Continuity and Harnack inequalities for local minimizers of non uniformly elliptic functionals with generalized Orlicz growth under the non-logarithmic conditions *

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Abstract

We study the qualitative properties of functions belonging to the corresponding De Giorgi classes

$$\int\limits_{B_{r(1-\sigma)}(x_0)} \Phi(x, |\nabla(u-k)_\pm|)\, dx \leqslant \gamma \int\limits_{B_r(x_0)} \Phi\Big(x, \frac{(u-k)_\pm}{\sigma r}\Big)\, dx,$$

where σ, $r \in (0,1)$, $k \in \mathbb{R}$ and the function Φ satisfies the non-logarithmic condition

$$\Big(r^{-n} \int\limits_{B_r(x_0)} [\Phi\big(x, \frac{v}{r}\big)]^s\, dx\Big)^{\frac{1}{s}} \Big(r^{-n} \int\limits_{B_r(x_0)} [\Phi\big(x, \frac{v}{r}\big)]^{-t}\, dx\Big)^{\frac{1}{t}} \leqslant c(K)\Lambda(x_0, r), \quad r \leqslant v \leqslant K\,\lambda(r),$$

under some assumptions on the functions $\lambda(r)$ and $\Lambda(x_0, r)$ and the numbers s, $t > 1$. These conditions generalize the known logarithmic, non-logarithmic and non uniformly elliptic conditions.

In particular, our results cover new cases of non uniformly elliptic double-phase, degenerate double-phase functionals and functionals with variable exponents.

Keywords: non-autonomous functionals, non-logarithmic conditions, continuity, Harnack's inequality.

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1 Introduction and main results

To explain the point of view of this research consider the energy integrals $\int\limits_\Omega \Phi_i(x, |\nabla u|) dx$,

$\Phi_1(x, v) = v^p + a_1(x) v^q, \quad a_1(x) \geqslant 0, \qquad \underset{B_r(x_0)}{\operatorname{osc}}\, a_1(x) \leqslant A\, r^{q-p}, \quad A > 0, \quad v > 0.$

$\Phi_2(x, v) = v^p\big(1 + a_2(x)\log(1+v)\big), \qquad a_2(x) \geqslant 0, \qquad \underset{B_r(x_0)}{\operatorname{osc}}\, a_2(x) \leqslant \dfrac{A}{\log\frac{1}{r}}, \quad A > 0, \quad v > 0.$

In particular, these conditions imply

$$\sup_{B_r(x_0)} \Phi_i\big(x, \frac{v}{r}\big) \leqslant \gamma(K) \inf_{B_r(x_0)} \Phi_i\big(x, \frac{v}{r}\big), \quad r \leqslant v \leqslant K, \quad i = 1, 2. \tag{1.1}$$

*Dedicated to Dr. Mykhailo Voitovych, missing in the hero-city of Mariupol

It is well known (see [9]) that the minimizers of the corresponding integrals of the calculus of variations satisfy Harnack's type inequality, or more generally (see [24]), Harnack's type inequality is valid under the conditions

$$\operatorname*{osc}_{B_r(x_0)} a_1(x) \leqslant A\big[\log\frac{1}{r}\big]^L r^{q-p}, \quad \operatorname*{osc}_{B_r(x_0)} a_2(x) \leqslant A\frac{\big[\log\log\frac{1}{r}\big]^L}{\log\frac{1}{r}},$$

if $L>0$ is sufficiently small. These conditions yield

$$\sup_{B_r(x_0)} \Phi_1\big(x,\frac{v}{r}\big) \leqslant \gamma(K) \inf_{B_r(x_0)} \Phi_1\big(x,\frac{v}{r}\big), \quad r\leqslant v\leqslant K\,\lambda(r), \quad \lambda(r)=\big[\log\frac{1}{r}\big]^{-\frac{L}{q-p}}, \tag{1.2}$$

and

$$\sup_{B_r(x_0)} \Phi_2\big(x,\frac{v}{r}\big) \leqslant \gamma(K)\,\Lambda(r) \inf_{B_r(x_0)} \Phi_2\big(x,\frac{v}{r}\big), \quad r\leqslant v\leqslant K, \quad \Lambda(r)=\big[\log\log\frac{1}{r}\big]^{L}. \tag{1.3}$$

To take into account the non-uniformly elliptic case, we set

$$a(x)=\big|\log\big|\log\frac{1}{|x-x_0|}\big|\big|^{L_1}, \quad x_0\in\Omega,$$

and let $\Phi_1(v)=v^p+v^q$, $\Phi_2(v)=v^p\big(1+\log(1+v)\big)$, then

$$\gamma^{-1}\,a(x)\Phi_i(v)\leqslant\Phi_i(x,v)\leqslant\gamma\,\Phi_i(v), \quad L_1<0, \quad i=1,2,$$

$$\gamma^{-1}\,\Phi_i(v)\leqslant\Phi_i(x,v)\leqslant\gamma\,a(x)\,\Phi_i(v), \quad L_1>0, \quad i=1,2$$

provided that $B_r(x_0)\subset B_R(x_0)\subset\Omega$ and R is sufficiently small and the bounded local solutions of the corresponding elliptic equations satisfy Harnack's type inequality [25] if

$$\frac{1}{a(x)}\in L^t(\Omega) \quad\text{and}\quad a(x)\in L^s(\Omega) \tag{1.4}$$

with some t, $s>1$, i.e. if L_1 is sufficiently small. In this paper, our aim is to combine logarithmic, non-logarithmic and non uniformly elliptic conditions (1.1)–(1.4). Obviously, conditions (1.1)–(1.4) imply for i=1,2

$$\Big(r^{-n}\int\limits_{B_r(x_0)}[\Phi_i\big(x,\frac{v}{r}\big)]^s\,dx\Big)^{\frac{1}{s}}\Big(r^{-n}\int\limits_{B_r(x_0)}[\Phi_i\big(x,\frac{v}{r}\big)]^{-t}\,dx\Big)^{\frac{1}{t}}\leqslant\gamma(K)\Lambda(x_0,r), \quad r\leqslant v\leqslant K\,\lambda(r), \tag{1.5}$$

with some s, t and the precise choice of $\lambda(r)$ and $\Lambda(x_0,r)$.

Another interesting example is the energy integral $\int\limits_\Omega\Phi_3(x,|\nabla u|)\,dx$,

$$\Phi_3(x,v)=v^{p(x)}, \quad \operatorname*{osc}_{B_r(x_0)} p(x)\leqslant\frac{\bar\mu(r)}{\log\frac{1}{r}}, \quad \lim_{r\to0}\bar\mu(r)=\infty, \quad \lim_{r\to0}\frac{\bar\mu(r)}{\log\frac{1}{r}}=0, \quad v>0.$$

It is known that the solutions of the corresponding equations and the minimizers of the corresponding integrals satisfy the Harnack type inequality ([1]) if $\mu(r)\equiv const$, or more generally (see [5–7, 49]) if $\mu(r)=L\log\log\log\frac{1}{r}$, i.e. under conditions (1.3). The bounded local solutions of the corresponding elliptic equations, as well as the minimizers of the corresponding integrals belong to the corresponding De Giorgi's classes, i.e. for $k \in \mathbb{R}$, σ, $r \in (0,1)$ the following inequalities hold

$$\int\limits_{B_{r(1-\sigma)}(x_0)} |\nabla(u-k)_\pm|^{p(x)}\,dx \leqslant \gamma \int\limits_{B_r(x_0)} \left(\frac{u-k}{\sigma r}\right)_\pm^{p(x)} dx.$$

Set $a_\pm(x,k,r) = \left(\dfrac{M_\pm(k,r)}{r}\right)^{p(x)-p_-}$, $\quad M_\pm(k,r) = \sup\limits_{B_r(x_0)} (u-k)_\pm$ and $p_- = \min\limits_{B_r(x_0)} p(x)$, then by the Young inequality

$$\int\limits_{B_{r(1-\sigma)}(x_0)} a_\pm(x,k,r)|\nabla(u-k)_\pm|^{p_-}\,dx \leqslant \gamma\,\sigma^{-\gamma}\left(\frac{M_\pm(k,r)}{r}\right)^{p_-} \int\limits_{B_r(x_0)\cap\{(u-k)_\pm>0\}} a_\pm(x,k,r)\,dx.$$

There are two possibilities how to use this inequality. The first one is almost standard, by our assumptions on the function $p(x)$ we obtain

$$\int\limits_{B_{r(1-\sigma)}(x_0)} |\nabla(u-k)_\pm|^{p_-}\,dx \leqslant \gamma\,\sigma^{-\gamma}\,\exp(\gamma\,\bar\mu(r))\left(\frac{M_\pm(k,r)}{r}\right)^{p_-} \big|B_r(x_0)\cap\{(u-k)_\pm>0\}\big|,$$

provided that $M_\pm(k,r) \geqslant r$. This estimate leads us to the condition (see e.g. [5–7, 49])

$$\int\limits_0 \exp\big(-\gamma_1\,\exp(\gamma_2\,\bar\mu(r)\,)\big)\frac{dr}{r} = \infty, \tag{1.6}$$

with some $\gamma_1, \gamma_2 > 0$. The function $\bar\mu(r) = L\log\log\log\frac{1}{r}$ satisfies (1.6) if $L > 0$ is sufficiently small.

This condition can be improved, namely, it turns out that the function $a_\pm(x,k,r)$ with $p(x) = p + L\dfrac{\log\log\frac{1}{|x-x_0|}}{\log\frac{1}{|x-x_0|}}$ satisfies the condition (see [48])

$$\left(r^{-n}\int\limits_{B_r(x_0)} [a_\pm(x,k,r)]^{-t}\,dx\right)^{\frac{1}{t}} \left(r^{-n}\int\limits_{B_r(x_0)} [a_\pm(x,k,r)]^{s}\,dx\right)^{\frac{1}{s}} \leqslant \gamma(t,s), \quad t,s>0, \tag{1.7}$$

provided that

$$\frac{1}{\log\log\frac{1}{16r}} + \bar\gamma\,L\frac{\log\log\frac{1}{16r}}{\log\frac{1}{16r}} \leqslant 1, \quad \text{and} \quad r \leqslant M_\pm(k,r) \leqslant 2M = 2\sup_\Omega|u|,$$

with sufficiently large $\bar\gamma > 0$. This condition leads us to the standard Harnack type inequality for solutions of the corresponding $p(x)$-Laplace equation. Obviously, inequalities (1.7) can be generalized by conditions (1.5).

In this paper we also consider the integrals of this type. And, of course, it would be interesting to unify our approach. More precisely, we will prove continuity and Harnack's inequality for functions belonging to the corresponding non uniformly elliptic De Giorgi classes $DG_\Phi(B_R(x_0))$.

We write $W^{1,\Phi}(B_R(x_0))$ for the class of functions $u \in W^{1,1}(B_R(x_0))$ with $\int\limits_{B_R(x_0)} \Phi(x,|\nabla u|)dx < \infty$ and we say that a measurable function $u : B_R(x_0) \to \mathbb{R}$ belongs to the elliptic class $DG^{\pm}_{\Phi}(B_R(x_0))$ if $u\in W^{1,\Phi}(B_R(x_0))\cap L^\infty(B_R(x_0))$ and there exist numbers $c>0$, $q>1$ such that for any ball $B_r(x_0)\subset B_R(x_0)$, any $k\in\mathbb{R}$ and any $\sigma\in(0,1)$ the following inequalities hold:

$$\int\limits_{A^{\pm}_{k,r}} \Phi\big(x,|\nabla u|\big)\zeta^q(x)dx \leqslant c \int\limits_{A^{\pm}_{k,r}} \Phi\bigg(x,\frac{(u-k)_\pm}{\sigma r}\bigg)dx, \tag{1.8}$$

here $(u-k)_\pm := \max\{\pm(u-k),0\}$, $A^{\pm}_{k,r} := B_r(x_0)\cap\{(u-k)_\pm>0\}$, $\zeta(x)\in C_0^\infty(B_r(x_0))$, $0\leqslant\zeta(x)\leqslant 1$, $\zeta(x)=1$ in $B_{(1-\sigma)r}(x_0)$ and $|\nabla\zeta(x)|\leqslant\dfrac{1}{\sigma r}$. We also say that $u\in DG^{\pm}_{\Phi}(\Omega)$ if $u\in DG^{\pm}_{\Phi}(B_R(x_0))$ for any $B_{8R}(x_0)\subset\Omega$. We set also $DG_{\Phi}(B_R(x_0)) = DG^{-}_{\Phi}(B_R(x_0))\cup DG^{+}_{\Phi}(B_R(x_0))$ and $DG_{\Phi}(\Omega)=DG^{-}_{\Phi}(\Omega)\cup DG^{+}_{\Phi}(\Omega)$.

Further we suppose that $\Phi(x,v): B_R(x_0)\times\mathbb{R}_+\to\mathbb{R}_+$ is a non-negative function satisfying the following properties: for any $x\in B_R(x_0)$ the function $v\to\Phi(x,v)$ is increasing and $\lim\limits_{v\to0}\Phi(x,v)=0$, $\lim\limits_{v\to+\infty}\Phi(x,v)=+\infty$. We also assume that

(Φ_0) There exists $c_0\geqslant 1$ such that for any $x\in B_R(x_0)$ there holds

$$c_0^{-1}\leqslant\Phi(x,1)\leqslant c_0.$$

(Φ) There exist $1<p<q$ such that for $x\in B_R(x_0)$ and for $w\geqslant v>0$ there holds

$$\Big(\frac{w}{v}\Big)^p\leqslant\frac{\Phi(x,w)}{\Phi(x,v)}\leqslant\Big(\frac{w}{v}\Big)^q.$$

($\Phi^{\lambda}_{\Lambda,x_0}$) There exist continuous, non-decreasing function $0<\lambda(r)\leqslant 1$ and continuous, non-increasing function $\Lambda_\lambda(x_0,r)\geqslant 1$ on the interval $(0,R)$ such that for any $B_r(x_0)\subset B_R(x_0)$, for any $K>0$ there holds

$$\sup_{r\leqslant v\leqslant K\,\lambda(r)}\Lambda_\Phi\big(x_0,r,\frac{v}{r}\big)\leqslant c_1(K)\Lambda_\lambda(x_0,r),$$

$$\frac{1}{tp}+\frac{1}{sp}<\frac{1}{n},\quad t\in\big(\max(1,\frac{1}{p-1}),\infty\big],\quad s\in(1,\infty],$$

here $c_1(K)$ is some fixed positive number depending on K, s and t and

$$\Lambda_\Phi\big(x_0,r,\frac{v}{r}\big):=\Lambda_{-,\Phi}\big(x_0,r,\frac{v}{r}\big)\ \Lambda_{+,\Phi}\big(x_0,r,\frac{v}{r}\big),$$

$$\Lambda_{-,\Phi}\big(x_0,r,\frac{v}{r}\big):=\bigg(r^{-n}\int\limits_{B_r(x_0)}\big[\Phi(x,\frac{v}{r})\big]^{-t}dx\bigg)^{\frac{1}{t}},\Lambda_{+,\Phi}\big(x_0,r,\frac{v}{r}\big):=\bigg(r^{-n}\int\limits_{B_r(x_0)}\big[\Phi(x,\frac{v}{r})\big]^{s}dx\bigg)^{\frac{1}{s}}.$$

We will also write (Φ^{λ}_{Λ}) if condition ($\Phi^{\lambda}_{\Lambda,x_0}$) holds for any $B_R(x_0)\subset B_{8R}(x_0)\subset\Omega$ and set

$$\Lambda_\lambda(r):=\sup_{x_0\in\Omega,B_{8R}(x_0)\subset\Omega}\Lambda_\lambda(x_0,r).$$

Remark 1.1. *Note that in the logarithmic case, i.e. if $\lambda(r)\equiv\Lambda_\lambda(r)\equiv 1$ functions from $DG_\Phi(\Omega)$ belong to the standard De Giorgi class $DG_{p\frac{t}{t+1}}(\Omega)$, $p\frac{t}{t+1}>1$ (see Lemma 2.2 below), so continuity and Harnack's inequality follow directly from results of [32] and [20].*

Remark 1.2. *We note that condition* $(\Phi^{\lambda}_{\Lambda})$ *generalizes known conditions on the function* Φ, *for example, that is condition* $(A1-n)$ *from [27–29] in the logarithmic case, i.e. if* $\lambda(r)\equiv\Lambda_1(r)\equiv$ $\equiv$ *const. This condition generalizes conditions* Φ_λ *and* Φ_μ *(see [24]) in the non-logarithmic case. Moreover, this condition generalizes the non-uniformly elliptic condition(see [25]). And finally, condition* $\Phi^{\lambda}_{\Lambda,x_0}$ *includes conditions of the type* (1.7).

Sometimes we will also need the following technical assumption

(λ) There exist positive constants c_2 and c_3 such that

$$\lambda(\rho)\leqslant\Big(\frac{\rho}{r}\Big)^{c_2}\lambda(r),\quad \Lambda_\lambda(x_0,r)\leqslant\Big(\frac{\rho}{r}\Big)^{c_3}\Lambda_\lambda(x_0,\rho),\quad 0<r\leqslant\rho.$$

We refer to the parameters n, p, q, t, s, c, c_0, $M(R):=\sup\limits_{B_R(x_0)}|u|$, $c_1(M(R))$, c_2 and c_3 as our structural data, and we write γ if it can be quantitatively determined a priory in terms of the above quantities. The generic constant γ may change from line to line. In general, we assume that $M:=\sup\limits_{\Omega}|u|$ and $c_1(M)$ are also the data. Our first result is the interior continuity of the functions belonging to the corresponding De Giorgi classes.

Theorem 1.1. *Let* $u\in DG_\Phi(B_R(x_0))$ *and let conditions* (Φ_0), (Φ), $(\Phi^{\lambda}_{\Lambda,x_0})$ *be fulfilled. There exist numbers* C_1, $\beta_1>0$ *depending only on the data such that if*

$$\int\limits_0 \exp\big(C_1\big[\Lambda_\lambda(x_0,r)\big]^{\beta_1}\big)\frac{dr}{\lambda(r)}<+\infty,\quad \int\limits_0 \lambda(r)\exp\big(-C_1\big[\Lambda_\lambda(x_0,r)\big]^{\beta_1}\big)\frac{dr}{r}=+\infty, \qquad (1.9)$$

then $u(x)$ *is continuous at point* x_0.

If additionally, $u\in DG_\Phi(\Omega)$, *condition* $(\Phi^{\lambda}_{\Lambda})$ *holds and*

$$\int\limits_0 \exp\big(C_1\big[\Lambda_\lambda(r)\big]^{\beta_1}\big)\frac{dr}{\lambda(r)}<+\infty,\quad \int\limits_0 \lambda(r)\exp\big(-C_1\big[\Lambda_\lambda(r)\big]^{\beta_1}\big)\frac{dr}{r}=+\infty, \qquad (1.10)$$

then $u(x)\in C(\Omega)$.

Here some typical examples of the function Φ which satisfies the conditions of the above theorem .

• The function $\Phi_1(x,v)=v^p+a(x)v^q$ satisfies condition (Φ^{λ}_{1,x_0}) with $\lambda(r)=[\log\frac{1}{r}]^{-L}$ and $\Lambda_\lambda(x_0,r)\equiv 1$ if $\operatorname*{osc}\limits_{B_r(x_0)} a(x)\leqslant Ar^{q-p}[\log\frac{1}{r}]^{L(q-p)}$ and $a(x_0)=0$. Condition (1.9) holds if $L\leqslant 1$. If $a(x_0)>0$, then $a(x)\asymp a(x_0)$, provided that R is small enough, condition (Φ^{1}_{1,x_0}) holds with $\lambda(r)\equiv\Lambda_1(x_0,r)\equiv 1$. Condition (1.9) is always satisfied.
The function $\Phi_1(x,v)$ satisfies condition (Φ^{1}_{Λ,x_0}) with $\lambda(r)\equiv 1$ and $\Lambda_1(x_0,r)=[\log\log\frac{1}{r}]^L$, $L>0$ provided that $[\log\log\frac{1}{|x-x_0|}]^{-L}\leqslant a(x)\leqslant 1$. Condition (1.9) holds if $L\,\beta_1<1$.

• The function $\Phi_2(x,v)=v^p\big(1+a(x)\log(1+v)\big)$ satisfies condition (Φ^{1}_{Λ,x_0}) with $\lambda(r)\equiv 1$ and $\Lambda_1(x_0,r)=[\log\log\frac{1}{r}]^L$ if $\operatorname*{osc}\limits_{B_r(x_0)} a(x)\leqslant A\dfrac{[\log\log\frac{1}{r}]^L}{\log\frac{1}{r}}$ and $a(x_0)=0$, provided that R is sufficiently small. Condition (1.9) holds if $L\beta_1<1$. If $a(x_0)>0$, then $a(x)\asymp a(x_0)$, provided that R is small enough, condition (Φ^{1}_{1,x_0}) holds with $\lambda(r)\equiv\Lambda_1(x_0,r)\equiv 1$. Condition (1.9) is

always satisfied.
The function $\Phi_2(x,v)$ satisfies condition (Φ^1_{Λ,x_0}) with $\lambda(r)\equiv 1$ and $\Lambda_1(x_0,r)=[\log\log\frac{1}{r}]^L$, $L>0$ provided that $[\log\log\frac{1}{|x-x_0|}]^{-L}\leqslant a(x)\leqslant 1$. Condition (1.9) holds if $L\,\beta_1<1$.

• The function $\Phi_3(x,v)=v^{p(x)}$ satisfies condition (Φ^1_{1,x_0}) with $\lambda(r)\equiv 1$ and $\Lambda_1(x_0,r)=[\log\log\frac{1}{r}]^L$ if $\underset{B_r(x_0)}{\text{osc}}\, p(x)\leqslant L\frac{\log\log\log\frac{1}{r}}{\log\frac{1}{r}}$. Condition (1.9) holds if $L\beta_1<1$.
The function $\Phi_3(x,v)$ satisfies condition (Φ^1_{1,x_0}) with $\lambda(r)\equiv\Lambda_1(x_0,r)\equiv 1$, if $p(x)=p\pm L\dfrac{\log\log\frac{1}{|x-x_0|}}{\log\frac{1}{|x-x_0|}}$, $\quad L>0$, provided that R is small enough. Condition (1.9) is always holds.

• The function $\Phi_4(x,v)=v^p\big(1+\log(1+a(x)v)\big)$ satisfies condition (Φ^λ_{1,x_0}) with $\lambda(r)=[\log\frac{1}{r}]^{-L}$ and $\Lambda_1(x_0,r)\equiv 1$ if $\underset{B_r(x_0)}{\text{osc}}\, a(x)\leqslant Ar[\log\frac{1}{r}]^L$ and $a(x_0)=0$. Condition (1.9) holds if $L\leqslant 1$. If $a(x_0)>0$, then $a(x)\asymp a(x_0)$, provided that R is small enough, condition (Φ^1_{1,x_0}) holds with $\lambda(r)\equiv\Lambda_1(x_0,r)\equiv 1$. Condition (1.9) is always satisfied.

Next result is the Harnack inequality. We will distinguish several cases, first we will assume that $\Lambda_\lambda(r)\leqslant const$, $0<r\leqslant R$. Note that the case $\lim\limits_{r\to 0}\Lambda_1(r)=\infty$ is possible.

Theorem 1.2. *Let $u\in DG^-_\Phi(\Omega)$, $u\geqslant 0$, let conditions (Φ_0), (Φ), (Φ^λ_1), (λ) be fulfilled. Then there exist numbers $C_2>0$, $\theta\in(0,1)$ depending only on the data such that*

$$\bigg(\rho^{-n}\int\limits_{B_\rho(x_0)} u^\theta\,dx\bigg)^{\frac{1}{\theta}}\leqslant\frac{C_2}{\lambda(\rho)}\Big\{\min_{B_{\frac{\rho}{2}}(x_0)} u+\rho\Big\},\quad 0<\rho\leqslant R, \tag{1.11}$$

provided that $B_{8R}(x_0)\subset\Omega$.

In addition, if $u\in DG_\Phi(\Omega)$ and condition (Φ^1_Λ) holds , then there exist numbers C_3, $\beta_2>0$ depending only on the data such that

$$\max_{B_{\frac{\rho}{2}}(x_0)} u\leqslant C_3\,\frac{[\Lambda_1(\rho)]^{\beta_2}}{\lambda(\rho)}\Big\{\min_{B_{\frac{\rho}{2}}(x_0)} u+\rho\Big\},\quad 0<\rho\leqslant R, \tag{1.12}$$

provided that $B_{8R}(x_0)\subset\Omega$.

We formulate our next theorem under the assumption $\lambda(r)\equiv 1$, moreover, its formulation requires more complicated conditions on the function $\Lambda_1(r)$, so we will prove it only in the model case, namely, we will assume that $\Lambda_1(r)=[\log\log\frac{1}{r}]^L, L>0$.

Theorem 1.3. *Let $u\in DG_\Phi(\Omega)\cap C(\Omega)$, $u\geqslant 0$ and let conditions (Φ_0), (Φ), (Φ^1_Λ) be fulfilled. Let $\Lambda_1(\rho)=\big[\log\log\frac{1}{\rho}\big]^L$, $\quad\rho\in(0,1)$, $L>0$. Then there exists number $C_4>0$, depending only on the data and L such that*

$$u(x_0)\leqslant C_4\,\log\frac{1}{\rho}\,\Big\{\min_{B_{\frac{\rho}{2}}(x_0)} u+\rho\Big\},\quad 0<\rho\leqslant R, \tag{1.13}$$

provided that $B_{8R}(x_0)\subset\Omega$ and L is small enough.

We prove our most general result only for solutions of the corresponding equations. More precisely, we are concerned with elliptic equations

$$div\bigg(\Phi(x,|\nabla u|)\frac{\nabla u}{|\nabla u|^2}\bigg)=0,\quad x\in\Omega. \tag{1.14}$$

We say that a function u is a weak sub(super)-solution to Eq. (1.14) if $u \in W^{1,\Phi}(\Omega)$ and the integral identity

$$\int\limits_{\Omega} \Phi(x, |\nabla u|) \frac{\nabla u}{|\nabla u|^2} \nabla \eta \, dx \leqslant (\geqslant) = 0, \tag{1.15}$$

holds for all non-negative test functions $\eta \in W_0^{1,\Phi}(\Omega)$.

The next result is Harnack's inequality under the point condition $(\Phi^{\lambda}_{\Lambda,x_0})$.

Theorem 1.4. *Let u be a non-negative bounded weak super-solution to Eq.* (1.14) *and let conditions (Φ_0), (Φ), $(\Phi^{\lambda}_{\Lambda,x_0})$ and (λ) be fulfilled. Assume also that*

$$\big(\Phi(x,|\xi|)\frac{\xi}{|\xi|^2} - \Phi(x,|\zeta|)\frac{\zeta}{|\zeta|^2}\big)(\xi - \zeta) > 0, \quad \xi, \zeta \in \mathbb{R}^n, \quad \xi \neq \zeta, \quad x \in \Omega. \tag{1.16}$$

Then there exist numbers C_5, $C_6 > 0$, $\theta \in (0,1)$ depending only on the data such that

$$\Big(\rho^{-n} \int\limits_{B_\rho(x_0)} u^\theta \, dx\Big)^{\frac{1}{\theta}} \leqslant \frac{1}{\lambda(\rho)} \exp\big(C_5\big[\Lambda_\lambda(x_0,\rho)\big]^{C_6}\big)\big\{ \min_{B_{\frac{\rho}{2}}(x_0)} u + \rho\big\}, \tag{1.17}$$

provided that $B_{8\rho}(x_0) \subset \Omega$.

In addition, if u is a non-negative bounded weak solution to Eq. (1.14), *then*

$$\max_{B_{\frac{\rho}{2}}(x_0)} u \leqslant \frac{1}{\lambda(\rho)} \exp\big(C_7\big[\Lambda_\lambda(x_0,\rho)\big]^{C_8}\big)\big\{ \min_{B_{\frac{\rho}{2}}(x_0)} u + \rho\big\}, \tag{1.18}$$

provided that $B_{8\rho}(x_0) \subset \Omega$. Here C_7, $C_8 > 0$ depend only on the data.

Before describing the method of proof, a few words about the history of the problem. Qualitative properties of functions belonging to the corresponding De Giorgi classes in the standard case, i.e. if $p = q$ are well known(we refer the reader to the well-known monograph of Ladyzhenskaya and Ural'tseva [32] and to the seminal paper of DiBenedetto and Trudinger [20]).Harnack's inequality for non uniformly elliptic equations has been known since the well-known paper of Trudinger [51].

The study of regularity of minima of functionals with non-standard growth has been initiated by Zhikov [52–55, 57], Marcellini [37, 38], and Lieberman [36], and in the last thirty years, the qualitative theory of second order elliptic equations with so-called log-condition (if $\lambda(r) \equiv \Lambda_1(x_0, r) \equiv 1$) has been actively developed.Moreover,many authors have established local boundedness, Harnack's inequality and continuity of solutions to such equations without or with singular lower order terms, as well as of local minimizers, Q-minimizers, and ω-minimizers of the corresponding minimization problems(see, e.g. [1–4, 8–17, 21–23, 26–30, 40, 45, 50] and references therein).

The case when conditions $(\Phi^{\lambda}_{\Lambda,x_0})$ hold differs substantially from the logarithmic case. To our knowledge there are few results in this direction. Zhikov [56] obtained a generalization of the logarithmic condition which guarantees the density of smooth functions in Sobolev space $W^{1,p(x)}(\Omega)$. This result holds if $1 < p \leqslant p(x)$ and

$$|p(x) - p(y)| \leqslant \frac{|\log|\log\mu(|x-y|)||}{|\log|x-y||}, \quad x, y \in \Omega, \quad x \neq y, \int\limits_0 (\mu(r))^{-\frac{n}{p}} \frac{dr}{r} = +\infty,$$

Particularly, the function $\mu(r) = (\log \frac{1}{r})^L$ satisfies the above condition if $L \leqslant \frac{p}{n}$.

Interior continuity, continuity up to the boundary and Harnack's inequality to the $p(x)$-Laplace equation were proved in [4, 7, 49] under the condition (1.1). These results were generalized in [41, 47] for a wide class of elliptic equations with non-logarithmic Orlicz growth. Particularly, Harnack's inequality was proved in [47] under condition (1.6). In the proof, the authors used Trudinger's ideas [51]. Qualitative properties for solutions of non uniformly elliptic equations with non-standard growth under the non-logarithmic conditions were considered in [25].

As it was mentioned, in this paper we cover the non-uniformly elliptic case and the case of variable exponent of the type (1.7).

The main difficulty arising in the proof of the main results is related to the so-called theorem on the expansion of positivity. Roughly speaking, having information on the measure of the "positivity set" of u over the ball $B_r(\bar{x})$:

$$|\{x \in B_r(\bar{x}) : u(x) \geqslant N\}| \geqslant \alpha(r)|B_r(\bar{x})|,$$

with some $r, N > 0$ and $\alpha(r) \in (0,1)$, we cannot use the classical approach of of Krylov and Safonov [31], DiBenedetto and Trudinger [20] as it was done in the logarithmic case, i.e. if α is independent of r(see e.g. [9]). Difficulties arise not only due to the presence of a constant $\alpha(r)$ depending on r, but also because in the process of iteration from $B_r(\bar{x})$ to $B_\rho(x_0)$ an additional factor arises, which can be estimated only under conditions of Theorems 1.2 and 1.3. So, first we prove the following expansion of positivity theorem.

Theorem 1.5. *Let $u \in DG_\Phi(\Omega), u \geqslant 0$, let $x_0 \in \Omega$ be such that $B_{8R}(x_0) \subset \Omega$,and let conditions (Φ_0), (Φ), (Φ_Λ^λ), (λ) be fulfilled , assume also that*

$$|\{B_r(y) : u > N\}| \geqslant \alpha|B_r(y)|, \tag{1.19}$$

with some $\alpha \in (0,1)$, some $0 < N < M$ and $B_r(y) \subset B_\rho(x_0) \subset B_R(x_0)$, then there exist numbers $\varepsilon_0, \gamma, c, \beta, \tau_1, \tau_2 > 0$ depending only on the data such that

$$N\lambda(\rho)\alpha^{\tau_1} \leqslant \gamma \left(\frac{\rho}{r}\right)^{\tau_2} \exp\big(c\int\limits_{\bar{r}}^{\rho} [\Lambda_\lambda(s)]^\beta \frac{ds}{s}\big)\big\{\min_{B_{\frac{\rho}{2}}(x_0)} u + \rho\big\}, \quad \bar{r} = \varepsilon_0\alpha^2 \frac{r}{\Lambda_\lambda(r)}. \tag{1.20}$$

The main step in the proof of Theorem 1.5 is the following local clustering lemma due to DiBenedetto,Gianazza and Vespri [18] (see also [19, 35, 50]).

Lemma 1.1. *Let $K_r(y)$ be a cube in $\mathbb{R}^n$ of edge r centered at y and let $u \in W^{1,1}(K_r(y))$ satisfies*

$$||(u-k)_-||_{W^{1,1}(K_r(y))} \leqslant \mathcal{K}\, k\, r^{n-1}, \quad \textit{and} \quad |\{K_r(y) : u \geqslant k\}| \geqslant \alpha|K_r(y)|, \tag{1.21}$$

with some $\alpha \in (0,1)$, $k \in \mathbb{R}^1$ and $\mathcal{K} > 0$.Then for any $\xi \in (0,1)$ and any $\nu \in (0,1)$ there exists $\bar{x} \in K_r(y)$ and $\varepsilon = \varepsilon(n) \in (0,1)$ such that

$$|\{K_{\bar{r}}(\bar{x}) : u \geqslant \xi\, k\}| \geqslant (1-\nu)|K_{\bar{r}}(y)|, \;\; \bar{r} := \varepsilon\alpha^2 \frac{(1-\xi)\nu}{\mathcal{K}}\, r. \tag{1.22}$$

As it was already mentioned during the iteration from $B_r(\bar{x})$ to $B_\rho(x_0)$ an additional factor arises, that even in the case $\Lambda(x_0,\rho) \leqslant const$ cannot be estimated. To overcome it and to prove Theorem 1.4 we use a workaround that goes back to Mazya [39] and Landis [33, 34]. For the proof of the following expansion of positivity theorem we use the potential-type auxiliary solutions. We also note that by the presence of the function $\lambda(r)$ in condition $(\mathit{\Phi}^{\lambda}_{\Lambda,x_0})$ we cannot use Moser's method, adapting the ideas of Trudinger [51] (see, e.g. [4, 7, 41, 49]).

Theorem 1.6. *Let u be a non-negative weak super-solution to Eq.* (1.14) *in Ω, let conditions $(\mathit{\Phi}_0)$, $(\mathit{\Phi})$ and $(\mathit{\Phi}^{\lambda}_{\Lambda,x_0})$ be fulfilled, assume also that condition* (1.16) *holds. Then there exist positive constants γ, β_3 and β_4 depending only on the data, such that for any $0 < N < M$ and any $B_{8\rho}(x_0) \subset \Omega$ there holds*

$$N\,\lambda(\rho)\left(\frac{|E(\rho,N)|}{\rho^n}\right)^{\beta_3} \leqslant \gamma\,\exp\big(\gamma\,[\Lambda_\lambda(x_0,\frac{\rho}{4})]^{\beta_4}\big)\big\{\min_{B_{\frac{\rho}{2}}(x_0)} u + \rho\big\}, \tag{1.23}$$

where $E(\rho,N) := B_\rho(x_0) \cap \{u(x) > N\}$.

To prove Theorem 1.6 we consider the solution w of the following problem

$$div\left(\mathit{\Phi}(x,|\nabla w|)\frac{\nabla w}{|\nabla w|^2}\right) = 0, \quad x \in D := B_{8\rho}(x_0) \setminus E, \quad w - m\psi \in W_0^{1,\mathit{\Phi}}(D), \tag{1.24}$$

where $E \subset B_\rho(x_0)$, $m \in (\rho, \lambda(\rho)M)$ is some fixed positive number and $\psi \in W_0^{1,\mathit{\Phi}}(B_{8\rho}(x_0))$, $\psi = 1$ on E.

In Section 4 we prove upper and lower bounds for solutions of problem (1.24), from which Theorem 1.6 is obtained as a simple corollary. Thanks to the use of auxiliary solutions of problem (1.24), it is possible to avoid the appearance of an additional factor during the iteration from $B_r(x_0)$ to $B_\rho(x_0)$.

The rest of the paper contains the proof of the above theorems. In Section 2 we collect some auxiliary propositions and required integral estimates of functions belonging to the corresponding De Giorgi classes. Section 3 contains the proof of continuity, Theorem 1.1, expansion of positivity, Theorem 1.5 and the proof of Harnack type inequalities, Theorems 1.2 and 1.3. Upper and lower bounds of auxiliary solutions are proved in Section 4. A variant of the expansion of the positivity theorem, Theorem 1.6 is also proved in Section 4. Finally, in Section 4 we sketch a proof of Harnack's inequality, Theorem 1.4, leaving the details to the reader.

2 Auxiliary material and integral estimates

2.1 Auxiliary Lemma

The following lemma will be used in the sequel, it is the well-known De Giorgi-Poincare lemma (see [32], Chapter 2).

Lemma 2.1. *Let $u \in W^{1,1}(B_r(y))$ for some $r > 0$, and $y \in \mathbb{R}^n$. Let k, l be real numbers such that $k < l$. Then there exists a constant γ depending only on n such that*

$$(l-k)|A^-_{k,r}||B_r(y) \setminus A^-_{l,r}| \leqslant \gamma r^{n+1} \int\limits_{A^-_{l,r}\setminus A^-_{k,r}} |\nabla u|\,dx,$$

where $A^-_{k,r} = B_r(y) \cap \{u < k\}$.

2.2 Local energy estimates

For $\theta \in (0,p]$ and $v > 0$ set $\varphi_\theta(x,v) := \dfrac{\Phi(x,v)}{v^\theta}$. The following lemma is a consequence of the definition of the De Giorgi class $DG_\Phi(B_R(x_0))$ and of the following analogue of the Young inequality

$$\varphi_\theta(x,a)\, b^\theta \leqslant \varepsilon^{-\theta}\, \Phi(x,a) + b^\theta\, \varphi_\theta(x,\varepsilon b), \quad \varepsilon, a, b > 0, \quad \theta \in (0,p], \tag{2.1}$$

indeed, if $b \leqslant \varepsilon^{-1} a$, then $\varphi_\theta(x,a)\, b^\theta \leqslant \varepsilon^{-\theta}\, a^\theta\, \varphi_\theta(x,a) = \varepsilon^{-\theta}\Phi(x,a)$, and if $b \geqslant \varepsilon^{-1}a$, then since by condition (Φ) $\varphi_\theta(x,\cdot)$ is non-decreasing $\varphi_\theta(x,a)\, b^\theta \leqslant b^\theta\, \varphi_\theta(x,\varepsilon b)$.

Lemma 2.2. *Let* $u \in DG_\Phi(B_R(x_0))$*, then for any* $r < R$ *, any* $k \in \mathbb{R}$ *, any* $\sigma \in (0,1)$ *and any* $\theta \in [1, p\frac{t}{t+1}]$ *next inequalities hold*

$$\int\limits_{A^\pm_{k,r}} |\nabla u|^\theta\, \zeta^q(x)\, dx \leqslant \frac{\gamma}{\sigma^{\theta\frac{q}{p}}} \left(\frac{M_\pm(k,r)}{r}\right)^\theta r^n \left[\Lambda_\Phi\big(x_0, r, \frac{M_\pm(k,r)}{r}\big)\right]^{\frac{\theta}{p}} \left(\frac{|A^\pm_{k,r}|}{|B_r(x_0)|}\right)^{1-\frac{\theta}{tp}-\frac{\theta}{sp}}. \tag{2.2}$$

Here $M_\pm(k,r) := \operatorname*{ess\,sup}_{B_r(x_0)} (u-k)_\pm$, $\zeta(x)$ *is the same as in* (1.8) *and* $\Lambda_\Phi\big(x_0, r, \frac{M_\pm(k,r)}{r}\big)$ *was defined in* ($\Phi^\lambda_{\Lambda,x_0}$).

Proof. We use the Hölder inequality and inequality (2.1) for the function $\varphi_p(x,\cdot)$ with $a = \dfrac{M_\pm(k,r)}{r}$, $b = |\nabla u|^p$ and $\varepsilon = 1$

$$\int\limits_{A^\pm_{k,r}} |\nabla u|^{p\frac{t}{t+1}}\, \zeta^q(x)\, dx \leqslant$$

$$\leqslant \left(\int\limits_{A^\pm_{k,r}} |\nabla u|^p\, \varphi_p\Big(x, \frac{M_\pm(k,r)}{r}\Big)\, \zeta^q(x)\, dx\right)^{\frac{t}{t+1}} \left(\int\limits_{A^\pm_{k,r}} \Big[\varphi_p\Big(x, \frac{M_\pm(k,r)}{r}\Big)\Big]^{-t} dx\right)^{\frac{1}{t+1}} \leqslant$$

$$\leqslant \gamma\Big(\frac{M_\pm(k,r)}{r}\Big)^{p\frac{t}{t+1}} \left(\int\limits_{A^\pm_{k,r}} \Phi(x, |\nabla u|)\, \zeta^q(x)\, dx + \int\limits_{A^\pm_{k,r}} \Phi\big(x, \frac{M_\pm(k,r)}{r}\big)\, \zeta^q(x)\, dx\right)^{\frac{t}{t+1}} \times$$

$$\times \left(\int\limits_{B_r(x_0)} \big[\Phi\big(x, \frac{M_\pm(k,r)}{r}\big)\big]^{-t} dx\right)^{\frac{1}{t+1}} \leqslant \frac{\gamma}{\sigma^{q\frac{t}{t+1}}} \Big(\frac{M_\pm(k,r)}{r}\Big)^{p\frac{t}{t+1}} \times$$

$$\times \left(\int\limits_{B_r(x_0)} \big[\Phi\big(x, \frac{M_\pm(k,r)}{r}\big)\big]^{-t} dx\right)^{\frac{1}{t+1}} \left(\int\limits_{B_r(x_0)} \big[\Phi\big(x, \frac{M_\pm(k,r)}{r}\big)\big]^{s} dx\right)^{\frac{t}{s(t+1)}} |A^\pm_{k,r}|^{\frac{s-1}{s}\frac{t}{t+1}},$$

from which by the Hölder inequality the required (2.2) follows. This proves Lemma 2.2. ◻

In what follows, we will use only inequalities (2.2), which can be taken as the definition of the corresponding De Giorgi $DG_\Phi(\Omega)$ classes.

2.3 A Variant of Expansion of the Positivity Lemma

The following lemma will be used in the sequel. In the proof we closely follow to [32, Chap. 2]. Let $M(r) \geqslant \sup\limits_{B_r(x_0)} u$, $m(r) \leqslant \inf\limits_{B_r(x_0)} u$, $\omega(r) := M(r) - m(r)$ and set $v_+(x) := M(r) - u(x)$, $v_-(x) := u(x) - m(r)$.

Lemma 2.3. *Let $u \in DG_\Phi(B_R(x_0))$ and let conditions (Φ_0), (Φ), $(\Phi^\lambda_{\Lambda,x_0})$ be fulfilled. Let $\xi \in (0,1)$ and assume that with some $\alpha_0 \in (0,1)$ there holds*

$$\Big|\Big\{x \in B_{3r/4}(x_0) : v_\pm(x) \leqslant \xi\, \omega(r)\Big\}\Big| \leqslant (1-\alpha_0)\, |B_{3r/4}(x_0)|. \tag{2.3}$$

Then for any $\nu \in (0,1)$ there exists number $C_* \geqslant 1$ depending only on the known data, α_0, ξ and ν such that either

$$\omega(r) \leqslant \frac{r}{\lambda(r)} \exp\big(C_* [\Lambda_\lambda(x_0,r)]^{\bar\beta_1}\big), \tag{2.4}$$

or

$$|\{B_{3/4r}(x_0) : v_\pm(x) \leqslant \omega(r)\, \lambda(r)\, \exp\big(-C_*[\Lambda_\lambda(x_0,r)]^{\bar\beta_1}\big)\}| \leqslant \nu |B_{3/4r}(x_0)|, \tag{2.5}$$

here $\bar\beta_1$ is some fixed positive number depending only on the data.

Proof. We provide the proof of (2.5) for v_+, while the proof for v_- is completely similar. We set $k_j := M(r) - \dfrac{\lambda(r)}{2^j}\omega(r)$, $j = [\log 1/\xi] + 1, 2, \dots, j_*$, where j_* to be chosen. We will assume for all $j \in [[\log 1/\xi] + 1, j_*]$ that $M_+(k_j, 3/4r) \geqslant \dfrac{\lambda(r)}{2^{j+1}}\, \omega(r)$, because if for some j this inequality is violated then the required (2.5) with $C_* \geqslant j_* + 1$ is evident. If (2.4) is violated, then $M_+(k_j, r) \geqslant M_+(k_j, 3/4r) \geqslant r$ and since $2^{-j-1}\, \omega(r)\, \lambda(r) \leqslant M_+(k_j, r) \leqslant 2^{-j}\, \omega(r)\, \lambda(r) \leqslant$ $\leqslant 2M(R)\lambda(r)$, by $(\Phi^\lambda_{\Lambda,x_0})$ we obtain that

$$\Lambda_\Phi\big(x_0, r, \frac{M_+(k_j,r)}{r}\big) \leqslant 2^q \Lambda_\Phi\big(x_0, r, \frac{\lambda(r)}{r 2^j}\omega(r)\big) \leqslant \gamma \Lambda_\lambda(x_0, r).$$

Therefore , if (2.4) is violated, inequality (2.2) with $\theta = p\frac{t}{t+1}$ can be rewritten as

$$\int\limits_{A^+_{k_j,r}} |\nabla u|^\theta\, \zeta^{\,q}\, dx \leqslant \gamma \left(\frac{\lambda(r)}{2^j r}\omega(r)\right)^\theta r^n\, [\Lambda_\lambda(x_0,r)]^{\frac{\theta}{p}} \left(\frac{|A^+_{k_j,r}|}{|B_r(x_0)|}\right)^{\frac{1}{\kappa_1}}$$

where $\dfrac{1}{\kappa_1} = 1 - \dfrac{\theta}{sp} - \dfrac{\theta}{tp}$, $\theta = p\frac{t}{t+1}$ and $\zeta \in C_0^\infty(B_r(x_0))$, $0 \leqslant \zeta \leqslant 1$, $\zeta = 1$ in $B_{3r/4}(x_0)$, $|\nabla\zeta| \leqslant 4/r$. From this by Lemma 2.1 we obtain

$$\begin{aligned}
\frac{\lambda(r)}{2^{j+1}}\omega(r)|A^+_{k_j,3/4r}| &\leqslant \frac{\gamma}{\alpha_0}\, r \int\limits_{A^+_{k_j,r}\setminus A^+_{k_{j+1},r}} |\nabla u|\, \zeta^{\,q}\, dx \leqslant \\
&\leqslant \frac{\gamma}{\alpha_0}\, r \Big(\int\limits_{A^+_{k_j,r}} |\nabla u|^\theta\, \zeta^{\,q}\, dx\Big)^{\frac{1}{\theta}} |A^+_{k_j,r} \setminus A^+_{k_{j+1},r}|^{1-\frac{1}{\theta}} \leqslant \\
&\leqslant \frac{\gamma}{\alpha_0}\, \frac{\lambda(r)}{2^j}\omega(r)\, [\Lambda_\lambda(x_0,r)]^{\frac{1}{p}} \left(\frac{|A^+_{k_j,r}\setminus A^+_{k_{j+1},r}|}{|B_r(x_0)|}\right)^{1-\frac{1}{\theta}} \left(\frac{|A^+_{k_j,r}|}{|B_r(x_0)|}\right)^{\frac{1}{\theta\kappa_1}} |B_r(x_0)|,
\end{aligned}$$

raising the left and right hand-sides to the power $\dfrac{\theta}{\theta-1}$ and summing up the resulting inequalities in j, $j=[\log 1/\xi]+1,2,\dots,j_*$,we conclude that

$$(j_*-[\log 1/\xi]-1)\bigg(\frac{|A^+_{k_{j_*},3/4r}|}{B_{3/4r}(x_0)|}\bigg)^{\frac{\theta}{\theta-1}} \leqslant \gamma \alpha_0^{-\frac{\theta}{\theta-1}}\,[\Lambda_\lambda\big(x_0,r\big)]^{\frac{\theta}{p(\theta-1)}}.$$

Choosing j_* by the condition

$$j_*-[\log 1/\xi]-1 \geqslant \gamma\,\nu^{-\frac{\theta}{\theta-1}}\,\alpha_0^{-\frac{\theta}{\theta-1}}[\Lambda_\lambda\big(x_0,r\big)]^{\frac{\theta}{p(\theta-1)}},\quad \theta=p\frac{t}{t+1},$$

we obtain inequality (2.5), which proves Lemma 2.3 with

$$C_*\geqslant 1+\log 1/\xi+\gamma\,\alpha_0^{-\frac{pt}{(p-1)t-1}}\,\nu^{-\frac{pt}{(p-1)t-1}}\quad\text{and}\quad \bar\beta_1=\frac{t}{(p-1)t-1}.$$

□

2.4 De Giorgi Type Lemma

The following theorem is the De Giorgi type Lemma, the proof is almost standard (see e.g. [32]).

Lemma 2.4. *Let $u\in DG_\Phi(B_R(x_0))$ and let conditions $(\Phi_0),(\Phi)$, $(\Phi^\lambda_{\Lambda,x_0})$ be fulfilled. Fix $\xi,\eta\in(0,1)$, there exists number $\nu_1\in(0,1)$ depending only on the data and η, such that if*

$$|\{x\in B_r(x_0): v_\pm(x)\leqslant \xi\,\omega(r)\}|\leqslant \nu_1\,[\Lambda_\lambda(x_0,r)]^{-\bar\beta_2}|B_r(x_0)|, \tag{2.6}$$

then either

$$\eta\,(1-\eta)\,\xi\,\omega(r)\leqslant \frac{r}{\lambda(r)}, \tag{2.7}$$

or

$$v_\pm(x)\geqslant (1-\eta)^2\,\xi\,\lambda(r)\,\omega(r),\quad x\in B_{\frac{r}{2}}(x_0). \tag{2.8}$$

Here $\bar\beta_2$ is some fixed positive number, depending only on the data.

Proof. For $j=0,1,2,\dots$ we set $r_j:=\dfrac{r}{2}(1+2^{-j})$, $\bar r_j=\dfrac{r_j+r_{j+1}}{2}$, $k_j:=M(r)-\eta\,(2-\eta)\xi\,\lambda(r)\,\omega(r)-\dfrac{(1-\eta)^2}{2^j}\xi\,\lambda(r)\,\omega(r)$, and let $\zeta_j\in C_0^\infty(B_{\bar r_j}(x_0))$, $0\leqslant\zeta_j\leqslant 1$, $\zeta_j=1$ in $B_{r_{j+1}}(x_0)$, and $|\nabla\zeta_j|\leqslant\gamma\dfrac{2^j}{r}$. We assume that $M_+(k_\infty,r/2)\geqslant \eta(1-\eta)\,\xi\,\lambda(r)\,\omega(r)$, because in the opposite case, the required (2.8) is evident. If (2.7) is violated then $M_+(k_\infty,r/2)\geqslant r$. In addition, since $\eta(1-\eta)\xi\,\lambda(r)\,\omega(r)\leqslant M_+(k_j,r)\leqslant$ $\leqslant \xi\,\lambda(r)\,\omega(r)$ for $j=0,1,2,\dots$, then by $(\Phi^\lambda_{\Lambda,x_0})$ we obtain that

$$\Lambda\big(x_0,r_j,\frac{M_+(k_j,r)}{r_j}\big)\leqslant \eta^{-q}\,(1-\eta)^{-q}\Lambda\big(x_0,r_j,\xi\,\frac{\lambda(r)}{r_j}\,\omega(r)\big)\leqslant \gamma\eta^{-q}\,(1-\eta)^{-q}\Lambda_\lambda(x_0,r).$$

Therefore inequality (2.2) with $\theta=1$ can be rewritten as

$$\int\limits_{A^+_{k_j,r_j}}|\nabla u|\,\zeta_j^q\,dx\leqslant \gamma\,2^{j\gamma}\,\eta^{-\frac{q}{p}}\,(1-\eta)^{-\frac{q}{p}}\,\xi\,\lambda(r)\,\omega(r)r^n[\Lambda_\lambda(x_0,r)]^{\frac1p}\,\bigg(\frac{|A^+_{k_j,r_j}|}{|B_r(x_0)|}\bigg)^{\frac{1}{\kappa_2}},\ \frac{1}{\kappa_2}=1-\frac{1}{sp}-\frac{1}{tp}.$$

From this, by the Sobolev embedding theorem we obtain

$$y_{j+1} = \frac{|A^{+}_{k_{j+1},r_{j+1}}|}{|B_r(x_0)|} \leqslant \gamma\, 2^{j\gamma}\, \eta^{-\frac{q}{p}}(1-\eta)^{-2-\frac{q}{p}}\, [\Lambda_\lambda(x_0,r)]^{\frac{1}{p}} y_j^{1+\kappa},\ j=0,1,2,...,$$

where $\kappa = \dfrac{1}{\kappa_2} - 1 + \dfrac{1}{n} = \dfrac{1}{n} - \dfrac{1}{sp} - \dfrac{1}{tp} > 0$. Choosing ν_1, $\bar\beta_2$ from the condition

$$\nu_1 = \gamma^{-1}\, \eta^{\frac{q}{p\kappa}}\, (1-\eta)^{\frac{2+\frac{q}{p}}{\kappa}}, \quad \bar\beta_2 = \frac{1}{p\kappa},$$

and iterating the previous inequality we arrive at the required (2.8), which completes the proof of the lemma. □

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3 Continuity and Harnack's Type Inequality , Proof of Theorems 1.1–1.3 and 1.5

3.1 Continuity

Let r,ρ be arbitrary such that $0<r<\rho<R$, where R is small enough. We assume that the following two alternative cases are possible:

$$\left|\left\{x\in B_{\frac{3}{4}r}(x_0) : u(x) \geqslant M(r) - \frac{1}{2}\,\omega(r)\right\}\right| \leqslant \frac{1}{2}\,|B_{\frac{3}{4}r}(x_0)|$$

or

$$\left|\left\{x\in B_{\frac{3}{4}r}(x_0) : u(x) \leqslant m(r) + \frac{1}{2}\,\omega(r)\right\}\right| \leqslant \frac{1}{2}\,|B_{\frac{3}{4}r}(x_0)|.$$

Assume, for example, the first one. Then Lemmas 2.3, 2.4, the choice of constant C_* in Lemma 2.3 and the choice of ν in Lemma 2.4 ensure the existence of $\bar\beta_3 = \bar\beta_1 + \bar\beta_2 \dfrac{pt}{(p-1)t-1}$ such that

$$\omega\Big(\frac{r}{2}\Big) \leqslant \left(1 - \frac{\lambda(r)}{4}\exp\big(-\gamma[\Lambda_\lambda(x_0,r)]^{\bar\beta_3}\big)\right)\omega(r) + \gamma\,\frac{r}{\lambda(r)}\,\exp\big(\gamma[\Lambda_\lambda(x_0,r)]^{\bar\beta_3}\big),$$

where $\bar\beta_1$, $\bar\beta_2$ were defined in Lemmas 2.3, 2.4 .

Iterating this inequality, we obtain

$$\omega(r) \leqslant 2\,M(R)\,\exp\left(-\gamma\int_r^{2\rho}\lambda(s)\,\exp\big(-\gamma[\Lambda_\lambda(x_0,s)]^{\bar\beta_3}\big)\,\frac{ds}{s}\right) + \int_{\frac{r}{2}}^{\rho}\exp\big(\gamma[\Lambda_\lambda(x_0,s)]^{\bar\beta_3}\big)\,\frac{ds}{\lambda(s)},$$

from which the required continuity follows. This completes the proof of Theorem 1.1.

3.2 Proof of Theorem 1.5

Let $B_r(y)\subset B_\rho(x_0)\subset B_{8\rho}(x_0)\subset\Omega$ and let the following inequality holds

$$|\{\,B_r(y) : u \geqslant N\,\}| \geqslant \alpha|B_r(y)|, \tag{3.1}$$

for some $0 < N < M$, $\alpha \in (0,1)$.

Let $\bar r < r$ be some number which we will fix later, we will assume that $\sup\limits_{B_r(y)} (u - N\lambda(\bar r))_- \geqslant$ $\geqslant \frac{N}{2}\lambda(\bar r)$, because in the opposite case the required inequality (1.20) is obvious. If $N \geqslant r$, then by $(\Phi^{\lambda}_{\Lambda})$, using the fact that $\lambda(\bar r) \leqslant \lambda(r)$ if $\bar r \leqslant r$, we have with any $0 < \bar r \leqslant r$

$$\Lambda_{\Phi}\big(y, r, \frac{M_-(N\lambda(\bar r), r)}{r}\big) \leqslant 2^q \Lambda_{\Phi}\big(y, r, \frac{\lambda(\bar r)}{r} N\big) \leqslant 2^q \Lambda_{\lambda}(y, r) \leqslant 2^q \Lambda_{\lambda}(r),$$

apply inequality (2.2) with $\theta = 1$ for $(u - \lambda(\bar r)\, N)_-$ over the pair of balls $B_r(y)$ and $B_{2r}(y)$ and with arbitrary $0 < \bar r \leqslant r$ to obtain

$$\int\limits_{B_r(y)} |\nabla (u - \lambda(\bar r)\, N)_-|\, dx \leqslant \gamma\, [\Lambda_{\lambda}(r)]^{\frac{1}{p}}\, \lambda(\bar r)\, N\, r^{n-1}.$$

The local clustering Lemma 1.1 with $k = \lambda(\bar r)\, N$, $\nu = \frac{1}{4}$, $\xi = \frac{1}{2}$, $\mathcal{K} = \gamma[\Lambda_{\lambda}(r)]^{\frac{1}{p}}$ implies the existence of a point $\bar x \in B_r(y)$ and $\varepsilon \in (0,1)$ depending only on the data such that

$$|\{B_{\bar r}(\bar x) : u > \frac{\lambda(\bar r)}{2} N\}| \geqslant \frac{1}{4} |B_{\bar r}(\bar x)|, \quad \bar r = \frac{\varepsilon \alpha^2}{8\gamma}\, \frac{r}{[\Lambda_{\lambda}(r)]^{\frac{1}{p}}}.$$

Set $\varepsilon_0 := \dfrac{\varepsilon}{8\gamma}$, then the previous inequality can be rewritten as

$$|\{B_{\bar r}(\bar x) : u > \frac{\lambda(\bar r)}{2} N\}| \geqslant \frac{1}{4} |B_{\bar r}(\bar x)|. \tag{3.2}$$

From this by Lemmas 2.3 and 2.4 we obtain

$$u(x) \geqslant \frac{\lambda(\bar r)}{2}\, N \exp\big(-\gamma[\Lambda_{\lambda}(2\bar r))]^{\bar\beta_3}\big), \quad x \in B_{2\bar r}(\bar x), \quad \bar\beta_3 = \bar\beta_1 + \bar\beta_2 \frac{pt}{(p-1)t - 1},$$

provided that

$$N \geqslant \frac{2\bar r}{\lambda(\bar r)} \exp\big(\gamma[\Lambda_{\lambda}(2\bar r)]^{\bar\beta_3}\big),$$

where $\bar\beta_1$, $\bar\beta_2$ are the constants defined in Lemmas 2.3, 2.4.

Repeating this procedure j-times we obtain

$$u(x) \geqslant 2^{-j}\, \lambda(\bar r)\, N\, \exp\big(-\gamma \int\limits_{2\bar r}^{2^{j+1}\bar r} [\Lambda_{\lambda}(s)]^{\bar\beta_3} \frac{ds}{s}\big),\ x \in B_{2^j \bar r}(\bar x),$$

provided that

$$N \geqslant \frac{2^j \bar r}{\lambda(\bar r)} \exp\big(\gamma \int\limits_{2\bar r}^{2^{j+1}\bar r} [\Lambda_{\lambda}(s)]^{\bar\beta_3} \frac{ds}{s}\big).$$

Choosing j from the condition $2^j \bar r = \rho$ and using condition (λ), from the previous we obtain

$$u(x) \geqslant \gamma^{-1} \frac{\lambda(\bar r)}{\Lambda_{\lambda}(r)}\, N\, \alpha^2 \frac{r}{\rho} \exp\big(-\gamma \int\limits_{2\bar r}^{2\rho} [\Lambda_{\lambda}(s)]^{\bar\beta_3} \frac{ds}{s}\big) \geqslant$$

$$\geqslant \gamma^{-1} \frac{\lambda(\rho)}{\Lambda_{\lambda}(\rho)}\, N\, \alpha^{\tau_1} \Big(\frac{r}{\rho}\Big)^{\tau_2} \exp\big(-\gamma \int\limits_{2\bar r}^{2\rho} [\Lambda_{\lambda}(s)]^{\bar\beta_3} \frac{ds}{s}\big) \geqslant$$

$$\geqslant \gamma^{-1}\, \lambda(\rho)\, N\, \alpha^{\tau_1} \Big(\frac{r}{\rho}\Big)^{\tau_2} \exp\big(-\gamma \int\limits_{2\bar r}^{2\rho} [\Lambda_{\lambda}(s)]^{\bar\beta_3} \frac{ds}{s}\big),\ x \in B_{\frac{\rho}{2}}(x_0),$$

provided that

$$N \geqslant \gamma \frac{\rho}{\lambda(\rho)} \alpha^{-\tau_1} \Big(\frac{\rho}{r}\Big)^{\tau_2} \exp\big(\gamma \int\limits_{2\bar{r}}^{2\rho} [\Lambda_\lambda(s)]^{\bar{\beta}_3} \frac{ds}{s}\big),$$

which completes the proof of the theorem.

3.3 Proof of Theorem 1.2

Inequality (1.11) of Theorem 1.2 follows immediately from Theorem 1.5 with $\Lambda_\lambda(\rho) \leqslant \gamma$, indeed set

$$\bar{m}(\rho) = \frac{1}{\lambda(\rho)} \{ \min_{B_{\frac{\rho}{2}}(x_0)} u(x) + \rho \}.$$

By Theorem 1.5 with $r = \rho$ we obtain for $\theta \in \big(0, \frac{1}{\tau_1}\big)$

$$\rho^{-n} \int\limits_{B_\rho(x_0)} u^\theta \, dx = \theta \, \rho^{-n} \int\limits_0^\infty |\{B_\rho(x_0) : u(x) > N\}| \, N^{\theta-1} \, dN \leqslant [\bar{m}(\rho)]^\theta +$$

$$+ \gamma [\bar{m}(\rho)]^{\frac{1}{\tau_1}} \int\limits_{\bar{m}(\rho)}^\infty N^{\theta - \frac{1}{\tau_1} - 1} \, dN \leqslant \frac{\gamma \tau_1}{1 - \theta \tau_1} [\bar{m}(\rho)]^\theta, \quad (3.3)$$

which proves inequality (1.11).

To prove (1.12) fix $\sigma \in (0, \frac{1}{8})$, $s \in (\frac{3}{4}\rho, \frac{7}{8}\rho)$ and let $M_0 := \sup\limits_{B_s(x_0)} u$, $M_\sigma := \sup\limits_{B_{s(1-\sigma)}(x_0)} u$. Fix $\bar{x} \in B_{s(1-\sigma)}(x_0)$ and for $j = 0, 1, 2, ...$ set $\rho_j := s\frac{\sigma}{2}(1 + 2^{-j})$, $B_j := B_{\rho_j}(\bar{x})$, $k_j = k(1 - 2^{-j})$, where $k > 0$ and suppose that $(u(\bar{x}) - k)_+ \geqslant \rho$, then $\sup\limits_{B_j}(u - k_j)_+ \geqslant \rho$. We will use inequality (2.2), by condition (Φ^1_Λ) with $K = M$ we have

$$\Lambda_\Phi\big(\bar{x}, \rho_j, \sup_{B_j}(u - k_j)_+/\rho_j\big) \leqslant \gamma(M) \Lambda_1(\bar{x}, \rho_j) \leqslant \gamma \Lambda_1(\rho).$$

Hence, inequality (2.2) can be rewritten as

$$\int\limits_{B_{j+1} \cap \{u > k_j\}} |\nabla u| \, dx \leqslant \frac{\gamma}{\sigma^\gamma} 2^{j\gamma} \, M_0 \, \rho^{n-1} [\Lambda_1(\rho)]^{\frac{1}{p}} \Big(\frac{|B_j \cap \{u > k_j\}|}{|B_j|}\Big)^{1 - \frac{1}{tp} - \frac{1}{sp}}.$$

Since $\bar{x}$ is an arbitrary point in $B_{s(1-\sigma)}(x_0)$ this inequality by standard arguments (see e.g. [32]) yields

$$M_\sigma^{1+\frac{1}{\kappa}} \leqslant \gamma \, \sigma^{-\gamma} \, M_0^{\frac{1}{\kappa}} \, [\Lambda_1(\rho)]^{\frac{1}{p\kappa}} \, \rho^{-n} \int\limits_{B_0} u \, dx + \gamma \, \rho^{1+\frac{1}{\kappa}}, \quad \kappa = \frac{1}{n} - \frac{1}{tp} - \frac{1}{sp} > 0,$$

which by the Young inequality implies for any $\varepsilon, \theta \in (0, 1)$ that

$$M_\sigma \leqslant \varepsilon \, M_0 + \gamma \, \sigma^{-\gamma} [\Lambda_1(\rho)]^{\frac{1}{p\kappa\theta}} \Big(\rho^{-n} \int\limits_{B_0} u^\theta \, dx\Big)^{\frac{1}{\theta}} + \gamma \, \rho.$$

Iterating this inequality we arrive at

$$\sup_{B_{\frac{\rho}{2}}(x_0)} u \leqslant \gamma\,[\Lambda_1(\rho)]^{\frac{1}{p\kappa\theta}} \Big(\rho^{-n} \int\limits_{B_0} u^\theta\, dx\Big)^{\frac{1}{\theta}} + \gamma\,\rho. \tag{3.4}$$

Collecting (1.11) and (3.4) with $\theta = \dfrac{1}{2\tau_1}$ we arrive at the required (1.12) with $\beta_2 = (2p\kappa\tau_1)^{-1}$, which completes the proof of Theorem 1.2.

3.4 Proof of Theorem 1.3

Construct the ball $B_{\rho\tau}(x_0)$, $\tau \in (0,1)$ and set $u_0 := u(x_0)$, $M_\tau := \max\limits_{B_{\rho\tau}(x_0)} u$,

$N_\tau := \dfrac{u_0}{2}\,(1-\tau)^{-l_1}\, \exp\Big(l_2\big[\log\log\frac{1}{(1-\tau)\rho}\big]^{Ll_3} - l_2\big[\log\log\frac{1}{\rho}\big]^{Ll_3}\Big)\, \exp\big(c \int\limits_{\psi((1-\tau)\rho)}^{\psi(\rho)} [\log\log\frac{1}{s}]^{L\beta}\frac{ds}{s}\big),$
where c and β the numbers defined in Theorem 1.5, $l_1, l_2, l_3 > 0$ will be chosen depending only on the known data and

$$\psi(r) = \varepsilon_0\, \alpha^2(r)\, r\, [\log\log\frac{1}{r}]^{-L},$$

here $\varepsilon_0 \in (0,1)$ is the number, defined in Theorem 1.5 and $\alpha(r) \in (0,1)$ is continuous non-increasing function, which will be defined later. Consider the equation

$$M_\tau = N_\tau. \tag{3.5}$$

Further we will assume that

$$u_0 \geqslant l_4\,\rho\, \log\frac{1}{\rho}, \tag{3.6}$$

with some $l_4 > 0$ to be fixed later.

Let $\tau_0 \in (0,1)$ be the maximal root of equation (3.5) and fix y by the condition $u(y) = \max\limits_{B_{\rho\tau_0}(x_0)} u$. Since $B_{\frac{\rho}{2}(1-\tau_0)}(y) \subset B_{\frac{\rho}{2}(1+\tau_0)}(x_0)$, setting $r = \frac{\rho}{2}(1-\tau_0)$ we have

$$\max_{B_r(y)} u \leqslant u(y)2^{l_1}\, \exp\Big(l_2\big(\big[\log\log\frac{1}{r}\big]^{Ll_3} - \big[\log\log\frac{1}{2r}\big]^{Ll_3}\big)\Big)\, \exp\big(c\int\limits_{\psi(r)}^{\psi(2r)} [\log\log\frac{1}{s}]^{L\beta}\frac{ds}{s}\big) \leqslant$$
$$\leqslant u(y)2^{l_1}\, \exp\Big(l_2\big(\big[\log\log\frac{1}{r}\big]^{Ll_3} - \big[\log\log\frac{1}{2r}\big]^{Ll_3}\big)\Big)\, \exp\big(c[\log\log\frac{1}{\psi(r)}]^{L\beta}\, \log\frac{\psi(2r)}{\psi(r)}\big).$$

Let us estimate the terms on the right-hand side of the previous inequality. If

$$Ll_3 < 1 \quad \text{and} \quad 2l_2\big[\log\frac{1}{2\rho}\big]^{-1} \leqslant 1, \tag{3.7}$$

then

$$l_2\big(\big[\log\log\frac{1}{r}\big]^{Ll_3} - \big[\log\log\frac{1}{2r}\big]^{Ll_3}\big) \leqslant 1.$$

By our assumption $\alpha(r)$ is non-increasing, therefore choosing ρ sufficiently small we have

$$\frac{\psi(2r)}{\psi(r)} = 2\frac{\alpha^2(2r)}{\alpha^2(r)}\, \frac{[\log\log\frac{1}{r}]^L}{[\log\log\frac{1}{2r}]^L} \leqslant 2\, \frac{[\log\log\frac{1}{r}]^L}{[\log\log\frac{1}{2r}]^L} \leqslant 4.$$

Moreover

$$\log\log\frac{1}{\psi(r)} = \log\log\frac{[\log\log\frac{1}{r}]^{L}}{\varepsilon_0 r\alpha(r)} \leqslant 2\log\log\frac{1}{r},$$

provided that ρ is small enough and

$$2\log\frac{1}{\alpha(r)} \leqslant \log\frac{1}{r}. \tag{3.8}$$

Therefore

$$\max_{B_r(y)} u \leqslant u(y)\, 2^{l_1+1}\, \exp\big(8^{L\beta}\, c\, [\log\log\frac{1}{r}]^{L\beta}\big).$$

Claim. There exists a positive number $\nu_1 \in (0,1)$ depending only on the known data and l_1 such that

$$\Big|\Big\{x\in B_r(y) : u(x) \geqslant \frac{u(y)}{4}\Big\}\Big| \geqslant \nu_1\,[\log\log\frac{1}{r}]^{-\bar\beta_2}\,\exp\Big(-\frac{q}{p\kappa}8^{L\beta}\,c\,[\log\log\frac{1}{r}]^{L\beta}\Big)\,|B_r(y)|,$$

where $\bar\beta_2$ is the number defined in Lemma 2.4 and $\kappa = \frac{1}{n} - \frac{1}{sp} - \frac{1}{tp} > 0$.

Indeed, in the opposite case we apply Lemma 2.4 with the choices

$$M(r) = u(y)\, 2^{l_1+1}\, \exp\big(8^{L\beta}\, c\, [\log\log\tfrac{1}{r}]^{L\beta}\big), \quad \xi\,\omega(r) = (M(r) - \tfrac{1}{4}u(y)),$$

$$1-\eta = \left(\frac{M(r) - \frac{11}{16}u(y)}{M(r)-\frac{1}{4}u(y)}\right)^{\frac{1}{2}} < 1 \text{ and } \eta \geqslant \frac{1}{\gamma(l_1)}\exp\big(-8^{L\beta}\,c\,[\log\log\tfrac{1}{r}]^{L\beta}\big).$$

By Lemma 2.4 it follows that if

$$\eta\,(1-\eta)\xi\,\omega(r) \geqslant r \tag{3.9}$$

then

$$u(y) \leqslant \max_{B_{\frac{r}{2}}(y)} u \leqslant M(r) - (1-\eta)^2\xi\,\omega(r) = \frac{11}{16}\,u(y),$$

reaching a contradiction, which proves the claim.

We note that inequality (3.9) is a consequence of (3.6), provided that ρ is small enough and

$$L\,\beta < 1 \quad \text{and} \quad l_4 \geqslant \gamma(l_1). \tag{3.10}$$

We set

$$\alpha(r) := \nu_1\,[\log\log\frac{1}{r}]^{-\bar\beta_2}\,\exp\Big(-\frac{q}{p\kappa}8^{L\beta}\,c\,[\log\log\frac{1}{r}]^{L\beta}\Big)$$

and apply Theorem 1.5 with $\alpha = \alpha(r)$, $\bar r = \varepsilon_0\alpha^2(r) r[\log\log\frac{1}{r}]^{-L}$ and $N = \frac{u(y)}{4}$, note that by our choices $\bar r = \psi(r)$, so we obtain

$$\min_{B_{\frac{\rho}{2}}(x_0)} u \geqslant \gamma^{-1}\,u(y)\,\alpha^{\tau_1}(r)\Big(\frac{r}{\rho}\Big)^{\tau_2}\,\exp\Big(-c\int\limits_{2\bar r}^{\rho}[\log\log\frac{1}{s}]^{L\,\beta}\frac{ds}{s}\Big) \geqslant$$

$$\geqslant \gamma^{-1}\nu_1^{\tau_1}\,u(y)\Big(\frac{r}{\rho}\Big)^{\tau_2}[\log\log\frac{1}{r}]^{-\bar\beta_2\tau_1}\,\exp\Big(-c\int\limits_{\psi(2r)}^{\rho}[\log\log\frac{1}{s}]^{L\beta}\frac{ds}{s} - c\frac{q}{p\kappa}8^{L\beta}[\log\log\frac{1}{r}]^{L\beta}\Big).$$

If ρ is small enough, then $\log\log\dfrac{1}{r}\leqslant\dfrac{\rho}{r}\log\log\dfrac{1}{\rho}$, hence from the previous we obtain

$$\min_{B_{\frac{\rho}{2}}(x_0)} u\geqslant\gamma^{-1}\,\nu_1^{\tau_1}\,u_0\left(\frac{\rho}{r}\right)^{l_1-1-\tau_2-\bar\beta_2\tau_1}[\log\log\frac{1}{\rho}]^{-\bar\beta_2\tau_1}\exp\big(-l_2[\log\log\frac{1}{\rho}]^{Ll_3}\big)\times$$
$$\times\exp\left(l_2[\log\log\frac{1}{r}]^{Ll_3}-c\frac{q}{p\kappa}8^{L\beta}[\log\log\frac{1}{r}]^{L\beta}\right)\exp\big(-c\int\limits_{\psi(\rho)}^{\rho}[\log\log\frac{1}{s}]^{L\beta}\frac{ds}{s}\big).$$

Fix l_1, l_2 and l_3 by the conditions

$$l_1=1+\tau_2+\bar\beta_2\tau_1,\quad l_2=c\frac{q}{p\kappa}8^{L\beta},\quad l_3=\beta,$$

then the last inequality can be rewritten as

$$\min_{B_{\frac{\rho}{2}}(x_0)} u\geqslant\gamma^{-1}\,\nu_1^{\tau_1}\,u_0\,[\log\log\frac{1}{\rho}]^{-\bar\beta_2\tau_1}\exp\left(-l_2[\log\log\frac{1}{\rho}]^{Ll_3}-c\int\limits_{\psi(\rho)}^{\rho}[\log\log\frac{1}{s}]^{L\beta}\frac{ds}{s}\right).$$

Using the fact that $\log\log\frac{1}{\psi(\rho)}\leqslant\gamma\log\log\frac{1}{\rho}$, $\log\frac{\rho}{\psi(\rho)}\leqslant\gamma[\log\log\frac{1}{\rho}]^{L\beta}$ and $[\log\log\frac{1}{\rho}]^{\bar\beta_2\tau_1}\leqslant$ $\leqslant\exp\big(\gamma[\log\log\frac{1}{\rho}]^{L\beta}\big)$ if ρ is small enough, from the previous we arrive at

$$\min_{B_{\frac{\rho}{2}}(x_0)} u\geqslant\gamma^{-1}\,\nu_1^{\tau_1}\,u_0\,\exp\big(-\gamma[\log\log\frac{1}{\rho}]^{L\beta}\big).$$

Choose l_4 sufficiently large, according to (3.10), choose L sufficiently small, according to (3.7), (3.10), choose ρ small enough, according to the second inequality in (3.7) and note that (3.8) holds by our choice of $\alpha(r)$, we arrive at

$$\min_{B_{\frac{\rho}{2}}(x_0)} u\geqslant\gamma^{-1}\,\nu_1^{\tau_1}\,\frac{u_0}{\log\frac{1}{\rho}},$$

provided that inequality (3.6) holds. This completes the proof of Theorem 1.3.

4 Pointwise Estimates of Auxiliary Solutions, Proof of Theorems 1.4 and 1.6

We will assume that the following integral identity holds:

$$\int\limits_D \varPhi(x,|\nabla w|)\,\frac{\nabla w}{|\nabla w|^2}\nabla\eta\,dx=0\quad\text{for any}\quad\eta\in W_0^{1,\varPhi}(D).\tag{4.1}$$

The existence of the solutions w follows from the general theory of monotone operators. Testing (4.1) by $\eta=(w-m)_+$ and by $\eta=w_-$ we obtain that $0\leqslant w\leqslant m\leqslant\lambda(\rho)M$.

To formulate our next result, we need the notion of the capacity. For this set

$$C_{\varPhi}(E,B_{8\rho}(x_0);m):=\frac{1}{m}\inf_{v\in\mathfrak{M}(E)}\int\limits_{B_{8\rho}(x_0)}\varPhi(x,m|\nabla v|)\,dx,$$

where the infimum is taken over the set $\mathfrak{M}(E)$ of all functions $v \in W_0^{1,\Phi}(B_{8\rho}(x_0))$ with $v \geqslant 1$ on E. If $m = 1$, this definition leads to the standard definition of $C_\Phi(E, B_{8\rho}(x_0))$ capacity (see, e.g. [27]).

Further we will assume that

$$\Lambda^{-1}_{+,x_0,8\rho}\left(\frac{C_\Phi(E, B_{8\rho}(x_0); m)}{\rho^{n-1}\,[\Lambda_\lambda(x_0,\rho)]^{\bar c_1}}\right) \geqslant \bar c, \tag{4.2}$$

where $\Lambda^{-1}_{+,x_0,8\rho}(\cdot)$ is the inverse function to $\Lambda_{+,\varphi}(x_0, 8\rho, \cdot)$, $\varphi(x,v) := \dfrac{\Phi(x,v)}{v}$, $v > 0$ and $\bar c, \bar c_1 > 0$ to be chosen later depending only on the data.

4.1 Upper bound for the function w

We note that in the standard case (i.e. if $p = q$) the upper bound for the function w was proved in [42] (see also [43, Chap. 8, Sec. 3], [44]).

Lemma 4.1. *There exists $\bar\beta > 0$ depending only on the data such that*

$$w(x) \leqslant \gamma\ \ \rho\,\Lambda^{-1}_{+,x_0,8\rho}\bigg(\big[\Lambda_\lambda(x_0,\rho)\big]^{\bar\beta}\,\frac{C_\Phi(E, B_{8\rho}(x_0); m)}{\rho^{n-1}}\bigg), \quad x \in K_{\frac{3}{2}\rho,8\rho} = B_{8\rho}(x_0) \setminus B_{\frac{3}{2}\rho}(x_0).$$

Proof. Fix $\sigma \in (0, \frac{1}{8})$, $s \in ((1+\sigma)\rho, (8-\sigma)\rho)$ and let $M_0 := \sup\limits_{K_{s,8\rho}} w$, $M_\sigma := \sup\limits_{K_{s(1+\sigma),8\rho}} w$. Fix $\bar x \in K_{s(1+\sigma),8\rho}$ and for $j = 0, 1, 2, ...$ set $\rho_j := s\frac{\sigma}{2}(1 + 2^{-j})$, $B_j := B_{\rho_j}(\bar x)$, $k_j = k(1 - 2^{-j})$, where $k > 0$ and suppose that $(w(\bar x) - k)_+ \geqslant \bar c\rho\big[\Lambda_\lambda(x_0,\rho)\big]^{\bar c_1}$, then $\sup\limits_{B_j}(w - k_j)_+ \geqslant \rho$. We will use inequality (2.2), by condition $(\Phi^\lambda_{\Lambda,x_0})$ with $K = M$, using the fact that $\sup\limits_{B_j}(w - k_j)_+ \leqslant$ $\leqslant m \leqslant \lambda(\rho)M$ we have

$$\Lambda_\Phi\big(\bar x, \rho_j, \sup_{B_j}(w - k_j)_+/\rho_j\big) \leqslant \gamma(M)\Lambda_\lambda(\bar x, \rho_j) \leqslant \gamma\Lambda_\lambda(x_0,\rho).$$

Hence, inequality (2.2) with $\theta = p\frac{t}{t+1}$ can be rewritten as

$$\int\limits_{B_{j+1}\cap\{u>k_j\}} |\nabla w|^\theta\,dx \leqslant \frac{\gamma}{\sigma^\gamma}2^{j\gamma}\left(\frac{M_0}{\rho}\right)^\theta \rho^n[\Lambda_\lambda(x_0,\rho)]^{\frac{\theta}{p}}\left(\frac{|B_j\cap\{w > k_j\}|}{|B_j|}\right)^{1-\frac{\theta}{tp}-\frac{\theta}{sp}}.$$

Since $\bar x$ is an arbitrary point in $\bar x \in K_{s(1+\sigma),8\rho}$ this inequality by standard arguments (see e.g. [32]) yields

$$M_\sigma^{\theta+\frac{1}{\kappa}} \leqslant \gamma\,\sigma^{-\gamma}\,M_0^{\frac{1}{\kappa}}\,[\Lambda_\lambda(x_0,\rho)]^{\frac{1}{p\kappa}}\,\rho^{-n}\int\limits_{K_{s,8\rho}} w^\theta\,dx + \gamma\,\sigma^{-\gamma}\rho^{\theta+\frac{1}{\kappa}}\big[\Lambda_\lambda(x_0,\rho)\big]^{(\theta+\frac{1}{\kappa})\bar c_1},$$

where $\kappa = \frac{1}{n} - \frac{1}{tp} - \frac{1}{sp} > 0$. Using the Young inequality we obtain for any $\varepsilon \in (0,1)$

$$M_\sigma \leqslant \varepsilon M_0 + \gamma\,\sigma^{-\gamma}\,\varepsilon^{-\gamma}[\Lambda_\lambda(x_0,\rho)]^{\frac{1}{p\theta\kappa}}\left(\rho^{-n}\int\limits_{K_{s,8\rho}} w^\theta\,dx\right)^{\frac{1}{\theta}} + \gamma\,\rho\big[\Lambda_\lambda(x_0,\rho)\big]^{\bar c_1}, \quad \theta = p\frac{t}{t+1}. \tag{4.3}$$

Let us estimate the second term on the right-hand side of (4.3). For this we assume that $M_\sigma \geqslant \bar{c}\sigma^{-\gamma}\rho\big[\Lambda_\lambda(x_0,\rho)\big]^{\bar{c}_1} \geqslant \rho$, because otherwise, by (4.2) the upper estimate is evident. Set $w_{M_0} := \min\{w, M_0\}$, by the Poincare and Hölder inequalities we have with arbitrary $\varepsilon_1 \in (0,1)$

$$\begin{aligned}
\Big(\rho^{-n}\int\limits_{K_{s,8\rho}} w^\theta\,dx\Big)^{\frac{1}{\theta}} &= \Big(\rho^{-n}\int\limits_{K_{s,8\rho}} w_{M_0}^\theta\,dx\Big)^{\frac{1}{\theta}} \leqslant \gamma\rho\Big(\rho^{-n}\int\limits_{D} |\nabla w_{M_0}|^\theta\,dx\Big)^{\frac{1}{\theta}} \leqslant \\
&\leqslant \gamma\rho\Big(\rho^{-n}\int\limits_{D} |\nabla w_{M_0}|^p\varphi_p(x,\varepsilon_1\frac{M_0}{\rho})\,dx\Big)^{\frac{1}{p}}\Big(\rho^{-n}\int\limits_{B_{8\rho}(x_0)} \big[\varphi_p(x,\varepsilon_1\frac{M_0}{\rho})\big]^{-t}\,dx\Big)^{\frac{1}{pt}} = \\
&= \gamma\varepsilon_1 M_0\Big(\rho^{-n}\int\limits_{D} |\nabla w_{M_0}|^p\varphi_p(x,\varepsilon_1\frac{M_0}{\rho})\,dx\Big)^{\frac{1}{p}}\Big(\rho^{-n}\int\limits_{B_{8\rho}(x_0)} \big[\Phi(x,\varepsilon_1\frac{M_0}{\rho})\big]^{-t}\,dx\Big)^{\frac{1}{pt}} \leqslant \\
&\leqslant \gamma\varepsilon_1 M_0\Big(\rho^{-n}\int\limits_{D} \Phi(x,|\nabla w_{M_0}|)dx + \rho^{-n}\int\limits_{B_{8\rho}(x_0)} \Phi(x,\varepsilon_1\frac{M_0}{\rho})\,dx\Big)^{\frac{1}{p}}\times \\
&\times\Big(\rho^{-n}\int\limits_{B_{8\rho}(x_0)} \big[\Phi(x,\varepsilon_1\frac{M_0}{\rho})\big]^{-t}\,dx\Big)^{\frac{1}{pt}}, \qquad (4.4)
\end{aligned}$$

above we also used inequality (2.1) with $\varepsilon = 1$. Fix ε_1 by the condition

$$\varepsilon_1 = \gamma^{-1}\varepsilon^{1+\gamma}\sigma^\gamma\big[\Lambda_\lambda(x_0,\rho)\big]^{-\frac{1}{p}(1+\frac{1}{\theta\kappa})}.$$

The second term on the right-hand side of (4.4) we estimate using condition $(\Phi^\lambda_{\Lambda,x_0})$. By our choice $\varepsilon_1 M_0 \geqslant \varepsilon_1\bar{c}\sigma^{-\gamma}\big[\Lambda(x_0,\rho)\big]^{\bar{c}_1}\rho \geqslant \rho$, provided that $\bar{c} \geqslant \varepsilon^{-1-\gamma}\,\gamma$ and $\bar{c}_1 \geqslant \frac{1}{p}(1+\frac{1}{\theta\kappa})$, moreover $\varepsilon_1 M_0 \leqslant m \leqslant \lambda(\rho)M$, therefore by condition $(\Phi^\lambda_{\Lambda,x_0})$

$$\begin{aligned}
&\Big(\rho^{-n}\int\limits_{B_{8\rho}(x_0)} \big[\Phi(x,\varepsilon_1\frac{M_0}{\rho})\big]^s\,dx\Big)^{\frac{1}{ps}}\Big(\rho^{-n}\int\limits_{B_{8\rho}(x_0)} \big[\Phi(x,\varepsilon_1\frac{M_0}{\rho})\big]^{-t}\,dx\Big)^{\frac{1}{pt}} \leqslant \\
&\leqslant \gamma\big[\Lambda_{+,\Phi}(x_0,8\rho,\varepsilon_1\frac{M_0}{\rho})\big]^{\frac{1}{p}}\,\big[\Lambda_{-,\Phi}(x_0,8\rho,\varepsilon_1\frac{M_0}{\rho})\big]^{\frac{1}{p}} \leqslant \gamma\big[\Lambda_\lambda(x_0,\rho)\big]^{\frac{1}{p}}. \qquad (4.5)
\end{aligned}$$

Combining estimates (4.3)–(4.5) and using condition (Φ) we arrive at

$$\begin{aligned}
M_\sigma &\leqslant 2\varepsilon M_0 + \gamma M_0\frac{\big[\Lambda_\lambda(x_0,\rho)\big]^{\frac{1}{p}}}{\big[\Lambda_{+,\Phi}(x_0,8\rho,\varepsilon_1\frac{M_0}{\rho})\big]^{\frac{1}{p}}}\Big(\rho^{-n}\int\limits_{D}\Phi(x,|\nabla w_{M_0}|\,dx\Big)^{\frac{1}{p}} + \gamma\rho\big[\Lambda_\lambda(x_0,\rho)\big]^{\bar{c}_1} \leqslant \\
&\leqslant 2\varepsilon M_0 + \gamma\varepsilon^{-\gamma}\sigma^{-\gamma}M_0\frac{\big[\Lambda_\lambda(x_0,\rho)\big]^{\bar{\beta}_4}}{\big[\Lambda_{+,\Phi}(x_0,8\rho,\frac{M_0}{\rho})\big]^{\frac{1}{p}}}\Big(\rho^{-n}\int\limits_{D}\Phi(x,|\nabla w_{M_0}|\,dx\Big)^{\frac{1}{p}} + \gamma\rho\big[\Lambda_\lambda(x_0,\rho)\big]^{\bar{c}_1},
\end{aligned} \qquad (4.6)$$

where $\bar{\beta}_4 = \frac{1}{p}(1+(q-1)\frac{1}{\theta\kappa})$.

We need to estimate the integral on the right-hand side of (4.6). Let $\psi \in W_0^{1,\Phi}(B_{8\rho}(x_0))$, $\psi = 1$ on E, be such that $\frac{1}{m}\int\limits_{B_{8\rho}(x_0)}\Phi(x,m|\nabla\psi|)\,dx \leqslant \gamma C_\Phi(E,B_{8\rho}(x_0);m) + \gamma\rho^n$, test identinty

(4.1) by $\eta = w - m\psi$, using inequality (2.1) with $\theta = 1$, $a = |\nabla w|$, $b = |\nabla\psi|$ and sufficiently small ε we obtaint

$$\int\limits_D \Phi(x,|\nabla w|)\,dx \leqslant \gamma \int\limits_{B_{8\rho}(x_0)} \Phi(x,m|\nabla\psi|)\,dx \leqslant \gamma m C_\Phi(E,B_{8\rho}(x_0);m) + \gamma m\rho^n.$$

Now testing identity (4.1) by $\eta = w_{M_0} - \dfrac{M_0}{m}w$, using (2.1) and the previous inequality we obtain

$$\int\limits_D \Phi(x,|\nabla w_{M_0}|)\,dx \leqslant \gamma\frac{M_0}{m}\int\limits_D \Phi(x,|\nabla w|)\,dx \leqslant \gamma M_0 C_\Phi(E,B_{8\rho}(x_0);m) + \gamma M_0\rho^n. \tag{4.7}$$

Combining estimates (4.6) and (4.7), using the fact that $\Lambda_{+,\Phi}(x_0,8\rho,\frac{M_0}{\rho}) = \frac{M_0}{\rho}\Lambda_{+,\varphi}(x_0,8\rho,\frac{M_0}{\rho})$ and using our assumption that $M_0 \geqslant \bar c\rho\big[\Lambda_\lambda(x_0,\rho)\big]^{\bar c_1}$ we obtain

$$M_\sigma \leqslant 2\varepsilon M_0 + \gamma\varepsilon^{-\gamma}\sigma^{-\gamma}M_0\frac{\big[\Lambda_\lambda(x_0,\rho)\big]^{\bar\beta_4}}{\big[\Lambda_{+,\Phi}(x_0,8\rho,\frac{M_0}{\rho})\big]^{\frac1p}}\Big(\rho^{-n}M_0C_\Phi(E,B_{8\rho}(x_0);m) + M_0\Big)^{\frac1p} +$$

$$+\gamma\rho\big[\Lambda_\lambda(x_0,\rho)\big]^{\bar c_1} \leqslant (2\varepsilon+\frac{\gamma}{\bar c})M_0 + \gamma\varepsilon^{-\gamma}\sigma^{-\gamma}M_0\frac{\big[\Lambda_\lambda(x_0,\rho)\big]^{\bar\beta_4}}{\big[\Lambda_{+,\varphi}(x_0,8\rho,\frac{M_0}{\rho})\big]^{\frac1p}}\Big(\rho^{1-n}C_\Phi(E,B_{8\rho}(x_0);m)+\rho\Big)^{\frac1p}$$

$$\leqslant (2\varepsilon+\frac{\gamma}{\bar c})M_0 + \gamma\varepsilon^{-\gamma}\sigma^{-\gamma}M_0\frac{\big[\Lambda_\lambda(x_0,\rho)\big]^{\bar\beta_4}}{\big[\Lambda_{+,\varphi}(x_0,8\rho,\frac{M_0}{\rho})\big]^{\frac1p}}\Big(\rho^{1-n}C_\Phi(E,B_{8\rho}(x_0);m)\Big)^{\frac1p}.$$

By the fact that $\Lambda_{+,\varphi}(x_0,\rho,\varepsilon_0\frac{M_0}{\rho}) \leqslant \varepsilon_0^{p-1}\Lambda_{+,\varphi}(x_0,\rho,\frac{M_0}{\rho})$, $\varepsilon_0 \in (0,1)$ and using (2.1) with $a = \frac{M_\sigma}{\rho}$, $b = 1$, ε replaced by $\varepsilon_0\frac{M_0}{\rho}$ and $\varphi_p(x,\cdot)$ replaced by $\big[\Lambda_{+,\varphi}(x_0,\rho,\cdot)\big]^{\frac1p}$ from this we obtain

$$\big[\Lambda_{+,\varphi}(x_0,8\rho,\frac{M_\sigma}{\rho})\big]^{\frac1p} \leqslant \frac{M_\sigma}{\varepsilon_0M_0}\big[\Lambda_{+,\varphi}(x_0,8\rho,\frac{M_\sigma}{\rho})\big]^{\frac1p} + \big[\Lambda_{+,\varphi}(x_0,8\rho,\varepsilon_0\frac{M_0}{\rho})\big]^{\frac1p} \leqslant$$

$$\leqslant \frac{M_\sigma}{\varepsilon_0M_0}\big[\Lambda_{+,\varphi}(x_0,8\rho,\frac{M_0}{\rho})\big]^{\frac1p} + \varepsilon_0^{\frac{p-1}{p}}\big[\Lambda_{+,\varphi}(x_0,8\rho,\frac{M_0}{\rho})\big]^{\frac1p} \leqslant$$

$$\leqslant (\varepsilon^{\frac{p-1}{p}} + 2\frac{\varepsilon}{\varepsilon_0} + \frac{\gamma}{\bar c})\big[\Lambda_{+,\varphi}(x_0,8\rho,\frac{M_0}{\rho})\big]^{\frac1p} + \gamma\varepsilon_0^{-1}\varepsilon^{-\gamma}\sigma^{-\gamma}\big[\Lambda_\lambda(x_0,\rho)\big]^{\bar\beta_4}\Big(\rho^{1-n}C_\Phi(E,B_{8\rho}(x_0);m)\Big)^{\frac1p}.$$

Fix ε, $\bar c$ by the conditions $\varepsilon = \frac14\varepsilon_0^{2+\frac{p}{p-1}}$, $\bar c \geqslant \frac{4\gamma}{\varepsilon_0}$, $\varepsilon_0 \in (0,1)$, then the last inequality yields

$$\big[\Lambda_{+,\varphi}(x_0,8\rho,\frac{M_\sigma}{\rho})\big]^{\frac1p} \leqslant \varepsilon_0\big[\Lambda_{+,\varphi}(x_0,8\rho,\frac{M_0}{\rho})\big]^{\frac1p} + \gamma\varepsilon_0^{-\gamma}\sigma^{-\gamma}\big[\Lambda_\lambda(x_0,\rho)\big]^{\bar\beta_4}\Big(\rho^{1-n}C_\Phi(E,B_{8\rho}(x_0);m)\Big)^{\frac1p}.$$

Iterating this inequality we arrive at

$$\big[\Lambda_{+,\varphi}(x_0,8\rho,\frac{M_0}{\rho})\big]^{\frac1p} \leqslant \gamma\big[\Lambda_\lambda(x_0,\rho)\big]^{\bar\beta_4}\Big(\rho^{1-n}C_\Phi(E,B_{8\rho}(x_0);m)\Big)^{\frac1p},$$

which proves the required upper bound with $\bar\beta = p\bar\beta_4 = 1 + (q-1)\frac{t+1}{pt\kappa}$. This completes the proof of the lemma. □

4.2 Lower bound for the function w

The main step in the proof of the lower bound is the following lemma.

Lemma 4.2. *There exist numbers ε, $\vartheta \in (0,1)$, $\bar\beta_5 > 0$ depending only on the data such that*

$$\Big|\Big\{K_{\frac{3}{2}\rho,\,4\rho} : w(x) \leqslant \varepsilon\,\rho\,\Lambda^{-1}_{+,x_0,8\rho}\Big(\frac{C_\Phi(E,B_{8\rho}(x_0);m)}{\rho^{n-1}}\Big)\Big\}\Big| \leqslant$$

$$\leqslant \Big(1-\vartheta\,\big[\Lambda_\lambda(x_0,\rho)\big]^{-\bar\beta_5}\Big)\,|K_{\frac{3}{2}\rho,\,4\rho}|. \quad (4.8)$$

Proof. Let $\zeta_1(x) \in C_0^\infty(B_{3\rho}(x_0))$, $0 \leqslant \zeta_1(x) \leqslant 1$, $\zeta_1(x) = 1$ in $B_{2\rho}(x_0)$ and $|\nabla\zeta_1| \leqslant \frac{\gamma}{\rho}$. Test (4.1) by $\eta = w - m\zeta^q$ and use the Young inequality (2.1) with $\theta = 1$ we obtain with any $\varepsilon_1 \in (\rho, M\lambda(\rho))$

$$\int\limits_D \Phi(x,|\nabla w|)\,dx \leqslant \gamma\frac{m}{\rho}\int\limits_{K_{2\rho,3\rho}} \varphi(x,|\nabla w|)\,\zeta_1^{q-1}\,dx \leqslant \gamma\frac{m}{\varepsilon_1}\int\limits_{K_{2\rho,3\rho}} \Phi(x,|\nabla w|)\,dx+$$

$$+\gamma\frac{m}{\rho}\int\limits_{K_{2\rho,3\rho}} \varphi(x,\frac{\varepsilon_1}{\rho})\,dx \leqslant \gamma\frac{m}{\varepsilon_1}\int\limits_{K_{2\rho,3\rho}} \Phi(x,|\nabla w|)\,dx + \gamma m\rho^{n-1}\Big(\rho^{-n}\int\limits_{B_{8\rho}(x_0)} \big[\varphi(x,\frac{\varepsilon_1}{\rho})\big]^s\,dx\Big)^{\frac{1}{s}} \leqslant$$

$$\leqslant \gamma\frac{m}{\varepsilon_1}\int\limits_{K_{2\rho,3\rho}} \Phi(x,|\nabla w|)\,dx + \gamma m\rho^{n-1}\Lambda_{+,\varphi}(x_0,8\rho,\frac{\varepsilon_1}{\rho}).$$

Let $\zeta_2(x) \in C_0^\infty(K_{\frac{3}{2}\rho,4\rho})$, $0 \leqslant \zeta_2(x) \leqslant 1$, $\zeta_2(x) = 1$ in $K_{\rho,2\rho}$ and $|\nabla\zeta_2| \leqslant \frac{\gamma}{\rho}$. Testing (4.1) by $\eta = w\,\zeta^q$ and using the Young inequality (2.1) we estimate the first term on the right-hand side of the previous inequality as follows

$$\int\limits_{K_{2\rho,3\rho}} \Phi(x,|\nabla w|)\,dx \leqslant \int\limits_{K_{\frac{3}{2}\rho,4\rho}} \Phi(x,|\nabla w|)\,\zeta_2^q\,dx \leqslant \gamma\int\limits_{K_{\frac{3}{2}\rho,4\rho}} \Phi\big(x,\frac{w}{\rho}\big)\,dx.$$

Combining the last two inequalities and using the definition of the capacity we obtain

$$C_\Phi(E,B_{8\rho}(x_0);m) \leqslant \frac{1}{m}\int\limits_D \Phi(x,|\nabla w|)\,dx \leqslant \frac{\gamma}{\varepsilon_1}\int\limits_{K_{\frac{3}{2}\rho,4\rho}} \Phi\big(x,\frac{w}{\rho}\big)\,dx+\gamma\rho^{n-1}\Lambda_{+,\varphi}(x_0,8\rho,\frac{\varepsilon_1}{\rho}). \quad (4.9)$$

Choose ε_1 by the condition $\gamma\rho^{n-1}\Lambda_{+,\varphi}(x_0,8\rho,\frac{\varepsilon_1}{\rho}) = \frac{1}{4}C_\Phi(E,B_{8\rho}(x_0);m)$, i.e.

$$\varepsilon_1 = \rho\Lambda^{-1}_{+,x_0,8\rho}\Big(\frac{C_\Phi(E,B_{8\rho}(x_0);m)}{4\gamma\,\rho^{n-1}}\Big) \geqslant (4\gamma)^{-\frac{1}{p-1}}\,\rho\,\Lambda^{-1}_{+,x_0,8\rho}\Big(\frac{C_\Phi(E,B_{8\rho}(x_0);m)}{\rho^{n-1}}\Big).$$

Hence inequality (4.9) implies

$$C_\Phi(E,B_{8\rho}(x_0);m) \leqslant \frac{\gamma}{\varepsilon_1}\int\limits_{K_{\frac{3}{2}\rho,4\rho}} \Phi\big(x,\frac{w}{\rho}\big)\,dx. \quad (4.10)$$

Let us estimate the integral on the right-hand side of inequality (4.10), for this we decompose $K_{\frac{3}{2}\rho,4\rho}$ as $K_{\frac{3}{2}\rho,4\rho} = K' \cup K''$, $K' := K_{\frac{3}{2}\rho,4\rho} \cap \big\{w \leqslant \varepsilon\,\rho\,\delta(\rho,m)\big\}$ and $K'' := K_{\frac{3}{2}\rho,4\rho} \setminus K'$,

$\delta(\rho,m)=\Lambda^{-1}_{+,x_0,8\rho}\left(\frac{C_\Phi(E,B_{8\rho}(x_0);m)}{\rho^{n-1}}\right)$. By ($\Phi$) and our choice of ε_1 we have

$$\frac{\gamma}{\varepsilon_1}\int\limits_{K'}\Phi\big(x,\frac{w}{\rho}\big)\,dx\leqslant\frac{\gamma\varepsilon^p}{\varepsilon_1}\delta(\rho,m)\,\rho^n\Bigg(\rho^{-n}\int\limits_{B_{8\rho}(x_0)}\big[\varphi(x,\delta(\rho,m))\big]^s\,dx\Bigg)^{\frac{1}{s}}\leqslant$$

$$\leqslant\gamma\varepsilon^p\rho^{n-1}\Lambda_{+,\varphi}(x_0,8\rho,\delta(\rho,m))\leqslant\gamma\varepsilon^pC_\Phi(E,B_{8\rho}(x_0);m).\quad(4.11)$$

Similarly, by (Φ), Lemma 4.1 and our choice of ε_1

$$\frac{\gamma}{\varepsilon_1}\int\limits_{K''}\Phi\big(x,\frac{w}{\rho}\big)\,dx\leqslant\frac{\gamma}{\varepsilon_1}\delta(\rho,m)\,\rho^n[\Lambda_\lambda(x_0,\rho)]^{\frac{q\bar\beta}{p}}\Bigg(\rho^{-n}\int\limits_{B_{8\rho}(x_0)}\big[\varphi(x,\delta(\rho,m))\big]^s\,dx\Bigg)^{\frac{1}{s}}\bigg(\frac{|K''|}{|K_{\frac{3}{2}\rho,4\rho}|}\bigg)^{1-\frac{1}{s}}$$

$$\leqslant\gamma C_\Phi(E,B_{8\rho}(x_0);m)[\Lambda_\lambda(x_0,\rho)]^{\frac{q\bar\beta}{p}}\bigg(\frac{|K''|}{|K_{\frac{3}{2}\rho,4\rho}|}\bigg)^{1-\frac{1}{s}}.\quad(4.12)$$

Collecting estimates (4.10)–(4.12) we obtain

$$1\leqslant\gamma\varepsilon^p+\gamma[\Lambda_\lambda(x_0,\rho)]^{\frac{q\bar\beta}{p}}\bigg(\frac{|K''|}{|K_{\frac{3}{2}\rho,4\rho}|}\bigg)^{1-\frac{1}{s}},$$

choosing ε from the condition $\gamma\varepsilon^p=\frac{1}{2}$, from the previous we arrive at the required (4.8) with $\bar\beta_5=\frac{s}{s-1}\frac{q\bar\beta}{p}$, which completes the proof of the lemma. □

The following lemma is the main result of this Paragraph.

Lemma 4.3. *There exist numbers $\bar\varepsilon\in(0,1)$, $\bar\beta_6$, $\bar\beta_7>0$ depending only on the data such that*

$$\Big|\Big\{K_{\frac{3}{2}\rho,4\rho}:w\leqslant\bar\varepsilon m\big[\Lambda_\lambda(x_0,\rho)\big]^{-\bar\beta_6}\bigg(\frac{|E|}{\rho^n}\bigg)^{\bar\beta_7}\Big\}\Big|\leqslant\big(1-\vartheta\,\big[\Lambda_\lambda(x_0,\rho)\big]^{-\bar\beta_5}\big)|K_{\frac{3}{2}\rho,\,4\rho}|,\quad(4.13)$$

provided that

$$m\bigg(\frac{|E|}{\rho^n}\bigg)^{\bar\beta_7}\geqslant\frac{\rho}{\bar\varepsilon}\big[\Lambda_\lambda(x_0,\rho)\big]^{\bar\beta_6},\quad(4.14)$$

where $\bar\beta_5$, $\vartheta>0$ were defined in Lemma 4.2.

Proof. To prove inequality (4.13) we need to estimate the term on the left-hand side of (4.8). Let $\psi\in W^{1,\Phi}_0(B_{8\rho}(x_0))$, $\psi(x)=1$ for $x\in E$ and fix $\theta\in(1+\frac{1}{t},p)$. By the Poincare and Hölder inequalities and using (2.1) with $\varepsilon\in(0,1)$ we have

$$m|E|\leqslant m\int\limits_{B_{8\rho}(x_0)}|\psi|\,dx\leqslant\gamma\,\rho\int\limits_{B_{8\rho}(x_0)}|\nabla(m\psi)|\,dx\leqslant$$

$$\leqslant\gamma\,\rho\Bigg(\int\limits_{B_{8\rho}(x_0)}|\nabla(m\psi)|^\theta\,\varphi_\theta(x,\frac{m}{\rho})\,dx\Bigg)^{\frac{1}{\theta}}\Bigg(\int\limits_{B_{8\rho}(x_0)}\big[\varphi_\theta(x,\frac{m}{\rho})\big]^{-\frac{1}{\theta-1}}\,dx\Bigg)^{1-\frac{1}{\theta}}\leqslant$$

$$\leqslant\gamma m\rho^n\Bigg(\varepsilon^{-\theta}\rho^{-n}\int\limits_{B_{8\rho}(x_0)}\Phi(x,|\nabla(m\psi)|)\,dx+\varepsilon^{p-\theta}\Bigg(\rho^{-n}\int\limits_{B_{8\rho}(x_0)}\big[\Phi(x,\frac{m}{\rho})\big]^s\,dx\Bigg)^{\frac{1}{s}}\Bigg)^{\frac{1}{\theta}}\times$$

$$\times\Bigg(\rho^{-n}\int\limits_{B_{8\rho}(x_0)}\big[\Phi(x,\frac{m}{\rho})\big]^{-t}\,dx\Bigg)^{\frac{1}{\theta t}}.\quad(4.15)$$

Let us estimate the terms on the right-hand side of (4.15). Since $\rho \leqslant m \leqslant \lambda(\rho)M$, by $(\Phi^{\lambda}_{\Lambda,x_0})$ we have

$$\Big(\rho^{-n}\int\limits_{B_{8\rho}(x_0)}\big[\Phi(x,\frac{m}{\rho})\big]^{s}\,dx\Big)^{\frac{1}{\theta s}}\Big(\rho^{-n}\int\limits_{B_{8\rho}(x_0)}\big[\Phi(x,\frac{m}{\rho})\big]^{-t}\,dx\Big)^{\frac{1}{\theta t}} \leqslant \gamma[\Lambda_\lambda(x_0,\rho)]^{\frac{1}{\theta}}, \qquad (4.16)$$

therefore, choosing ε from the condition

$$\gamma\rho^{n}\varepsilon^{\frac{p-\theta}{\theta}}[\Lambda_\lambda(x_0,\rho)]^{\frac{1}{\theta}} = \frac{1}{2}|E|, \quad \text{i.e.} \quad \varepsilon = (2\gamma)^{-\frac{\theta}{p-\theta}}[\Lambda_\lambda(x_0,\rho)]^{-\frac{1}{p-\theta}}\Big(\frac{|E|}{\rho^n}\Big)^{\frac{\theta}{p-\theta}} < 1,$$

we obtain from (4.15), (4.16)

$$\gamma^{-1}\,\varepsilon^{\theta}\Big(\frac{|E|}{\rho^n}\Big)^{\theta} \leqslant \frac{\Lambda_\lambda(x_0,\rho)}{\Big(\rho^{-n}\int\limits_{B_{8\rho}(x_0)}\big[\Phi(x,\frac{m}{\rho})\big]^{s}\,dx\Big)^{\frac{1}{s}}}\,\rho^{-n}\int\limits_{B_{8\rho}(x_0)}\Phi(x,|\nabla(m\psi)|)\,dx.$$

From this, using the definition of capacity and using the fact that $\Phi(x,v) = v\varphi(x,v)$, $v > 0$, we obtain

$$\Lambda_{+,\varphi}(x_0,8\rho,\frac{m}{\rho})\Big(\frac{|E|}{\rho^n}\Big)^{\theta} \leqslant \gamma\varepsilon^{-\theta}\Lambda_\lambda(x_0,\rho)\rho^{1-n}C_{\Phi}(E,B_{8\rho}(x_0);m),$$

which yields by our choice of ε

$$\rho\Lambda^{-1}_{+,x_0,8\rho}\Big(\frac{C_{\Phi}(E,B_{8\rho}(x_0);m)}{\rho^{n-1}}\Big) \geqslant \gamma^{-1}\,m\,\Big(\varepsilon^{\theta}\big[\Lambda_\lambda(x_0,\rho)\big]^{-1}\Big(\frac{|E|}{\rho^n}\Big)^{\theta}\Big)^{\frac{1}{p-1}} =$$

$$= \gamma^{-1}\,m\,\big[\Lambda_\lambda(x_0,\rho)\big]^{-\bar\beta_6}\Big(\frac{|E|}{\rho^n}\Big)^{\bar\beta_7}, \quad \bar\beta_6 = \frac{p}{(p-\theta)(p-1)}, \bar\beta_7 = \frac{p\theta}{(p-\theta)(p-1)}. \qquad (4.17)$$

Therefore, inequality (4.13) is a consequence of (4.8), provided that (4.2) is valid. By (4.17), inequality (4.2), in turn, is a consequence of (4.14), provided that $\bar c$ is large enough and $\bar c_1 \geqslant \bar\beta_6$. This completes the proof of the lemma.

4.3 Proof of Theorems 1.4 and 1.6

Let $u \geqslant 0$ be a super-solution to Eq. (1.14) in Ω and construct the sets $E(\rho,N) := B_\rho(x_0) \cap \cap \{u > N\}$ and $E_\lambda(\rho,N) := B_\rho(x_0) \cap \{u > \lambda(\rho)\,N\}$, $0 < N < M$, $E(\rho,N) \subset E_\lambda(\rho,N)$. Let w be an auxiliary solution to the problem (1.24) in $D = B_{8\rho} \setminus E_\lambda(\rho,N)$. Since $u \geqslant w$ on ∂D, by the monotonicity condition (1.16) $u \geqslant w$ in D and Lemma 4.3 with $m = \lambda(\rho)N$ implies

$$\big|\big\{B_{2\rho}(x_0) : u \leqslant \bar\varepsilon\lambda(\rho)N\big[\Lambda_\lambda(x_0,\rho)\big]^{-\bar\beta_6}\Big(\frac{|E_\lambda(\rho)|}{\rho^n}\Big)^{\bar\beta_7}\big\}\big| \leqslant \big(1-\vartheta\,\big[\Lambda_\lambda(x_0,\rho)\big]^{-\bar\beta_5}\big)|B_{2\rho}(x_0)|, \qquad (4.18)$$

provided that

$$\lambda(\rho)\,N\Big(\frac{|E_\lambda(\rho,m)|}{\rho^n}\Big)^{\bar\beta_7} \geqslant \frac{\rho}{\bar\varepsilon}\big[\Lambda_\lambda(x_0,\rho)\big]^{\bar\beta_6}, \qquad (4.19)$$

with some positive $\bar\beta_5$, $\bar\beta_6$, $\bar\beta_7$ and $\vartheta \in (0,1)$ depending only on the data.

We use Lemmas 2.3, 2.4 with $\alpha_0 = \vartheta\big[\Lambda_\lambda(x_0,\rho)\big]^{-\bar\beta_5}$ and ν defined in Lemma 2.4. These lemmas ensure the existence of C, $\bar\beta_8 = \bar\beta_1 + p\bar\beta_1\bar\beta_5 + p\bar\beta_1\bar\beta_2 > 0$, where $\bar\beta_1$, $\bar\beta_2 > 0$ were defined in Lemmas 2.3 and 2.4 depending only on the data, such that

$$\lambda(\rho)N\Big(\frac{|E(\rho,m)|}{\rho^n}\Big)^{\bar\beta_7} \leqslant \lambda(\rho)N\Big(\frac{|E_\lambda(\rho,m)|}{\rho^n}\Big)^{\bar\beta_7} \leqslant$$
$$\leqslant \big[\Lambda_\lambda(x_0,\rho)\big]^{-\bar\beta_6}\exp\big(C\big[\Lambda_\lambda(x_0,\rho)\big]^{\bar\beta_8}\big)\{\inf_{B_{\frac{\rho}{2}}(x_0)} u+\rho\} \leqslant$$
$$\leqslant \exp\big(C\big[\Lambda_\lambda(x_0,\rho)\big]^{1+\bar\beta_8}\big)\{\inf_{B_{\frac{\rho}{2}}(x_0)} u+\rho\}.$$

This completes the proof of Theorem 1.6.

The proof of the weak Harnack-type inequality (1.17) and the upper bound (1.18) is almost the same as for Theorem 1.2, inequalities (1.11) and (1.12), see Section 3.3 for details, we leave them to the reader.

□

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