\|_p \\ &\leq -\mathcal{M}_1^p \|Z\zeta(t)\|_p + \mathcal{M}_2^p \sup_{s\in[t-\eta,t]_{\mathbb{T}}} \|Z\zeta(s)\|_p.\end{aligned}$$

Hence, using Definition 2.1 again, we get the same inequality as (17). Thus, from the above two steps, for any $t \in \mathbb{T}$, we have

$$D_\Delta^+ V(\zeta(t)) = -\mathcal{M}_1^p V(\zeta(t)) + \mathcal{M}_2^p \sup_{s\in[t-\eta,t]_{\mathbb{T}}} V(\zeta(s)).$$

Therefore, from Lemma 4.1, we get

$$V(\zeta(t)) \leq \sup_{s\in[t-\eta,t]_{\mathbb{T}}} V(\zeta(s)) e_{\ominus\lambda}(t, t_0),$$

where λ is the solution of $\lambda + \mathcal{M}_2^p \exp(\lambda\eta) \le \mathcal{M}_1^p$. Further, it is clear that

$$\begin{aligned}\|\zeta(t)\|_p &= \|Z^{-1}Z\zeta(t)\|_p \\ &\le \|Z^{-1}\|_p\|V(\zeta(t))\|_p \\ &\le \|Z^{-1}\|_p \sup_{s\in[t-\eta,t]_{\mathbb{T}}} V(\zeta(s))e_{\ominus\lambda}(t,t_0) \\ &\le Ce_{\ominus\lambda}(t,t_0),\end{aligned}$$

where $C = \|Z\|_p\|Z^{-1}\|_p \sup_{s\in[t-\eta,t]_{\mathbb{T}}} \|\zeta(s)\| > 0$. Hence, from Definition 3.1, the error dynamic (10) is exponentially stable, and hence, the drive system (1) and the response system (2) are exponentially lag-synchronized. □

Remark 4.1. *One could notice that in Theorem 4.1, the matrix Z is non-singular, so by choosing $Z = \mathrm{Id}$, the constants $\mathcal{M}_1^p$ and $\mathcal{M}_2^p$ of Theorem 4.1 become*

$$\begin{aligned}\mathcal{M}_1^p &= -(M_p(-K,\mathbb{T}) + (M_\Gamma L_\Upsilon + M_\Upsilon L_\Gamma) + (M_\Gamma L_\mathcal{F} + M_\mathcal{F} L_\Gamma)\|P\|_p + L_\Gamma\|I\|_p), \\ \mathcal{M}_2^p &= (M_\Gamma L_\mathcal{F} + M_\mathcal{F} L_\Gamma)(\|Q\|_p + \eta\|R\|_p).\end{aligned}$$

Next, we consider a particular case of the considered problem by setting $\Gamma(x(t)) = \mathrm{Id}$ and $\Upsilon(x(t)) = Ax(t)$, where $A = \mathrm{diag}\{q_1, q_2, \ldots, q_n\} \in \mathbb{R}^{n\times n}$ with $q_i > 0, i = 1, 2, \ldots, n$, then the drive system (1) and the response system (2) become

$$\begin{cases} x^\Delta(t) &= -Ax(t) + P\mathcal{F}(x(t)) + Q\mathcal{F}(x(t-\eta_1)) + R\int_{t-\eta_2}^t \mathcal{F}(x(s))\Delta s + I,\ t \in [0,\infty)_{\mathbb{T}}, \\ x(s) &= \phi(s),\ s \in [-\eta, 0]_{\mathbb{T}} \end{cases} \tag{18}$$

and

$$\begin{cases} y^\Delta(t) &= -Ay(t) + P\mathcal{F}(y(t)) + Q\mathcal{F}(y(t-\eta_1)) + R\int_{t-\eta_2}^t \mathcal{F}(y(s))\Delta s + I + u(t),\ t \in [0,\infty)_{\mathbb{T}}, \\ y(s) &= \psi(s),\ s \in [-\eta, 0]_{\mathbb{T}}, \end{cases} \tag{19}$$

respectively. Also, the error system (10) becomes

$$\zeta^\Delta(t) = -(A+K)\zeta(t) + P\hat{\mathcal{F}}(\zeta(t)) + Q\hat{\mathcal{F}}(\zeta(t-\eta_1)) + R\int_{t-\eta_2}^t \hat{\mathcal{F}}(\zeta(s))\Delta s, \tag{20}$$

where $\hat{\mathcal{F}}(\zeta(\cdot)) = \mathcal{F}(y(\cdot)) - \mathcal{F}(x(\cdot - \beta))$.

Remark 4.2. *One could have a remark similar to Remark 3.1 for the drive system (18) and the response system (19).*

Now, we will give some sufficient conditions for the exponential lag synchronization for the systems (18)-(19) as follows.

Theorem 4.2. *Let Assumption 1 hold. Then the drive system (18) and response system (19) are exponentially lag synchronized if for some $p \in \{1, 2, \infty\}$, there exist a non-singular matrix Z, a control gain matrix K such that $\mathcal{M}_3^p - \mathcal{M}_4^p > 0$ and $-\mathcal{M}_3^p \in \mathcal{R}^+$, where*

$$\begin{aligned}\mathcal{M}_3^p &= -(M_p(-Z(A+K)Z^{-1},\mathbb{T}) + \|Z\|_p\|Z^{-1}\|_p\|P\|_p L_\mathcal{F}), \\ \mathcal{M}_4^p &= \|Z\|_p\|Z^{-1}\|_p L_\mathcal{F}(\|Q\|_p + \eta\|R\|_p).\end{aligned}$$

Proof. For any non-singular matrix Z, we define

$$V(\zeta(t)) = \|Z\zeta(t)\|_p.$$

Now, consider the following two steps:
Step 1: When $\mu(t) > 0$, then for any $t \in \mathbb{T}$, we have

$$\begin{aligned}
\frac{\|Z\zeta(\sigma(t))\|_p - \|Z\zeta(t)\|_p}{\mu(t)} &= \frac{1}{\mu(t)}\Big\{\|Z\zeta(t) + \mu(t)Z\zeta^{\Delta}(t)\|_p - \|Z\zeta(t)\|_p\Big\} \\
&= \frac{1}{\mu(t)}\Big\{\Big\|Z\zeta(t) + \mu(t)Z(-(A+K)\zeta(t) + P\hat{\mathcal{F}}(\zeta(t)) + Q\hat{\mathcal{F}}(\zeta(t-\eta_1)) \\
&\quad + R\int_{t-\eta_2}^{t}\hat{\mathcal{F}}(\zeta(s))\Delta s)\Big\|_p - \|Z\zeta(t)\|_p\Big\} \\
&\leq \frac{1}{\mu(t)}\big\{\|Z\zeta(t) + \mu(t)(-Z(A+K))\zeta(t)\|_p - \|Z\zeta(t)\|_p\big\} + \|ZP\hat{\mathcal{F}}(\zeta(t))\|_p \\
&\quad + \|ZQ\hat{\mathcal{F}}(\zeta(t-\eta_1))\|_p + \Big\|ZR\int_{t-\eta_2}^{t}\hat{\mathcal{F}}(\zeta(s))\Delta s\Big\|_p \\
&\leq -\mathcal{M}_3^p\|Z\zeta(t)\|_p + \mathcal{M}_4^p \sup_{s\in[t-\eta,t]_{\mathbb{T}}}\|Z\zeta(s)\|_p.
\end{aligned}$$

Hence, from Definition 2.1, we get

$$D_{\Delta}^{+}V(\zeta(t)) = -\mathcal{M}_3^p V(\zeta(t)) + \mathcal{M}_4^p \sup_{s\in[t-\eta,t]_{\mathbb{T}}} V(\zeta(s)). \tag{21}$$

Step 2: When $\mu(t) = 0$, then using the same analysis as in Step 1, we get

$$\begin{aligned}
\lim_{h\to 0^+}\frac{\|Z\zeta(t+h)\|_p - \|Z\zeta(t)\|_p}{h} &= \lim_{h\to 0^+}\frac{1}{h}\Big\{\|Z\zeta(t) + hZ\zeta^{\Delta}(t) + o(h)\|_p - \|Z\zeta(t)\|_p\Big\} \\
&\leq \lim_{h\to 0^+}\frac{1}{h}\Big\{\|Z\zeta(t) + hZ(-(A+K)\zeta(t) + P\hat{\mathcal{F}}(\zeta(t)) \\
&\quad + Q\hat{\mathcal{F}}(\zeta(t-\eta_1)) + R\int_{t-\eta_2}^{t}\hat{\mathcal{F}}(\zeta(s))\Delta s) + o(h)\|_p - \|Z\zeta(t)\|_p\Big\} \\
&\leq -\mathcal{M}_3^p\|Z\zeta(t)\|_p + \mathcal{M}_4^p \sup_{s\in[t-\eta,t]_{\mathbb{T}}}\|Z\zeta(s)\|_p.
\end{aligned}$$

Hence, using Definition 2.1 again, we get the same inequality as (21). Thus, from the above two steps, for any $t \in \mathbb{T}$, we have

$$D_{\Delta}^{+}V(\zeta(t)) = -\mathcal{M}_3^p V(\zeta(t)) + \mathcal{M}_4^p \sup_{s\in[t-\eta,t]_{\mathbb{T}}} V(\zeta(s)).$$

Therefore, from Lemma 4.1, we get $V(\zeta(t)) \leq \sup_{s\in[t-\eta,t]_{\mathbb{T}}} V(\zeta(s))e_{\ominus\lambda}(t,t_0)$, where λ is the solution of $\lambda + \mathcal{M}_4^p\exp(\lambda\eta) \leq \mathcal{M}_3^p$. Further, it is clear that $\|\zeta(t)\|_p = \|Z^{-1}Z\zeta(t)\|_p \leq Ce_{\ominus\lambda}(t,t_0)$, where $C = \|Z\|_p\|Z^{-1}\|_p\sup_{s\in[t-\eta,t]_{\mathbb{T}}}\|\zeta(s)\| > 0$. Hence, from Definition 3.1, the error dynamics (10) is exponentially stable, and hence, the drive system (1) and the response system (2) are exponentially lag-synchronized. □

Remark 4.3. *Similar to Remark 4.1, by choosing* $Z = \mathrm{Id}$, *the constants* $\mathcal{M}_3^p$ *and* $\mathcal{M}_4^p$ *of Theorem 4.2 become*

$$\mathcal{M}_3^p = -(M_p(-(A+K),\mathbb{T}) + \|P\|_p L_{\mathcal{F}}),\ \mathcal{M}_4^p = L_{\mathcal{F}}(\|Q\|_p + \eta\|R\|_p).$$

Remark 4.4. *In the case when there is no distributed time-delay in the systems (1)-(2) (or (18)-(19)), i.e., when* $\eta_2 = 0$, *then one can establish all the above results by setting the corresponding terms to zero in the computation of the constants* $\mathcal{M}_1^p$ *and* $\mathcal{M}_2^p$ *(or* $\mathcal{M}_3^p$ *and* $\mathcal{M}_4^p$*).*

Remark 4.5. *The results of Theorem 4.1 and Theorem 4.2 cover the problem in all generality, therefore, one can obtain the results for particular time domains, such as the continuous-time domain (when $\mathbb{T} = \mathbb{R}$) and discrete-time domain (when $\mathbb{T} = \mathbb{Z}$), by replacing the matrix-measures evolves in the constants $\mathcal{M}_1^p, \mathcal{M}_2^p, \mathcal{M}_3^p$ and $\mathcal{M}_4^p$ from the known Definition 2.5.*

Remark 4.6. *For the continuous-time domain, few authors reported the synchronization results for the C-GNNs with mixed delays [1, 10, 12, 29]. Particularly, in [1], the authors considered a class of C-GNNs with mixed delays and studied the exponential lag synchronization via periodically intermittent control and mathematical induction technique. In [29], the authors studied finite-time synchronization of C-GNNs with mixed delays by using the Lyapunov-Krasovskii functional approach. Furthermore, there are only a few authors who studied the synchronization problem of the discrete-time C-GNNs [18, 31]. In particular, the authors in [18], studied the exponential synchronization results for an array of coupled discrete-time C-GNNs with time-dependent delay by applying the Lyapunov-Krasovskii functional approach while in [31], the authors investigated the existence of a bounded unique solution, exponential stability, and synchronization by using some fixed point techniques and inequality techniques.*

Remark 4.7. *All the results obtained on continuous-time [1, 10, 12, 29] and discrete-time [18, 31] C-GNNs are studied separately. The continuous-time or discrete-time C-GNNs results cannot be directly applied and easily extended to the case of arbitrary time C-GNNs. And, there is no manuscript on the continuous-time or discrete-time domain which discussed the exponential lag synchronization results for the C-GNNs with mixed delays by using the matrix-measure and Halanay inequality, therefore, the results of this manuscript are completely new even for the continuous case ($\mathbb{T} = \mathbb{R}$) and discrete case ($\mathbb{T} = \mathbb{Z}$).*

5 An Example

Example 5.1. *Consider the drive system (1) and response system (2) with the following coefficients*

$$\Gamma(x(t)) = \begin{bmatrix} 0.4 + 0.2\cos(x_1(t)) & 0.0 \\ 0.0 & 0.4 - 0.2\sin(x_2(t)) \end{bmatrix}, \quad \Upsilon(x(t)) = \begin{bmatrix} 0.3 + 0.2\sin(x_1(t)) \\ 0.3 - 0.2\cos(x_2(t)) \end{bmatrix},$$
$$P = \begin{bmatrix} 0.8 & 0.0 \\ -0.2 & -0.7 \end{bmatrix}, \quad Q = \begin{bmatrix} -0.4 & 0.1 \\ -0.2 & 0.5 \end{bmatrix}, \quad R = \begin{bmatrix} -0.5 & 0.6 \\ -0.6 & 0.5 \end{bmatrix}, \quad I = \begin{bmatrix} 0.4 \\ 0.3 \end{bmatrix},$$
$$\mathcal{F} = \begin{bmatrix} 0.8\tanh \\ 0.8\tanh \end{bmatrix}, \quad \phi(s) = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \; \psi(s) = \begin{bmatrix} -1 \\ -0.5 \end{bmatrix} \quad \textit{for } s \in [-\eta, 0]_{\mathbb{T}}, \; Z = \mathrm{Id}.$$

One can confirm that for Example 5.1, Γ, Υ, and $\mathcal{F}$ satisfy Assumption 1 with $L_\Gamma = L_\Upsilon = 0.2, L_\mathcal{F} = M_\mathcal{F} = 0.8, M_\Gamma = 0.6, M_\Upsilon = 0.5$. Now, we consider the following three different time domains as follows.
Case 1. $\mathbb{T} = \mathbb{R}$. Let $\eta_1 = 0.5, \eta_2 = 0.8$ and $\beta = 0.4$. Here, $\eta = 0.8$ and the graininess function $\mu(t) = 0$ for all $t \in \mathbb{R}$. The state trajectories and the error trajectories of the systems (1)-(2) without feedback control are shown in Figure 1 and Figure 2, respectively. Clearly, from Figure 1 and Figure 2, the drive system (1) and the response system (2) are not synchronized. However, for the control gain matrix

$$K = \begin{bmatrix} 2.2 & 0.0 \\ 0.0 & 2.2 \end{bmatrix},$$

we can calculate

$$\mathcal{M}_2^1 = 0.9472, \; \mathcal{M}_2^2 = 0.9542, \; \mathcal{M}_2^\infty = 1.0112$$

and

$$\Lambda_1(-K) = -2.2000, \; \Lambda_2(-K) = -4.4000, \; \Lambda_\infty(-K) = -2.2000.$$

Hence,

$$\mathcal{M}_1^1 = 0.7800, \quad \mathcal{M}_1^2 = 3.2242, \quad \text{and } \mathcal{M}_1^\infty = 1.0840.$$

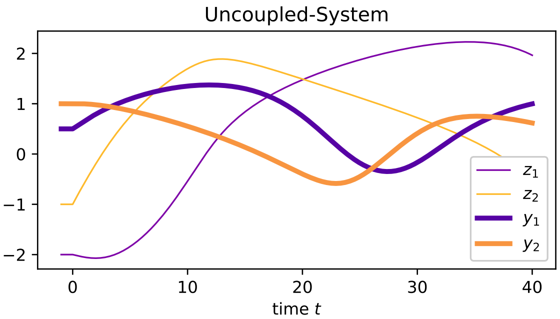

Figure 1: Uncoupled synchronization curves when $\mathbb{T} = \mathbb{R}$

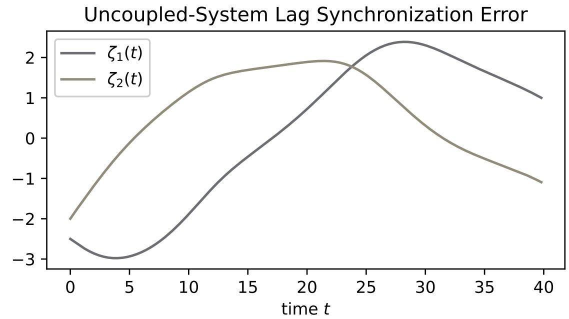

Figure 2: Uncoupled synchronization error curves when $\mathbb{T} = \mathbb{R}$

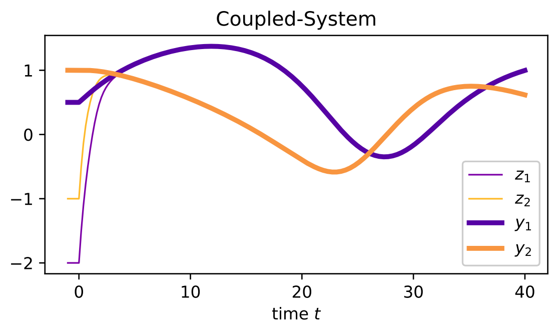

Figure 3: Coupled synchronization curves when $\mathbb{T} = \mathbb{R}$

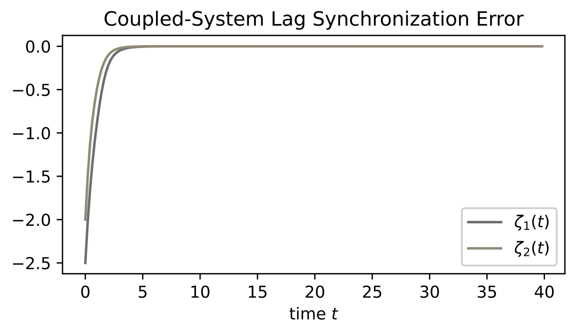

Figure 4: Coupled synchronization error curves when $\mathbb{T} = \mathbb{R}$

Therefore, we can see that $\mathcal{M}_1^1 - \mathcal{M}_2^1 = -0.1672 < 0$, $\mathcal{M}_1^2 - \mathcal{M}_2^2 = 2.2700 > 0$, and $\mathcal{M}_1^\infty - \mathcal{M}_2^\infty = 0.0728 > 0$. Also, $-\mathcal{M}_1^2, -\mathcal{M}_1^\infty \in \mathcal{R}^+$. Hence, for $p = 2, \infty$, all the conditions of Theorem 4.1 hold, and thus, the systems (1)-(2) with feedback control (3) are exponentially lag synchronized with the maximum rate of convergence for $p = 2, \infty$ are 1.0366 and 0.0394, respectively. The synchronized curves and synchronized errors curves with feedback control are shown in Figure 3 and Figure 4, respectively.

Case 2. $\mathbb{T} = 0.5\mathbb{Z}$. Let $\eta_1 = \eta_2 = \beta = 0.5$. Here, $\eta = 0.5$ and the graininess function $\mu(t) = 0.5$ for all $t \in \mathbb{R}$. The state trajectories and the error trajectories of the systems (1)-(2) without feedback control are shown in Figure 5 and Figure 6, respectively which are clearly not synchronized. However, for the control gain matrix

$$K = \begin{bmatrix} 2.0 & 0.0 \\ 0.0 & 2.0 \end{bmatrix},$$

we can calculate

$$\mathcal{M}_2^1 = 0.7360, \ \mathcal{M}_2^2 = 0.7430, \ \mathcal{M}_2^\infty = 0.8000$$

and

$$\Lambda_1(-K) = -2.000, \ \Lambda_2(-K) = -2.000, \ \Lambda_\infty(-K) = -2.000.$$

Hence,

$$\mathcal{M}_1^1 = 0.5800, \quad \mathcal{M}_1^2 = 0.8242, \quad \text{and } \mathcal{M}_1^\infty = 0.8840.$$

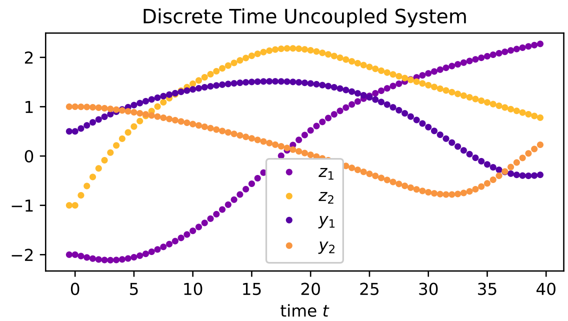

Figure 5: Uncoupled synchronization curves when $\mathbb{T} = \frac{1}{2}\mathbb{Z}$

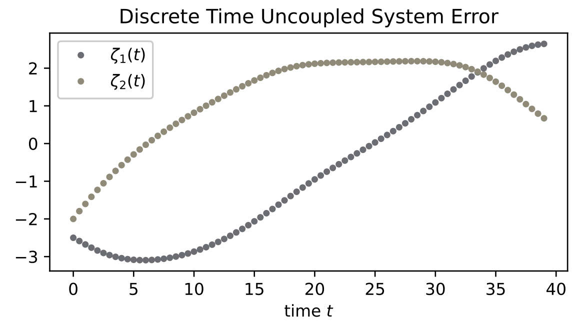

Figure 6: Uncoupled synchronization error curves when $\mathbb{T} = \frac{1}{2}\mathbb{Z}$

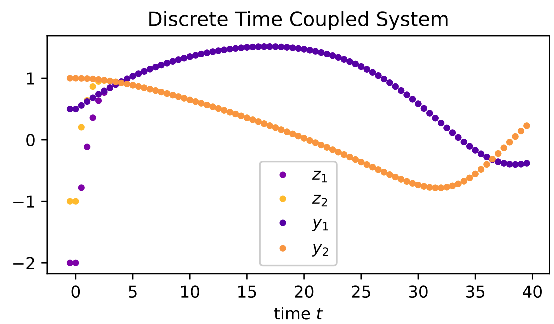

Figure 7: Coupled synchronization curves when $\mathbb{T} = \frac{1}{2}\mathbb{Z}$

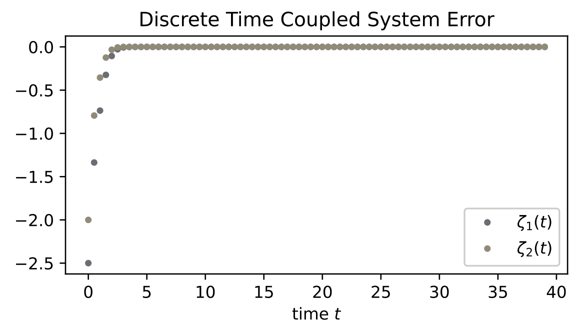

Figure 8: Coupled synchronization error curves when $\mathbb{T} = \frac{1}{2}\mathbb{Z}$

Therefore, we can see that $\mathcal{M}_1^1 - \mathcal{M}_2^1 = -0.1560 < 0$, $\mathcal{M}_1^2 - \mathcal{M}_2^2 = 0.0812 > 0$, and $\mathcal{M}_1^\infty - \mathcal{M}_2^\infty = 0.0840 > 0$. Also, $-\mathcal{M}_1^2, -\mathcal{M}_1^\infty \in \mathcal{R}^+$. Hence, for $p = 2, \infty$, all the conditions of Theorem 4.1 hold, and thus, the systems (1)-(2) with feedback control (3) are exponentially lag synchronized with the maximum rate of convergence for $p = 2, \infty$ are 0.0583 and 0.0590, respectively. The synchronized curves and synchronized errors curves with feedback control are shown in Figure 7 and Figure 8, respectively.

Case 3. $\mathbb{T} = \mathcal{P} = [-1, 0] \cup_{i=0}^{\infty} [i, i + 0.7]$. Let $\eta_1 = \eta_2 = \beta = 1$. Here, $\eta = 1$ and the graininess function $\mu(t)$ is given by

$$\mu(t) = \begin{cases} 0, \ t \in [-1, 0] \cup_{i=0}^{\infty} [i, i + 0.7), \\ 0.3, \ t = \cup_{i=0}^{\infty} \{i + 0.7\}. \end{cases}$$

The state trajectories and the error trajectories of the systems (1)-(2) without feedback control are shown in Figure 9 and Figure 10, respectively which are clearly not synchronized. However, for the control gain matrix

$$K = \begin{bmatrix} 2.4 & 0.0 \\ 0.0 & 2.4 \end{bmatrix},$$

we can calculate

$$\mathcal{M}_2^1 = 1.0880, \ \mathcal{M}_2^2 = 1.0950, \ \mathcal{M}_2^\infty = 1.1520$$

and

$$\Lambda_1(-K) = -2.4000, \ \Lambda_2(-K) = -2.4000, \ \Lambda_\infty(-K) = -2.4000.$$

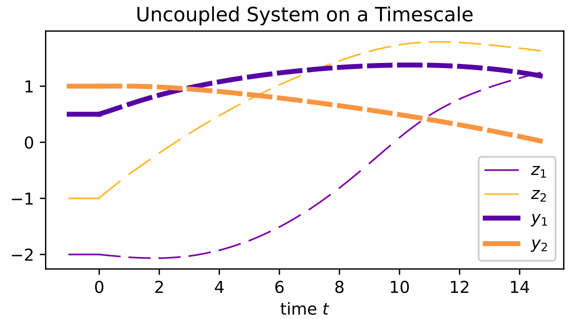

Figure 9: Uncoupled synchronization curves when $\mathbb{T} = \mathcal{P}$

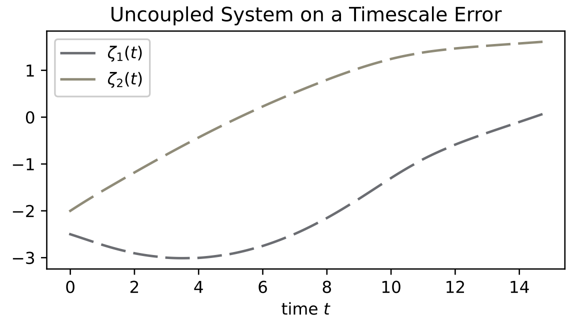

Figure 10: Uncoupled synchronization error curves when $\mathbb{T} = \mathcal{P}$

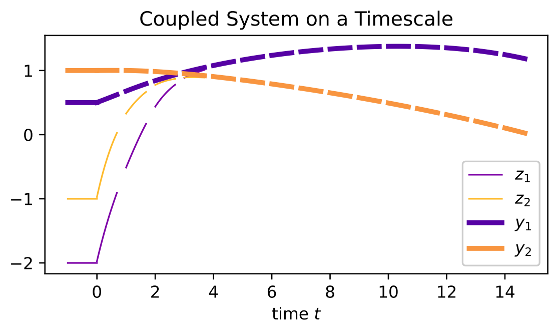

Figure 11: Coupled synchronization curves when $\mathbb{T} = \frac{1}{2}\mathbb{Z}$

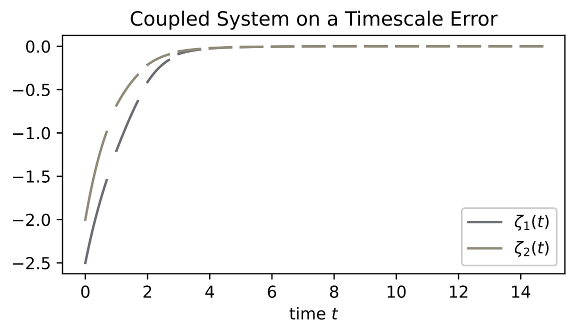

Figure 12: Coupled synchronization error curves when $\mathbb{T} = \frac{1}{2}\mathbb{Z}$

Hence,

$$\mathcal{M}_1^1 = 0.9800, \quad \mathcal{M}_1^2 = 1.2242, \quad \text{and } \mathcal{M}_1^\infty = 1.2840.$$

Therefore, we can see that $\mathcal{M}_1^1 - \mathcal{M}_2^1 = -0.1080 < 0$, $\mathcal{M}_1^2 - \mathcal{M}_2^2 = 0.1292 > 0$, and $\mathcal{M}_1^\infty - \mathcal{M}_2^\infty = 0.1320 > 0$. Also, $-\mathcal{M}_1^2, -\mathcal{M}_1^\infty \in \mathcal{R}^+$. Hence, for $p = 2, \infty$, all the conditions of Theorem 4.1 hold and thus, the systems (1)-(2) with feedback control (3) are exponentially lag synchronized with the maximum rate of convergence for $p = 2, \infty$ are 0.0602 and 0.0599, respectively. The synchronized curves and synchronized errors curves with feedback control are shown in Figure 11 and Figure 12, respectively.

Conclusion

We have successfully established the exponential lag synchronization results for a new class of C-GNNs with discrete and distributed time delays on arbitrary time domains by using the theory of time scales and feedback control law. We have also studied some special cases of the considered problem. We mainly used a unified matrix-measure theory and Halanay inequality to establish these results. The obtained results are verified by providing a simulated example for different time domains including the continuous-time domain (case 1 of Example 5.1), discrete-time domain (case 2 of Example 5.1), and non-overlapping time domain (case 3 of Example 5.1).

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