Constructible reality condition of pseudo entropy

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Dedicated to Hermann Nicolai on the occasion of his 70^{th} *birthday.*

Abstract

As a generalization of entanglement entropy, pseudo entropy is not always real. The real-valued pseudo entropy has promising applications in holography and quantum phase transition. We apply the notion of pseudo-Hermticity to formulate the reality condition of pseudo entropy. We find the general form of the transition matrix for which the eigenvalues of the reduced transition matrix possess real or complex pairs of eigenvalues. Further, we construct a class of transition matrices for which the pseudo (Rényi) entropies are positive. Some known examples which give real pseudo entropy in quantum field theories can be explained in our framework. Our results offer a novel method to generate the transition matrix with real pseudo entropy. Finally, we show the reality condition for pseudo entropy is related to the Tomita-Takesaki modular theory for quantum field theory.

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1 Introduction

Quantum entanglement is a fundamental concept in various branches of physics[1]. Entanglement entropy (EE) as entanglement measure has been investigated at many aspects[2]-[9]. Specially, in the context of AdS/CFT [10][11][12], entanglement plays an important role to understand the emergence of geometry[13][14], subregion/subregion duality[15][16] and information paradox of black hole[17][18]. Our paper will focus on a generalization of EE, called pseudo entropy, which may bring us a new understanding of the role of entanglement in quantum field theory (QFT) or gravity.

Assume the density matrix of the system is ρ. One could divide the system into two subsystems A and $\bar{A}$. The reduced density matrix of A is defined by $\rho_A := \mathrm{tr}_{\bar{A}}\rho$. The Von Neumann entropy gives the EE

$$S(\rho_A) := -\mathrm{tr}(\rho_A \log \rho_A). \tag{1}$$

In the framework of AdS/CFT, the EE in CFTs can be evaluated by the area of the extremal surface in bulk, known as Ryu-Takayanagi formula[8][9]. The pseudo entropy is a generalization of entanglement entropy by introducing the transition matrix between two pure states $|\psi\rangle$ and $|\phi\rangle$ as

$$\mathcal{T}^{\psi|\phi} := \frac{|\psi\rangle\langle\phi|}{\langle\phi|\psi\rangle}. \tag{2}$$

Similarly, one could define the reduced transition matrix

$$\mathcal{T}_A^{\psi|\phi} := \mathrm{tr}_{\bar{A}}\mathcal{T}^{\psi|\phi}. \tag{3}$$

The pseudo Rényi entropy is defined as

$$S^{(n)}(\mathcal{T}_A^{\psi|\phi}) = \frac{1}{1-n}\log \mathrm{tr}[(\mathcal{T}_A^{\psi|\phi})^n]. \tag{4}$$

The idea of pseudo entropy was first introduced in [19], see also [20]-[25] for recent studies. In this paper, we will also consider a generalization of the transition matrix for the mixed state, which can be expressed as

$$X := \sum_{i,j} X_{ij}|\psi_i\rangle\langle\phi_j|, \tag{5}$$

where X_{ij} are the constant coefficients, $|\psi_i\rangle$ and $|\phi_j\rangle$ are pure states. One could define the reduced transition matrix $X_A := \mathrm{tr}_{\bar{A}}X$ and the corresponding pseudo Rényi entropy as Eq. (4). The pseudo Rényi entropy can be evaluated using the replica method [7] in path integral formulation in quantum field theories. According to the AdS/CFT, the path integral in CFTs can be translated to the gravitational path integral in AdS. It is proposed in [19] that the pseudo entropy can also be calculated by generalizing the RT formula to Euclidean time-dependent gravity background, which is associated with the states $|\psi\rangle$ and $|\phi\rangle$. Thus it depends both on two different states. Compared with EE, this new quantity includes more information on the relation between quantum information in CFTs and geometries in AdS. It is one of the motivations of [19] to propose this novel quantity. Moreover, pseudo entropy can also be used in quantum many-body systems. It can be taken as a new order parameter as EE[3][4].

However, some properties of the pseudo entropy are still unclear. In general, X and X_A are not Hermitian. The eigenvalues of X_A are not necessarily real. Thus the pseudo entropy may not be real. The most interesting class of transition matrices is the one that gives real pseudo entropy. The real-valued pseudo entropy can have a holographic counterpart. Further, pseudo entropy could be a new order parameter to capture the phase transition[20][21]. The reality condition of the pseudo entropy is still unknown. It is the central motivation of our paper to find such conditions. The problem is closely related to non-Hermitian physics, which has been extensively studied recently; see the review [26]. It is found the eigenvalues of the non-Hermitian Hamiltonian can be real. The system satisfying parity-time (PT) symmetry is one of the most important classes [27][28]. The notion of pseudo-Hermiticity is handy to character a class of non-Hermitian matrices having real eigenvalues[29][30][31] if the matrix M has a complete bi-orthonormal eigenbasis.

An operator M is said to be η-pseudo-Hermitian if there exists a Hermitian invertible operator η such that

$$M^{\dagger} = \eta M \eta^{-1}. \tag{6}$$

If η is the identity, the pseudo-Hermitian condition reduces to Hermiticity. The necessary and sufficient conditions for the pseudo-Hermiticity of M are given by the following theorem[29].

Theorem 1: *An operator M with a complete biorthonormal eigenbasis and a discrete spectrum is pseudo-Hermitian if and only if one of the following conditions hold:*

1. *The eigenvalues of M are real.*

2. *The complex eigenvalues come in complex conjugate pairs, and the degeneracy of the eigenvalues are same.*

Remark 1: The existence of a bi-orthonormal eigenbasis in Hilbert space with respect to M is equivalent to M being diagonalizable. It follows that, according to the Theorem 1, when the reduced transition matrix X_A is diagonalizable, the necessary and sufficient condition of $\mathrm{tr}[(X_A)^n]$ being real is that X_A is η_A-pseudo-Hermitian.[4] Throughout this paper, unless otherwise stated, we will assume that the reduced transition matrix is diagonalizable. The pseudo Rényi entropy $S^{(n)}(X_A)$, however, may not be real in this case. Apparently, to guarantee the reality of $S^{(n)}(X_A)$, one should require $\mathrm{tr}[(X_A)^n] > 0$, which gives more constraints on η_A.

2 General form of transition matrix

Our motivation is to construct transition matrix X (5) with positive $\mathrm{tr}[(X_A)^n]$ using the notion of pseudo-Hermiticity. But step by step, let's first look at what constraints the real $\mathrm{tr}[(X_A)^n]$

[4]When X_A is non-diagonalizable, a pseudo-Hermitian X_A can still give a real $\mathrm{tr}[(X_A)^n]$, but the converse is not necessarily true. We show an example in Appendix A where $\mathrm{tr}[(X_A)^n]$ is real, but X_A is not pseudo-Hermitian.

conditions place on the transition matrix. If the reduced transition matrix $X_{A(\bar{A})}$ is $\eta_{A(\bar{A})}$-pseudo-Hermitian, in general, the transition matrix X is not pseudo-Hermitian. However, we would show that X can be written as a linear combination of pseudo-Hermitian operators.

Proposition 1: *Any operator $\mathcal{O}$ can be decomposed as*

$$\mathcal{O} = \mathcal{O}_1 + i\mathcal{O}_2, \tag{7}$$

where $\mathcal{O}_1$ and $\mathcal{O}_2$ are η-pseudo-Hermitian operators, η can be any Hermitian invertible operator.

Proof: For any operator $\mathcal{O}$, we can divide it into two parts

$$\mathcal{O} = \frac{\mathcal{O} + \eta^{-1}\mathcal{O}^\dagger\eta}{2} + i\frac{\mathcal{O} - \eta^{-1}\mathcal{O}^\dagger\eta}{2i}, \tag{8}$$

where η is any Hermitian invertible operator. For η being identity, (8) is the well-known result that any operator can be decomposed as linear combinations of two Hermitian operators. Where $\mathcal{O}_1 = \frac{\mathcal{O}+\eta^{-1}\mathcal{O}^\dagger\eta}{2}$ and $\mathcal{O}_2 = \frac{\mathcal{O}-\eta^{-1}\mathcal{O}^\dagger\eta}{2i}$. □

Proposition 2: *$X_{A(\bar{A})}$ is $\eta_{A(\bar{A})}$-pseudo-Hermitian, if and only if the transition matrix X can be written as*

$$X = X_1 + iX_2, \tag{9}$$

where X_1 and X_2 are both η-pseudo-Hermitian with $\eta = \eta_A \otimes \eta_{\bar{A}}$. Further, X_2 satisfies $tr_{\bar{A}(A)}X_2 = 0$.

Proof: Using the result of Proposition 1, let's define the operator

$$\begin{aligned} X_1 &:= \frac{1}{2}(X + \eta^{-1}X^\dagger\eta), \\ X_2 &:= \frac{i}{2}(\eta^{-1}X^\dagger\eta - X). \end{aligned} \tag{10}$$

Since $X_{A(\bar{A})}$ is $\eta_{A(\bar{A})}$-pseudo-Hermitian, we have

$$\mathrm{tr}_{\bar{A}}X_2 = \frac{i}{2}[\eta_A^{-1}(\mathrm{tr}_{\bar{A}}X^\dagger)\eta_A - \mathrm{tr}_{\bar{A}}X] = \frac{1}{2}(\eta_A^{-1}X_A^\dagger\eta_A - X_A) = 0. \tag{11}$$

Similarly, one could show $\mathrm{tr}_A X_2 = 0$.

If X can be written as (9) and $\mathrm{tr}_{\bar{A}}X_2 = 0$, we have $X_A = \mathrm{tr}_{\bar{A}}X_1$ and

$$X_A^\dagger = \mathrm{tr}_{\bar{A}}X^\dagger = \mathrm{tr}_{\bar{A}}(\eta X_1\eta^{-1}) = \eta_A(\mathrm{tr}_{\bar{A}}X_1)\eta_A^{-1} = \eta_A X_A \eta_A^{-1}. \tag{12}$$

Thus X_A is η_A-pseudo-Hermitian. Similarly, we can show $X_{\bar{A}}$ is $\eta_{\bar{A}}$-pseudo-Hermitian. □

An obvious corollary of Proposition 2 is that a η-pseudo-Hermitian transition matrix with $\eta = \eta_A \otimes \eta_{\bar{A}}$ generates a pseudo-Hermitian reduced transition matrix. Let's first focus on the pure states transition matrix $\mathcal{T}^{\psi|\phi}$ and show how to construct the pseudo-Hermitian transition matrices. The η-pseudo-Hermiticity gives constraints on the pure states $|\psi\rangle$ and $|\phi\rangle$. We have the following theorem.

Theorem 2: *The transition matrix $\mathcal{T}^{\psi|\phi}$ is η-pseudo-Hermitian , if and only if it can be written as follows.*

$$\mathcal{T}^{\psi|\phi} = \frac{|\psi\rangle\langle\psi|\eta}{\langle\psi|\eta|\psi\rangle}. \tag{13}$$

Proof: Now assume the transition matrix is η-pseudo-Hermitian, we have

$$\frac{\eta|\psi\rangle\langle\phi|\eta^{-1}}{\langle\phi|\psi\rangle} = \frac{|\phi\rangle\langle\psi|}{\langle\psi|\phi\rangle}. \tag{14}$$

This leads to

$$\eta|\psi\rangle = \frac{\langle\psi|\eta|\psi\rangle}{\langle\psi|\phi\rangle}|\phi\rangle. \tag{15}$$

Taking the above formula into (2), one can show the transition matrix $\mathcal{T}^{\psi|\phi}$ is given by (13). □

Remark 2: In the Theorem 2, the pure states $|\psi\rangle$ and $|\phi\rangle$ of the transition matrix $\mathcal{T}^{\psi|\phi}$ are assumed to be non-orthogonal. For the case $\langle\phi|\psi\rangle = 0$, one could consider the matrix $\mathcal{T}'^{\psi|\phi} = |\psi\rangle\langle\phi|$. If it is η-pseudo-Hermitian, we have

$$\eta|\psi\rangle\langle\phi|\eta^{-1} = |\phi\rangle\langle\psi|. \tag{16}$$

To satisfy the above condition it is necessary that $\langle\psi|\eta|\phi\rangle = \langle\phi|\eta|\psi\rangle \neq 0$, otherwise we would have $\eta|\psi\rangle = 0$, which is impossible for $|\psi\rangle \neq 0$. Further, we have the relation

$$|\phi\rangle = \frac{\eta|\psi\rangle}{\langle\psi|\eta|\phi\rangle}. \tag{17}$$

Therefore, the transition matrix $|\psi\rangle\langle\phi|$ is given by

$$\mathcal{T}'^{\psi|\phi} = \frac{|\psi\rangle\langle\psi|\eta}{\langle\psi|\eta|\phi\rangle}, \tag{18}$$

for the case $\langle\psi|\phi\rangle = 0$.

Consider a linear combination of the transition matrices $\{\mathcal{T}^{\psi_i|\phi_i}\}$,

$$\mathcal{T} := \sum_i t_i \mathcal{T}^{\psi_i|\phi_i}. \tag{19}$$

Corollary 1: *If $\{\mathcal{T}^{\psi_i|\phi_i}\}$ are all η-pseudo-Hermitian, $\mathcal{T}$ can be expressed as*

$$\mathcal{T} = \mathcal{T}_1 + i\mathcal{T}_2, \tag{20}$$

where $\mathcal{T}_1$ and $\mathcal{T}_2$ are η-pseudo-Hermitian.

The above results can be shown by writing $t_i = t_i^R + it_i^I$, where t_i^R and t_i^I are real and imaged part of t_i.

Corollary 2: *Any diagonalizable η-pseudo-Hermitian matrix M can be expressed as*

$$M = \sum_i m_i \mathcal{T}^{\psi_i|\phi_i}, \tag{21}$$

where m_i are real, $\mathcal{T}^{\psi_i|\phi_i}$ are η-pseudo-Hermitian transition matrices between two pure states $|\psi_i\rangle$ and $|\phi_i\rangle$.

The proof can be found in Appendix B. As a summary of our results, we have the following theorem.

Theorem 3: *Suppose that there is a Hermitian invertible matrix $\eta = \eta_A \otimes \eta_{\bar{A}}$ such that $X \pm \eta^{-1}X^\dagger\eta$ are diagonalizable, then X_A and $X_{\bar{A}}$ are η_A- and $\eta_{\bar{A}}$-pseudo-Hermitian, if and only if the transition matrix X is given by*

$$X = \sum_i x_i^1 \mathcal{T}^{\psi_i|\phi_i} + i \sum_j x_j^2 \mathcal{T}^{\psi_j|\phi_j}, \tag{22}$$

where x_i^1 and x_j^2 are real, $\mathcal{T}^{\psi_i|\phi_i}$ are η-pseudo-Hermitian with $\eta = \eta_A \otimes \eta_{\bar{A}}$, which take the form (13) or (18). Further, we have the constraint $tr_{A(\bar{A})} \sum_j x_j^2 \mathcal{T}^{\psi_j|\phi_j} = 0$.

Remark 3: The above theorem provides a method to check whether the reduced transition matrices X_A and $X_{\bar{A}}$ are pseudo-Hermitian or not. On the other hand, it also gives us a way to generate the transition matrices X, for which their reduced transition matrices are pseudo-Hermitian. We will use some examples below to show the applications of our results. The pseudo-Hermiticity of X_A and $X_{\bar{A}}$ only guarantees $\mathrm{tr}[(X_{A(\bar{A})})^n]$ is real, not necessarily positive. One should give more constraints on $\eta_{A(\bar{A})}$ to make $\mathrm{tr}[(X_{A(\bar{A})})^n] > 0$. One special set of the transition matrices is the one that gives X_A and $X_{\bar{A}}$ whose eigenvalues are a positive real number. The following result provides a sufficient condition for this set.

Corollary 3: *$\mathcal{T}^{\psi|\phi}$ is η-pseudo-Hermitian with $\eta = \eta_A \otimes \eta_{\bar{A}}$.*

1. *If $\eta_{A(\bar{A})}$ is positive or negative definite operator, the eigenvalues of $\mathcal{T}_{A(\bar{A})}$ are real.*
2. *If η_A is positive or negative and $\eta_{\bar{A}}$ is positive or negative too, then the eigenvalues of $\mathcal{T}_{A(\bar{A})}$ are non-negative.*

Proof: In general, $\mathcal{T}_{A(\bar{A})}$ are expected to have complex eigenvalues for arbitrary Hermitian $\eta_{A(\bar{A})}$. Let us define

$$\tilde{\mathcal{T}}_A := \frac{\mathrm{tr}_{\bar{A}}\left(|\psi\rangle\langle\psi|\eta_{\bar{A}}\right)}{\langle\psi|\eta|\psi\rangle}. \tag{23}$$

It is obvious that $\tilde{\mathcal{T}}_A$ is a Hermitian operator. By using (13) we have

$$\mathcal{T}_A = \mathrm{tr}_{\bar{A}}\mathcal{T}^{\psi|\phi} = \tilde{\mathcal{T}}_A \eta_A. \tag{24}$$

Assume that η_A is a positive (negative) operator. There exists a Hermitian (skew-Hermitian) invertible operator $\eta_A^{1/2}$ such that $(\eta_A^{1/2})^2 = \eta_A$. Thus we have

$$\eta_A^{1/2}\mathcal{T}_A\eta_A^{-1/2} = \eta_A^{1/2}\tilde{\mathcal{T}}_A\eta_A^{1/2}. \tag{25}$$

$\mathcal{T}_A$ is similar to the operator on the right hand side of the above equation, which is Hermitian. Thus, the eigenvalues of $\mathcal{T}_A$ are real.

If further assuming $\eta_{\bar{A}}$ is positive (negative) definite, we have

$$\tilde{\mathcal{T}}_A = \frac{\mathrm{tr}_{\bar{A}}\left(\eta_{\bar{A}}^{1/2}|\psi\rangle\langle\psi|\eta_{\bar{A}}^{1/2}\right)}{\langle\psi|\eta|\psi\rangle}, \tag{26}$$

where we define the Hermitian (skew-Hermitian) invertible operator $\eta_{\bar{A}}^{1/2}$ and use the cyclic property of partial trace. It is not hard to show $\eta_A^{1/2}\tilde{\mathcal{T}}_A\eta_A^{1/2}$ is always positive semi-definite, the eigenvalues of which are non-negative. Therefore, using (25), we have proved the eigenvalues of $\mathcal{T}_A$ are non-negative. By the similar process one could show the eigenvalues of $\mathcal{T}_{\bar{A}}$ are all non-negative. □

Corollary 4: *The transition matrix X taking the form (22) is η-pseudo-Hermitian with $\eta = \eta_A \otimes \eta_{\bar{A}}$. Assume $\eta_{A(\bar{A})}$ are positive or negative definite operators. If $x_i^1 > 0$, the eigenvalues of X_A and $X_{\bar{A}}$ are non-negative.*

Remark 4: One could show this by using those linear combinations of positive semi-definite operators with positive coefficients that are also positive semi-definite. One could slightly generalize the above result. Suppose some coefficients, say the set $\{x_a\}$ with $x_a < 0$, is negative. If the transition matrix $\mathcal{T}^{\psi_a|\phi_a}$ satisfies $\mathrm{tr}_{A(\bar{A})}\sum_a \mathcal{T}^{\psi_a|\phi_a} = 0$, one could also show the eigenvalues of $X_{A(\bar{A})}$ are non-negative.

If the eigenvalues of X_A and $X_{\bar{A}}$ are positive, one could show $0 < \mathrm{tr}[(X_{A(\bar{A})})^n] < 1$. Therefore, Theorem 3 provides a method to generate transition matrix X, for which the pseudo Rényi entropy of $X_{A(\bar{A})}$ is positive. Note that the above condition is only sufficient to have pseudo Rényi entropy been positive. In Appendix E.2 we show an example of finite dimension. $\eta_{A(\bar{A})}$ are neither positive nor negative, but the pseudo Rényi entropy is positive for integers $n \geq 2$.

3 Construction of the transition matrix

According to (13) one could construct the η-pseudo-Hermitian transition matrix by fixing $|\psi\rangle$ and η. In QFTs, the Hilbert space can be built by acting operators in a vacuum. We would like to review the algebraic view of QFTs.

Firstly, let's start with finite-dimension examples. The Hilbert space of the total system is denoted by $\mathcal{H}$. Considering a bipartite quantum system with $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$. Assume the dimension of $\mathcal{H}_A$ and $\mathcal{H}_{\bar{A}}$ are same. Let's denote $\mathcal{R}_{A(\bar{A})}$ to be the algebra of operators working on $\mathcal{H}_{A(\bar{A})}$. The algebra of operators for the total system is given by $\mathcal{R} = \mathcal{R}_A \otimes \mathcal{R}_{\bar{A}}$. Choosing a reference state $|\Psi\rangle$, one could contruct the states in $\mathcal{H}$ by acting the operators in $\mathcal{R}_{A(\bar{A})}$ on $|\Psi\rangle$. A state $|\Psi\rangle$ is said to be cyclic for the algebra such as $\mathcal{R}_A$ with respect to $\mathcal{H}$ if the set $\{a|\Psi\rangle\}$, $a \in \mathcal{R}_A$ is dense in $\mathcal{H}$. For the finite-dimensional case one could choose the state

$$|\Psi\rangle := \sum_k c_k |k\rangle_A \otimes |k\rangle_{\bar{A}}, \tag{27}$$

with $c_k \neq 0$, $|k\rangle_{A(\bar{A})}$ are basis of $\mathcal{H}_{A(\bar{A})}$. These states are entangled. We would say a state $|\Psi\rangle$ is cyclic, if for any state $|\psi\rangle$ in $\mathcal{H}$ there exists operators $a \in \mathcal{R}_A$ or $\bar{a} \in \mathcal{R}_{\bar{A}}$

$$|\psi\rangle = a|\Psi\rangle = \bar{a}|\Psi\rangle. \tag{28}$$

It is not hard to show the above state $|\Psi\rangle$ is cyclic for the algebra $\mathcal{R}_A$ and $\mathcal{R}_{\bar{A}}$ with respect to $\mathcal{H}$. In the framework of algebraic QFTs, one could also construct the local algebra $\mathcal{R}(A)$ consisting of the local operators supported in the open region A. The Reeh-Schlieder theorem states that the vacuum state $|0\rangle$ is cyclic for the algebra $\mathcal{A}$ associated with any bounded open region A. Therefore, one could construct any pure state $|\psi\rangle$ by only using the operators in $\mathcal{R}(A)$ or $\mathcal{R}(\bar{A})$. In Appendix C, we briefly review the algebraic QFTs. One could also refer to [35] or [36].

Assume the state $|\Psi\rangle$ is cyclic for the algebra $\mathcal{R}(A)$, there exist a and $\tilde{a}$ such that

$$a|\Psi\rangle = |\psi\rangle, \quad \tilde{a}|\Psi\rangle = |\phi\rangle. \tag{29}$$

By using the result (13), the η-pseudo-Hermitian transition matrices can be expressed as

$$\mathcal{T}^{\psi|\phi} = \frac{a|\Psi\rangle\langle\Psi|a^\dagger \eta}{\langle\Psi|a^\dagger \eta a|\Psi\rangle}. \tag{30}$$

If the state $|\Psi\rangle$ is also a cyclic state for the algebra $\mathcal{R}_{\bar{A}}$, so there exists an operator $\bar{a} \in \mathcal{R}_{\bar{A}}$ such that the pure state $|\phi\rangle = \bar{a}|\Psi\rangle$. It seems the transition matrix $\mathcal{T}^{\psi|\phi} \propto a|\Psi\rangle\langle\Psi|\bar{a}^\dagger$ is not like the form (30). However, there always exists an operator $a' \in \mathcal{R}_A$ such that $a'|\Psi\rangle = \bar{a}|\Psi\rangle$ by using the cyclic property of $|\Psi\rangle$. Therefore, the transition matrix can also be written as (30).

3.1 Finite dimension examples

Consider the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. $\mathcal{H}_1$ and $\mathcal{H}_2$ are of the same dimension d. We can choose a reference state

$$|\Psi\rangle := \sum_k c_k |k\rangle_A |k\rangle_{\bar{A}}, \tag{31}$$

where $\sum_{k=1}^d c_k = 1$, $\{|k\rangle_{A(\bar{A})}\}$ are the bases of $\mathcal{H}_{A(\bar{A})}$. If all the c_k are non-vanishing, $|\Psi\rangle$ is cyclic state for the algebras $\mathcal{R}_A$ and $\mathcal{R}_{\bar{A}}$. For simplicity we will choose c_k to be positive.

Suppose $a = \sum_{ij} a_{ij}|i\rangle_A \ {}_A\langle j|$, $\eta_A = \eta_{mn}|m\rangle_A \ {}_A\langle n|$ and $\eta_{\bar{A}} = \bar{\eta}_{mn}|m\rangle_{\bar{A}} \ {}_{\bar{A}}\langle n|$, where the matrices η_{mn} and $\bar{\eta}_{mn}$ are invertible and Hermitian. With this we could construct the transition matrix $\mathcal{T}^a$ by using the formula (30). With some calculations we obtain

$$\mathcal{T}^a_A = \mathcal{N} \sum_{\substack{j,k \\ i',j',k'}} a_{jk'} c_{k'} a^*_{j'i'} c_{i'} \eta_{j'k} \bar{\eta}_{i'k'} |j\rangle_A \ {}_A\langle k|, \tag{32}$$

where $\mathcal{N}$ is the normalization. We show some numerical results in the Appendix E.1.

3.2 Examples in QFTs

3.2.1 2-dimensional free scalar theory

In [19] the authors studied some examples in two-dimensional free CFT. In the Euclidean space the coordinates are τ, x or $w = x - i\tau$, $\bar{w} = x + i\tau$. The action is given by $S = \int dw d\bar{w} \partial_w \phi \partial_{\bar{w}} \phi$. The vertex operator $\mathcal{V}_\alpha :=: e^{i\alpha\phi}$: is primary operator with conformal dimension $h_\alpha = h_{\bar{\alpha}} = \frac{\alpha^2}{2}$. One could also define the momentum operator[37]

$$\pi_0 := \frac{i}{4\pi} \int dx \partial_\tau \phi, \tag{33}$$

which satisfies the following commutation relation

$$[\pi_0, \mathcal{V}_\alpha] = \alpha \mathcal{V}_\alpha. \tag{34}$$

One can also show π_0 is a Hermitian operator and commutes with Hamiltonian H. Consider the operators

$$\mathcal{O} = e^{\frac{i}{2}\phi} + e^{-\frac{i}{2}\phi}, \quad \tilde{\mathcal{O}} := e^{\frac{i}{2}\phi} + e^{i\theta} e^{-\frac{i}{2}\phi}, \tag{35}$$

where $\theta \in [-\pi, \pi]$. Define the states

$$|\psi\rangle = e^{-aH} \tilde{\mathcal{O}}(x=0)|0\rangle, \quad |\phi\rangle = e^{-a'H} \mathcal{O}(x=0)|0\rangle, \tag{36}$$

where H is the Hamiltonian of CFTs, a and a' are cutoff to avoid UV divergence. The transition matrix is

$$\mathcal{T}^{\psi|\phi} = e^{-aH} \tilde{\mathcal{T}}^{\psi|\phi} e^{-a'H}, \tag{37}$$

with

$$\tilde{\mathcal{T}}^{\psi|\phi} := \frac{(e^{\frac{i}{2}\phi(0)} e^{-\frac{i}{2}\theta} + e^{-\frac{i}{2}\phi(0)} e^{\frac{i}{2}\theta})|0\rangle\langle 0|(e^{\frac{i}{2}\phi(0)} + e^{-\frac{i}{2}\phi(0)})}{2\cos\frac{\theta}{2} \langle e^{\frac{i}{2}\phi(0)} e^{-(a+a')H} e^{-\frac{i}{2}\phi(0)} \rangle}, \tag{38}$$

where $\phi(0) := \phi(x=0)$.

Firstly, let us consider the special case $\theta = 0$. In this case $\tilde{\mathcal{T}}^{\psi|\phi}$ is Hermitian. Thus, the transition matrix is η-pseudo-Hermitian with $\eta = e^{-(a'-a)H}$. Generally, the Hamiltonian can be written as integration with local energy density, that is

$$H = H_A + H_{\bar{A}}, \quad H_{A(\bar{A})} = \int_{A(\bar{A})} dx T_{00}, \tag{39}$$

where T_{00} is the energy density operator. $H_{A(\bar{A})}$ are Hermitian operators. Therefore, we have $\eta = \eta_A \otimes \eta_{\bar{A}}$ with $\eta_{A(\bar{A})} := e^{-(a'-a)H_{A(\bar{A})}}$. $\eta_{A(\bar{A})}$ is an invertible and positive definite operator. According to Corollary 3 we conclude the eigenvalues of $\mathcal{T}_{A(\bar{A})}$ are non-negative. The pseudo Rényi entropy should be positive, consistent with the result in [19]. Note that in [19], the authors only calculate the 2nd and 3rd pseudo Rényi entropy. Our results predict the pseudo Rényi entropy should be real for all n.

One could construct other examples similar to the above example. For any local operators $\mathcal{O}$ in a given QFTs, we could construct the following transition matrix

$$\mathcal{T}^{\mathcal{O}} := \frac{\mathcal{O}(\vec{x}_0, ia)|0\rangle\langle 0|\mathcal{O}^\dagger(\vec{x}_0, ia')}{\langle \mathcal{O}(\vec{x}_0, ia)\mathcal{O}^\dagger(\vec{x}_0, ia')\rangle}, \tag{40}$$

where $\vec{x}_0$ denotes the space position, $\mathcal{O}(\vec{x}_0, ia) := e^{-aH}\mathcal{O}(\vec{x}_0, 0)e^{aH}$. One could show the eigenvalues of the reduced transition matrix $\mathcal{T}^{\mathcal{O}}_{A(\bar{A})}$ should be positive, thus the pseudo Rényi entropy should be real.

For $\theta \neq 0$ the operator $\tilde{\mathcal{T}}^{\psi|\phi}$ (38) is non-Hermitian. However, we can also show the transition matrix $\mathcal{T}^{\psi|\phi}$ (37) is η-pseudo-Hermitian. By using the commutator (34) and the Baker–Campbell–Hausdorff formula we have

$$e^{\lambda\pi_0}\mathcal{V}_\alpha e^{-\lambda\pi_0} = e^{\alpha\lambda}\mathcal{V}_\alpha. \tag{41}$$

Therefore, we can rewrite $\tilde{\mathcal{T}}^{\psi|\phi}$ (38) as

$$\tilde{\mathcal{T}}^{\psi|\phi} = \frac{(e^{\frac{i}{2}\phi(0)}e^{-\frac{i}{2}\theta} + e^{-\frac{i}{2}\phi(0)}e^{\frac{i}{2}\theta})|0\rangle\langle 0|(e^{\frac{i}{2}\phi(0)}e^{-\frac{i}{2}\theta} + e^{-\frac{i}{2}\phi(0)}e^{\frac{i}{2}\theta})}{2\cos\frac{\theta}{2}\langle e^{\frac{i}{2}\phi(0)}e^{-(a+a')H}e^{-\frac{i}{2}\phi(0)}\rangle}e^{-i\theta\pi_0}. \tag{42}$$

The transition matrix (37) can be written as

$$\mathcal{T}^{\psi|\phi} = \frac{|\Phi\rangle\langle\Phi|\eta_\Phi}{\langle\Phi|\eta_\Phi|\Phi\rangle}, \tag{43}$$

with

$$\eta_\Phi := e^{-i\theta\pi_0}e^{-(a'-a)H}, \quad |\Phi\rangle := e^{-aH}(e^{\frac{i}{2}\phi(0)}e^{-\frac{i}{2}\theta} + e^{-\frac{i}{2}\phi(0)}e^{\frac{i}{2}\theta})|0\rangle. \tag{44}$$

Since the momentum operator π_0 is Hermitian, $e^{-i\theta\pi_0}$ is a unitary operator. Thus, η_Φ is not a Hermitian operator. The transition matrix (43) seems to be different from the general form of the pseudo-Hermitian operator (13). But this doesn't mean the transition matrix cannot be pseudo-Hermitian. We show an example of a two-qubit system in Appendix.F. Here our problem is similar to the two-qubit system. Actually, in the limit $a, a' \to 0$, the state (36) is similar with the qubit system in the quasi-particle picture[19][32]. The transition matrix (42) takes the form $|\Phi\rangle\langle\Phi|U$ with the unitary operator $U := e^{-i\pi_0\theta}$ in the limit $a, a' \to 0$. Motivated by the example in Appendix.F This transition matrix can be pseudo-Hermitian if the unitary operator U and the state $|\Phi\rangle$ satisfy some constraints. It is necessary that there exists a Hermitian and invertible operator η such that $U\eta|\Phi\rangle = |\Phi\rangle$, where $|\Phi\rangle$ is given by (44). Equally, the operator $\eta' := U\eta$ should satisfy the condition (97) (98) or a weaker condition (99).

One could show the transition matrix (37) is $\eta e^{-(a'-a)H}$-pseudo-Hermitian if the transition matrix (43) in the limit $a, a' \to 0$ is η-pseudo-Hermitian and η commutes with H. Assume the transition matrix (43) in the limit $a, a' \to 0$ is η-pseudo-Hermitian, we have

$$\eta \tilde{\mathcal{T}}^{\psi|\phi} \eta^{-1} = \tilde{\mathcal{T}}^{\psi|\phi\dagger}. \tag{45}$$

By using the above result and the definition (37) one could show

$$\begin{aligned}
&(\eta e^{-(a'-a)H)}) \mathcal{T}^{\psi|\phi} (\eta e^{-(a'-a)H)})^{-1} \\
&= \eta e^{-a'H} \tilde{\mathcal{T}}^{\psi|\phi} e^{-aH} \eta^{-1} \\
&= e^{-a'H} (\tilde{\mathcal{T}}^{\psi|\phi})^\dagger e^{-aH} = \mathcal{T}^{\psi|\phi^\dagger},
\end{aligned} \tag{46}$$

where in the second step we use η commutes with H and (45).

In [19], the authors calculate the 2nd and 3rd pseudo Rényi entropy for the reduced transition matrix of (43), and it is found that they are real. It indicates the transition matrix (43) should be pseudo-Hermitian. According to the above discussions, it is expected that there exists an operator η' satisfying the condition (99), or equally $\eta'\pi_0\eta' = -\pi_0$. A candidate for η' is the time reflection operator Θ[33][34], which gives the transformation $\Theta\phi(\tau, x)\Theta = \phi(-\tau, x)$ and $\Theta\partial_\tau\phi(\tau)\Theta = -\partial_\tau\phi(\tau)$. Thus we have $\Theta\pi_0\Theta = -\pi_0$. Also, note that Θ commutes with H.

3.2.2 2 dimensional rational CFTs

Another example is the time evolution of pseudo Rényi entropy in 2-dimensional CFTs considered in [25]. The subsystem A is taken to be $[-L, L]$ $(L > 0)$[5]. Consider the transition matrix

$$\mathcal{T}^{\mathcal{O}} := \frac{\mathcal{O}(x_1, t_1)|0\rangle\langle 0|\mathcal{O}(x_2, t_2)}{\langle 0|\mathcal{O}(x_2, t_2)\mathcal{O}(x_1.t_1)|0\rangle}, \tag{47}$$

[5]Note that A is chosen to be $[0, L]$ in [25], which, according to the spatial translation symmetry, does not affect the final conclusion.

where $\mathcal{O}$ is assumed to be Hermitian primary operator. Consider the case with $t_1 = t_2 = -t$ and $x_1 = -x_2$. We find for some rational CFT models $\mathrm{tr}[(\mathcal{T}_A^{\mathcal{O}})^2]$ is always real [25]. Assume the Hamiltonian H commutes with the parity P. The transition matrix can be written as

$$\mathcal{T}^{\mathcal{O}} = \frac{\mathcal{O}(x_1, -t)|0\rangle\langle 0|\mathcal{O}^\dagger(x_1, -t)\mathrm{P}}{\langle 0|\mathcal{O}^\dagger(x_1, -t)\mathrm{P}\mathcal{O}(x_1, -t)|0\rangle}. \tag{48}$$

It shows that $\mathcal{T}^{\mathcal{O}}$ is P-pseudo-Hermitian. Since the subsystem A and $\bar{A}$ are invariant under parity P, we could decompose $\mathrm{P} = \mathrm{P}_A \otimes \mathrm{P}_{\bar{A}}$, where $\mathrm{P}_{A(\bar{A})}$ only works on $A(\bar{A})$. We can schematically write $|\phi(x)\rangle$ $(x \in A)$ and $|\phi(\bar{x})\rangle$ $(\bar{x} \in \bar{A})$ as the basis for subsystem A and $\bar{A}$. The action of $\mathrm{P}_{A(\bar{A})}$ is understood as $\mathrm{P}_A|\phi(x)\rangle = |\phi(-x)\rangle$. By definition $-x \in A$ if $x \in A$. P_A maps the basis of A into itself. The parity operators are invertible. Therefore, $\mathcal{T}^{\mathcal{O}}_{A(\bar{A})}$ is $\mathrm{P}_{A(\bar{A})}$-pseudo-Hermitian. This leads to $\mathrm{tr}[(\mathcal{T}^{\mathcal{O}}_{A(\bar{A})})^2]$ is real, which is consistent with the results in [25]. Again, our results in this paper predict $\mathrm{tr}[(\mathcal{T}^{\mathcal{O}}_{A(\bar{A})})^n]$ should be real for any positive integer $n \geq 2$.

3.2.3 More general examples

Consider a QFT living in d-dimensional Minkowski spacetimes. The metric is $ds^2 = -dt^2 + dx^2 + d\vec{y}^2$, where $\vec{y}$ are coordinates of $(d-2)$-dimensional Euclidean space. Let the subsystem A be the half-space $x < 0$. Its complement $\bar{A}$ is $x > 0$. The local algebra $\mathcal{R}_A$ is given by operators located at the left Rindler wedge $\mathcal{W}_A := \{(t, x, \vec{y})|x < -|t|\}$, which is the causal domain of A. The operators can be constructed by the smeared field $\int d^d x f(x^\mu)\phi(x^\mu)$ with the functions f supported in $\mathcal{W}_A$. Similarly, the algebra $\mathcal{R}_{\bar{A}}$ is associated with the right wedge $\mathcal{W}_{\bar{A}} := \{(t, x, \vec{y})|x > |t|\}$.

The vacuum state $|0\rangle$ is cyclic for the algebra $\mathcal{R}_{A(\bar{A})}$. There exists an antiunitary operator J_Ω that exchanges the algebras $\mathcal{W}_A$ and $\mathcal{W}_{\bar{A}}$ according to the Tomita-Takesaki theory. We briefly review the Tomita-Takesaki theory and its application in algebraic QFTs in Appendix C. J_Ω is called modular conjugation. For a given Hermitian operator $\phi(t, x, \vec{y})$, J_Ω acts as

$$J_\Omega \phi(t, x, \vec{y}) J_\Omega = \phi(-t, -x, \vec{y}). \tag{49}$$

It has been proved that $J_\Omega = \mathrm{CRT}$, where C and T are charge and time reversal operators, and R is the reflection $x \to -x$ while keeping other coordinates invariant[35][38]. For any operator $\mathcal{O}_A \in \mathcal{R}_A$ we can define the operator $\mathcal{O}_{\bar{A}} := J_\Omega \mathcal{O}_A J_\Omega \in \mathcal{R}_{\bar{A}}$. For any pure state $|\psi\rangle$ there exists $\mathcal{O}_A \in \mathcal{R}_A$ such that $|\psi\rangle$ can be approximated by $\mathcal{O}_A|0\rangle$ by the cyclic property of the vacuum state $|0\rangle$ for the algebra $\mathcal{R}_A$. Define the transition matrix

$$\mathcal{T}^{\mathcal{O}_A} = \frac{\mathcal{O}_A|0\rangle\langle 0|\mathcal{O}_{\bar{A}}}{\langle 0|\mathcal{O}_A \mathcal{O}_{\bar{A}}|0\rangle}, \tag{50}$$

where $\mathcal{O}_{\bar{A}} = J_\Omega \mathcal{O}_A J_\Omega$. It can be shown the spectra of $\mathcal{T}_A^{\mathcal{O}_A}$ are non-negative. Thus the pseudo Rényi entropy is positive. To show this result, we need to use the modular theory of QFTs. By using the modular theory

$$\mathcal{T}^{\mathcal{O}_A} = \frac{\mathcal{O}_A|0\rangle\langle 0|\mathcal{O}_A^\dagger \Delta_\Omega^{1/2}}{\langle 0|\mathcal{O}_A^\dagger \Delta_\Omega^{1/2}\mathcal{O}_A|0\rangle}, \tag{51}$$

where Δ_Ω is the modular operator, a positive Hermitian operator. $\mathcal{T}^{\mathcal{O}_A}$ takes the same form as (30). Thus it is $\Delta_\Omega^{1/2}$-pseudo-Hermitian. Further, $\Delta_\Omega^{1/2} = e^{-\pi K_A}\otimes e^{\pi K_{\bar{A}}}$. By using Corollary 3 and the fact that $e^{-\pi K_A} = e^{-\pi K_A/2}e^{-\pi K_A/2}$ and $e^{\pi K_{\bar{A}}} = e^{\pi K_{\bar{A}}/2}e^{\pi K_{\bar{A}}/2}$ we conclude the eigenvalues of $\mathcal{T}_A^{\mathcal{O}_A}$ are all positive. In Appendix D we show more general examples with positive pseudo Rényi entropy.

One could check the above result by evaluating the pseudo Rényi entropy by QFT methods. To move on, let's focus on 2D CFTs.[6] Consider the transition matrix (47). If $x_1 = -x_2$ and $t_1 = -t_2$, $J_\Omega \mathcal{O}(x_1, t_1) J_\Omega = \mathcal{O}(x_2, t_2)$, thus the pseudo Rényi entropy is expected to be real in this case. The results are shown in Figure 1 for models in rational CFTs.

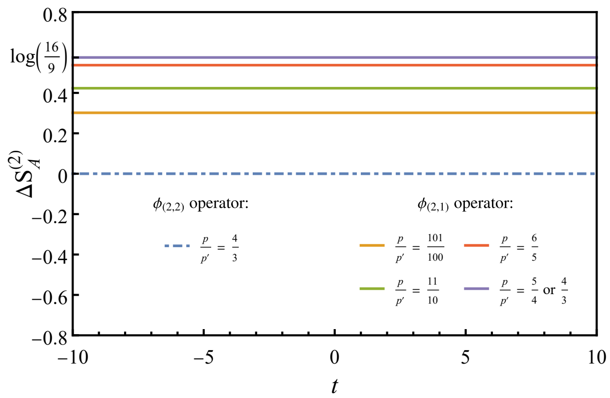

Figure 1: The excess of the 2nd pseudo Rényi entropy $\Delta S_A^{(2)}$ ($\Delta S_A^{(2)} \equiv S_A^{(2)} - S_{A;vac}^{(2)}$, where $S_{A;vac}^{(2)}$ denotes the 2nd Rényi entropy of A when the total system is in the vacuum) of the transition matrix $\mathcal{T}_A \equiv \mathrm{tr}_{\bar{A}} \frac{\mathcal{O}(x,t)|0\rangle\langle 0|\mathcal{O}(-x,-t)}{\langle 0|\mathcal{O}(-x,-t)\mathcal{O}(x,t)|0\rangle}$ in the minimal models $\mathcal{M}(p,p')$. We study the case of $\mathcal{O} = \phi_{(2,2)}$ (dot-dashed line) and $\mathcal{O} = \phi_{(2,1)}$ (solid line), respectively. One novel feature is that the 2nd pseudo entropy is real and time independent.

4 Discussion

In this paper, we apply the notion of pseudo-Hermitian to construct the reality condition of pseudo entropy since the real-valued pseudo entropy has robust potential application to study holograph and quantum phase transition. Our results provide a useful method to diagnose whether the pseudo Rényi entropies for a given transition matrix X are real-valued or not without evaluating the eigenvalues or pseudo Rényi entropies of all the index n. Especially, in QFT this is a very practical method as we have shown in section.3.2. The operator η seems to be associated with the symmetry of the states $|\psi\rangle$, $|\phi\rangle$ and the subsystem A. Moreover, one

[6] In Appendix G, we give an overview of the replica method to compute the nth pseudo Rényi entropy in 2D CFTs.

may use the results in this paper as the guidance to construct the transition matrix X with real-valued pseudo entropy.

The notion of pseudo-Hermiticity originates from non-Hermitian matrices, which are diagonalizable and own a complete bi-orthonormal eigenbasis. We mainly focus on the pseudo-Hermiticity of the reduced transition matrix to construct the real-valued pseudo entropy. If the transition matrix is not diagonalizable, the results in our paper are unavailable. It would be interesting to find the condition for reality by writing the matrix into Jordan form [26].

As a generalization of EE, the pseudo entropy has a more complicated structure. Thus it includes more information on the underlying theory, which is still unknown. As shown in the examples of QFTs, the reality condition is closely related to the modular theory. For some particular examples of CFTs, the pseudo entropy depends on the conformal blocks in Minkowski spacetime. It would be an interesting future problem to explore the relation between the reality of pseudo entropy and the structure of conformal blocks.

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A Non-pseudo-Hermitian X_A with real tr$[(X_A)^n]$

Let us consider a 4 qubits system $\mathcal{S}$ (2 qubits each for A and $\bar{A}$) and a transition matrix $X = \frac{|\psi\rangle\langle\phi|}{\langle\phi|\psi\rangle}$ acting on its Hilbert space $\mathcal{H}_\mathcal{S} \equiv H_A \otimes H_{\bar{A}}$, where $|\psi\rangle$ and $|\phi\rangle$ are two non-orthogonal quantum states living in $H_\mathcal{S}$,

$$\begin{aligned}|\psi\rangle =&\frac{1}{2}|00\rangle_A|00\rangle_{\bar{A}} + \frac{1}{2}|01\rangle_A|01\rangle_{\bar{A}} + \frac{1}{2}|10\rangle_A|10\rangle_{\bar{A}} + \frac{1}{2}|11\rangle_A|11\rangle_{\bar{A}},\\ |\psi_\perp\rangle =&\frac{i}{4}|00\rangle_A|00\rangle_{\bar{A}} + \frac{i}{4}|01\rangle_A|01\rangle_{\bar{A}} - \frac{i}{4}|10\rangle_A|10\rangle_{\bar{A}} - \frac{i}{4}|11\rangle_A|11\rangle_{\bar{A}} + \frac{\sqrt{3}}{2}|11\rangle_A|10\rangle_{\bar{A}},\\ |\phi\rangle \equiv&\frac{\sqrt{2}}{2}|\psi\rangle + \frac{\sqrt{2}}{2}|\psi_\perp\rangle, \qquad (\langle\phi|\phi\rangle = \langle\psi|\psi\rangle = \langle\psi_\perp|\psi_\perp\rangle = 1,\ \langle\psi|\psi_\perp\rangle = 0). \end{aligned} \tag{52}$$

The reduced transition matrix of the subsystem A, obtained by tracing out the d.o.f. of $\bar{A}$, is given by

$$X_A \equiv \mathrm{tr}_{\bar{A}} X = \Big(\frac{1}{4}-\frac{i}{8}\Big)\big(|00\rangle_A\,{}_A\langle\, 00| + |01\rangle_A\,{}_A\langle\, 01|\big) + \Big(\frac{1}{4}+\frac{i}{8}\Big)\big(|10\rangle_A\,{}_A\langle\, 10| + |11\rangle_A\,{}_A\langle\, 11|\big) + \frac{\sqrt{3}}{4}|01\rangle\langle 11|. \tag{53}$$

Building on (53), it's more useful to write down the matrix formulation of X_A,

$$X_A = \begin{pmatrix} \frac{1}{4}-\frac{i}{8} & & & \\ & \frac{1}{4}-\frac{i}{8} & & \\ & & \frac{1}{4}+\frac{i}{8} & \frac{\sqrt{3}}{4} \\ & & & \frac{1}{4}+\frac{i}{8} \end{pmatrix}, \tag{54}$$

which is an upper triangular 4×4 matrix and cannot be diagonalized. It can be found from (54) that the eigenvalues of X_A consist of two complex conjugate pairs, which renders $\mathrm{tr}[(X_A)^n]$ real. On the other hand, we have

$$X_A^\dagger = \begin{pmatrix} \frac{1}{4}+\frac{i}{8} & & & \\ & \frac{1}{4}+\frac{i}{8} & & \\ & & \frac{1}{4}-\frac{i}{8} & \\ & & \frac{\sqrt{3}}{4} & \frac{1}{4}-\frac{i}{8} \end{pmatrix}. \tag{55}$$

Although $X_A^\dagger$ has the same eigenvalues as X_A, they are not similar. This is because they have different Jordan standard forms, which is read from the fact that the Jordan blocks of the same eigenvalue of two matrices are different. Therefore, we know that X_A is non-pseudo-Hermitian.

B A proof of Corollary 2

The result of Corollary 2 follows from the spectral decomposition of pseudo-Hermitian matrices. For any diagonalizable η-pseudo-Hermitian matrix M, we can write the spectral decomposition of M as

$$M = \sum_i \lambda_{0,i}|\psi_{0,i}\rangle\langle\phi_{0,i}| + \sum_j \Big(\lambda_{+,j}|\psi_{+,j}\rangle\langle\phi_{+,j}| + \lambda_{-,j}|\psi_{-,j}\rangle\langle\phi_{-,j}|\Big), \tag{56}$$

where λ, $|\psi\rangle$ and $\langle\phi|$ represent the eigenvalue, right eigenvector, and left eigenvector of M, respectively. [7] Since we can always choose the bi-orthonormal eigenbasis such that

$$|\phi_{0,i}\rangle = \eta|\psi_{0,i}\rangle, \quad |\phi_{\pm,j}\rangle = \eta|\psi_{\mp,j}\rangle \tag{57}$$

[7] We use the subscript 0 to stand for real eigenvalues and the corresponding basis eigenvectors and the subscript $\pm$ to stand for the complex eigenvalues with $\pm$ imaginary part and the corresponding basis eigenvectors.

hold [29], the spectrum decomposition becomes

$$\begin{aligned}
M =& \sum_i \lambda_{0,i}|\psi_{0,i}\rangle\langle\psi_{0,i}|\eta \\
&+ \sum_j \left(\lambda_{+,j}|\psi_{+,j}\rangle\langle\psi_{-,j}|\eta + \lambda_{-,j}|\psi_{-,j}\rangle\langle\psi_{+,j}|\eta\right) \\
=& \sum_i \lambda_{0,i}|\psi_{0,i}\rangle\langle\psi_{0,i}|\eta \\
&+ \sum_j \left(\lambda_{+,j}|\psi_{+,j}\rangle\langle\psi_{-,j}|\eta + \lambda_{-,j}|\psi_{-,j}\rangle\langle\psi_{+,j}|\eta\right) \\
=& \sum_i \lambda_{0,i}|\psi_{0,i}\rangle\langle\psi_{0,i}|\eta \\
&+ \sum_j \lambda^R_{+,j}\big[(|\psi_{+,j}\rangle + |\psi_{-,j}\rangle)(\langle\psi_{-,j}| + \langle\psi_{+,j}|)\eta \\
&- |\psi_{-,j}\rangle\langle\psi_{-,j}|\eta - |\psi_{+,j}\rangle\langle\psi_{+,j}|\eta\big] \\
&+ \sum_j \lambda^I_{+,j}\big[(|\psi_{+,j}\rangle - i|\psi_{-,j}\rangle)(\langle\psi_{+,j}| + i\langle\psi_{-,j}|)\eta \\
&- |\psi_{-,j}\rangle\langle\psi_{-,j}|\eta - |\psi_{+,j}\rangle\langle\psi_{+,j}|\eta\big],
\end{aligned} \tag{58}$$

where $\lambda^R_{+,j}$ and $\lambda^I_{+,j}$ are the real and imaged part of $\lambda_{+,j}$. Note that every term in the summation is η-pseudo-Hermitian.

C Brief review of modular theory in QFTs

For any given open subsystem A in spacetimes, the local algebra $\mathcal{R}_A$ consists of all the operators supported in A. The algebra can also be associated with the domain of dependence of A, denoted by $\mathcal{D}(A)$. The reason is that the operators located in $\mathcal{D}(A)$ can be determined by the ones in A according to the dynamical time evolution of the theory. If A' is another subsystem that is spacelike with A, we expect the operators in A' would commute with the ones in A, that is $[\mathcal{R}_A, \mathcal{R}_{A'}] = 0$.

Denote the algebra associated with the whole spacetime as $\mathcal{R}$. The full Hilbert space $\mathcal{H}_0$ could be constructed by acting the operators in $\mathcal{R}$ on the vacuum state $|0\rangle$. The Reeh-Schlieder theorem says that the set $\{a|0\rangle, a \in \mathcal{R}_A\}$ is also dense in $\mathcal{H}_0$. For any given state $|\psi\rangle$, the theorem means that there exist operator $a \in \mathcal{R}_A$ such that $a|0\rangle$ can be arbitrarily close to $|\psi\rangle$. Thus we could construct the transition matrix $\mathcal{T}^{\psi|\phi}$ only by using the operators located in a subsystem. The above results can also be generalized to any cyclic state $|\Psi\rangle$.

The Tomita operator S_Ψ for the state $|\Psi\rangle$ is antilinear and satisfies

$$S_\Psi a|\Psi\rangle = a^\dagger|\Psi\rangle, \tag{59}$$

for any $a \in \mathcal{R}_A$. By definition it is obvious that $S_\Psi^2 = 1$. S_Ψ has a unique polar decomposition

$$S_\Psi = J_\Psi \Delta_\Psi^{1/2}, \tag{60}$$

where J_Ψ is antiunitary, $\Delta_\Psi^{1/2}$ is a positive Hermitian operator. J_Ψ is called the modular conjugation satisfying $J_\Psi^2 = 1$ and $J_\Psi^\dagger = J_\Psi$. Δ_Ψ is the modular operator associated with $\mathcal{R}_A$ and $|\Psi\rangle$. Similarly, one could define the modular operator $\bar{S}_\Psi$ associated with $\mathcal{R}_{\bar{A}}$. By using $S_\Psi^2 = 1$ we have

$$J_\Psi \Delta_\Psi^{1/2} J_\Psi = \Delta_\Psi^{-1/2}. \tag{61}$$

It can be shown that

$$\bar{S}_\Psi = S_\Psi^\dagger = \Delta_\Psi^{1/2} J_\Psi = J_\Psi \Delta^{-1/2}. \tag{62}$$

Consider the d-dimensional Minkowski spacetime. The metric is $ds^2 = -dt^2 + dx^2 + d\vec{y}^2$, where $\vec{y}$ are coordinates of $(d-2)$-dimensional Euclidean space. Let the subsystem A be $x > 0$. The domain of dependence of A is known as the Rindler wedge $\mathcal{W}_A$. For the vacuum state $|0\rangle$ the modular conjugation J_Ω is given by

$$J_\Omega = \text{CRT}, \tag{63}$$

which is first proved by Bisognano and Wichmann [38]. The modular operator Δ_Ω can be formly written as

$$\Delta_\Omega = \rho_A \otimes \rho_{\bar{A}}^{-1}, \tag{64}$$

where $\rho_A := e^{-2\pi K_A}$ and $\rho_{\bar{A}} := e^{-2\pi K_{\bar{A}}}$ are the reduced density matrices of A and $\bar{A}$. K_A and $K_{\bar{A}}$ are known as the modular Hamiltonian of A and $\bar{A}$. The density matrices are positive operators. For any positive function $f(x)$, the operators $f(\rho_A)$ or $f(\rho_{\bar{A}})$ are also positive. For example, one could define the operator $\rho_A^{1/2} = e^{-\pi K_A}$, $\rho_A^{1/4} = e^{-\pi K_A/2}$. It is obvious the modular operator $\Delta_\Omega = e^{-2\pi(K_A - K_{\bar{A}})}$ is a positive Hermitian operator.

For the Rindler wedge K_A and $K_{\bar{A}}$ are associated with the Lorentz boost generators

$$\begin{aligned} K_A &= \int_{t=0, x\geq 0} dx d^{d-2}y x T_{00}, \\ K_{\bar{A}} &= -\int_{t=0, x\leq 0} dx d^{d-2}y x T_{00}. \end{aligned} \tag{65}$$

For any Hermitian operator $\mathcal{O}(t, x, \vec{y})$, according to the definition of S_Ω we have

$$S_\Omega \mathcal{O}(t, x, \vec{y})|0\rangle = \mathcal{O}(t, x, \vec{y})|0\rangle, \tag{66}$$

which leads to

$$\Delta_\Omega^{1/2} \mathcal{O}(t, x, \vec{y})|0\rangle = J_\Omega \mathcal{O}(t, x, \vec{y})|0\rangle|0\rangle = \mathcal{O}(-t, -x, \vec{y})|0\rangle, \tag{67}$$

where we use the fact $J_\Omega^2 = 1$ and $J_\Omega|0\rangle = |0\rangle$.

D Details of the example in QFTs

The transition matrix (50) is related to the operators $\mathcal{O}_A$ and $\mathcal{O}_{\bar{A}}$. In Figure 2, we show the positions of the two operators.

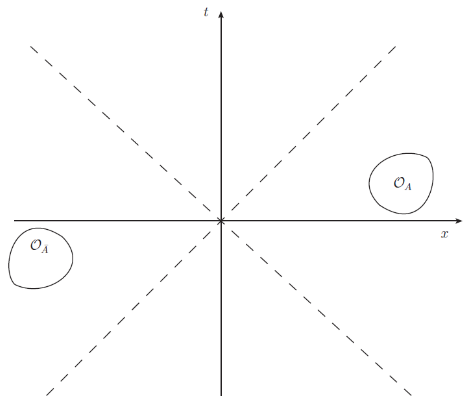


Figure 2: Illustration of the operators $\mathcal{O}_A$ and $\mathcal{O}_{\bar{A}}$.

Taking $\mathcal{O}_{\bar{A}}$ into (50) we obtain

$$\mathcal{T}^{\mathcal{O}_A} = \frac{\mathcal{O}_A|0\rangle\langle 0|\mathcal{O}_A J_\Omega}{\langle 0|\mathcal{O}_A J_\Omega \mathcal{O}_A|0\rangle}. \tag{68}$$

By the definition of Tomita operator we have

$$\begin{aligned} S_\Omega \mathcal{O}_A^\dagger|0\rangle &= J_\Omega \Delta_\Omega^{1/2}\mathcal{O}_A|0\rangle \\ &= J_\Omega \Delta_\Omega^{1/2} J_\Omega J_\Omega \mathcal{O}_A^\dagger|0\rangle \\ &= \Delta_\Omega^{-1/2} J_\Omega \mathcal{O}_A^\dagger|0\rangle, \end{aligned} \tag{69}$$

where in the second step we use the fact $J_\Omega^2 = 1$,in the third step we use (61). Therefore, we have

$$J_\Omega \mathcal{O}_A^\dagger|0\rangle = \Delta_\Omega^{1/2}\mathcal{O}_A|0\rangle. \tag{70}$$

The transition matrix (68) is reduced to

$$\mathcal{T}^{\mathcal{O}_A} = \frac{\mathcal{O}_A|0\rangle\langle 0|\mathcal{O}_A^\dagger \Delta_\Omega^{1/2}}{\langle 0|\mathcal{O}_A^\dagger \Delta_\Omega^{1/2}\mathcal{O}_A|0\rangle}. \tag{71}$$

In the main text, we only discuss the special case (47), for which the eigenvalues of $\mathcal{T}_A^{\mathcal{O}_A}$ are positive real. More generally, one could choose $\mathcal{O}_A = \sum_j C_j \mathcal{O}_j(x_1, t_1)$, where $\mathcal{O}_j$ are Hermitian operators (not necessarily be primary), C_j are arbitrary constants. By $\mathcal{O}_{\bar{A}} = J_\Omega \mathcal{O}_{\bar{A}} J_\Omega$ we have $\mathcal{O}_{\bar{A}} = \sum_j C_j^* \mathcal{O}_j(-x_1, -t_1)$. It is expected that the pseudo Rényi entropy for the transition matrix associated with these operators will also be real.

The result of Theorem 2 can be used to construct the η-pseudo-Hermitian transition matrix in QFTs. Assume $|\Psi\rangle$ is a cyclic state for the algebra $\mathcal{R}_A$. The general η-pseudo-Hermitian transition matrices in QFTs are

$$\mathcal{T}^{\mathcal{O}_A} = \frac{\mathcal{O}_A|\Psi\rangle\langle\Psi|\mathcal{O}_A^\dagger\eta}{\langle\Psi|\mathcal{O}_A^\dagger\eta\mathcal{O}_A|\Psi\rangle}. \tag{72}$$

If η is taken to be identity, the transition matrix reduces to the Hermitian case. By using (59) one could rewrite the above formula as

$$\begin{aligned}\mathcal{T}^{\mathcal{O}_A} &= \frac{\mathcal{O}_A|\Psi\rangle\langle\Psi|\mathcal{O}_A S_\Psi^\dagger\eta}{\langle\Psi|\mathcal{O}_A^\dagger\eta\mathcal{O}_A|\Psi\rangle} \\ &= \frac{\mathcal{O}_A|\Psi\rangle\langle\Psi|\mathcal{O}_A J_\Omega \Delta_\Psi^{-1/2}\eta}{\langle\Psi|\mathcal{O}_A^\dagger\eta\mathcal{O}_A|\Psi\rangle}\end{aligned} \tag{73}$$

This paper only focuses on the vacuum state $|0\rangle$. Our example (68) is a special case $\eta = \Delta_\Omega^{1/2}$. In general, one could choose $\eta = \eta_A \otimes \eta_{\bar{A}}$, where η_A and $\eta_{\bar{A}}$ are invertible positive operators. Using Corollary 3, one could show that the pseudo Rényi entropy is also real in this case.

E Finite dimension example

Assume the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$, the dimension of $\mathcal{H}_{A(\bar{A})}$ is d. (30) provides us a way to generate the η-pseudo-Hermitian transition matrices with $\eta = \eta_A \otimes \eta_{\bar{A}}$. One could arbitrarily choose the reference state $|\Psi\rangle$, i.e., the coefficients c_k and the operators a, $\eta_{A(\bar{A})}$. With a given basis $|k\rangle_{A(\bar{A})}$, we have the expansion

$$\begin{aligned} a &= \sum_{ij} a_{ij}|i\rangle_A\ {}_A\langle j|, \\ \eta_A &= \sum_{m,n} \eta_{mn}|m\rangle_A\ {}_A\langle n|, \\ \eta_{\bar{A}} &= \sum_{m,n} \bar{\eta}_{mn}|m\rangle_{\bar{A}}\ {}_{\bar{A}}\langle n|. \end{aligned} \tag{74}$$

The matrices η_{mn} and $\bar{\eta}_{mn}$ should be Hermitian and invertible.

In finite dimension, it is easy to show the Reeh-Schlieder theorem. Any state $|\psi\rangle$ can be constructed by only local operations on A or $\bar{A}$. The reference state $|\Psi\rangle := \sum_k c_k|k\rangle_A \otimes |k\rangle_{\bar{A}}$ is cyclic if the coefficients c_k are all non-vanishing. For any given state $|\psi\rangle$, we can expand it as

$$|\psi\rangle = \sum_{i,j} \psi_{ij}|i\rangle_A \otimes |j\rangle_{\bar{A}}. \tag{75}$$

It is enough to show that the basis $|i\rangle_A \otimes |j\rangle_{\bar{A}}$ of $\mathcal{H}$ can be obtained only by local operations on $|\Psi\rangle$. One could achieve this by acting an operator $|i\rangle_A\ {}_A\langle j|$ on $|\Psi\rangle$.

Taking (74) into (30) one could obtain the transition matrix $\mathcal{T}^a$. (32) can be obtained by partial trace $tr_{\bar{A}}\mathcal{T}^a := \sum_k\ {}_{\bar{A}}\langle k|\mathcal{T}^a|k\rangle_{\bar{A}}$. One could generate random matrices a_{ij}, η_{mn} and $\bar{\eta}_{mn}$

by software, e.g., Mathematica. Then we can construct the matrices $\mathcal{T}_A^a(32)$ and evaluate the eigenvalues of them. According to Theorem 3 the transition matrices by linear combinations of $\mathcal{T}^a$ can also have positive eigenvalues.

We have the following three different cases.

Case I: $\eta_{A(\bar{A})}$ is Hermitian and invertible matrices.

Generally, in this case, the eigenvalues are expected to come in complex conjugate pairs or be real.

Case II: $\eta_{A(\bar{A})} = \mathcal{O}_{A(\bar{A})}\mathcal{O}^\dagger_{A(\bar{A})}$. $\mathcal{O}_{A(\bar{A})}$ is an arbitrary invertible operator.

The eigenvalues, in this case, are expected to be real and positive. By considering the normalization of $\mathcal{T}_A^a$ the eigenvalues should belong to $[0,1]$. Thus the pseudo Rényi entropy should be real.

Case III: The linear combinations of $\mathcal{T}^{a^I}$,

$$\mathcal{T} := \sum_I x_I \mathcal{T}^{a^I}, \tag{76}$$

where x_I are positive numbers satisfying $\sum_I x_I = 1$, $\mathcal{T}^{a^I}$ is $\eta_A \otimes \eta_{\bar{A}}$-pseudo-Hermitian transition matrices with $\eta_{A(\bar{A})} = \mathcal{O}_{A(\bar{A})}\mathcal{O}^\dagger_{A(\bar{A})}$. In this case, the eigenvalues of $\mathcal{T}_A$ are positive.

E.1 Numerical result with $d = 3$

We show examples for these three cases in the main text, obtained by randomly choosing the matrices. In the following, we would like to show an example with $d = 3$.

Case I. The matrices a_{ij}, η_{mn} and $\bar{\eta}_{mn}$ are randomly generated by Mathematica,

$$\eta_A = \begin{pmatrix} -12.7085 & 24.1113 + 2.50006i & 12.752 - 7.64134i \\ 24.1113 - 2.50006i & -34.9796 & 16.159 + 12.3798i \\ 12.752 + 7.64134i & 16.159 - 12.3798i & 6.06277 \end{pmatrix}$$
$$\eta_{\bar{A}} = \begin{pmatrix} -18.2979 & -5.89479 - 25.5118i & 5.61273 + 21.1508i \\ -5.89479 + 25.5118i & -32.9428 & 12.504 - 10.931i \\ 5.61273 - 21.1508i & 12.504 + 10.931i & -25.1785 \end{pmatrix}$$
$$a = \begin{pmatrix} 17.5055 - 19.3962i & 8.29301 + 13.4073i & -13.3458 + 5.79992i \\ -2.34212 + 12.1545i & -19.0161 + 8.64625i & 17.1027 + 20.3801i \\ -5.46605 - 24.0534i & -0.924333 + 21.9112i & -19.6201 + 22.0798i \end{pmatrix}$$

The reference state $|\Psi\rangle = \frac{1}{\sqrt{3}}\sum_{k=1}^3 |k\rangle_A |k\rangle_{\bar{A}}$. One could evaluate the reduced transition matrix $\mathcal{T}_A^a$ by using (32), it is given by

$$\mathcal{T}_A^a = \begin{pmatrix} 1.06475 + 0.82173i & -2.58944 - 1.89593i & -0.414598 + 1.47507i \\ 5.81552 + 1.36514i & -2.51542 - 3.45208i & 3.90801 + 1.03913i \\ 4.96095 + 0.0703348i & -5.61733 - 0.740827i & 2.45067 + 2.63035i \end{pmatrix}$$

It is obvious that $\mathcal{T}_A^a$ is non-Hermitian. The eigenvalues of it are

$$\lambda_1 = 0.7053 - 6.27836i,\ \lambda_2 = 0.7053 + 6.27836i,\ \lambda_3 = -0.410601. \tag{77}$$

The pseudo Rényi entropy may not be real in this case. e.g., $S^{(2)}(\mathcal{T}_A^a) = -4.3525 + 3.14159i$.

Case III. We take $x_1 = 0.932007$, $x_2 = 0.0679932$. η_{mn}, $\bar{\eta}_{mn}$ and a_{ij} are given by

$$\eta_A = \begin{pmatrix} 2241.0 & -1009.73 + 735.915i & 286.517 + 572.134i \\ -1009.73 - 735.915i & 1007.3 & 58.5703 - 617.441i \\ 286.517 - 572.134i & 58.5703 + 617.441i & 1399.02 \end{pmatrix}$$

$$\eta_{\bar{A}} = \begin{pmatrix} 967.287 & -307.565 - 126.497i & -129.349 + 149.126i \\ -307.565 + 126.497i & 1336.39 & 209.52 + 1520.81i \\ -129.349 - 149.126i & 209.52 - 1520.81i & 2269.12 \end{pmatrix}$$

$$a^1 = \begin{pmatrix} -12.9325 + 0.0289028i & -7.24499 + 4.48426i & -14.4313 + 15.8304i \\ -4.09857 - 26.938i & -14.2456 - 2.55161i & 10.7265 - 5.71364i \\ 4.30869 + 11.8775i & -19.1378 + 9.46391i & 1.32846 + 4.07899i \end{pmatrix}$$

$$a^2 = \begin{pmatrix} -9.48366 + 25.7059i & -6.14031 + 23.5242i & -13.0021 - 20.8661i \\ -3.87512 + 5.57805i & 4.9788 - 6.5475i & 1.21723 + 7.54634i \\ -10.6898 + 13.5806i & 11.563 - 1.35289i & -14.61 + 21.6139i \end{pmatrix}$$

The reference state $|\Psi\rangle = \frac{1}{\sqrt{3}}\sum_{k=1}^{3}|k\rangle_A|k\rangle_{\bar{A}}$. We have the reduced transition matrix $\mathcal{T}_A := x_1 T_A^{a^1} + x_2 T_A^{a^2}$

$$\mathcal{T}_A = \begin{pmatrix} 0.530706 - 0.0443678i & -0.249067 + 0.220004i & 0.16275 + 0.257015i \\ -0.0842444 - 0.152933i & 0.129589 + 0.0451508i & 0.0460981 - 0.13474i \\ 0.213683 - 0.402251i & 0.0403375 + 0.312084i & 0.339705 - 0.000782992i \end{pmatrix} \quad (78)$$

The eigenvalues are

$$\lambda_1 = 0.938253,\ \ \lambda_2 = 0.0533309,\ \ \lambda_3 = 0.00841637. \quad (79)$$

The pseudo Rényi entropy is real. The result is shown in Figure 3.

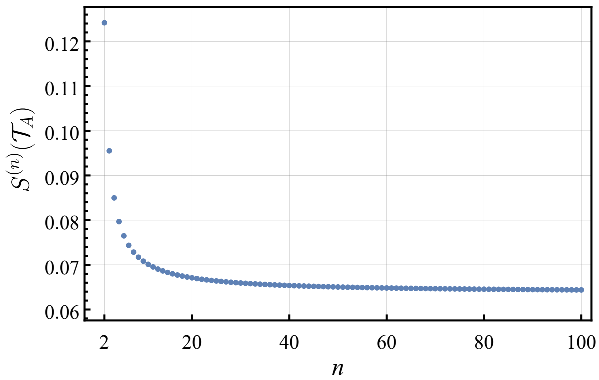


Figure 3: The plot of $S^{(n)}(\mathcal{T}_A)$.

E.2 Example: $S^{(n)}(\mathcal{T}_A) > 0$, η_A is not positive definite

Corollary 3 only gives a sufficient condition for $S^{(n)}(\mathcal{T}_A) > 0$. In this section, we would like to use a numerical example to show it is not a necessary condition. We will focus on a three-dimensional example. Choosing the matrices

$$\eta_A = \begin{pmatrix} 13.9359 & -17.8554 + 8.22163i & 0.740751 - 0.860494i \\ -17.8554 - 8.22163i & 11.7561 & 3.87722 + 0.527719i \\ 0.740751 + 0.860494i & 3.87722 - 0.527719i & 4.32501 \end{pmatrix},$$

$$\eta_{\bar{A}} = \begin{pmatrix} 2.68826 + 0.i & -2.76297 + 6.09204i & -13.4254 - 5.89942i \\ -2.76297 - 6.09204i & 23.4288 + 0.i & 2.24652 - 1.6307i \\ -13.4254 + 5.89942i & 2.24652 + 1.6307i & 6.07729 + 0.i \end{pmatrix},$$

$$a = \begin{pmatrix} 2.79442 + 26.2305i & 14.4042 - 1.54735i & 1.27623 + 2.29185i \\ 17.0343 + 21.4595i & 6.13678 - 4.72818i & -6.82378 + 24.1677i \\ -6.55401 + 2.08772i & -6.0073 - 29.8274i & -7.59207 - 24.0165i \end{pmatrix}. \tag{80}$$

The reference state $|\Psi\rangle = \frac{1}{\sqrt{3}} \sum_{k=1}^{3} |k\rangle_A |k\rangle_{\bar{A}}$. The eigenvalues of η_A and $\eta_{\bar{A}}$ are

$$\eta_A \to \{32.6819, -7.87014, 5.20516\},$$
$$\eta_{\bar{A}} \to \{26.3549, 17.3493, -11.5099\}. \tag{81}$$

Thus they are not positive operators. The eigenvalues of $\mathcal{T}_A$ are

$$\lambda_1 = 0.849706, \quad \lambda_2 = 0.075147 - 0.106401i, \quad \lambda_3 = 0.075147 + 0.106401i. \tag{82}$$

The pseudo Rényi entropy is positive in this example as shown in Figure 4

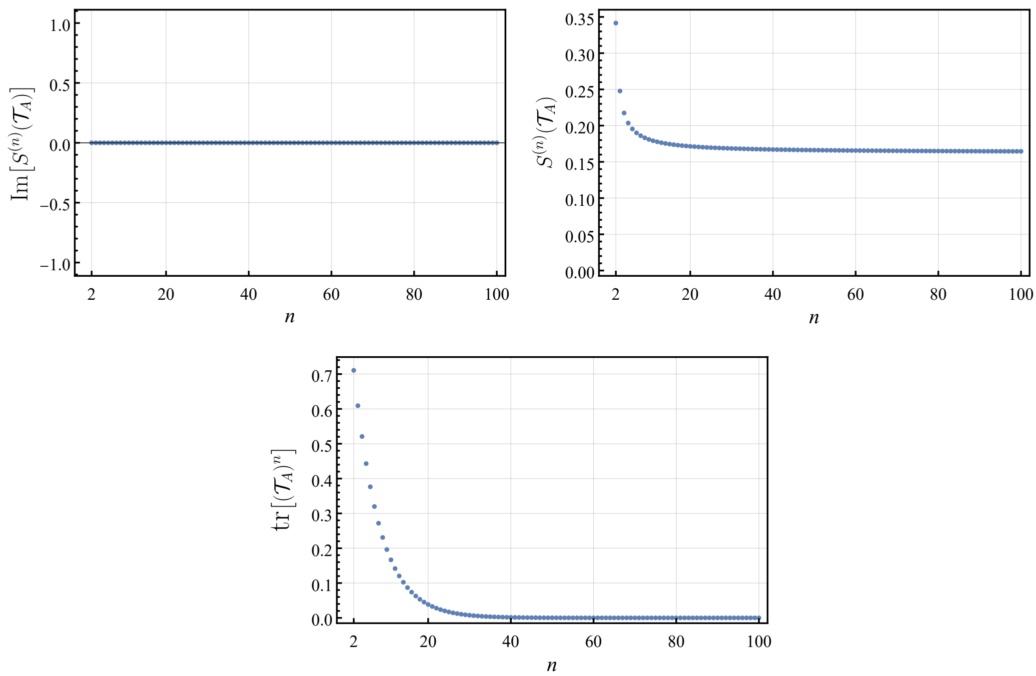


Figure 4: The plot of $S^{(n)}(\mathcal{T}_A)$ and tr$[(\mathcal{T}_A)^n]$. The upper left plot shows the imaginary part of $S^{(n)}(\mathcal{T}_A)$, which are vanishing. The upper right plot shows $S^{(n)}(\mathcal{T}_A)$. The lower plot shows tr$[(\mathcal{T}_A)^n]$, which are in the region $(0, 1)$.

E.3 Example: $S^{(n)}(\mathcal{T}_A) < 0$

In this section we show an example for which $S^{(n)}(\mathcal{T}_A) < 0$. Choosing the matrices

$$\eta_A = \begin{pmatrix} 13.9359 & -17.8554 + 8.22163i & 0.740751 - 0.860494i \\ -17.8554 - 8.22163i & 11.7561 & 3.87722 + 0.527719i \\ 0.740751 + 0.860494i & 3.87722 - 0.527719i & 4.32501 \end{pmatrix},$$

$$\eta_{\bar{A}} = \begin{pmatrix} 2.68826 & -2.76297 + 6.09204i & -13.4254 - 5.89942i \\ -2.76297 - 6.09204i & 23.4288 & 2.24652 - 1.6307i \\ -13.4254 + 5.89942i & 2.24652 + 1.6307i & 6.07729 \end{pmatrix},$$

$$a = \begin{pmatrix} 2.79442 + 26.2305i & 14.4042 - 1.54735i & 1.27623 + 2.29185i \\ 17.0343 + 21.4595i & 6.13678 - 4.72818i & -6.82378 + 24.1677i \\ -6.55401 + 2.08772i & -6.0073 - 29.8274i & -7.59207 - 24.0165i \end{pmatrix}. \tag{83}$$

The reference state $|\Psi\rangle = \frac{1}{\sqrt{3}} \sum_{k=1}^{3} |k\rangle_A |k\rangle_{\bar{A}}$. The eigenvalues of η_A, $\eta_{\bar{A}}$ and $\mathcal{T}_A$ are given by

$$\begin{aligned} &\eta_A \to \{85.7965, -45.7377, -0.637431\}, \\ &\eta_{\bar{A}} \to \{-51.9884, -40.48, 28.8633\}, \\ &\mathcal{T}_A \to \{1.37237, -0.368265, -0.00410468\}. \end{aligned} \tag{84}$$

The pseudo Rényi entropy is negative in this example. The result is shown in Figure 5.

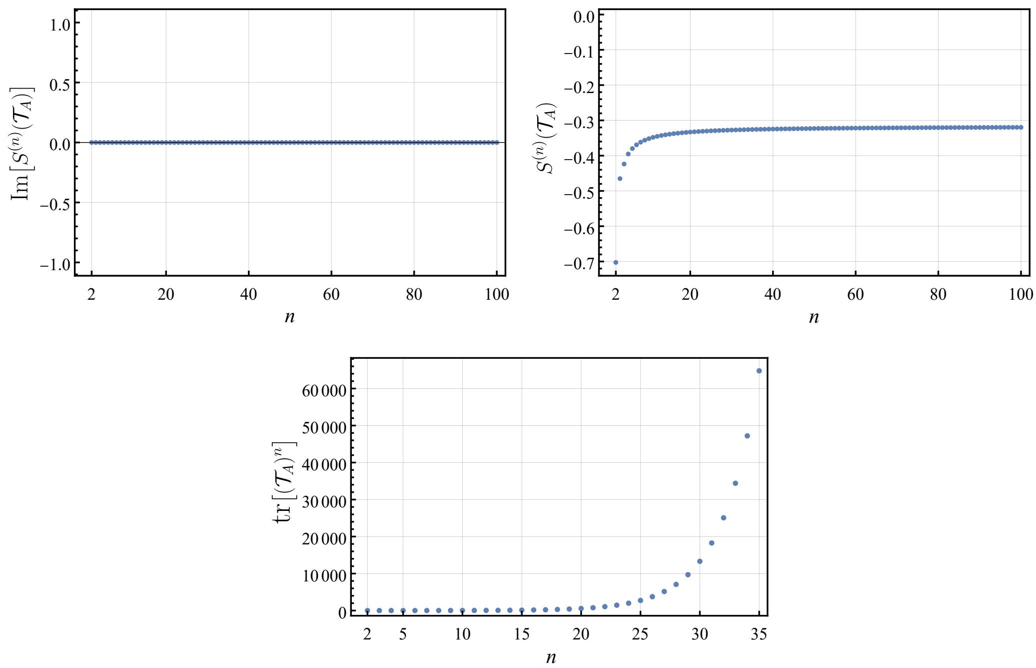


Figure 5: The plot of $S^{(n)}(\mathcal{T}_A)$ and $\mathrm{tr}[(\mathcal{T}_A)^n]$. The upper left plot shows the imaginary part of $S^{(n)}(\mathcal{T}_A)$, which are vanishing. The upper right plot shows $S^{(n)}(\mathcal{T}_A)$, which are negative. The lower plot shows $\mathrm{tr}[(\mathcal{T}_A)^n]$.

E.4 Example with $d = 2$

Let the reference state be $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_{\bar{A}} + |1\rangle_A|1\rangle_{\bar{A}})$. Let the operators η_A and $\eta_{\bar{A}}$ be diagonal, a be arbitrary. They are given by

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix}, \quad \eta_{\bar{A}} = \begin{pmatrix} \bar{\eta}_1 & 0 \\ 0 & \bar{\eta}_2 \end{pmatrix}. \tag{85}$$

Assume η_A and $\eta_{\bar{A}}$ to be positive, thus $\eta_{1(2)} > 0$, $\bar{\eta}_{1(2)} > 0$. One could construct the transition matrix $\mathcal{T}^a$ with these operators. According to Corollary 3 we know the eigenvalues of $\mathcal{T}^a_A$ are positive. With some calculations, we have

$$\mathcal{T}^a_A = \begin{pmatrix} \frac{|a_{11}|^2\eta_1\bar{\eta}_1+|a_{12}|^2\eta_1\bar{\eta}_2}{|a_{11}|^2\eta_1\bar{\eta}_1+|a_{12}|^2\eta_1\bar{\eta}_2+|a_{21}|^2\bar{\eta}_1\eta_2+|a_{22}|^2\eta_2\bar{\eta}_2} & \frac{a_{11}a^*_{21}\eta_2\bar{\eta}_1+a_{12}a^*_{22}\eta_2\bar{\eta}_2}{|a_{11}|^2\eta_1\bar{\eta}_1+|a_{12}|^2\eta_1\bar{\eta}_2+|a_{21}|^2\bar{\eta}_1\eta_2+|a_{22}|^2\eta_2\bar{\eta}_2} \\ \frac{a^*_{11}a_{21}\eta_1\bar{\eta}_1+a^*_{12}a_{22}\eta_1\bar{\eta}_2}{|a_{11}|^2\eta_1\bar{\eta}_1+|a_{12}|^2\eta_1\bar{\eta}_2+|a_{21}|^2\bar{\eta}_1\eta_2+|a_{22}|^2\eta_2\bar{\eta}_2} & \frac{|a_{21}|^2\bar{\eta}_1\eta_2+|a_{22}|^2\eta_2\bar{\eta}_2}{|a_{11}|^2\eta_1\bar{\eta}_1+|a_{12}|^2\eta_1\bar{\eta}_2+|a_{21}|^2\bar{\eta}_1\eta_2+|a_{22}|^2\eta_2\bar{\eta}_2} \end{pmatrix}. \tag{86}$$

The pseudo Rényi entropy of the 2-qubit system is studied in [19]. They claim the eigenvalues of $\mathcal{T}^a_A$ are positive if and only if $0 \leq det[\mathcal{T}^a_A] \leq 1/4$. With some calculations, we have

$$\begin{aligned} \det[\mathcal{T}^a_A] &= \frac{|a_{12}a_{21} - a_{11}a_{22}|^2\eta_1\bar{\eta}_1\eta_2\bar{\eta}_2}{(|a_{11}|^2\eta_1\bar{\eta}_1 + |a_{12}|^2\eta_1\bar{\eta}_2 + |a_{21}|^2\bar{\eta}_1\eta_2 + |a_{22}|^2\eta_2\bar{\eta}_2)^2} \\ &\leq \frac{|a_{12}a_{21} - a_{11}a_{22}|^2\eta_1\bar{\eta}_1\eta_2\bar{\eta}_2}{(2|a_{11}||a_{22}|\sqrt{\eta_1\bar{\eta}_1\eta_2\bar{\eta}_2} + 2|a_{12}||a_{21}|\sqrt{\eta_1\bar{\eta}_1\eta_2\bar{\eta}_2})^2} \leq \frac{1}{4}. \end{aligned} \tag{87}$$

The above result can be generalized to arbitrary positive $\tilde{\eta}_A$ and $\tilde{\eta}_{\bar{A}}$. Since they are Hermitian operators, there exists unitary operator U_A and $U_{\bar{A}}$ such that

$$\tilde{\eta}_A = U_A\eta_A U_A^\dagger, \quad \tilde{\eta}_{\bar{A}} = U_{\bar{A}}\eta_{\bar{A}}U_{\bar{A}}^\dagger, \tag{88}$$

where η_A and $\eta_{\bar{A}}$ are digonal. The transition matrix $\mathcal{T}^a$ with a given reference state $|\Psi'\rangle$ is given by

$$\mathcal{T}^a \propto a|\Psi'\rangle\langle\Psi'|a^\dagger U_A\eta_A U_A^\dagger U_{\bar{A}}\eta_{\bar{A}}U_{\bar{A}}^\dagger. \tag{89}$$

Taking partial trace we have

$$\mathcal{T}^a_A = tr_{\bar{A}}\mathcal{T}^a \propto a(tr_{\bar{A}}U_{\bar{A}}^\dagger|\Psi'\rangle\langle\Psi'|U_{\bar{A}}\eta_{\bar{A}})a^\dagger U_A\eta_A U_A^\dagger. \tag{90}$$

It is always possible to make the operator $tr_{\bar{A}}U_{\bar{A}}^\dagger|\Psi'\rangle\langle\Psi'|U_{\bar{A}}\eta_{\bar{A}} = tr_{\bar{A}}|\Psi\rangle\langle\Psi|\eta_{\bar{A}}$ by choosing suitable $|\Psi'\rangle$. With this choice one can show $det[\mathcal{T}^a_A]$ is equal to (87). Therefore, the transition matrix $\mathcal{T}^a_A$ having positive eigenvalues satisfies that $det[\mathcal{T}^a_A] \leq 1/4$, which is consistent with the result in [19].

F Details of the example of free scalar with $\theta \neq 0$

In the main text we discuss the transition matrix (37), which can be written as the form

$$\mathcal{T}^{\psi|\phi} \propto |\Phi\rangle\langle\Phi|U, \tag{91}$$

where U is a unitary operator, it seems the above transition matrix is not like the general form (13) for the pure pseudo-Hermitian transition matrix. In this section, we will show the transition matrix (91) can be pseudo-Hermitian for some particular unitary operator U and pure state $|\Phi\rangle$.

Let's start with two-qubit example in [19] with

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + e^{i\theta}|11\rangle), \quad |\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \tag{92}$$

where we label the states $|i\rangle_A|j\rangle_{\bar{A}}$ $(i,j=0,1)$ as $|ij\rangle$. The transition matrix $\mathcal{T}^{\psi|\phi}$ is given by

$$\mathcal{T}^{\psi|\phi} = \frac{1}{2\cos\frac{\theta}{2}}|\Phi\rangle\langle\Phi|U, \tag{93}$$

where we define the state $|\Phi\rangle = e^{-\frac{i}{2}\theta}|00\rangle + e^{\frac{i}{2}\theta}|11\rangle$ and the unitary operator

$$U = e^{-\frac{i}{2}\theta}|00\rangle\langle 00| + e^{\frac{i}{2}\theta}|11\rangle\langle 11 + |01\rangle\langle 10| + |10\rangle\langle 01|. \tag{94}$$

In fact, we can also show it is η-pseudo-Hermitian with

$$\eta := e^{-\frac{i}{2}\theta}|00\rangle\langle 11| + |01\rangle\langle 10| + |10\rangle\langle 01| + e^{\frac{i}{2}\theta}|11\rangle\langle 00|. \tag{95}$$

To satisfy the pseudo Hermitian condition we should require

$$\eta|\Phi\rangle\langle\Phi|U\eta^{-1} = U^\dagger|\Phi\rangle\langle\Phi|. \tag{96}$$

Define the operator $\eta' := U\eta$. This condition is given by

$$\eta'|\Phi\rangle = |\Phi\rangle, \quad \langle\Phi|(\eta'^{-1})^\dagger = \langle\Phi|. \tag{97}$$

which can be transformed to the operator relation

$$\eta' = \eta'^{-1} + \alpha P_\perp^\Phi, \tag{98}$$

where α is some constant, $P_\perp^\Phi$ satisfies the condition $P_\perp^\Phi|\Phi\rangle = 0$. One special case is taking $\alpha = 0$. One would have the following relations:

$$(\eta')^2 = 1, \quad \eta' = (\eta')^{-1}. \tag{99}$$

One could check the above two qubits example satisfies the constraints (97) and (99). For the example of free scalar theory with $\theta \neq 0$, one could show the transition matrix is pseudo-Hermitian by proving the existence of the operator η' which satisfies the conditions (97) (98) or (99).

It is not hard to show the Hermitian matrix η (95) can be written as $\eta = \eta_A \otimes \eta_{\bar{A}}$. Thus the reduced transition matrix $\mathcal{T}^{\psi|\phi}_{A(\bar{A})}$ is also pseudo-Hermitian. $\eta_{A(\bar{A})}$ is not a positive or negative definite matrix. The eigenvalues should not be real. In fact the eigenvalues are $\{\frac{1}{1+e^{i\theta}}, \frac{e^{i\theta}}{1+e^{i\theta}}\}$, which is consistent with the above discussion.

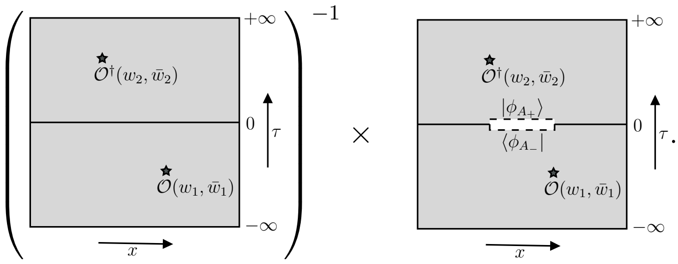
G Calculation of pseudo Rényi entropy by replica method

We outline the replica method in QFTs to compute the pseudo Rényi entropy in this appendix. In particular, we focus on 2D CFTs as the correlation functions in the replica manifold are easy to obtain by conformal mapping. Let's consider a 2D CFT with Lagrangian $\mathcal{L}(\phi, \partial\phi)$ dwells on a Euclidean plane Σ_1 ($ds^2 = dwd\bar{w}$, $(w, \bar{w}) = (x + i\tau, x - i\tau)$) and a transition matrix generated by a local operator $\mathcal{O}(w, \bar{w}) \equiv e^{\tau H}\mathcal{O}(x, 0)e^{-\tau H}$,

$$\mathcal{T}^{\mathcal{O}}_E = \frac{\mathcal{O}(w_1, \bar{w}_1)|0\rangle\langle 0|\mathcal{O}^\dagger(w_2, \bar{w}_2)}{\langle 0|\mathcal{O}^\dagger(w_2, \bar{w}_2)\mathcal{O}(w_1, \bar{w}_1)|0\rangle}, \tag{100}$$

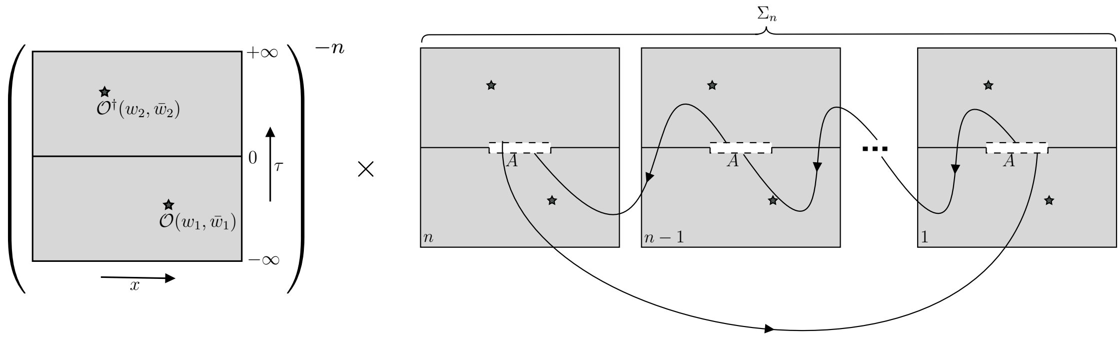
where $w_1 = x_1 - i\tau_1$ and $w_2 = x_2 + i\tau_2$, $(\tau_1, \tau_2 > 0)$. The reduced transition matrix of a subsystem A, $\mathcal{T}^{\mathcal{O}}_{E,A} := \mathrm{tr}_{\bar{A}}\mathcal{T}^{\mathcal{O}}_E$, can be expressed by path integral with operators inserted at $(w_1, \bar{w}_1)$ and $(w_2, \bar{w}_2)$ on the w-plane with a cut on A

$$\langle \phi_{A_-}|\mathcal{T}^{\mathcal{O}}_{E,A}|\phi_{A_+}\rangle = \frac{\int_{\phi(x\in A,\tau=0_-)=\phi_{A_-}(x)}^{\phi(x\in A,\tau=0_+)=\phi_{A_+}(x)}[d\phi]\mathcal{O}^\dagger(w_2, \bar{w}_2)\mathcal{O}(w_1, \bar{w}_1)\exp\left\{-\int_{\mathbb{R}^2}\mathcal{L}(\phi, \partial\phi)\right\}}{\int[d\phi]\mathcal{O}^\dagger(w_2, \bar{w}_2)\mathcal{O}(w_1, \bar{w}_1)\exp\left\{-\int_{\mathbb{R}^2}\mathcal{L}(\phi, \partial\phi)\right\}}$$

$=$  (101)

Building on (101), $\mathrm{tr}[(\mathcal{T}^{\mathcal{O}}_{E,A})^n]$ is given by a $2n$-point correlation function on a n-sheet Riemann surface Σ_n,

$$\mathrm{tr}[(\mathcal{T}^{\mathcal{O}}_{E,A})^n] = \frac{\mathcal{Z}_n}{\mathcal{Z}_1^n} \cdot \frac{\langle \mathcal{O}(w_1, \bar{w}_1)\mathcal{O}^\dagger(w_2, \bar{w}_2)...\mathcal{O}(w_{2n-1}, \bar{w}_{2n-1})\mathcal{O}^\dagger(w_{2n}, \bar{w}_{2n})\rangle_{\Sigma_n}}{\langle \mathcal{O}^\dagger(w_2, \bar{w}_2)\mathcal{O}(w_1, \bar{w}_1)\rangle^n_{\Sigma_1}}$$

$=$  , (102)

where $\mathcal{Z}_1$ and $\mathcal{Z}_n$ are partition functions on Σ_1 and Σ_n, respectively, and $\mathcal{O}(w_{2k-1}, \bar{w}_{2k-1})$ and $\mathcal{O}^\dagger(w_{2k}, \bar{w}_{2k})$ denote the operators inserted at kth sheet. The nth pseudo Rényi entropy of $\mathcal{T}^{\mathcal{O}}_{E,A}$ turns out to be

$$S^{(n)}(\mathcal{T}^{\mathcal{O}}_{E,A}) = S^{(n)}_{A;vac} + \Delta S^{(n)}(\mathcal{T}^{\mathcal{O}}_{E,A}), \tag{103}$$

where $S^{(n)}_{A;vac} \equiv \frac{1}{1-n}\log\frac{\mathcal{Z}_n}{\mathcal{Z}_1^n}$ is the nth Rényi entropy of A when the total system is in the vacuum, and $\Delta S^{(n)}(\mathcal{T}^{\mathcal{O}}_{E,A})$ we refer to as the excess of nth pseudo Rényi entropy of A,

$$\Delta S^{(n)}(\mathcal{T}^{\mathcal{O}}_{E,A}) = \frac{1}{1-n}\log\frac{\langle\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)...\mathcal{O}(w_{2n-1},\bar{w}_{2n-1})\mathcal{O}^\dagger(w_{2n},\bar{w}_{2n})\rangle_{\Sigma_n}}{\langle\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\rangle^n_{\Sigma_1}}. \quad (104)$$

For our purposes, we only focus on the 2nd pseudo Rényi entropy,

$$\Delta S^{(2)}(\mathcal{T}^{\mathcal{O}}_{E,A}) = -\log\frac{\langle\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\mathcal{O}(w_3,\bar{w}_3)\mathcal{O}^\dagger(w_4,\bar{w}_4)\rangle_{\Sigma_2}}{\langle\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\rangle^2_{\Sigma_1}}. \quad (105)$$

Meanwhile, $\mathcal{O}$ is assumed to be a primary with chiral and anti-chiral conformal dimension $\Delta_{\mathcal{O}}$. By conformal symmetry, the 2- and 4-point function of $\mathcal{O}$ on Σ_1 can be expressed as

$$\langle\mathcal{O}(z_1,\bar{z}_1)\mathcal{O}^\dagger(z_2,\bar{z}_2)\rangle_{\Sigma_1} = \frac{c_{12}}{|z_{12}|^{4\Delta_{\mathcal{O}}}}, \quad (106)$$

$$\langle\mathcal{O}(z_1,\bar{z}_1)\mathcal{O}^\dagger(z_2,\bar{z}_2)\mathcal{O}(z_3,\bar{z}_3)\mathcal{O}^\dagger(z_4,\bar{z}_4)\rangle_{\Sigma_1} = |z_{13}z_{24}|^{-4\Delta_{\mathcal{O}}}G(\eta,\bar{\eta}), \quad (107)$$

respectively, where $\eta := \frac{z_{12}z_{34}}{z_{13}z_{24}}$ and $\bar{\eta} := \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{24}}$ are the cross ratios. Since there are conformal mappings

$$z = \begin{cases} w^{1/n}, & A = [0,\infty), \\ \left(\frac{w+L}{w-L}\right)^{1/n}, & A = [-L,L], \end{cases}$$

from Σ_n to Σ_1, the 4-point function on Σ_2 is obtained by applying the above conformal mappings with $n = 2$

$$\langle\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\mathcal{O}(w_3,\bar{w}_3)\mathcal{O}^\dagger(w_4,\bar{w}_4)\rangle_{\Sigma_2} = \begin{cases} \left|\frac{64L^2z_1^2z_2^2}{(z_1^2-1)^2(z_2^2-1)^2}\right|^{-4\Delta_{\mathcal{O}}}G(\eta,\bar{\eta}), & A = [-L,L], \\ \\ \left|16z_1^2z_2^2\right|^{-4\Delta_{\mathcal{O}}}G(\eta,\bar{\eta}), & A = [0,+\infty). \end{cases} \quad (108)$$

Substituting (106) and (108) into (105) and after some algebra, we obtain

$$\Delta S^{(2)}(\mathcal{T}^{\mathcal{O}}_{E,A}) = \log\frac{c_{12}^2}{|\eta(1-\eta)|^{4\Delta_{\mathcal{O}}}G(\eta,\bar{\eta})}, \quad (109)$$

which only depends on the cross ratios η and $\bar{\eta}$. The 2nd pseudo Rényi entropy with regard to the real-time dependent transition matrix can be obtained by applying the analytic continuation to τ_1 and τ_2 in the above result. When $\tau_1 \to \epsilon + it$ and $\tau_2 \to \epsilon - it$, we meet the case studied in [25]. As we mentioned in the previous section, we would like to focus on the case of $\mathcal{T}^{\mathcal{O}} = \frac{\mathcal{O}(x,t)|0\rangle\langle 0|\mathcal{O}(-x,-t)}{\langle 0|\mathcal{O}(-x,-t)\mathcal{O}(x,t)|0\rangle}$. Thus we have the analytic continuation $\tau_1 = \tau_2 \to \epsilon - it$. An infinitesimally small regularization parameter ϵ is introduced to suppress the high energy modes [39].

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