Spin contributions to the gravitational-waveform modes for spin-aligned binaries at the 3.5PN order

Quentin Henry[1,*] Sylvain Marsat,[1,‡] and Mohammed Khalil[2,†]

[1]*Max Planck Institute for Gravitational Physics (Albert Einstein Institute), D-14476 Potsdam, Germany*
[2]*Laboratoire des 2 Infinis - Toulouse (L2IT-IN2P3), Université de Toulouse,*
CNRS, UPS, F-31062 Toulouse Cedex 9, France

We complete the post-Newtonian (PN) prediction at the 3.5PN order for the spin contributions to the gravitational waveforms emitted by inspiraling compact binaries, in the case of quasicircular, equatorial orbits, where both spins are aligned with the orbital angular momentum. Using results from the multipolar post-Minkowskian wave generation formalism, we extend previous works that derived the dynamics and gravitational-wave energy flux and phasing, by computing the full waveform decomposed in spin-weighted spherical harmonics. This new calculation requires the computation of multipolar moments of higher multipolar order, new quadratic-in-spin contributions to the hereditary tail terms entering at the 3.5PN order, as well as other nonlinear interactions between moments. When specialized to the test-mass limit, our results are equivalent to those obtained in the literature for the waveform emitted by a test-mass in equatorial, circular orbits around a Kerr black hole. We also compute the factorized modes for use in effective-one-body waveform models, correcting the 2.5PN nonspinning and 3PN quadratic-in-spin terms in the (2,1) mode used in current models.

I. INTRODUCTION

Since the first gravitational-wave (GW) detection in 2015 [1], the LIGO-Virgo-KAGRA collaboration has observed over 90 GW signals from stellar-mass compact binary objects [2–4]. Future GW detectors, such as LISA [5] and the Einstein Telescope [6], will also widen the range in parameter space of detectable systems, including signals from extreme-mass-ratio inspirals and supermassive binary black holes, in addition to improving our knowledge of the deformability of neutron stars.

The use of accurate waveform templates for data analysis is crucial and requires constant improvement to match the increasing sensitivity of GW detectors. The post-Newtonian (PN) scheme is well suited to describe the inspiral of compact binaries, but its accuracy deteriorates in the strong-field regime. The gravitational self-force (GSF) framework [7–13] is valid in the strong field, but for small-mass ratios.

However, by combining different analytical approximation methods and numerical-relativity results, the effective-one-body (EOB) formalism [14,15] produces accurate waveforms over the entire parameter space.

Spin has a significant effect on the binary dynamics; thus, improving the spin description in waveform models is necessary to obtain accurate parameter estimations, which helps in improving our understanding of the properties of compact binaries and their astrophysical formation channels [16–18]. To model spin effects in general relativity, we use an effective field theory (EFT) approach, which constructs an effective action defined on the worldline of the bodies [19–35]. In the conservative sector, many approaches were used to tackle the problem of the spin contributions in the equations of motion (EOM) which now reaches the 4.5PN accuracy [33–66]. On the other hand, the 3.5PN radiated flux and phase, including all spin contributions, were derived for quasicircular orbits and nonprecessing spins [35–37,67–72]. Recently, this computation has been pushed to 4PN by EFT methods [73]. However, the polarizations for nonprecessing spins are only known to 2.5PN order [58,74,75]. In the case of precessing spins, which complicate the computations, the waveform amplitude is known to 1.5PN order, and the phase to 2PN order [74].

In the present article, we compute the spin effects in the full waveform up to the 3.5PN order. The dynamics and phase evolution are already known to that order, and the remaining piece is the computation of the amplitude of the

[*]quentin.henry@aei.mpg.de
[†]mohammed.khalil@aei.mpg.de
[‡]sylvain.marsat@l2it.in2p3.fr

waveform, decomposed as a sum of spin-weighted spherical harmonics. These harmonics complete our knowledge of the post-Newtonian waveform, and they are a crucial ingredient of waveform models such as the EOB approach [14,76–87].

At the 3.5PN order, three interactions arise: the spin-orbit (SO), the spin-induced quadrupole and the spin-induced octupole interactions. We use the post-Newtonian-multipolar-post-Minkowskian (PN-MPM) formalism [88], where the observables of the system are parametrized in terms of the so-called radiative multipole moments, defined in the radiative zone. The PN-MPM formalism allows linking these moments to another set of moments called the source multipole moments, which are defined over the source of the system and are linked to the metric and stress-energy tensor. The PN approximation is an expansion in the dimensionless quantity $v^2/c^2 \sim GM/c^2 r \ll 1$, where v is the relative velocity and M is the total mass of the binary. To make PN expansions clear, we write $1/c$ explicitly in this paper, and rescale the physical spin variable S_{physical} as

$$S = c S_{\mathrm{physical}} = G m^2 \chi, \qquad (1.1)$$

where χ is the dimensionless spin, whose value is one for an extremal Kerr black hole.

The paper is organized as follows. In Sec. II, we give some definitions and conventions, and recall the relation between the spherical modes and the radiative multipole moments. Section III contains the technical aspects regarding the computation of the radiative multipole moments. First in Sec. III A, we recall some aspects of the effective action describing spinning effects up to the 3.5PN order. In Sec. III B, we give the expression of the stress-energy tensor as well as its $3 + 1$ decomposition. Section III C gives the general definition of the source multipole moments. In Sec. III D we give the PN metric and compute the potentials that parametrize it. Section III E is dedicated to the integration of the source multipole moments using the potentials and the decomposed stress-energy tensor. In Sec. III F, we express the source moments in the center of mass (CM) in the case of aligned spins and quasicircular orbits. Finally, for this technical part, in Sec. III G, we derive the radiative multipole moments by computing the nonlinear interactions in the GW field. In Sec. IV, we give the results for the spinning contributions to the amplitude modes, written in the conventional way in Sec. IV A and factorized conveniently for EOB usage in Sec. IV B. Appendices A and B contain the lengthy expressions for the source densities and the factorized modes. We also provide our results for the waveform modes as *Mathematica* files in the Supplemental Material [89].

II. SUMMARY OF THE FORMALISM

A. Definitions and conventions

In this paper, we compute the amplitude modes of the GW emitted by a binary system of spinning compact objects in the inspiral phase. We restrict ourselves to the nonprecessing case which implies that, as for the nonspinning case, the motion of the system remains planar. Let us define the radiative coordinate system $X^\mu = (cT, X)$, in which $R = |X|$ is the distance of the observer from the source and $N = X/R$ is the direction of propagation of the GW. We denote retarded time $T_R = T - R/c$, where c is the speed of light in vacuum. We define the spatial unitary basis (e_X, e_Y, e_Z) such that the orbital plane lies within the (e_X, e_Y) plane and N in the (e_Y, e_Z) plane. The two GW polarizations h_+ and $h_\times$, defined in Eq. (2.4), propagate in the plane orthogonal to the direction of propagation N. It is convenient to introduce the orthonormal triad (P, Q, N) by choosing $P = e_X$ and $Q = N \times P$.

We use the following conventions henceforth: $\mathcal{O}(n)$ means $\mathcal{O}(1/c^n)$, i.e., represents a contribution of the order $(n/2)$PN at least. Greek indices denote spacetime coordinates, i.e., $\mu = 0, 1, 2, 3$, while Latin indices are used for spatial coordinates, i.e., $i = 1, 2, 3$. We use the multi-index notation $L = i_1 \dots i_\ell$. Symmetrization and antisymmetrization are represented by, respectively, parenthesis and brackets around indices while the symmetric trace-free (STF) projection is denoted by $\langle \rangle$. We adopt the signature $(-, +, +, +)$ and keep explicit both Newton's constant G and the speed of light c, unless explicitly specified. Finally the covariant derivative along the worldline is written as $\mathrm{D}/(c\mathrm{D}\tau) = u^\mu \nabla_\mu$, where u^μ is the four-velocity of the particle.

The problem is parametrized using the following notations: we use the coordinate-time t parametrization. The quantities m_A, y_A, $v_A = \mathrm{d}y_A/\mathrm{d}t$ and $a_A = \mathrm{d}^2 y_A/\mathrm{d}t^2$ refer respectively to the mass, position, velocity and acceleration of body $A = 1, 2$. The notation SO refers to the spin-orbit interaction, SS to the quadratic-in-spin interaction and finally SSS to the cubic-in-spin terms.

B. Spherical harmonics decomposition

The Einstein field equations can be exactly written, by imposing the harmonic gauge condition $\partial_\nu h^{\mu\nu} = 0$, as

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4} \tau^{\mu\nu}, \qquad (2.1)$$

where $\Box$ is the flat d'Alembertian operator defined with respect to the inverse Minkowski metric $\eta^{\mu\nu}$, $h^{\mu\nu}$ is the deviation to the gothic metric $h^{\mu\nu} = \sqrt{-g} g^{\mu\nu} - \eta^{\mu\nu}$, $g = \det(g_{\mu\nu})$ is the determinant of the metric. Finally, $\tau^{\mu\nu}$ is the stress-energy pseudo tensor

$$\tau^{\mu\nu} = |g| T^{\mu\nu} + \frac{c^4}{16\pi G} \Lambda^{\mu\nu}, \qquad (2.2)$$

where $T^{\mu\nu}$ is the stress-energy tensor and $\Lambda^{\mu\nu}$ is a function of derivatives of at least quadratic terms in the perturbed metric $h^{\mu\nu}$. Its expression is given in Eq. (24) of Ref. [88].

In the coordinate system $X^\mu = (cT, \boldsymbol{X})$, the transverse-traceless (TT) projection h_{ij}^{TT} of the gravitational field $h^{\mu\nu}$ can be, at leading order in $1/R$, uniquely decomposed in terms of a set of STF multipole moments U_L and V_L, called radiative multipole moments, as [90]

$$
h_{ij}^{\mathrm{TT}} = \frac{4G}{c^2 R} \mathcal{P}_{ijkl}(\boldsymbol{N}) \sum_{\ell=2}^{+\infty} \frac{1}{c^\ell \ell!} \Big\{ N_{L-2} U_{klL-2}(T_R)
$$
$$
- \frac{2\ell}{c(\ell+1)} N_{aL-2} \varepsilon_{ab(k} V_{l)bL-2}(T_R) \Big\}, \qquad (2.3)
$$

where we introduced the TT projection operator $\mathcal{P}_{ijkl} = \mathcal{P}_{i(k}\mathcal{P}_{l)j} - \frac{1}{2}\mathcal{P}_{ij}\mathcal{P}_{kl}$, with $\mathcal{P}_{ij} = \delta_{ij} - N_i N_j$ the projector orthogonal to the unit direction $\boldsymbol{N}$. We can define the usual polarization waveforms in the orthonormal triad $(\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{N})$ as

$$
h_+ = \frac{1}{2}(P_i P_j - Q_i Q_j) h_{ij}^{\mathrm{TT}}, \qquad (2.4a)
$$

$$
h_\times = \frac{1}{2}(P_i Q_j + Q_i P_j) h_{ij}^{\mathrm{TT}}. \qquad (2.4b)
$$

We can then decompose the quantity $h_+ - i h_\times$ in a spin-weighted spherical harmonics basis of weight -2 [91]

$$
h \equiv h_+ - i h_\times = \sum_{l=0}^{\infty} \sum_{m=-\ell}^{\ell} h_{\ell m} Y_{-2}^{\ell m}(\Theta, \Phi), \qquad (2.5)
$$

where the two angles (Θ, Φ) characterize the direction of propagation $\boldsymbol{N}$ in the coordinate system X^μ. In this paper, we follow the conventions of Ref. [92]. Notably, the explicit expression of the spin-weighted spherical harmonics are given in Eqs. (2.4) of that reference. The amplitude modes $h_{\ell m}$ are then linked to the radiative moments U_L and V_L through

$$
h_{\ell m} = -\frac{G}{\sqrt{2} R c^{\ell+2}} \left(U^{\ell m} - \frac{i}{c} V^{\ell m} \right). \qquad (2.6)
$$

where $U^{\ell m}$ and $V^{\ell m}$ read[1]

$$
U^{\ell m} = \frac{4}{\ell!} \sqrt{\frac{(\ell+1)(\ell+2)}{2\ell(\ell-1)}} \alpha_L^{\ell m} U_L, \qquad (2.7a)
$$

$$
V^{\ell m} = -\frac{8}{\ell!} \sqrt{\frac{\ell(\ell+2)}{2(\ell+1)(\ell-1)}} \alpha_L^{\ell m} V_L. \qquad (2.7b)
$$

---

[1] Note that the choice of definitions in Refs. [79,91] on $h_{\ell m}$ differs from ours by a global minus sign for each ℓ and m due to a different definition of the vector basis $(\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{N})$.

TABLE I.   Leading and required (relative to the leading) orders of the spin contributions in the radiative moments for the full gravitational waveform at 3.5PN. The order corresponds to the power in $1/c$ and not the PN order.

		Leading order			Required relative order		
ℓ	Moment	SO	SS	SSS	SO	SS	SSS
2	U_{ij}	3	4	7	4	3	0
3	U_{ijk}	3	4	7	3	2	$\cdots$
4	U_{ijkl}	3	4	7	2	1	$\cdots$
5	U_{ijklm}	3	4	7	1	0	$\cdots$
6	U_{ijklmp}	3	4	7	0	$\cdots$	$\cdots$
2	V_{ij}	1	4	5	5	2	1
3	V_{ijk}	1	4	5	4	1	0
4	V_{ijkl}	1	4	5	3	0	$\cdots$
5	V_{ijklm}	1	4	5	2	$\cdots$	$\cdots$
6	V_{ijklmp}	1	4	5	1	$\cdots$	$\cdots$
7	$V_{ijklmpq}$	1	4	5	0	$\cdots$	$\cdots$

The STF tensorial coefficient $\alpha_L^{\ell m} \equiv \int d\Omega\, \hat{N}_L \bar{Y}^{\ell m}$ is defined from the ordinary spherical harmonics $Y^{\ell m}$ (or in fact its complex conjugate $\bar{Y}^{\ell m}$) and its explicit expression is given in Eq. (4.7) of Ref. [93].

The main result of this paper is the spin contributions in h up to the 3.5PN order for a nonprecessing, quasicircular motion. In order to derive them, we have to compute the spin part of the radiative moments to consistent order. Table I shows at which order the different multipoles need to be computed for each spin interaction. The SO terms in the radiated flux and GW phase have been derived in the PN-MPM formalism up to 4PN while the SS contributions are known to 3.5PN [35,70–72], however the radiative moments are required to a higher order for the modes than for the flux. This can be understood by comparing the expression (2.3) of the full waveform, with a $1/c$ increment for each multipolar order, to the expression of the energy flux emitted in gravitational waves (e.g., (4.2) in Ref. [72]) as sum of squares of radiative moments with a $1/c^2$ increment per multipolar order.

III. COMPUTATION OF THE RADIATIVE MULTIPOLE MOMENTS

In this paper, we use the PN-MPM formalism, see Ref. [88] for a review. Within this approach, the procedure to derive the radiative multipole moments is well known. For completeness, we give a summarized version of the method, recalling the different steps: (i) we first compute the stress-energy tensor from the effective skeletonized action modeling spinning particles in Sec. III A; (ii) in Sec. III B, we decompose the spatial and temporal indices of the stress-energy tensor to compute the source densities; (iii) in Sec. III D, we compute the PN metric parametrized by a set of potentials; (iv) in Sec. III E, we derive the source multipole moments defined in Sec. III C; (v) we then

express the source moments in the CM frame for aligned spins and quasicircular orbits in Sec. III F; (vi) finally, we compute the nonlinear effects to deduce the radiative multipole moments in Sec. III G.

A. Effective skeletonized action of spinning particles

In this section, we start by recalling the method of the effective action approach to the spin-induced effects. We follow here the presentation of [35,72], but note that a more general formalism in the language of effective field theories can be found in Ref. [33]. Here, we drop the particle's label A and set $c = 1$ for more clarity.

The matter action is constructed on the worldline of the individual body z^μ. The affine parameter along this worldline is called τ and will be associated after variation of the action to the proper time of the body. We define the usual 4-velocity of the particle $u^\mu = dz^\mu/d\tau$ such that $u^\mu u_\mu = -1$. Attaching an orthonormal tetrad $\epsilon_A{}^\mu$ to the moving body, we also introduce antisymmetric rotation coefficients $\Omega^{\mu\nu} = \epsilon^{A\mu} D\epsilon_A{}^\nu/d\tau$ representing the rotational degrees of freedom. The matter action then reads symbolically

$$S_M = \int d^4x \sqrt{-g} \int d\tau\, L_M[u^\mu, \Omega^{\mu\nu}, g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \nabla_\lambda R_{\mu\nu\rho\sigma}]$$
$$\times \frac{\delta^{(4)}(x^\mu - z^\mu(\tau))}{\sqrt{-g}}, \tag{3.1}$$

where the matter Lagrangian includes finite-size effects in the form of couplings to the Riemann tensor and its derivative, necessary to represent spin-induced effects up to the cubic order in spin. We also introduce the following quantities

$$p_\mu \equiv \frac{\partial L}{\partial u^\mu}, \qquad S_{\mu\nu} \equiv 2\frac{\partial L}{\partial \Omega^{\mu\nu}},$$
$$J^{\mu\nu\rho\sigma} \equiv -6\frac{\partial L}{\partial R_{\mu\nu\rho\sigma}}, \qquad J^{\lambda\mu\nu\rho\sigma} \equiv -12\frac{\partial L}{\partial \nabla_\lambda R_{\mu\nu\rho\sigma}}, \tag{3.2}$$

that represent the couplings entering the action. The quantities p^μ and $S^{\mu\nu}$ are the linear momentum and the antisymmetric spin tensor. The moments $J^{\mu\nu\rho\sigma}$ and $J^{\lambda\mu\nu\rho\sigma}$ are Dixon-type moments [23,24,27] representing the quadrupole and octupole, assumed to be purely spin-induced in our case (no other finite-size effects such as tidal effects). The general form of the stress-energy tensor will be expressed in terms of these quantities in the following section.

Varying the action with respect to the worldline and rotational degrees of freedom (together with a scalarity condition for the Lagrangian, see (2.8) in Ref. [35]) yields evolution equations for the 4-momentum as well as the spin tensor:

$$\frac{Dp_\mu}{D\tau} = -\frac{1}{2}R_{\mu\nu\rho\sigma}u^\nu S^{\rho\sigma} - \frac{1}{6}J^{\lambda\nu\rho\sigma}\nabla_\mu R_{\lambda\nu\rho\sigma}$$
$$- \frac{1}{12}J^{\tau\lambda\nu\rho\sigma}\nabla_\mu\nabla_\tau R_{\lambda\nu\rho\sigma} + \mathcal{O}(S^4), \tag{3.3a}$$

$$\frac{DS^{\mu\nu}}{D\tau} = 2p^{[\mu}u^{\nu]} + \frac{4}{3}R^{[\mu}{}_{\lambda\rho\sigma}J^{\nu]\lambda\rho\sigma} + \frac{2}{3}\nabla^\lambda R^{[\mu}{}_{\tau\rho\sigma}J_\lambda^{\nu]\tau\rho\sigma}$$
$$+ \frac{1}{6}\nabla^{[\mu}R_{\lambda\tau\rho\sigma}J^{\nu]\lambda\tau\rho\sigma} + \mathcal{O}(S^4). \tag{3.3b}$$

However, the introduction of the spin tensor adds 3 degrees of freedom to the problem. In order to close the system of equations describing the dynamics, we impose a spin supplementary condition (SSC). In particular we choose to impose the Tulczyjew-Dixon SSC [22,27] which reads

$$S^{\mu\nu}p_\nu = 0. \tag{3.4}$$

Note that the mass $p_\mu p^\mu = -\tilde{m}^2$ is not conserved at $\mathcal{O}(S^2)$. Together with the equations of motion, the SSC allows us to find a conserved mass, that we use henceforth in this paper,

$$m \equiv \tilde{m} - \frac{1}{6}R_{\mu\nu\rho\sigma}J^{\mu\nu\rho\sigma}, \tag{3.5}$$

and to relate p^μ and the 4-velocity as

$$p^\mu = mu^\mu + \frac{1}{6}u^\mu R_{\rho\lambda\mu\nu}J^{\rho\lambda\mu\nu} - \frac{1}{2m}S^{\mu\nu}R_{\nu\lambda\rho\sigma}u^\lambda S^{\rho\sigma}$$
$$+ \frac{4}{3}R^{[\mu}{}_{\lambda\rho\sigma}J^{\nu]\lambda\rho\sigma}u_\nu + \frac{2}{3}u_\nu\nabla^\lambda R^{[\mu}{}_{\tau\rho\sigma}J_\lambda{}^{\nu]\tau\rho\sigma}$$
$$+ \frac{1}{6}u_\nu\nabla^{[\mu}R_{\lambda\tau\rho\sigma}J^{\nu]\lambda\tau\rho\sigma} - \frac{1}{6m}S^{\mu\nu}J^{\lambda\tau\rho\sigma}\nabla_\nu R_{\lambda\tau\rho\sigma}$$
$$+ \mathcal{O}(S^4). \tag{3.6}$$

The norm of the spin tensor

$$s^2 \equiv \frac{1}{2}S_{\mu\nu}S^{\mu\nu}, \tag{3.7}$$

is conserved at the spin-cubic order that we consider: $ds/d\tau = 0$.

Imposing that the quadrupole and octupole moments are purely spin-induced with the appropriate symmetries, we find

$$J^{\mu\nu\rho\sigma} = \frac{3\kappa}{m}u^{[\mu}S^{\nu]\lambda}S_\lambda{}^{[\rho}u^{\sigma]}, \tag{3.8a}$$

$$J^{\lambda\mu\nu\rho\sigma} = \frac{\lambda}{4m^2}[\Theta^{\lambda[\mu}u^{\nu]}S^{\rho\sigma} + \Theta^{\lambda[\rho}u^{\sigma]}S^{\mu\nu}$$
$$- \Theta^{\lambda[\mu}S^{\nu][\rho}u^{\sigma]} - \Theta^{\lambda[\rho}S^{\sigma][\mu}u^{\nu]} - S^{\lambda[\mu}\Theta^{\nu][\rho}u^{\sigma]}$$
$$- S^{\lambda[\rho}\Theta^{\sigma][\mu}u^{\nu]}], \tag{3.8b}$$

where we used the notation $\Theta^{\mu\nu} = S^{\mu\lambda}S^{\nu}{}_{\lambda}$, with quadrupolar and octupolar parameters κ, λ normalized to unity for a Kerr black hole.

With the SSC (3.4), one can construct a spin vector S^{μ} from the spin tensor $S^{\mu\nu}$, that also has a conserved norm by construction. They are linked using the orthogonal unit vector $p^{\mu}/\tilde{m}$ (the mass correction (3.5) is irrelevant at our PN order in spin terms) through the relations

$$S^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} \frac{p_{\rho}}{\tilde{m}} S_{\sigma}, \tag{3.9a}$$

$$S^{\mu} = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \frac{p_{\nu}}{\tilde{m}} S_{\rho\sigma}, \tag{3.9b}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita tensor. One can further construct a 3-dimensional spin vector with conserved Euclidean norm S (see Sec. II C in Ref. [72]), which we use in our final results.

For the two-body problem, the equations of motion (EOM) of the individual bodies as well as the precession evolution of the spins have been derived in harmonic coordinates in Refs. [33,35,51,60,72] and known at the considered PN order of the problem so that we do not need to solve them here. The precession equation in terms of the 3-dimensional spin vector reads $\dot{\mathbf{S}} = \mathbf{\Omega} \times \mathbf{S}$ where $\mathbf{\Omega}$ starts at $\mathcal{O}(2)$. We recall that in our problem, we have rescaled the spin variable as in Eq. (1.1) to explicitly write the c factors.

B. Effective stress-energy tensor and source densities

Equations (2.23)–(2.25) of Ref. [35] give the general form of the stress-energy tensor in terms of p^{μ}, $S^{\mu\nu}$, $J^{\mu\nu\rho\sigma}$ and $J^{\lambda\mu\nu\rho\sigma}$. One can rewrite this expression as

$$T^{\mu\nu} = \sum_{A} [U_A^{\mu\nu} \delta_A + \nabla_{\alpha}(U_A^{\mu\nu\alpha}\delta_A) + \nabla_{\alpha}\nabla_{\beta}(U_A^{\mu\nu\alpha\beta}\delta_A) + \nabla_{\alpha}\nabla_{\beta}\nabla_{\gamma}(U_A^{\mu\nu\alpha\beta\gamma}\delta_A)], \tag{3.10}$$

where $\delta_A \equiv \delta^{(3)}[\mathbf{x} - \mathbf{y}_A(t)]$ is the usual three-dimensional Dirac distribution. In our case, the expression of the U's, up to the octupolar level, are given by

$$U_A^{\mu\nu} = \frac{1}{u_A^0 \sqrt{-g}} \left(p_A^{(\mu} u_A^{\nu)} + \frac{1}{3} R_A{}^{(\mu}{}_{\lambda\rho\sigma} J_A^{\nu)\lambda\rho\sigma} + \frac{1}{6} \nabla_{\lambda} R_A{}^{(\mu}{}_{\xi\rho\sigma} J_A^{\lambda\nu)\xi\rho\sigma} + \frac{1}{12} \nabla^{(\mu} R_{A\xi\tau\rho\sigma} J_A^{\nu)\xi\tau\rho\sigma} \right), \tag{3.11a}$$

$$U_A^{\mu\nu\alpha} = \frac{1}{3u_A^0 \sqrt{-g}} \left(3u_A^{(\mu} S_A^{\nu)\alpha} - \frac{1}{2} R_A{}^{(\mu}{}_{\xi\lambda\sigma} J_A^{\alpha\nu)\xi\lambda\sigma} - R_A{}^{(\mu}{}_{\xi\lambda\sigma} J_A^{\nu)\alpha\xi\lambda\sigma} + R_A{}^{\alpha}{}_{\xi\lambda\sigma} J_A^{(\mu\nu)\xi\lambda\sigma} \right), \tag{3.11b}$$

$$U_A^{\mu\nu\alpha\beta} = -\frac{2}{3u_A^0 \sqrt{-g}} J_A^{\alpha(\mu\nu)\beta}, \tag{3.11c}$$

$$U_A^{\mu\nu\alpha\beta\gamma} = -\frac{1}{3u_A^0 \sqrt{-g}} J_A^{\gamma\beta(\mu\nu)\alpha}, \tag{3.11d}$$

where $u_A^0 = \mathrm{d}t/\mathrm{d}\tau_A$. The goal of this section is to decompose the spatial and temporal indices of the stress-energy tensor in order to make the factors in c explicit. After restoring the c powers in the expressions, we first express the covariant derivatives in terms of the partial derivatives and the Christoffel symbols, then separate spatial and temporal indices. The practical formulas to perform this decomposition are given in Sec. II B. of Ref. [94]. Then, the source densities are defined by

$$\sigma = \frac{T^{00} + T^{ii}}{c^2}, \qquad \sigma_i = \frac{T^{0i}}{c}, \qquad \sigma_{ij} = T^{ij}. \tag{3.12}$$

These quantities source the PN potentials as well as the source multipole moments. We provide in Eqs. (A1a) the values of their spin contribution. As we will see later on, σ is required up to $\mathcal{O}(7)$, σ_i to $\mathcal{O}(6)$ and σ_{ij} to $\mathcal{O}(4)$.

C. Source multipole moments

In the PN-MPM approach, a crucial step to compute the radiative moments is the derivation of the so-called source multipole moments, which are expressed as integrals over the source. The mass and current source moments are given by [88]

$$I_L(t) = \operatorname*{FP}_{B=0} \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \int_{-1}^{1} \mathrm{d}z \left\{ \delta_{\ell} \hat{x}_L \Sigma - \frac{4(2\ell+1)}{c^2(\ell+1)(2\ell+3)} \delta_{\ell+1} \hat{x}_{iL} \Sigma_i^{(1)} \right. + \frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)} \delta_{\ell+2} \hat{x}_{ijL} \Sigma_{ij}^{(2)} \right\} (\mathbf{x}, t + z\, r/c), \tag{3.13a}$$

$$J_L(t) = \underset{B=0}{\text{FP}} \, \varepsilon_{ab<i_\ell} \int d^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \int_{-1}^{1} dz \Big\{ \delta_\ell \hat{x}_{L-1>a} \Sigma_b$$

$$- \frac{2\ell+1}{c^2(\ell+2)(2\ell+3)} \delta_{\ell+1} \hat{x}_{L-1>ac} \Sigma_{bc}^{(1)} \Big\}(\mathbf{x}, t + zr/c), \tag{3.13b}$$

where we have $\delta_\ell(z) = a_\ell(1-z^2)^\ell$ and $a_\ell = (2\ell+1)!!/2^{\ell+1}\ell!$ and

$$\Sigma = \frac{\tau^{00} + \tau^{ii}}{c^2}, \qquad \Sigma_i = \frac{\tau^{0i}}{c}, \qquad \Sigma_{ij} = \tau^{ij}. \tag{3.14}$$

The explicit expressions of the source moments in terms of the source densities and the PN metric are given in Eqs. (4.7) of Ref. [95]. Furthermore, the computation of the general expressions of the source moments requires the introduction of a regularization. In this paper, we use the Hadamard *partie finie* regularization, denoted by FP in which we introduce the scale constant r_0 [96]. The integral over z can be easily integrated after performing a PN expansion

$$\int_{-1}^{1} dz \, \delta_\ell(z) \Sigma(\mathbf{x}, t + zr/c)$$

$$= \sum_{k=0}^{+\infty} \frac{(2\ell+1)!!}{(2k)!!(2\ell+2k+1)!!} \left(\frac{r}{c}\right)^{2k} \Sigma^{(2k)}(\mathbf{x}, t). \tag{3.15}$$

For the computation of the radiative moments at the order we aim at, we also need to compute another set of moments called gauge moments (W_L, X_L, Y_L, Z_L). They are required to low order and admit similar definitions that can be found, e.g., in Eqs. (125) of Ref. [88].

From Eq. (2.2), we see for example that $\Sigma_{ij} = |g|\sigma_{ij} + \frac{c^4}{16\pi G}\Lambda_{ij}$. This means that the multipoles are sourced by the densities derived in the previous section, as well as the nonlinearities of the gravitational field $\Lambda^{\mu\nu}$ through the PN metric. The relation between the source and radiative moments, given in Sec. III G, shows that the source moments are to be computed at the same orders as the ones displayed in Table I. This implies that, looking at (3.13), we require Σ up to $\mathcal{O}(7)$, Σ_i to $\mathcal{O}(6)$ and Σ_{ij} to $\mathcal{O}(4)$, which is why the source densities were required to these orders.

D. Post-Newtonian metric and potentials

In the post-Newtonian framework, the metric is parametrized by a set of elementary retarded-type potentials that satisfy sourced wave equations. In this problem, we need the 3PN metric, which is given by [97]

$$g_{00} = -1 + \frac{2}{c^2}V - \frac{2}{c^4}V^2 + \frac{8}{c^6}\left(\hat{X} + V_kV_k + \frac{V^3}{6}\right)$$

$$+ \frac{16}{c^8}\left(-\frac{V^4}{24} - VV_kV_k - V\hat{X} + 2\hat{R}_kV_k + 2\hat{T}\right)$$

$$+ \mathcal{O}(10), \tag{3.16a}$$

$$g_{0i} = -\frac{4}{c^3}V_i - \frac{8}{c^5}\hat{R}_i - \frac{8}{c^7}(V^2V_i + V_k\hat{W}_{ik} + 2\hat{Y}_i) + \mathcal{O}(9), \tag{3.16b}$$

$$g_{ij} = \delta_{ij}\left[1 + \frac{2}{c^2}V + \frac{2}{c^4}V^2 + \frac{8}{c^6}\left(\hat{X} + V_kV_k + \frac{V^3}{6}\right)\right]$$

$$+ \frac{4}{c^4}\hat{W}_{ij} + \frac{8}{c^6}(-2V_iV_j + V\hat{W}_{ij} + 2\hat{Z}_{ij}) + \mathcal{O}(8), \tag{3.16c}$$

where the potentials are defined below. Note that an extension of this parametrization is known at 4PN together with the definition of the potentials required to this order. They are displayed in Appendix A of Ref. [98].

However, in the present paper, we do not need to use the full metric. Instead, we need to know the source of the multipole moments to consistent order. It turns out that the only potentials that have a nonzero contribution to the spin effects in the source multipole moments are the following

$$V = \Box_\mathcal{R}^{-1}[-4\pi G\sigma], \tag{3.17a}$$

$$V_i = \Box_\mathcal{R}^{-1}[-4\pi G\sigma_i], \tag{3.17b}$$

$$\hat{W}_{ij} = \Box_\mathcal{R}^{-1}[-4\pi G(\sigma_{ij} - \delta_{ij}\sigma_{kk}) - \partial_i V\partial_j V], \tag{3.17c}$$

$$\hat{R}_i = \Box_\mathcal{R}^{-1}\left[-4\pi G(V\sigma_i - V_i\sigma) - 2\partial_k V\partial_i V_k - \frac{3}{2}\partial_t V\partial_i V\right], \tag{3.17d}$$

$$\hat{X} = \Box_\mathcal{R}^{-1}\left[-4\pi G \, V\sigma_{ii} + \hat{W}_{ij}\partial_{ij}V + 2V_i\partial_t\partial_i V \right.$$

$$\left. + V\partial_t^2 V + \frac{3}{2}(\partial_t V)^2 - 2\partial_i V_j\partial_j V_i\right], \tag{3.17e}$$

TABLE II. Orders at which the potentials are required for the spin contributions in the full gravitational waveform at 3.5PN. LO refers to the leading order of the contribution of the interaction while RO refers to the relative order required.

Potential	SO		SO	
	LO	RO	LO	RO
V	3	2	0	0
V_i	1	3	0	0
$\hat{W}_{ij}$	1	2	…	…
$\hat{R}_i$	1	1	…	…
$\hat{X}$	1	0	…	…
$\hat{Z}_{ij}$	1	0	…	…

$$
\hat{Z}_{ij} = \Box_{\mathcal{R}}^{-1} \Big[-4\pi G(\sigma_{ij} - \delta_{ij}\sigma_{kk})V - 2\partial_{(i}V\partial_t V_{j)} + \partial_i V_k \partial_j V_k
$$
$$
+ \partial_k V_i \partial_k V_j - 2\partial_{(i}V_k \partial_k V_{j)}
$$
$$
- \delta_{ij}\partial_k V_l (\partial_k V_l - \partial_l V_k) - \frac{3}{4}\delta_{ij}(\partial_t V)^2 \Big], \qquad (3.17\text{f})
$$

where $\Box_{\mathcal{R}}^{-1}$ refers to the retarded flat d'Alembertian. More specifically, we are interested in the spin contributions to these potentials as their point-particle part are already known. As we can see, the potentials are sourced by the source densities derived in the previous section as well as other simpler potentials.

To find the specific orders at which we require the potentials, one has to look at the order at which the potentials enter in Σ, Σ_i, and Σ_{ij} including also the order of appearance in the source densities σ, σ_i, and σ_{ij}. The orders required for each interaction are displayed in Table II.

Note that the SSS interaction does not appear in Table II because it will play no role in the expressions of the source moments. More details regarding this affirmation are given in the following section. The regularization scheme used to compute the potentials is the Hadamard regularization [99]. The methods used to compute the spin part of the potentials to these orders are detailed in, e.g., Sec. IV of Ref. [50]. An interesting feature of the SS interaction is that, as explained in Ref. [72], the potentials V and V_i contain distributional terms which are crucial to take into account for the computation of the source multipole moments.

E. Computation of the source multipole moments

This step in the overall project is the most technical one. However, no new methods were required and we used the same computational techniques as in, e.g., Refs. [50,72,98,100]. Thus, we refer to these articles for more details.

As discussed above, the Σ's are composed of the source densities and the nonlinearities of the gravitational field whose dependency comes through the potentials. Thus, the multipole moments are integrals sourced by three types of terms: the compact terms, the noncompact terms and the surface terms. Note that the sources of the moments can be written in different ways. For example a term of the form $\int d^3\mathbf{x}\, \hat{x}_L \partial_k V \partial_k V$ can be rewritten, dropping here the FP notation, as

$$
\int d^3\mathbf{x}\, \hat{x}_L \partial_k V \partial_k V = -4\pi G \int d^3\mathbf{x} \hat{x}_L \sigma V
$$
$$
+ \frac{1}{2}\int d^3\mathbf{x}\hat{x}_L \Delta(V^2) + \mathcal{O}(2). \quad (3.18)
$$

We see that the left-hand side, which is a noncompact term, can be turned into a sum of a compact and a so-called surface term.[2] The multipole moments have been independently computed and double checked using different formulations for their sources. It is crucial to take into account the distributional derivatives induced in the left-hand side of (3.18) in order to recover the value of the right-hand side.

At this stage, after integrating Eqs. (3.13) for required ℓ to consistent order, we obtain the expressions of the source moments in a general frame for arbitrary orbits. In particular, we assumed neither the aligned-spin nor the quasicircular orbits conditions. The lengthy expressions of the source moments are not displayed in this paper, however we recall here their leading PN order for the spin contributions [35]

$$
(I_L)_{\text{NS}} = m_1 y_1^{\langle L \rangle} + 1 \leftrightarrow 2 + \mathcal{O}(2), \qquad (3.19\text{a})
$$

$$
(J_L)_{\text{NS}} = y_1^a v_1^b \varepsilon^{ab\langle i_\ell} y_1^{L-1\rangle} + 1 \leftrightarrow 2 + \mathcal{O}(2), \qquad (3.19\text{b})
$$

$$
(I_L)_{\text{SO}} = \frac{2\ell}{c^3(\ell+1)}[\ell v_1^a S_1^b \varepsilon^{ab\langle i_\ell} y_1^{L-1\rangle}
$$
$$
- (\ell-1)y_1^a S_1^b \varepsilon^{ab\langle i_\ell} v_1^{i_{\ell-1}} y_1^{L-2\rangle}] + 1
$$
$$
\leftrightarrow 2 + \mathcal{O}(5), \qquad (3.19\text{c})
$$

$$
(J_L)_{\text{SO}} = \frac{\ell+1}{2c}S_1^{\langle i_\ell} y_1^{L-1\rangle} + 1 \leftrightarrow 2 + \mathcal{O}(3), \qquad (3.19\text{d})
$$

$$
(I_L)_{\text{SS}} = -\frac{\ell(\ell-1)\kappa_1}{2m_1 c^4}S_1^{\langle i_\ell} S_1^{i_{\ell-1}} y_1^{L-2\rangle} + 1 \leftrightarrow 2 + \mathcal{O}(6),
$$
$$
(3.19\text{e})
$$

---

[2]The second term of the right-hand side is called a surface term because we only need to know the expansion in r of V^2 when $r \to \infty$ to compute it.

$$(J_L)_{\mathrm{SS}} = \frac{(\ell-1)\kappa_1}{2m_1c^4}[2v_1^a S_1^b \varepsilon^{ab\langle i_\ell} S_1^{i_{\ell-1}} y_1^{L-2\rangle}$$
$$- (\ell-2)y_1^a v_1^b \varepsilon^{ab\langle i_\ell} S_1^{i_{\ell-1}} S_1^{i_{\ell-2}} y_1^{L-3\rangle}]$$
$$+ 1 \leftrightarrow 2 + \mathcal{O}(6), \tag{3.19f}$$

$$(I_L)_{\mathrm{SSS}} = \frac{\ell(\ell-1)(\ell-2)\lambda_1}{3(\ell+1)m_1^2c^7}[-\ell v_1^a S_1^b \varepsilon^{ab\langle i_\ell} S_1^{i_{\ell-1}} S_1^{i_{\ell-2}} y_1^{L-3\rangle}$$
$$+ (\ell-3)y_1^a S_1^b \varepsilon^{ab\langle i_\ell} v_1^{i_{\ell-1}} S_1^{i_{\ell-1}} S_1^{i_{\ell-2}} v_1^{i_{\ell-3}} y_1^{L-4\rangle}]$$
$$+ 1 \leftrightarrow 2 + \mathcal{O}(8), \tag{3.19g}$$

$$(J_L)_{\mathrm{SSS}} = -\frac{(\ell+1)(\ell-1)(\ell-2)\lambda_1}{12m_1^2c^5}S_1^{\langle i_\ell} S_1^{i_{\ell-1}} S_1^{i_{\ell-2}} y_1^{L-3\rangle}$$
$$+ 1 \leftrightarrow 2 + \mathcal{O}(6), \tag{3.19h}$$

where NS refers to the nonspinning contributions. As stated in the previous section, the SSS interaction plays no role at the level of the radiative moments. Indeed, the only mass-type moment in which the SSS appears is for $\ell=2$ which vanishes due to the $\ell-2$ factor and similarly for the current quadrupole. The source current octupole has a nonzero contribution. However, to obtain the radiative current octupole, one has to perform a time derivative which operates on the spin vector. Since $\dot{S}_A = \mathcal{O}(2)$, the SSS contribution to the radiative moment is higher order.

F. Reduction to quasicircular orbits in the CM frame for aligned spins

In this section, we reduce the expressions of the source moments in the CM frame in the quasicircular orbits approximation without precession. The first step is to express them in the CM frame. We define as usual the quantities $M = m_1 + m_2$, $\delta = (m_1 - m_2)/M$, $\nu = m_1 m_2/M^2$, $r = |y_1 - y_2|$, $n = (y_1 - y_2)/r$, $v = v_1 - v_2$, $\kappa_\pm = \kappa_1 \pm \kappa_2$ and $\lambda_\pm = \lambda_1 \pm \lambda_2$. The CM frame is defined as the frame in which the CM position of the system G^i vanishes. It allows us to express the positions and velocities of the two compact objects in terms of the dynamical variables of the system

$$y_1^{\mathrm{CM}} = \frac{m_2}{M}rn + z \qquad , \tag{3.20a}$$

$$y_2^{\mathrm{CM}} = -\frac{m_1}{M}rn + z, \tag{3.20b}$$

and similarly for the velocities. The function z is a higher order quantity known from previous works [35,51,72]. We also introduce the following combinations of the individual spins

$$S = S_1 + S_2, \tag{3.21a}$$

$$\Sigma = \frac{M}{m_2}S_2 - \frac{M}{m_1}S_1. \tag{3.21b}$$

With these relations in hand, we obtain the source multipoles in the CM frame.

The next step is to impose the aligned-spins condition which drastically simplifies the computations. In particular, this implies that the orbital motion remains planar and that the two individual spins are aligned with the total angular momentum of the system. We can define the unitary vector $\ell = n \times v/|n \times v|$. In the absence of precession, ℓ is constant and coincides with e_Z defined in Sec. II A. The spin combinations S and Σ are also directed along ℓ such that

$$S = S_\ell \ell, \tag{3.22a}$$

$$\Sigma = \Sigma_\ell \ell. \tag{3.22b}$$

The last step is the reduction to quasicircular orbits. As for the nonspinning case, the acceleration $a = a_1 - a_2$ is directed along n as

$$a = -r\omega^2 n, \tag{3.23}$$

which defines the orbital frequency ω. Note that we neglect the radiation reaction force in the EOM because we are interested in the spin contributions to the acceleration. The spin contributions to the radiation reaction term is at least of order $\mathcal{O}(8)$, which is a higher order than required. The EOM (3.23) are known from previous works [35,72]. In this approximation, $v = -r\omega\lambda$ where λ completes the time-dependent orthonormal basis (n, λ, ℓ) as $\lambda = \ell \times n$. With this parametrization,

$$n(t) = \cos\phi(t)e_X + \sin\phi(t)e_Y, \tag{3.24a}$$

$$\lambda(t) = -\sin\phi(t)e_X + \cos\phi(t)e_Y, \tag{3.24b}$$

$$\ell(t) = e_Z, \tag{3.24c}$$

where ϕ is the phase and is given by $\phi = \int dt\,\omega$. After defining the usual PN quantities

$$\gamma = \frac{GM}{rc^2}, \qquad x = \left(\frac{GM\omega}{c^3}\right)^{2/3}, \tag{3.25}$$

we can read the expression of ω^2 in terms of γ. Then, we invert this relation to obtain γ in terms of ω and thus x which leads to

$$\gamma = x\left[1 + xg_{\mathrm{NS}} + x^{3/2}\frac{g_{\mathrm{SO}}}{GM^2} + x^2\frac{g_{\mathrm{SS}}}{G^2M^4} + x^{7/2}\frac{g_{\mathrm{SSS}}}{G^3M^6} + \mathcal{O}(8)\right], \tag{3.26}$$

where g_{NS} and g_{SO} are given in Eq. (4.3) of Ref. [51], g_{SS} in Eq. (3.32) of Ref. [72] and g_{SSS} in Eq. (6.15) of Ref. [35] at

consistent order. This allows us to express the source multipoles in terms of the orbital frequency and thus x.

G. Nonlinear contributions to the radiative moments

Once the source multipoles are known in the CM, we follow rigorously the procedure in Ref. [92] to compute the radiative moments. Without precession, there are no additional technicalities and the problem is equivalent to point-particle.

To link the radiative to the source moments, one has to introduce the canonical moments (M_L, S_L) which are related to the set of source and gauge moments $(I_L, J_L, W_L, X_L, Y_L, Z_L)$ (see [92] for an account of the procedure). The computation of the canonical moments is detailed in Sec. III G 1. Once the canonical moments are known, we can deduce the radiative moments through the following relations

$$U_L = M_L^{(\ell)} + \text{(nonlinear terms)}, \qquad (3.27a)$$

$$V_L = S_L^{(\ell)} + \text{(nonlinear terms)}, \qquad (3.27b)$$

where the nonlinear terms are at least of order $\mathcal{O}(3)$, so for radiative multipoles that are required at a low order, their contribution vanish. They are composed of three different types of contributions: the instantaneous terms, the tail terms and the memory terms. The computation of these different terms are detailed below.

1. Link between source and canonical moments

The relations between the canonical and source moments take the form

$$M_L = I_L + \delta I_L, \qquad (3.28a)$$

$$S_L = J_L + \delta J_L, \qquad (3.28b)$$

where δI_L and δJ_L are nonlinear corrections made of products of source and gauge moments, and starting at the 2.5PN order. The full expressions of δI_L and δJ_L up to the 3.5PN order are displayed in Sec. III. B. of Ref. [101]. Recently, the expression for δI_{ij} has been derived up to the 4PN order [102]. For the spin contributions, we have explicitly

$$[\delta I_{ij}]_S = \mathcal{O}(8), \qquad (3.29a)$$

$$[\delta I_{ijk}]_S = \frac{12G}{c^5}\left[I_{\langle ij}Y_{k\rangle}^{(1)}\right]_S + \mathcal{O}(8), \qquad (3.29b)$$

$$[\delta J_{ij}]_S = \frac{2G}{c^5}\left[\epsilon_{ab\langle i}(-I_{j\rangle b}^{(3)}W_a - 2I_{j\rangle b}Y_a^{(2)} + I_{j\rangle b}^{(1)}Y_a^{(1)})\right.$$
$$\left. + 3J_{\langle i}W_{j\rangle}^{(1)}\right]_S + \mathcal{O}(8), \qquad (3.29c)$$

where the spin parts of W_i, Y_i and J_i for aligned spins are given by $[J_i]_S = S_\ell \ell^i/c$, $[W_i]_S = \nu r \Sigma_\ell \lambda^i/4c$ and $[Y_i]_S = \nu r \omega \Sigma_\ell n^i/4c$. With these relations in hand, it is very straightforward to compute the canonical moments. They have been computed at the same orders as those displayed in Table I. After obtaining their expression, one has to perform a time differentiation to obtain the linear part of the radiative moments.

2. Instantaneous terms

At the 3.5PN order, the instantaneous terms are quadratic interactions of the canonical moments and their complete expressions are displayed in Sec. III. A. 1. of Ref. [92]. The relevant spin contributions of these terms are the following

$$[U_{ij}^{\text{inst}}]_S = \frac{2G}{c^5}\left[\frac{1}{3}\epsilon_{ab\langle i}M_{j\rangle a}^{(4)}S_b\right]_S + \mathcal{O}(8), \qquad (3.30a)$$

$$[U_{ijk}^{\text{inst}}]_S = \frac{G}{c^5}\left[\frac{1}{5}\epsilon_{ab\langle i}(-12S_{j\underline{a}}^{(2)}M_{k\rangle b}^{(3)} - 8M_{j\underline{a}}^{(2)}S_{k\rangle b}^{(3)} - 3S_{j\underline{a}}^{(1)}M_{k\rangle b}^{(4)} - 27M_{j\underline{a}}^{(1)}S_{k\rangle b}^{(4)}\right.$$
$$\left. - S_{j\underline{a}}M_{k\rangle b}^{(5)} - 9M_{j\underline{a}}S_{k\rangle b}^{(5)} - \frac{9}{4}S_{\underline{a}}M_{jk\rangle b}^{(5)}) + \frac{12}{5}S_{\langle i}S_{jk\rangle}^{(4)}\right]_S + \mathcal{O}(7), \qquad (3.30b)$$

$$[V_{ij}^{\text{inst}}]_S = \frac{G}{7c^5}\left[4S_{a\langle i}^{(2)}M_{j\rangle a}^{(3)} + 8M_{a\langle i}^{(2)}S_{j\rangle a}^{(3)} + 17S_{a\langle i}^{(1)}M_{j\rangle a}^{(4)} - 3M_{a\langle i}^{(1)}S_{j\rangle a}^{(4)}\right.$$
$$\left. + 9S_{a\langle i}M_{j\rangle a}^{(5)} - 3M_{a\langle i}S_{j\rangle a}^{(5)} - \frac{1}{4}S_a M_{ija}^{(5)} - 7\epsilon_{ab\langle i}S_{\underline{a}}S_{j\rangle b}^{(4)}\right]_S + \mathcal{O}(7), \qquad (3.30c)$$

$$[V_{ijk}^{\text{inst}}]_S = -\frac{2G}{c^3}\left[S_{\langle i}M_{jk\rangle}^{(4)}\right]_S + \mathcal{O}(6), \qquad (3.30d)$$

$$[V_{ijkl}^{\text{inst}}]_S = \frac{G}{c^3}\left[-\frac{35}{3}S_{\langle ij}^{(2)}M_{kl\rangle}^{(3)} - \frac{25}{3}M_{\langle ij}^{(2)}S_{kl\rangle}^{(3)} - \frac{65}{6}S_{\langle ij}^{(1)}M_{kl\rangle}^{(4)} - \frac{25}{6}M_{\langle ij}^{(1)}S_{kl\rangle}^{(4)}\right.$$
$$\left.-\frac{19}{6}S_{\langle ij}M_{kl\rangle}^{(5)} - \frac{11}{6}M_{\langle ij}S_{kl\rangle}^{(5)} - \frac{11}{12}S_{\langle i}M_{jkl\rangle}^{(5)}\right]_S + \mathcal{O}(5). \tag{3.30e}$$

They are treated as δI_L and δJ_L. In these equations, S_i and S_{ij} correspond to the current canonical moments and not the spin vector or tensor of the individual bodies. Note that these interactions are to be evaluated at the retarded time $T_R = T - R/c$.

3. Tail terms

The tails correspond to time integrals over the past of the source. Their expressions are known for each ℓ. For the spin effects at the order considered, we only need to compute the following ones

$$[U_{ij}^{\text{tail}}]_S = \frac{2G\mathcal{M}}{c^3}\int_0^{+\infty}d\tau\left[\ln\left(\frac{\tau}{2b}\right) + \frac{11}{12}\right][M_{ij}^{(4)}(T_R - \tau)]_S, \tag{3.31a}$$

$$[U_{ijk}^{\text{tail}}]_S = \frac{2G\mathcal{M}}{c^3}\int_0^{+\infty}d\tau\left[\ln\left(\frac{\tau}{2b}\right) + \frac{97}{60}\right][M_{ijk}^{(5)}(T_R - \tau)]_S, \tag{3.31b}$$

$$[V_{ij}^{\text{tail}}]_S = \frac{2G\mathcal{M}}{c^3}\int_0^{+\infty}d\tau\left[\ln\left(\frac{\tau}{2b}\right) + \frac{7}{6}\right][S_{ij}^{(4)}(T_R - \tau)]_S, \tag{3.31c}$$

$$[V_{ijk}^{\text{tail}}]_S = \frac{2G\mathcal{M}}{c^3}\int_0^{+\infty}d\tau\left[\ln\left(\frac{\tau}{2b}\right) + \frac{5}{3}\right][S_{ijk}^{(5)}(T_R - \tau)]_S, \tag{3.31d}$$

$$[V_{ijkl}^{\text{tail}}]_S = \frac{2G\mathcal{M}}{c^3}\int_0^{+\infty}d\tau\left[\ln\left(\frac{\tau}{2b}\right) + \frac{119}{60}\right][S_{ijkl}^{(6)}(T_R - \tau)]_S, \tag{3.31e}$$

where we introduced an arbitrary timescale constant b. The mass monopole $\mathcal{M}$, or ADM mass, differs from the total constant mass M through the relation $\mathcal{M} = M + \bar{E}/c^2$ where $\bar{E}$ is the conservative binding energy of the system. This implies that the spin contributions in $\mathcal{M}$ start at $\mathcal{O}(5)$ and thus do not need to be taken into account in the computation of the tail terms as they are of higher order.

To compute the tail integrals, we consider the aligned-spin case. There are no precession effects to consider here and the evolution of the dynamics of the binary is qualitatively the same as for the usual quasicircular orbits, with the aligned conserved norm spins acting simply as constant

vectors. This is to be contrasted with the more general case of binaries on quasicircular but precessing orbits (as defined for instance in Ref. [74]), where one must solve analytically the dynamics consistently with the order at which the analysis is carried, to be able to compute these integrals. The idea to compute these integrals in the nonprecessing case is to project the moments in the spatial basis (e_X, e_Y, e_Z) defined in Sec. II A using the relations (3.24) and assuming that the separation r is constant over time.[3] By doing so, we find one-dimensional integrals of the type $\int_0^\infty dy\ \ln(y)e^{i\alpha y}$ that are computable analytically [105].

4. Memory terms

The memory terms, as well as the tail terms, are called hereditary effects in the sense that they are integrals over the past of the source. They are integrals of quadratic interactions of canonical moments and only appear in the mass-type multipoles [106]. In our case, only the mass octupole moment contains a spin contribution to the memory terms according to

$$[U_{ijk}^{\text{mem}}]_S = -\frac{4G}{5c^5}\int_0^{+\infty}d\tau[\epsilon_{ab\langle i}M_{j\underline{a}}^{(3)}(T_R - \tau)S_{k\rangle b}^{(3)}(T_R - \tau)]_S + \mathcal{O}(7). \tag{3.32}$$

For the mass quadrupole, the spin contribution to the memory effects are of order $\mathcal{O}(8)$ and thus do not need to be considered.

The computation of the memory terms differ from the one of the tails because one cannot assume that the separation r is constant. We have to take into account the radiation reaction. As for the tails, we project the values of the moments on the basis (e_X, e_Y, e_Z). By doing so, we encounter integrals of the type

$$\int_{-\infty}^{T_R}d\tau\frac{e^{in\phi(\tau)}}{r^p(\tau)}, \tag{3.33}$$

where n is a nonzero integer, p is a half-integer, and ϕ is the phase variable defined in Sec. III F, which satisfies $\dot{\phi} = \omega$. As we can see, if we assumed the separation to be constant over time, this integral would diverge. At leading order, the

---

[3]It has been shown [103,104] that for the tail integrals, such an approximation is valid because the remote past of the source is negligible when compared to its recent past. On this timescale, r does not have the time to vary significantly and can be assumed constant.

separation scales as $r(\tau) \sim (-\tau)^{1/4}$ and the phase as $\phi(\tau) \sim (-\tau)^{5/8}$, which allows us to compute this integral, as detailed in Refs. [92,104].

One can also encounter integrals of this type with $n = 0$, which can be computed analytically as well. However, the main difference from the case $n \neq 0$ is that they induce a factor c^5. This means that their contribution is of order 2.5PN lower than the initial order of the memory interaction. Thus, when using the PN-MPM formalism, we cannot be consistent to the 3.5PN order for these terms since it requires the knowledge of the general memory interactions at higher orders. Fortunately, these integrals only contribute to the modes for $m = 0$, which we do not derive in this paper. More details about these peculiar modes are given in Sec. IV A.

After combining all the previous intermediate results, we derived the radiative multipole moments in the quasicircular, spin-aligned approximations at the orders displayed in Table I. To obtain the amplitude spherical modes, we insert the obtained expressions in Eq. (2.7) and then in Eq. (2.6).

IV. RESULTS

We now present the waveform modes. In Sec. IV A, we express them in a PN expansion following, e.g., [92,101], whereas in Sec. IV B, we factorize them in a way that is suitable for the EOB approach and notably template building.

A. Spin-weighted spherical modes

The amplitude modes defined in Eq. (2.6) can be written in terms of the phase variable ϕ defined in Sec. III F. However it is convenient to introduce a new phase variable ψ that allows factoring out the logarithm dependence on the orbital frequency induced by the tail terms in the radiative moments. The new phase variable reads

$$\psi \equiv \phi - \frac{2G\mathcal{M}\omega}{c^3}\ln\left(\frac{\omega}{\omega_0}\right), \tag{4.1}$$

where the constant ω_0 is linked to the timescale constant b introduced in Eqs. (3.31) through $\omega_0 = \frac{1}{4b}\exp[\frac{11}{12} - \gamma_{\mathrm{E}}]$ and we recall that $\mathcal{M}$ is the ADM mass. The amplitude modes then read

$$h_{\ell m} = \frac{2GM\nu x}{Rc^2}\sqrt{\frac{16\pi}{5}}\hat{H}_{\ell m}e^{-im\psi}, \tag{4.2}$$

where we recall that

$$x = \left(\frac{GM\omega}{c^3}\right)^{2/3}. \tag{4.3}$$

The spin part of the $\hat{H}_{\ell m}$ are given by

$$
\begin{aligned}
\hat{H}_{22}^{\mathrm{S}} = \frac{x^{3/2}}{GM^2}&\left[-2S_\ell - \frac{2}{3}\Sigma_\ell\delta + \left(S_\ell\left(-\frac{163}{63} - \frac{92}{63}\nu\right) + \Sigma_\ell\delta\left(-\frac{1}{21} + \frac{20}{63}\nu\right)\right)x\right.\\
&+ \left(\left(-\frac{4}{3}\mathrm{i} - 4\pi\right)S_\ell - \frac{4}{3}\pi\Sigma_\ell\delta\right)x^{3/2}\\
&\left.+ \left(S_\ell\left(\frac{1061}{84} + \frac{4043}{84}\nu + \frac{499}{84}\nu^2\right) + \Sigma_\ell\delta\left(\frac{3931}{756} + \frac{7813}{378}\nu + \frac{1025}{252}\nu^2\right)\right)x^2\right]\\
+ \frac{x^2}{G^2M^4}&\left[S_\ell^2(2 + \kappa_+) + S_\ell\Sigma_\ell(2\delta - \kappa_- + \delta\kappa_+) + \Sigma_\ell^2\left(-\frac{1}{2}\delta\kappa_- + \frac{1}{2}\kappa_+ - 2\nu - \kappa_+\nu\right)\right.\\
&+ \left(S_\ell^2\left(-\frac{404}{63} + \frac{55}{42}\delta\kappa_- - \frac{31}{42}\kappa_+ + \frac{68}{21}\nu + \frac{34}{21}\kappa_+\nu\right)\right.\\
&+ S_\ell\Sigma_\ell\left(-\frac{481}{63}\delta + \frac{43}{21}\kappa_- - \frac{43}{21}\delta\kappa_+ + \frac{68}{21}\delta\nu - \frac{48}{7}\kappa_-\nu + \frac{34}{21}\delta\kappa_+\nu\right)\\
&+ \left.\Sigma_\ell^2\left(-\frac{5}{3} + \frac{43}{42}\delta\kappa_- - \frac{43}{42}\kappa_+ + \frac{172}{21}\nu - \frac{89}{42}\delta\kappa_-\nu + \frac{25}{6}\kappa_+\nu - \frac{68}{21}\nu^2 - \frac{34}{21}\kappa_+\nu^2\right)\right)x\\
&\left.+ \pi(S_\ell^2(4 + 2\kappa_+) + S_\ell\Sigma_\ell(4\delta - 2\kappa_- + 2\delta\kappa_+) + \Sigma_\ell^2(-\delta\kappa_- + \kappa_+ - 4\nu - 2\kappa_+\nu))x^{3/2}\right]\\
+ \frac{x^{7/2}}{G^3M^6}&\left[S_\ell^3\left(\frac{32}{3} - \frac{2}{3}\kappa_+ - 2\lambda_+\right) + S_\ell^2\Sigma_\ell\left(\frac{52}{3}\delta - \frac{7}{3}\kappa_- - \frac{1}{3}\delta\kappa_+ + 3\lambda_- - 3\delta\lambda_+\right)\right.\\
&+ S_\ell\Sigma_\ell^2\left(\frac{20}{3} - 3\delta\kappa_- + 3\kappa_+ + 3\delta\lambda_- - 3\lambda_+ - \frac{112}{3}\nu - \frac{2}{3}\kappa_+\nu + 6\lambda_+\nu\right)\\
&\left.+ \Sigma_\ell^3\left(-\frac{5}{3}\kappa_- + \frac{5}{3}\delta\kappa_+ + \lambda_- - \delta\lambda_+ - \frac{20}{3}\delta\nu + \frac{11}{3}\kappa_-\nu - \frac{1}{3}\delta\kappa_+\nu - 3\lambda_-\nu + \delta\lambda_+\nu\right)\right],
\end{aligned}
\tag{4.4a}
$$

$$\hat{H}^{S}_{21} = \frac{ix}{2GM^2}\left[\Sigma_\ell + \left(-\frac{86}{21}S_\ell\delta + \Sigma_\ell\left(-\frac{79}{21} + \frac{139}{21}\nu\right)\right)x + \Sigma_\ell\left(-\frac{i}{2} + \pi - 2i\ln(2)\right)x^{3/2}\right.$$

$$+ \left(S_\ell\delta\left(-\frac{331}{378} + \frac{772}{189}\nu\right) + \Sigma_\ell\left(\frac{293}{378} - \frac{2615}{756}\nu - \frac{1723}{189}\nu^2\right)\right)x^2$$

$$+ \left(S_\ell\delta\left(\frac{181}{105}i - \frac{86}{21}\pi + \frac{172}{21}i\ln(2)\right) + \Sigma_\ell\left(\frac{79}{42}i - \frac{79}{21}\pi - \frac{1951}{140}i\nu + \frac{257}{42}\pi\nu + \frac{158}{21}i\ln(2) - \frac{257}{21}i\nu\ln(2)\right)\right)x^{5/2}\right]$$

$$+ \frac{ix^{5/2}}{G^2M^4}\left[S_\ell^2\left(\delta - \frac{1}{3}\kappa_- + \frac{1}{2}\delta\kappa_+\right) + S_\ell\Sigma_\ell\left(-\frac{1}{3} - \frac{5}{6}\delta\kappa_- + \frac{5}{6}\kappa_+ - 4\nu - 2\kappa_+\nu\right)\right.$$

$$+ \Sigma_\ell^2\left(-\frac{1}{2}\delta - \frac{5}{12}\kappa_- + \frac{5}{12}\delta\kappa_+ - \delta\nu + \frac{4}{3}\kappa_-\nu - \frac{1}{2}\delta\kappa_+\nu\right) + \left(S_\ell^2\left(\frac{41}{42}\delta + \frac{23}{48}\kappa_- + \frac{47}{336}\delta\kappa_+ - \frac{1}{7}\delta\nu - \frac{191}{72}\kappa_-\nu - \frac{1}{14}\delta\kappa_+\nu\right)\right.$$

$$+ S_\ell\Sigma_\ell\left(-\frac{29}{21} + \frac{19}{56}\delta\kappa_- - \frac{19}{56}\kappa_+ + \frac{100}{21}\nu - \frac{1301}{504}\delta\kappa_-\nu + \frac{1019}{504}\kappa_+\nu + \frac{4}{7}\nu^2 + \frac{2}{7}\kappa_+\nu^2\right)$$

$$+ \Sigma_\ell^2\left(-\frac{6}{7}\delta + \frac{19}{112}\kappa_- - \frac{19}{112}\delta\kappa_+ + \frac{59}{21}\delta\nu - \frac{751}{504}\kappa_-\nu + \frac{145}{126}\delta\kappa_+\nu + \frac{1}{7}\delta\nu^2 + \frac{1265}{504}\kappa_-\nu^2 + \frac{1}{14}\delta\kappa_+\nu^2\right)\right)x\right]$$

$$+ \frac{ix^3}{2G^3M^6}\left[S_\ell^2\Sigma_\ell\left(1 + \frac{1}{2}\kappa_+\right) + S_\ell\Sigma_\ell^2\left(\delta - \frac{1}{2}\kappa_- + \frac{1}{2}\delta\kappa_+\right) + \Sigma_\ell^3\left(-\frac{1}{4}\delta\kappa_- + \frac{1}{4}\kappa_+ - \nu - \frac{1}{2}\kappa_+\nu\right)\right], \tag{4.4b}$$

$$\hat{H}^{S}_{33} = \frac{3i\sqrt{15}x^2}{8\sqrt{14}GM^2}\left[7S_\ell\delta + \Sigma_\ell(3 - 9\nu) + \left(S_\ell\delta\left(-\frac{139}{15} + \frac{83}{15}\nu\right) + \Sigma_\ell\left(-\frac{43}{5} + 24\nu + 5\nu^2\right)\right)x\right.$$

$$+ \left(S_\ell\delta\left(-\frac{213}{10}i + 21\pi - 42i\ln(2) + 42i\ln(3)\right)\right.$$

$$+ \left.\Sigma_\ell\left(-\frac{63}{5}i + 9\pi + \frac{8797}{270}i\nu - 27\pi\nu - 18i\ln(2) + 54i\nu\ln(2) + 18i\ln(3) - 54i\nu\ln(3)\right)\right)x^{3/2}\right]$$

$$+ \frac{3i\sqrt{15}x^{5/2}}{8\sqrt{14}G^2M^4}\left[S_\ell^2\delta(-6 - 3\kappa_+) + S_\ell\Sigma_\ell(-6 + 3\delta\kappa_- - 3\kappa_+ + 24\nu + 12\kappa_+\nu)\right.$$

$$+ \Sigma_\ell^2\left(\frac{3}{2}\kappa_- - \frac{3}{2}\delta\kappa_+ + 6\delta\nu - 6\kappa_-\nu + 3\delta\kappa_+\nu\right) + \left(S_\ell^2\left(23\delta - \frac{7}{2}\kappa_- + \frac{15}{2}\delta\kappa_+ - 12\delta\nu + 16\kappa_-\nu - 6\delta\kappa_+\nu\right)\right.$$

$$+ S_\ell\Sigma_\ell(26 - 11\delta\kappa_- + 11\kappa_+ - 128\nu + 22\delta\kappa_-\nu - 52\kappa_+\nu + 48\nu^2 + 24\kappa_+\nu^2)$$

$$+ \left.\Sigma_\ell^2\left(3\delta - \frac{11}{2}\kappa_- + \frac{11}{2}\delta\kappa_+ - 32\delta\nu + \frac{59}{2}\kappa_-\nu - \frac{37}{2}\delta\kappa_+\nu + 12\delta\nu^2 - 28\kappa_-\nu^2 + 6\delta\kappa_+\nu^2\right)\right)x\right], \tag{4.4c}$$

$$\hat{H}^{S}_{32} = \frac{2\sqrt{5}x^{3/2}}{3\sqrt{7}GM^2}\left[S_\ell + \Sigma_\ell\delta + \left(S_\ell\left(-\frac{13}{2} + \frac{73}{6}\nu\right) + \Sigma_\ell\delta\left(-\frac{31}{6} + 5\nu\right)\right)x + (S_\ell(-i + 2\pi) + \Sigma_\ell\delta(-3i + 2\pi))x^{3/2}\right.$$

$$+ \left.\left(S_\ell\left(\frac{4859}{1320} - \frac{15413}{792}\nu - \frac{419}{88}\nu^2\right) + \Sigma_\ell\delta\left(\frac{19241}{3960} - \frac{808}{55}\nu - \frac{16153}{3960}\nu^2\right)\right)x^2\right]$$

$$+ \frac{8\sqrt{5}x^3}{9\sqrt{7}G^2M^4}\left[S_\ell^2\left(-1 - \frac{3}{8}\delta\kappa_- + \frac{9}{8}\kappa_+ - \frac{9}{2}\nu - \frac{9}{4}\kappa_+\nu\right) + S_\ell\Sigma_\ell\left(-\frac{5}{2}\delta - \frac{3}{2}\kappa_- + \frac{3}{2}\delta\kappa_+ - \frac{9}{2}\delta\nu + \frac{15}{4}\kappa_-\nu - \frac{9}{4}\delta\kappa_+\nu\right)\right.$$

$$+ \left.\Sigma_\ell^2\left(-\frac{3}{2} - \frac{3}{4}\delta\kappa_- + \frac{3}{4}\kappa_+ + 3\nu + \frac{3}{2}\delta\kappa_-\nu - 3\kappa_+\nu + \frac{9}{2}\nu^2 + \frac{9}{4}\kappa_+\nu^2\right)\right]$$

$$+ \frac{4\sqrt{5}x^{7/2}}{3\sqrt{7}G^3M^6}\left[S_\ell^3\left(1 + \frac{1}{2}\kappa_+\right) + S_\ell^2\Sigma_\ell\left(2\delta - \frac{1}{2}\kappa_- + \delta\kappa_+\right) + S_\ell\Sigma_\ell^2\left(1 - \frac{3}{4}\delta\kappa_- + \frac{3}{4}\kappa_+ - 5\nu - \frac{5}{2}\kappa_+\nu\right)\right.$$

$$+ \left.\Sigma_\ell^3\left(-\frac{1}{4}\kappa_- + \frac{1}{4}\delta\kappa_+ - \delta\nu + \kappa_-\nu - \frac{1}{2}\delta\kappa_+\nu\right)\right], \tag{4.4d}$$

$$\hat{H}_{31}^{S} = \frac{ix^2}{24\sqrt{14}GM^2}\left[S_\ell\delta + \Sigma_\ell(5-15\nu) + \left(S_\ell\delta\left(-\frac{79}{9}+\frac{443}{9}\nu\right) + \Sigma_\ell\left(-\frac{149}{9}+\frac{700}{9}\nu-\frac{841}{9}\nu^2\right)\right)x\right.$$

$$+ \left.\left(S_\ell\delta\left(\frac{47}{10}i+\pi-2i\ln(2)\right) + \Sigma_\ell\left(-7i+5\pi+\frac{11}{10}i\nu-15\pi\nu-10i\ln(2)+30i\nu\ln(2)\right)\right)x^{3/2}\right]$$

$$+ \frac{ix^{5/2}}{4\sqrt{14}G^2M^4}\left[S_\ell^2\left(\delta-\frac{4}{3}\kappa_-+\frac{1}{2}\delta\kappa_+\right) + S_\ell\Sigma_\ell\left(1-\frac{11}{6}\delta\kappa_-+\frac{11}{6}\kappa_+-4\nu-2\kappa_+\nu\right)\right.$$

$$+ \Sigma_\ell^2\left(-\frac{11}{12}\kappa_-+\frac{11}{12}\delta\kappa_+-\delta\nu+\frac{7}{3}\kappa_-\nu-\frac{1}{2}\delta\kappa_+\nu\right) + \left(S_\ell^2\left(-\frac{149}{18}\delta+\frac{13}{4}\kappa_--\frac{53}{36}\delta\kappa_+-\frac{22}{9}\delta\nu-\frac{16}{9}\kappa_-\nu-\frac{11}{9}\delta\kappa_+\nu\right)\right.$$

$$+ S_\ell\Sigma_\ell\left(-\frac{115}{9}+\frac{85}{18}\delta\kappa_--\frac{85}{18}\kappa_++40\nu-\frac{5}{9}\delta\kappa_-\nu+\frac{58}{9}\kappa_+\nu+\frac{88}{9}\nu^2+\frac{44}{9}\kappa_+\nu^2\right)$$

$$+ \left.\left.\Sigma_\ell^2\left(-\frac{9}{2}\delta+\frac{85}{36}\kappa_--\frac{85}{36}\delta\kappa_++\frac{100}{9}\delta\nu-\frac{233}{36}\kappa_-\nu+\frac{7}{4}\delta\kappa_+\nu+\frac{22}{9}\delta\nu^2-\frac{2}{3}\kappa_-\nu^2+\frac{11}{9}\delta\kappa_+\nu^2\right)\right)x\right], \tag{4.4e}$$

$$\hat{H}_{44}^{S} = \frac{32x^{5/2}}{9\sqrt{35}GM^2}\left[S_\ell\left(\frac{19}{3}-19\nu\right) + \Sigma_\ell\delta(3-6\nu)\right.$$

$$+ \left.\left(S_\ell\left(-\frac{437}{22}+\frac{10063}{132}\nu-\frac{971}{44}\nu^2\right) + \Sigma_\ell\delta\left(-\frac{153}{11}+\frac{2165}{66}\nu+\frac{67}{44}\nu^2\right)\right)x\right]$$

$$+ \frac{8\sqrt{5}x^3}{9\sqrt{7}G^2M^4}[S_\ell^2(-4-2\kappa_++12\nu+6\kappa_+\nu) + S_\ell\Sigma_\ell(-4\delta+2\kappa_--2\delta\kappa_++12\delta\nu-6\kappa_-\nu+6\delta\kappa_+\nu)$$

$$+ \Sigma_\ell^2(\delta\kappa_--\kappa_++4\nu-3\delta\kappa_-\nu+5\kappa_+\nu-12\nu^2-6\kappa_+\nu^2)], \tag{4.4f}$$

$$\hat{H}_{43}^{S} = \frac{9i\sqrt{5}x^2}{8\sqrt{14}GM^2}\left[-S_\ell\delta + \Sigma_\ell(-1+3\nu) + \left(S_\ell\delta\left(\frac{1303}{165}-\frac{1451}{165}\nu\right) + \Sigma_\ell\left(\frac{361}{55}-\frac{284}{11}\nu+\frac{163}{11}\nu^2\right)\right)x\right.$$

$$+ \left(S_\ell\delta\left(\frac{53}{10}i-3\pi+6i\ln(2)-6i\ln(3)\right)\right.$$

$$+ \left.\left.\Sigma_\ell\left(\frac{32}{5}i-3\pi-\frac{6007}{270}i\nu+9\pi\nu+6i\ln(2)-18i\nu\ln(2)-6i\ln(3)+18i\nu\ln(3)\right)\right)x^{3/2}\right]$$

$$+ \frac{9i\sqrt{5}x^{7/2}}{8\sqrt{14}G^2M^4}\left[S_\ell^2\left(3\delta+\frac{3}{5}\kappa_--\frac{8}{5}\delta\kappa_++4\delta\nu-\frac{6}{5}\kappa_-\nu+2\delta\kappa_+\nu\right)\right.$$

$$+ S_\ell\Sigma_\ell\left(6+\frac{11}{5}\delta\kappa_--\frac{11}{5}\kappa_+-12\nu-\frac{16}{5}\delta\kappa_-\nu+\frac{48}{5}\kappa_+\nu-16\nu^2-8\kappa_+\nu^2\right)$$

$$+ \left.\Sigma_\ell^2\left(3\delta+\frac{11}{10}\kappa_--\frac{11}{10}\delta\kappa_+-4\delta\nu-\frac{27}{5}\kappa_-\nu+\frac{16}{5}\delta\kappa_+\nu-4\delta\nu^2+\frac{26}{5}\kappa_-\nu^2-2\delta\kappa_+\nu^2\right)\right], \tag{4.4g}$$

$$\hat{H}_{42}^{S} = \frac{4x^{5/2}}{21\sqrt{5}GM^2}\left[S_\ell\left(-\frac{1}{9}+\frac{1}{3}\nu\right) + \Sigma_\ell\delta(1-2\nu)\right.$$

$$+ \left.\left(S_\ell\left(-\frac{43}{22}+\frac{6653}{396}\nu-\frac{1387}{44}\nu^2\right) + \Sigma_\ell\delta\left(-\frac{313}{66}+\frac{3349}{198}\nu-\frac{725}{44}\nu^2\right)\right)x\right]$$

$$+ \frac{4x^3}{21\sqrt{5}G^2M^4}\left[S_\ell^2\left(\frac{5}{3}-\frac{5}{4}\delta\kappa_-+\frac{25}{12}\kappa_+-5\nu-\frac{5}{2}\kappa_+\nu\right)\right.$$

$$+ S_\ell\Sigma_\ell\left(\frac{5}{3}\delta-\frac{10}{3}\kappa_-+\frac{10}{3}\delta\kappa_+-5\delta\nu+\frac{15}{2}\kappa_-\nu-\frac{5}{2}\delta\kappa_+\nu\right)$$

$$+ \left.\Sigma_\ell^2\left(-\frac{5}{3}\delta\kappa_-+\frac{5}{3}\kappa_+-\frac{5}{3}\nu+\frac{5}{2}\delta\kappa_-\nu-\frac{35}{6}\kappa_+\nu+5\nu^2+\frac{5}{2}\kappa_+\nu^2\right)\right], \tag{4.4h}$$

$$
\hat{H}_{41}^{S} = \frac{i\sqrt{5}x^2}{168\sqrt{2}GM^2}\left[S_\ell\delta + \Sigma_\ell(1-3\nu) + \left(S_\ell\delta\left(-\frac{1147}{165}+\frac{1139}{165}\nu\right) + \Sigma_\ell\left(-\frac{309}{55}+\frac{232}{11}\nu - \frac{111}{11}\nu^2\right)\right)x\right.
$$
$$
+ \left(S_\ell\delta\left(-\frac{53}{30}i + \pi - 2i\ln(2)\right) + \Sigma_\ell\left(-\frac{32}{15}i + \pi + \frac{181}{30}i\nu - 3\pi\nu - 2i\ln(2) + 6i\nu\ln(2)\right)\right)x^{3/2}\right]
$$
$$
+ \frac{i\sqrt{5}x^{7/2}}{168\sqrt{2}G^2M^4}\left[S_\ell^2\left(-3\delta - \frac{7}{5}\kappa_- + \frac{12}{5}\delta\kappa_+ - 4\delta\nu + \frac{14}{5}\kappa_-\nu - 2\delta\kappa_+\nu\right)\right.
$$
$$
+ S_\ell\Sigma_\ell\left(-6 - \frac{19}{5}\delta\kappa_- + \frac{19}{5}\kappa_+ + 12\nu + \frac{24}{5}\delta\kappa_-\nu - \frac{72}{5}\kappa_+\nu + 16\nu^2 + 8\kappa_+\nu^2\right)
$$
$$
+ \Sigma_\ell^2\left(-3\delta - \frac{19}{10}\kappa_- + \frac{19}{10}\delta\kappa_+ + 4\delta\nu + \frac{43}{5}\kappa_-\nu - \frac{24}{5}\delta\kappa_+\nu + 4\delta\nu^2 - \frac{34}{5}\kappa_-\nu^2 + 2\delta\kappa_+\nu^2\right)\right], \tag{4.4i}
$$

$$
\hat{H}_{55}^{S} = \frac{3125ix^3}{36\sqrt{66}GM^2}\left[S_\ell\delta\left(-\frac{1}{2}+\nu\right) + \Sigma_\ell\left(-\frac{1}{4}+\frac{5}{4}\nu - \frac{5}{4}\nu^2\right)\right]
$$
$$
+ \frac{3125ix^{7/2}}{96\sqrt{66}G^2M^4}\left[S_\ell^2\delta\left(1+\frac{1}{2}\kappa_+ - 2\nu - \kappa_+\nu\right)\right.
$$
$$
+ S_\ell\Sigma_\ell\left(1 - \frac{1}{2}\delta\kappa_- + \frac{1}{2}\kappa_+ - 6\nu + \delta\kappa_-\nu - 3\kappa_+\nu + 8\nu^2 + 4\kappa_+\nu^2\right)
$$
$$
+ \Sigma_\ell^2\left(-\frac{1}{4}\kappa_- + \frac{1}{4}\delta\kappa_+ - \delta\nu + \frac{3}{2}\kappa_-\nu - \delta\kappa_+\nu + 2\delta\nu^2 - 2\kappa_-\nu^2 + \delta\kappa_+\nu^2\right)\right], \tag{4.4j}
$$

$$
\hat{H}_{54}^{S} = \frac{32x^{5/2}}{3\sqrt{165}GM^2}\left[S_\ell(-1+3\nu) + \Sigma_\ell\delta(-1+2\nu)\right.
$$
$$
+ \left(S_\ell\left(\frac{241}{26} - \frac{2939}{78}\nu + \frac{633}{26}\nu^2\right) + \Sigma_\ell\delta\left(\frac{619}{78} - \frac{300}{13}\nu + \frac{817}{78}\nu^2\right)\right)x\right], \tag{4.4k}
$$

$$
\hat{H}_{53}^{S} = \frac{3i\sqrt{3}x^3}{4\sqrt{110}GM^2}\left[S_\ell\delta\left(\frac{1}{2}-\nu\right) + \Sigma_\ell\left(-\frac{3}{4}+\frac{15}{4}\nu - \frac{15}{4}\nu^2\right)\right]
$$
$$
+ \frac{9i\sqrt{15}x^{7/2}}{32\sqrt{22}G^2M^4}\left[S_\ell^2\left(-\delta + \frac{4}{5}\kappa_- - \frac{13}{10}\delta\kappa_+ + 2\delta\nu - \frac{8}{5}\kappa_-\nu + \delta\kappa_+\nu\right)\right.
$$
$$
+ S_\ell\Sigma_\ell\left(-1 + \frac{21}{10}\delta\kappa_- - \frac{21}{10}\kappa_+ + 6\nu - \frac{13}{5}\delta\kappa_-\nu + \frac{39}{5}\kappa_+\nu - 8\nu^2 - 4\kappa_+\nu^2\right)
$$
$$
+ \Sigma_\ell^2\left(\frac{21}{20}\kappa_- - \frac{21}{20}\delta\kappa_+ + \delta\nu - \frac{47}{10}\kappa_-\nu + \frac{13}{5}\delta\kappa_+\nu - 2\delta\nu^2 + \frac{18}{5}\kappa_-\nu^2 - \delta\kappa_+\nu^2\right)\right], \tag{4.4l}
$$

$$
\hat{H}_{52}^{S} = \frac{2x^{5/2}}{9\sqrt{55}GM^2}\left[S_\ell(1-3\nu) + \Sigma_\ell\delta(1-2\nu)\right.
$$
$$
+ \left(S_\ell\left(-\frac{213}{26} + \frac{2519}{78}\nu - \frac{493}{26}\nu^2\right) + \Sigma_\ell\delta\left(-\frac{535}{78} + \frac{244}{13}\nu - \frac{565}{78}\nu^2\right)\right)x\right], \tag{4.4m}
$$

$$\hat{H}^{S}_{51} = \frac{ix^3}{216\sqrt{385}GM^2}\left[S_\ell\delta(1-2\nu) + \Sigma_\ell\left(\frac{7}{2} - \frac{35}{2}\nu + \frac{35}{2}\nu^2\right)\right]$$
$$+ \frac{i\sqrt{5}x^{7/2}}{288\sqrt{77}G^2M^4}\left[S^2_\ell\left(\delta - \frac{6}{5}\kappa_- + \frac{17}{10}\delta\kappa_+ - 2\delta\nu + \frac{12}{5}\kappa_-\nu - \delta\kappa_+\nu\right)\right.$$
$$+ S_\ell\Sigma_\ell\left(1 - \frac{29}{10}\delta\kappa_- + \frac{29}{10}\kappa_+ - 6\nu + \frac{17}{5}\delta\kappa_-\nu - \frac{51}{5}\kappa_+\nu + 8\nu^2 + 4\kappa_+\nu^2\right)$$
$$\left.+ \Sigma^2_\ell\left(-\frac{29}{20}\kappa_- + \frac{29}{20}\delta\kappa_+ - \delta\nu + \frac{63}{10}\kappa_-\nu - \frac{17}{5}\delta\kappa_+\nu + 2\delta\nu^2 - \frac{22}{5}\kappa_-\nu^2 + \delta\kappa_+\nu^2\right)\right], \tag{4.4n}$$

$$\hat{H}^{S}_{66} = -\frac{3132x^{7/2}}{35\sqrt{143}GM^2}\left[S_\ell(1-5\nu+5\nu^2) + \Sigma_\ell\delta\left(\frac{15}{29} - \frac{60}{29}\nu + \frac{45}{29}\nu^2\right)\right], \tag{4.4o}$$

$$\hat{H}_{65} = \frac{3125ix^3}{144\sqrt{429}GM^2}[S_\ell\delta(1-2\nu) + \Sigma_\ell(1-5\nu+5\nu^2)], \tag{4.4p}$$

$$\hat{H}^{S}_{64} = \frac{256\sqrt{2}x^{7/2}}{385\sqrt{39}GM^2}\left[S_\ell(1-5\nu+5\nu^2) + \Sigma_\ell\delta\left(-\frac{5}{9} + \frac{20}{9}\nu - \frac{5}{3}\nu^2\right)\right], \tag{4.4q}$$

$$\hat{H}^{S}_{63} = -\frac{81ix^3}{176\sqrt{65}GM^2}[S_\ell\delta(1-2\nu) + \Sigma_\ell(1-5\nu+5\nu^2)], \tag{4.4r}$$

$$\hat{H}^{S}_{62} = \frac{4x^{7/2}}{693\sqrt{65}GM^2}\left[S_\ell(1-5\nu+5\nu^2) + \Sigma_\ell\delta\left(\frac{17}{3} - \frac{68}{3}\nu + 17\nu^2\right)\right], \tag{4.4s}$$

$$\hat{H}^{S}_{61} = \frac{ix^3}{2376\sqrt{26}GM^2}[S_\ell\delta(1-2\nu) + \Sigma_\ell(1-5\nu+5\nu^2)], \tag{4.4t}$$

$$\hat{H}^{S}_{77} = 0, \tag{4.4u}$$

$$\hat{H}^{S}_{76} = \frac{324\sqrt{3}x^{7/2}}{35\sqrt{143}GM^2}[S_\ell(1-5\nu+5\nu^2) + \Sigma_\ell\delta(1-4\nu+3\nu^2)], \tag{4.4v}$$

$$\hat{H}^{S}_{75} = 0, \tag{4.4w}$$

$$\hat{H}^{S}_{74} = -\frac{512\sqrt{2}x^{7/2}}{1365\sqrt{33}GM^2}[S_\ell(1-5\nu+5\nu^2) + \Sigma_\ell\delta(1-4\nu+3\nu^2)], \tag{4.4x}$$

$$\hat{H}^{S}_{73} = 0, \tag{4.4y}$$

$$\hat{H}^{S}_{72} = \frac{4x^{7/2}}{3003\sqrt{3}GM^2}[S_\ell(1-5\nu+5\nu^2) + \Sigma_\ell\delta(1-4\nu+3\nu^2)], \tag{4.4z}$$

$$\hat{H}^{S}_{71} = 0. \tag{4.4aa}$$

As stated in Sec. III G 4, we do not derive the $h_{\ell 0}$ because the PN-MPM formalism does not allow computing the memory terms to consistent orders. However, other methods were used in the literature to compute these nonoscillatory modes and their spin contribution are displayed to the 2PN order in Appendix A of Ref. [107].

For planar motions, the modes for negative m are related to those for positive m through $h_{\ell,-m}(\phi) = (-1)^{\ell+m} h_{\ell m}^* (\phi + \pi)$ which translates for the amplitude to $\hat{H}_{\ell,-m} = (-1)^{\ell} \hat{H}_{\ell m}^*$ where the star notation refers to the complex conjugate. The modes displayed above are in agreement with the literature, and notably with the modes derived for a test particle around a Kerr black hole [79,108].

B. Effective-one-body factorized modes

Conveniently for EOB waveform models, we write the PN-expanded waveform given by Eqs. (4.2)–(4.4) in a factorized, resummed form as [77–80]

$$h_{\ell m}^{\mathrm{F}} = h_{\ell m}^{(N,\epsilon_p)} \hat{S}_{\mathrm{eff}}^{(\epsilon_p)} T_{\ell m} e^{i\delta_{\ell m}} f_{\ell m}, \qquad (4.5)$$

where ϵ_p is the parity of $\ell + m$: $\epsilon_p = 0$ if $\ell + m$ is even, and $\epsilon_p = 1$ if $\ell + m$ is odd. The first term $h_{\ell m}^{(N,\epsilon_p)}$ is the leading (Newtonian) order waveform, which is known for any (ℓ, m) [90,91], and its explicit expression is given in, e.g., Eq. (3) of Ref. [79]. Note that the convention for the definition of the waveform modes differs by a global minus sign between this paper and Ref. [79]. However this difference only affects $h_{\ell m}^{(N,\epsilon_p)}$, while the other factors in Eq. (4.5) are not altered.

The (dimensionless) effective source term $\hat{S}_{\mathrm{eff}}$ is given by either the effective energy E_{eff} or the orbital angular momentum p_ϕ, both expressed as functions of $v \equiv (M\omega)^{1/3} = \sqrt{x}$, such that[4]

$$\hat{S}_{\mathrm{eff}} = \begin{cases} \dfrac{E_{\mathrm{eff}}(v)}{\mu}, & \ell + m \text{ even} \\[2mm] v \dfrac{p_\phi(v)}{\mu M}, & \ell + m \text{ odd} \end{cases}, \qquad (4.6)$$

where $\mu \equiv m_1 m_2 / M$ is the reduced mass, and E_{eff} is related to the total energy E via the EOB energy map $E = M\sqrt{1 + 2\nu(E_{\mathrm{eff}}/\mu - 1)}$. This source term is motivated by the Regge-Wheeler-Zerilli equation [109,110], whose source depends on the stress-energy tensor for a test body in a Schwarzschild background.

The factor $T_{\ell m}$ resums the infinite number of "leading logarithms" entering the tail effects [111–113], and is given by

---

[4]In this section, we use units in which $c = G = 1$ to simplify the notation.

$$T_{\ell m} = \frac{\Gamma(\ell + 1 - 2i\hat{k})}{\Gamma(\ell + 1)} e^{\pi\hat{k}} e^{2i\hat{k}\ln(2m\omega r_0)}, \qquad (4.7)$$

where $\Gamma(\ldots)$ is the Euler gamma function, $\hat{k} \equiv m\omega E$, ω is the orbital frequency, and the constant r_0 takes the value $2M/\sqrt{e}$ to give agreement with waveforms computed in the test-body limit [79].

The remaining part of the factorized modes is expressed as an amplitude $f_{\ell m}$ and a phase $\delta_{\ell m}$, which are computed such that the expansion of $h_{\ell m}^{\mathrm{F}}$ agrees with the PN-expanded modes in Eq. (4.2). To improve the agreement with numerical-relativity waveforms, $f_{\ell m}$ is further resummed as [79,80] $f_{\ell m} = (\rho_{\ell m})^\ell$ to reduce the magnitude of the 1PN nonspinning coefficient, which grows linearly with ℓ. For spinning binaries, the nonspinning and spin contributions are separated for the odd m modes, such that

$$f_{\ell m} = \begin{cases} \rho_{\ell m}^\ell, & m \text{ even} \\[2mm] (\rho_{\ell m}^{\mathrm{NS}})^\ell + f_{\ell m}^{\mathrm{S}}, & m \text{ odd} \end{cases}, \qquad (4.8)$$

where $\rho_{\ell m}^{\mathrm{NS}}$ is the nonspinning part of $\rho_{\ell m}$, while $f_{\ell m}^{\mathrm{S}}$ is the spin part of $f_{\ell m}$.

To simplify the expressions for the factorized modes, and to be consistent with the notation used in the literature, we introduce the dimensionless symmetric and antisymmetric spin parameters

$$\chi_S \equiv \frac{1}{2}(\chi_1 + \chi_2) = \frac{S_\ell + \delta\Sigma_\ell}{2\nu G M^2}, \qquad (4.9a)$$

$$\chi_A \equiv \frac{1}{2}(\chi_1 - \chi_2) = -\frac{\delta S_\ell + (1 - 2\nu)\Sigma_\ell}{2\nu G M^2}, \qquad (4.9b)$$

and define the following combinations of the spin-multipole constants and spins:

$$\tilde{\kappa}_S \equiv \frac{1}{2}[\chi_1^2(\kappa_1 - 1) + \chi_2^2(\kappa_2 - 1)], \qquad (4.10a)$$

$$\tilde{\kappa}_A \equiv \frac{1}{2}[\chi_1^2(\kappa_1 - 1) - \chi_2^2(\kappa_2 - 1)], \qquad (4.10b)$$

$$\tilde{\lambda}_S \equiv \frac{1}{2}[\chi_1^3(\lambda_1 - 1) + \chi_2^3(\lambda_2 - 1)], \qquad (4.10c)$$

$$\tilde{\lambda}_A \equiv \frac{1}{2}[\chi_1^3(\lambda_1 - 1) - \chi_2^3(\lambda_2 - 1)], \qquad (4.10d)$$

which equal zero for black holes.

For the (2, 2) mode, we obtain

$$\rho_{22} = 1 + v^2 \left(\frac{55}{84}\nu - \frac{43}{42}\right) + v^3 \left[\left(\frac{2}{3}\nu - \frac{2}{3}\right)\chi_S - \frac{2}{3}\delta\chi_A\right] + v^4 \left[\frac{19583}{42336}\nu^2 - \frac{33025}{21168}\nu - \frac{20555}{10584}\right.$$

$$+ \left(\frac{1}{2} - 2\nu\right)\chi_A^2 + \delta\chi_A\chi_S + \frac{1}{2}\chi_S^2 + \frac{1}{2}\delta\kappa_A + \kappa_S\left(\frac{1}{2} - \nu\right)\bigg]$$

$$+ v^5 \left[\delta\left(-\frac{19}{42}\nu - \frac{34}{21}\right)\chi_A + \left(\frac{209}{126}\nu^2 + \frac{49}{18}\nu - \frac{34}{21}\right)\chi_S\right]$$

$$+ v^6 \left[\delta\left(\frac{89}{126} - \frac{781}{252}\nu\right)\chi_A\chi_S + \left(-\frac{27}{14}\nu^2 - \frac{457}{504}\nu + \frac{89}{252}\right)\chi_A^2 + \left(\frac{10}{9}\nu^2 - \frac{1817}{504}\nu + \frac{89}{252}\right)\chi_S^2\right.$$

$$+ \left(\frac{67}{84} - \frac{139}{168}\nu\right)\delta\tilde{\kappa}_A + \left(-\frac{27}{28}\nu^2 - \frac{407}{168}\nu + \frac{67}{84}\right)\tilde{\kappa}_S\bigg]$$

$$+ v^7 \left[\delta\left(\frac{97865}{63504}\nu^2 + \frac{50140}{3969}\nu + \frac{18733}{15876}\right)\chi_A + \left(\frac{50803}{63504}\nu^3 - \frac{245717}{63504}\nu^2 + \frac{74749}{5292}\nu + \frac{18733}{15876}\right)\chi_S\right.$$

$$+ \delta\chi_A^3\left(\frac{1}{3} - \frac{4}{3}\nu\right) + \delta(2\nu + 1)\chi_A\chi_S^2 + (-4\nu^2 - 3\nu + 1)\chi_A^2\chi_S + \left(\nu + \frac{1}{3}\right)\chi_S^3$$

$$+ \tilde{\kappa}_S\left[\left(\frac{1}{3}\nu + \frac{4}{3}\right)\delta\chi_A + \left(-2\nu^2 - \frac{14}{3}\nu + \frac{4}{3}\right)\chi_S\right] + \tilde{\kappa}_A\left[\left(\frac{4}{3} - \frac{7}{3}\nu\right)\chi_A + \left(\frac{4}{3} - 2\nu\right)\delta\chi_S\right]$$

$$+ (\nu - 1)\delta\tilde{\lambda}_A + (3\nu - 1)\tilde{\lambda}_S\bigg], \tag{4.11a}$$

$$\delta_{22} = \frac{7}{3}\omega E + (\omega E)^2\left[\left(\frac{8}{3}\nu - \frac{4}{3}\right)\chi_S - \frac{4}{3}\delta\chi_A\right], \tag{4.11b}$$

where we only write the nonspinning part to the order needed for the 3.5PN spin contributions. The energy E in $\delta_{\ell m}$ is replaced by the Hamiltonian in EOB waveform models.

For the (2,1) mode, we obtain

$$\rho_{21}^{\text{NS}} = 1 + \left(\frac{23}{84}\nu - \frac{59}{56}\right)v^2 + \left(\frac{617}{4704}\nu^2 - \frac{10993}{14112}\nu - \frac{47009}{56448}\right)v^4, \tag{4.12a}$$

$$f_{21}^S = v\left(-\frac{3}{2}\frac{\chi_A}{\delta} - \frac{3}{2}\chi_S\right) + v^3\left[\left(\frac{131}{84}\nu + \frac{61}{12}\right)\frac{\chi_A}{\delta} + \left(\frac{79}{84}\nu + \frac{61}{12}\right)\chi_S\right]$$

$$+ v^4\left[(-2\nu - 3)\chi_A^2 + \left(\frac{21}{2}\nu - 6\right)\frac{\chi_A\chi_S}{\delta} + \left(\frac{1}{2}\nu - 3\right)\chi_S^2 + \left(-\nu - \frac{1}{2}\right)\tilde{\kappa}_S - \frac{\tilde{\kappa}_A}{2\delta}\right]$$

$$+ v^5\left[\left(-\frac{703}{112}\nu^2 + \frac{8797}{1008}\nu - \frac{81}{16}\right)\frac{\chi_A}{\delta} + \left(\frac{613}{1008}\nu^2 + \frac{1709}{1008}\nu - \frac{81}{16}\right)\chi_S\right.$$

$$+ \left(\frac{3}{4} - 3\nu\right)\frac{\chi_A^3}{\delta} + \left(\frac{9}{4} - 6\nu\right)\frac{\chi_A\chi_S^2}{\delta} + \left(\frac{9}{4} - 3\nu\right)\chi_A^2\chi_S + \frac{3}{4}\chi_S^3$$

$$+ \frac{3}{4}\tilde{\kappa}_A\chi_A + \left(\frac{3}{4} - 3\nu\right)\frac{\tilde{\kappa}_A\chi_S}{\delta} + \left(\frac{3}{4} - \frac{3}{2}\nu\right)\frac{\tilde{\kappa}_S\chi_A}{\delta} + \left(\frac{3}{4} - \frac{3}{2}\nu\right)\tilde{\kappa}_S\chi_S\bigg]$$

$$+ v^6\left[\left(\frac{5}{7}\nu^2 - \frac{9287}{1008}\nu + \frac{4163}{252}\right)\chi_A^2 + \left(\frac{139}{72}\nu^2 - \frac{2633}{1008}\nu + \frac{4163}{252}\right)\chi_S^2\right.$$

$$+ \left(\frac{9487}{504}\nu^2 - \frac{1636}{21}\nu + \frac{4163}{126}\right)\frac{\chi_A\chi_S}{\delta} + \left(\frac{473}{84}\nu^2 + \frac{8}{21}\nu + \frac{1}{21}\right)\frac{\tilde{\kappa}_A}{\delta} + \left(\frac{5}{14}\nu^2 + \frac{10}{21}\nu + \frac{1}{21}\right)\tilde{\kappa}_S\bigg], \tag{4.12b}$$

$$\delta_{21} = \frac{2}{3}\omega E - \frac{25}{2}\nu v^5 + (\omega E)^2\left[\left(-\frac{69}{140}\nu - \frac{17}{35}\right)\frac{\chi_A}{\delta} + \left(-\frac{41}{28}\nu - \frac{17}{35}\right)\chi_S\right]. \tag{4.12c}$$

In the odd m modes, the functions $f_{\ell m}$ and $\delta_{\ell m}$ depend on $1/\delta$, which diverges for equal masses. However, since the leading order of these modes is proportional to δ, the PN-expanded modes do not diverge. Thus, in EOB models, one needs to treat the equal-mass limit separately, as discussed in Appendix A of Ref. [87].

We note that the $\mathcal{O}(v^6\chi^2\nu^2)$ terms in the (2,1) mode disagree with those used in the SEOBNRv4HM model [87].[5] Those terms were used in SEOBNRv4HM based on unpublished results by one of the authors of this paper. However, we checked that the SS contributions to the (2,1) mode should be given by Eq. (4.12).

Interestingly, we also find a discrepancy with literature [80,114] in the NS part of δ_{21}, which was required to derive the spin terms in the factorized waveform. The difference is in the radiation reaction term $\mathcal{O}(\nu v^5)$ which, in these papers, has a coefficient $-493/42$. After investigation, we found out that this value came from a wrong expression for the (2,1) mode in Ref. [92], which was later corrected in an erratum, but the factorized mode was never corrected. The coefficient $-493/42$ should read $-25/2$ as we see in Eq. (4.12c).

Explicit expressions for the other modes are given in Appendix B. We also provide all expressions as a *Mathematica* file in the Supplemental Material [89].

V. SUMMARY

We computed the spin contributions to the spherical-harmonics modes of the GW polarizations to the 3.5PN order, for nonprecessing spins in quasicircular orbits. We used the PN-MPM formalism to tackle the computation of the radiative multipole moments, which were required to a higher multipolar order than what was known in the literature. We also derived the spin contributions to the hereditary tail terms, as well as other nonlinear interactions between the moments.

Our results include all spin terms, i.e., all SO, SS and SSS terms to that order. We wrote the waveform modes in two forms: in the conventional PN-expanded form, as well as a factorized form convenient for the EOB approach. The factorized modes we obtained are in agreement with Refs. [79,87] except for the three terms $\mathcal{O}(v^6\chi^2\nu^2)$ in the (2,1) mode used in SEOBNRv4HM. As stated in Ref. [87], these terms came from private communications and are now corrected. We also corrected a NS term in the quantity δ_{21}.

The results derived in this paper can be useful in improving analytical waveform models. Interestingly,

---

[5]The difference, for black holes, is given by

$$f_{21}^{\mathrm{S,(this\,paper)}} - f_{21}^{\mathrm{S,(SEOBNRv4HM)}} = v^6\nu^2\left(\frac{165}{112}\chi_A^2 + \frac{87}{56}\frac{\chi_A\chi_S}{\delta} + \frac{165}{112}\chi_S^2\right).$$
(4.13)

preliminary implementation of some of our results in an EOB waveform model showed a good improvement when compared to numerical relativity [115].

Future work will focus on relaxing the nonprecessing approximation as well as the quasicircular-orbits condition, which is important to improve waveform models for eccentric orbits and precessing spins.

APPENDIX A: SOURCE DENSITIES

The explicit expressions of the spin part of source densities are given by

$$\sigma^{\mathrm{SO}} = -\frac{4}{c^7\sqrt{-g}}\partial_t(\delta_1 S_1^{ab}v_1^a V^b)$$
$$+ \frac{1}{\sqrt{-g}}\partial_i\left\{\delta_1\left[-\frac{2}{c^3}S_1^{ia}v_1^a + \frac{4}{c^5}(S_1^{ia}V^a - S_1^{ia}Vv_1^a)\right.\right.$$
$$+ \frac{4}{c^7}(2\hat{R}^a S_1^{ia} + S_1^{ia}(v_1^k V^k)v_1^a - 2S_1^{ia}V^2v_1^a$$
$$\left.\left.+ 2S_1^{ia}VV^a - S_1^{ia}v_1^b\hat{W}_{ab})\right]\right\}$$
$$+ \delta_1\left\{\frac{2}{c^5}[S_1^{ab}v_1^a\partial^b V + 2S_1^{ab}\partial^a V^b] + \frac{4}{c^7}[S_1^{ab}v_1^a\partial_t V^b\right.$$
$$+ S_1^{bk}v_1^a v_1^b\partial_a V^k + S_1^{ab}v_1^k\partial_b\hat{W}_{ka} - S_1^{ab}V^a\partial^b V$$
$$\left.- 2S_1^{ab}\partial^b\hat{R}^a - 2S_1^{ab}v_1^a V\partial^b V + 4S_1^{ab}V\partial^b V^a]\right\}$$
$$+ 1 \leftrightarrow 2 + \mathcal{O}(8),$$
(A1a)

$$\sigma^{\mathrm{SS}} = \frac{\kappa_1}{m_1\sqrt{-g}}\partial_{kl}\left\{\delta_1\left[\frac{S_1^{ka}S_1^{la}}{2c^4} + \frac{1}{c^6}\left(S_1^{ka}S_1^{la}\left(\frac{3}{4}v_1^2 + \frac{3}{2}V\right)\right.\right.\right.$$
$$- \frac{1}{2}S_1^{ka}S_1^{lb}v_1^a v_1^b + \frac{1}{2}(S_1^{ab}S_1^{ab})v_1^k v_1^l$$
$$\left.\left.\left.+ S_1^{ab}(-S_1^{lb}v_1^a v_1^k - S_1^{kb}v_1^a v_1^l)\right)\right]\right\}$$
$$- \frac{\kappa_1}{2m_1}S_1^{ba}S_1^{bi}\partial_{ia}V\delta_1$$
$$+ \frac{\kappa_1}{m_1 c^6\sqrt{-g}}\partial_k[\delta_1(S_1^{ab}S_1^{kb}\partial_a V - (S_1^{ab}S_1^{ab})\partial_k V)]$$
$$+ \frac{\kappa_1}{2m_1 c^6\sqrt{-g}}\partial_t^2(\delta_1 S_1^{ab}S_1^{ab})$$
$$+ \frac{\kappa_1}{m_1 c^6\sqrt{-g}}\partial_t\partial_k[\delta_1(-S_1^{ab}S_1^{kb}v_1^a + (S_1^{ab}S_1^{ab})v_1^k)]$$
$$+ 1 \leftrightarrow 2 + \mathcal{O}(8),$$
(A1b)

$$\sigma^{\mathrm{SSS}} = \frac{\lambda_1}{3m_1^2 c^7} v_1^m S_1^{mi} S_1^{jl} S_1^{kl} \partial_{ijk}\delta_1 + 1 \leftrightarrow 2 + \mathcal{O}(8), \tag{A1c}$$

$$\sigma_i^{\mathrm{SO}} = \frac{1}{\sqrt{-g}}\partial_k\left\{\delta_1\left[\frac{S_1^{ik}}{2c} + \frac{S_1^{ak}v_1^a v_1^i}{2c^3} + \frac{2}{c^5}\left(S_1^{ka}v_1^i V^a - S_1^{ka}v_1^a V\right)\right]\right\}$$

$$+ \frac{1}{\sqrt{-g}}\partial_t\left\{\delta_1\left[\frac{S_1^{ia}v_1^a}{2c^3} + \frac{2}{c^5}\left(S_1^{ia}v_1^a V - S_1^{ia}V^a\right)\right]\right\}$$

$$+ \delta_1\left\{-\frac{S_1^{ia}\partial^a V}{2c^3} + \frac{1}{c^5}\left[-S_1^{ia}v_1^k\partial^k V v_1^a + \frac{1}{2}v_1^i S_1^{ab}v_1^a\partial^b V - \frac{1}{2}S_1^{ia}v_1^a\partial_t V + S_1^{ib}v_1^a\partial^a V^b\right.\right.$$

$$\left.\left.+ S_1^{ia}V\partial^a V + S_1^{ib}v_1^a\partial^b V^a - S_1^{ab}v_1^a\partial^b V^i + S_1^{ab}v_1^a\partial^i V^b\right]\right\} + 1 \leftrightarrow 2 + \mathcal{O}(8), \tag{A1d}$$

$$\sigma_i^{\mathrm{SS}} = \frac{\delta_1}{2m_1 c^6}[2\kappa_1 S_1^{ab}S_1^{bj}\partial^{j(a}V v_1^{i)} - S_1^{ab}S_1^{ib}\partial_t\partial_a V + S_1^{bj}S_1^{ib}v_1^a\partial_{aj}V + 2S_1^{bj}S_1^{ia}\partial_{aj}V^b$$

$$- S_1^{aj}S_1^{ib}v_1^a\partial_{bj}V] + \frac{\kappa_1}{m_1 c^6\sqrt{-g}}\partial_t\left[\delta_1\left(\frac{1}{2}S_1^{ab}S_1^{ab}\partial^i V - S_1^{ab}S_1^{ib}\partial^a V\right)\right]$$

$$+ \frac{\kappa_1}{c^4 m_1\sqrt{-g}}\partial_{kl}\left\{v_1^{[i}S_1^{k]a}S_1^{la}\delta_1 + \frac{v_1^{[i}S_1^{k]a}}{c^2}\delta_1\left[\frac{1}{2}S_1^{la}v_1^2 + 3S_1^{la}V - S_1^{lb}v_1^a v_1^b + S_1^{ab}v_1^b v_1^l\right]\right\}$$

$$- \frac{\kappa_1}{2m_1 c^4\sqrt{-g}}\partial_t\partial_k\left\{S_1^{ia}S_1^{ka}\delta_1 + \frac{\delta_1}{c^2}\left[S_1^{ia}S_1^{ka}\left(\frac{v_1^2}{2} + 3V\right) - 2v_1^{(i}S_1^{k)b}S_1^{ab}v_1^a - S_1^{ia}S_1^{kb}v_1^a v_1^b\right]\right\}$$

$$+ \frac{\kappa_1}{m_1 c^6\sqrt{-g}}\partial_k\left\{\delta_1[S_1^{ia}S_1^{ka}v_1^l\partial^l V + \frac{1}{2}S_1^{ia}S_1^{ka}\partial_t V + S_1^{ab}S_1^{kb}v_1^i\partial^a V - 2S_1^{ab}S_1^{ib}v_1^k\partial^a V\right.$$

$$- S_1^{ab}S_1^{kb}\partial^a V^i + S_1^{ab}S_1^{ib}\partial^a V^k - \frac{1}{2}S_1^{ab}S_1^{kb}v_1^a\partial^i V + \frac{1}{2}S_1^{ab}S_1^{ab}v_1^k\partial^i V + S_1^{ab}S_1^{kb}\partial^i V^a$$

$$+ \frac{1}{2}S_1^{ab}S_1^{ib}v_1^a\partial^k V - \frac{1}{2}S_1^{ab}S_1^{ab}v_1^i\partial^k V - S_1^{ab}S_1^{ib}\partial^k V^a\right]\right\} + 1 \leftrightarrow 2 + \mathcal{O}(7), \tag{A1e}$$

$$\sigma_i^{\mathrm{SSS}} = \frac{\lambda_1}{12m_1^2 c^5} S_1^{ij} S_1^{km} S_1^{lm} \partial_{jkl}\delta_1 + 1 \leftrightarrow 2 + \mathcal{O}(7), \tag{A1f}$$

$$\sigma_{ij}^{\mathrm{SO}} = \frac{1}{c\sqrt{-g}}\partial_k(\delta_1 v_1^{(i}S_1^{j)k}) + \frac{1}{c^3\sqrt{-g}}\partial_t(\delta_1 v_1^{(i}S_1^{j)k}v_1^k)$$

$$+ 2\frac{\delta_1}{c^3}S_1^{k(i}\left(\partial_k V^{j)} + \partial^{j)}V v_1^k - \partial^{j)}V^k - \frac{1}{2}v_1^{j)}\partial_k V\right) + 1 \leftrightarrow 2 + \mathcal{O}(5), \tag{A1g}$$

$$\sigma_{ij}^{\mathrm{SS}} = \frac{\kappa_1}{m_1 c^4\sqrt{-g}}\left\{-\delta_1\sqrt{-g}S_1^{ab}S_1^{a(i}\partial^{j)b}V + \partial_t\partial_k[\delta_1(S_1^{ia}S_1^{ja}v_1^k + S_1^{ka}S_1^{a(i}v_1^{j)})]\right.$$

$$+ \frac{1}{2}\partial_t^2(\delta_1 S_1^{ia}S_1^{ja}) - \partial_k\left[\delta_1\left(S_1^{ka}S_1^{a(i}\partial^{j)}V + \frac{1}{2}S_1^{ia}S_1^{ja}\partial^k V\right)\right]$$

$$\left.+ \frac{1}{2}\partial_{kl}[\delta_1(S_1^{ka}S_1^{la}v_1^i v_1^j - 2v_1^{(i}S_1^{j)a}S_1^{ka}v_1^l + S_1^{ia}S_1^{ja}v_1^k v_1^l)]\right\} + 1 \leftrightarrow 2 + \mathcal{O}(5), \tag{A1h}$$

$$\sigma_{ij}^{\mathrm{SSS}} = \mathcal{O}(5), \tag{A1i}$$

where $v_1^2 = v_1^k v_1^k$.

APPENDIX B: EXPLICIT QUANTITIES OF THE FACTORIZED MODES

In this Appendix, we write the explicit expressions for the factorized modes (see Sec. IV B). The (2,2) mode is given by Eq. (4.11), the (2,1) mode by Eq. (4.12), while the other subdominant modes read

$$\rho_{33}^{\rm NS} = 1 + \left(\frac{2}{3}\nu - \frac{7}{6}\right)v^2, \tag{B1a}$$

$$
\begin{aligned}
f_{33}^{\rm S} = v^3 & \left[\left(\frac{19}{2}\nu - 2\right)\frac{\chi_A}{\delta} + \left(\frac{5}{2}\nu - 2\right)\chi_S\right] + v^4\left[\left(\frac{3}{2} - 6\nu\right)\chi_A^2 + (3 - 12\nu)\frac{\chi_A\chi_S}{\delta} + \frac{3}{2}\chi_S^2\right.\\
& \left. + \left(\frac{3}{2} - 6\nu\right)\frac{\tilde{\kappa}_A}{\delta} + \left(\frac{3}{2} - 3\nu\right)\tilde{\kappa}_S\right] + v^5\left[\left(\frac{407}{30}\nu^2 - \frac{593}{60}\nu + \frac{2}{3}\right)\frac{\chi_A}{\delta} + \left(\frac{241}{30}\nu^2 + \frac{11}{20}\nu + \frac{2}{3}\right)\chi_S\right]\\
& + v^6\left[\left(-12\nu^2 + \frac{11}{2}\nu - \frac{7}{4}\right)\chi_A^2 + \left(44\nu^2 - \nu - \frac{7}{2}\right)\frac{\chi_A\chi_S}{\delta} + \left(6\nu^2 - \frac{27}{2}\nu - \frac{7}{4}\right)\chi_S^2\right.\\
& \left. + \left(-12\nu^2 + \frac{11}{2}\nu - \frac{7}{4}\right)\chi_A^2 + \left(44\nu^2 - \nu - \frac{7}{2}\right)\frac{\chi_A\chi_S}{\delta} + \left(6\nu^2 - \frac{27}{2}\nu - \frac{7}{4}\right)\chi_S^2\right],
\end{aligned}
\tag{B1b}
$$

$$\delta_{33} = \frac{13}{10}\omega E + (\omega E)^2\left[\left(\frac{7339}{540}\nu - \frac{81}{20}\right)\frac{\chi_A}{\delta} + \left(\frac{593}{108}\nu - \frac{81}{20}\right)\chi_S\right], \tag{B1c}$$

$$
\begin{aligned}
\rho_{32} = 1 & + v\frac{4\nu\chi_S}{3(1-3\nu)} + v^2\left[\frac{-\frac{32}{27}\nu^2 + \frac{223}{54}\nu - \frac{164}{135}}{1-3\nu} - \frac{16\nu^2\chi_S^2}{9(1-3\nu)^2}\right]\\
& + v^3\left[\left(\frac{13}{9}\nu + \frac{2}{9}\right)\frac{\delta\chi_A}{1-3\nu} + \left(\frac{607}{81}\nu^3 + \frac{503}{81}\nu^2 - \frac{1478}{405}\nu + \frac{2}{9}\right)\frac{\chi_S}{(1-3\nu)^2} + \frac{320\nu^3\chi_S^3}{81(1-3\nu)^3}\right]\\
& + v^4\left[\frac{\frac{77141}{40095}\nu^4 - \frac{508474}{40095}\nu^3 - \frac{945121}{320760}\nu^2 + \frac{1610009\nu}{320760} - \frac{180566}{200475}}{(1-3\nu)^2} + \left(4\nu^2 - 3\nu + \frac{1}{3}\right)\frac{\chi_A^2}{1-3\nu}\right.\\
& + \left(-\frac{50}{27}\nu^2 - \frac{88}{27}\nu + \frac{2}{3}\right)\frac{\delta\chi_A\chi_S}{(1-3\nu)^2} + \left(-\frac{2452}{243}\nu^4 - \frac{1997}{243}\nu^3 + \frac{1435}{243}\nu^2 - \frac{43}{27}\nu + \frac{1}{3}\right)\frac{\chi_S^2}{(1-3\nu)^3}\\
& \left. + \left(\frac{1}{3} - \frac{1}{3}\nu\right)\frac{\delta\tilde{\kappa}_A}{1-3\nu} + \left(2\nu^2 - \nu + \frac{1}{3}\right)\frac{\tilde{\kappa}_S}{1-3\nu}\right]\\
& + v^5\left[\left(-\frac{1184225}{96228}\nu^5 - \frac{40204523}{962280}\nu^4 + \frac{101706029}{962280}\nu^3 - \frac{14103833}{192456}\nu^2 + \frac{20471053}{962280}\nu - \frac{2788}{1215}\right)\frac{\chi_S}{(1-3\nu)^3}\right.\\
& + \left(\frac{889673}{106920}\nu^3 - \frac{75737}{5346}\nu^2 + \frac{376177}{35640}\nu - \frac{2788}{1215}\right)\frac{\delta\chi_A}{(1-3\nu)^2} + \left(\frac{608}{81}\nu^3 + \frac{736}{81}\nu^2 - \frac{16}{9}\nu\right)\frac{\delta\chi_A\chi_S^2}{(1-3\nu)^3}\\
& + \left(\frac{96176}{2187}\nu^5 + \frac{43528}{2187}\nu^4 - \frac{40232}{2187}\nu^3 + \frac{376}{81}\nu^2 - \frac{8\nu}{9}\right)\frac{\chi_S^3}{(1-3\nu)^4}\\
& + \left(-\frac{32}{3}\nu^3 + 8\nu^2 - \frac{8}{9}\nu\right)\frac{\chi_A^2\chi_S}{(1-3\nu)^2} + \left(\frac{8}{9}\delta\nu^2 - \frac{8}{9}\delta\nu\right)\frac{\chi_S\tilde{\kappa}_A}{(1-3\nu)^2}\\
& \left. + \left(-\frac{16}{3}\nu^3 + \frac{8}{3}\nu^2 - \frac{8}{9}\nu\right)\frac{\chi_S\tilde{\kappa}_S}{(1-3\nu)^2}\right],
\end{aligned}
\tag{B2a}
$$

$$\delta_{32} = \left(\frac{11}{5}\nu + \frac{2}{3}\right)\frac{\omega E}{1 - 3\nu} + v^4\left[\frac{4\delta\chi_A}{1 - 3\nu} + \left(\frac{36}{5}\nu^2 - 20\nu + 4\right)\frac{\chi_S}{(1 - 3\nu)^2}\right]$$

$$+ v^5\left[\left(-\frac{144}{5}\nu^3 + 80\nu^2 - 16\nu\right)\frac{\chi_S^2}{(1 - 3\nu)^3} - \frac{16\delta\nu\chi_A\chi_S}{(1 - 3\nu)^2}\right], \tag{B2b}$$

$$\rho_{31}^{\mathrm{NS}} = 1 + v^2\left(-\frac{2}{9}\nu - \frac{13}{18}\right), \tag{B3a}$$

$$f_{31}^{\mathrm{S}} = v^3\left[\left(\frac{11}{2}\nu - 2\right)\frac{\chi_A}{\delta} + \left(\frac{13}{2}\nu - 2\right)\chi_S\right]$$

$$+ v^4\left[\left(-6\nu - \frac{5}{2}\right)\chi_A^2 + (4\nu - 5)\frac{\chi_A\chi_S}{\delta} - \frac{5}{2}\chi_S^2 + \left(2\nu - \frac{5}{2}\right)\frac{\tilde{\kappa}_A}{\delta} + \left(-3\nu - \frac{5}{2}\right)\tilde{\kappa}_S\right]$$

$$+ v^5\left[\left(-\frac{931}{18}\nu^2 + \frac{25}{36}\nu + \frac{38}{9}\right)\frac{\chi_A}{\delta} + \left(-\frac{5}{2}\nu^2 - \frac{35}{12}\nu + \frac{38}{9}\right)\chi_S\right]$$

$$+ v^6\left[\left(\frac{44}{3}\nu^2 + \frac{25}{6}\nu + \frac{43}{12}\right)\chi_A^2 + \left(\frac{452}{3}\nu^2 - \frac{83}{3}\nu + \frac{43}{6}\right)\frac{\chi_A\chi_S}{\delta} + \left(22\nu^2 - \frac{35}{2}\nu + \frac{43}{12}\right)\chi_S^2\right.$$

$$\left. + \left(\frac{76}{3}\nu^2 - \frac{47}{6}\nu + \frac{43}{12}\right)\frac{\tilde{\kappa}_A}{\delta} + \left(\frac{22}{3}\nu^2 - \frac{2}{3}\nu + \frac{43}{12}\right)\tilde{\kappa}_S\right], \tag{B3b}$$

$$\delta_{31} = \frac{13}{30}\omega E + (\omega E)^2\left[\left(-\frac{9}{4}\nu + \frac{61}{20}\right)\frac{\chi_A}{\delta} + \left(\frac{77}{20}\nu + \frac{61}{20}\right)\chi_S\right], \tag{B3c}$$

$$\rho_{44} = 1 + \frac{v^2}{1 - 3\nu}\left(-\frac{175}{88}\nu^2 + \frac{587}{132}\nu - \frac{269}{220}\right) + \frac{v^3}{1 - 3\nu}\left[\left(\frac{13}{5}\nu - \frac{2}{3}\right)\delta\chi_A + \left(-\frac{14}{5}\nu^2 + \frac{41}{15}\nu - \frac{2}{3}\right)\chi_S\right]$$

$$+ v^4\left[\frac{1}{2}\delta\tilde{\kappa}_A + \left(\frac{1}{2} - \nu\right)\tilde{\kappa}_S + \left(\frac{1}{2} - 2\nu\right)\chi_A^2 + \delta\chi_A\chi_S + \frac{1}{2}\chi_S^2\right]$$

$$+ \frac{v^5}{(1 - 3\nu)^2}\left[\delta\left(\frac{597}{440}\nu^3 - \frac{1933}{220}\nu^2 + \frac{10679}{1650}\nu - \frac{69}{55}\right)\chi_A\right.$$

$$\left. + \left(\frac{591}{44}\nu^4 + \frac{8539}{440}\nu^3 - \frac{2673}{100}\nu^2 + \frac{16571}{1650}\nu - \frac{69}{55}\right)\chi_S\right], \tag{B4}$$

$$\rho_{43}^{\mathrm{NS}} = 1 + \frac{v^2}{1 - 2\nu}\left(-\frac{10}{11}\nu^2 + \frac{547}{176}\nu - \frac{111}{88}\right), \tag{B5a}$$

$$f_{43}^{\mathrm{S}} = \frac{v}{1 - 2\nu}\left(\frac{5}{2}\nu\chi_S - \frac{5}{2}\nu\frac{\chi_A}{\delta}\right) + \frac{v^3}{1 - 2\nu}\left[\left(\frac{887}{44}\nu - \frac{3143}{132}\nu^2\right)\frac{\chi_A}{\delta} + \left(-\frac{529}{132}\nu^2 - \frac{667}{44}\nu\right)\chi_S\right]$$

$$+ \frac{v^4}{1 - 2\nu}\left[\left(12\nu^2 - \frac{37}{3}\nu + \frac{3}{2}\right)\chi_A^2 + \left(\frac{137}{6}\nu^2 - 18\nu + 3\right)\frac{\chi_A\chi_S}{\delta} + \left(\frac{35}{6}\nu^2 + \frac{1}{3}\nu + \frac{3}{2}\right)\chi_S^2\right.$$

$$\left. + \left(6\nu^2 - 9\nu + \frac{3}{2}\right)\frac{\tilde{\kappa}_A}{\delta} + \left(6\nu^2 - 6\nu + \frac{3}{2}\right)\tilde{\kappa}_S\right], \tag{B5b}$$

$$\delta_{43} = \left(\frac{4961}{810}\nu + \frac{3}{5}\right)\frac{\omega E}{1-2\nu} + \frac{v^4}{(1-2\nu)^2}\left[\left(\frac{17999}{324}\nu^2 - \frac{2605}{108}\nu + \frac{11}{4}\right)\frac{\chi_A}{\delta} + \left(\frac{2569}{324}\nu^2 - \frac{2011}{108}\nu + \frac{11}{4}\right)\chi_S\right], \quad \text{(B5c)}$$

$$\rho_{42} = 1 + \frac{v^2}{1-3\nu}\left(-\frac{19}{88}\nu^2 + \frac{353}{132}\nu - \frac{191}{220}\right)$$
$$+ \frac{v^3}{1-3\nu}\left[\delta\left(\frac{7}{5}\nu - \frac{2}{3}\right)\chi_A + \left(-\frac{26}{5}\nu^2 + \frac{59}{15}\nu - \frac{2}{3}\right)\chi_S\right]$$
$$+ \frac{v^4}{1-3\nu}\left[\left(6\nu^2 - 2\nu + \frac{1}{2}\right)\chi_A^2 + \delta\chi_A\chi_S + \frac{1}{2}\chi_S^2 + \frac{1}{2}\delta\tilde{\kappa}_A + 3\nu^2\tilde{\kappa}_S - \nu\tilde{\kappa}_S + \frac{1}{2}\tilde{\kappa}_S + \frac{1}{2}\chi_S^2\right]$$
$$+ \frac{v^5}{(1-3\nu)^2}\left[\left(\frac{2463}{88}\nu^3 - \frac{4309}{220}\nu^2 + \frac{2977}{825}\nu - \frac{7}{110}\right)\delta\chi_A\right.$$
$$+ \left.\left(\frac{393}{220}\nu^4 + \frac{6437}{440}\nu^3 - \frac{12443}{1100}\nu^2 + \frac{1873}{825}\nu - \frac{7}{110}\right)\chi_S\right], \quad \text{(B6)}$$

$$\rho_{41}^{\text{NS}} = 1 + \frac{v^2}{1-2\nu}\left(-\frac{6}{11}\nu^2 + \frac{1385}{528}\nu - \frac{301}{264}\right), \quad \text{(B7a)}$$

$$f_{41}^{\text{S}} = \frac{v}{1-2\nu}\left(\frac{5}{2}\nu\chi_S - \frac{5}{2}\nu\frac{\chi_A}{\delta}\right) + \frac{v^3}{1-2\nu}\left[\left(\frac{783}{44}\nu - \frac{2207}{132}\nu^2\right)\frac{\chi_A}{\delta} + \left(-\frac{841}{132}\nu^2 - \frac{563}{44}\nu\right)\chi_S\right]$$
$$+ \frac{v^4}{1-2\nu}\left[\left(12\nu^2 - \frac{37}{3}\nu + \frac{3}{2}\right)\chi_A^2 + \left(\frac{41\nu^2}{6} - 18\nu + 3\right)\frac{\chi_A\chi_S}{\delta} + \left(\frac{35}{6} + \frac{1}{3}\nu + \frac{3}{2}\right)\chi_S^2\right.$$
$$+ \left.\left(-2\nu^2 - 9\nu + \frac{3}{2}\right)\frac{\tilde{\kappa}_A}{\delta} + \left(6\nu^2 - 6\nu + \frac{3}{2}\right)\tilde{\kappa}_S\right], \quad \text{(B7b)}$$

$$\delta_{41} = \left(\frac{507}{10}\nu + \frac{1}{5}\right)\frac{\omega E}{1-2\nu} + \frac{v^4}{(1-2\nu)^2}$$
$$\times \left[\left(\frac{533}{4}\nu^2 - \frac{55}{12}\nu + \frac{11}{12}\right)\frac{\chi_A}{\delta} + \left(-\frac{1511}{12}\nu^2 - \frac{11}{4}\nu + \frac{11}{12}\right)\chi_S\right], \quad \text{(B7c)}$$

$$\rho_{55}^{\text{NS}} = 1, \quad \text{(B8a)}$$

$$f_{55}^{\text{S}} = \frac{v^3}{1-2\nu}\left[\left(-\frac{110}{3}\nu^2 + \frac{70}{3}\nu - \frac{10}{3}\right)\frac{\chi_A}{\delta} + \left(-10\nu^2 + 10\nu - \frac{10}{3}\right)\chi_S\right]$$
$$+ v^4\left[\left(\frac{5}{2} - 10\nu\right)\chi_A^2 + (5 - 20\nu)\frac{\chi_A\chi_S}{\delta} + \frac{5}{2}\chi_S^2 + \left(\frac{5}{2} - 10\nu\right)\frac{\tilde{\kappa}_A}{\delta} + \left(\frac{5}{2} - 5\nu\right)\tilde{\kappa}_S\right], \quad \text{(B8b)}$$

$$\rho_{54} = 1 + \frac{3\nu v(-\delta\chi_A - 2\nu\chi_S + \chi_S)}{5(5\nu^2 - 5\nu + 1)} + v^2 \left\{ \frac{33320\nu^3 - 127610\nu^2 + 96019\nu - 17448}{13650(5\nu^2 - 5\nu + 1)} \right.$$

$$\left. - \frac{18\nu^2[(1-4\nu)\chi_A^2 + 2\delta(2\nu-1)\chi_A\chi_S + (1-2\nu)^2\chi_S^2]}{25(5\nu^2 - 5\nu + 1)^2} \right\}$$

$$+ v^3 \left\{ \frac{\delta(-1268785\nu^4 + 2296805\nu^3 - 1386497\nu^2 + 268024\nu - 9100)\chi_A}{68250(5\nu^2 - 5\nu + 1)^2} \right.$$

$$+ \frac{(1690430\nu^5 - 1436855\nu^4 - 174699\nu^3 + 239045\nu^2 + 426\nu - 9100)\chi_S}{68250(5\nu^2 - 5\nu + 1)^2}$$

$$\left. - \frac{162\nu^3}{125(5\nu^2 - 5\nu + 1)^3}[(1-4\nu)\delta\chi_A^3 + 3(1-2\nu)^2\delta\chi_A\chi_S^2 + (-24\nu^2 + 18\nu - 3)\chi_S\chi_A^2 - (1-2\nu)^3\chi_S^3] \right\}, \quad \text{(B9)}$$

$$\rho_{53}^{\text{NS}} = 1, \tag{B10a}$$

$$f_{53}^{\text{S}} = \frac{v^3}{1-2\nu}\left[\left(-\frac{62}{3}\nu^2 + 18\nu - \frac{10}{3}\right)\frac{\chi_A}{\delta} + \left(-\frac{46}{3}\nu^2 + \frac{46}{3}\nu - \frac{10}{3}\right)\chi_S\right]$$

$$+ \frac{v^4}{1-2\nu}\left[\left(20\nu^2 - 15\nu + \frac{5}{2}\right)\chi_A^2 + (8\nu^2 - 30\nu + 5)\frac{\chi_A\chi_S}{\delta} + \left(\frac{5}{2} - 5\nu\right)\chi_S^2\right.$$

$$\left. + \left(4\nu^2 - 15\nu + \frac{5}{2}\right)\frac{\tilde{\kappa}_A}{\delta} + \left(10\nu^2 - 10\nu + \frac{5}{2}\right)\tilde{\kappa}_S\right], \tag{B10b}$$

$$\rho_{52} = 1 + \frac{3\nu v(-\delta\chi_A - 2\nu\chi_S + \chi_S)}{5(5\nu^2 - 5\nu + 1)} + v^2\left\{\frac{21980\nu^3 - 104930\nu^2 + 84679\nu - 15828}{13650(5\nu^2 - 5\nu + 1)}\right.$$

$$\left. - \frac{18\nu^2[(1-4\nu)\chi_A^2 + 2\delta(2\nu-1)\chi_A\chi_S + (1-2\nu)^2\chi_S^2]}{25(5\nu^2 - 5\nu + 1)^2}\right\}$$

$$+ v^3\left\{\frac{(-963865\nu^4 + 1907465\nu^3 - 1213877\nu^2 + 243364\nu - 9100)\delta\chi_A}{68250(5\nu^2 - 5\nu + 1)^2}\right.$$

$$+ \frac{(1859270\nu^5 - 2079455\nu^4 + 471681\nu^3 + 17105\nu^2 + 25086\nu - 9100)\chi_S}{68250(5\nu^2 - 5\nu + 1)^2}$$

$$- \frac{162\nu^3}{125(5\nu^2 - 5\nu + 1)^3}[(1-4\nu)\delta\chi_A^3 + 3(1-2\nu)^2\delta\chi_A\chi_S^2$$

$$\left. + (-24\nu^2 + 18\nu - 3)\chi_S\chi_A^2 - (1-2\nu)^3\chi_S^3]\right\}, \tag{B11}$$

$$\rho_{51}^{\text{NS}} = 1, \tag{B12a}$$

$$f_{51}^{\text{S}} = \frac{v^3}{1-2\nu}\left[\left(-\frac{38}{3}\nu^2 + \frac{46}{3}\nu - \frac{10}{3}\right)\frac{\chi_A}{\delta} + \left(-18\nu^2 + 18\nu - \frac{10}{3}\right)\chi_S\right]$$

$$+ \frac{v^4}{1-2\nu}\left[\left(20\nu^2 - 15\nu + \frac{5}{2}\right)\chi_A^2 + (-8\nu^2 - 30\nu + 5)\frac{\chi_A\chi_S}{\delta} + \left(\frac{5}{2} - 5\nu\right)\chi_S^2\right.$$

$$\left. + \left(-4\nu^2 - 15\nu + \frac{5}{2}\right)\frac{\tilde{\kappa}_A}{\delta} + \left(10\nu^2 - 10\nu + \frac{5}{2}\right)\tilde{\kappa}_S\right], \tag{B12b}$$

$$\rho_{66} = 1 + v^3 \frac{\delta(-100\nu^2 + 85\nu - 14)\chi_A + (110\nu^3 - 150\nu^2 + 83\nu - 14)\chi_S}{21(5\nu^2 - 5\nu + 1)}, \tag{B13}$$

$$\rho_{65}^{\mathrm{NS}} = 1, \tag{B14a}$$

$$f_{65}^{\mathrm{S}} = -\frac{7\nu v[(1 - 3\nu)\chi_A + \delta(\nu - 1)\chi_S]}{2\delta(\nu - 1)(3\nu - 1)}, \tag{B14b}$$

$$\rho_{64} = 1 + v^3 \frac{\delta(-60\nu^2 + 65\nu - 14)\chi_A + (150\nu^3 - 230\nu^2 + 103\nu - 14)\chi_S}{21(5\nu^2 - 5\nu + 1)}, \tag{B15}$$

$$\rho_{63}^{\mathrm{NS}} = 1, \tag{B16a}$$

$$f_{63}^{\mathrm{S}} = -\frac{7\nu v[(1 - 3\nu)\chi_A + \delta(\nu - 1)\chi_S]}{2\delta(\nu - 1)(3\nu - 1)}, \tag{B16b}$$

$$\rho_{62} = 1 + v^3 \frac{\delta(-36\nu^2 + 53\nu - 14)\chi_A + (174\nu^3 - 278\nu^2 + 115\nu - 14)\chi_S}{21(5\nu^2 - 5\nu + 1)}, \tag{B17}$$

$$\rho_{61}^{\mathrm{NS}} = 1, \tag{B18a}$$

$$f_{61}^{\mathrm{S}} = -\frac{7\nu v[(1 - 3\nu)\chi_A + \delta(\nu - 1)\chi_S]}{2\delta(\nu - 1)(3\nu - 1)}, \tag{B18b}$$

$$\rho_{76} = 1 - \frac{4\nu v[\delta(2\nu - 1)\chi_A + (2\nu^2 - 4\nu + 1)\chi_S]}{7(7\nu^3 - 14\nu^2 + 7\nu - 1)}, \tag{B19}$$

$$\rho_{74} = 1 - \frac{4\nu v[\delta(2\nu - 1)\chi_A + (2\nu^2 - 4\nu + 1)\chi_S]}{7(7\nu^3 - 14\nu^2 + 7\nu - 1)}, \tag{B20}$$

$$\rho_{72} = 1 - \frac{4\nu v[\delta(2\nu - 1)\chi_A + (2\nu^2 - 4\nu + 1)\chi_S]}{7(7\nu^3 - 14\nu^2 + 7\nu - 1)}. \tag{B21}$$

[1] B. P. Abbott *et al.* (LIGO Scientific Collaboration and Virgo Collaboration), Observation of Gravitational Waves from a Binary Black Hole Merger, Phys. Rev. Lett. **116,** 061102 (2016).

[2] B. P. Abbott *et al.* (LIGO Scientific Collaboration and Virgo Collaboration), GWTC-1: A Gravitational-Wave Transient Catalog of Compact Binary Mergers Observed by LIGO and Virgo During the First and Second Observing Runs, Phys. Rev. X **9,** 031040 (2019).

[3] R. Abbott, T. Abbott, S. Abraham, F. Acernese, K. Ackley, A. Adams, C. Adams, R. Adhikari, V. Adya, C. Affeldt *et al.*, GWTC-2: Compact Binary Coalescences Observed by LIGO and Virgo During the First Half of the Third Observing Run, Phys. Rev. X **11,** 021053 (2021).

[4] R. Abbott *et al.* (LIGO Scientific, VIRGO, and KAGRA Collaborations), GWTC-3: Compact binary coalescences observed by LIGO and Virgo during the second part of the third observing run, arXiv:2111.03606.

[5] Amaro-Seoane *et al.*, Laser interferometer space antenna, arXiv:1702.00786.

[6] M. Punturo *et al.*, The Einstein telescope: A third-generation gravitational wave observatory, Classical Quantum Gravity **27**, 194002 (2010).

[7] Y. Mino, M. Sasaki, and T. Tanaka, Gravitational radiation reaction to a particle motion, Phys. Rev. D **55**, 3457 (1997).

[8] T. C. Quinn and R. M. Wald, An axiomatic approach to electromagnetic and gravitational radiation reaction of particles in curved space-time, Phys. Rev. D **56**, 3381 (1997).

[9] L. Barack and A. Ori, Mode sum regularization approach for the self-force in black hole space-time, Phys. Rev. D **61**, 061502 (2000).

[10] S. L. Detweiler and B. F. Whiting, Selfforce via a Green's function decomposition, Phys. Rev. D **67**, 024025 (2003).

[11] A. Pound, Second-Order Gravitational Self-Force, Phys. Rev. Lett. **109**, 051101 (2012).

[12] L. Barack and A. Pound, Self-force and radiation reaction in general relativity, Rep. Prog. Phys. **82**, 016904 (2019).

[13] A. Pound and B. Wardell, Black hole perturbation theory and gravitational self-force, arXiv:2101.04592.

[14] A. Buonanno and T. Damour, Effective one-body approach to general relativistic two-body dynamics, Phys. Rev. D **59**, 084006 (1999).

[15] A. Buonanno and T. Damour, Transition from inspiral to plunge in binary black hole coalescences, Phys. Rev. D **62**, 064015 (2000).

[16] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Binary black hole population properties inferred from the first and second observing runs of Advanced LIGO and Advanced Virgo, Astrophys. J. Lett. **882**, L24 (2019).

[17] R. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Population properties of compact objects from the second LIGO-Virgo gravitational-wave transient catalog, Astrophys. J. Lett. **913**, L7 (2021).

[18] R. Abbott *et al.* (LIGO Scientific, VIRGO, and KAGRA Collaborations), The population of merging compact binaries inferred using gravitational waves through GWTC-3, arXiv:2111.03634.

[19] M. Mathisson, Republication of: New mechanics of material systems, Gen. Relativ. Gravit. **42**, 1011 (2010).

[20] A. Papapetrou, Spinning test-particles in general relativity. I, Proc. R. Soc. A **209**, 248 (1951).

[21] E. Corinaldesi and A. Papapetrou, Spinning test-particles in general relativity. II, Proc. R. Soc. A **209**, 259 (1951).

[22] W. Tulczyjew, Motion of multipole particles in general relativity theory, Acta Phys. Pol. **18**, 37 (1959).

[23] W. G. Dixon, A covariant multipole formalism for extended test bodies in general relativity, Nuovo Cimento (1955–1965) **34**, 317 (1964).

[24] W. G. Dixon, The definition of multipole moments for extended bodies, Gen. Relativ. Gravit. **4**, 199 (1973).

[25] B. Barker and R. O'Connell, Gravitational two-body problem with arbitrary masses, spins, and quadrupole moments, Phys. Rev. D **12**, 329 (1975).

[26] B. Barker and R. O'Connell, The gravitational interaction: Spin, rotation, and quantum effects—a review, Gen. Relativ. Gravit. **11**, 149 (1979).

[27] W. G. Dixon, Extended bodies in general relativity: Their description and motion, in *Isolated Gravitating Systems in General Relativity*, edited by J. Ehlers (North Holland, Amsterdam, 1979).

[28] I. Bailey and W. Israel, Lagrangian dynamics of spinning particles and polarized media in general relativity, Commun. Math. Phys. **42**, 65 (1975).

[29] R. Porto, Post-Newtonian corrections to the motion of spinning bodies in NRGR, Phys. Rev. D **73**, 104031 (2006).

[30] J. Steinhoff and D. Puetzfeld, Multipolar equations of motion for extended test bodies in general relativity, Phys. Rev. D **81**, 044019 (2010).

[31] A. I. Harte, Mechanics of extended masses in general relativity, Classical Quantum Gravity **29**, 055012 (2012).

[32] M. Levi and J. Steinhoff, Leading order finite size effects with spins for inspiralling compact binaries, J. High Energy Phys. 06 (2015) 059.

[33] M. Levi and J. Steinhoff, Spinning gravitating objects in the effective field theory in the post-Newtonian scheme, J. High Energy Phys. 09 (2015) 219.

[34] M. Levi and J. Steinhoff, Next-to-next-to-leading order gravitational spin-squared potential via the effective field theory for spinning objects in the post-Newtonian scheme, J. Cosmol. Astropart. Phys. 01 (2016) 008.

[35] S. Marsat, Cubic order spin effects in the dynamics and gravitational wave energy flux of compact object binaries, Classical Quantum Gravity **32**, 085008 (2015).

[36] L. Kidder, C. Will, and A. Wiseman, Spin effects in the inspiral of coalescing compact binaries, Phys. Rev. D **47**, R4183 (1993).

[37] L. Kidder, Coalescing binary systems of compact objects to $(\text{post})^{5/2}$-Newtonian order. v. spin effects, Phys. Rev. D **52**, 821 (1995).

[38] H. Tagoshi, A. Ohashi, and B. Owen, Gravitational field and equations of motion of spinning compact binaries to 2.5-post-Newtonian order, Phys. Rev. D **63**, 044006 (2001).

[39] G. Faye, L. Blanchet, and A. Buonanno, Higher-order spin effects in the dynamics of compact binaries i. equations of motion, Phys. Rev. D **74**, 104033 (2006).

[40] J. Steinhoff, S. Hergt, and G. Schäfer, Next-to-leading order gravitational spin(1)-spin(2) dynamics in Hamiltonian form, Phys. Rev. D **77**, 081501(R) (2008).

[41] J. Steinhoff, G. Schäfer, and S. Hergt, Adm canonical formalism for gravitating spinning objects, Phys. Rev. D **77**, 104018 (2008).

[42] J. Steinhoff, S. Hergt, and G. Schäfer, Spin-squared Hamiltonian of next-to-leading order gravitational interaction, Phys. Rev. D **78**, 101503(R) (2008).

[43] R. A. Porto and I. Z. Rothstein, Spin(1)Spin(2) effects in the motion of inspiralling compact binaries at third order in the post-Newtonian expansion, Phys. Rev. D **78**, 044012 (2008).

[44] R. A. Porto and I. Z. Rothstein, Next to leading order spin (1)spin(1) effects in the motion of inspiralling compact binaries, Phys. Rev. D **78,** 044013 (2008).

[45] R. A. Porto, A. Ross, and I. Z. Rothstein, Spin induced multipole moments for the gravitational wave flux from binary inspirals to third post-Newtonian order, J. Cosmol. Astropart. Phys. 03 (2011) 009.

[46] R. A. Porto, Next-to-leading-order spin orbit effects in the motion of inspiralling compact binaries, Classical Quantum Gravity **27,** 205001 (2010).

[47] M. Levi, Next-to-leading order gravitational spin-orbit coupling in an effective field theory approach, Phys. Rev. D **82,** 104004 (2010).

[48] M. Levi, Next-to-leading order gravitational spin1-spin2 coupling with Kaluza-Klein reduction, Phys. Rev. D **82,** 064029 (2010).

[49] M. Levi, Binary dynamics from spin1-spin2 coupling at fourth post-Newtonian order, Phys. Rev. D **85,** 064043 (2012).

[50] S. Marsat, A. Bohé, G. Faye, and L. Blanchet, Next-to-next-to-leading order spin-orbit effects in the equations of motion of compact binary systems, Classical Quantum Gravity **30,** 055007 (2013).

[51] A. Bohé, S. Marsat, G. Faye, and L. Blanchet, Next-to-next-to-leading order spin-orbit effects in the near-zone metric and precession equations of compact binary systems, Classical Quantum Gravity **30,** 075017 (2013).

[52] T. Damour, P. Jaranowski, and G. Schäfer, Hamiltonian of two spinning compact bodies with next-to-leading order gravitational spin-orbit coupling, Phys. Rev. D **77,** 064032 (2008).

[53] S. Hergt, J. Steinhoff, and G. Schäfer, The reduced Hamiltonian for next-to-leading-order spin-squared dynamics of general compact binaries, Classical Quantum Gravity **27,** 135007 (2010).

[54] J. Hartung and J. Steinhoff, Next-to-leading order spin-orbit and spin(a)-spin(b) Hamiltonians for *n* gravitating spinning compact objects, Phys. Rev. D **83,** 044008 (2011).

[55] J. Hartung and J. Steinhoff, Next-to-next-to-leading order post-Newtonian spin-orbit Hamiltonian for self-gravitating binaries, Ann. Phys. (Berlin) **523,** 783 (2011).

[56] J. Hartung and J. Steinhoff, Next-to-next-to-leading order post-Newtonian spin(1)-spin(2) Hamiltonian for self-gravitating binaries, Ann. Phys. (Berlin) **523,** 919 (2011).

[57] J. Hartung, J. Steinhoff, and G. Schäfer, Next-to-next-to-leading order post-Newtonian linear-in-spin binary Hamiltonians, Ann. Phys. (Berlin) **525,** 359 (2013).

[58] A. Buonanno, G. Faye, and T. Hinderer, Spin effects on gravitational waves from inspiraling compact binaries at second post-Newtonian order, Phys. Rev. D **87,** 044009 (2013).

[59] M. Levi and J. Steinhoff, Equivalence of ADM Hamiltonian and effective field theory approaches at next-to-next-to-leading order spin1-spin2 coupling of binary inspirals, J. Cosmol. Astropart. Phys. 12 (2014) 003.

[60] M. Levi and J. Steinhoff, Next-to-next-to-leading order gravitational spin-orbit coupling via the effective field theory for spinning objects in the post-Newtonian scheme, J. Cosmol. Astropart. Phys. 01 (2016) 011.

[61] S. Marsat, A. Bohe, G. Faye, and L. Blanchet, Next-to-next-to-leading order spin-orbit effects in the equations of motion of compact binary systems, Classical Quantum Gravity **30,** 055007 (2013).

[62] M. Levi and J. Steinhoff, Complete conservative dynamics for inspiralling compact binaries with spins at the fourth post-Newtonian order, J. Cosmol. Astropart. Phys. 09 (2021) 029.

[63] M. Levi, S. Mougiakakos, and M. Vieira, Gravitational cubic-in-spin interaction at the next-to-leading post-Newtonian order, J. High Energy Phys. 01 (2021) 036.

[64] J. Vines and J. Steinhoff, Spin-multipole effects in binary black holes and the test-body limit, Phys. Rev. D **97,** 064010 (2018).

[65] A. Antonelli, C. Kavanagh, M. Khalil, J. Steinhoff, and J. Vines, Gravitational spin-orbit and aligned spin$_1$-spin$_2$ couplings through third-subleading post-Newtonian orders, Phys. Rev. D **102,** 124024 (2020).

[66] A. Antonelli, C. Kavanagh, M. Khalil, J. Steinhoff, and J. Vines, Gravitational Spin-Orbit Coupling Through Third-Subleading Post-Newtonian Order: From First-Order Self-Force to Arbitrary Mass Ratios, Phys. Rev. Lett. **125,** 011103 (2020).

[67] B. Mikóczi, M. Vasúth, and L. Gergely, Self-interaction spin effects in inspiralling compact binaries, Phys. Rev. D **71,** 124043 (2005).

[68] L. Blanchet, A. Buonanno, and G. Faye, Higher-order spin effects in the dynamics of compact binaries II. Radiation field, Phys. Rev. D **74,** 104034 (2006); Erratum, Phys. Rev. D **75,** 049903 (2007).

[69] L. Blanchet, A. Buonanno, and G. Faye, Tail-induced spin-orbit effect in the gravitational radiation of compact binaries, Phys. Rev. D **84,** 064041 (2011).

[70] A. Bohé, S. Marsat, and L. Blanchet, Next-to-next-to-leading order spin-orbit effects in the gravitational wave flux and orbital phasing of compact binaries, Classical Quantum Gravity **30,** 135009 (2013).

[71] S. Marsat, A. Bohé, L. Blanchet, and A. Buonanno, Next-to-leading tail-induced spin-orbit effects in the gravitational radiation of compact binaries, Classical Quantum Gravity **31,** 025023 (2013).

[72] A. Bohé, G. Faye, S. Marsat, and E. Porter, Quadratic-in-spin effects in the orbital dynamics and gravitational-wave energy flux of compact binaries at the 3PN order, Classical Quantum Gravity **32,** 195010 (2015).

[73] G. Cho, R. A. Porto, and Z. Yang, Gravitational radiation from inspiralling compact objects: Spin effects to fourth post-Newtonian order, Phys. Rev. D **106,** L101501 (2022).

[74] K. Arun, A. Buonanno, G. Faye, and E. Ochsner, Higher-order spin effects in the amplitude and phase of gravitational waveforms emitted by inspiraling compact binaries: Ready-to-use gravitational waveforms, Phys. Rev. D **79,** 104023 (2009).

[75] R. A. Porto, A. Ross, and I. Z. Rothstein, Spin induced multipole moments for the gravitational wave amplitude from binary inspirals to 2.5 post-Newtonian order, J. Cosmol. Astropart. Phys. 09 (2012) 028.

[76] A. Buonanno and T. Damour, Effective one-body approach to general relativistic two-body dynamics, Phys. Rev. D **59,** 084006 (1999).

[77] T. Damour and A. Nagar, Faithful effective-one-body waveforms of small-mass-ratio coalescing black-hole binaries, Phys. Rev. D **76,** 064028 (2007).

[78] T. Damour and A. Nagar, Comparing effective-one-body gravitational waveforms to accurate numerical data, Phys. Rev. D **77,** 024043 (2008).

[79] Y. Pan, A. Buonanno, R. Fujita, E. Racine, and H. Tagoshi, Post-Newtonian factorized multipolar waveforms for spinning, non-precessing black-hole binaries, Phys. Rev. D **83,** 064003 (2011).

[80] T. Damour, B. Iyer, and A. Nagar, Improved resummation of post-Newtonian multipolar waveforms from circularized compact binaries, Phys. Rev. D **79,** 064004 (2009).

[81] A. Taracchini, Y. Pan, A. Buonanno, E. Barausse, M. Boyle, T. Chu, G. Lovelace, H. P. Pfeiffer, and M. A. Scheel, Prototype effective-one-body model for nonprecessing spinning inspiral-merger-ringdown waveforms, Phys. Rev. D **86,** 024011 (2012).

[82] A. Bohé *et al.*, Improved effective-one-body model of spinning, nonprecessing binary black holes for the era of gravitational-wave astrophysics with advanced detectors, Phys. Rev. D **95,** 044028 (2017).

[83] A. Nagar and A. Shah, Factorization and resummation: A new paradigm to improve gravitational wave amplitudes, Phys. Rev. D **94,** 104017 (2016).

[84] A. Nagar *et al.*, Time-domain effective-one-body gravitational waveforms for coalescing compact binaries with nonprecessing spins, tides and self-spin effects, Phys. Rev. D **98,** 104052 (2018).

[85] A. Nagar, G. Pratten, G. Riemenschneider, and R. Gamba, Multipolar effective one body model for nonspinning black hole binaries, Phys. Rev. D **101,** 024041 (2020).

[86] F. Messina, A. Maldarella, and A. Nagar, Factorization and resummation: A new paradigm to improve gravitational wave amplitudes. II: the higher multipolar modes, Phys. Rev. D **97,** 084016 (2018).

[87] R. Cotesta, A. Buonanno, A. Bohé, A. Taracchini, I. Hinder, and S. Ossokine, Enriching the symphony of gravitational waves from binary black holes by tuning higher harmonics, Phys. Rev. D **98,** 084028 (2018).

[88] L. Blanchet, Gravitational radiation from post-Newtonian sources and inspiralling compact binaries, Living Rev. Relativity **17,** 2 (2014).

[89] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevD.106.124018, for the waveform modes with 3.5PN spin contributions.

[90] K. Thorne, Multipole expansions of gravitational radiation, Rev. Mod. Phys. **52,** 299 (1980).

[91] L. Kidder, Using full information when computing modes of post-Newtonian waveforms from inspiralling compact binaries in circular orbits, Phys. Rev. D **77,** 044016 (2008).

[92] L. Blanchet, G. Faye, B. R. Iyer, and S. Sinha, The third post-Newtonian gravitational wave polarisations and associated spherical harmonic modes for inspiralling compact binaries in quasi-circular orbits, Classical Quantum Gravity **25,** 165003 (2008); Erratum, Classical Quantum Gravity **29,** 239501 (2012).

[93] Q. Henry, G. Faye, and L. Blanchet, The current-type quadrupole moment and gravitational-wave mode $(\ell, m) = (2, 1)$ of compact binary systems at the third post-Newtonian order, Classical Quantum Gravity **38,** 185004 (2021).

[94] Q. Henry, G. Faye, and L. Blanchet, Tidal effects in the gravitational-wave phase evolution of compact binary systems to next-to-next-to-leading post-Newtonian order, Phys. Rev. D **102,** 044033 (2020).

[95] L. Blanchet, T. Damour, G. Esposito-Farèse, and B. R. Iyer, Dimensional regularization of the third post-Newtonian gravitational wave generation of two point masses, Phys. Rev. D **71,** 124004 (2005).

[96] L. Blanchet and G. Faye, Hadamard regularization, J. Math. Phys. (N.Y.) **41,** 7675 (2000).

[97] L. Blanchet and G. Faye, General relativistic dynamics of compact binaries at the third post-Newtonian order, Phys. Rev. D **63,** 062005 (2001).

[98] T. Marchand, Q. Henry, F. Larrouturou, S. Marsat, G. Faye, and L. Blanchet, The mass quadrupole moment of compact binary systems at the fourth post-Newtonian order, Classical Quantum Gravity **37,** 215006 (2020).

[99] L. Blanchet and B. R. Iyer, Hadamard regularization of the third post-Newtonian gravitational wave generation of two point masses, Phys. Rev. D **71,** 024004 (2004).

[100] L. Blanchet, G. Faye, and B. Ponsot, Gravitational field and equations of motion of compact binaries to 5/2 post-Newtonian order, Phys. Rev. D **58,** 124002 (1998).

[101] G. Faye, L. Blanchet, and B. R. Iyer, Non-linear multipole interactions and gravitational-wave octupole modes for inspiralling compact binaries to third-and-a-half post-Newtonian order, Classical Quantum Gravity **32,** 045016 (2015).

[102] L. Blanchet, G. Faye, and F. Larrouturou, The quadrupole moment of compact binaries to the fourth post-Newtonian order: From source to canonical moment, Classical Quantum Gravity **39,** 195003 (2022).

[103] L. Blanchet and G. Schäfer, Gravitational wave tails and binary star systems, Classical Quantum Gravity **10,** 2699 (1993).

[104] K. Arun, L. Blanchet, B. Iyer, and M. Qusailah, Inspiralling compact binaries in quasi-elliptical orbits: The complete 3PN energy flux, Phys. Rev. D **77,** 064035 (2008).

[105] I. Gradshteyn and I. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, New York, 1980).

[106] M. Favata, Post-Newtonian corrections to the gravitational-wave memory for quasicircular, inspiralling compact binaries, Phys. Rev. D **80,** 024002 (2009).

[107] K. Mitman *et al.*, Fixing the BMS frame of numerical relativity waveforms with BMS charges, Phys. Rev. D **106,** 084029 (2022).

[108] T. Tanaka, Y. Mino, M. Sasaki, and Shibata, Gravitational waves from a spinning particle in circular orbits around a rotating black hole, Phys. Rev. D **54,** 3762 (1996).

[109] T. Regge and J. A. Wheeler, Stability of a Schwarzschild singularity, Phys. Rev. **108,** 1063 (1957).

[110] F. J. Zerilli, Effective Potential for Even Parity Regge-Wheeler Gravitational Perturbation Equations, Phys. Rev. Lett. **24,** 737 (1970).

[111] E. Poisson, Gravitational radiation from a particle in circular orbit around a black hole. 1: Analytical results for the nonrotating case, Phys. Rev. D **47,** 1497 (1993).

[112] L. Blanchet, Gravitational wave tails of tails, Classical Quantum Gravity **15**, 113 (1998); Erratum, Classical Quantum Gravity **22**, 3381 (2005).

[113] H. Asada and T. Futamase, Propagation of gravitational waves from slow motion sources in Coulomb type potential, Phys. Rev. D **56**, R6062 (1997).

[114] Y. Pan, A. Buonanno, M. Boyle, L. T. Buchman, L. E. Kidder, H. P. Pfeiffer, and M. A. Scheel, Inspiral-merger-ringdown multipolar waveforms of nonspinning black-hole binaries using the effective-one-body formalism, Phys. Rev. D **84**, 124052 (2011).

[115] Sergei Ossokine (private communication).