

Complete gravitational-waveform amplitude modes for quasicircular compact binaries to the 3.5PN order

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We complete the computation of the gravitational-waveform amplitude for nonspinning compact binaries to the third and a half post-Newtonian (PN) order in the quasicircular case. This is done by performing a spin-weighted spherical harmonics decomposition of the amplitude. This computation is achieved using the post-Newtonian-multipolar-post-Minkowskian formalism. The amplitude modes are written in a PN-expanded way as well as a factorized form suitable for effective-one-body template building. Combining this work with previous ones, we provide in Supplemental Material the expressions of the modes including all nonspinning and spinning effects to the 3.5PN order for nonprecessing quasicircular binaries.

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I. INTRODUCTION

The gravitational wave (GW) detections by current detectors LIGO-Virgo-KAGRA [1–3], as well as next-generation detectors, such as LISA [4] and Einstein Telescope [5], require precise modeling of the waveform in order to extract the signal from the noise. Although only numerical relativity can provide accurate waveforms close to the merger, several analytical methods produce theoretical predictions of the inspiral phase. To model this phase, three main types of expansion exist. First of all, the post-Newtonian (PN) one consists in performing a Taylor expansion of the equations using the small parameter v^2/c^2 , where v is the relative velocity of the two companions of a binary system and c the speed of light. In the standard nomenclature, an n PN quantity is of order $\mathcal{O}(1/c^{2n})$. The second one is the post-Minkowskian (PM) expansion, in which we consider the small parameter GM/rc^2 , where G is Newton's constant, M the total mass of the system and r the separation. The last one consists in performing an expansion using the small parameter is the mass ratio $q = m_1/m_2$. Different frameworks using these types of approximations provide analytical solutions for physical observables such as the radiated energy flux or the GW amplitude and phase. These partial results overlap and can be combined to numerical relativity notably using

the effective-one-body (EOB) formalism [6,7] which produces accurate waveforms over the entire parameter space.

In the radiative sector, the energy flux as well as the GW phase are known to the 3.5PN order beyond the Einstein quadrupole formula [8,9] and are now being pushed to 4PN [10–15]. However, only partial results are currently available for the amplitude of the gravitational waveform to the 3.5PN order. The amplitude can be computed in the form of a spin-weighted spherical harmonics mode decomposition, parametrized as usual by two numbers (ℓ, m) .

The test-mass limit approximation provides extremely useful results in the amplitude modes at the leading order in the mass ratio. In the parameter space where both the PN and the test-mass limit approximations are valid, these results should agree, which constitutes a very powerful check. The amplitude modes in the test-mass limit are displayed in, e.g., [16–18].

For nonspinning binaries, the amplitude modes in the PN approximation have been derived consistently to the 3PN order [19]. Regarding the 3.5PN accuracy, the $(\ell, m) = (2, 2)$ mode was published in [20], the $(3, 3)$ and $(3, 1)$ modes in [21] and the $(2, 1)$ mode in [11]. Some other modes were derived in [22], especially some for high ℓ at the relative 1PN order, matching the 3.5PN accuracy in the waveform. These modes have been used for EOB template building. It has been found that factorizing their expressions in several blocks increases the convergence of the EOB model when compared to numerical relativity [23]. Thus, the PN-expanded modes at the third PN order were factorized in, e.g., [24,25] to give more accurate EOB templates. Although the factorized form of the $(2, 2)$, $(3, 3)$ and $(3, 1)$ modes are known for the 3.5PN accuracy in the waveform, the one of the $(2, 1)$ mode was never published.

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Furthermore, it was found out recently that an error in a published version of the (2,1) mode at the 2.5PN order propagated in the EOB models (see [26] for justification). The PN-expanded form of the (2,1) mode eventually got corrected in the erratum of [19] but the mistake in the EOB models remained at the time of publication of this work.

In order to prevent new propagation of mistakes and to avoid multiplying references for different terms in the PN-expanded modes as well as their factorized version, this paper aims at completing the full gravitational waveform to the 3.5PN order, including spin effects for a quasicircular and nonprecessing motion. The spin effects in the waveform modes to this order were derived in the recent work [26] and we now complete the missing nonspinning coefficients. We also write the modes in a convenient way for EOB template building. Due to the length of the expressions, the spin contributions to the amplitude modes are not be displayed here but can be found in [26]. However, Supplemental Material attached to this paper contains both nonspinning and spinning coefficients to the 3.5PN order [27].

In the present paper, we use the post-Newtonian-multipolar-post-Minkowskian (PNMPPM) formalism; see [28] for a detailed review of this approach. This method combines the PN and the PM expansions, so that we consider the small parameters $v^2/c^2 \sim GM/c^2 r \ll 1$. We will qualitatively describe this method in Sec. II B.

The paper is organized as follows. In Sec. II, we define the spherical modes and recall their relation with the radiative multipole moments. We also explain qualitatively how to derive the radiative moments within the PNMPPM scheme. We refer to previous works for technical details of this derivation. In Sec. III, we give the results for the nonspinning terms in amplitude modes, written in a conventional Taylor expansion form in Sec. III A and factorized conveniently for EOB usage in Sec. III B. Appendix A contains generalized formulas to compute some particular hereditary effects. Appendix B contains the different quantities appearing in the factorized modes. We also provide our results for the full waveform modes including nonspinning and spinning effects to the 3.5PN order as *Mathematica* files in Supplemental Material [27].

II. BRIEF OVERVIEW OF THE METHOD

A. Spherical harmonics decomposition of the gravitational field

The transverse-traceless (TT) projection h_{ij}^{TT} of the gravitational field can be uniquely decomposed in terms of a set of symmetric trace-free (STF) multipole moments U_L and V_L , called radiative multipole moments, as¹ [29]

¹We use the signature $(-, +, +, +)$. Latin indices stand for spatial coordinates, i.e. $i = 1, 2, 3$ and the multi-index notation $L = i_1 \dots i_\ell$. The weighted symmetrization operator is noted by parentheses around indices.

$$h_{ij}^{\text{TT}} = \frac{4G}{c^2 R} \mathcal{P}_{ijkl}(N) \sum_{\ell=2}^{+\infty} \frac{1}{c^\ell \ell!} \left\{ N_{L-2} U_{kL-2}(T_R) - \frac{2\ell}{c(\ell+1)} N_{aL-2} \varepsilon_{ab(k} V_{l)bL-2}(T_R) \right\} + \mathcal{O}\left(\frac{1}{R^2}\right), \quad (2.1)$$

where R is the distance between the source and the observer, N is the direction of propagation of the GW and $T_R = T - R/c$ is the retarded time in some radiative gauge in which T_R is asymptotically null. The quantity $\mathcal{P}_{ijkl} = \mathcal{P}_{i(k} \mathcal{P}_{l)j} - \frac{1}{2} \mathcal{P}_{ij} \mathcal{P}_{kl}$ is the TT projection operator, where $\mathcal{P}_{ij} = \delta_{ij} - N_i N_j$. The polarization waveforms are defined by

$$h_+ = \frac{1}{2} (P_i P_j - Q_i Q_j) h_{ij}^{\text{TT}}, \quad (2.2a)$$

$$h_\times = \frac{1}{2} (P_i Q_j + Q_i P_j) h_{ij}^{\text{TT}}, \quad (2.2b)$$

where the vectors (P, Q, N) form an orthonormal triad properly defined in, e.g., Sec. II. A of [26]. As usual, we decompose $h_+ - ih_\times$ in a spin-weighted spherical harmonics basis of weight -2 [30]:

$$h \equiv h_+ - ih_\times = \sum_{l=0}^{\infty} \sum_{m=-\ell}^{\ell} h_{\ell m} Y_{-2}^{\ell m}(\Theta, \Phi), \quad (2.3)$$

where the two angles (Θ, Φ) characterize the direction of propagation N . The amplitude modes $h_{\ell m}$ are then linked to the radiative multipole moments through

$$h_{\ell m} = -\frac{G}{\sqrt{2} R c^{\ell+2}} \left(U^{\ell m} - \frac{i}{c} V^{\ell m} \right). \quad (2.4)$$

In our convention, $U^{\ell m}$ and $V^{\ell m}$ are given by²

$$U^{\ell m} = \frac{4}{\ell!} \sqrt{\frac{(\ell+1)(\ell+2)}{2\ell(\ell-1)}} \alpha_L^{\ell m} U_L, \quad (2.5a)$$

$$V^{\ell m} = -\frac{8}{\ell!} \sqrt{\frac{\ell(\ell+2)}{2(\ell+1)(\ell-1)}} \alpha_L^{\ell m} V_L, \quad (2.5b)$$

with $\alpha_L^{\ell m} \equiv \int d\Omega \hat{N}_L \bar{Y}^{\ell m}$ being defined from the complex conjugate of the ordinary spherical harmonics $Y^{\ell m}$. Its explicit expression is displayed in Eq. (4.7) of Ref. [11].

As we can see from Eqs. (2.4) and (2.5), one needs to derive the radiative multipole moments U_L and V_L to

²Note that the choice of definitions in Refs. [30,31] on $h_{\ell m}$ differs from ours by a global minus sign for each ℓ and m due to a different definition of the vector basis (P, Q, N) .

consistent order to obtain the amplitude modes $h_{\ell m}$. The derivation of these moments are described in the following section.

B. Computational method

In this section, we describe qualitatively how to derive the radiative multipole moments required for the knowledge of the full waveform at the 3.5PN order. We will not give the technical details of the intermediate computations as they were broadly described in other papers cited below. A more detailed overview of each step can be found in, e.g., Sec. III of [26].

The PNMPM formalism provides an efficient way to compute the radiative multipole moments defined in (2.1); see Ref. [28] for a review. The first step of this computation consists in deriving the equations of motion (EOM) of the individual compact bodies. They are displayed to the 3.5PN order in [20] in harmonic coordinates and have been derived more recently to the 4PN order in [32] in the same gauge. In the case of quasicircular orbits, one can extract the orbital frequency of the system ω from the EOM which allows defining the orbital phase through

$$\phi = \int dt \omega(t). \quad (2.6)$$

To derive the phase to a given PN order, one needs to know both the conserved energy and the radiated energy flux at the same order. As mentioned in the introduction, this phase is known to the 3.5PN order. However, the PN precision required on the radiative multipole moments to derive the radiated flux, and hence the phase, is lower than the one required to derive the amplitude modes. In other words, one needs more PN information to derive the modes than the energy flux at a given PN order.

In this project, we computed additional PN terms in the U_L for $\ell \in \{4, \dots, 9\}$ and V_L for $\ell \in \{3, \dots, 8\}$. The mass quadrupole, mass octupole and current quadrupole were already obtained in previous works to consistent order [11,20,21].

For this project, we first computed the so-called source multipole moments. They are defined by constructing the most general solution of Einstein's equations in vacuum in the form of a multipolar post-Minkowskian expansion outside the system. They are then matched to the PN metric, valid in a spatial zone near the system. By doing so, we find that the source multipole moments are integrals over quantities that contain the stress-energy tensor and the PN metric. The explicit expressions of the source moments are given in, e.g., Eqs. (2.31) and (2.33) of [11]. The metric as well as the stress-energy tensor were already known from previous works (see, e.g., [10]), in which these quantities were derived for the 4PN accuracy of the source mass quadrupole.

In order to perform this matching between different representations of the gravitational field, we must resort to a regularization scheme. We used in this paper the *Hadamard partie finie* regularization. However, it was found out that this regularization starts to fail at high PN orders. In fact, only dimensional regularization has been proved to be satisfactory. It was shown in previous works that, for nonspinning binaries, these two regularizations start to differ to the 3PN order [9,12,13,33–35]. In this project, the missing multipoles to complete the full nonspinning waveform at the 3.5PN order were at most required to the 2.5PN order so that we could use the *Hadamard partie finie* regularization safely. The integration methods for this regularization scheme are provided in, e.g., [36].

This integration leads to the expression of the source multipole moments in an arbitrary frame for general orbits. The next step consists in reducing the expressions to the center of mass (c.m.) frame and impose a quasicircular motion. To do so, we use the c.m. position displayed in, e.g., [37] and neglect eccentricities.

As a last step, we deduce the radiative moments from the source moments. Due to the nonlinearity of the gravitational field, they differ by some nonlinear interactions of different moments. We find two main types of contributions: the so-called instantaneous interactions, which correspond to at least quadratic products of source multipoles evaluated at retarded time; and hereditary interactions, which contain, for example, tail and memory integrals over the past history of the source preceding the retarded time. The general expressions of the interactions were derived entirely to the 3.5PN order in [21]. By using these general formulas and the integration methods presented in Sec. VII of [19], we obtained all the radiative multipole moments that were needed to compute the full waveform to the 3.5PN order. In Appendix A, we give the generalization to higher PN order of some integrals required for computing the memory interactions.

Finally, we inserted the values of these moments in Eqs. (2.4) and (2.5) to obtain the GW modes displayed in the following section.

III. RESULTS

In this section, we display the results for the waveform modes written in two different ways. In Sec. III A, we give their expressions in the conventional PN-expanded way in the form of, e.g., [19]. In Sec. III B, we write them in a factorized form convenient for the EOB method.

A. Spin-weighted spherical modes

The amplitude modes defined in Eq. (2.4) can be written in terms of the phase variable ϕ defined in (2.6). In order to simplify the explicit expressions of the modes, we introduce the new phase variable

$$\psi \equiv \phi - \frac{2GM\omega}{c^3} \ln\left(\frac{\omega}{\omega_0}\right), \quad (3.1)$$

where \mathcal{M} is the Arnowitt-Deser-Misner mass and where the scale constant ω_0 is introduced in the definition of the tail integrals [19]. This redefinition absorbs most of the logarithms of the orbital frequency in the oscillatory part of the modes. The GW modes appearing in (2.3) read

$$h_{\ell m} = \frac{2GM\nu x}{Rc^2} \sqrt{\frac{16\pi}{5}} \hat{H}_{\ell m} e^{-im\psi}, \quad (3.2)$$

where M is the total mass of the system, $\nu = m_1 m_2 / M^2$ is the symmetric mass ratio and where we introduced the convenient PN parameter

$$x = \left(\frac{GM\omega}{c^3}\right)^{2/3}. \quad (3.3)$$

The values of the $\hat{H}_{\ell m}$'s for the full waveform produced by binary a binary system of nonspinning compact objects to the 3.5PN order are given by

$$\begin{aligned} \hat{H}_{22} = & 1 + x \left(-\frac{107}{42} + \frac{55\nu}{42} \right) + 2\pi x^{3/2} + x^2 \left(-\frac{2173}{1512} - \frac{1069\nu}{216} + \frac{2047\nu^2}{1512} \right) \\ & + x^{5/2} \left(-\frac{107\pi}{21} - 24i\nu + \frac{34\pi\nu}{21} \right) + x^3 \left(\frac{27027409}{646800} - \frac{856\gamma_E}{105} + \frac{428i\pi}{105} + \frac{2\pi^2}{3} \right. \\ & + \left(-\frac{278185}{33264} + \frac{41\pi^2}{96} \right) \nu - \frac{20261\nu^2}{2772} + \frac{114635\nu^3}{99792} - \frac{428}{105} \ln(16x) \Big) \\ & + x^{7/2} \left(-\frac{2173}{756} \pi + \left(-\frac{2495}{378} \pi + \frac{14333}{162} i \right) \nu + \left(\frac{40}{27} \pi - \frac{4066}{945} i \right) \nu^2 \right), \end{aligned} \quad (3.4a)$$

$$\begin{aligned} \hat{H}_{21} = & \frac{1}{3} i \delta \left[x^{1/2} + x^{3/2} \left(-\frac{17}{28} + \frac{5\nu}{7} \right) + x^2 \left(\pi + i \left(-\frac{1}{2} - 2 \ln 2 \right) \right) \right. \\ & + x^{5/2} \left(-\frac{43}{126} - \frac{509\nu}{126} + \frac{79\nu^2}{168} \right) + x^3 \left(-\frac{17\pi}{28} + \frac{3\pi\nu}{14} + i \left(\frac{17}{56} + \nu \left(-\frac{353}{28} - \frac{3 \ln 2}{7} \right) + \frac{17 \ln 2}{14} \right) \right) \\ & + x^{7/2} \left(\frac{15223771}{1455300} + \frac{\pi^2}{6} - \frac{214}{105} \gamma_E - \frac{107}{105} \ln(4x) - \ln 2 - 2(\ln 2)^2 \right. \\ & \left. \left. + \nu \left(-\frac{102119}{2376} + \frac{205}{128} \pi^2 \right) - \frac{4211}{8316} \nu^2 + \frac{2263}{8316} \nu^3 + i\pi \left(\frac{109}{210} - 2 \ln 2 \right) \right) \right], \end{aligned} \quad (3.4b)$$

$$\hat{H}_{20} = -\frac{5}{14\sqrt{6}} \left[1 + x \left(-\frac{4075}{4032} + \frac{67\nu}{48} \right) \right] + \mathcal{O}\left(\frac{1}{c^4}\right), \quad (3.4c)$$

$$\begin{aligned} \hat{H}_{33} = & -\frac{3}{4} i \sqrt{\frac{15}{14}} \delta \left[x^{1/2} + x^{3/2} (-4 + 2\nu) + x^2 \left(3\pi + i \left(-\frac{21}{5} + 6 \ln(3/2) \right) \right) \right. \\ & + x^{5/2} \left(\frac{123}{110} - \frac{1838\nu}{165} + \frac{887\nu^2}{330} \right) + x^3 \left(-12\pi + \frac{9\pi\nu}{2} + i \left(\frac{84}{5} - 24 \ln(3/2) + \nu \left(-\frac{48103}{1215} + 9 \ln(3/2) \right) \right) \right) \\ & + x^{7/2} \left(\frac{19388147}{280280} + \frac{492}{35} \ln(3/2) - 18 \ln^2(3/2) - \frac{78}{7} \gamma_E + \frac{3}{2} \pi^2 + 6i\pi \left(-\frac{41}{35} + 3 \ln(3/2) \right) \right. \\ & \left. \left. + \frac{\nu}{8} \left(-\frac{7055}{429} + \frac{41}{8} \pi^2 \right) - \frac{318841}{17160} \nu^2 + \frac{8237}{2860} \nu^3 - \frac{39}{7} \ln(16x) \right) \right], \end{aligned} \quad (3.4d)$$

$$\begin{aligned} \hat{H}_{32} = & \frac{1}{3} \sqrt{\frac{5}{7}} \left[x(1 - 3\nu) + x^2 \left(-\frac{193}{90} + \frac{145\nu}{18} - \frac{73\nu^2}{18} \right) + x^{5/2} \left(2\pi - 6\pi\nu + i \left(-3 + \frac{66\nu}{5} \right) \right) \right. \\ & \left. + x^3 \left(-\frac{1451}{3960} - \frac{17387\nu}{3960} + \frac{5557\nu^2}{220} - \frac{5341\nu^3}{1320} \right) + x^{7/2} \left(\frac{193}{30} i - \frac{193}{45} \pi + \nu \left(-\frac{258929}{5400} i + \frac{136}{9} \pi \right) + \nu^2 \left(\frac{33751}{450} i - \frac{46}{9} \pi \right) \right) \right], \end{aligned} \quad (3.4e)$$

$$\begin{aligned}\hat{H}_{31} = & \frac{i\delta}{12\sqrt{14}} \left[x^{1/2} + x^{3/2} \left(-\frac{8}{3} - \frac{2\nu}{3} \right) + x^2 \left(\pi + i \left(-\frac{7}{5} - 2 \ln 2 \right) \right) \right. \\ & + x^{5/2} \left(\frac{607}{198} - \frac{136\nu}{99} - \frac{247\nu^2}{198} \right) + x^3 \left(-\frac{8\pi}{3} - \frac{7\pi\nu}{6} + i \left(\frac{56}{15} + \frac{16 \ln 2}{3} + \nu \left(-\frac{1}{15} + \frac{7 \ln 2}{3} \right) \right) \right) \\ & + x^{7/2} \left(\frac{10753397}{1513512} - 2 \ln 2 \left(\frac{212}{105} + \ln 2 \right) - \frac{26}{21} \gamma_E + \frac{\pi^2}{6} - 2i\pi \left(\frac{41}{105} + \ln 2 \right) \right. \\ & \left. \left. + \frac{\nu}{8} \left(-\frac{1738843}{19305} + \frac{41}{8} \pi^2 \right) + \frac{327059}{30888} \nu^2 - \frac{17525}{15444} \nu^3 - \frac{13}{21} \ln x \right) \right],\end{aligned}\quad (3.4f)$$

$$\hat{H}_{30} = -\frac{2}{5} i \sqrt{\frac{6}{7}} \nu \left[x^{5/2} + x^{7/2} \left(-\frac{5017}{1296} - \frac{25}{108} \nu \right) \right], \quad (3.4g)$$

$$\begin{aligned}\hat{H}_{44} = & -\frac{8}{9} \sqrt{\frac{5}{7}} \left[x(1-3\nu) + x^2 \left(-\frac{593}{110} + \frac{1273\nu}{66} - \frac{175\nu^2}{22} \right) + x^{5/2} \left(4\pi - 12\pi\nu + i \left(-\frac{42}{5} + \nu \left(\frac{1193}{40} - 24 \ln 2 \right) + 8 \ln 2 \right) \right) \right. \\ & + x^3 \left(\frac{1068671}{200200} - \frac{1088119\nu}{28600} + \frac{146879\nu^2}{2340} - \frac{226097\nu^3}{17160} \right) + x^{7/2} \left(\frac{12453}{275} i - \frac{1186}{55} \pi - \frac{2372}{55} i \ln 2 \right. \\ & \left. + \nu \left(-\frac{31525499}{140800} i + \frac{2480}{33} \pi + \frac{4960}{33} i \ln 2 \right) + \nu^2 \left(\frac{4096237}{21120} i - \frac{284}{11} \pi - \frac{568}{11} i \ln 2 \right) \right],\end{aligned}\quad (3.4h)$$

$$\begin{aligned}\hat{H}_{43} = & -\frac{9i\delta}{4\sqrt{70}} \left[x^{3/2}(1-2\nu) + x^{5/2} \left(-\frac{39}{11} + \frac{1267\nu}{132} - \frac{131\nu^2}{33} \right) + x^3 \left(3\pi - 6\pi\nu + i \left(-\frac{32}{5} + \nu \left(\frac{16301}{810} - 12 \ln(3/2) \right) \right. \right. \right. \\ & \left. \left. + 6 \ln(3/2) \right) \right) + x^{7/2} \left(\frac{7206}{5005} - \frac{82869}{5720} \nu + \frac{104839}{3432} \nu^2 - \frac{2987}{572} \nu^3 \right) \right],\end{aligned}\quad (3.4i)$$

$$\begin{aligned}\hat{H}_{42} = & \frac{1}{63} \sqrt{5} \left[x(1-3\nu) + x^2 \left(-\frac{437}{110} + \frac{805\nu}{66} - \frac{19\nu^2}{22} \right) + x^{5/2} \left(2\pi - 6\pi\nu + i \left(-\frac{21}{5} + \frac{84\nu}{5} \right) \right) \right. \\ & + x^3 \left(\frac{1038039}{200200} - \frac{606751\nu}{28600} + \frac{400453\nu^2}{25740} + \frac{25783\nu^3}{17160} \right) \\ & \left. + x^{7/2} \left(\frac{9177}{550} i - \frac{437}{55} \pi + \nu \left(-\frac{83029}{880} i + \frac{772}{33} \pi \right) + \nu^2 \left(\frac{93081}{1100} i + \frac{14}{11} \pi \right) \right) \right],\end{aligned}\quad (3.4j)$$

$$\begin{aligned}\hat{H}_{41} = & \frac{i\delta}{84\sqrt{10}} \left[x^{3/2}(1-2\nu) + x^{5/2} \left(-\frac{101}{33} + \frac{337\nu}{44} - \frac{83\nu^2}{33} \right) + x^3 \left(\pi - 2\pi\nu + i \left(-\frac{32}{15} - 2 \ln 2 + \nu \left(\frac{1661}{30} + 4 \ln 2 \right) \right) \right) \right. \\ & \left. + x^{7/2} \left(\frac{42982}{15015} - \frac{513989}{51480} \nu + \frac{196957}{10296} \nu^2 - \frac{1195}{572} \nu^3 \right) \right],\end{aligned}\quad (3.4k)$$

$$\hat{H}_{40} = -\frac{1}{504\sqrt{2}} \left[1 + x \left(-\frac{180101}{29568} + \frac{27227}{1056} \nu \right) \right] + \mathcal{O}\left(\frac{1}{c^4}\right), \quad (3.4l)$$

$$\begin{aligned}\hat{H}_{55} = & \frac{625i\delta}{96\sqrt{66}} \left[x^{3/2}(1-2\nu) + x^{5/2} \left(-\frac{263}{39} + \frac{688\nu}{39} - \frac{256\nu^2}{39} \right) \right. \\ & + x^3 \left(5\pi - 10\pi\nu + i \left(-\frac{181}{14} + \nu \left(\frac{105834}{3125} - 20 \ln(5/2) \right) + 10 \ln(5/2) \right) \right) \\ & \left. + x^{7/2} \left(\frac{9185}{819} - \frac{188765}{3276} \nu + \frac{54428}{819} \nu^2 - \frac{10567}{819} \nu^3 \right) \right],\end{aligned}\quad (3.4m)$$

$$\hat{H}_{54} = -\frac{32}{9\sqrt{165}} \left[x^2(1-5\nu+5\nu^2) + x^3 \left(-\frac{4451}{910} + \frac{3619\nu}{130} - \frac{521\nu^2}{13} + \frac{339\nu^3}{26} \right) \right. \\ \left. + x^{7/2} \left(-\frac{52}{5}i + 4\pi + 8i \ln 2 + \nu \left(\frac{3351011}{53760}i - 20\pi - 40i \ln 2 \right) + \nu^2 \left(-\frac{10923}{128}i + 20\pi + 40i \ln 2 \right) \right) \right], \quad (3.4n)$$

$$\hat{H}_{53} = -\frac{9}{32}i\sqrt{\frac{3}{110}}\delta \left[x^{3/2}(1-2\nu) + x^{5/2} \left(-\frac{69}{13} + \frac{464\nu}{39} - \frac{88\nu^2}{39} \right) \right. \\ \left. + x^3 \left(3\pi - 6\pi\nu + i \left(-\frac{543}{70} + \nu \left(\frac{83702}{3645} - 12 \ln(3/2) \right) + 6 \ln(3/2) \right) \right) \right. \\ \left. + x^{7/2} \left(\frac{12463}{1365} - \frac{56969}{1820}\nu + \frac{2172}{91}\nu^2 - \frac{365}{273}\nu^3 \right) \right], \quad (3.4o)$$

$$\hat{H}_{52} = \frac{2}{27\sqrt{55}} \left[x^2(1-5\nu+5\nu^2) + x^3 \left(-\frac{3911}{910} + \frac{3079\nu}{130} - \frac{413\nu^2}{13} + \frac{231\nu^3}{26} \right) \right. \\ \left. + x^{7/2} \left(-\frac{26}{5}i + 2\pi + \nu \left(\frac{16237}{336}i - 10\pi \right) + \nu^2 \left(-\frac{1861}{20}i + 10\pi \right) \right) \right], \quad (3.4p)$$

$$\hat{H}_{51} = \frac{i\delta}{288\sqrt{385}} \left[x^{3/2}(1-2\nu) + x^{5/2} \left(-\frac{179}{39} + \frac{352\nu}{39} - \frac{4\nu^2}{39} \right) \right. \\ \left. + x^3 \left(\pi - 2\pi\nu + i \left(-\frac{181}{70} - 2 \ln 2 + \nu \left(\frac{626}{5} + 4 \ln 2 \right) \right) \right) + x^{7/2} \left(\frac{5023}{585} - \frac{49447}{2340}\nu + \frac{68}{9}\nu^2 + \frac{287}{117}\nu^3 \right) \right], \quad (3.4q)$$

$$\hat{H}_{50} = \frac{4117i\nu}{7560\sqrt{462}} x^{7/2} \left[1 - \frac{21588}{4117}\nu \right], \quad (3.4r)$$

$$\hat{H}_{66} = \frac{54}{5\sqrt{143}} \left[x^2(1-5\nu+5\nu^2) + x^3 \left(-\frac{113}{14} + \frac{91\nu}{2} - 64\nu^2 + \frac{39\nu^3}{2} \right) \right. \\ \left. + x^{7/2} \left(-\frac{249}{14}i + 6\pi + 12i \ln 3 + \nu \left(\frac{21787499}{217728}i - 30\pi - 60i \ln 3 \right) + \nu^2 \left(-\frac{323903}{2592}i + 30\pi + 60i \ln 3 \right) \right) \right], \quad (3.4s)$$

$$\hat{H}_{65} = \frac{3125i\delta}{504\sqrt{429}} \left[x^{5/2}(1-4\nu+3\nu^2) + x^{7/2} \left(-\frac{149}{24} + \frac{349}{12}\nu - \frac{409}{12}\nu^2 + \frac{29}{3}\nu^3 \right) \right], \quad (3.4t)$$

$$\hat{H}_{64} = -\frac{128}{495}\sqrt{\frac{2}{39}} \left[x^2(1-5\nu+5\nu^2) + x^3 \left(-\frac{93}{14} + \frac{71\nu}{2} - 44\nu^2 + \frac{19\nu^3}{2} \right) \right. \\ \left. + x^{7/2} \left(-\frac{83}{7}i + 4\pi + 8i \ln 2 + \nu \left(\frac{3432215}{49152}i - 20\pi - 40i \ln 2 \right) + \nu^2 \left(-\frac{382365}{4096}i + 20\pi + 40i \ln 2 \right) \right) \right], \quad (3.4u)$$

$$\hat{H}_{63} = -\frac{81i\delta}{616\sqrt{65}} \left[x^{5/2}(1-4\nu+3\nu^2) + x^{7/2} \left(-\frac{133}{24} + \frac{301}{12}\nu - \frac{329}{12}\nu^2 + 7\nu^3 \right) \right], \quad (3.4v)$$

$$\hat{H}_{62} = \frac{2}{297\sqrt{65}} \left[x^2(1-5\nu+5\nu^2) + x^3 \left(-\frac{81}{14} + \frac{59\nu}{2} - 32\nu^2 + \frac{7\nu^3}{2} \right) \right. \\ \left. + x^{7/2} \left(-\frac{83}{14}i + 2\pi + \nu \left(\frac{799789}{13440}i - 10\pi \right) + \nu^2 \left(-\frac{19193}{160}i + 10\pi \right) \right) \right], \quad (3.4w)$$

$$\hat{H}_{61} = \frac{i\delta}{8316\sqrt{26}} \left[x^{5/2}(1-4\nu+3\nu^2) + x^{7/2} \left(-\frac{125}{24} + \frac{277}{12}\nu - \frac{289}{12}\nu^2 + \frac{17}{3}\nu^3 \right) \right], \quad (3.4x)$$

$$\hat{H}_{60} = \frac{4195x}{1419264\sqrt{273}} \left[1 - \frac{3612}{839}\nu \right] + \mathcal{O}\left(\frac{1}{c^4}\right), \quad (3.4y)$$

$$\hat{H}_{77} = -\frac{16807i\delta}{1440} \sqrt{\frac{7}{858}} \left[x^{5/2}(1 - 4\nu + 3\nu^2) + x^{7/2} \left(-\frac{319}{34} + \frac{2225}{51}\nu - \frac{2558}{51}\nu^2 + \frac{230}{17}\nu^3 \right) \right], \quad (3.4z)$$

$$\hat{H}_{76} = \frac{81}{35} \sqrt{\frac{3}{143}} x^3 [1 - 7\nu + 14\nu^2 - 7\nu^3], \quad (3.4aa)$$

$$\hat{H}_{75} = \frac{15625i\delta}{26208\sqrt{66}} \left[x^{5/2}(1 - 4\nu + 3\nu^2) + x^{7/2} \left(-\frac{271}{34} + \frac{1793}{51}\nu - \frac{1838}{51}\nu^2 + \frac{134}{17}\nu^3 \right) \right], \quad (3.4bb)$$

$$\hat{H}_{74} = -\frac{128x^3}{1365} \sqrt{\frac{2}{33}} [1 - 7\nu + 14\nu^2 - 7\nu^3], \quad (3.4cc)$$

$$\hat{H}_{73} = -\frac{243i\delta}{160160} \sqrt{\frac{3}{2}} \left[x^{5/2}(1 - 4\nu + 3\nu^2) + x^{7/2} \left(-\frac{239}{34} + \frac{1505}{51}\nu - \frac{1358}{51}\nu^2 + \frac{70}{17}\nu^3 \right) \right], \quad (3.4dd)$$

$$\hat{H}_{72} = \frac{x^3}{3003\sqrt{3}} [1 - 7\nu + 14\nu^2 - 7\nu^3], \quad (3.4ee)$$

$$\hat{H}_{71} = \frac{i\delta}{864864\sqrt{2}} \left[x^{5/2}(1 - 4\nu + 3\nu^2) + x^{7/2} \left(-\frac{223}{34} + \frac{1361}{51}\nu - \frac{1118}{51}\nu^2 + \frac{38}{17}\nu^3 \right) \right], \quad (3.4ff)$$

$$\hat{H}_{70} = \mathcal{O}\left(\frac{1}{c^8}\right), \quad (3.4gg)$$

$$\hat{H}_{88} = -\frac{16384}{63} \sqrt{\frac{2}{85085}} x^3 [1 - 7\nu + 14\nu^2 - 7\nu^3], \quad (3.4hh)$$

$$\hat{H}_{87} = -\frac{117649i\delta}{5184} \sqrt{\frac{7}{24310}} x^{7/2} [1 - 6\nu + 10\nu^2 - 4\nu^3], \quad (3.4ii)$$

$$\hat{H}_{86} = \frac{243}{35} \sqrt{\frac{3}{17017}} x^3 [1 - 7\nu + 14\nu^2 - 7\nu^3], \quad (3.4jj)$$

$$\hat{H}_{85} = \frac{78125i\delta}{36288\sqrt{4862}} x^{7/2} [1 - 6\nu + 10\nu^2 - 4\nu^3], \quad (3.4kk)$$

$$\hat{H}_{84} = -\frac{128}{4095} \sqrt{\frac{2}{187}} x^3 [1 - 7\nu + 14\nu^2 - 7\nu^3], \quad (3.4ll)$$

$$\hat{H}_{83} = -\frac{81i\delta}{5824} \sqrt{\frac{3}{1870}} x^{7/2} [1 - 6\nu + 10\nu^2 - 4\nu^3], \quad (3.4mm)$$

$$\hat{H}_{82} = \frac{x^3}{9009\sqrt{85}} [1 - 7\nu + 14\nu^2 - 7\nu^3], \quad (3.4nn)$$

$$\hat{H}_{81} = \frac{i\delta}{741312\sqrt{238}} x^{7/2} [1 - 6\nu + 10\nu^2 - 4\nu^3], \quad (3.4oo)$$

$$\hat{H}_{80} = \mathcal{O}\left(\frac{1}{c^4}\right), \quad (3.4pp)$$

$$\hat{H}_{99} = \frac{1594323i\delta}{7168\sqrt{20995}}x^{7/2}[1 - 6\nu + 10\nu^2 - 4\nu^3], \quad (3.4qq)$$

$$\hat{H}_{97} = -\frac{5764801i\delta}{1410048\sqrt{1235}}x^{7/2}[1 - 6\nu + 10\nu^2 - 4\nu^3], \quad (3.4rr)$$

$$\hat{H}_{95} = \frac{390625i\delta}{4935168\sqrt{247}}x^{7/2}[1 - 6\nu + 10\nu^2 - 4\nu^3], \quad (3.4ss)$$

$$\hat{H}_{93} = -\frac{81i\delta}{113152}\sqrt{\frac{3}{665}}x^{7/2}[1 - 6\nu + 10\nu^2 - 4\nu^3], \quad (3.4tt)$$

$$\hat{H}_{91} = \frac{i\delta}{9165312\sqrt{2090}}x^{7/2}[1 - 6\nu + 10\nu^2 - 4\nu^3]. \quad (3.4uu)$$

The modes written without a remainder are all known up to order $\mathcal{O}(1/c^7)$. Note that some modes for $(\ell, 0)$ are written with a lower order remainder. As discussed in [26], the contributions to these modes only arise from the memory integrals in the mass-type radiative moments. Although the PNMPM does not allow to compute them to high orders, they have been derived to the 3PN order beyond their leading contribution in [22]. Other modes not appearing in (3.4) are of order $\mathcal{O}(1/c^8)$ and are thus neglected here. Finally, the modes for negative m can be recovered using $\hat{H}_{\ell, -m} = (-1)^\ell \hat{H}_{\ell m}^*$ where the star notation refers to the complex conjugate operation.

The modes (2,2), (2,1), (3,3) and (3,1) were already known for the 3.5PN waveform and derived in [11,20,21]. We recover these published results. Furthermore, the computation has been pushed 0.5PN higher for the other modes which completes the full waveform at the 3.5PN order. The overlaps agree with the existing literature [19,30] and notably with the modes derived for a test particle around a Schwarzschild black hole [16–18,25,31].

B. Effective-one-body factorized modes

In this section, we recall the conventions and definitions of the resummed form of the modes convenient for EOB application. Following Refs. [23,24,31,38], we rewrite the PN-expanded waveform modes of Eqs. (3.2)–(3.4) in a factorized, resummed form as³

$$h_{\ell m}^F = h_{\ell m}^{(N, \epsilon_p)} \hat{S}_{\text{eff}}^{(\epsilon_p)} T_{\ell m} e^{i\delta_{\ell m}} f_{\ell m}, \quad (3.5)$$

where ϵ_p is the parity of $\ell + m$: $\epsilon_p = 0$ if $\ell + m$ is even, and $\epsilon_p = 1$ if $\ell + m$ is odd. We see that the modes are factorized in five blocks. The first one, $h_{\ell m}^{(N, \epsilon_p)}$, is the leading PN order contribution. It is known analytically for each ℓ and m [29,30].

³In this section, we pose $c = G = 1$.

The second factor is the effective source term \hat{S}_{eff} which, depending on ϵ_p , is either the effective energy E_{eff} or the orbital angular momentum p_ϕ :

$$\hat{S}_{\text{eff}} = \begin{cases} \frac{E_{\text{eff}}(v)}{\mu}, & \ell + m \text{ even}, \\ v \frac{p_\phi(v)}{\mu M}, & \ell + m \text{ odd}, \end{cases} \quad (3.6)$$

where $\mu \equiv m_1 m_2 / M$ is the reduced mass, $v \equiv (M\omega)^{1/3} = \sqrt{x}$ and E_{eff} is related to the total energy E via the EOB energy map $E = M\sqrt{1 + 2\nu(E_{\text{eff}}/\mu - 1)}$.

Similarly to the phase redefinition (3.1), the factor $T_{\ell m}$ allows absorption of the “leading logarithms” induced by tail effects [39–41]. It is given by

$$T_{\ell m} = \frac{\Gamma(\ell + 1 - 2i\hat{k})}{\Gamma(\ell + 1)} e^{\pi\hat{k}} e^{2i\hat{k} \ln(2m\omega r_0)}, \quad (3.7)$$

where $\Gamma(\dots)$ is the Euler gamma function, $\hat{k} \equiv m\omega E$ and the constant r_0 , related to ω_0 , takes the value $2M/\sqrt{e}$.

The remaining part of the factorized modes is expressed as an amplitude $f_{\ell m}$ and a phase $\delta_{\ell m}$, which are computed such that the expansion of $h_{\ell m}^F$ agrees with the PN-expanded modes in Eq. (3.2). We can further resum the amplitude term by introducing $\rho_{\ell m} = (f_{\ell m})^{1/\ell}$ which improves the agreement with numerical-relativity waveforms [24,31]. For spinning binaries, the nonspinning and spin contributions are separated for the odd m modes, such that

$$f_{\ell m} = \begin{cases} \rho_{\ell m}^\ell, & m \text{ even}, \\ (\rho_{\ell m}^{\text{NS}})^\ell + f_{\ell m}^S, & m \text{ odd}, \end{cases} \quad (3.8)$$

where $\rho_{\ell m}^{\text{NS}}$ is the nonspinning part of $\rho_{\ell m}$, while $f_{\ell m}^S$ is the spinning part of $f_{\ell m}$.

The explicit results for $\rho_{\ell m}$, $f_{\ell m}$ and $\delta_{\ell m}$ are displayed in Appendix B. The energy E in $\delta_{\ell m}$ is replaced by the Hamiltonian in EOB waveform models.

As discussed in [26], we remind that an error in the literature [24,25] has been found in the $\mathcal{O}(\nu v^5)$ term of δ_{21} . The coefficient $-493/42$ in these papers, coming from the radiation reaction contribution, should read $-25/2$, as we see in Eq. (B3b). We also provide new PN terms: although the (2,1) PN-expanded mode was published in [11], its factorized form was not. We finally obtain new coefficients in the (3,2) and $\ell \geq 4$ modes. These expressions are available in a *Mathematica* file in Supplemental Material [27] including spinning effects.

IV. SUMMARY

In this paper, we computed the full gravitational waveform for nonspinning binaries in quasicircular motion to the 3.5PN order. To do so, we used the PNMPM formalism to derive the radiative multipole moments. All the computational methods were already developed in previous works (except for the generalization to next-to-leading order of some integration formulas, displayed in Appendix A, in order to compute the memory terms).

The waveform modes have been written in the conventional PN-expanded way as well as a factorized form convenient for EOB template building. We already derived in [26] all spin effects in the full waveform to the 3.5PN order in the quasicircular and nonprecessing case. Combining these results, we obtain the waveform modes to the 3.5PN order including nonspinning and spinning effects.

We recall that, in [26], we found some mistakes in the literature, notably some spin coefficients $\mathcal{O}(v^6 \chi^2 \nu^2)$ in the (2,1) mode. We also corrected a nonspinning term in δ_{21} . In the present work, we agree fully on the overlap with the literature, except for this discrepancy in δ_{21} .

We provide in Supplemental Material [27] the full expressions of the modes for nonspinning and spinning effects. The notations and details regarding the spinning part of the modes are given in [26].

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APPENDIX A: GENERALIZATION OF FORMULAS TO COMPUTE MEMORY TERMS

The memory integrals are of the form

$$\int_{-\infty}^{T_R} d\tau A_L(\tau) B_K(\tau), \quad (\text{A1})$$

where A_L and B_L are time derivatives of STF multipole moments. To compute them, one needs to know the value of integrals of the type

$$\int_{-\infty}^{T_R} d\tau \frac{e^{in\phi(\tau)}}{r^p(\tau)}, \quad (\text{A2})$$

where we recall that ϕ is the orbital phase, r is the separation, n is an integer and p is an integer or half-integer. The memory integrals have been derived in the 3PN full nonspinning waveform paper [19]. At this order, only the leading PN order is required. However, for the 3.5PN waveform, we need to generalize the formula to the next-to-leading order (NLO). To do so, we introduce the adiabatic parameter evaluated at retarded time T_R :

$$\xi(T_R) \equiv \frac{1}{(T_c - T_R)\omega(T_R)}, \quad (\text{A3})$$

where T_c is the coalescence time. This quantity is a small parameter of order $\mathcal{O}(c^{-5})$. At NLO, the adiabatic parameter reads

$$\xi(T_R) = \frac{256}{5} \nu x^{5/2}(T_R) [1 + \kappa x(T_R) + \mathcal{O}(x^{3/2})], \quad (\text{A4})$$

where $\kappa = -\frac{743}{252} - \frac{11}{3}\nu$. Next, we perform the following change of variable:

$$y \equiv \frac{T_R - \tau}{T_c - T_R}, \quad (\text{A5})$$

and we can express the phase and the separation in terms of y at the NLO:

$$r(\tau) = r(T_R)(1+y)^{1/4} [1 + \eta x(T_R)((1+y)^{-1/4} - 1)], \quad (\text{A6a})$$

$$\begin{aligned} \phi(\tau) = \phi(T_R) - \frac{8}{5\xi(T_R)} \{[(1+y)^{5/8} - 1] + x(T_R) \\ \times [\zeta[(1+y)^{5/8} - 1] + \lambda[(1+y)^{3/8} - 1]]\}, \end{aligned} \quad (\text{A6b})$$

where $\eta = -\frac{1751}{1008} - \frac{7}{12}\nu$, $\zeta = -\frac{743}{672} - \frac{11}{8}\nu$ and $\lambda = \frac{3715}{2016} + \frac{55}{24}\nu$. After inserting these expressions in (A2), performing the new change of variable $z = \frac{8}{5}[(1+y)^{5/8} - 1]$ and truncating the integrand to the NLO, we find

$$\begin{aligned} \int_{-\infty}^{T_R} d\tau \frac{e^{in\phi(\tau)}}{r^p(\tau)} = \frac{e^{in\phi(T_R)}}{r^p(T_R)\omega(T_R)\xi(T_R)} \\ \times \int_0^\infty dz \frac{e^{-\frac{in}{\xi}z}}{(1 + \frac{5z}{8})^{\frac{2p-3}{5}}} [1 + x(T_R)f(z) + \mathcal{O}(x^{3/2})], \end{aligned} \quad (\text{A7})$$

where f is a function of z , straightforward to compute but lengthy to display. Finally, by performing two integrations

by part and using the fact that $\xi = \mathcal{O}(c^{-5})$, we find for $n \neq 0$

$$\begin{aligned} & \int_{-\infty}^{T_R} d\tau \frac{e^{in\phi(\tau)}}{r^p(\tau)} \\ &= \frac{e^{in\phi(T_R)}}{inr^p(T_R)\omega(T_R)} \left[1 - \underbrace{x(T_R) \left(\zeta + \frac{3}{5}\lambda \right)}_{=0} + \mathcal{O}(x^{3/2}) \right]. \quad (\text{A8}) \end{aligned}$$

Interestingly, the combination $\zeta + \frac{3}{5}\lambda$ vanishes, which implies that we do not need to take into account the 1PN correction in Eqs. (A6). Thus, the 1PN correction only comes from the one of $r^p(T_R)$ which leads to the final formula

$$\begin{aligned} & \int_{-\infty}^{T_R} d\tau \frac{e^{in\phi(\tau)}}{r^p(\tau)} \\ &= \frac{c^{2p-3}}{(GM)^{p-1}} \frac{e^{in\phi(T_R)}}{in} x^{p-3/2} \left[1 + p \left(1 - \frac{\nu}{3} \right) x + \mathcal{O}(x^{3/2}) \right]. \quad (\text{A9}) \end{aligned}$$

This formula is valid for $n \neq 0$, but when computing the memory integrals, we also encounter integrals with $n = 0$. The same kind of reasoning allows us to compute

$$\begin{aligned} & \int_{-\infty}^{T_R} \frac{d\tau}{r^p(\tau)} = \frac{5}{64\nu} \frac{c^{2p-3}}{(GM)^{p-1}} \frac{x^{p-4}}{p-4} \\ & \times \left\{ 1 + \left[-\kappa + p \left(\frac{\eta}{p-3} + 1 - \frac{\nu}{3} \right) \right] x + \mathcal{O}(x^{3/2}) \right\}. \quad (\text{A10}) \end{aligned}$$

Note that, in this case, the 1PN correction of $r^p(\tau)$ plays a role in the result. In our computations, the case $p = 3$ and $p = 4$ do not appear. With (A9) and (A10), we have everything in hand to compute the memory integrals required for the full 3.5PN waveform.

APPENDIX B: EXPLICIT QUANTITIES OF THE FACTORIZED MODES

In this appendix, we write the explicit expressions for the factorized modes (see Sec. III B):

$$\begin{aligned} f_{22} = & 1 + v^2 \left(-\frac{43}{21} + \frac{55}{42}\nu \right) + v^4 \left(-\frac{536}{189} - \frac{6745}{1512}\nu + \frac{2047}{1512}\nu^2 \right) \\ & + v^6 \left(\frac{21428357}{727650} - \frac{856}{105}(\gamma_E + \ln 4\nu) + \left(-\frac{34625}{3696} + \frac{41}{96}\pi^2 \right)\nu - \frac{227875}{33264}\nu^2 + \frac{114635}{99792}\nu^3 \right), \quad (\text{B1a}) \end{aligned}$$

$$\begin{aligned} f_{21} = & 1 + v^2 \left(-\frac{59}{28} + \frac{23}{42}\nu \right) + v^4 \left(-\frac{5}{9} - \frac{269}{126}\nu + \frac{85}{252}\nu^2 \right) \\ & + v^6 \left(\frac{88404893}{11642400} - \frac{214}{105}(\gamma_E + \ln 2\nu) + \left(\frac{86699}{66528} - \frac{41}{384}\pi^2 \right)\nu - \frac{37241}{66528}\nu^2 + \frac{9365}{49896}\nu^3 \right), \quad (\text{B1b}) \end{aligned}$$

$$\begin{aligned} f_{33} = & 1 + v^2 \left(-\frac{7}{2} + 2\nu \right) + v^4 \left(-\frac{443}{440} - \frac{3401}{330}\nu + \frac{887}{330}\nu^2 \right) \\ & + v^6 \left(\frac{147471561}{2802800} - \frac{78}{7}(\gamma_E + \ln 6\nu) + \left(-\frac{17161}{2860} + \frac{41}{64}\pi^2 \right)\nu - \frac{27409}{1560}\nu^2 + \frac{8237}{2860}\nu^3 \right), \quad (\text{B1c}) \end{aligned}$$

$$f_{32} = 1 + \frac{v^2}{1-3\nu} \left(-\frac{164}{45} + \frac{223}{18}\nu - \frac{32}{9}\nu^2 \right) + \frac{v^4}{1-3\nu} \left(\frac{854}{495} - \frac{23443}{2376}\nu + \frac{31721}{1485}\nu^2 - \frac{4943}{1485}\nu^3 \right), \quad (\text{B1d})$$

$$\begin{aligned} f_{31} = & 1 + v^2 \left(-\frac{13}{6} - \frac{2}{3}\nu \right) + v^4 \left(\frac{1273}{792} - \frac{371}{198}\nu - \frac{247}{198}\nu^2 \right) \\ & + v^6 \left(\frac{400427563}{75675600} - \frac{26}{21}(\gamma_E + \ln 2\nu) + \left(-\frac{788399}{77220} + \frac{41}{64}\pi^2 \right)\nu + \frac{311225}{30888}\nu^2 - \frac{17525}{15444}\nu^3 \right), \quad (\text{B1e}) \end{aligned}$$

$$f_{44} = 1 + \frac{v^2}{1-3\nu} \left(-\frac{269}{55} + \frac{587}{33}\nu - \frac{175}{22}\nu^2 \right) + \frac{v^4}{1-3\nu} \left(\frac{63002}{25025} - \frac{9163}{325}\nu + \frac{381541}{6435}\nu^2 - \frac{226097}{17160}\nu^3 \right), \quad (\text{B1f})$$

$$f_{43} = 1 + \frac{v^2}{1-2\nu} \left(-\frac{111}{22} + \frac{547}{44}\nu - \frac{40}{11}\nu^2 \right) + \frac{v^4}{1-2\nu} \left(\frac{225543}{40040} - \frac{132529}{5720}\nu + \frac{100003}{3432}\nu^2 - \frac{3889}{858}\nu^3 \right), \quad (\text{B1g})$$

$$f_{42} = 1 + \frac{v^2}{1-3\nu} \left(-\frac{191}{55} + \frac{353}{33}\nu - \frac{19}{22}\nu^2 \right) + \frac{v^4}{1-3\nu} \left(\frac{76918}{25025} - \frac{53297}{3575}\nu + \frac{100552}{6435}\nu^2 + \frac{25783}{17160}\nu^3 \right), \quad (\text{B1h})$$

$$f_{41} = 1 + \frac{v^2}{1-2\nu} \left(-\frac{301}{66} + \frac{1385}{132}\nu - \frac{24}{11}\nu^2 \right) + \frac{v^4}{1-2\nu} \left(\frac{760181}{120120} - \frac{815329}{51480}\nu + \frac{163313}{10296}\nu^2 - \frac{1409}{858}\nu^3 \right), \quad (\text{B1i})$$

$$f_{55} = 1 + \frac{v^2}{1-2\nu} \left(-\frac{487}{78} + \frac{649}{39}\nu - \frac{256}{39}\nu^2 \right) + \frac{v^4}{1-2\nu} \left(\frac{50569}{6552} - \frac{39899}{819}\nu + \frac{4001}{63}\nu^2 - \frac{10567}{819}\nu^3 \right), \quad (\text{B1j})$$

$$f_{54} = 1 + \frac{v^2}{1-5\nu+5\nu^2} \left(-\frac{2908}{455} + \frac{13717}{390}\nu - \frac{1823}{39}\nu^2 + \frac{476}{39}\nu^3 \right), \quad (\text{B1k})$$

$$f_{53} = 1 + \frac{v^2}{1-2\nu} \left(-\frac{125}{26} + \frac{425}{39}\nu - \frac{88}{39}\nu^2 \right) + \frac{v^4}{1-2\nu} \left(\frac{69359}{10920} - \frac{34493}{1365}\nu + \frac{6299}{273}\nu^2 - \frac{365}{273}\nu^3 \right), \quad (\text{B1l})$$

$$f_{52} = 1 + \frac{v^2}{1-5\nu+5\nu^2} \left(-\frac{2638}{455} + \frac{12097}{390}\nu - \frac{1499}{39}\nu^2 + \frac{314}{39}\nu^3 \right), \quad (\text{B1m})$$

$$f_{51} = 1 + \frac{v^2}{1-2\nu} \left(-\frac{319}{78} + \frac{313}{39}\nu - \frac{4}{39}\nu^2 \right) + \frac{v^4}{1-2\nu} \left(\frac{28859}{4680} - \frac{9673}{585}\nu + \frac{917}{117}\nu^2 + \frac{287}{117}\nu^3 \right), \quad (\text{B1n})$$

$$f_{66} = 1 + \frac{v^2}{1-5\nu+5\nu^2} \left(-\frac{53}{7} + 43\nu - \frac{123}{2}\nu^2 + \frac{39}{2}\nu^3 \right), \quad (\text{B1o})$$

$$f_{65} = 1 + \frac{v^2}{1-4\nu+3\nu^2} \left(-\frac{185}{24} + \frac{419}{12}\nu - \frac{455}{12}\nu^2 + \frac{55}{6}\nu^3 \right), \quad (\text{B1p})$$

$$f_{64} = 1 + \frac{v^2}{1-5\nu+5\nu^2} \left(-\frac{43}{7} + 33\nu - \frac{83}{2}\nu^2 + \frac{19}{2}\nu^3 \right), \quad (\text{B1q})$$

$$f_{63} = 1 + \frac{v^2}{1-4\nu+3\nu^2} \left(-\frac{169}{24} + \frac{371}{12}\nu - \frac{125}{4}\nu^2 + \frac{13}{2}\nu^3 \right), \quad (\text{B1r})$$

$$f_{62} = 1 + \frac{v^2}{1-5\nu+5\nu^2} \left(-\frac{37}{7} + 27\nu - \frac{59}{2}\nu^2 + \frac{7}{2}\nu^3 \right), \quad (\text{B1s})$$

$$f_{61} = 1 + \frac{v^2}{1-4\nu+3\nu^2} \left(-\frac{161}{24} + \frac{347}{12}\nu - \frac{335}{12}\nu^2 + \frac{31}{6}\nu^3 \right), \quad (\text{B1t})$$

$$f_{77} = 1 + \frac{v^2}{1-4\nu+3\nu^2} \left(-\frac{151}{17} + \frac{2123}{51}\nu - \frac{4963}{102}\nu^2 + \frac{230}{17}\nu^3 \right), \quad (\text{B1u})$$

$$f_{75} = 1 + \frac{v^2}{1-4\nu+3\nu^2} \left(-\frac{127}{17} + \frac{1691}{51}\nu - \frac{3523}{102}\nu^2 + \frac{134}{17}\nu^3 \right), \quad (\text{B1v})$$

$$f_{73} = 1 + \frac{v^2}{1-4\nu+3\nu^2} \left(-\frac{111}{17} + \frac{1403}{51}\nu - \frac{2563}{102}\nu^2 + \frac{70}{17}\nu^3 \right), \quad (\text{B1w})$$

$$f_{71} = 1 + \frac{v^2}{1-4\nu+3\nu^2} \left(-\frac{103}{17} + \frac{1259}{51}\nu - \frac{2083}{102}\nu^2 + \frac{38}{17}\nu^3 \right), \quad (\text{B1x})$$

and at the leading order, $f_{\ell m} = 1$. Furthermore, the $\rho_{\ell m}$ read

$$\rho_{22} = 1 + v^2 \left(-\frac{43}{42} + \frac{55}{84} \nu \right) + v^4 \left(-\frac{20555}{10584} - \frac{33025}{21168} \nu + \frac{19583}{42336} \nu^2 \right) + v^6 \left(\frac{1556919113}{122245200} - \frac{428}{105} (\gamma_E + \ln 4 v) + \left(-\frac{48993925}{9779616} + \frac{41}{192} \pi^2 \right) \nu - \frac{6292061}{3259872} \nu^2 + \frac{10620745}{39118464} \nu^3 \right), \quad (\text{B2a})$$

$$\rho_{21} = 1 + v^2 \left(-\frac{59}{56} + \frac{23}{84} \nu \right) + v^4 \left(-\frac{47009}{56448} - \frac{10993}{14112} \nu + \frac{617}{4704} \nu^2 \right) + v^6 \left(\frac{7613184941}{2607897600} - \frac{107}{105} (\gamma_E + \ln 2 v) + \left(\frac{1024181}{17385984} - \frac{41}{768} \pi^2 \right) \nu + \frac{622373}{8692992} \nu^2 + \frac{2266171}{39118464} \nu^3 \right), \quad (\text{B2b})$$

$$\rho_{33} = 1 + v^2 \left(-\frac{7}{6} + \frac{2}{3} \nu \right) + v^4 \left(-\frac{6719}{3960} - \frac{1861}{990} \nu + \frac{149}{330} \nu^2 \right) + v^6 \left(\frac{3203101567}{227026800} - \frac{26}{7} (\gamma_E + \ln 6 v) + \left(-\frac{129509}{25740} + \frac{41}{192} \pi^2 \right) \nu - \frac{274621}{154440} \nu^2 + \frac{12011}{46332} \nu^3 \right), \quad (\text{B2c})$$

$$\rho_{32} = 1 + \frac{v^2}{1-3\nu} \left(-\frac{164}{135} + \frac{223}{54} \nu - \frac{32}{27} \nu^2 \right) + \frac{v^4}{(1-3\nu)^2} \left(-\frac{180566}{200475} + \frac{1610009}{320760} \nu - \frac{945121}{320760} \nu^2 - \frac{508474}{40095} \nu^3 + \frac{77141}{40095} \nu^4 \right), \quad (\text{B2d})$$

$$\rho_{31} = 1 + v^2 \left(-\frac{13}{18} - \frac{2}{9} \nu \right) + v^4 \left(\frac{101}{7128} - \frac{1685}{1782} \nu - \frac{829}{1782} \nu^2 \right) + v^6 \left(\frac{11706720301}{6129723600} - \frac{26}{63} (\gamma_E + \ln 2 v) + \left(-\frac{9688441}{2084940} + \frac{41}{192} \pi^2 \right) \nu + \frac{174535}{75816} \nu^2 - \frac{727247}{1250964} \nu^3 \right), \quad (\text{B2e})$$

$$\rho_{44} = 1 + \frac{v^2}{1-3\nu} \left(-\frac{269}{220} + \frac{587}{132} \nu - \frac{175}{88} \nu^2 \right) + \frac{v^4}{(1-3\nu)^2} \left(-\frac{14210377}{8808800} + \frac{32485357}{4404400} \nu - \frac{1401149}{1415700} \nu^2 - \frac{801565}{37752} \nu^3 + \frac{3976393}{1006720} \nu^4 \right), \quad (\text{B2f})$$

$$\rho_{43} = 1 + \frac{v^2}{1-2\nu} \left(-\frac{111}{88} + \frac{547}{176} \nu - \frac{10}{11} \nu^2 \right) + \frac{v^4}{(1-2\nu)^2} \left(-\frac{6894273}{7047040} + \frac{22211989}{7047040} \nu + \frac{1032509}{1098240} \nu^2 - \frac{181867}{25168} \nu^3 + \frac{19379}{18876} \nu^4 \right), \quad (\text{B2g})$$

$$\rho_{42} = 1 + \frac{v^2}{1-3\nu} \left(-\frac{191}{220} + \frac{353}{132} \nu - \frac{19}{88} \nu^2 \right) + \frac{v^4}{(1-3\nu)^2} \left(-\frac{3190529}{8808800} + \frac{4108813}{4404400} \nu + \frac{21506941}{5662800} \nu^2 - \frac{3628549}{377520} \nu^3 - \frac{1204847}{1006720} \nu^4 \right), \quad (\text{B2h})$$

$$\rho_{41} = 1 + \frac{v^2}{1-2\nu} \left(-\frac{301}{264} + \frac{1385}{528} \nu - \frac{6}{11} \nu^2 \right) + \frac{v^4}{(1-2\nu)^2} \left(-\frac{7775491}{21141120} + \frac{117238909}{63423360} \nu - \frac{332099}{1098240} \nu^2 - \frac{917167}{226512} \nu^3 + \frac{7075}{18876} \nu^4 \right), \quad (\text{B2i})$$

$$\rho_{55} = 1 + \frac{v^2}{1-2\nu} \left(-\frac{487}{390} + \frac{649}{195} \nu - \frac{256}{195} \nu^2 \right) + \frac{v^4}{(1-2\nu)^2} \left(-\frac{3353747}{2129400} + \frac{4038803}{1064700} \nu + \frac{925493}{266175} \nu^2 - \frac{932171}{88725} \nu^3 + \frac{456206}{266175} \nu^4 \right), \quad (\text{B2j})$$

$$\rho_{54} = 1 + \frac{v^2}{1 - 5\nu + 5\nu^2} \left(-\frac{2908}{2275} + \frac{13717}{1950}\nu - \frac{1823}{195}\nu^2 + \frac{476}{195}\nu^3 \right), \quad (\text{B2k})$$

$$\begin{aligned} \rho_{53} = 1 + \frac{v^2}{1 - 2\nu} \left(-\frac{25}{26} + \frac{85}{39}\nu - \frac{88}{195}\nu^2 \right) \\ + \frac{v^4}{(1 - 2\nu)^2} \left(-\frac{410833}{709800} + \frac{279697}{354900}\nu + \frac{928009}{266175}\nu^2 - \frac{296117}{53235}\nu^3 + \frac{33934}{266175}\nu^4 \right), \end{aligned} \quad (\text{B2l})$$

$$\rho_{52} = 1 + \frac{v^2}{1 - 5\nu + 5\nu^2} \left(-\frac{2638}{2275} + \frac{12097}{1950}\nu - \frac{1499}{195}\nu^2 + \frac{314}{195}\nu^3 \right), \quad (\text{B2m})$$

$$\begin{aligned} \rho_{51} = 1 + \frac{v^2}{1 - 2\nu} \left(-\frac{319}{390} + \frac{313}{195}\nu - \frac{4}{195}\nu^2 \right) \\ + \frac{v^4}{(1 - 2\nu)^2} \left(-\frac{31877}{304200} - \frac{79387}{152100}\nu + \frac{112613}{38025}\nu^2 - \frac{31849}{12675}\nu^3 - \frac{37342}{38025}\nu^4 \right), \end{aligned} \quad (\text{B2n})$$

$$\rho_{66} = 1 + \frac{v^2}{1 - 5\nu + 5\nu^2} \left(-\frac{53}{42} + \frac{43}{6}\nu - \frac{41}{4}\nu^2 + \frac{13}{4}\nu^3 \right), \quad (\text{B2o})$$

$$\rho_{65} = 1 + \frac{v^2}{1 - 4\nu + 3\nu^2} \left(-\frac{185}{144} + \frac{419}{72}\nu - \frac{455}{72}\nu^2 + \frac{55}{36}\nu^3 \right), \quad (\text{B2p})$$

$$\rho_{64} = 1 + \frac{v^2}{1 - 5\nu + 5\nu^2} \left(-\frac{43}{42} + \frac{11}{2}\nu - \frac{83}{12}\nu^2 + \frac{19}{12}\nu^3 \right), \quad (\text{B2q})$$

$$\rho_{63} = 1 + \frac{v^2}{1 - 4\nu + 3\nu^2} \left(-\frac{169}{144} + \frac{371}{72}\nu - \frac{125}{24}\nu^2 + \frac{13}{12}\nu^3 \right), \quad (\text{B2r})$$

$$\rho_{62} = 1 + \frac{v^2}{1 - 5\nu + 5\nu^2} \left(-\frac{37}{42} + \frac{9}{2}\nu - \frac{59}{12}\nu^2 + \frac{7}{12}\nu^3 \right), \quad (\text{B2s})$$

$$\rho_{61} = 1 + \frac{v^2}{1 - 4\nu + 3\nu^2} \left(-\frac{161}{144} + \frac{347}{72}\nu - \frac{335}{72}\nu^2 + \frac{31}{36}\nu^3 \right), \quad (\text{B2t})$$

$$\rho_{77} = 1 + \frac{v^2}{1 - 4\nu + 3\nu^2} \left(-\frac{151}{119} + \frac{2123}{357}\nu - \frac{709}{102}\nu^2 + \frac{230}{119}\nu^3 \right), \quad (\text{B2u})$$

$$\rho_{75} = 1 + \frac{v^2}{1 - 4\nu + 3\nu^2} \left(-\frac{127}{119} + \frac{1691}{357}\nu - \frac{3523}{714}\nu^2 + \frac{134}{119}\nu^3 \right), \quad (\text{B2v})$$

$$\rho_{73} = 1 + \frac{v^2}{1 - 4\nu + 3\nu^2} \left(-\frac{111}{119} + \frac{1403}{357}\nu - \frac{2563}{714}\nu^2 + \frac{10}{17}\nu^3 \right), \quad (\text{B2w})$$

$$\rho_{71} = 1 + \frac{v^2}{1 - 4\nu + 3\nu^2} \left(-\frac{103}{119} + \frac{1259}{357}\nu - \frac{2083}{714}\nu^2 + \frac{38}{119}\nu^3 \right). \quad (\text{B2x})$$

As for the $f_{\ell m}$, at leading order $\rho_{\ell m} = 1$. Finally, we have

$$\delta_{22} = \frac{7}{3}\omega E - 24\nu v^5 + \frac{428\pi}{105}(\omega E)^2 + v^7\nu \left(\frac{30995}{1134} + \frac{3662}{135}\nu \right), \quad (\text{B3a})$$

$$\delta_{21} = \frac{2}{3}\omega E - \frac{25}{2}\nu v^5 + \frac{107\pi}{105}(\omega E)^2, \quad (\text{B3b})$$

$$\delta_{33} = \frac{13}{10}\omega E - \frac{80897}{2430}\nu v^5 + \frac{39\pi}{7}(\omega E)^2, \quad (\text{B3c})$$

$$\delta_{32} = \left(\frac{2}{3} + \frac{11}{5}\nu\right) \frac{\omega E}{1-3\nu} + \frac{v^5\nu}{(1-3\nu)^2} \left(-\frac{87893}{5400} + \frac{66871}{600}\nu - \frac{28691}{150}\nu^2\right), \quad (\text{B3d})$$

$$\delta_{31} = \frac{13}{30}\omega E - \frac{17}{10}\nu v^5 + \frac{13\pi}{21}(\omega E)^2, \quad (\text{B3e})$$

$$\delta_{44} = \left(\frac{14}{15} + \frac{73}{40}\nu\right) \frac{\omega E}{1-3\nu} + \frac{v^5\nu}{(1-3\nu)^2} \left(-\frac{5794139}{140800} + \frac{35353757}{140800}\nu - \frac{2740989}{7040}\nu^2\right), \quad (\text{B3f})$$

$$\delta_{43} = \left(\frac{3}{5} + \frac{4961}{810}\nu\right) \frac{\omega E}{1-2\nu}, \quad (\text{B3g})$$

$$\delta_{42} = \left(\frac{7}{15} + \frac{14}{5}\nu\right) \frac{\omega E}{1-3\nu} + \frac{v^5\nu}{(1-3\nu)^2} \left(-\frac{125569}{4400} + \frac{764879}{4400}\nu - \frac{291003}{1100}\nu^2\right), \quad (\text{B3h})$$

$$\delta_{41} = \left(\frac{1}{5} + \frac{507}{10}\nu\right) \frac{\omega E}{1-2\nu}, \quad (\text{B3i})$$

$$\delta_{55} = \left(\frac{31}{42} + \frac{61252}{9375}\nu\right) \frac{\omega E}{1-2\nu}, \quad (\text{B3j})$$

$$\delta_{54} = \left(\frac{8}{15} + \frac{137377}{17920}\nu - \frac{11777}{384}\nu^2\right) \frac{\omega E}{1-5\nu+5\nu^2}, \quad (\text{B3k})$$

$$\delta_{53} = \left(\frac{31}{70} + \frac{23924}{3645}\nu\right) \frac{\omega E}{1-2\nu}, \quad (\text{B3l})$$

$$\delta_{52} = \left(\frac{4}{15} + \frac{2351}{112}\nu - \frac{3943}{60}\nu^2\right) \frac{\omega E}{1-5\nu+5\nu^2}, \quad (\text{B3m})$$

$$\delta_{51} = \left(\frac{31}{210} + \frac{1796}{15}\nu\right) \frac{\omega E}{1-2\nu}, \quad (\text{B3n})$$

$$\delta_{66} = \left(\frac{43}{70} + \frac{1756523}{217728}\nu - \frac{85439}{2592}\nu^2\right) \frac{\omega E}{1-5\nu+5\nu^2}, \quad (\text{B3o})$$

$$\delta_{64} = \left(\frac{43}{105} + \frac{417559}{49152}\nu - \frac{393431}{12288}\nu^2\right) \frac{\omega E}{1-5\nu+5\nu^2}, \quad (\text{B3p})$$

$$\delta_{62} = \left(\frac{43}{210} + \frac{387629}{13440}\nu - \frac{42859}{480}\nu^2\right) \frac{\omega E}{1-5\nu+5\nu^2}, \quad (\text{B3q})$$

while the values for the others $\delta_{\ell m}$ are 0 at the 3.5PN order.

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