BRIEF REPORT

Comments on: "Every variance function … can be produced by any location-scale family …"

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I here present a few comments on, and mathematical corrections to, the article (Cohen 2020) by J. E. Cohen, and its recently published corrections note (Cohen 2022). The central claim of Cohen (2020), that no strong inference about the underlying process can be made solely on the basis of the slope of a Taylor's law (or generalization thereof), remains nonetheless perfectly valid.

In the corrections (Cohen 2022), equation (1) correctly implies (2), which however states that the variance—*not the log-variance*—is quadratic in the log-expectations, which is not Theorem 1 of Cohen (2020). The generalization (3) correctly relates log-variance to log-expectation, but then the construction (4) implies not (3) but again that *variance as opposed to log-variance* is a polynomial in the log-expectation. The upshot of correcting the updated proof in Cohen (2022) is that there is no need choose some fine-tuned $gp = \log x$ and to restrict the real coefficients b_n so as to ensure that $\sqrt{\log a + \sum_n b_n \log x}$ is real, because the prefactor to $(Z - \mu)/\sigma$ should be $\exp[(\log a + \sum_n b_n \log x)/2]$ which is necessarily real.

In any case, Theorem 3 of Cohen (2020) is correct, from which Theorems 1 and 2 follow without restrictions on the b_n. Without affecting the existence results contained in these theorems, I note that the construction of the $X(p)$ that furnishes the results can be slightly generalized to

$$X(p) = x(p) + f(x(p))^{1/2} \left(\frac{Z - \mu}{\sigma} \right),$$

for arbitrary positive function $x(p)$, and the set $P(\ni p)$ does not have to be a subset of $\mathbb{R}$. In the proof of Theorem 3, $x(p)$

was taken to be p. A related construction to generate a distribution satisfying Taylor's law with slope b and intercept a from an underlying infinitely divisible distribution was noted by Cohen and Huillet (Cohen and Huillet 2022, Sect. 7–8).

In essence, knowing that $X(p)$ has mean $x(p)$ and variance $V(p)$ ($= f(x(p))$ in the special cases that produce Taylor-like laws) leaves $\widetilde{Z}(p) := (X(p) - x(p))/V(p)$ undetermined up to having zero mean and unit variance—not even its independence of p can be generally inferred! Thus, as emphasized in Cohen (2020) and Cohen and Huillet (2022), the measured slope of Taylor's law in general only allows us to reject models that lead to unobserved slopes. Alternatively, if by some means *independent of measuring Taylor's law* we could ascertain (or hypothesize) the mechanism by which it arises in a given system, then the measured slope could fix parameters of the underlying model.

As a recent example of the latter scenario, consider the macroecological laws proposed for microbial communities by Grilli (2020). It was found empirically that the relative abundance X of a species p across samples is Gamma distributed, $X(p) \sim \text{Gamma}(\alpha_p, \beta_p)$. As noted in Cohen and Huillet (2022), the Gamma distribution can generate Taylor's law with different slopes depending on which combination of α_p and β_p is held constant, as $x(p) = \alpha_p \beta_p$ and $V(p) = \alpha_p \beta_p^2$; for instance, $V(p) = \alpha_p^{-1}[x(p)]^2$. As $\{(x(p), V(p))\}_{p=1,2,3,...}$ was found to satisfy a Taylor's law of slope two for the microbial communities, one concludes $\alpha_p = \alpha$, a constant independent of the species. Grilli proposed a stochastic logistic growth model to generate the macroecological laws, for which $\alpha_p = \alpha$ has the meaning that each species is affected by noise of the same statistical properties (up to irrelevant time-scale differences). For other communities, or in non-ecological examples, Taylor's law may be associated with a different distribution than the Gamma; Cohen and Huillet (2022) details the relationship between the Taylor's law slope and the model parameters in a number of useful cases.

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Declarations

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