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Singular Rouquier complexes

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Abstract

We generalize the construction of Rouquier complexes to the setting of one-sided singular Soergel bimodules. Singular Rouquier complexes are defined by taking minimal complexes of restricted Rouquier complexes. We show that they retain many of the properties of ordinary Rouquier complexes: they are Δ-split, they satisfy a vanishing formula, and when Soergel's conjecture holds they are perverse. As an application, we establish Hodge theory for singular Soergel bimodules.

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1 | INTRODUCTION

Consider a complex reductive algebraic group G with Borel subgroup B and Weyl group W. The category of B-equivariant parity sheaves on the flag variety $X = G/B$ provides a categorification of the Hecke algebra $\mathcal{H}$ of W. Soergel [14, 15] an alternative categorification of the Hecke algebra $\mathcal{H}$ via certain graded bimodules over $R = \mathrm{Sym}^{\bullet}_{\mathbb{K}}(\mathfrak{h}^*)$, where $\mathfrak{h}^*$ is a (well-behaved) representation of W over a field $\mathbb{K}$. A major advantage of using Soergel bimodules is that their construction is completely algebraic, in particular their definition makes sense for an arbitrary Coxeter group W.

The situation is very similar when we consider a parabolic subgroup P of G containing B and the partial flag variety G/P. Let I be the subset of the simple reflections $S \subset W$ corresponding to P. Let W_I denote the subgroup of W generated by I. Then, B-equivariant parity sheaves on G/P categorify the left ideal $\mathcal{H}^I := \mathcal{H}\underline{\mathbf{H}}_I$ of the Hecke algebra $\mathcal{H}$, where $\underline{\mathbf{H}}_I \in \mathcal{H}$ is the Kazhdan–Lusztig basis element corresponding to the longest element in W_I. In this case an algebraic replacement is provided by the category of singular Soergel bimodules, introduced by Williamson in [16].

The construction of (one-sided) singular Soergel bimodules is algebraic and works for any Coxeter group W and any subset $I \subset S$ such that W_I is finite. Singular Soergel bimodules are graded (R, R^I) bimodules, where R^I denotes the subring of W_I-invariants in R. The indecomposable singular Soergel bimodules P_w are parameterized (up to grading shifts) by elements $w \in W^I$, where W^I is the set of elements of W which are minimal in their right W_I-coset.

For any $I \subset S$ such that W_I is finite, we denote by $\mathbb{S}Bim^I$ the corresponding category of singular Soergel bimodules. We simply write $\mathbb{S}Bim$ for $\mathbb{S}Bim^\emptyset$, the category of (ordinary) Soergel bimodules.

For a Coxeter group W, let B_W denote the corresponding Artin braid group. In [13], Rouquier introduced, inside the homotopy category of Soergel bimodules, a categorification of B_W: the 2-braid group $\mathfrak{B}_W$. Let us briefly recall its construction. For any element $s \in S$, let $B_s = R \otimes_{R^s} R(1)$ be the corresponding indecomposable Soergel bimodule and consider the complexes

$$F_s := [0 \to B_s \to R(1) \to 0]$$

$$E_s := [0 \to R(-1) \to B_s \to 0].$$

We can regard F_s and E_s as objects in $\mathcal{K}^b(\mathbb{S}Bim)$, the bounded homotopy category of Soergel bimodules. Then, E_s and F_s are inverse to each other with respect to the tensor product operation, so we can also write $E_s = (F_s)^{-1}$. To any word $w = s_1^{\varepsilon_1} s_2^{\varepsilon_2} \dots s_k^{\varepsilon_k} \in B_W$ (where $\varepsilon_i = \pm 1$) we associate the complex

$$\overline{F}_w := (F_{s_1})^{\varepsilon_1}(F_{s_2})^{\varepsilon_2} \dots (F_{s_k})^{\varepsilon_k} \in \mathcal{K}^b(\mathbb{S}Bim)$$

(where concatenation indicates the tensor product of complexes). Then the objects in $\mathfrak{B}_W$ are the complexes $\overline{F}_w$, for $w \in B_W$. If W is a finite group then $\mathfrak{B}_W$ is a faithful categorification of B_W [1, 6, 7]: we have $\overline{F}_w \cong \overline{F}_v$ if and only if $w = v$.

Any elements of the Coxeter group W has two distinguished lifts to B_W, and hence to $\mathfrak{B}_W$. If $w = s_1 s_2 \dots s_k \in W$ we define $F_{s_1} F_{s_2} \dots F_{s_k}$ to be the positive lift and $E_{s_1} E_{s_2} \dots E_{s_k}$ to be the negative lift of w in $\mathfrak{B}_W$. Let F_w be the minimal complex of $F_{s_1} F_{s_2} \dots F_{s_k}$, that is, F_w is the complex in $C^b(\mathbb{S}Bim)$ obtained by removing all the contractible summands from $F_{s_1} F_{s_2} \dots F_{s_k}$. Similarly, let E_w be the minimal complex of $E_{s_1} E_{s_2} \dots E_{s_k}$. The complexes F_w and E_w are called the (minimal) *Rouquier complexes*.

One can easily repeat Rouquier's construction in the world of singular Soergel bimodules by restricting a complex of (R, R)-bimodules to a complex of (R, R^I)-bimodules. For any $w \in W^I$ we define the *singular Rouquier complex* F_w^I to be the minimal complex of $\mathrm{res}_{R,R}^{R,R^I}(F_w)$ in the category of complexes of I-singular Soergel bimodules $C^b(\mathbb{S}Bim^I)$. We show that *singular 2-braid group* retains some of the important properties of the 2-braid group.

In [8], Libedinsky and Williamson showed that the 2-braid groups have standard and costandard objects. More precisely, they showed that we have the following vanishing property:

$$\mathrm{Hom}(F_w, E_v[i]) = \begin{cases} \mathbb{K} & \text{if } v = w \text{ and } i = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(If W is a Weyl group and $\mathbb{K} = \mathbb{C}$, this statement is equivalent to the existence of standard and costandard objects in category $\mathcal{O}$.) The main result of this paper is the generalization of the results in [8] to singular Rouquier complexes. In particular, we prove the singular version of (1):

$$\mathrm{Hom}(F^I_w, E^I_v[i]) = \begin{cases} \mathbb{K} & \text{if } v = w \text{ and } i = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

It follows that also singular 2-braid groups have standard and costandard objects. We discuss now two applications of this generalization.

- In [11], we restrict ourselves to the case of Grassmannians, that is, we consider the case when W is the symmetric group S_n and W_I is a maximal parabolic subgroup. In this setting, summands in singular Rouquier complexes can be understood using the combinatorics of Dyck partitions. A careful study of the first two terms in singular Rouquier complexes allows us to deduce some crucial relations involving maps of degree one. In turn, these relations allow us to explicitly construct bases of the morphisms spaces between singular Soergel bimodules. In particular, we also obtain bases for the intersection cohomology of Schubert varieties that naturally extend the Schubert basis.
- When Soergel's conjecture holds, for example, when we work over the real numbers and we consider the same representation of W as in [15, Prop. 2.1], then indecomposable Soergel bimodules categorify the Kazhdan–Lusztig basis in the Hecke algebra. In this case, Rouquier complexes are perverse and they categorify the inverse Kazhdan–Lusztig polynomials (as in [5, Remark 6.10]). We show that the same is true for singular Rouquier complexes: they are perverse and from the multiplicities of its summands we can reconstruct the inverse parabolic Kazhdan–Lusztig polynomial.

  In [5], Rouquier complexes are a crucial tool in establishing Hodge theory for Soergel bimodules, and hence in proving Soergel's conjecture. Elias and Williamson's idea is to emulate the geometric proof of de Cataldo and Migliorini [3] of the hard Lefschetz theorem and of the bilinear Hodge–Riemann relations. Here the Rouquier complexes have the decisive role of providing a surrogate for a smooth hyperplane section. After having shown that singular Rouquier complexes are perverse, it is rather straightforward to adapt the arguments in [5] to singular Soergel bimodules. Hence, we obtain a proof of the hard Lefschetz theorem (Theorem 5.2) and of the Hodge–Riemann bilinear relations (Theorem 5.3) for singular Soergel bimodules.

  We remark that using the Hodge theory of singular Soergel bimodules we can give a (slightly) different proof of Soergel's conjecture (cf. Remark 5.6), which is closer to the geometric proof of the decomposition theorem discussed in [3].

In [16], Williamson also developed the theory of two-sided singular Soergel bimodules. These are graded (R^J, R^I)-bimodules where $I, J \subset S$ are subsets such that W_I and W_J are finite. However, we only treat here the case of one-sided bimodules. In fact, we do not expect that singular Rouquier complexes can be nicely generalized to the two-sided case. As we explain in Remark 4.18, two-sided Rouquier complexes cannot be perverse even when Soergel's conjecture holds. Moreover, for applications to Hodge theory, two-sided bimodules are unnecessary, since the Soergel modules obtained starting from one-sided or two-sided bimodules coincide (see [9, Remark 4.2.5]).

2 | HECKE ALGEBRA

We recall some basic notation about Coxeter groups and their Hecke algebras from [5, § 3.2] and [16, § 2].

Let (W, S) be a Coxeter system. For $s, t \in S$, let m_{st} denote the order of (st). We denote the length function by ℓ and the Bruhat order by $\leq$.

The Hecke algebra $\mathcal{H} := \mathcal{H}(W, S)$ is the unital associative $\mathbb{Z}[v, v^{-1}]$ algebra with generators $\mathbf{H}_s$, for $s \in S$, subject to the following relations, for any $s, t \in S$:

$$\overbrace{\mathbf{H}_s\mathbf{H}_t \dots}^{m_{st}} = \overbrace{\mathbf{H}_t\mathbf{H}_s \dots}^{m_{st}}$$

$$\mathbf{H}_s^2 = -(v - v^{-1})\mathbf{H}_s + 1.$$

For any $x \in W$ the element $\mathbf{H}_x$ is defined as $\mathbf{H}_x := \mathbf{H}_{s_1} \dots \mathbf{H}_{s_l}$ where $x = s_1 s_2 \dots s_l$ is any reduced expression for x. The set $\{\mathbf{H}_x\}_{x \in W}$ is a $\mathbb{Z}[v, v^{-1}]$-basis of $\mathcal{H}$, called the *standard basis*.

We denote by $\overline{(-)} : \mathcal{H} \to \mathcal{H}$ the involution defined by $\overline{\mathbf{H}_s} = \mathbf{H}_s^{-1}$ and $\overline{v} = v^{-1}$. For any $x \in W$ the Kazhdan–Lusztig basis element $\underline{\mathbf{H}}_x$. This is the unique element in $\mathcal{H}$ such that the following two conditions hold:

- $\overline{\underline{\mathbf{H}}_x} = \underline{\mathbf{H}}_x$,
- $\underline{\mathbf{H}}_x = \mathbf{H}_x + \sum_{y<x} h_{y,x}(v)\mathbf{H}_y$, for some polynomials $h_{y,x}(v) \in v\mathbb{Z}[v]$.

The polynomials $h_{y,x}(v)$ are called the *Kazhdan–Lusztig polynomials*. The set $\{\underline{\mathbf{H}}_x\}_{x \in W}$ is a $\mathbb{Z}[v, v^{-1}]$-basis of $\mathcal{H}$, called the *Kazhdan–Lusztig basis*.

There exists an anti-involution a of $\mathcal{H}$ defined by $a(\mathbf{H}_x) = \mathbf{H}_{x^{-1}}$ for $x \in W$ and $a(v) = v$. The trace ε is the $\mathbb{Z}[v, v^{-1}]$-linear map defined by $\varepsilon(\mathbf{H}_w) = \delta_{w,id}$. We define a $\mathbb{Z}[v, v^{-1}]$-bilinear pairing

$$(-,-) : \mathcal{H} \times \mathcal{H} \to Z[v, v^{-1}] \tag{3}$$

by $(h, h') = \varepsilon(a(h)h')$.

For a subset $I \subset S$, let W_I be the parabolic subgroup of W generated by I. A subset $I \subseteq S$ is said to be *finitary* if the group W_I is finite. We denote by W^I the set of right I-minimal elements, that is, the set of elements $x \in W$ such that $xs \geqslant x$ for all $s \in I$.

Let $q : W \to W/W_I$ denote the projection map. For $y \in W/W_I$ we denote by y_- the minimal element in the coset y. The bijection $W^I \cong W/W_I$ induces a partial order on W/W_I by restricting the Bruhat order of W, that is, for $y, z \in W/W_I$ we say $y \leqslant z$ if and only if $y_- \leqslant z_-$. The projection q is a strict morphism of posets:

Lemma 2.1 [4, Lemma 2.2]. *Let $w \geqslant v$ in W. Then $q(w) \geqslant q(v)$.*

Let I be finitary and let w_I be the longest element in W_I. We define

$$\underline{\mathbf{H}}_I := \underline{\mathbf{H}}_{w_I} = \sum_{x \in W_I} v^{\ell(w_I) - \ell(x)}\mathbf{H}_x.$$

Consider the left ideal $\mathcal{H}^I := \mathcal{H}\underline{\mathbf{H}}_I$ of $\mathcal{H}$. We recall a few basis facts about $\mathcal{H}^I$ from [16, § 2.3]. For $x \in W^I$ we define $\mathbf{H}_x^I = \mathbf{H}_x\underline{\mathbf{H}}_I$. The Kazhdan–Lusztig basis element $\underline{\mathbf{H}}_y$ belongs to $\mathcal{H}^I$ if and only if y is maximal in its right W_I-coset. Thus, for $x \in W^I$, we can define $\underline{\mathbf{H}}_x^I = \underline{\mathbf{H}}_{xw_I}$. The set $\{\underline{\mathbf{H}}_x^I\}_{x \in W^I}$ forms a $\mathbb{Z}[v, v^{-1}]$-basis of $\mathcal{H}^I$, called the *I-parabolic Kazhdan–Lusztig basis* of $\mathcal{H}^I$. For any $x \in W^I$ we can write

$$\underline{\mathbf{H}}^I_x = \mathbf{H}^I_x + \sum_{W^I \ni y < x} h^I_{y,x}(v)\mathbf{H}^I_y.$$

The polynomials $h^I_{y,x}(v)$ are called the *I-parabolic Kazhdan–Lusztig polynomials* and are related to the ordinary Kazhdan–Lusztig polynomials by the formula $h^I_{y,x}(v) = h_{yw_I,xw_I}(v)$.

3 | ONE-SIDED SINGULAR SOERGEL BIMODULES

The main reference for this section is [16, § 7]. We fix a field $\mathbb{K}$ and a reflection faithful representation $\mathfrak{h}^*$ of W over $\mathbb{K}$ (in the sense of [15, Definition 1.7]). Let R denote the polynomial ring $\mathrm{Sym}_{\mathbb{K}}(\mathfrak{h}^*)$. We regard R as a graded ring by setting $\deg(\alpha) = 2$ for any $\alpha \in \mathfrak{h}^*$.

We fix now a finitary subset $I \subseteq S$. We use the abbreviations $(\mathfrak{h}^*)^I := (\mathfrak{h}^*)^{W_I}$ and $R^I := R^{W_I}$ to denote the corresponding subspaces of W_I-invariants. We work in the category of graded (R, R^I)-bimodules. We denote by (1) the grading shift on graded bimodules; in $R(1)$ the identity appears in degree -1. If B is a graded (R, R)-bimodule we denote by B_I the restriction of B to a graded (R, R^I)-bimodule.

We make the following assumption: the ring R regarded as a R^I-module is free of graded rank $\widetilde{\pi}(I)$. This is always the case if we make one the following two assumptions:

- $\mathrm{char}(\mathbb{K}) = 0$,
- W is a Weyl group, $\mathfrak{h}^* = X \otimes_{\mathbb{Z}} \mathbb{K}$ is the representation obtained by extending scalars on the action of W on the weight lattice and $\mathrm{char}(\mathbb{K})$ is not a torsion prime for W (cf. [16, Remark 4.1.2]).

For $s \in S$, let $B_s := R \otimes_{R^s} R(1)$. For any sequence of simple reflections $\underline{w} = (s_1, \dots, s_k)$ we consider the corresponding Bott–Samelson bimodule

$$BS(\underline{w}) := B_{s_1} \otimes_R B_{s_2} \otimes_R \dots \otimes_R B_{s_k}.$$

Definition 3.1. The category of I-singular Soergel bimodules $\mathbb{S}Bim^I$ is the smallest full subcategory of graded (R, R^I)-bimodules which contains all Bott–Samelson bimodules $BS(\underline{w})_I$ and which is closed under direct sums, grading shifts, and taking direct summands.

Morphisms in $\mathbb{S}Bim^I$ are the morphisms of graded (R, R^I)-bimodules of degree 0 and are denoted by $\mathrm{Hom}(-,-)$.

If $I = \emptyset$ then $\mathbb{S}Bim^\emptyset$ is simply denoted by $\mathbb{S}Bim$ and called the category of Soergel bimodules.

For any $P, P' \in \mathbb{S}Bim^I$ and $i \in \mathbb{Z}$ we set $\mathrm{Hom}^i(P, P') = \mathrm{Hom}(P, P'(i))$ and

$$\mathrm{Hom}^\bullet(P, P') = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}(P, P'(i)).$$

There is a duality functor $\mathbb{D}$ with $\mathbb{D}P = \mathrm{Hom}^\bullet_{R-}(P, R)$ on $\mathbb{S}Bim^I$. The (R, R^I)-bimodule structure on $\mathbb{D}P$ is given by

$$(rfr')(b) = f(rbr') = rf(br') \qquad \text{for any } f \in \mathbb{D}P, b \in P, r \in R, r' \in R^I.$$

Theorem 3.2 (Soergel–Williamson categorification theorem [16, Theorem 1]). *There exists a bijection*

$$W^I \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{isomorphism classes of} \\ \textit{indecomposable self-dual} \\ I\textit{-singular Soergel bimodules} \end{array} \right\}.$$

We denote by P^I_x the indecomposable self-dual bimodule corresponding to x. Every indecomposable I-singular Soergel bimodule is isomorphic up to a shift to some P^I_x.

Let $x = s_1 s_2 \dots s_k$ be a reduced expression for $x \in W^I$. Then P^I_x is the unique direct summand of $BS(\underline{s_1 s_2 \dots s_k})_I$ which is not a direct summand of any Bott–Samelson bimodule of smaller length.

Notation. In [16], the indecomposable Soergel bimodules are denoted by B^I_x, for $x \in W^I$. When the finitary set I is clear from the context, we will remove the I from the notation and simply denote the indecomposable self-dual singular Soergel bimodule by P_x, for $x \in W^I$. If $I = \emptyset$, we denote the indecomposable bimodules $P^{\emptyset}_y$ simply by B_y, for $y \in W$.

In general, to help the reader distinguish between ordinary and singular Soergel bimodules, we adopt the following convention: objects in $\mathbb{S}Bim$ are denoted by the letter B while objects in $\mathbb{S}Bim^I$ are denoted by the letter P.

Given two bimodules $P_1, P_2 \in \mathbb{S}Bim^I$ and $x \in W^I$, consider the subspace

$$\operatorname{Hom}^\bullet_{<x}(P_1, P_2) \subseteq \operatorname{Hom}^\bullet(P_1, P_2)$$

spanned by all the maps $\varphi : P_1 \to P_2(k)$ which factor through $P_1 \to P_y(k') \to P_2(k)$ for some $y < x$ and $k' \in \mathbb{Z}$. Let

$$\operatorname{Hom}^\bullet_{\not< x}(P_1, P_2) := \operatorname{Hom}^\bullet(P_1, P_2) / \operatorname{Hom}^\bullet_{<x}(P_1, P_2).$$

Let $[\mathbb{S}Bim^I]$ denote the split Grothendieck group of $\mathbb{S}Bim^I$. We regard it as a $\mathbb{Z}[v, v^{-1}]$-module via $v \cdot [P] = [P(1)]$. If $V = \bigoplus_{i \in \mathbb{Z}} R(-i)^{m_i}$ is a graded free R-module we define the *graded rank* of V as:

$$\operatorname{grrk}(V) := \sum_{i \in \mathbb{Z}} m_i v^i.$$

The *character map* is a morphism of $\mathbb{Z}[v, v^{-1}]$-modules $\operatorname{ch} : [\mathbb{S}Bim^I] \to \mathcal{H}^I$ defined by

$$\operatorname{ch}([P]) = \sum_{x \in W^I} \operatorname{grrk} \operatorname{Hom}^\bullet_{\not< x}(P, P_x) \mathbf{H}^I_x \tag{4}$$

for any $P \in \mathbb{S}Bim^I$.† It follows from Theorem 3.2 that ch is an isomorphism. Moreover, the following diagram is commutative:

$$\begin{array}{ccc} [\mathbb{S}Bim] \times [\mathbb{S}Bim^I] & \xrightarrow{\;- \otimes_R -\;} & [\mathbb{S}Bim^I] \\ {\scriptstyle \operatorname{ch} \times \operatorname{ch}} \downarrow & & {\scriptstyle \operatorname{ch}} \downarrow \\ \mathcal{H} \times \mathcal{H}^I & \xrightarrow{\quad m \quad} & \mathcal{H}^I \end{array}$$

† The R-module $\operatorname{Hom}^\bullet_{\not< x}(P, P_x)$ is free: this follows from [16, Theorem 7.2.2] and the fact that $\operatorname{Hom}^\bullet_{\not< x}(P, P_x) \cong \operatorname{Hom}^\bullet_{\not< x}(P, R_{x,I}(\ell(x)))$, where $R_{x,I} = (R_x)_I$ is a standard module (cf. Remark 4.8).

(here m is the multiplication in $\mathcal{H}$). Hence $\mathbb{S}Bim^I$ *categorifies* the ideal $\mathcal{H}^I$ as a module over $\mathcal{H}$. We can use the isomorphism ch to compute the dimension of the space of morphisms in the category $\mathbb{S}Bim^I$.

Theorem 3.3 (Soergel's Hom formula for singular Soergel bimodules [16, Theorem 7.4.1]). *Let $P_1, P_2 \in \mathbb{S}Bim^I$. Then* $\mathrm{Hom}^\bullet(P_1, P_2)$ *is a free graded left R-module and*

$$\mathrm{grrk}\,\mathrm{Hom}^\bullet_{R\otimes R^I}(P_1,P_2) = \frac{1}{\widetilde{\pi}(I)}(\overline{\mathrm{ch}(P_1)}, \mathrm{ch}(P_2)).$$

Here $(-,-)$ is the pairing in the Hecke algebra, defined in (3), and $\widetilde{\pi}(I)$ is the Poincaré polynomial of W_I, defined as

$$\widetilde{\pi}(I) := \sum_{w\in W_I} v^{2\ell(w)}.$$

We can identify $R\otimes_{\mathbb{R}} R^I$ with the ring of regular functions on $\mathfrak{h}\times(\mathfrak{h}/W_I)$. Hence a Soergel bimodule $P\in\mathbb{S}Bim^I$ can be thought as a quasi-coherent sheaf on $\mathfrak{h}\times(\mathfrak{h}/W_I)$. The inclusion $R\otimes_{\mathbb{R}} R^I \hookrightarrow R\otimes_{\mathbb{R}} R$ corresponds to the projection map $\pi : \mathfrak{h}\times\mathfrak{h}\to\mathfrak{h}\times(\mathfrak{h}/W_I)$.

For $x\in W$ we denote the twisted graph of x by $Gr(x)$, that is,

$$Gr(x) = \{(x\cdot\lambda,\lambda) | \lambda\in\mathfrak{h}\}\subseteq\mathfrak{h}\times\mathfrak{h}.$$

If $C\subseteq W$, let $Gr(C)=\bigcup_{x\in C} Gr(x)$. For a coset $y\in W/W_I$ let $Gr^I(y) := \pi(Gr(y))$. Notice that $Gr^I(y)=\pi(Gr(\widetilde{y}))$ for any $\widetilde{y}\in y$. Similarly, if $C\subseteq W/W_I$, let $Gr^I(C) := \bigcup_{p\in C} Gr^I(p)$.

The support of every Soergel bimodule $P\in\mathbb{S}Bim^I$ is contained in $Gr(W/W_I)$. For $C\subseteq W/W_I$ we define

$$\Gamma^I_C P = \{b\in P \mid \mathrm{supp}\, b\subseteq Gr^I(C)\}.$$

We will simply write Γ_C for $\Gamma^{\emptyset}_C$. For any $B\in\mathbb{S}Bim$ and any $C\subseteq W/W_I$ we have by [16, Prop 6.1.6]

$$(\Gamma_{q^{-1}(C)}B)_I = \Gamma^I_C(B_I). \tag{5}$$

Remark 3.4. We would like to draw attention to a few slight differences with the definitions given in [16]. Our definition of the duality functor $\mathbb{D}$ contains a different shift, and thus our self-dual indecomposable bimodules P_x coincide with $B^I_x(-\ell(w_I))$ in Williamson's notation. The advantage of our definition of $\mathbb{D}$ is that it guarantees that the singular Soergel modules $\overline{P_x}=\mathbb{K}\otimes_R P_x$ have symmetric Betti numbers. This is more natural in the geometric setting where these modules are isomorphic to intersection cohomology of Schubert varieties in a partial flag variety. This choice of the shift is particularly convenient when dealing with Hodge theoretic properties (cf. Section 5).

We point out that with our definition of the duality $\mathbb{D}$ we have $\mathrm{ch}(B_I)=\mathrm{ch}(B)\underline{\mathbf{H}}_{w_I}$ and if $x\in W^I$ we have

$$P_x\otimes_{R^I} R(\ell(w_I)) \cong B_{xw_I}\in\mathbb{S}Bim.$$

4 | SINGULAR ROUQUIER COMPLEXES

Let $C^b(\mathbb{S}Bim^I)$ be the bounded category of complexes of I-singular Soergel bimodules and let $\mathcal{K}^b(\mathbb{S}Bim^I)$ be the corresponding bounded homotopy category.

Following the notation of [5, § 6], we indicate the homological degree of an object $F \in C^b(\mathbb{S}Bim^I)$ on the left as follows:

$$F = [\dots \to {}^{i-1}F \to {}^{i}F \to {}^{i+1}F \to \dots].$$

We denote by $[-]$ the homological shift, so that ${}^i(F[1]) = {}^{i+1}F$.

For $s \in S$ let F_s denote the complex[†]

$$F_s = [0 \to \overset{0}{B_s} \xrightarrow{d_s} R(1) \to 0]$$

where d_s is the map defined by $f \otimes g \mapsto fg$. Then, tensoring with F_s on the left induces an equivalence on the category $\mathcal{K}^b(\mathbb{S}Bim^I)$. In fact, tensoring on the left with the complex

$$E_s = [0 \to R(-1) \xrightarrow{d'_s} \overset{0}{B_s} \to 0]. \tag{6}$$

gives an inverse. Here the map d'_s is the morphism of R-bimodules which sends $1 \in R(-1)$ to $c_s := \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$.

Given $x \in W^I$ and any reduced expression $x = s_1 \dots s_k$, we consider the complex $F_{s_1} \dots F_{s_k}$, where concatenation indicates the tensor product of complexes. As an object in $\mathcal{K}^b(\mathbb{S}Bim)$, the complex $F_{s_1} \dots F_{s_k}$ does not depend on the chosen reduced expression up to canonical isomorphism [13, Proposition 9.2]. Hence, $(F_{s_1} \dots F_{s_k})_I$ also does not depend on the reduced expression up to canonical isomorphism as an object in $\mathcal{K}^b(\mathbb{S}Bim^I)$.

Definition 4.1. For $x \in W^I$, we denote by $F^I_x \in C^b(\mathbb{S}Bim^I)$ the minimal subcomplex of $(F_{s_1} \dots F_{s_k})_I$, where $s_1 \dots s_k$ is a reduced expression for x (cf. [5, § 6.1]). This means that F^I_x is a summand of $(F_{s_1} \dots F_{s_k})_I \in C^b(\mathbb{S}Bim^I)$ such that F^I_x does not contain any contractible direct summand and $F^I_x \cong (F_{s_1} \dots F_{s_k})_I$ in $\mathcal{K}^b(\mathbb{S}Bim^I)$. We call F^I_x the *I-singular Rouquier complex* of x.

Observe that if $F_x \in C^b(\mathbb{S}Bim)$ is the Rouquier complex for x, that is, if F_x is the minimal subcomplex for $F_{s_1} \dots F_{s_k}$, then F^I_x can also be obtained as the minimal subcomplex of $F_{x,I} := (F_x)_I$ in $C^b(\mathbb{S}Bim^I)$.

4.1 | Singular Rouquier complexes are Δ-split

If $x \in W^I$ we write $\Gamma^I_{\geqslant x}$ for the functor $\Gamma^I_{\{y \in W^I | y \geqslant x\}}$ on $\mathbb{S}Bim^I$. We define similarly $\Gamma^I_{>x}$, $\Gamma^I_{<x}$, and $\Gamma^I_{\leqslant x}$. For $P \in \mathbb{S}Bim^I$, let $\Gamma^I_{\geqslant x/>x}P := (\Gamma^I_{\geqslant x}P)/(\Gamma^I_{>x}P)$ and $\Gamma^I_{\leqslant x/<x}P := (\Gamma^I_{\leqslant x}P)/(\Gamma^I_{<x}P)$. Recall the projection $q : W \to W/W_I$. If $y \in W/W_I$ we have $q^{-1}(\geqslant y) = \{x \in W \mid x \geqslant y_-\}$. By (5), for any

[†] We use here the notation $\overset{0}{-}$ to indicate where the object in homological degree 0 is placed.

$B \in \mathbb{S}Bim$ we have

$$(\Gamma_{\geqslant (y_-)} B)_I = \Gamma^I_{\geqslant y}(B_I). \tag{7}$$

We choose an enumeration $y_1, y_2, y_3, \ldots$ of W^I refining the Bruhat order on W^I and an enumeration $w_1, w_2, \ldots w_{|W_I|}$ of W_I refining the Bruhat order of W_I. Let

$$z_1 = y_1 w_1,\ z_2 = y_1 w_2,\ \ldots,\ z_{|W_I|} = y_1 w_{|W_I|},\ z_{|W_I|+1} = y_2 w_1,\ z_{|W_I|+2} = y_2 w_2 \ldots .$$

Using Lemma 2.1 we can see that also $z_1, z_2, z_3 \ldots$ is an enumeration of W which refines the Bruhat order.

We denote by $\Gamma^I_{\geqslant m}$ the functor $\Gamma^I_{\{y_i : i \geqslant m\}}$ on $\mathbb{S}Bim^I$ and by $\Gamma_{\geqslant m}$ the functor $\Gamma_{\{z_i : i \geqslant m\}}$ on $\mathbb{S}Bim$. For $l \geqslant k$, let

$$\Gamma^I_{\geqslant k/\geqslant l} B := (\Gamma^I_{\geqslant k} B)/(\Gamma^I_{\geqslant l} B).$$

We define similarly $\Gamma_{\geqslant k/\geqslant l}$, $\Gamma^I_{\leqslant k/\leqslant l}$ and $\Gamma_{\leqslant k/\leqslant l}$.

All the functors above ($\Gamma^I_{\geqslant x}, \Gamma^I_{\geqslant x/>x}$, etc.), extend to functors between the respective homotopy categories, for example, the functor $\Gamma^I_{\geqslant k/\geqslant l}$ extends to a functor

$$\mathcal{K}^b(\mathbb{S}Bim^I) \to \mathcal{K}^b(R\text{-Mod-}R^I)$$

which we denote again simply by $\Gamma^I_{\geqslant k/\geqslant l}$.

For $x \in W$, let R_x denote the corresponding standard bimodule (cf. [5]) and for $x \in W^I$ let $R_{x,I} := (R_x)_I$. Fix $y = y_m W_I \in W/W_I$ and $x \in W^I$. Let $k = |W_I|(m-1)+1$, so that we have $y_m = z_k$ and $y_{m+1} = z_{k+|W_I|}$. Then, we have

$$\Gamma^I_{\geqslant y/>y}(F^I_x) \cong \Gamma^I_{\geqslant y/>y}(F_{x,I}) \cong \Gamma^I_{\geqslant m/\geqslant m+1}(F_{x,I}) \cong (\Gamma_{\geqslant k/\geqslant k+|W_I|} F_x)_I \in \mathcal{K}^b(R\text{-Mod-}R^I) \tag{8}$$

where the second isomorphism follows from the hin-und-her Lemma for singular Soergel bimodules [16, Lemma 6.3.2] and the first and third isomorphism from (7).

For any i such that $0 \leqslant i \leqslant |W_I| - 1$ we have an exact sequence of complexes of R-bimodules.

$$0 \to \Gamma_{\geqslant k+i/\geqslant k+i+1} F_x \to \Gamma_{\geqslant k/\geqslant k+i+1} F_x \to \Gamma_{\geqslant k/\geqslant k+i} F_x \to 0. \tag{9}$$

Notice that, in general, a short exact sequence of complexes does not induce a distinguished triangle in $\mathcal{K}^b(R\text{-Mod-}R)$, as the following example illustrates.

Example 4.2. Let for $s \in S$ be a simple reflection. Consider the following exact sequence of complexes

$$0 \to \Gamma_{\geqslant s} F_s \xrightarrow{\iota} F_s \to F_s/\Gamma_s F_s \to 0. \tag{10}$$

After we expand it, we obtain the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma_{\geq s} B_s \cong R_s(-1) & \longrightarrow & B_s & \longrightarrow & R(1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & R(1) & \longrightarrow & R(1) & \longrightarrow & 0 \end{array} \tag{11}$$

Notice that $F_s/\Gamma_s F_s$ is trivial as an object in the homotopy category $\mathcal{K}^b(R\text{-Mod-}R)$ and the map $\iota : \Gamma_{\geqslant s} F_s \to F_s$ is a quasi-isomorphism. However, it is easy to see that the map ι is not a homotopy equivalence, that is, ι does not induce an isomorphism in $\mathcal{K}^b(R\text{-Mod-}R)$.

After we restrict to $R\text{-Mod-}R^s$, the rows in (11) become split. Hence, the map ι induces an isomorphism in $\mathcal{K}^b(R\text{-Mod-}R^s)$ and the short exact sequence (10) induces a distinguished triangle in $\mathcal{K}^b(R\text{-Mod-}R^s)$.

As the next Lemma shows, the situation in general is similar to Example 4.2: after restricting to $\mathcal{K}^b(R\text{–Mod-}R^I)$, sequence (9) does indeed induce a distinguished triangle.

Lemma 4.3. *The restriction to R-Mod-R^I of the exact sequence of complexes* (9) *is termwise split (i.e., every row is split exact).*

Notice that for $I = \emptyset$ this statement is trivial since we assumed $0 \leqslant i \leqslant |W_I| - 1$.

Proof. Each term in $\Gamma_{\geqslant k+i/\geqslant k+i+1} F_x$ is isomorphic to direct sums of shifts of $R_{y_m w_i}$. By induction on i, each term in $\Gamma_{\geqslant k/\geqslant k+i} F_x$ can be obtained as an extensions of the standard modules $R_{y_m w_j}$, with $j < i$. By [16, Lemma 6.2.4], all the extensions between $R_{y_m w_i}$ and $R_{y_m w_j}$ with $j \neq i$ become split after restricting to R-Mod-R^I. It follows that the exact sequence (9) becomes termwise split after restricting to R-Mod-R^I. □

Hence, we have the following distinguished triangle in $\mathcal{K}^b(R\text{-Mod-}R^I)$:

$$(\Gamma_{\geqslant k+i/\geqslant k+i+1} F_x)_I \to (\Gamma_{\geqslant k/\geqslant k+i+1} F_x)_I \to (\Gamma_{\geqslant k/\geqslant k+i} F_x)_I \xrightarrow{[1]} . \tag{12}$$

Recall from [8, Prop 3.7] the following crucial statement about Rouquier complexes. For any $x, y \in W$ we have

$$\Gamma_{\geqslant y/>y}(F_x) = \begin{cases} 0 & \text{if } y \neq x, \\ R_x(-\ell(x)) & \text{if } y = x. \end{cases} \tag{13}$$

We can now prove the singular analogue of (13).

Lemma 4.4. *Let $x, y \in W^I$. Then*

$$\Gamma^I_{\geqslant y/>y}(F^I_x) = \begin{cases} 0 & \textit{if } y \neq x, \\ R_{x,I}(-\ell(x)) & \textit{if } y = x. \end{cases}$$

Proof. Let m and k be such that $y_m = z_k = y$. By (8) we have

$$\Gamma^I_{\geqslant y/>y}(F^I_x) \cong (\Gamma_{\geqslant k/\geqslant k+|W_I|} F_x)_I \in \mathcal{K}^b(R\text{-Mod-}R^I).$$

First assume $x \neq y$. Then $x = z_j$ with $j < k$ or $j \geqslant k + |W_I|$. For any i with $0 \leqslant i \leqslant |W_I| - 1$, we have $(\Gamma_{\geqslant k+i/\geqslant k+i+1} F_x)_I \cong 0$ by (13). Then, using (12) we obtain by induction $\Gamma_{\geqslant k/\geqslant k+|W_I|} F_x \cong 0$.

Assume now $x = y$, so that $x = z_k$. In this case, by (13) we have

$$(\Gamma_{\geqslant k/\geqslant k+1} F_x)_I \cong R_{x,I}(-\ell(x)) \quad \text{and} \quad (\Gamma_{\geqslant k+i/\geqslant k+i+1} F_x)_I \cong 0 \;\text{ for any } 0 < i < |W_I|.$$

Again, we can use (12) to obtain $(\Gamma_{\geqslant k/\geqslant k+|W_I|} F_x)_I \cong R_{x,I}(-\ell(x))$. □

Dually, given $x \in W^I$ we can define the complexes E^I_x as the minimal complex of $(E_{s_1}E_{s_2}\dots E_{s_k})_I$, where $s_1s_2\dots s_k$ is any reduced expression of x (the complex E_s is defined in (6)). Similar arguments to those above show that for any $x, y \in W^I$ we have

$$\Gamma^I_{\leqslant y,<y}(E^I_x) = \begin{cases} 0 & \text{if } y \neq x, \\ R_{x,I}(\ell(x)) & \text{if } y = x. \end{cases}$$

As in [8], we can define the *augmented singular Rouquier complexes* as

$$\widetilde{F^I_x} := \operatorname{cone}(f_x) \quad \text{where} \quad f_x : R_{x,I}(-\ell(x)) = H^0(F^I_x) \to F^I_x,$$

$$\widetilde{E^I_x} := \operatorname{cone}(e_x) \quad \text{where} \quad e_x : E^I_x \to R_{x,I}(\ell(x)) = H^0(E^I_x).$$

We write $\operatorname{Hom}_{\mathcal{K}}(-,-)$ to denote the morphisms in $\mathcal{K}^b(R\text{-Mod-}R^I)$. Combining [16, Theorem 7.4.1] and Lemma 4.4 we obtain, by the same argument of [8, Corollary 3.10], the following result.

Corollary 4.5. *For any $H \in \mathcal{K}^b(\mathbb{S}Bim^I)$ we have*

$$\operatorname{Hom}_{\mathcal{K}}(H, \widetilde{E^I_x}) = 0 = \operatorname{Hom}_{\mathcal{K}}(\widetilde{F^I_x}, H).$$

We also obtain a generalization of [8, Theorem 1.1].

Lemma 4.6. *For any $x, y \in W^I$ and $m \in \mathbb{Z}$ we have*

$$\operatorname{Hom}_{\mathcal{K}}(F^I_x, E^I_y[m]) \cong \begin{cases} R_I & \textit{if } x = y \textit{ and } m = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We apply the homological functor to $\operatorname{Hom}_{\mathcal{K}}(F^I_x, -)$ to the triangle

$$E^I_y \to R_{y,I}(\ell(y)) \to \widetilde{E^I_y} \xrightarrow{[1]} .$$

It follows from Corollary 4.5 that for any $m \in \mathbb{Z}$ we have

$$\operatorname{Hom}_{\mathcal{K}}(F^I_x, R_{y,I}(\ell(y))[m]) \cong \operatorname{Hom}_{\mathcal{K}}(F^I_x, E^I_y[m]).$$

Similarly, applying the cohomological functor $\operatorname{Hom}_{\mathcal{K}}(-, E^I_y[m])$ we also obtain

$$\operatorname{Hom}_{\mathcal{K}}(R_{x,I}(-\ell(x)), E^I_y[m]) \cong \operatorname{Hom}_{\mathcal{K}}(F^I_x, E^I_y[m]).$$

Notice that all the summands of ${}^iF^I_x$ and of ${}^iE^I_x$ are of the form $P_z(m_z)$ for some $z \leqslant x$ and, moreover, we have $z < x$ if $i \neq 0$. In particular, if $y \nleqslant x$ we have $\operatorname{Hom}({}^iF^I_x, R_{y,I}(\ell(y)) = 0$ for all i, hence

$$\operatorname{Hom}_{\mathcal{K}}(F^I_x, R_{y,I}(\ell(y)[m]) = 0$$

for all m. Dually, if $x \nleqslant y$ we have $\operatorname{Hom}(R_{x,I}(-\ell(x)), {}^iE^I_y) = 0$ for all i, hence

$$\operatorname{Hom}_{\mathcal{K}}(R_{x,I}(-\ell(x)), E^I_y[m]) = 0$$

for all $m \in \mathbb{Z}$.

It remains to consider the case $x = y$. If $m \neq 0$, we have $\mathrm{Hom}_{\mathcal{K}}(R_{x,I}(-\ell(x)), E_x^I[m]) = 0$ since all the summands in ${}^mE_x^I$ are smaller than P_x. If $m = 0$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{K}}(R_{x,I}(-\ell(x)), E_x^I) &\cong \mathrm{Hom}(R_{x,I}(-\ell(x)), {}^0E_x^I)\\ &\cong \mathrm{Hom}(R_{x,I}(-\ell(x)), \Gamma^I_{\leqslant x/<x}({}^0E_x^I))\\ &= \mathrm{Hom}(R_{x,I}(-\ell(x)), R_{x,I}(-\ell(x))) = R_I.\end{aligned}$$

□

4.2 | Singular Rouquier complexes and the support filtration

The homological properties of (singular) Rouquier complexes observed in the last section turn out to be useful to understand the support filtration of (singular) Rouquier complexes.

Let $x \in W^I$ and consider a reduced expression $\underline{x} = s_1 s_2 \dots s_k$. The bimodule P_x is a direct summand of $BS(\underline{x})_I = {}^0(F_{s_1}F_{s_2}\dots F_{s_k})_I$, but it is not a direct summand of ${}^i(F_{s_1}F_{s_2}\dots F_{s_k})_I$ for any $i > 0$. Hence P_x must also be a direct summand of ${}^0F_x^I$. Similarly P_x is a direct summand of ${}^0E_x^I$.

Lemma 4.7. *Let $x, y \in W^I$ with $y < x$ and $m \in \mathbb{Z}$. Then every map $P_y(m) \xrightarrow{\varphi} P_x$ factors through ${}^{-1}E_x^I$.*

Proof. After choosing a decomposition ${}^0E_x^I = P_x \oplus ({}^0E_x^I)'$, the map φ induces a map $\varphi : P_y(m) \to {}^0E_x^I$. By Corollary 4.5 we have an exact sequence

$$\mathrm{Hom}(P_y(m), {}^{-1}E_x^I) \to \mathrm{Hom}(P_y(m), {}^0E_x^I) \to \mathrm{Hom}(P_y(m), R_{x,I}(\ell(x))) \to 0.$$

The claim now follows since $\mathrm{Hom}(P_y(m), R_{x,I}(\ell(x))) = 0$ for $y < x$. □

Remark 4.8. Now let $x, y \in W^I$ be arbitrary. Choose a decomposition ${}^0E_x^I = P_x \oplus ({}^0E_x^I)'$ as above. Since $\mathrm{Hom}^\bullet(({}^0E_x^I)', R_{x,I}) = 0$, by Corollary 4.5 we also have an exact sequence of graded rings

$$\mathrm{Hom}^\bullet(P_y, {}^{-1}E_x^I) \xrightarrow{\vartheta} \mathrm{Hom}^\bullet(P_y, P_x) \to \mathrm{Hom}^\bullet(P_y, R_{x,I}(\ell(x))) \to 0.$$

We claim that the image of the map ϑ is $\mathrm{Hom}^\bullet_{<x}(P_y, P_x)$. In fact, if a map $P_y \to P_x(k)$ factors through $P_z(k')$ for some $z < x$, then by Lemma 4.7 it also factors through ${}^{-1}E_x^I(k)$. As a consequence, we have

$$\mathrm{Hom}^\bullet_{\not< x}(P_y, P_x) \cong \mathrm{Hom}^\bullet(P_y, R_{x,I}(\ell(x))) \cong \mathrm{Hom}^\bullet(\Gamma^I_{\geqslant x/>x} P_y, R_{x,I})(-\ell(x))$$

where the second isomorphism is [16, Theorem 7.3.5 (ii)]. So we can give an equivalent definition of the character map $\mathrm{ch} : [\mathbb{S}Bim^I] \to \mathcal{H}^I$ defined in (4) via

$$\mathrm{ch}([P]) = \sum_{x \in W^I} \mathrm{grrk}\, \mathrm{Hom}^\bullet(P, R_{x,I}) v^{-\ell(x)} \mathbf{H}_x^I = \sum_{x \in W^I} \overline{\mathrm{grrk}(\Gamma^I_{\geqslant x/>x} P)} v^{\ell(x)} \mathbf{H}_x^I \tag{14}$$

where $\overline{(-)} : \mathbb{Z}[v, v^{-1}] \to \mathbb{Z}[v, v^{-1}]$ is the automorphism defined by $\overline{v} = v^{-1}$.

We can use Lemma 4.4 to give a useful characterization of the support filtration. For $x \in W^I$, the elements in P_x of degree $-\ell(x)$ form a one-dimensional vector space. Let $c_{\mathrm{bot}} \in P_x$ be a non-zero element of this vector space.

Lemma 4.9. *Let $P \in \mathbb{S}Bim^I$ and $y \in W^I$. Then*

$$\Gamma^I_{\leqslant y} P = \mathrm{span}_R \langle \varphi(c_{bot}) \mid \varphi \in \mathrm{Hom}^\bullet(P_y, P)\rangle. \tag{15}$$

Proof. For $b \in P_y$, we clearly have $\mathrm{supp}\, \varphi(b) \subseteq \mathrm{supp}\, b \subseteq \{\leqslant y\}$, hence the inclusion $\supseteq$ in (4.9) follows. We show now the reverse inclusion.

If $y \geqslant z$ we claim that there exists a morphism $\psi : P_y \to P_z(\ell(y) - \ell(z))$ such that $\psi(c_{\mathrm{bot}}) = c_{\mathrm{bot}}$.

Let $\underline{y}$ and $\underline{z}$ be reduced words for y and z. Then the inclusion $P_y \hookrightarrow BS(\underline{y})$ (respectively the projection $BS(\underline{z}) \twoheadrightarrow P_z$) is an isomorphism in degree $-\ell(y)$ (respectively $-\ell(z)$). We can define $\psi : P_y \to P_z(\ell(y) - \ell(z))$ as the composition

$$P_y \hookrightarrow BS(\underline{y}) \xrightarrow{\widetilde{\psi}} BS(\underline{z})(\ell(y) - \ell(z)) \twoheadrightarrow P_z(\ell(y) - \ell(z)),$$

where $\widetilde{\psi} : BS(\underline{y}) \to BS(\underline{z})(\ell(y) - \ell(z))$ is the morphism corresponding to the unique light leaf with only ups in its stroll (see, e.g., [10, Lemma 30] for a construction of the map $\widetilde{\psi}$).

So, we can now replace the RHS in (15) with

$$\mathrm{span}_R \langle \varphi(c_{\mathrm{bot}}) \mid \varphi \in \mathrm{Hom}^\bullet(P_z, P) \text{ for some } z \leqslant y\rangle.$$

It is enough to show the claim for P indecomposable, that is, $P = P_x$ for some $x \in W^I$. Since

$$\Gamma^I_{\leqslant y}(P_x) = \bigcup_{z \leqslant x \text{ and } z \leqslant y} \Gamma^I_{\leqslant z} P_x,$$

it is enough to consider the case $y \leqslant x$. Let $b \in \Gamma^I_{\leqslant y}(P_x)$. Consider the singular Rouquier complex E^I_x. If $y < x$, from $\Gamma_{\leqslant y} E^I_x \cong 0$ we deduce that $\Gamma_{\leqslant y}({}^{-1}E_x) \to \Gamma_{\leqslant y}(P_x)$ is surjective. Moreover, every direct summand in ${}^{-1}E_x$ is of the form $P_z(k)$ with $z < x$, so the claim easily follows by induction on $\ell(x)$.

If $y = x$ we have $\Gamma_{\leqslant x/<x} P_x \cong R_{x,I}(\ell(x))$, and it is generated by the image of c_{bot}. Hence for any $b \in P_x$ there exists $f \in R$ such that $b - f c_{\mathrm{bot}} \in \Gamma^I_{<y} P_x$. The claim now follows from the previous case. □

4.3 | Soergel's conjecture and the perverse filtration

For some of our applications we need Soergel's conjecture to hold for our representation $\mathfrak{h}^*$. To ensure this, we require that the results of [5] are available, that is, we require that $\mathbb{K} = \mathbb{R}$ and assume that $\mathfrak{h}^*$ is a reflection faithful representation of W with a good notion of positive roots (cf. [2, § 2]). Such a representation always exist: see, for example, the construction given in [15, Prop 2.1] or in [12, Prop 1.1]. By [16, Theorem 3], Soergel's conjecture for Soergel bimodule [5] implies the corresponding result for singular Soergel bimodules:

Theorem 4.10. *Assume $\mathbb{K} = \mathbb{R}$ and $\mathfrak{h}^*$ as above. Then for $x \in W^I$ we have* $\mathrm{ch}(P_x) = \underline{\mathbf{H}}^I_x$.

With these assumptions, it follows from Theorem 3.3 that for $x > y$ we have

$$\mathrm{grrk}\,\mathrm{Hom}^\bullet_{\not< y}(P_x, P_y) = h^I_{y,x}(v)$$

and, as a consequence, for any $x, y \in W^I$

$$\mathrm{Hom}^i(P_x, P_y) \cong \begin{cases} 0 & \text{if } i < 0, \text{ or } i = 0 \text{ and } x \neq y \\ \mathbb{R} & \text{if } i = 0 \text{ and } x = y. \end{cases} \tag{16}$$

For any bimodule $P \in \mathbb{S}Bim^I$ we have a (non-canonical) decomposition

$$P = \bigoplus (P_x(i))^{\oplus m_{x,i}}, \tag{17}$$

and we can define the *perverse filtration* τ on P as

$$\tau_{\leqslant j} P = \bigoplus_{i \geqslant -j} (P_x(i))^{\oplus m_{x,i}}.$$

As a consequence of the vanishing of homomorphisms of negative degree (16), the perverse filtration does not depend on the choice of the decomposition in (17).

A bimodule $P \in \mathbb{S}Bim^I$ is said to be *perverse* if we can write $\mathrm{ch}([P]) = \sum_{x \in W^I} m_x \underline{\mathbf{H}}^I_x$ with $m_x \in \mathbb{Z}_{\geqslant 0}$ or, equivalently, if $\tau_{\leqslant -1} P = 0$ and $\tau_{\leqslant 0} B = B$.

Definition 4.11. We define ${}^p\mathcal{K}^{\geqslant 0}$ to be the full subcategory of $\mathcal{K}^b(\mathbb{S}Bim^I)$ with objects complexes in $\mathcal{K}^b(\mathbb{S}Bim^I)$ which are isomorphic to a complex F satisfying $\tau_{\leqslant -i-1}{}^i F = 0$ for all $i \in \mathbb{Z}$.

Similarly, we define ${}^p\mathcal{K}^{\leqslant 0}$ to be the full subcategory whose objects are complexes in $\mathcal{K}^b(\mathbb{S}Bim^I)$ which are isomorphic to a complex F satisfying ${}^i F = \tau_{\leqslant -i}{}^i F$ for all $i \in \mathbb{Z}$.

Let ${}^p\mathcal{K}^0 := {}^p\mathcal{K}^{\geqslant 0} \cap {}^p\mathcal{K}^{\leqslant 0}$.

It follows from Theorems 4.10 and 3.3 that the pair $({}^p\mathcal{K}^{\leqslant 0}, {}^p\mathcal{K}^{\geqslant 0})$ defines a non-degenerate t-structure on $\mathcal{K}^b(\mathbb{S}Bim^I)$, called the *perverse t-structure*. We denote by ${}^p\mathcal{K}^0$ the *heart* of this t-structure. One should regard ${}^p\mathcal{K}^0$ as the category of equivariant mixed perverse sheaves on the (possibly non-existent) partial flag variety associated with I.

It is clear that the following statement analogous to [5, Lemma 6.1] holds in the singular setting: for a distinguished triangle

$$F' \to F \to F'' \xrightarrow{[1]}$$

in $\mathcal{K}^b(\mathbb{S}Bim^I)$, if $F', F'' \in {}^p\mathcal{K}^{\geqslant 0}$ (respectively $\mathcal{K}^{\leqslant 0}$), then $F \in {}^p\mathcal{K}^{\geqslant 0}$ (respectively $\mathcal{K}^{\leqslant 0}$).

Lemma 4.12. *Given a Rouquier complex $F_x \in \mathcal{K}^b(\mathbb{S}Bim)$, the functor*

$$F_x \otimes (-) : \mathcal{K}^b(\mathbb{S}Bim^I) \to \mathcal{K}^b(\mathbb{S}Bim^I)$$

is left t-exact with respect to the perverse t-structure, that is, it restricts to a functor ${}^p\mathcal{K}^{\geqslant 0} \to {}^p\mathcal{K}^{\geqslant 0}$.

Proof. We can assume $x = s \in S$. Since the category ${}^p\mathcal{K}^{\geqslant 0}$ is generated under extensions by the objects $P_y(m)[n]$, with $y \in W^I$ and $m + n \leqslant 0$ it is enough to show that $F_s P_y \in {}^p\mathcal{K}^{\geqslant 0}$ for all $y \in W^I$. We divide the proof into two cases:

(i) Assume $syw_I > yw_I$. We have $\mathrm{ch}(B_s P_y) = \underline{\mathbf{H}}_s \underline{\mathbf{H}}^I_y = \underline{\mathbf{H}}^I_{sy} + \sum_{\substack{z \in W^I \\ z < ys}} m_z \underline{\mathbf{H}}^I_z$ with $m_z \in \mathbb{Z}_{\geqslant 0}$. From Theorem 4.10, we get

$$B_s P_y \cong B^I_{sy} \oplus \bigoplus_{\substack{z \in W^I \\ z < sy}} (P_z)^{\oplus m_z}$$

and the complex

$$F_s P_y = [0 \to \overset{0}{B_s P_y} \to P_y(1) \to 0]$$

is manifestly in ${}^p\mathcal{K}^{\geqslant 0}$.

ii) Assume $syw_I < yw_I$. Then we have $\mathrm{ch}(B_s P_y) = \underline{\mathbf{H}}_s \underline{\mathbf{H}}^I_y = \underline{\mathbf{H}}_s \underline{\mathbf{H}}_{yw_I} = (v + v^{-1}) \underline{\mathbf{H}}^I_y$. Therefore $B_s P_y \cong P_y(1) \oplus P_y(-1)$ and

$$F_s P_y = [0 \to P_y(1) \overset{0}{\oplus} P_y(-1) \to P_y(1) \to 0].$$

Tensoring with F_s induces an equivalence on the category $\mathcal{K}^b(\mathbb{S}Bim^I)$. Since P_y is indecomposable, the complex $F_s P_y$ must also be indecomposable. Therefore, the map $\overset{0}{P_y(1)} \to P_y(1)$ cannot be trivial, otherwise $P_y(1)$ would be a non-trivial direct summand of $F_s P_y$. Since $P_y(1) \to P_y(1)$ is non-zero, it is an isomorphism by (16) and $P_y(1) \to P_y(1)$ is a contractible direct summand. Removing this contractible summand we obtain $F_s P_y \cong P_y(-1) \in {}^p\mathcal{K}^{\geqslant 0}$.

□

Corollary 4.13. *For any $x \in W^I$ we have $F^I_x \in {}^p\mathcal{K}^{\geqslant 0}$.*

Proof. This easily follows from Lemma 4.12 since $R_I \in {}^p\mathcal{K}^{\geqslant 0}$ and $F^I_x \cong F_x \otimes R_I$ in $\mathcal{K}^b(\mathbb{S}Bim^I)$. □

4.4 | Singular Rouquier complexes are linear

When Soergel's conjecture holds, we can describe quite explicitly the singular Rouquier complexes. (This explicit description is a crucial tool in [11] where the case of Grassmannians is studied in detail.)

Lemma 4.14. *Let $x \in W^I$ and $i > 0$. If ${}^iF^I_x$ contains a direct summand isomorphic to $P_z(j)$, then ${}^{i-1}F^I_x$ contains a direct summand isomorphic to $P_{z'}(j')$ with $z' > z$ and $j' < j$.*

Proof. The proof is basically the same as in [5, Lemma 6.11]. From Theorem 4.10 and (14) we see that, for any $y, z \in W^I$ such that $y > z$, the bimodule $\Gamma^I_{\geqslant z/>z}(P_y)$ is generated in degree $< \ell(z)$. Moreover, we have $\Gamma^I_{\geqslant y/>y}(P_y) \cong R_{y,I}(-\ell(y))$.

The image of $P_z(j)$ in ${}^{i+1}F^I_x$ is contained in $\tau_{<-j}({}^{i+1}F^I_x)$ because of (16): in fact any non-zero homomorphism in degree 0 is an isomorphism and thus yields a contractible direct summand.

Applying $\Gamma^I_{\geqslant z/>z}$ to F^I_x, the direct summand $P_z(j)$ returns a summand $R_{z,I}(j-\ell(z))$. This cannot be a direct summand in $\Gamma^I_{\geqslant z/>z}(\tau_{<-j}{}^{i+1}F^I_x)$, and cannot survive in the cohomology of the complex because of Lemma 4.4. Thus $R_{z,I}(j-\ell(z))$ must be the image of a direct summand $R_{z,I}(j-\ell(z))$ in $\Gamma_{\geqslant z/>z}(\tau_{>-j}({}^{i-1}F_x))$.

This implies that there is a direct summand $P_{z'}(j')$ in ${}^{i-1}F_x$ with $z'>z$ and $j'<j$. □

Theorem 4.15. *Let $x\in W^I$ and let F^I_x be a singular Rouquier complex. Then:*

(i) ${}^0F^I_x = P_x$.
(ii) *For $i\geqslant 1$, ${}^iF^I_x=\bigoplus(P_z(i))^{\oplus m_{z,i}}$ with $z<x$, $z\in W^I$ and $m_{z,i}\in\mathbb{Z}_{\geqslant 0}$.*

In particular, $F^I_x\in {}^p\mathcal{K}^0$.

Proof. We can use the same argument as in Lemma 4.14 to deduce that, since ${}^{-1}(F^I_x)=0$ and $\Gamma_{\geqslant x/>x}F^I_x\cong R_{x,I}(-\ell(x))$, we must have ${}^0(F^I_x)\cong P_x$. By induction on i we get ${}^iF^I_x=\tau_{\leqslant -i}F^I_x$ for any $i>0$. Now ii) follows since we already know $F^I_x\in{}^p\mathcal{K}^{\geqslant 0}$ from Corollary 4.13. □

Remark 4.16. We can define the character of a complex $F\in\mathcal{K}^b(\mathbb{S}Bim^I)$ by

$$\mathrm{ch}(F)=\sum_{i\in\mathbb{Z}}(-1)^i\,\mathrm{ch}({}^iF)\in\mathcal{H}.$$

If $x\in W^I$ and $\underline{x}=s_1s_2\dots s_k$ is a reduced expression we have

$$\mathrm{ch}(F^I_x)=\mathrm{ch}((F_{s_1}F_{s_2}\dots F_{s_k})_I)=\mathbf{H}_x\underline{\mathbf{H}}_I=:\mathbf{H}^I_x.$$

An immediate consequence of Theorem 4.10 is that there is a non-trivial morphism of degree i between P_x and P_y for $x,y\in W^I$ only if i and $\ell(x)-\ell(y)$ have the same parity. Therefore for all summands $P_y(i)\Subset^i F^I_x$ the number $i-\ell(y)+\ell(x)$ is even. Because of Theorem 4.15 we can write

$$\mathbf{H}^I_x=\sum_{i\geqslant 0}(-1)^i\,\mathrm{ch}({}^iF_x)=\sum_{y\leqslant x}(-1)^{\ell(y)-\ell(x)}g^I_{y,z}(v)\underline{\mathbf{H}}^I_y \tag{18}$$

with $g_{x,x}(v)=1$ and $g_{y,x}(v)=\sum_{i>0}m_{y,i}v^i\in v\mathbb{N}[v]$. The polynomials $g^I_{x,y}$ are called the *I-parabolic inverse Kazhdan–Lusztig polynomials*, and they are also determined by the following *inversion formula*:

$$\sum_{y\in W^I}(-1)^{\ell(y)-\ell(x)}g^I_{x,y}(v)h^I_{y,z}(v)=\delta_{x,z}. \tag{19}$$

One can use (18) to deduce that the I-parabolic inverse Kazhdan–Lusztig polynomials $g^I_{x,y}(v)$ have non-negative coefficients.

By a dual argument we have that ${}^iE^I_x(i)\cong{}^iF^I_x(-i)$ for all i, so in particular also $E^I_x\in{}^p\mathcal{K}^0$.

By looking at the coefficient of v in (19) we see that $m_{z,1}$, the coefficient of v in $g^I_{z,x}(v)$, equals the coefficient of v in $h^I_{z,x}(v)$, hence they both coincide with $\dim\mathrm{Hom}^1(P_z,P_x)$. In particular, we have

$$
{}^{-1}E_x^I = \bigoplus_z (P_z(-1))^{\oplus m_{z,1}}. \tag{20}
$$

We denote by d_x^i, for $i < 0$, the differentials in the complex E_x^I. We now describe the first differential d_x^{-1}.

Lemma 4.17. *Let $x \in W^I$. For any $z \in W^I$ fix a basis $\{\varphi_i^z\}_{i=1}^{m_{z,1}}$ of $\mathrm{Hom}^1(P_z, P_x)$. Then there exists an isomorphism $K : (P_z(-1))^{\oplus m_{z,1}} \xrightarrow{\sim} {}^{-1}E_x^I$ such that the following diagram*

$$
\begin{array}{ccc}
{}^{-1}E_x^I & \xleftarrow[{}^0K]{\sim} & \bigoplus_z (P_z(-1))^{\oplus m_{z,1}} \\
\Big\downarrow d_x^{-1} & \swarrow \bigoplus_{z,i} \varphi_i^z & \\
{}^0E_x^I = P_x & &
\end{array}
$$

commutes.

Proof. Let $P := \bigoplus_z (P_z(-1))^{\oplus m_{z,1}}$ and consider the map $\varphi := \bigoplus_{z,i} \varphi_i^z : P \to P_x$. Then φ induces a map of complexes concentrated in homological degree 0

$$
\varphi : F := \bigoplus_{z,i} F_z^I(-1)^{\oplus m_{z,1}} \to E_x^I.
$$

From Lemma 4.6 we see that φ is homotopic to 0. Let K be a chain homotopy between ϕ and 0, with ${}^iK : {}^iF \to {}^{i-1}E_x^I$. Then ${}^1K : {}^1F \to {}^0E_x^I = P_x$ must be trivial by (16) since ${}^1F$ is perverse. It follows that $d_x^{-1} \circ {}^0K = \varphi$, where ${}^0K : P \to {}^{-1}E_x^I$.

Since $\{\varphi_i^z\}_i$ is a basis, the map ${}^0K$ cannot vanish on any direct summand of P. Notice that ${}^0K$ is of degree 0, therefore ${}^0K$ is a split injection. Since $P \cong {}^{-1}E_x$ by (20), we conclude that ${}^0K$ is an isomorphism. □

Remark 4.18. In [16], Williamson also developed the theory of two-sided (I, J)-singular Soergel bimodules ${}^J\mathbb{S}Bim^I$ for any pair of finitary subsets $I, J \subset S$. It seems natural to define a two-sided singular Rouquier complex ${}^IF_x^J$ as the minimal subcomplex of the restriction of an ordinary Rouquier complex to a (R^J, R^I)-bimodule. However, there is no immediate analogue of Theorem 4.15 holding in this case as the following example shows.

We adopt the notation of [16]. Let W be a Weyl group of type A_3 with simple reflections $S = \{s, t, u\}$. Let $I = \{s, u\}$ and $J = \{t\}$. There are four classes of indecomposable of indecomposable (I, J)-singular Soergel bimodules, corresponding to the double cosets $\{id, st, ut, sut\} = W_J \backslash W / W_I$. We have

$$
\mathrm{ch}({}^JB_{id}^I) = \underline{\mathbf{H}}_{tus},\ {}^JB_{st}^I = \underline{\mathbf{H}}_{stus},\ {}^JB_{ut}^I = \underline{\mathbf{H}}_{utus} \text{ and } {}^JB_{sut}^I = \underline{\mathbf{H}}_{stusts}.
$$

From the Hom formula [16, Theorem 7.4.1], we see that

$$\mathrm{Hom}({}^J B^I_{sut}, P(1)) = 0$$

for all the perverse $P \in {}^J\mathbb{S}Bim^I$. Hence, if ${}^0({}^J F^I_{sut}) = {}^J B^I_{sut}$ and ${}^1 d_{sut} \neq 0$, the bimodule ${}^1({}^J F^I_{sut})$ cannot be perverse.

5 | HODGE THEORY OF SINGULAR SOERGEL BIMODULES

Once we have Lemma 4.1 at disposal, we can adapt almost word by word the arguments of [5] to the setting of singular Soergel bimodules. As the proof of the results in this section are completely analogous to [5, § 6.6] (but nevertheless rather long and technical) we do not carry out the details in this paper, but we refer to [9, Chapter 4] for exhaustive proofs.

We assume that we are in the setting of Section 4.3, so $\mathbb{K} = \mathbb{R}$ and Soergel's conjecture holds for $\mathfrak{h}^*$. We denote by $(\mathfrak{h}^*)^I \subset \mathfrak{h}^*$ the subspace of W_I-invariants. Let $\rho \in (\mathfrak{h}^*)^I \subseteq R^I$.

Definition 5.1. We say that $\rho \in (\mathfrak{h}^*)^I$ is *ample* if $\rho(\alpha_s^\vee) > 0$ for any $s \in S \setminus I$.

Note that there always exists an ample $\rho \in (\mathfrak{h}^*)^I$ since the set $\{\alpha_s^\vee\}_{s\in S}$ is linearly independent in $\mathfrak{h}^*$.

Theorem 5.2 (Hard Lefschetz theorem for singular Soergel bimodules). *Let $\rho \in (\mathfrak{h}^*)^I$ be ample. Then right multiplication by ρ induces a degree 2 map on $\overline{P_x} := \mathbb{R} \otimes_R P_x$ such that, for any $i > 0$ we have an isomorphism*

$$\rho^i : (\overline{P_x})^{-i} \to (\overline{P_x})^i.$$

Here $(\overline{P_x})^i$ denotes the degree i component of P_x.

The indecomposable bimodules P_x are self-dual, and moreover $\mathrm{Hom}(P_x, P_x) \cong \mathbb{R}$. This implies that there exists a unique (up to scalar) bilinear form

$$\langle -, - \rangle_{P_x} : P_x \times P_x \to R$$

such that for any $b, b' \in P_x$, $f \in R$ and $g \in R^I$ we have

$$\langle fb, b' \rangle_{P_x} = \langle b, fb' \rangle_{P_x} = f\langle b, b' \rangle_{P_x},$$

$$\langle bg, b' \rangle_{P_x} = \langle b, b'g \rangle_{P_x}.$$

Let $\rho \in (\mathfrak{h}^*)^I$ be ample. Then we fix the sign by requiring that $\langle b, b \cdot \rho^{\ell(x)} \rangle_{P_x} > 0$ for any $0 \neq b \in (P_x)^{-\ell(x)}$. We call $\langle -, - \rangle_{P_x}$ the *intersection form* of P_x.

The intersection form induces a real valued symmetric and R^I-invariant form $\langle -, - \rangle_{\overline{P_x}}$ on $\overline{P_x}$. For $i \geqslant 0$ we define the *Lefschetz form*

$$(-, -)^{-i}_\rho := \langle -, - \cdot \rho^i \rangle_{\overline{P_x}} : \overline{P_x}^{-i} \times \overline{P_x}^{-i} \to \mathbb{R}.$$

Theorem 5.3 (Hodge–Riemann bilinear relations for singular Soergel modules). *Let $x \in W^I$. For all $i \geqslant 0$ the restriction of Lefschetz form $(-,-)^{-i}_{\rho}$ to $\mathrm{Prim}(\rho, i) := \ker(\rho^{i+1}) \subseteq (\overline{P_x})^{-i}$ is $(-1)^{(-\ell(x)+i)/2}$-definite.*

Theorems 5.2 and 5.3 have also consequences for non-singular bimodules, allowing us to extend the hard Lefschetz theorem and the Hodge–Riemann relations "on the walls."

Let $x \in W$ and $s \in S$ be such that $xs > x$. Let $B_x \in \mathbb{S}Bim$ be the corresponding indecomposable (non-singular) Soergel bimodule. Assume $I = \{s\}$, so that $w_I = s$. Then $(B_x)_I$ is a perverse singular Soergel bimodule, in fact we have:

$$\mathrm{ch}((B_x)_I) = \underline{\mathbf{H}}_x \underline{\mathbf{H}}_s = \underline{\mathbf{H}}^I_x + \sum_{\substack{ys>y \\ y<x}} m_y \underline{\mathbf{H}}^I_y \qquad \text{with } m_y \in \mathbb{Z}_{\geqslant 0} \tag{21}$$

We obtain the following:

Corollary 5.4. *Let $I = \{s\}$ and $x \in W^I$, that is, $x \in W$ such that $xs > x$. Assume that $\rho \in (\mathfrak{h}^*)^s$ is ample, that is, $\rho(\alpha_s^\vee) = 0$ and $\rho(\alpha_t^\vee) > 0$ for all $t \neq s$. Then, right multiplication by ρ on $\overline{B_x}$ satisfies the hard Lefschetz theorem and the Hodge–Riemann bilinear relations.*

Proof. We have

$$(B_x)_I \cong P_x \oplus \bigoplus_{\substack{ys>y \\ y<x}} (P_y)^{\oplus m_y}. \tag{22}$$

By Theorem 5.2, the hard Lefschetz for ρ holds for $\overline{P_y}$ such that P_y is a direct summand in (22). It follows that multiplication by ρ satisfies the hard Lefschetz theorem on $\overline{B_{x,I}}$, hence on $\overline{B_x}$ since B_x and $B_{x,I}$ have the same underlying left R-module structure.

Let ϖ_s be a fundamental weight for s and let $\rho_\zeta = \rho + \zeta\varpi_s$ for $\zeta \geqslant 0$. We know from the non-singular case [5, Theorem 1.4], multiplication by ρ_ζ satisfies hard Lefschetz on B_x for all $\zeta \geqslant 0$ and Hodge–Riemann for every $\zeta > 0$. If the hard Lefschetz theorem holds, for a continuous family of operators, then the Hodge–Riemann bilinear relations are equivalent to a statement about the signature of the Lefschetz forms (see [5, Lemma 2.1]). Since the signature of a family of non-degenerate forms does not change, we deduce the Hodge–Riemann bilinear relations for $\rho_0 = \rho$. □

Remark 5.5. Corollary 5.4 has the following geometric motivation. Assume that W is the Weyl group of a complex semisimple group G. Let $x \in W$ be such that $xs > x$ for $s \in S$ and let $X_x \subseteq G/B$ be the corresponding Schubert variety. Let $\mathbf{P}_s$ be the minimal parabolic subgroup of G containing s. Then the restriction of the projection $G/B \to G/\mathbf{P}_s$ to X_x is semismall. It follows from [3, Theorem 2.3.1] that the pull-back of any ample class on $G/\mathbf{P}_s$ satisfies hard Lefschetz and Hodge–Riemann on X_x.

Remark 5.6. We can obtain from Corollary 5.4 an alternative proof of Soergel's conjecture, that translates more closely de Cataldo and Migliorini's proof of the decomposition theorem in [3].

Assume $w \in W$ such that $ws > w$ and assume $\mathrm{ch}(B_x) = \underline{\mathbf{H}}_x$ for all $x < ws$. Let $I = \{s\}$ and fix ρ ample in $(\mathfrak{h}^*)^s$. Let $x < w \in W$ be such that $xs > x$. Consider the primitive subspace

$$\mathrm{Prim}(\rho, k) := \ker(\rho^{k+1}) \cap (\overline{B_w})^{-k}.$$

We have a symmetric form

$$(-,-) : \mathrm{Hom}(P_x, (B_w)_I) \times \mathrm{Hom}(P_x, (B_w)_I) \to \mathrm{End}(P_x) \cong \mathbb{R}$$

defined by $(f, g) = g^* \circ f$, where g^* denotes the map adjoint to g with respect of the intersection forms. Then we can show, as in [5, Theorem 4.1], that the map

$$\iota : \mathrm{Hom}(P_x, (B_w)_I) \to \mathrm{Prim}(\rho, \ell(x))$$

defined by $f \mapsto f(c_{\mathrm{bot}})$ is injective. Moreover, if we equip $\mathrm{Prim}(\rho, \ell(x))$ with the Lefschetz form, then ι is an isometry (up to a positive scalar) and, by the Hodge–Riemann bilinear relations, the form $(-,-)$ is definite on $\mathrm{Hom}(P_x, (B_w)_I)$ and, in particular, non-degenerate. If $d = \dim \mathrm{Hom}(P_x, (B_w)_I)$, it follows that $(P_x)^{\oplus d}$ is a direct summand of $(B_w)_I$, hence $(B_{xs})^{\oplus d}$ is a direct summand of $B_w B_s$.

Example 5.7. In Corollary 5.4 it is crucial that $(B_x)_I$ is perverse when $I = \{s\}$ and $x < xs$ (cf. (21)). The analogous statement is false for larger parabolic subgroup, as this example illustrates.

Let W be the Weyl group of type A_3 with simple reflections labeled s, t, u. Let $I = \{s, t\}$, so that $w_I = sts$. Then $stu \in W^I$ but a simple computation in the Hecke algebra shows that

$$\underline{\mathbf{H}}_{stu}\underline{\mathbf{H}}_{sts} = \underline{\mathbf{H}}^I_{stu} + \underline{\mathbf{H}}^I_u + (v + v^{-1})\underline{\mathbf{H}}^I_{id}.$$

Therefore, the singular Soergel bimodule $(B_{stu})_I$ is not perverse, and there is no $\rho \in (\mathfrak{h}^*)^I$ which satisfies hard Lefschetz on $\overline{B_{stu}}$.

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JOURNAL INFORMATION

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