

Tidal Deformability of Neutron Stars in Scalar-Tensor Theories of Gravity for Gravitational Wave Analysis

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ABSTRACT

Gravitational waves from compact binary coalescence are valuable for testing theories of gravity in the strong field regime. By measuring neutron star tidal deformability in gravitational waves from binary neutron stars, stringent constraints were placed on the equation of state of matter at extreme densities. Tidal Love numbers in alternative theories of gravity may differ significantly from their general relativistic counterparts. Understanding exactly how the tidal Love numbers change will enable scientists to untangle effects from physics beyond general relativity from the uncertainty in the equation of state measurement. In this work, we explicitly calculate the fully relativistic $l \geq 2$ tidal love numbers for neutron stars in scalar-tensor theories of gravitation. We use several realistic equations of state to explore how the mass, radius, and tidal deformability relations differ from those of general relativity. We find that tidal Love numbers and tidal deformabilities can differ significantly from those in general relativity ($> 200\%$ in strong scalarization cases) in certain regimes. This difference suggests that using the tidal Love numbers from general relativity could lead to significant errors in tests of general relativity using the gravitational waves from binary neutron star and neutron-star–black-hole mergers.

1. INTRODUCTION AND MOTIVATION

Compact objects such as neutron stars and black holes are essential for testing general relativity in the strong field regime. Gravitational waves emitted by compact objects by LIGO-Virgo have improved our understanding of gravity in the strong field regime. The LIGO/Virgo collaboration has detected almost one-hundred compact binary coalescences to date: two binary neutron star mergers, two neutron star-black hole mergers, and more than eighty binary black hole mergers [Abbott et al. \(2019a, 2021a,b\)](#). An independent analysis of the available data found even more events [Nitz et al. \(2020, 2021a,b\)](#). Analysis of these events has already placed limits on possible deviations from general relativity (GR) [Abbott et al. \(2021c, 2019b\)](#); [Mehta et al. \(2022\)](#); [Wang et al. \(2021, 2022\)](#). Recently, waveforms for various alternate theories of gravity have been developed and applied to parameter estimation. These waveforms allow for stringent tests of various theories of gravity and more general tests for physics beyond general relativity, such as scalar and tensor propagation modes [Chatziioannou et al. \(2021\)](#); [Nair et al. \(2019\)](#); [Mirshekari & Will \(2013\)](#).

Neutron stars are also unique laboratories for studying nuclear physics at ultra-high densities. Information about neutron star matter is encoded in gravitational waves from binary neutron star and neutron star-black hole mergers. Neutron stars contain vital information needed to understand phases of matter encountered in Quantum Chromodynamics. The tidal deformability encodes information about the nuclear equation of state in gravitational wave (GW)s. Studies of binary neutron star merger GW170817 have improved our knowledge of the nuclear equation of state (EOS) [Capano et al. \(2020\)](#); [Radice & Dai \(2019\)](#); [Raaijmakers et al. \(2021\)](#); [Abbott et al. \(2018\)](#). Despite this, the nuclear EOS is still unknown. Studying neutron stars in alternative theories of gravity is challenging because deviations in neutron star properties caused by non-GR effects are of the same order of magnitude as the uncertainty in the EOS. Understanding how the mass-radius-tidal deformability relationships deviate from-GR is essential to untangling these differences.

Tidal deformability connects gravitational waves and the nuclear equation of state. Tidal deformabilities and the associated tidal Love numbers relate an applied external tidal field to the induced internal multipole moment, measuring the magnitude of deformation under a given tidal force. Love numbers were initially defined in Newtonian gravity [Love \(1909\)](#); [Shida \(1912\)](#) and then expanded to general relativity by Flanagan and Hinderer [Flanagan & Hinderer \(2008\)](#); [Hinderer \(2008\)](#). The concept was

further expanded and made more concrete in several follow-up papers, including Binnington and Poisson [Binnington & Poisson \(2009\)](#) and Damour and Nagar [Damour & Nagar \(2009\)](#).

This work focuses on scalar-tensor theory, one of the most natural and best studied alternate theories of gravity. The theory was initially motivated partly by Mach’s principle [Brans & Dicke \(1961\)](#) and partly in an attempt to expand GR to five dimensions [Jordan \(1955\)](#). However, it is still of interest today. Scalar degrees of freedom are critical for string theory, superstring theory, and other supergravity theories [Fujii & Maeda \(2003\)](#). Therefore scalar-tensor theories can sometimes be used as a phenomenological proxy for more complex extensions of GR. Furthermore, scalar fields have been proposed as an alternative solution to the dark energy problem [Garcí-Bellido & Quirós \(1990\)](#); [Boisseau et al. \(2000\)](#); [Clifton et al. \(2012\)](#).

Scalar-tensor theories add a massless scalar field (φ) to the standard GR metric ($g_{\mu\nu}$). The metric and the scalar field are coupled into an ‘effective metric’ $\tilde{g}_{\mu\nu} = A^2(\varphi)g_{\mu\nu}$. The earliest versions of this theory were presented more than half a century ago by Fierz, Jordan, and Brans and Dicke [Fierz \(1956\)](#); [Jordan \(1955\)](#); [Brans & Dicke \(1961\)](#). In the simplest scalar-tensor theory, known as FJBD (Fierz, Jordan, and Brans and Dicke), the scalar field is coupled to the metric by the coupling function $A(\varphi) = e^{\alpha\varphi}$. Solar system experiments have placed stringent constraints on the value of α . These constraints also significantly limit the strong-field behavior. Damour and Esposito-Ferè discovered the ‘spontaneous scalarization’ effect, which allows large deviations from GR in the strong field regime without violating the strict solar system constraints. Damour and Esposito-Ferè defined $A(\varphi) = e^{\beta\varphi^2/2}$ and found that scalarization occurs for $\beta \lesssim -4.5$ [Damour & Esposito-Farèse \(1993\)](#). A follow-up study showed that scalarization occurs for $\beta \lesssim -4.35$ [Harada \(1998\)](#).

In this work, we calculate the tidal Love numbers of neutron stars in scalar-tensor theories of gravity, focusing on the spontaneous scalarization case. [Sec. 2](#) presents the equilibrium configuration for neutron stars in scalar-tensor theory. [Sec. 3](#) discusses the first order linear time-independent perturbations upon which the tidal deformabilities depend. [Sec. 4](#) details the method for deriving the various tidal Love numbers. [Sec. 5](#) presents the results and demonstrates how the Love Numbers in scalar-tensor theories differ from those in GR. The paper concludes with [Sec. 6](#), which discusses the results.

2. NEUTRON STARS IN SCALAR-TENSOR THEORY

Scalar-tensor theories are a straightforward alternative to GR. They depend on both a metric tensor ($g_{\mu\nu}$) and a massless scalar field (φ) and are typically expressed in one of two conformal frames: the Einstein frame and the Jordan frame. Historically, there has been much debate over the correct choice of frame [Deruelle & Sasaki \(2011\)](#), but it is now agreed that experiments measure Jordan frame quantities even though the field equations simplify in the Einstein frame [Pani & Berti \(2014\)](#); [Palenzuela et al. \(2014\)](#); [Doneva et al. \(2013\)](#); [Barausse et al. \(2013\)](#); [Crisostomi & Koyama \(2018\)](#).

In the Jordan frame, the action is

$$S = \frac{1}{16\pi} \int \sqrt{-g} \left(\phi \tilde{R} - \frac{\omega(\phi)}{\phi} \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + 2\lambda(\phi) \right) + S_m[\Psi_m, \tilde{g}_{\mu\nu}] . \quad (1)$$

where the tilde denotes Jordan frame quantities, ϕ is the Jordan frame scalar field, $\tilde{g}_{\mu\nu}$ is metric, \tilde{R} is the Ricci scalar, and $\lambda(\phi)$ is the scalar potential. S_m denotes the action of the matter, which is a function of the matter fields Ψ_m and the Jordan metric $\tilde{g}_{\mu\nu}$. Due to the $\phi \tilde{R}$ term, the gravitational constant G becomes a function of the scalar field $\tilde{G} = G(\phi)$. Throughout this work, we will continue to denote Jordan frame quantities with a tilde.

The Jordan frame is the physical frame, but the field equations are typically expressed in the Einstein frame, where the metric and scalar decouple. A conformal transformation relates the two frames:

$$\tilde{g}_{\mu\nu} = A^2(\varphi)g_{\mu\nu} . \quad (2)$$

Using this transformation, the action can be re-written in a way that resembles the Einstein-Hilbert action:

$$S = \frac{1}{16\pi G_*} \int \sqrt{-g} (R - 2g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - 2\lambda(\varphi)) d^4x + S_m[\Psi_m, A^2(\varphi)g_{\mu\nu}] \quad (3)$$

where all quantities are related to the Einstein metric $g_{\mu\nu}$. R is the scalar curvature, G_* is the bare gravitational coupling constant which is set to 1, along with c , from here on. This paper will focus on the $\lambda(\varphi) = 0$ case.

Much of the work presented here is applicable for any $A(\varphi)$, but when necessary, the spontaneous scalarization coupling function [Damour & Esposito-Farèse \(1993\)](#) is used:

$$A(\varphi) = e^{1/2\beta\varphi^2} . \quad (4)$$

The modified field equations, derived from the Einstein frame action have the form

$$G_{\mu\nu} = 8\pi G_* T_{\mu\nu} + T_{\mu\nu}^{(\varphi)} \quad (5a)$$

$$\square_g \varphi = -4\pi G_* \alpha(\varphi) T, \quad (5b)$$

where $\alpha(\varphi) \equiv d \ln A(\varphi) / d\varphi$ and $T_{\mu\nu}^{(\varphi)}$ can be considered the stress-energy of the massless scalar field and has the form

$$T_{\mu\nu}^{(\varphi)} \equiv 2\partial_\mu \varphi \partial_\nu \varphi - g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi. \quad (6)$$

$T_{\mu\nu}$ is the stress-energy tensor in the Einstein frame, and T is the contracted stress-energy tensor $T = T^\mu_\mu = g^{\mu\nu} T_{\mu\nu}$. $T_{\mu\nu}$ is related to the Jordan frame stress-energy tensor ($\tilde{T}_{\mu\nu}$) in the following manner

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{|g|}} \frac{\delta S_m}{\delta g^{\mu\nu}} = A^2(\varphi) \tilde{T}_{\mu\nu} \quad (7)$$

Note setting $\alpha(\varphi)$ to zero, retrieves the GR field equations.

We model neutron stars as static, spherically symmetric, non-rotating objects and assume that neutron star matter can be described as a perfect fluid. The stress-energy tensor for a perfect fluid is defined in the physical frame

$$\tilde{T}_{\mu\nu} = (\tilde{\rho} + \tilde{p}) \tilde{u}_\mu \tilde{u}_\nu - \tilde{p} \tilde{g}_{\mu\nu} \quad (8)$$

where \tilde{u}_μ is the four-velocity of the fluid and $\tilde{\rho}$ and \tilde{p} are the energy density and pressure in the Jordan frame. We assume that $\tilde{\rho}$ and \tilde{p} are related by some barotropic equation of state so that

$$\delta \tilde{\rho} = \frac{d\tilde{p}}{d\tilde{\rho}} \delta \tilde{p}. \quad (9)$$

As the star is static, only the t component of the four-velocity is non-zero:

$$u^\mu = (e^{\nu/2}, 0, 0, 0) \quad (10)$$

Conservation of energy and momentum is defined in the physical or Jordan frame i.e., $\tilde{\nabla}_\mu \tilde{T}^\mu_\nu = 0$. Transforming to the Einstein Frame gives

$$\nabla_\nu T^\nu_\mu = \alpha(\varphi) T \nabla_\mu \varphi. \quad (11)$$

The metric for a static, spherically symmetric, self-gravitating object is

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -e^\nu dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \quad (12)$$

where ν and λ are functions of r and $e^{-\lambda} = 1 - 2\mu(r)/r$.

The modified Tolman–Oppenheimer–Volkoff (TOV) or structure equations, which can be derived from the field equations and the equation for conservation of energy, have the form

$$\frac{d\mu}{dr} = 4\pi G_* r^2 A^4(\varphi) \tilde{\rho} + \frac{1}{2} r(r - 2\mu) \psi^2 \quad (13a)$$

$$\frac{d\nu}{dr} = 8\pi G_* \frac{r^2 A^4(\varphi) \tilde{p}}{r - 2\mu} + r\psi^2 + \frac{2\mu}{r(r - 2\mu)} \quad (13b)$$

$$\frac{d\varphi}{dr} = \psi \quad (13c)$$

$$\frac{d\psi}{dr} = 4\pi G_* \frac{r A^4(\varphi)}{r - 2\mu} [\alpha(\varphi)(\tilde{\rho} - 3\tilde{p}) + r\psi(\tilde{\rho} - \tilde{p})] - \frac{2(r - \mu)}{r(r - 2\mu)} \psi \quad (13d)$$

$$\frac{d\tilde{p}}{dr} = -(\tilde{\rho} + \tilde{p}) \left[4\pi G_* \frac{r^2 A^4(\varphi) \tilde{p}}{r - 2\mu} + \frac{1}{2} r\psi^2 + \frac{\mu}{r(r - 2\mu)} + \alpha(\varphi)\psi \right] \quad (13e)$$

where μ is the mass function. $\psi = \partial_r \varphi$ is used throughout this paper for improved readability.

3. STATIONARY PERTURBATIONS

In this section, we compute the linear, time-independent scalar and spacetime perturbations following the method initially laid out by [Thorne & Campolattaro \(1967\)](#). The complete system of time-dependent perturbations in scalar-tensor theory was calculated in [Sotani & Kokkotas \(2005\)](#), and the perturbation equations in this section have been cross checked with the extant results.

We use the Regge-Wheeler gauge [Regge & Wheeler \(1957\)](#), which separates the metric perturbation $h_{\mu\nu}$ into its even and odd parity components $h_{\mu\nu} = h_{\mu\nu}^+ + h_{\mu\nu}^-$. [Sotani & Kokkotas \(2005\)](#) demonstrates that the metric in both frames can be written in the Regge-Wheeler gauge using the proper redefinition of the metric components between frames.

For this analysis, as we are interested in time-independent perturbations, all functions (H_0, H_2, K, h_0 , and h_1) are functions of r only. Furthermore, the tr term H_1 typically present in the Regge-Wheeler gauge vanishes.

The Einstein metric can be written in the following way:

$$h_{\mu\nu} = h_{\mu\nu}^+ + h_{\mu\nu}^- \quad (14)$$

where

$$h_{\mu\nu}^+ = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \begin{bmatrix} e^\nu H_{0,\ell m} & 0 & 0 & 0 \\ 0 & e^\lambda H_{2,\ell m} & 0 & 0 \\ 0 & 0 & r^2 K_{\ell m} & 0 \\ 0 & 0 & 0 & K_{\ell m} r^2 \sin^2 \theta \end{bmatrix} Y_{\ell m}(\theta, \phi) \quad (15)$$

and

$$h_{\mu\nu}^- = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \begin{bmatrix} 0 & 0 & -h_{0,\ell m} \sin^{-1} \theta \partial_\phi & h_{0,\ell m} \sin \theta \partial_\theta \\ 0 & 0 & -h_{1,\ell m} \sin^{-1} \theta \partial_\phi & h_{1,\ell m} \sin \theta \partial_\theta \\ sym & sym & 0 & 0 \\ sym & sym & 0 & 0 \end{bmatrix} Y_{\ell m}(\theta, \phi) \quad (16)$$

where $Y_{\ell m}(\theta, \phi)$ is the spherical harmonic function for l, m , and *sym* indicates that the metric is symmetric.

The explicit form of the conformal transformation between the Jordan and Einstein frame perturbation ($\tilde{h}_{\mu\nu} \rightarrow h_{\mu\nu}$) is needed to determine the Jordan frame tidal deformability. The conformal transformation is obtained by perturbing Eq. (2) and substituting in the Regge-Wheeler metric [Sotani & Kokkotas \(2005\)](#). This gives

$$h_{\mu\nu} = \frac{1}{A^2(\varphi)} \tilde{h}_{\mu\nu} - \frac{2}{A(\varphi)} g_{\mu\nu} \delta A. \quad (17)$$

where δA is the variation of the conformal factor; it is a function of the scalar field perturbation $\delta\varphi = \delta\varphi(r)$. The relationship between δA and $\delta\varphi$ depends on the functional form of the conformal factor. In the case of spontaneous scalarization $\delta A = \beta A \delta\varphi$. The explicit relationships between the individual metric perturbations are

$$\tilde{H}_{0,2} = \frac{1}{A^2(\varphi)} (H_{0,2} + 2A\delta A) \quad (18a)$$

$$\tilde{K} = \frac{1}{A^2(\varphi)} (K + 2A\delta A) \quad (18b)$$

$$\tilde{h}_{0,1} = \frac{1}{A^2(\varphi)} h_{0,1} \quad (18c)$$

We dropped the ℓm subscripts from H_0, H_2, K, h_0 , and h_1 for readability and will continue to do so throughout this work.

The complete set of perturbation equations needed to calculate the tidal deformability are laid out in Appendix B.

In GR, the full system of time-independent perturbed equations can be reduced to one differential equation for each parity: one for the even parity tensor perturbation (H) and one for the odd parity tensor perturbation (h). Scalar-tensor theories have an additional equation for the scalar field, which is of even parity. The metric and the scalar field are decoupled in the Einstein frame; therefore, the equations for H and $\delta\varphi$ decouple. The Jordan frame perturbation \tilde{H} depends on both H and $\delta\varphi$.

4. NEUTRON STAR TIDAL DEFORMABILITY

We derive and compute the scalar tensor tidal Love numbers and tidal deformabilities using the method developed by [Hinderer \(2008\)](#) and extended in [Binnington & Poisson \(2009\)](#); [Damour & Nagar \(2009\)](#).

Tidal deformabilities (e.g., λ) relate an applied external tidal field ($\mathcal{E}_{ij\dots\ell}$) to the induced multipole moment ($Q_{ij\dots\ell}$). To linear order in $\mathcal{E}_{ij\dots\ell}$, the tidal deformability is a proportionality constant between the two [Hinderer \(2008\)](#) i.e.,

$$Q_{ij\dots\ell} = -\lambda \mathcal{E}_{ij\dots\ell} . \quad (19)$$

Both $\mathcal{E}_{ij\dots\ell}$ and $Q_{ij\dots\ell}$ can be decomposed into tensor harmonics

$$\mathcal{E}_{ij\dots\ell} = \sum_{m=-\ell}^{\ell} \mathcal{E}_m Y_{\ell m}(\theta, \phi) \quad (20)$$

$$Q_{ij\dots\ell} = \sum_{m=-\ell}^{\ell} Q_m Y_{\ell m}(\theta, \phi) \quad (21)$$

where $Y_{\ell m}(\theta, \phi)$ are the even parity tensor spherical harmonics. This means that the tensor relation in Eq. (19) can be expressed as a scalar relation

$$Q_m = -\lambda \mathcal{E}_m . \quad (22)$$

To calculate λ it is sufficient to calculate one non-vanishing \mathcal{E}_m [Hinderer \(2008\)](#).

The external tidal field and the induced multipole moment affect space-time in and around the neutron star. Outside the star, the large r behavior of the metric can be written in terms of \mathcal{E}_{ij} and Q_{ij} [Hinderer \(2008\)](#); [Thorne \(1998\)](#). For example, the metric expansion for a spherically symmetric star of mass M in a quadrupolar tidal field \mathcal{E}_{ij} for large r is

$$g_{tt} = g_{tt}^0 + h_{tt} = -\left(1 - \frac{2\mu}{r}\right) + \frac{3Q_{ij}n^i n^j}{r^3} + \mathcal{O}(r^{-4}) - \mathcal{E}_{ij}n^i n^j r^2 + \mathcal{O}(r^3) \quad (23)$$

where n^i is the unit radial vector.

4.1. Electric Type Love Numbers

In general relativity, the electric or even parity Love numbers are calculated from the $g_{tt} = g_{tt}^{(0)} + h_{tt}$ component of the metric and are based on a single, second order linear differential equation for $H = H_0 = H_2$. However, in scalar-tensor theory, there are two types of even parity perturbations: scalar and tensor. The even parity metric tidal Love numbers k_ℓ define how the body responds to a change in the *metric*. The scalar tidal Love numbers κ_ℓ define how the body responds to a change in the applied *scalar field*. As the scalar field and the metric are not coupled in the Einstein frame, a change in the matter field does not induce a scalar perturbation, and vice-versa¹. The perturbation equations for the tidal Love number calculation must be derived carefully to first order in either the scalar perturbation or the metric perturbation but not both. This approach differs from previous approaches in [Yazadjiev et al. \(2018\)](#) and [Pani & Berti \(2014\)](#).

There are two master equations: one second order linear differential equation for the tensor perturbation $H = H_0 = H_2$, which comes from the perturbation of the field equation Eq. (5a), and one for the scalar perturbation $\delta\varphi$ which comes from the scalar wave equation Eq. (5b).

While the differential equation for φ can be derived directly from the scalar wave equation (see Eq. (B35)), the differential equation for H is derived from the a system of equations Eqs. (B29) to (B34) and is obtained by the following steps (which have been widely used in general relativity [Hinderer \(2008\)](#); [Damour & Nagar \(2009\)](#); [Binnington & Poisson \(2009\)](#)):

- Eq. (B29) $\longrightarrow H_0 = H_2 \equiv H$
- Eq. (B30) $\longrightarrow K_r = H_r + \nu_r H - 4\delta\varphi$
- Eq. (B31) $\longrightarrow K'' = H'' + \nu_r H' + \nu_{rr} H - 4\varphi'' \delta\varphi - 4\varphi' \delta\varphi'$
- Eq. (B33) $\longrightarrow \delta p = \frac{1}{2}(p + \rho)H$

¹ We thank Gastón Creci for his insights on this issue.

• Eq. (B34) $\longrightarrow H'' + c_1 H' + c_2 H = 0$

This gives

$$H'' + \left(\frac{2}{r} + \frac{1}{2}(\nu_r - \lambda_r) \right) H' - \left(e^\lambda \frac{l(l+1)}{r^2} + \left(\frac{2\mu}{r(r-2\mu)} + r\psi^2 \right) \right) H = 0 \quad (24)$$

$$\delta\varphi'' + \left(\frac{2}{r} + \frac{1}{2}(\nu_r - \lambda_r) \right) \delta\varphi' + e^\lambda \left(-\frac{l(l+1)}{r^2} \delta\varphi + 16A^3\pi\alpha(3p-\rho)\delta A + 4A^4\pi(3p-\rho)\delta\alpha \right) = 0. \quad (25)$$

In the case of spontaneous scalarization, Eq. (25) becomes

$$\delta\varphi'' + \left(\frac{2}{r} + \frac{1}{2}(\nu_r - \lambda_r) \right) \delta\varphi' + e^\lambda \left(-\frac{l(l+1)}{r^2} + 4A^4\pi\alpha(3p-\rho)(1+4\alpha\varphi) \right) \delta\varphi = 0. \quad (26)$$

External to the star, Eqs. (24) and (26), reduce to

$$H'' + \frac{2(r-\mu)}{r(r-2\mu)} H' - \left(e^\lambda \frac{l(l+1)}{r^2} + \left(\frac{2\mu}{r(r-2\mu)} + r\psi^2 \right) \right) H = 0 \quad (27a)$$

$$\delta\varphi'' + \frac{2(r-\mu)}{r(r-2\mu)} \delta\varphi' - \left(e^\lambda \frac{l(l+1)}{r^2} \right) \delta\varphi = 0. \quad (27b)$$

Eq. (27a) depends on the ψ , and so is coupled to the scalar wave equation (Eq. (5b)). As long as $\psi > 0$, there is no analytical solution to Eq. (27a). Only approximate solutions exist at the surface of the star ($\tilde{r} = 0$) because $\psi \neq 0$. Since φ asymptotically approaches a constant value φ_∞ , the derivative ψ vanishes at large r . In this regime, Eq. (27a) has an exact solution. The solutions to Eqs. (27a) and (27b) are

$$H = c_1 Q_\ell^2 \left(\frac{r}{m} - 1 \right) + c_2 P_\ell^2 \left(\frac{r}{m} - 1 \right) \quad (28a)$$

$$\delta\varphi = d_1 Q_\ell \left(\frac{r}{m} - 1 \right) + d_2 P_\ell \left(\frac{r}{m} - 1 \right), \quad (28b)$$

where P_ℓ^m and Q_ℓ^m are associated Legendre functions of the third kind.

In order to determine $c_1, c_2, d_1,$ and d_2 , we match the asymptotic behavior of the two solutions i.e.

$$H = \frac{8}{5} c_1 \left(\frac{r}{\mu} \right)^{-3} + \mathcal{O} \left(\left(\frac{r}{\mu} \right)^{-4} \right) + 3c_2 \left(\frac{r}{\mu} \right)^2 + \mathcal{O} \left(\frac{r}{\mu} \right) \quad (29a)$$

$$\delta\varphi = \frac{2}{15} d_1 \left(\frac{r}{\mu} \right)^{-3} + \mathcal{O} \left(\left(\frac{r}{\mu} \right)^{-4} \right) + \frac{3}{2} d_2 \left(\frac{r}{\mu} \right)^2 + \mathcal{O} \left(\frac{r}{\mu} \right) \quad (29b)$$

to the expansion of the g_{tt} component of the metric (Eq. (23)).

This gives us $c_1, c_2, d_1,$ and d_2 in terms of the tensor tidal deformability λ and the scalar tidal deformability λ_s respectively. For example, in the $\ell = 2$ case, we have

$$c_1 = \frac{15}{8} \frac{1}{\mu^3} \lambda \mathcal{E}, \quad c_2 = \frac{1}{3} \mu^2 \mathcal{E} \quad (30)$$

$$d_1 = \frac{45}{2} \frac{1}{\mu^3} \lambda \mathcal{E}, \quad d_2 = \frac{2}{3} \mu^2 \mathcal{E}. \quad (31)$$

By requiring continuity of the logarithmic derivatives

$$y = \frac{rH'}{H} \quad \& \quad w = \frac{r\delta\varphi'}{\varphi} \quad (32)$$

and thus of $H, \delta\varphi$, and their derivatives at the surface of the star, it is possible to determine λ and λ_s in terms of μ, r , and either y or w respectively.

The tidal Love numbers are connected to the tidal deformabilities by the following equations

$$k_\ell = \frac{(2\ell - 1)!!}{2} \lambda R^{-(2\ell+1)} \quad \& \quad \kappa_\ell = \frac{(2\ell - 1)!!}{2} \lambda_s R^{-(2\ell+1)} \quad (33)$$

While this approach is sufficient to *define* λ , difficulty arises in numerically calculating λ and k_ℓ because Eq. (28a) is only a solution to Eq. (27a) in the large r limit. It is not a solution near the surface of the star where numerical matching is typically done. In this region, an exact solution does not exist. While there is not an exact solution, an approximate series solution to Eq. (27a) can be constructed order by order in powers of r/μ . The leading order behavior of H is

$$H \approx a_\ell \left(\frac{r}{\mu}\right)^\ell + a_{-(\ell+1)} \left(\frac{r}{\mu}\right)^{-(\ell+1)} \quad (34)$$

From here higher order terms are added to construct a solution with the form

$$H \approx a_\ell \left(\left(\frac{r}{\mu}\right)^\ell + \alpha_1^+ \left(\frac{r}{\mu}\right)^{\ell-1} + \alpha_2^+ \left(\frac{r}{\mu}\right)^{\ell-2} + \dots + \alpha_n^+ \left(\frac{r}{\mu}\right)^{\ell-n} \right) \\ + a_{-(\ell+1)} \left(\left(\frac{r}{\mu}\right)^{-(\ell+1)} + \alpha_1^- \left(\frac{r}{\mu}\right)^{-(\ell+2)} + \alpha_2^- \left(\frac{r}{\mu}\right)^{-(\ell+3)} + \dots + \alpha_n^- \left(\frac{r}{\mu}\right)^{-(\ell+1+n)} \right) \quad (35)$$

To ensure sufficient accuracy, terms are added to the solution until the addition of the n and $(n - 1)$ terms changes the solution by $< 0.5\%$.

Note that H has only two degrees of freedom ($a_\ell, a_{-(\ell+1)}$). All other constants, α^+ and α^- , are functions of these two and are determined by Eq. (27a).

By matching a_ℓ and $a_{-(\ell+1)}$ to Eq. (23), the approximate tidal deformability λ can be derived in the same method described above.

4.2. Magnetic Type Love numbers

The odd parity or magnetic Love numbers j_l and their associated tidal deformabilities σ_l are functions of the odd parity metric perturbation $h_{\mu\nu}^- (h_0, h_1)$. The odd parity metric perturbations h_0 and h_1 (Eq. (16)) are coupled only to the explicit fluid velocity perturbation $U(r)Y_{\ell m}$:

$$\delta u^\mu = [4\pi e^{-\nu/2} r^2 A^4 (\rho + p)]^{-1} \left(0, 0, \partial_\phi Y_{\ell m}(\theta, \phi), \partial_\theta Y_{\ell m}(\theta, \phi) \right) \sin^{-1} \theta U(r) . \quad (36)$$

The odd parity metric perturbations can be constrained by three equations, which come from the $t\phi$, $r\phi$, and $\theta\phi$ components of the perturbation equations (see Eqs. (B36) to (B38)). There are multiple approaches to the magnetic Love number in the literature, but two are worthy of note [Pani et al. \(2018\)](#). The earliest two publications on magnetic tidal deformabilities [Binnington & Poisson \(2009\)](#); [Damour & Nagar \(2009\)](#) have approaches that are fundamentally different and whose results do not agree. The first approach developed by Binnington and Poisson [Binnington & Poisson \(2009\)](#) assumes a *strictly static* fluid i.e., $h_{0t} = h_{1t} = U = 0$. The second approach from Damour and Nagar [Damour & Nagar \(2009\)](#) assumes an *irrotational* fluid. Instead of initially setting $h_{0t} = h_{1t} = U = 0$, this approach calculates the full Regge-Wheeler equation and then takes the static limit ($\omega \rightarrow 0$). Note that these approaches seem equivalent at a surface level but do not lead to the same answer because the irrotational approach picks up a non-vanishing term from the fluid velocity perturbation. This chapter lays out both approaches and clarifies the subtle differences between them.

4.2.1. Static Approach

In this section, we apply the static method derived in [Binnington & Poisson \(2009\)](#) to scalar tensor theory; we assume that the perturbations are strictly static i.e., $h_{0t} = h_{1t} = U = 0$ and consider the odd parity perturbation equations Eqs. (B36) to (B38). Under this assumption, Eq. (B37) becomes

$$h_1 = 0 , \quad (37)$$

and Eq. (B38) becomes independent of h_0 and constrains only h_1 .

The final remaining equation, Eq. (B36), yields a second order differential equation for h_0 .

$$e^{-\lambda} h_0'' - \left[4\pi r A^4(\varphi)(p + \rho) + e^{-\lambda} r \psi^2 \right] h_0' - \left[\frac{l(l+1)}{r^2} - \frac{4\mu}{r^3} + 8\pi A^4(\varphi)(p + \rho) - 2e^{-\lambda} \psi^2 \right] h_0 = 0, \quad (38)$$

where the prime (') denotes a derivative with respect to r . This equation is consistent with Eq. B7 in [Sotani & Kokkotas \(2005\)](#).

In the region exterior to the star, $\mu(r) = \mu$ and $\tilde{p} = \tilde{\rho} = 0$ and Eq. (38) takes on a simpler form.

$$e^{-\lambda} h_0'' - \left[\frac{l(l+1)}{r^2} - \frac{4\mu(r)}{r^3} - 2e^{-\lambda} \psi^2 \right] h_0 = 0 \quad (39)$$

This equation differs from the GR equation by the factor of $-2e^{-\lambda} \psi^2$ [Binnington & Poisson \(2009\)](#). Eq. (39) is now coupled to the scalar wave equation (Eq. (5)) and no longer has an exact solution. This differs result from the $f(R)$ results [Yazadjiev et al. \(2018\)](#) because, in that case, $\psi \rightarrow 0$ as $r \rightarrow \infty$, and Eq. (39) reduces to its GR counterpart.

Since Eq. (39) is true for all $r < r_s$, including the large r regime where $r \gg r_s$ and $\psi \rightarrow 0$, there is an exact solution in the large r limit. This is sufficient to define the tidal Love numbers. However, tidal Love numbers are calculated by matching the interior and exterior solutions at or near the surface of the star where the static solution near the surface can only be approximated. Therefore, this work focuses on the irrotational results.

4.2.2. Irrotational Approach

In the *irrotational* approach, which was initially presented in [Damour & Nagar \(2009\)](#), it is assumed that the perturbations have a standard $e^{-i\omega t}$ time dependence i.e. $h_i(r, t) = h_i(r) e^{-i\omega t}$.

Previous authors [Cunningham et al. \(1978\)](#); [Andrade & H. Price \(1999\)](#); [Kojima \(1992\)](#); [Damour & Nagar \(2009\)](#) have noted that Eq. (B38) can be solved for h_0 in terms of h_1 unless one assumes that $h_{0t} = 0$ (for that case see Sec. 4.2.1).

Under this assumption, Eq. (B38) can be rewritten as

$$h_{0t} = e^{(\nu-\lambda)/2} (\Psi r)' \quad (40)$$

where Ψ is defined such that

$$h_1 = e^{(\lambda-\nu)/2} \Psi r. \quad (41)$$

Assuming that $h_0(r, t) = h_0(r) e^{-i\omega t}$, Eq. (40) can be used to define h_0 :

$$h_0 = \frac{i}{\omega} e^{(\nu-\lambda)/2} (\Psi r)'. \quad (42)$$

It is evident from this equation that h_0 is not well defined in the $\omega \rightarrow 0$ limit [Pani et al. \(2018\)](#). Substituting Eq. (42) and Eq. (41) into Eq. (B36) gives the following master equation:

$$\Psi'' + \frac{e^\lambda}{r^2} [2m + 4\pi r^3 A^4(p - e)] \Psi' + e^\lambda \left[e^{-\nu} \omega^2 + \frac{6m}{r^3} - \frac{l(l+1)}{r^2} + 4\pi A^4(p - \rho) \right] = 0. \quad (43)$$

This agrees with Equation 40 in [Sotani & Kokkotas \(2005\)](#).

Since it is assumed that the neutron star is static, we are interested in the $\omega \rightarrow 0$ limit. The master equation becomes

$$\Psi'' + \frac{e^\lambda}{r^2} [2m + 4\pi r^3 A^4(p - e)] \Psi' + e^\lambda \left[\frac{6m}{r^3} - \frac{l(l+1)}{r^2} + 4\pi A^4(p - \rho) \right] = 0. \quad (44)$$

Outside of the star, this equation simplifies further:

$$\Psi'' + e^\lambda \frac{2m}{r^2} \Psi' + e^\lambda \left[\frac{6m}{r^3} - \frac{l(l+1)}{r^2} \right] = 0. \quad (45)$$

Interestingly, this equation, unlike the static master equation (Eq. (38)), does not depend explicitly in φ or ψ . Therefore external to the star, the solution to Eq. (44) is known and identical to the GR solution. All non-GR effects arise from matching the internal and external solutions at the surface of the star.

We briefly demonstrate the difference in the static and irrotational solutions in scalar-tensor theory. The method presented in [Pani et al. \(2018\)](#) is applied to the scalar-tensor problem.

Using the axial component of the stress-energy tensor conservation equation (Eq. (11)) and assuming $\omega \neq 0$, one finds that

$$U(r) = -4\pi A^3(\rho + p)e^{-\nu}h_0. \quad (46)$$

Substituting Eq. (46) into Eq. (B36) and then taking the $\omega \rightarrow 0$ limit, the following differential equation for h_0 is obtained

$$e^{-\lambda}h_0'' - \left[4\pi A^4(\varphi)(p + \rho)r + e^{-\lambda}r\psi^2\right]h_0' - \left[\frac{l(l+1)}{r^2} - \frac{4m}{r^3} - 8\pi A^4(\varphi)(p + \rho) - 2e^{-\lambda}\varphi_r^2\right]h_0 = 0. \quad (47)$$

There is a sign change in the $8\pi A^4(\varphi)(p + \rho)$ term between Eq. (38) and Eq. (47). The difference occurs because $U(r) = 0$ for the static approach and $U(r) \neq 0$ in the irrotational case. So while there is irrotational fluid motion in one case, the other has a completely static fluid. As a result of this difference, the irrotational tidal Love numbers are negative while the static Love numbers are positive.

Returning to the main goal of this work, calculating j_ℓ and σ_ℓ , we use the similarity between the irrotational master equation and its GR counterpart to define the solution. Eq. (45) has an exact solution of the form

$$\psi^{ext}(R) = b_p\psi_p(R) + b_q\psi_q(R) = b_pR^{l+1} - \frac{b_q}{4}R^3\partial_R \left[(R)^{-4}F\left(l-1, l+2; 2l+2; \frac{2M}{r}\right) \right] \quad (48)$$

where $R = r/\mu$ and F is a hypergeometric function. For $l = 2$, F is expressible in terms of simple functions.

b_q and b_p are determined by the boundary conditions at the surface of the star. Since both ψ and ψ' are required to be continuous at the surface of the star, the logarithmic derivative $y^{odd} = r\psi'/\psi$ must also be continuous at the surface of the star.

j_ℓ and σ_ℓ are therefore defined to be

$$j_\ell \equiv -C^{2\ell+1} \frac{\psi_p'(R_s) - C y^{odd} \psi_p(R_s)}{\psi_q'(R_s) - C y^{odd} \psi_q(R_s)} \quad (49)$$

and

$$\sigma_\ell = \frac{\ell-1}{4(l+2)} \frac{j_\ell}{(2\ell-1)!!} C^{2\ell+1}, \quad (50)$$

where $R_s = r_s/\mu$ and C is the compactness.

j_ℓ and σ_ℓ are, at a glance, identical to their GR counterparts, but the scalar-tensor and GR values differ because all non-GR effects are contained in the value of y_s^{odd} calculated by integrating Eq. (44) along with the modified TOV equations (Eq. (13)) inside the star.

5. RESULTS

5.1. Electric Love Numbers

This section presents the electric tidal Love numbers and the associated tidal deformabilities and compare them to the GR results. There are two degrees of freedom needed to define a specific case of spontaneous scalarization: β and φ_∞ . β is constrained by binary pulsar experiments to $\beta < -5$ at the 1σ level Freire et al. (2012). In this work we use two values of β to demonstrate the results; $\beta = -4.5$ and $\beta = -6$. This gives us two sets of results, one conservative and one optimistic. The value of the scalar field at infinity, φ_∞ , is tightly constrained by the Cassini experiment Bertotti et al. (2003). That experiment directly constrains the Brans-Dicke parameter ω_{BD} to be $> 4 \times 10^4$. The value of the scalar field at infinity is related to the Brans-Dicke parameter by the equation

$$\varphi_\infty = \frac{2}{|\beta|} \sqrt{\frac{\pi}{3 + 2\omega_{BD}}}. \quad (51)$$

This constrains φ_∞ to $< 2.7 \times 10^{-3}$ and $< 2.0 \times 10^{-3}$ for $\beta = -4.5$ and $\beta = -6$ respectively. We use $\varphi_\infty = 10^{-3}$ for all results presented.

Using Eq. (28a) and Eq. (33), it is possible to define the tidal Love number in the large r limit. The scalar tidal Love number can be similarly calculated.

The $\ell = 2$ tidal Love numbers are defined as follows

$$k_2 = \frac{8(2C-1)^2 C^5 (2+2C(y-1)-y)}{5(2C(6+C^2(26-22y)-3y+4C^4(1+y)+3C(-8+5y)+C^3(6y-4))-3(1-2C)^2(2+2C(y-1)-y)\ln(1-2C))} \quad (52a)$$

$$\kappa_2 = \frac{4C^5(2C-1)(2C^2w-6Cw+3w+6C-6)}{45(2C(C(12-9w)+3(-2+w)+C^2(-2+6w))+(-1+2C)(-6+6C+3w-6Cw+2C^2w)\ln[1-2C])}, \quad (52b)$$

where $y = rH'/H$ and $w = r\delta\varphi'/\delta\varphi$.

y is traditionally evaluated at the star's surface for numerical applications. However, Eq. (52a) is not valid when $r = r_s$. Close to the star $\psi \neq 0$, and the solution to Eq. (27a) can only be approximated. After constructing a series solution that is accurate to better than 0.5% for even the largest values of ψ considered (see Sec. 4.1), we compared values from the exact solution (Eq. (52a)) evaluated at the surface to the values of k_2 calculated from the approximate solution. The tidal Love numbers agreed to better than $10^{-10}\%$ for both $\beta = -6$ and $\beta = -4.5$. Tidal deformabilities agreed within $\lesssim 3\%$.

Figure 1 shows how the electric tidal Love numbers and tidal deformabilities differ in scalar-tensor theory and GR. It is clear that the spontaneous scalarization effect can lead to significant deviations from the GR tidal deformabilities. Fig. 1 plots the physical or Jordan frame values, which are related to their Einstein frame counterparts by Eqs. (C56) and (C57).

$\tilde{\lambda}_2$ and \tilde{k}_2 are plotted against the neutron star's physical compactness, in this case defined as the ADM mass over the Jordan frame radius. The ADM mass can be determined by matching the interior and exterior metrics. The definition of the ADM mass in the Einstein frame is

$$M_{ADM} = \frac{r_s^2 \nu'_s}{2G_*} \left(1 - \frac{2\mu}{r_s}\right)^{1/2} \exp\left[-\frac{\nu'_s}{(\nu_s'^2 + 4\psi_s'^2)^{1/2}} \operatorname{arctanh}\left(\frac{(\nu_s'^2 + 4\psi_s'^2)^{1/2}}{\nu_s'^2 + 2/r_s}\right)\right]. \quad (53)$$

where s denotes the value at the surface, and the Jordan frame value can be calculated using Eq. (C42)

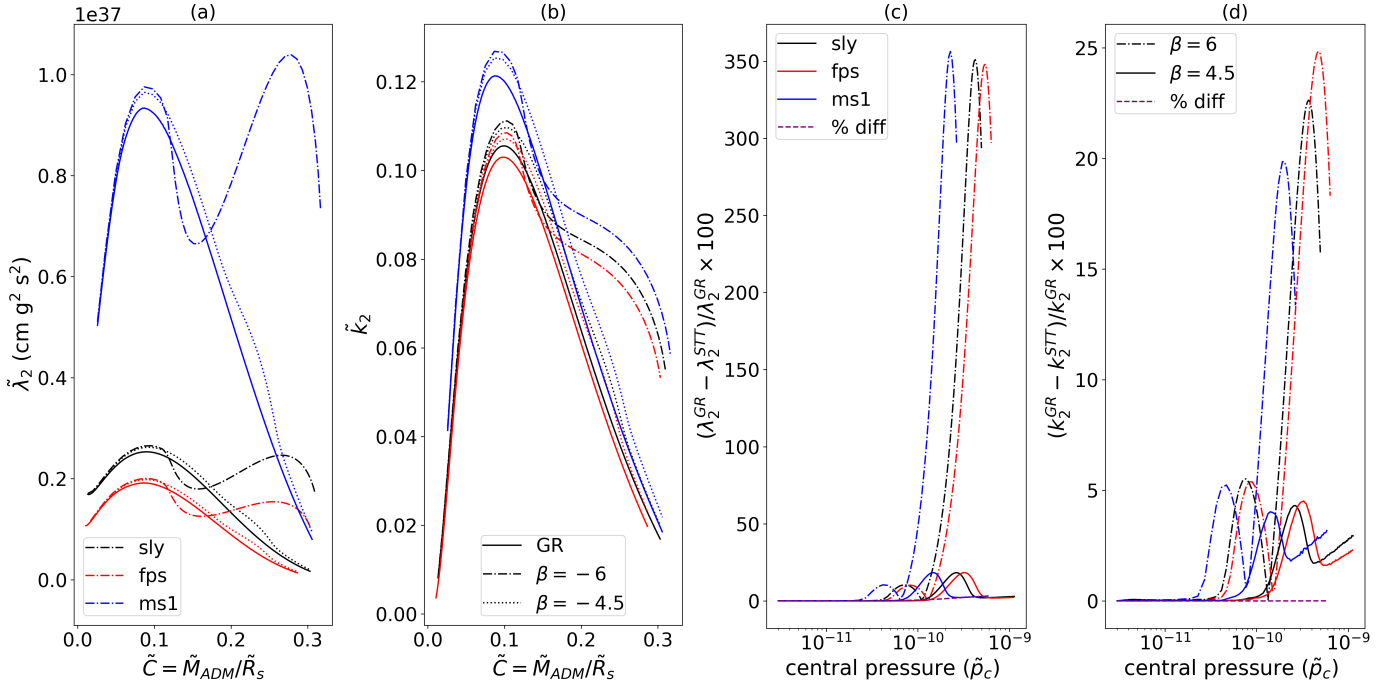


Figure 1. Panel (a) shows the $\ell = 2$ Einstein frame tidal deformability as a function of compactness, (b) shows the tidal Love number again as a function of compactness, (c) shows the percent difference between the tidal deformability in scalar-tensor theory and GR as a function of the pressure at the center of the star, and (d) shows the percent difference between the tidal Love numbers in scalar-tensor theory and GR as a function of the pressure at the center of the star. The value of the scalar field at infinity φ_∞ for all cases presented here is 10^{-3} . Three realistic nuclear equations of state are shown: SLy, FPS, and MS1. In both figures the results for $\beta = -6$ and $\beta = -4.5$ are shown.

For the case where $\beta = -6$, the tidal Love number differs by $\sim 20 - 25\%$ and the tidal deformability differs by $\sim 350\%$. In the more conservative case where $\beta = -4.5$, this reduces to $\sim 5\%$ and $\sim 20\%$ for the tidal Love number and tidal deformability, respectively. As the difference between scalar-tensor theory and GR is much greater than the difference between the two methods of calculating k_ℓ , we consider Eq. (52a) evaluated at r_s to be sufficiently accurate for GW parameter estimation with current detectors.

We show the Jordan frame $\ell = 2$ scalar tidal Love numbers and tidal deformabilities in Fig. 2. The $\ell = 3, 4$ tidal Love numbers are in Appendix D.

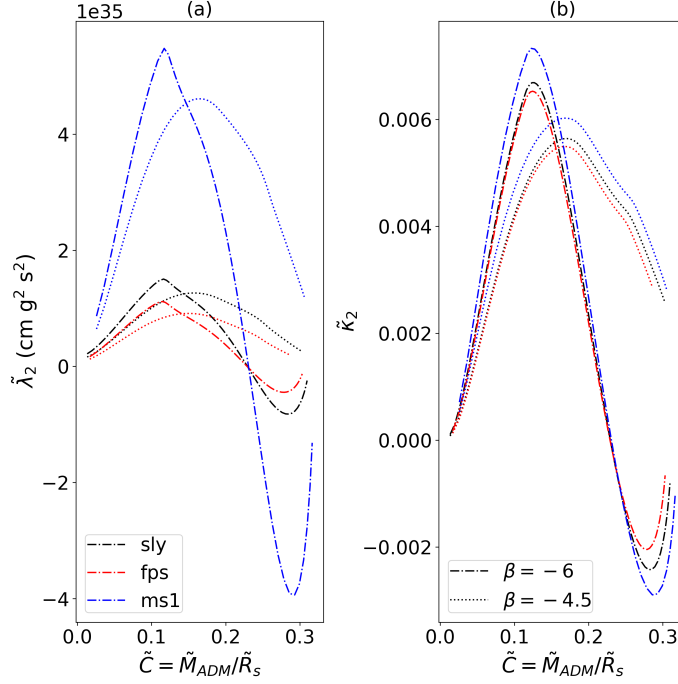


Figure 2. Panel (a) shows the $\ell = 2$ Jordan frame scalar tidal deformability as a function of compactness, (b) shows the scalar tidal Love number again as a function of compactness. The value of the scalar field at infinity φ_∞ for all cases presented here is 10^{-3} . In both figures the results for $\beta = -6$ and $\beta = 4.5$ are shown. Three realistic nuclear equations of state are shown: SLy, FPS, and MS1.

5.2. Magnetic Love Numbers

This section presents the magnetic tidal Love number and the associated tidal deformabilities in scalar-tensor theory and compare them to the GR results.

The exact equations for the magnetic tidal Love numbers j_ℓ and tidal deformabilities σ_ℓ can be determined by substituting Eq. (48) into Eq. (49) and Eq. (50).

The explicit equation for the $\ell = 2$ or quadrupolar tidal Love number is

$$j_2 = \frac{96C^5(2C-1)(y-3)}{5(2C(12(y+1)C^4 + 2(y-3)C^3 + 2(y-3)C^2 + 3(y-3)C - 3y + 9) + 3(2C-1)(y-3)\ln(1-2C))} \quad (54)$$

where $C = \mu/r_s$ is the Einstein frame compactness and $y = y^{odd}(r_s) = r_s \Psi'/\Psi$ is the logarithmic derivative at the surface.

Fig. 3 shows the Jordan frame $l = 2$ love numbers and tidal deformabilities and the difference between the GR and scalar-tensor tidal effects. The Jordan frame values are related to their Einstein frame counterparts by Eq. (C48a) and Eq. (C48b). The conformal transformations are derived in Appendix C.

It is clear that tidal Love numbers and tidal deformabilities differ between GR and scalar-tensor theory. For the optimistic case where $\beta = -6$, the tidal Love number has a maximum deviation of $\pm 12\%$ and the tidal deformability has a maximum deviation of $\sim 200\%$. In the more physical case where $\beta = -4.5$, we find that this changes to $\sim 1\%$ and $\sim 15\%$ respectively.

In GR empirical relationships between the $\ell = 2$ dimensionless magnetic and electric tidal deformabilities have been found Forteza et al. (2018). The dimensionless magnetic tidal deformability Σ_2 and the dimensionless electric tidal deformability Λ_2 have a quasi equation of state independent relationship:

$$\ln(-\Sigma_2) = \sum_{n=0}^5 a_n (\ln \Lambda_2)^n. \quad (55)$$

We find that the scalar tensor tidal deformabilities can be fit to a similar relationship, with the coefficients depending on the value of β .

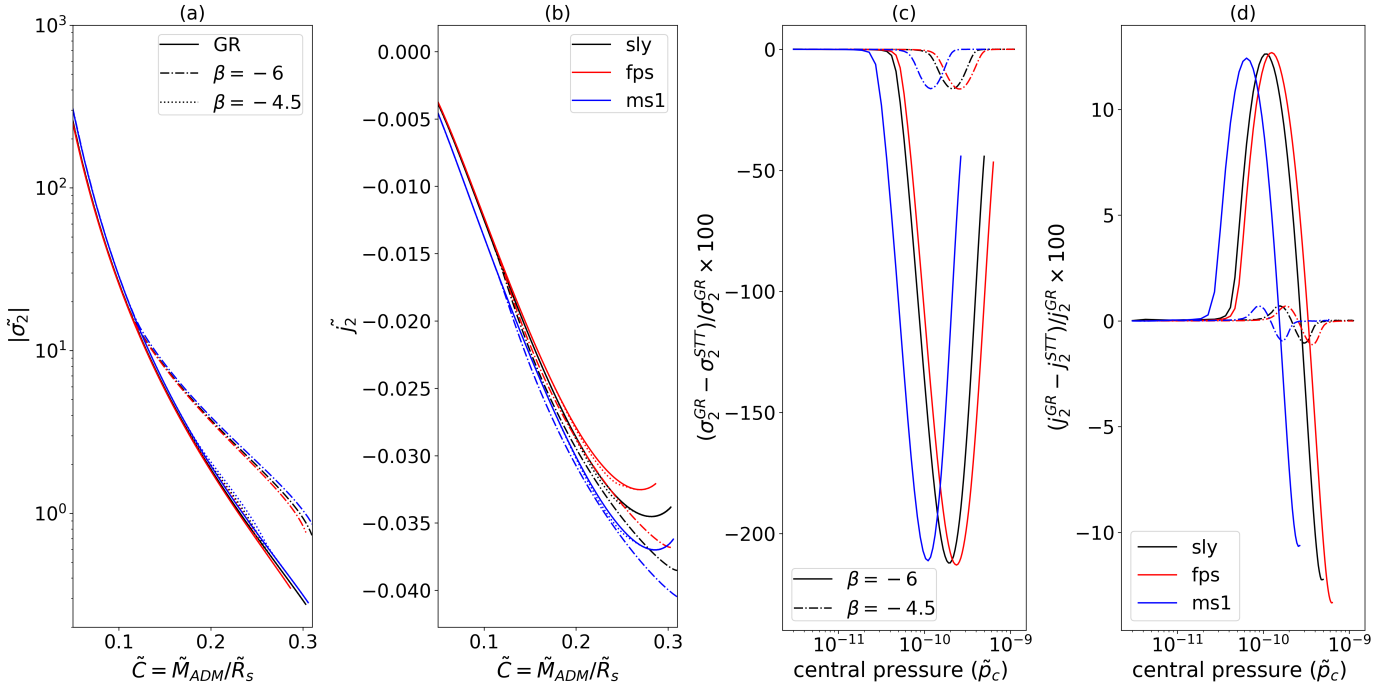


Figure 3. Panel (a) shows the absolute value of the magnetic $\ell = 2$ Jordan frame tidal deformability as a function of the neutron star compactness, (b) shows the tidal Love number again as a function of the neutron star compactness, (c) shows the percent difference between the percent difference between the tidal deformability in scalar-tensor theory and GR as a function of the pressure at the center of the star, and (d) shows the percent difference between the tidal Love numbers in scalar-tensor theory and GR as a function of the pressure at the center of the star. The value of the scalar field at infinity φ_∞ for all cases presented here is 10^{-3} . Three realistic nuclear equations of state are shown: SLy, FPS, and MS1. In both figures the results for $\beta = -6$ and $\beta = -4.5$ are shown.

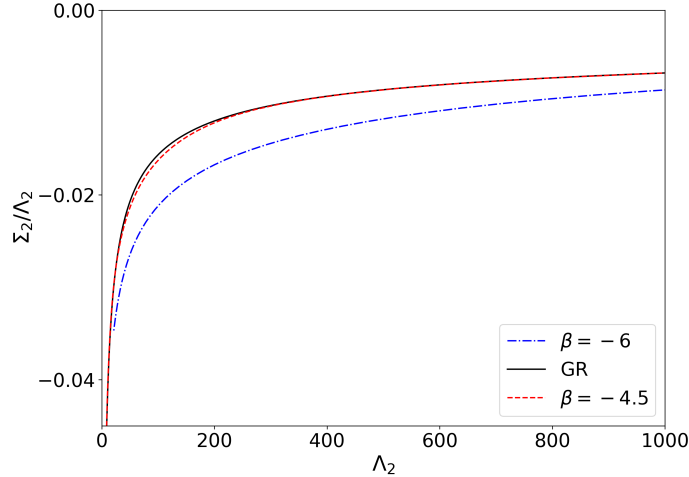


Figure 4. Quasiuniversal relations between the dimensionless magnetic tidal deformability (Σ_2) and the dimensionless electric quadrupolar tidal deformability Λ_2 . Only the irrotational magnetic tidal deformability is shown in this figure.

There does not appear to be a similar relationship between the scalar deformability and the electric tidal deformability. The scalar tidal deformability depends very strongly on β and $\Lambda_{(\varphi)}/\Lambda$ can take on very different shapes. It may be possible to construct a relationship between the scalar deformability and the electric tidal deformability for a specific β but there is no general relationship.

6. DISCUSSION

We present an analytical expression for the magnetic tidal Love numbers in scalar-tensor theory for the first time. The results establish that the magnetic Love numbers are only implicitly dependent on the scalar field and have an analytical solution. This is in agreement with Ref. [Sotani & Kokkotas \(2005\)](#), which shows that the time-dependent perturbation equation is only implicitly on the scalar field. However this was discussed only in the context of perturbations and not of tidal Love numbers.

The magnetic Love numbers in this paper can be compared to their $F(R)$ counterparts because $F(R)$ theory and scalar-tensor theory are mathematically similar. When calculating tidal Love numbers a difference arises in part due to the behavior of the scalar field at infinity. In $F(R)$ theory, the scalar field and its derivative go to zero at infinity, and the tidal deformability can be evaluated at some distance away from the neutron star where both the scalar field and its derivative are sufficiently small. This is different from scalar-tensor theories considered in this paper where the scalar field asymptotically approaches a constant. Additionally, the coupling function differs between theories with $A(\varphi) \propto e^{\alpha\varphi}$ in $F(R)$. Despite this, the perturbation equations inside the star should agree when the correct substitutions for $A(\varphi)$ and $\alpha(\varphi)$ have been made because they are mathematically similar. However, our perturbation equation differs from Eq. (23) in [Yazadjiev et al. \(2018\)](#). Comparing the tidal Love numbers themselves, shown in Fig. 3, with the results from [Yazadjiev et al. \(2018\)](#), it is clear that the qualitative features seem to agree, with the deviation from GR increasing with compactness. However, the difference between GR and scalar-tensor theory are smaller than those between GR and $F(R)$, at least for physically allowed values of β and φ_∞ .

This paper also includes the even parity tidal Love numbers and tidal deformabilities. The $\ell = 2$ electric tidal Love numbers in scalar-tensor theory were initially presented in [Pani & Berti \(2014\)](#) in the context of the so-called ‘‘I-Love-Q’’ relations. The methods in this paper differ significantly from [Pani & Berti \(2014\)](#).

There are both metric and scalar perturbations in the even parity case, and the relationship between them is not trivial. In the Einstein Frame, the metric tensor and the scalar field are not coupled. As a change in the metric should not affect the scalar-field and vice-versa, it is important to construct two independent first-order perturbation equations. One for the metric perturbation and one for the scalar. This differs from the approach in [Pani & Berti \(2014\)](#), where the two even parity equations are coupled. It is unsurprising, then, that Eqs. (24) and (25) are different from the equations presented in [Pani & Berti \(2014\)](#). The resulting tidal Love numbers must also differ. The definition for the scalar tidal Love number in this paper also differs from that in [Pani & Berti \(2014\)](#). [Pani & Berti \(2014\)](#) does not include a source term in their definition of the scalar tidal Love number and we do. This is because they are considering the perturbation in the scalar field produced by a change in the metric rather than by a change in the scalar field. This paper also includes the $\ell = 3, 4$ even parity Love numbers and tidal deformabilities in Sec. D, which have not been presented before.

This paper approaches tidal effects through the lens of gravitational wave parameter estimation. The mass-radius-tidal deformability relationships explored in this paper can be applied directly to GW parameter estimation of GWs from binary neutron star and neutron star-black hole systems.

The results demonstrate that tidal Love numbers and tidal deformabilities can differ significantly between scalar-tensor theory and GR. This is consistent with other results in the literature, which show that tidal Love numbers in $F(r)$ theory and scalar-Gauss-Bonnet gravity [Yazadjiev et al. \(2018\)](#); [Saffer & Yagi \(2021\)](#) also differ significantly from their general relativistic counterparts. As GWs emitted by neutron stars depend on the tidal deformability, it is essential to take the changes in the mass, radius, and tidal deformability into account when studying GWs from neutron stars in theories beyond GR. The deviations from GR in the GWs are smaller than or similar to the uncertainty in the tidal deformability measurement. By taking the modified tidal deformability into account, the small deviations from GR in the waveform can be more accurately determined.

REFERENCES

- Abbott, B. P., et al. 2018, Physical Review Letters, 121, doi: [10.1103/physrevlett.121.161101](https://doi.org/10.1103/physrevlett.121.161101)
- . 2019a, Physical Review X, 9, doi: [10.1103/physrevx.9.031040](https://doi.org/10.1103/physrevx.9.031040)
- . 2019b, Physical Review D, 100, doi: [10.1103/physrevd.100.104036](https://doi.org/10.1103/physrevd.100.104036)
- . 2021a, Physical Review X, 11, doi: [10.1103/physrevx.11.021053](https://doi.org/10.1103/physrevx.11.021053)
- Abbott, R., et al. 2021b, arXiv e-prints, doi: [10.48550/ARXIV.2111.03606](https://doi.org/10.48550/ARXIV.2111.03606)
- . 2021c, Phys. Rev. D, 103, 122002, doi: [10.1103/PhysRevD.103.122002](https://doi.org/10.1103/PhysRevD.103.122002)
- AEI. 2017, The Atlas Computing Cluster, <https://www.aei.mpg.de/43564/atlas-computing-cluster>. <https://www.aei.mpg.de/43564/atlas-computing-cluster>
- Andrade, Z., & H. Price, R. 1999, Physical Review D, 60, doi: [10.1103/physrevd.60.104037](https://doi.org/10.1103/physrevd.60.104037)
- Barausse, E., Palenzuela, C., Ponce, M., & Lehner, L. 2013, Physical Review D, 87, doi: [10.1103/physrevd.87.081506](https://doi.org/10.1103/physrevd.87.081506)
- Bertotti, B., Iess, L., & Tortora, P. 2003, Nature, 425, 374, doi: [10.1038/nature01997](https://doi.org/10.1038/nature01997)
- Binnington, T., & Poisson, E. 2009, Physical Review D, 80, doi: [10.1103/physrevd.80.084018](https://doi.org/10.1103/physrevd.80.084018)

- Boisseau, B., Esposito-Farèse, G., Polarski, D., & Starobinsky, A. A. 2000, *Physical Review Letters*, 85, 2236, doi: [10.1103/physrevlett.85.2236](https://doi.org/10.1103/physrevlett.85.2236)
- Brans, C., & Dicke, R. H. 1961, *Physical Review*, 124, 925
- Capano, C. D., Tews, I., Brown, S. M., et al. 2020, *Nature Astron.*, 4, 625, doi: [10.1038/s41550-020-1014-6](https://doi.org/10.1038/s41550-020-1014-6)
- Chatziioannou, K., Isi, M., Haster, C.-J., & Littenberg, T. B. 2021, *Phys. Rev. D*, 104, 044005, doi: [10.1103/PhysRevD.104.044005](https://doi.org/10.1103/PhysRevD.104.044005)
- Clifton, T., Ferreira, P. G., Padilla, A., & Skordis, C. 2012, *Physics Reports*, 513, 1, doi: <https://doi.org/10.1016/j.physrep.2012.01.001>
- Crisostomi, M., & Koyama, K. 2018, *Physical Review D*, 97, doi: [10.1103/physrevd.97.084004](https://doi.org/10.1103/physrevd.97.084004)
- Cunningham, C. T., Price, R. H., & Moncrief, V. 1978, *The Astrophysical Journal*, 224, 643, doi: [10.1086/156413](https://doi.org/10.1086/156413)
- Damour, T., & Esposito-Farèse, G. 1993, *Phys. Rev. Lett.*, 70, 2220, doi: [10.1103/PhysRevLett.70.2220](https://doi.org/10.1103/PhysRevLett.70.2220)
- Damour, T., & Nagar, A. 2009, *Physical Review D*, 80, doi: [10.1103/physrevd.80.084035](https://doi.org/10.1103/physrevd.80.084035)
- Deruelle, N., & Sasaki, M. 2011, in *Springer Proceedings in Physics (Springer Berlin Heidelberg)*, 247–260, doi: [10.1007/978-3-642-19760-4_23](https://doi.org/10.1007/978-3-642-19760-4_23)
- Doneva, D. D., Yazadjiev, S. S., Stergioulas, N., & Kokkotas, K. D. 2013, *Phys. Rev. D*, 88, 084060, doi: [10.1103/PhysRevD.88.084060](https://doi.org/10.1103/PhysRevD.88.084060)
- Douchin, F., & Haensel, P. 2001, *Astronomy & Astrophysics*, 380, 151, doi: [10.1051/0004-6361:20011402](https://doi.org/10.1051/0004-6361:20011402)
- Fierz, M. 1956, *Helvetica Physica Acta*, 29, 128, doi: <http://doi.org/10.5169/seals-112699>
- Flanagan, E. E., & Hinderer, T. 2008, *Physical Review D*, 77, doi: [10.1103/physrevd.77.021502](https://doi.org/10.1103/physrevd.77.021502)
- Forteza, X. J., Abdelsalhin, T., Pani, P., & Gualtieri, L. 2018, *Physical Review D*, 98, doi: [10.1103/physrevd.98.124014](https://doi.org/10.1103/physrevd.98.124014)
- Freire, P. C. C., Wex, N., Esposito-Farèse, G., et al. 2012, *Monthly Notices of the Royal Astronomical Society*, 423, 3328, doi: [10.1111/j.1365-2966.2012.21253.x](https://doi.org/10.1111/j.1365-2966.2012.21253.x)
- Friedman, B., & Pandharipande, V. R. 1981, *Nucl. Phys. A*, 361, 502, doi: [10.1016/0375-9474\(81\)90649-7](https://doi.org/10.1016/0375-9474(81)90649-7)
- Fujii, Y., & Maeda, K.-i. 2003, *The Scalar-Tensor Theory of Gravitation*, Cambridge Monographs on Mathematical Physics (Cambridge University Press), doi: [10.1017/CBO9780511535093](https://doi.org/10.1017/CBO9780511535093)
- García-Bellido, J., & Quirós, M. 1990, *Phys. Lett. B*, 243, 45, doi: [10.1016/0370-2693\(90\)90954-5](https://doi.org/10.1016/0370-2693(90)90954-5)
- Harada, T. 1998, *Phys. Rev. D*, 57, 4802, doi: [10.1103/PhysRevD.57.4802](https://doi.org/10.1103/PhysRevD.57.4802)
- Hinderer, T. 2008, *The Astrophysical Journal*, 677, 1216, doi: [10.1086/533487](https://doi.org/10.1086/533487)
- Jordan, P. 1955, *Schwerkraft und Weltall (Friedrich Vieweg und Sohn, Braunschweig)*
- Kojima, Y. 1992, *Phys. Rev. D*, 46, 4289, doi: [10.1103/PhysRevD.46.4289](https://doi.org/10.1103/PhysRevD.46.4289)
- Lattimer, J. M., & Prakash, M. 2001, *The Astrophysical Journal*, 550, 426–442, doi: [10.1086/319702](https://doi.org/10.1086/319702)
- Love, A. E. H. 1909, *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 82, 73, doi: [10.1098/rspa.1909.0008](https://doi.org/10.1098/rspa.1909.0008)
- Manolidis, D. 2014, PhD thesis, Washington University in St. Louis
- Mehta, A. K., Buonanno, A., Cotesta, R., et al. 2022, <https://arxiv.org/abs/2203.13937>
- Mirshekari, S., & Will, C. M. 2013, *Physical Review D*, 87, doi: [10.1103/physrevd.87.084070](https://doi.org/10.1103/physrevd.87.084070)
- Müller, H., & Serot, B. D. 1996, *Nuclear Physics A*, 606, 508, doi: [10.1016/0375-9474\(96\)00187-x](https://doi.org/10.1016/0375-9474(96)00187-x)
- Nair, R., Perkins, S., Silva, H. O., & Yunes, N. 2019, *Phys. Rev. Lett.*, 123, 191101, doi: [10.1103/PhysRevLett.123.191101](https://doi.org/10.1103/PhysRevLett.123.191101)
- Nitz, A. H., Capano, C. D., Kumar, S., et al. 2021a, *The Astrophysical Journal*, 922, 76, doi: [10.3847/1538-4357/ac1c03](https://doi.org/10.3847/1538-4357/ac1c03)
- Nitz, A. H., Kumar, S., Wang, Y.-F., et al. 2021b, doi: [10.48550/ARXIV.2112.06878](https://doi.org/10.48550/ARXIV.2112.06878)
- Nitz, A. H., Dent, T., Davies, G. S., et al. 2020, *The Astrophysical Journal*, 891, 123, doi: [10.3847/1538-4357/ab733f](https://doi.org/10.3847/1538-4357/ab733f)
- Palenzuela, C., Barausse, E., Ponce, M., & Lehner, L. 2014, *Phys. Rev. D*, 89, 044024, doi: [10.1103/PhysRevD.89.044024](https://doi.org/10.1103/PhysRevD.89.044024)
- Pani, P., & Berti, E. 2014, *Physical Review D*, 90, doi: [10.1103/physrevd.90.024025](https://doi.org/10.1103/physrevd.90.024025)
- Pani, P., Gualtieri, L., Abdelsalhin, T., & Jiménez-Forteza, X. 2018, *Physical Review D*, 98, doi: [10.1103/physrevd.98.124023](https://doi.org/10.1103/physrevd.98.124023)
- Press, W. H., Teukolsky, S. A., Vetterling, W. T., & Flannery, B. P. 2007, *Numerical Recipes 3rd Edition: The Art of Scientific Computing*, 3rd edn. (USA: Cambridge University Press)
- Raaismakers, G., Greif, S. K., Hebel, K., et al. 2021, <https://arxiv.org/abs/2105.06981>
- Radice, D., & Dai, L. 2019, *The European Physical Journal A*, 55, doi: [10.1140/epja/i2019-12716-4](https://doi.org/10.1140/epja/i2019-12716-4)
- Read, J. S., Lackey, B. D., Owen, B. J., & Friedman, J. L. 2009, *Phys. Rev. D*, 79, 124032, doi: [10.1103/PhysRevD.79.124032](https://doi.org/10.1103/PhysRevD.79.124032)
- Regge, T., & Wheeler, J. A. 1957, *Phys. Rev.*, 108, 1063, doi: [10.1103/PhysRev.108.1063](https://doi.org/10.1103/PhysRev.108.1063)
- Saffer, A., & Yagi, K. 2021, *Physical Review D*, 104, doi: [10.1103/physrevd.104.124052](https://doi.org/10.1103/physrevd.104.124052)
- Shida, T. 1912, *Proceedings of the Tokyo Mathematico-Physical Society. 2nd Series*, 6, 242, doi: [10.11429/ptmps1907.6.16.242](https://doi.org/10.11429/ptmps1907.6.16.242)
- Sotani, H., & Kokkotas, K. D. 2005, *Physical Review D*, 71, doi: [10.1103/physrevd.71.124038](https://doi.org/10.1103/physrevd.71.124038)
- Thorne, K. S. 1998, *Physical Review D*, 58, doi: [10.1103/physrevd.58.124031](https://doi.org/10.1103/physrevd.58.124031)
- Thorne, K. S., & Campolattaro, A. 1967, *The Astrophysical Journal*, 149, 591, doi: [10.1086/149288](https://doi.org/10.1086/149288)

Virtanen, P., Gommers, R., Oliphant, T. E., et al. 2020, *Nature Methods*, 17, 261, doi: [10.1038/s41592-019-0686-2](https://doi.org/10.1038/s41592-019-0686-2)

Wang, Y.-F., Brown, S. M., Shao, L., & Zhao, W. 2022, *Phys. Rev. D*, 106, 084005, doi: [10.1103/PhysRevD.106.084005](https://doi.org/10.1103/PhysRevD.106.084005)

Wang, Y.-F., Niu, R., Zhu, T., & Zhao, W. 2021, *The Astrophysical Journal*, 908, 58, doi: [10.3847/1538-4357/abd7a6](https://doi.org/10.3847/1538-4357/abd7a6)

Yazadjiev, S. S., Doneva, D. D., & Kokkotas, K. D. 2018, *The European Physical Journal C*, 78, doi: [10.1140/epjc/s10052-018-6285-z](https://doi.org/10.1140/epjc/s10052-018-6285-z)

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APPENDIX

A. NUMERICAL METHODS

This section lays out the numerical techniques used to calculate the mass, radius, and tidal deformability relations shown in Sec. 5, Figs. 1 to 3

First, the structure equations presented in Sec. 2 are solved numerically using Scipy's `solve_ivp` with the 'DOP853' option, which is an eighth order Runge-Kutta method. To validate the results, the 'DOP853' results are compared to those from the older `odeint` solver and `solve_ivp`'s 'RK45' option, which is a Runge-Kutta solver that uses a fifth order accurate formula but calculates the accuracy using the fourth order method [Virtanen et al. \(2020\)](#). Next, the perturbation equations are added to the TOV solver, and the tidal deformabilities and Love numbers are calculated from Eqs. (49), (52a) and (52b).

As discussed in Sec. 2, in scalar-tensor theories, the structure equations can be expressed either in the Jordan Frame or the Einstein Frame. The code takes advantage of the relative simplicity of the Einstein frame structure equations to numerically construct the neutron star model. The Jordan frame quantities are calculated at the end of the code, using the Einstein frame values and the conformal transformations in Appendix C.

A.1. Background Configuration

For numerical integration, the equations need to be posed as first order ordinary differential equations of the form

$$y'_i(x) = f(x, y_i, y'_i) . \quad (\text{A1})$$

where $x = r$ is the independent variable and $y = M, \nu, \varphi, \psi$, and p are the dependent variables.

It is important to consider that the values of r, M, ν, ψ , and p may vary greatly in magnitude, which can lead to numerical errors and instabilities. Codes often use scale factors to mitigate the numerical errors. In [Manolidis \(2014\)](#) it is claimed that by choosing $\rho = \rho_0 \hat{\rho}$, $p = \rho_0 \hat{p}$, $M = r_0 \hat{M}$, and $r = r_0 \hat{r}$ the form of the modified TOV equations remains unchanged so long as $\rho_0 r_0^2 = 1$. However, this is not the case.

This code uses a new set of scale factors. Specifically, we scale only $\tilde{\rho}$ and \tilde{p} and not M or r . With $\rho = \tilde{\rho}_0 \hat{\rho}$ and $p = \tilde{p}_0 \hat{p}$, we are able to lay out the structure equations used in the code:

$$\frac{d\mu}{dr} = 4\pi G_* r^2 A^4(\varphi) \tilde{\rho}_0 \hat{\rho} + \frac{1}{2} r (r - 2\mu) \psi^2 \quad (\text{A2a})$$

$$\frac{d\nu}{dr} = 8\pi G_* \frac{r^2 A^4(\varphi) \tilde{p}_0 \hat{p}}{r - 2\mu} + r \psi^2 + \frac{2\mu}{r(r - 2\mu)} \quad (\text{A2b})$$

$$\frac{d\varphi}{dr} = \psi \quad (\text{A2c})$$

$$\frac{d\psi}{dr} = 4\pi G_* \frac{r A^4(\varphi)}{r - 2\mu} \left[\alpha(\varphi) (\tilde{\rho}_0 \hat{\rho} - 3\tilde{p}_0 \hat{p}) + r \psi (\tilde{\rho}_0 \hat{\rho} - \tilde{p}_0 \hat{p}) \right] - \frac{2(r - \mu)}{r(r - 2\mu)} \psi \quad (\text{A2d})$$

$$\frac{d\hat{p}}{dr} = -(\tilde{\rho}_0 \hat{\rho} + \tilde{p}_0 \hat{p}) \left[4\pi G_* \frac{r^2 A^4(\varphi) \tilde{p}_0 \hat{p}}{r - 2\mu} + \frac{1}{2} r \psi^2 + \frac{\mu}{r(r - 2\mu)} + \alpha(\varphi) \psi \right] \quad (\text{A2e})$$

In order to solve these equations, the numerical solver requires initial conditions. In this case, the initial conditions are defined near the center of the star ($r \approx 0$). Due to numerical instabilities at $r = 0$, the code starts at some small, but finite radius (e.g. $r_0 = 10^{-5}$ m). We used a convergence test to ensure that r_0 was sufficiently small and would not effect the final results.

We know that

$$\mu(r = 0) = 0 \quad (\text{A3a})$$

$$\nu(r=0) = 0 \quad (\text{A3b})$$

$$\psi(r=0) = 0 \quad (\text{A3c})$$

and the initial pressure p varies. However, the scalar field is defined at infinity $\varphi(\infty) = \varphi_0$ and not at $r = 0$. The shooting method is employed to convert the boundary value problem into a initial value problem.

The process begins with an initial guess for $\varphi(r=0) = \varphi_c$. The system of equations is then integrated outward to the star's surface, which is defined to be where the pressure vanishes ($\tilde{p} = 0$). The code then calculates the value of the scalar field at infinity φ_∞ using the relationship between φ_s , the value of φ at the surface, and φ_∞ . The connection between φ_s and φ_∞ can be found by solving the scalar wave equation outside of the star and matching the interior and exterior solutions:

$$\varphi_\infty = \varphi_s + \frac{2\psi_s}{\sqrt{\nu_s'^2 + 4\psi_s^2}} \operatorname{arctanh}\left(\frac{\sqrt{\nu_s'^2 + 4\psi_s^2}}{\nu_s' + 2/r_s}\right), \quad (\text{A4})$$

where subscript s indicates values evaluated at the surface and the prime ($'$) denotes derivative with respect to r .

The code then compares the calculated value of the scalar field at infinity φ_∞ to the actual value of the scalar field at infinity φ_0 . The parameter $\Delta\varphi = \varphi_\infty - \varphi_c$ is calculated, and if $\Delta\varphi$ is greater than some tolerance (here $\Delta\varphi \leq 10^{-5}$), then the value of φ_c is updated and the process is repeated. The initial guess is updated and then the process is repeated until the φ_∞ agrees with φ_0 within some tolerance. For a more in depth discussion on the shooting method see, for instance, [Press et al. \(2007\)](#).

In order to solve the TOV equations, it is necessary to provide an equation of state $\tilde{p}(\tilde{\rho})$ which relates pressure density. In this work, we consider a variety of equations of state. All equations of state are defined in the physical frame. In order to include realistic equations of state, our code takes in equation of state data from external data files. The code obtains the density at any point by taking the given pressure and the data from the file and interpolating.

The Sly [Douchin & Haensel \(2001\)](#), FPS [Friedman & Pandharipande \(1981\)](#), and MS1 [Müller & Serot \(1996\)](#) equation of state are considered because they are commonly used in literature and useful for comparison with previous results [Lattimer & Prakash \(2001\)](#); [Read et al. \(2009\)](#).

A.2. Tidal Deformability

The definitions of the tidal deformabilities were derived in Sec. 5, now we focus on calculating them. First, we must integrate the perturbation equations for H , Ψ , and $\delta\varphi$ along with the scalar-tensor TOV equations. The initial value problem solver requires that we recast the second order differential equations Eqs. (24), (26) and (44) into first order differential equations. There are two ways to do this. One, any second order differential equation can be recast as a system of two first order differential equations. Two, a single first order differential equation for the logarithmic derivative (e.g. $y = rH'/H$) can be obtained from the original equation. Both approaches are valid. As the definitions of the tidal deformabilities and tidal love numbers Eqs. (49), (52a) and (52b) depend on the logarithmic derivative, we recast Eqs. (24), (26) and (44) into first order differential equations for the logarithmic derivative. These equations now have form:

$$\frac{dy(r)}{dr} = -\frac{1}{r} \left(y^2(r) + y(r)F(r) + r^2Q(r) \right) \quad (\text{A5})$$

For the magnetic perturbations

$$F(r) = \left(1 - \frac{2\mu}{r} \right)^{-1} \left(\frac{2\mu}{r} + 4\pi A^4(\varphi)r^2(\tilde{p} - \tilde{\rho}) \right) - 1 \quad (\text{A6a})$$

$$r^2Q(r) = \left(1 - \frac{2\mu}{r} \right)^{-1} \left(\frac{\ell(\ell+1)\mu}{r} + 4\pi A^4(\varphi)r^2(\tilde{p} - \tilde{\rho}) - 6 \right) \quad (\text{A6b})$$

For even parity tensor perturbations

$$F(r) = \left(1 - \frac{2\mu}{r} \right)^{-1} (1 + 4\pi A^4(\varphi)r^2(\tilde{p} - \tilde{\rho})) \quad (\text{A7a})$$

$$r^2 Q(r) = \left(1 - \frac{2\mu}{r}\right)^{-1} \left(-\ell(\ell+1) + \frac{4\pi A^4(\varphi) r^2 (\tilde{\rho} + \bar{\rho})}{d\rho/dp} + 4\pi A^4(\varphi) r^2 (9\tilde{\rho} + 5\bar{\rho})\right) - \left(\left(1 - \frac{2\mu}{r}\right)^{-1} (2\mu + 8\pi A^4(\varphi) r^3 \tilde{\rho}) + r^3 \psi^2\right)^2 \quad (\text{A7b})$$

For the scalar perturbations

$$F(r) = \left(1 - \frac{2\mu}{r}\right)^{-1} (1 + 4\pi A^4(\varphi) r^2 (\tilde{\rho} - \bar{\rho})) \quad (\text{A8a})$$

$$r^2 Q(r) = \left(1 - \frac{2\mu}{r}\right)^{-1} (4\pi A^4(\varphi) r^2 (1 + 4\varphi\alpha(\varphi))\beta(3\tilde{\rho} - \bar{\rho}) - \ell(\ell+1)) \quad (\text{A8b})$$

The initial conditions are

$$y_{\text{even}}(r=0) = 2 \quad (\text{A9a})$$

$$y_{\text{scalar}}(r=0) = 2 \quad (\text{A9b})$$

$$y_{\text{odd}}(r=0) = 3 \quad (\text{A9c})$$

The values of y_{odd} , y_{even} , and y_{scalar} at the surface are then determined, and the Love numbers can be calculated. Lastly, the Jordan frame values are calculated using then conformal transformation derived in appendix C.

B. PERTURBATION EQUATIONS

B.1. Perturbed Energy-Momentum Tensor

In this section of the appendix, the exact forms of the fluid stress-energy tensor perturbations are given. Subscripts are used to denote derivatives.

The pressure and density perturbations are defined in the physical frame to be $\delta\tilde{\rho}(r)Y_{\ell m}$ and $\delta\tilde{p}(r)Y_{\ell m}$. The fluid velocity and its perturbations are also written in the Jordan frame. In the case of static tides, the fluid velocity perturbation is generally a function only of the metric perturbations and does not have explicit velocity perturbations. Furthermore, as the tides are static, the total perturbed four velocity has the form:

$$\hat{u}^\mu = u^\mu + \delta u^\mu = (\hat{u}^0, 0, 0, 0) \quad (\text{B10})$$

where the tilde has been dropped for readability. The time component of \hat{u}^μ differs from u^μ because the perturbed metric differs from the unperturbed metric.

The even parity velocity perturbations are

$$\delta\tilde{u}^t = -\frac{1}{2A} e^{-\nu/2} H_0 Y_{\ell m} \quad (\text{B11a})$$

$$\delta\tilde{u}^i = 0, \quad i = 1, 2, 3 \quad (\text{B11b})$$

While time independent perturbations do not depend explicitly on fluid velocity perturbations, the time-dependent equations do. The two methods presented in Sec. 4.2 differ in the way that the fluid velocity term $U(r)$ is treated. In both cases, the explicit dependence vanishes, but their results differ because of how they treat this term. The time-dependent odd parity velocity perturbations are

$$\delta\tilde{u}^\phi = \frac{e^{\nu/2} U(r) e^{-i\omega t}}{4\pi A^4(\varphi) (\tilde{\rho} + \bar{\rho})} \csc\theta \partial_\theta Y_{\ell m} \quad (\text{B12a})$$

$$\delta\tilde{u}^\nu = 0, \quad \nu = 0, 1, 2 \quad (\text{B12b})$$

The components of \hat{u}_μ are calculated by lowering the contravariant four-velocity \hat{u}^μ with the total metric $g_{\mu\nu} = (g_{\mu\nu}^0 + h_{\mu\nu})$.

Additionally the perturbed matter stress-energy tensor depends on the energy density and pressure variations $\delta\tilde{\rho}(r)Y_{\ell m}$ and $\delta\tilde{p}(r)Y_{\ell m}$ respectively. We assume a barotropic equation of state, and so

$$\delta\tilde{\rho} = \frac{\partial\tilde{\rho}}{\partial\tilde{p}}\delta\tilde{p}. \quad (\text{B13})$$

Using these definitions and assuming that by symmetry $\partial_\phi Y_{\ell m} = 0$, the non-zero components of the perturbed matter stress-energy tensor are as follows:

$$\delta T_t^t = - (4A^3(\varphi)\tilde{\rho}\delta A + A^4(\varphi)\delta\tilde{\rho}) Y_{\ell m} \quad (\text{B14})$$

$$\delta T_\phi^t = - \left(A^4(\varphi)\tilde{\rho}h_0 + \frac{A(\varphi)}{4\pi}e^\nu U \right) \sin\theta\partial_\theta Y_\ell \quad (\text{B15})$$

$$\delta T_r^r = (4A^3(\varphi)\tilde{p}\delta A + A^4(\varphi)\delta\tilde{p}) Y_{\ell m} \quad (\text{B16})$$

$$\delta T_\phi^r = A^4(\varphi)\tilde{p} h_1 \sin\theta\partial_\theta Y_{\ell m} \quad (\text{B17})$$

$$\delta T_\theta^\theta = (4A^3(\varphi)\tilde{p}\delta A + A^4(\varphi)\delta\tilde{p}) Y_{\ell m} \quad (\text{B18})$$

$$\delta T_t^\phi = - \left(A^4(\varphi)\tilde{\rho}h_0 + \frac{A(\varphi)}{4\pi}e^\nu U \right) \sin\theta\partial_\theta Y_\ell \quad (\text{B19})$$

$$\delta T_r^\phi = A^4(\varphi)\tilde{p} h_1 \sin\theta\partial_\theta Y_{\ell m} \quad (\text{B20})$$

$$\delta T_\phi^\phi = (4A^3(\varphi)\tilde{p}\delta A + A^4(\varphi)\delta\tilde{p}) Y_{\ell m} \quad (\text{B21})$$

The nonzero components of the perturbed energy momentum tensor for the scalar field $T_{\mu\nu}^{(\varphi)}$ have the following form

$$\delta T_{00}^{(\varphi)} = -2e^{\nu-\lambda}[H\psi^2 - \psi\delta\varphi']Y_{\ell m} \quad (\text{B22})$$

$$\delta T_{03}^{(\varphi)} = -e^\lambda\psi^2 h_0 \sin\theta\partial_\theta Y_{\ell m} \quad (\text{B23})$$

$$\delta T_{11}^{(\varphi)} = 2\psi'\delta\varphi'Y_{\ell m} \quad (\text{B24})$$

$$\delta T_{12}^{(\varphi)} = 2\psi\delta\varphi\partial_\theta Y_{\ell m} \quad (\text{B25})$$

$$\delta T_{13}^{(\varphi)} = -e^{-\lambda}\psi^2 h_1 \sin\theta\partial_\theta Y_{\ell m} \quad (\text{B26})$$

$$\delta T_{22}^{(\varphi)} = r^2e^{-\lambda}[(H - K)\psi^2 - 2\psi\delta\varphi']Y_{\ell m} \quad (\text{B27})$$

$$\delta T_{33}^{(\varphi)} = r^2e^{-\lambda}[(H - K)\psi^2 - 2\psi\delta\varphi']\sin^2\theta Y_{\ell m} \quad (\text{B28})$$

B.2. Equations for Even Parity

The following equations are derived from the even parity metric perturbation equations. The first six come from perturbing the Einstein equation: Eq. (5).

- Eq B29 is $\delta G_2^2 - \delta G_3^3 = 8\pi G_* (\delta T_2^2 - \delta T_3^3) + (\delta T_2^{(\varphi)2} - \delta T_3^{(\varphi)3})$
- Eq B30 is $\delta G_1^2 = 8\pi G_* \delta T_1^2 + \delta T_1^{(\varphi)2}$
- Eq B31 is $\partial_r \left(\delta G_1^2 = 8\pi G_* \delta T_1^2 + \delta T_1^{(\varphi)2} \right)$
- Eq B32 is $\delta G_1^1 = 8\pi G_* \delta T_1^1 + \delta T_1^{(\varphi)1}$
- Eq B33 is $\delta G_2^2 + \delta G_3^3 = 8\pi G_* (\delta T_2^2 + \delta T_3^3) + (\delta T_2^{(\varphi)2} + \delta T_3^{(\varphi)3})$
- Eq B34 is $\delta G_0^0 - \delta G_1^1 = 8\pi G_* (\delta T_0^0 - \delta T_1^1) + (\delta T_0^{(\varphi)0} - \delta T_1^{(\varphi)1})$.

$$H_0 = H_2 \quad (\text{B29})$$

$$K' = H_{0,r} + \nu' H_0 - 4\psi \delta\varphi \quad (\text{B30})$$

$$K'' = H_0'' + \nu'' H_0 + \nu' H_{0,r} - 4\psi' \delta\varphi - 4\psi \delta\varphi' \quad (\text{B31})$$

$$(l(l+1) + 2)K = (l(l+1) - 2e^{-\lambda}(1 + r\nu' - r^2\psi^2)) H_0 - 2e^{-\lambda} r H_{0r} + e^{-\lambda} r(2 + r\nu') K' - 16\pi A^4(\varphi) r^2 \delta\tilde{p} \quad (\text{B32})$$

$$\begin{aligned} 16\pi A^4(\varphi) r^2 \delta\tilde{p} = & e^{-\lambda} \left(\frac{-4 + r\lambda' - 3r\nu'}{2r} \right) H_{0r} + e^{-\lambda} \left(\frac{4 - r\lambda' + r\nu'}{2r} \right) K' \\ & - e^{-\lambda} H_{0rr} + e^{-\lambda} K'' - e^{-\lambda} \left(\frac{4r\psi^2 - (\lambda' - \nu')(2 + r\nu') + 2r\nu''}{2r} \right) \end{aligned} \quad (\text{B33})$$

$$e^{-\lambda} K'' - e^{-\lambda} \left(\frac{-4 + r(\lambda' + \nu')}{2r} \right) K' + \left(\frac{e^{-\lambda} r(\lambda' + \nu' - 2r\psi^2) - l(l+1)}{r^2} \right) + 8\pi A^4(\varphi) \left(1 + \frac{d\rho}{dp} \right) \delta p = 0 \quad (\text{B34})$$

The equation for the scalar perturbation $\delta\varphi$ is derived by perturbing scalar wave equation Eq. (B35)).

$$\delta\varphi'' = \left(\frac{-4 + r\lambda' - r\nu'}{2r} \right) \delta\varphi' + e^\lambda \frac{l(l+1)}{r^2} + 16\pi A^3(\varphi) e^\lambda \alpha (\tilde{\rho} - 3\bar{\rho}) \delta A + 4\pi A^4(\varphi) e^\lambda (\tilde{\rho} - 3\bar{\rho}) \delta\alpha \quad (\text{B35})$$

B.3. Equations for Odd Parity

The static and irrotational methods used in this paper differ in their treatment of time derivatives. Even though the tidal Love numbers themselves are time-independent, we present the time-dependent equations in this section.

Combining the $\delta G_{\mu\nu}$ with the matter stress-energy tensor and scalar stress-energy tensor terms results in the following three equations:

- Equation B36 is $\delta G_{t\phi} = 8\pi \delta T_{t\phi} + T_{t\phi}^{(\varphi)}$
- Equation B37 is $\delta G_{r\phi} = 8\pi \delta T_{r\phi} + T_{r\phi}^{(\varphi)}$
- Equation B38 is $\delta G_{\theta\phi} = 8\pi \delta T_{\theta\phi} + T_{\theta\phi}^{(\varphi)}$

$$e^{-\lambda}(h_{0rr} - h_{1rt}) - \frac{2e^{-\lambda}}{r}h_{1t} + \left[4\pi r A^4(\varphi)(p + \rho) + e^{-\lambda}r\psi^2\right](h_{1t} - h_{0r}) - \frac{1}{r^3} \left[l(l+1)r - 4m + 8\pi A^4(\varphi)(p + \rho)r^3 - 2r^3 e^{-\lambda}\psi^2\right]h_0 - 4A(\varphi)e^\nu U = 0 \quad (\text{B36})$$

$$e^{-\nu}(h_{0rt} - h_{1tt}) - 2\frac{e^{-\nu}}{r}h_{0t} - \left[\frac{l(l+1) - 2}{r^2}\right]h_1 = 0 \quad (\text{B37})$$

$$e^{-\nu}h_{0t} - \frac{1}{r^2} \left[2m - 4\pi r^3 A^4(\varphi)(\rho - p)\right]h_1 - e^{-\lambda}h_{1r} \quad (\text{B38})$$

C. CONFORMAL TRANSFORMATIONS

The tidal Love numbers in this paper were derived in the Einstein frame; however, as experiments measure Jordan frame quantities, it is necessary to obtain the Jordan frame quantities using a conformal transformation. We assume here that the Jordan frame metric $\tilde{g}_{\mu\nu}$ is related to the Einstein frame metric $g_{\mu\nu}$ by a conformal factor $A(\varphi)$:

$$\tilde{g}_{\mu\nu} = A^2(\varphi)g_{\mu\nu} , \quad (\text{C39})$$

where $A(\varphi) = e^{\frac{1}{2}\beta\varphi^2}$. By construction, the Einstein frame metric is asymptotically flat. This implies that

$$\tilde{g}_{\mu\nu} \rightarrow A^2(\varphi)\eta_{\mu\nu} . \quad (\text{C40})$$

where $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric. As the Jordan frame metric is also asymptotically flat or Minkowskian, the \tilde{r} and \tilde{t} components must be related to their Einstein frame counterparts in the following way: $\tilde{r} = A(\varphi)r$ and $\tilde{t} = A(\varphi)t$. Furthermore, the effective gravitational constant \tilde{G} is no longer a constant in the Jordan frame and is not necessarily equal to the bare gravitational constant G which appears in the Einstein frame equations. The relationship between the two is known [Palenzuela et al. \(2014\)](#):

$$\tilde{G} = e^{\beta\varphi_\infty^2} \left[G + \frac{\beta\varphi_\infty^2}{4\pi} \right] . \quad (\text{C41})$$

The relationship between the Jordan frame ADM mass and its Einstein frame counterpart can be found by comparing the asymptotic expansions of $\tilde{g}_{\mu\nu}$ and $g_{\mu\nu}$ as $r \rightarrow \infty$ [Pani & Berti \(2014\)](#):

$$\tilde{M} = M_{ADM} e^{-\frac{1}{2}\beta\varphi_\infty^2}(\varphi_\infty) \left(M_{ADM} - \frac{\beta\varphi_\infty\psi_s}{2\nu'_s} \right) \quad (\text{C42})$$

where s denotes the value at the surface of the star.

We need the conformal transformations for the perturbations between the two frames to transform the tidal Love numbers and tidal deformabilities from the Einstein frame to the Jordan frame. These are presented in [Sec. 3](#).

C.1. Odd Parity

The odd parity perturbation in the Einstein frame h_0 is related to the odd parity perturbation in the Jordan frame by

$$\tilde{h}_0 = A^2(\varphi)h_0 . \quad (\text{C43})$$

From the definition of Ψ ([Eq. \(41\)](#)), it is straightforward to show that it transforms as

$$\tilde{\Psi}(\tilde{r}) = A^2(\varphi)\Psi(r) . \quad (\text{C44})$$

To properly define the magnetic tidal deformability in the Jordan frame, $\tilde{\Psi}$ must have the same leading order behavior as Ψ i.e.

$$\tilde{\Psi}^{ext}(\tilde{R}) = \tilde{b}_p \tilde{R}^{\ell+1} + \tilde{b}_q \tilde{R}^{-\ell} \quad (\text{C45})$$

where $\tilde{R} = \tilde{r}/\tilde{\mu} = A^2(\varphi)r/\mu$. [Eqs. \(C44\)](#) and [\(C45\)](#), can be used to relate $\tilde{b}_{q,p}$ to their Einstein frame counterparts:

$$\tilde{b}_p = A^{-4}(\varphi)b_p \quad \& \quad \tilde{b}_q = A^6(\varphi)b_q . \quad (\text{C46})$$

The Jordan frame tidal Love number \tilde{j}_ℓ is defined to be

$$\tilde{j}_\ell = \tilde{C}^{2\ell+1} \frac{\tilde{b}_q}{b_q}. \quad (\text{C47})$$

From Eqs. (C46) and (C47) it follows that

$$\tilde{j}_\ell = j_\ell \quad (\text{C48a})$$

$$\tilde{\sigma}_\ell = (A^2(\varphi_\infty))^{2\ell+1} \sigma_\ell \quad (\text{C48b})$$

C.2. Electric Love Number

To transform the scalar Love number between frames, it is only necessary to know the relationship between the scalar field in the Einstein (φ) and Jordan frames (ϕ)

$$\phi = e^{-\beta\varphi^2}. \quad (\text{C49})$$

By perturbing this equation, the relationship between the Jordan frame tidal deformability λ_ϕ and its Einstein frame counterpart λ_φ can be derived:

$$\lambda_\phi = (A^2(\varphi_\infty))^{2\ell+1} \lambda_\varphi. \quad (\text{C50})$$

The tidal Love numbers are related by

$$\kappa_\phi = \kappa_\varphi. \quad (\text{C51})$$

In the case of the even parity tensor tidal Love number, the transformation between frames is more complex due to mixing of the scalar and tensor perturbations. The relationship between the even parity metric perturbations in the two frames is constrained by the choice of gauge. Taking equation relating the time-time component of the metric perturbation in the Jordan frame \tilde{H} to the Einstein frame metric perturbation H and the Einstein frame scalar perturbation $\delta\varphi$ from Sec. 3, we have

$$\tilde{H} = A^2(\varphi)H - 2A(\varphi)\delta A. \quad (\text{C52})$$

In the spontaneous scalarization case, this becomes

$$\tilde{H} = A^2(\varphi)(H - 2\beta\varphi\delta\varphi). \quad (\text{C53})$$

Combining this with the leading order behavior of the perturbations, which are known to be

$$\begin{aligned} H &= -\mathcal{E}_{ij}r^2 + \mathcal{O}(r) + \frac{3Q_{ij}}{r^3} + \mathcal{O}(r^{-4}) \\ &= -\mathcal{E}_{ij}r^2 + \mathcal{O}(r) - \frac{3\lambda\mathcal{E}_{ij}}{r^3} + \mathcal{O}(r^{-4}) \end{aligned} \quad (\text{C54})$$

it is possible to define the Jordan frame tidal deformability $\tilde{\lambda}_J$:

$$\tilde{\lambda}_J = \frac{A^{2\ell+1}(\lambda_E\mathcal{E}_{ij}^E - 2\beta\varphi_\infty\lambda_\varphi\mathcal{E}_{ij}^\varphi)}{\mathcal{E}_{ij}^E - 2\beta\varphi_\infty\mathcal{E}_{ij}^\varphi} \quad (\text{C55})$$

where E denotes Einstein frame tensor quantities and φ denotes Einstein frame scalar quantities. From this equation, it is clear that the Jordan frame tidal deformability is related linearly to the even parity scalar and tensor tidal deformabilities. The exact relationship is

$$\tilde{\lambda}_J = A^{2\ell+1}(\varphi_\infty)(\lambda_E + \lambda_\varphi). \quad (\text{C56})$$

Finally, we determine the tidal Love numbers to have the following relationship

$$\tilde{k}_\ell = k_\ell + \kappa_\ell \quad (\text{C57})$$

D. $\ell = 3, 4$ LOVE NUMBERS

Using Eq. (28a), Eq. (28b), and the methods presented in Sec. 4.1, we determine the equations for the $\ell = 3, 4$ tidal Love numbers and tidal deformabilities at large r .

The $\ell = 3, 4$ even parity tensor tidal Love numbers are defined as

$$k_3 = 8(1 - 2C)^2 C^7 (-3 - 3C(-2 + y) + 2C^2(-1 + y) + y) \times \quad (D58)$$

$$\left[7(2C(15(-3 + y) + 4C^5(1 + y) - 45C(-5 + 2y) - 20C^3(-9 + 7y) + 2C^4(-2 + 9y) + 5C^2(-72 + 37y)) + 15(1 - 2C)^2(-3 - 3C(-2 + y) + 2C^2(-1 + y) + y) \ln(1 - 2C)) \right]^{-1}$$

$$k_4 = 32(1 - 2C)^2 C^9 (-7(-4 + y) + 28C(-3 + y) - 34C^2(-2 + y) + 12C^3(-1 + y)) \times \quad (D59)$$

$$\left[147(2C(C^2(5360 - 1910y) + C^4(1284 - 996y) - 105(-4 + y) + 8C^6(1 + y) + 105C(-24 + 7y) + 40C^3(-116 + 55y) + C^5(-8 + 68y)) + 15(1 - 2C)^2(-7(-4 + y) + 28C(-3 + y) - 34C^2(-2 + y) + 12C^3(-1 + y)) \ln(1 - 2C)) \right]^{-1}$$

and the scalar tidal Love numbers are

$$\kappa_3 = 12C^7 (-1 + 2C)(-5(-3 + w) + 15C(-2 + w) - 12C^2(-1 + w) + 2C^3w) \times \quad (D60)$$

$$\left[175(-2C(C^2(96 - 71w) - 15(-3 + w) + 15C(-9 + 4w) + C^3(-6 + 22w)) + 3(-1 + 2C)(-5(-3 + w) + 15C(-2 + w) - 12C^2(-1 + w) + 2C^3w) \ln(1 - 2C)) \right]^{-1}$$

$$\kappa_4 = (64C^9(-1 + 2C)(35(-4 + w) - 140C(-3 + w) + 180C^2(-2 + w) - 80C^3(-1 + w) + 8C^4w)) \times \quad (D61)$$

$$\left[3675(-2C(105(-4 + w) - 190C^3(-4 + 3w) - 105C(-16 + 5w) + 4C^4(-6 + 25w) + 10C^2(-206 + 89w)) + 3(-1 + 2C)(35(-4 + w) - 140C(-3 + w) + 180C^2(-2 + w) - 80C^3(-1 + w) + 8C^4w) \ln(1 - 2C)) \right]^{-1}.$$