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Abstract

We study the Masur–Veech volumes $MV_{g,n}$ of the principal stratum of the moduli space of quadratic differentials of unit area on curves of genus g with n punctures. We show that the volumes $MV_{g,n}$ are the constant terms of a family of polynomials $MV_{g,n}(L_1, \ldots, L_n)$ governed by the topological recursion/Virasoro constraints. This is equivalent to a formula giving these polynomials as a sum over stable graphs, and retrieves a result of [11] proved by combinatorial arguments. Our method is different: it relies on the geometric recursion and its application to statistics of hyperbolic lengths of simple multicurves developed in [3]. We also obtain an expression of the area Siegel–Veech constants in terms of hyperbolic geometry. The topological recursion allows numerical computations of Masur–Veech volumes, and thus of area Siegel–Veech constants for low g and n, which leads us to propose conjectural formulas for low g but all n.

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Contents

1 Introduction

We consider two facets of the geometry of surfaces. On the one hand, hyperbolic geometry with associated Teichmüller space and Weil–Petersson metric, and on the other hand, flat geometry associated with quadratic differentials and the Masur–Veech measure. We will show that invariants of the flat geometry of surfaces, namely the Masur–Veech volumes and the area Siegel–Veech constants, can be expressed as asymptotics of certain statistics of multicurves on hyperbolic surfaces. Using the geometric recursion developed in [3] for these statistics, we prove that the Masur–Veech volumes satisfy some form of the topological recursion à la Eynard–Orantin [18].

1.1 Notations and facts

We will let Σ denote a smooth, compact, oriented, not necessarily connected surface, which can be closed, punctured or bordered. We consider those cases to be mutually exclusive and we shall indicate which situation is considered when necessary. When Σ is not closed, the punctures or boundary components are labelled $\partial_1\Sigma, \ldots, \partial_n\Sigma$. We assume that Σ is stable, i.e. the Euler characteristic of each connected component is negative. We say that Σ has type (g, n) if it is connected of genus g with n boundary components. We use P (respectively T) to refer to surfaces with the topology of a pair of pants (resp. of a torus with one boundary component).

The Teichmüller space $\mathcal{T}_\Sigma$ of a bordered Σ is the set of hyperbolic metrics on Σ such that the boundary components are geodesic, modulo diffeomorphisms of Σ that restrict to the identity on $\partial\Sigma$ and which are isotopic to Id_Σ among such. The Teichmüller space $\mathcal{T}_\Sigma$ fibers over $\mathbb{R}_+^n$ and we denote the fiber over $L = (L_1, \ldots, L_n) \in \mathbb{R}_+^n$ by $\mathcal{T}_\Sigma(L)$. For a surface of type (g, n), $\mathcal{T}_\Sigma(L)$ is a smooth manifold of dimension $6g - 6 + 2n$. Here $\mathbb{R}_+$ is the positive real axis, excluding 0. In several places, we will also consider $L_i = 0$, which means that the i-th boundary corresponds to a cusp for the hyperbolic metric. The slice $\mathcal{T}_\Sigma(0, \ldots, 0) = \mathfrak{T}_\Sigma$ is the Teichmüller space of complete hyperbolic metric of finite area on $\Sigma - \partial\Sigma$, which is then considered as a punctured surface. $\mathfrak{T}_\Sigma$ can also be seen as the space of Riemann structures on the punctured surface. The cotangent bundle to $\mathfrak{T}_\Sigma$ is isomorphic to the bundle $Q\mathfrak{T}_\Sigma$ of holomorphic integrable quadratic differentials on the punctured surface. For any $(q, \sigma) \in Q\mathfrak{T}_\Sigma$, the quadratic differential q has either a removable singularity or a simple pole at each puncture of Σ. These spaces also exist for closed surfaces.

The mapping class group Mod_Σ is the group of isotopy classes of orientation-preserving diffeomorphisms of Σ. It admits as subgroup the pure mapping class group $\mathrm{Mod}_\Sigma^\partial$, consisting of the isotopy classes of diffeomorphisms that restrict to the identity on $\partial\Sigma$. The mapping class group acts on the Teichmüller spaces $\mathcal{T}_\Sigma(L)$ and on $\mathfrak{T}_\Sigma$ and on the space of quadratic differentials $Q\mathfrak{T}_\Sigma$. This action is properly discontinuous and the quotient spaces $\mathcal{M}_\Sigma(L)$, $\mathfrak{M}_\Sigma$ and $Q\mathfrak{M}_\Sigma$ are smooth orbifolds, called respectively the moduli space of bordered surfaces, the moduli space of punctured surfaces, and the moduli space of quadratic differentials. The moduli spaces for all surfaces of given type (g, n) are all canonically isomorphic and simply denoted by $\mathcal{M}_{g,n}(L)$, $\mathfrak{M}_{g,n}$ and $Q\mathfrak{M}_{g,n}$.

The spaces $\mathcal{T}_\Sigma(L)$ for $L \in \mathbb{R}_+^n$ and $\mathfrak{T}_\Sigma$ are endowed with the Weil–Petersson measures μ_{WP}. These measures are invariant under the action of the mapping class group and descend to the quotients $\mathcal{M}_{g,n}(L)$ and $\mathfrak{M}_{g,n}$. If Y_Σ is a $\mathrm{Mod}_\Sigma^\partial$-invariant function on $\mathcal{T}_\Sigma$, we denote by $Y_{g,n}$ the function it induces on $\mathcal{M}_{g,n}$ and we introduce

$$VY_{g,n}(L) = \int_{\mathcal{M}_{g,n}(L)} Y_{g,n}(\sigma)\, d\mu_{\mathrm{WP}}(\sigma) \tag{1.1}$$

if this integral makes sense.

Likewise, if Σ is a closed or punctured surface, $Q\mathfrak{M}_\Sigma$ is endowed with the Masur–Veech measure μ_{MV} coming from its piecewise linear integral structure. The function which associates to a quadratic differential q on Σ its area $\int_\Sigma |q|$ provides a natural way to define an induced measure on the space $Q^1\mathfrak{M}_{g,n}$ of quadratic differentials of unit area (see Section 3.1). By a theorem of Masur and Veech [28, 38] the total mass of this measure is finite. Its value is, by definition, the Masur–Veech volume and it is denoted by $MV_{g,n}$. Its computation is relevant in the study of the geometry of moduli spaces and the dynamics of measured foliations and has been the object of numerous investigations [5, 11, 16, 23, 33].

1.2 Overview

In Section 2, we review the definition and main properties of the geometric and topological recursion, mainly taken from [3].

In Section 3, for each connected bordered surface Σ of genus g with $n > 0$ boundaries, we construct a $\mathrm{Mod}_{\Sigma}^{\partial}$- invariant continuous function $\Omega_{\Sigma}^{MV} \colon \mathcal{T}_{\Sigma} \to \mathbb{R}$. It is such that the integral $V\Omega_{g,n}^{MV}(L_1,\ldots,L_n)$ is a polynomial function in the variables $L_1,\ldots,L_n$ and the Masur–Veech volume $MV_{g,n}$ of $Q^1\mathfrak{M}_{g,n}$ consisting of unit area quadratic differentials satisfies

$$MV_{g,n} = \frac{2^{4g-2+n}(4g-4+n)!}{(6g-7+2n)!}\, V\Omega_{g,n}^{MV}(0,\ldots,0). \tag{1.2}$$

The family of functions Ω^{MV} can be defined via the geometric recursion, with initial data found in Proposition 3.7. The polynomials $V\Omega^{MV}$, which we call Masur–Veech polynomials, have four different descriptions:

(**1**) they are sums over stable graphs (Section 3.3), which we reproduce in (1.3) below;

(**2**) they encode the asymptotic growth of the integral (against μ_{WP}) of additive statistics of the hyperbolic lengths of simple multicurves on a surface of type (g,n) with large boundaries, see Section 3.2 for the precise statement;

(**3**) they are obtained by integration of Ω^{MV}, in coherence with the notation (1.1);

(**4**) they satisfy the topological recursion, which is equivalent to the Virasoro constraints stated in Theorem 1.2 below;

The identity between (**1**) and (**2**) is proved in Theorem 3.5, which is the crux of our argument. The identity between (**1**), (**3**) and (**4**) is proved in Proposition 3.7 and follows from the general properties of the geometric and the topological recursion. In Corollary 3.6, we prove the relation (1.2) between the constant term of these polynomials and the Masur–Veech volume. Lemma 3.4 implies that the value of the Masur-Veech volumes for closed surfaces of genus $g \geqslant 2$ can be retrieved from $V\Omega_{g,1}^{MV}$.

In Section 4, we extend these arguments to show in Corollary 4.5 that the area Siegel–Veech constants can be expressed in terms of asymptotics of certain derivative statistics of hyperbolic lengths of simple multicurves. Our current proof of Corollary 4.5 uses Goujard's recursion [22] (here quoted in Theorem 4.1) for the area Siegel–Veech constants of the principal stratum in $Q^1\mathfrak{M}_{g,n}$ in terms of Masur–Veech volumes. It would be more satisfactory if one could obtain an independent proof of the identity of Corollary 4.5, as our Section 4 would then give a new proof of Goujard's recursion for the principal stratum.

Main results for the computation of Masur–Veech volumes and polynomials

Concretely, our results lead to two ways of computing the Masur–Veech volumes. Firstly, the Masur–Veech polynomials are expressed as a sum over the set $\mathbf{G}_{g,n}$ of stable graphs (see Definition 2.9). Stable graphs encode topological types of simple multicurves without multiplicities, which naturally appear via (**2**). Let us introduce the polynomials

$$V\Omega_{g,n}^{K}(L_1,\ldots,L_n) = \int_{\overline{\mathfrak{M}}_{g,n}} \exp\left(\sum_{i=1}^{n} \frac{L_i^2}{2}\,\psi_i\right),$$

which from Kontsevich's work [26] compute the volume of the combinatorial moduli spaces. The application of Theorem 3.5 to the computation of Masur–Veech volumes can be summarised as follows.

Theorem 1.1. For $g, n \geqslant 0$ such that $2g-2+n>0$, the Masur–Veech polynomials can be expressed as

$$V\Omega^{MV}_{g,n}(L_1,\ldots,L_n) = \sum_{\Gamma \in \mathbf{G}_{g,n}} \frac{1}{|\mathrm{Aut}\,\Gamma|} \int_{\mathbb{R}_+^{E_\Gamma}} \prod_{v \in V_\Gamma} V\Omega^K_{g(v),k(v)}\big((\ell_e)_{e \in E(v)}, (L_\lambda)_{\lambda \in \Lambda(v)}\big) \prod_{e \in E_\Gamma} \frac{\ell_e d\ell_e}{e^{\ell_e}-1}. \tag{1.3}$$

where V_Γ is the set of vertices of Γ and $E(v)$ (respectively, $\Lambda(v)$) is the set of edges (respectively, leaves) incident to v. In particular the Masur–Veech volumes can be computed as

$$\begin{aligned} MV_{g,n} = {} & \frac{2^{4g-2+n}(4g-4+n)!}{(6g-7+2n)!} \\ & \times \sum_{\Gamma \in \mathbf{G}_{g,n}} \frac{1}{|\mathrm{Aut}\,\Gamma|} \int_{\mathbb{R}_+^{E_\Gamma}} \prod_{v \in V_\Gamma} V\Omega^K_{g(v),k(v)}\big((\ell_e)_{e \in E(v)}, (0)_{\lambda \in \Lambda(v)}\big) \prod_{e \in E_\Gamma} \frac{\ell_e d\ell_e}{e^{\ell_e}-1}. \end{aligned} \tag{1.4}$$

⋆

The formula (1.4) was obtained prior to our work in [11] by combinatorial methods. It was presented by V.D. in a reading group organised by A.G. and D.L. The discussions which followed led to the present work where, in particular, we give a new proof of the formula (1.4).

Secondly, the coefficients of the Masur–Veech polynomials satisfy Virasoro constraints, expressed in terms of values of the Riemann zeta function at even integers. This is summarized by the following theorem, which combines the results of Corollary 3.6, Theorem 3.7, Lemma 3.4 and Section 5.2 of this paper.

Theorem 1.2. For any $g \geqslant 0$ and $n > 0$ such that $2g-2+n>0$, we have a decomposition

$$V\Omega^{MV}_{g,n}(L_1,\ldots,L_n) = \sum_{\substack{d_1,\ldots,d_n \geqslant 0 \\ d_1+\cdots+d_n \leqslant 3g-3+n}} F_{g,n}[d_1,\ldots,d_n] \prod_{j=1}^n \frac{L_j^{2d_j}}{(2d_j+1)!}.$$

Let us set $F_{0,1}[d_1] = F_{0,2}[d_1,d_2] = 0$ for all $d_1, d_2 \geqslant 0$. The base cases

$$F_{0,3}[d_1,d_2,d_3] = \delta_{d_1,d_2,d_3,0}, \qquad F_{1,1}[d] = \delta_{d,0}\,\frac{\zeta(2)}{2} + \delta_{d,1}\,\frac{1}{8}$$

determine uniquely all other coefficients via the following recursion on $2g-2+n \geqslant 2$, for $d_1,\ldots,d_n \geqslant 0$

$$\begin{aligned} F_{g,n}[d_1,\ldots,d_n] = {} & \sum_{m=2}^n \sum_{a \geqslant 0} B^{d_1}_{d_m,a}\, F_{g,n-1}[a,d_2,\ldots,\widehat{d_m},\ldots,d_n] + \\ & + \frac{1}{2}\sum_{a,b \geqslant 0} C^{d_1}_{a,b}\Big(F_{g-1,n+1}[a,b,d_2,\ldots,d_n] + \sum_{\substack{h+h'=g \\ J \sqcup J' = \{d_2,\ldots,d_n\}}} F_{h,1+|J|}[a,J]\,F_{h',1+|J'|}[b,J']\Big), \end{aligned}$$

where

$$\begin{aligned} B^i_{j,k} &= (2j+1)\,\delta_{i+j,k+1} + \delta_{i,j,0}\,\zeta(2k+2), \\ C^i_{j,k} &= \delta_{i,j+k+2} + \tfrac{(2j+2a+1)!\zeta(2j+2a+2)}{(2j+1)!(2a)!}\,\delta_{i+a,k+1} + \tfrac{(2k+2a+1)!\zeta(2k+2a+2)}{(2k+1)!(2a)!}\,\delta_{i+a,j+1} + \zeta(2j+2)\zeta(2k+2)\delta_{i,0}. \end{aligned}$$

For surfaces of genus g with $n > 0$ boundaries, the Masur–Veech volumes are identified as

$$MV_{g,n} = \frac{2^{4g-4+n}(4g-4+n)!}{(6g-7+2n)!}\,F_{g,n}[0,\ldots,0],$$

while for closed surfaces of genus $g \geqslant 2$ they are obtained through

$$MV_{g,0} = \frac{2^{4g-2}(4g-4)!}{(6g-6)!}\,F_{g,1}[1].$$

⋆

We use Theorems 1.2 to compute many Masur–Veech volumes and Masur–Veech polynomials for low g and n (Section 5). Based on numerical evidence, we propose conjectural formulas for $MV_{g,n}$ for all n and fixed $g \leqslant 6$ (Conjecture 5.4). Conditionally on this conjecture, we discuss the consequences for area Siegel–Veech constants in Corollary 5.5 and for the $n \to \infty$ asymptotics in Section 5.6.

The paper is supplemented with three appendices. In Appendix A, we establish a closed formula for all ψ classes intersections in genus one, which we have not found in the literature and which we use for computations of $V\Omega^{MV}_{1,n}$ via stable graphs. In Appendix B, we illustrate the computation of Masur–Veech polynomials using the original formulation of the topological recursion à la Eynard–Orantin, via residues on a spectral curve. Appendix C contains tables of coefficients for the Masur–Veech polynomials and area Siegel–Veech constants.

Throughout the paper we make use of the symbol ■ at the end of those statements whose proof is not part of the paper, whereas the symbol $\star$ is used if the proof is included, but not immediately after the statement. No symbol is used if the proof follows the statement.

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2 Review of geometric and topological recursion

We review the aspects of the formalism of geometric recursion developed in [3] and its relation to the topological recursion which are directly relevant for the analysis carried out in the present paper, in Section 3 and onwards.

2.1 Preliminaries

For a given point in $\mathcal{T}_\Sigma$, the systole is the length of the shortest closed geodesic on Σ (it could possibly be a boundary component). The ϵ-thick part of the Teichmüller space is denoted by $\mathcal{T}_\Sigma^{(\epsilon)}$: it consists of those classes of hyperbolic metrics for which the systole is bounded below by ϵ.

Let S_Σ be the set of isotopy classes of simple closed curves in the interior of Σ, M_Σ the set of simple multicurves possibly with multiplicities (i.e. isotopy classes of finite disjoint unions of simple closed curves which are not homotopic to boundary components of Σ) and M'_Σ the subset of simple multicurves without multiplicities (the components of the multicurve must be pairwise non-homotopic). By convention M_Σ and M'_Σ contain the empty multicurve, but S_Σ does not contain the empty closed curve. In particular

$$M_\Sigma \cong \left\{ (\gamma, m) \;\middle|\; \gamma \in M'_\Sigma,\;\; m \in \mathbb{Z}_+^{\pi_0(\gamma)} \right\},$$

where $\mathbb{Z}_+$ is the set of positive integers (it does not include 0).

2.2 Geometric recursion

In the present context, the geometric recursion (in brief, GR) is a recipe to construct $\mathrm{Mod}_\Sigma^\partial$-invariant functions Ω_Σ on $\mathcal{T}_\Sigma$ for bordered surfaces Σ of all topologies, by induction on the Euler characteristic of Σ. The initial data for GR is a quadruple (A, B, C, D) where A, B, C are functions on the Teichmüller space of a pair of pants, and D is a function on the Teichmüller space of a torus with one boundary component. Since $\mathcal{T}_P \cong \mathbb{R}_+^3$, the functions A, B and C are just functions of three positive variables. We further require that A and C are invariant under exchange of their two last variables. In the construction we need that initial data satisfy some decay conditions. Let $[x]_+ = \max(x, 0)$.

Definition 2.1. We say that an initial data (A, B, C, D) is *admissible* if

- A is bounded on $\mathcal{T}_P$ and D is bounded on $\mathcal{T}_T$.
- For any $s > 0$ and some $\eta \in [0, 2)$,

$$\sup_{L_1, L_2, \ell \geqslant 0} \left(1 + [\ell - L_1 - L_2]_+\right)^s |B(L_1, L_2, \ell)|\, \ell^\eta < +\infty,$$

$$\sup_{L_1, \ell, \ell' \geqslant 0} \left(1 + [\ell + \ell' - L_1]_+\right)^s |C(L_1, \ell, \ell')|\, (\ell\ell')^\eta < +\infty.$$

Let us now briefly recall the recursion introduced in [3], which relies on successive excisions of pairs of pants. Assume that Σ has genus g and n boundary components such that $2g - 2 + n \geqslant 2$. We consider the set of homotopy classes of embedded pairs of pants $\phi \colon P \hookrightarrow \Sigma$ such that

- $\partial_1 P$ is mapped to $\partial_1 \Sigma$,
- $\partial_2 P$ is either mapped to a boundary component of Σ, or mapped to a curve that is not null-homotopic neither homotopic to a boundary component of Σ.

Let $\mathcal{P}_\Sigma$ the set of homotopy class of such embeddings. It is partitioned into the subsets $\mathcal{P}_\Sigma^\emptyset$ and $\mathcal{P}_\Sigma^m$ for $m \in \{2, \ldots, n\}$, consisting respectively of those classes of embeddings such that $\partial_2 P$ is mapped to the interior of Σ, resp. mapped to $\partial_m \Sigma$. Given a hyperbolic metric σ with geodesic boundaries on Σ, each element of $\mathcal{P}_\Sigma$ has a representative P such that $\phi(P)$ has geodesic boundaries. We denote by $\vec{\ell}_\sigma(\partial P)$ the ordered triple of lengths of $\phi(P)$ for the metric σ. Removing this embedded pair of pants from Σ gives a bordered surface $\Sigma - P$. Our assumptions imply that $\Sigma - P$ is stable. It is also equipped with a hyperbolic metric $\sigma|_{\Sigma - P}$ with geodesic boundaries. We decide to label the boundary components of $\Sigma - P$ by putting first the boundary components that came from those of P (respecting the order in which they appeared in ∂P) and then the boundary components that came from those of Σ (with the order in which they appeared in Σ).

The GR amplitudes Ω_Σ are now defined as follows. For surfaces with Euler characteristic -1, we declare

$$\Omega_P = A, \qquad \Omega_T = D.$$

For disconnected surfaces, we use the identification $\mathcal{T}_{\Sigma_1 \cup \Sigma_2} \cong \mathcal{T}_{\Sigma_1} \times \mathcal{T}_{\Sigma_2}$ to set

$$\Omega_{\Sigma_1 \cup \Sigma_2}(\sigma_1, \sigma_2) = \Omega_{\Sigma_1}(\sigma_1)\Omega_\Sigma(\sigma_2),$$

and for connected surfaces with Euler characteristic $\leqslant -2$, we set

$$\Omega_\Sigma(\sigma) = \sum_{m=2}^{n} \sum_{[P] \in \mathcal{P}_\Sigma^m} B(\vec{\ell}_\sigma(\partial P))\, \Omega_{\Sigma - P}(\sigma|_{\Sigma - P}) + \frac{1}{2} \sum_{[P] \in \mathcal{P}_\Sigma^\emptyset} C(\vec{\ell}_\sigma(\partial P))\, \Omega_{\Sigma - P}(\sigma|_{\Sigma - P}). \tag{2.1}$$

The latter is a countable sum and its absolute convergence was addressed[1] in [3]. We recall the main construction theorem of that paper here. Let $\mathcal{F}(\mathcal{T}_\Sigma, \mathbb{C})$ denote the set of complex valued functions on $\mathcal{T}_\Sigma$.

Theorem 2.2. If (A, B, C, D) is an admissible initial data, then $\Sigma \mapsto \Omega_\Sigma \in \mathcal{F}(\mathcal{T}_\Sigma, \mathbb{C})$ is a well-defined assignment. More precisely

- the series (2.1) is absolutely convergent for the supremum norm over any compact subset of $\mathcal{T}_\Sigma$;
- for any $\epsilon > 0$, Ω_Σ is bounded on any ϵ-thick part of $\mathcal{T}_\Sigma$;
- Ω_Σ is invariant under all mapping classes in Mod_Σ which preserve $\partial_1 \Sigma$;
- if the initial data is continuous (or measurable), Ω_Σ is also continuous (or measurable).

■

[1] The notion of admissibility adopted in the present paper is more restrictive than the notion of (strong) admissibility in [3], but is sufficient for our purposes.

2.3 Two examples

We describe two examples of initial data which play a special role for us. The first one appears in [30] in Mirzakhani's generalisation of McShane identity [29], which is a prototype of GR and which we can formulate in GR terms as follows.

Theorem 2.3 (Mirzakhani & McShane)**.** The initial data

$$\begin{aligned} A^M(L_1, L_2, L_3) &= 1, \\ B^M(L_1, L_2, \ell) &= 1 - \frac{1}{L_1} \ln\left(\frac{\cosh\left(\frac{L_2}{2}\right) + \cosh\left(\frac{L_1+\ell}{2}\right)}{\cosh\left(\frac{L_2}{2}\right) + \cosh\left(\frac{L_1-\ell}{2}\right)}\right), \\ C^M(L_1, \ell, \ell') &= \frac{2}{L_1} \ln\left(\frac{e^{\frac{L_1}{2}} + e^{\frac{\ell+\ell'}{2}}}{e^{-\frac{L_1}{2}} + e^{\frac{\ell+\ell'}{2}}}\right), \\ D^M_T(\sigma) &= \sum_{c \in S_T} C^M\big(\ell_\sigma(\partial T), \ell_\sigma(c), \ell_\sigma(c)\big), \end{aligned} \tag{2.2}$$

are admissible, and for any bordered Σ the corresponding GR amplitude Ω^M_Σ is the constant function 1 on $\mathcal{T}_\Sigma$. ■

The second example is obtained by rescaling all length variables in Mirzakhani initial data as follows

$$X^K(L_1, L_2, L_3) = \lim_{\beta \to \infty} X^M(\beta L_1, \beta L_2, \beta L_3), \qquad X \in \{A, B, C\}. \tag{2.3}$$

More explicitly

$$\begin{aligned} A^K(L_1, L_2, L_3) &= 1, \\ B^K(L_1, L_2, \ell) &= \frac{1}{2L_1}\big([L_1 - L_2 - \ell]_+ - [-L_1 + L_2 - \ell]_+ + [L_1 + L_2 - \ell]_+\big), \\ C^K(L_1, \ell, \ell') &= \frac{1}{L_1}\,[L_1 - \ell - \ell']_+, \\ D^K_T(\sigma) &= \sum_{c \in S_T} C^K(L_1, \ell_\sigma(c), \ell_\sigma(c)). \end{aligned} \tag{2.4}$$

It is easy to check that these initial data are admissible, and we call them the Kontsevich initial data. Unlike the previous situation, the resulting GR amplitudes Ω^K_Σ are non-trivial functions on $\mathcal{T}_\Sigma$. Their geometric interpretation and basic properties are studied in [1].

2.4 Hyperbolic length statistics and twisting of initial data

Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disk. Let $f\colon \mathbb{R}_+ \to \mathbb{C}$ and $\widetilde{f}\colon \mathbb{R}_+ \to \mathbb{D}$ be two functions related by

$$f(\ell) = \sum_{k \geqslant 1} (\widetilde{f}(\ell))^k = \frac{\widetilde{f}(\ell)}{1 - \widetilde{f}(\ell)}. \tag{2.5}$$

Definition 2.4. We call $f\colon \mathbb{R}_+ \to \mathbb{C}$ an *admissible test function* if f is Riemann-integrable on $\mathbb{R}_+$ and for any $s > 0$

$$\sup_{\ell > 0}\ (1+\ell)^s\,|f(\ell)| < +\infty. \tag{2.6}$$

This condition is stronger than what is needed in [3], but is sufficient here.

Following [3], we consider multiplicative statistics of hyperbolic lengths of simple multicurves

$$N_\Sigma(f;\sigma) = \sum_{c \in M'_\Sigma} \prod_{\gamma \in \pi_0(c)} f(\ell_\sigma(\gamma)) = \sum_{c \in M_\Sigma} \prod_{\gamma \in \pi_0(c)} \widetilde{f}(\ell_\sigma(\gamma)). \tag{2.7}$$

It can be written either as a sum over simple multicurves without multiplicities, or as a sum over simple multicurves with multiplicities, the two expressions being related via the geometric series (2.5). According to our conventions, the empty multicurve gives a term equal to 1 in this sum.

In fact, these statistics satisfy the geometric recursion. If (A, B, C, D) are some initial data, we define its twisting

$$\begin{aligned} A[f](L_1, L_2, L_3) &= A(L_1, L_2, L_3), \\ B[f](L_1, L_2, \ell) &= B(L_1, L_2, \ell) + A(L_1, L_2, \ell)\, f(\ell), \\ C[f](L_1, \ell, \ell') &= C(L_1, \ell, \ell') + B(L_1, \ell, \ell')f(\ell) + B(L_1, \ell', \ell)f(\ell') + A(L_1, \ell, \ell')f(\ell)f(\ell'), \\ D[f]_T(\sigma) &= \sum_{c \in S_T} A\big(\ell_\sigma(\partial T), \ell_\sigma(c), \ell_\sigma(c)\big)\, f(\ell_\sigma(c)). \end{aligned} \tag{2.8}$$

Theorem 2.5. [3] If we choose (A, B, C, D) to be Mirzakhani initial data (2.2) and f is an admissible test function, the twisted initial data (2.8) are admissible and the resulting GR amplitudes equal the assignment $\Sigma \mapsto N_\Sigma(f; \cdot)$. ■

The idea of the proof is, for each $c \in M'_\Sigma$, to multiply the product in (2.7) by 1, seen as a function on the Teichmüller space of $\Sigma - c$. Then, one decomposes 1 using Mirzakhani's identity on $\mathcal{T}_{\Sigma - c}$, and interchanges the summation over simple multicurves with the summation over embedded pairs of pants. As the curves do not intersect the pair of pants, the structure of the geometric recursion (2.1) appears again, but the initial data are modified as in (2.8). It is important to consider only *simple* closed curves, as otherwise $\Sigma - c$ would not be anymore a bordered surface and the recursive procedure could not be carried out in this way.

The result of [3] is in fact more general. It says that for any choice of admissible initial data (A, B, C, D), the GR amplitudes resulting from their twist are statistics of hyperbolic lengths of simple multicurves biased by the GR amplitudes associated to (A, B, C, D).

2.5 Relation to the topological recursion

Being invariant under the pure mapping class group, the GR amplitudes Ω_Σ descend to functions on the moduli space $\mathcal{M}_{g,n}$, and we denote them by $\Omega_{g,n}$. The structure of the geometric recursion is compatible with factorisations of the Weil–Petersson volume form μ_{WP} when excising pairs of pants. This means that, if we integrate GR amplitude against μ_{WP}, the outcome will again be governed by a recursion with respect to the Euler characteristic, which is called the topological recursion (TR for short). The (countable) sum over homotopy classes of pairs of pants is replaced with a sum over the (finitely many) diffeomorphism class of embeddings of pair of pants.

Recall the notation

$$V\Omega_{g,n}(L_1, \ldots, L_n) = \int_{\mathcal{M}_{g,n}(L_1,\ldots,L_n)} \Omega_{g,n}(\sigma)\, d\mu_{WP}(\sigma),$$

whenever the integral on the right-hand side makes sense; by convention, we set $V\Omega_{g,n} = 0$ whenever $2g - 2 + n \leqslant 0$.

Theorem 2.6 (From GR to TR, [3])**.** If (A, B, C, D) are admissible, $V\Omega_{g,n}$ is well-defined as the integrand is Riemann-integrable, and it satisfies the topological recursion, that is for any $g \geqslant 0$ and $n \geqslant 1$ such that $2g - 2 + n \geqslant 2$

$$\begin{aligned} &V\Omega_{g,n}(L_1, L_2, \ldots, L_n) \\ &= \sum_{m=2}^{n} \int_{\mathbb{R}_+} B(L_1, L_m, \ell) V\Omega_{g,n-1}(\ell, L_2, \ldots, \widehat{L_m}, \ldots, L_n)\ell\, d\ell \\ &+ \frac{1}{2}\int_{\mathbb{R}_+^2} C(L_1, \ell, \ell')\bigg(V\Omega_{g-1,n+1}(\ell, \ell', L_2, \ldots, L_n) + \sum_{\substack{h+h'=g \\ J \sqcup J' = \{L_2,\ldots,L_n\}}} V\Omega_{h,1+|J|}(\ell, J)V\Omega_{h',1+|J'|}(\ell', J')\bigg)\ell\ell'\, d\ell\, d\ell' \end{aligned} \tag{2.9}$$

The base cases are

$$V\Omega_{0,3}(L_1, L_2, L_3) = A(L_1, L_2, L_3), \qquad V\Omega_{1,1}(L_1) = VD(L_1) := \int_{\mathcal{M}_{1,1}(L)} D(\sigma)\, d\mu_{WP}(\sigma).$$

■

We call any sequence of functions $V\Omega_{g,n}$ satisfying a recursion of the form (2.9) TR amplitudes. Let us come back to the two examples of Section 2.3.

According to Theorem 2.3, $V\Omega^M$ are the Weil–Petersson volumes of $\mathcal{M}_{g,n}(L_1, \ldots, L_n)$, and the topological recursion (2.9) in this case is Mirzakhani's recursion for these volumes [30]. To be complete, we should record the Weil–Petersson volume for tori with one boundary

$$VD^M(L_1) = \frac{\pi^2}{6} + \frac{L_1^2}{48},$$

which is also mentioned in [30]. Mirzakhani also expressed the Weil–Petersson volumes via intersection theory on Deligne–Mumford compactified moduli space of punctured surfaces $\overline{\mathfrak{M}}_{g,n}$.

Theorem 2.7. [31] The Weil–Petersson volumes satisfy

$$V\Omega^M_{g,n}(L_1, \ldots, L_n) = \int_{\overline{\mathfrak{M}}_{g,n}} \exp\left(2\pi^2\kappa_1 + \sum_{i=1}^{n} \frac{L_i^2}{2}\psi_i\right).$$

■

Actually, the topological recursion for $V\Omega^K$ is equivalent to the set of Virasoro constraints for the intersection of ψ classes on $\overline{\mathfrak{M}}_{g,n}$.

Theorem 2.8 (Conjecture [41], theorem of [26] and [12])**.**

$$V\Omega^K_{g,n}(L_1, \ldots, L_n) = \int_{\overline{\mathfrak{M}}_{g,n}} \exp\left(\sum_{i=1}^{n} \frac{L_i^2}{2}\psi_i\right).$$

In particular, $VD^K(L_1) = \frac{L_1^2}{48}$. ■

This is also a corollary of Theorem 2.7, as can be seen if we multiply all length variables by β in Mirzakhani initial data, let $\beta \to \infty$ and recall the definition (2.3). The main analysis carried out in this paper consists in rescaling length variables by $\beta \to \infty$ in the twisted GR amplitudes to understand properties of the asymptotic number of simple multicurves.

There are several other ways to see that Theorem 2.7 implies or is implied by Theorem 2.8, see [7, 13, 35]. They will also be discussed in the broader context of the geometric recursion in [1].

Symmetry issues

The GR amplitudes Ω_Σ are a priori invariant under mapping classes that preserve the first labelled boundary (see Theorem 2.2). Therefore, after integration, the TR amplitudes $V\Omega_{g,n}(L_1, \ldots, L_n)$ are symmetric functions of $L_2, \ldots, L_n$. In fact the topological recursion gives a special role to the length L_1 of the first boundary.

The framework of quantum Airy structures [27] provides sufficient conditions for the invariance of the TR amplitudes under all permutations of $(L_1, \ldots, L_n)$. These conditions are quadratic constraints on (A, B, C, VD) which are explicitly written down in [2, Section 2.2]. They are satisfied by the Mirzakhani and Kontsevich initial data obtained from spectral curves in the Eynard–Orantin description (Section 2.7.3), and they are stable under the twisting operation [2]. All TR amplitudes that will be considered in this article have the full $\mathfrak{S}_n$-symmetry.

The situation is different at the level of GR amplitudes. One can formulate a natural refinement of these conditions which implies invariance of the GR amplitudes under all mapping classes, including the ones exchanging $\partial_1\Sigma$ with another boundary component [3, Section 5]. However, these refined conditions are not satisfied by Kontsevich initial data, as one can prove [1] that Ω^K_Σ is *not* always invariant under mapping classes that do not respect $\partial_1\Sigma$. We do not know if the refined condition are satisfied by Mirzakhani initial data. Since Ω^M_Σ is the constant function 1, it is obviously Mod_Σ invariant, but if we decide to ignore this fact it remains mysterious why the recursion (2.1) for Mirzakhani's initial data produces functions that are fully Mod_Σ-invariant.

2.6 Twisting and stable graphs

If (A, B, C, D) are admissible initial data, the upper bound on the number of simple multicurves of bounded length directly implies that the twisted initial data $(A[f], B[f], C[f], D[f])$ remain admissible when f is an admissible test function (2.6). Therefore, the integrals

$$V\Omega_{g,n}(f; L_1, \ldots, L_n) := \int_{\mathcal{M}_{g,n}(L_1,\ldots,L_n)} \Omega_{g,n}(f;\sigma)\, d\mu_{WP}(\sigma)$$

of the twisted GR amplitudes $\Omega_{g,n}(f;\cdot)$ satisfy TR (2.9) for the initial data $(A[f], B[f], C[f])$ completed by

$$VD(f; L_1) = VD(L_1) + \frac{1}{2}\int_{\mathbb{R}_+} f(\ell)\, A(L_1, \ell, \ell)\, \ell d\ell.$$

The $V\Omega_{g,n}(f;\cdot)$ can also be evaluated by direct integration, exploiting the factorisation of the Weil–Petersson volume form when cutting along simple closed curves – which is clear from its expression in Fenchel–Nielsen coordinates. The result is that, while $\Omega_{g,n}(f;\cdot)$ is a (countable) sum over simple multicurves without multiplicities, its integral $V\Omega_{g,n}(f;\cdot)$ is a sum over the (finitely many) topological types of such multicurves. The latter is described by stable graphs.

Definition 2.9. A stable graph Γ of type (g, n) consists of the data

$$(V_\Gamma, H_\Gamma, \Lambda_\Gamma, h, \mathfrak{v}, i)$$

satisfying the following properties.

1. V_Γ is the set of vertices, equipped with a function $h\colon V_\Gamma \to \mathbb{N}$, called the genus.
2. H_Γ is the set of half-edges, $\mathfrak{v}\colon H_\Gamma \to V_\Gamma$ associate to each half-edge the vertex it is incident to, and $i\colon H_\Gamma \to H_\Gamma$ is the involution
3. E_Γ is the set of edges, consisting of the 2-cycles of i in H_Γ (loops at vertices are permitted).
4. Λ_Γ is the set of leaves, consisting of the fixed points of i, which are equipped with a labelling from 1 to n.
5. The pair (V_Γ, E_Γ) defines a connected graph.
6. If v is a vertex, $\overline{E}(v)$ (resp. $E(v)$) is the set of edges incident to v including (resp. excluding) the leaves and $k(v) = |\overline{E}(v)|$ is the valency of v. We require that for each vertex v, the stability condition holds:
$$2h(v) - 2 + k(v) > 0.$$
7. The genus condition
$$g = \sum_{v \in V_\Gamma} h(v) + b_1(\Gamma)$$
holds. Here $b_1(\Gamma)$ is the first Betti number of the graph Γ.

An automorphism of Γ consists of bijections of the sets V_Γ and H_Γ which leave invariant the structures $\mathfrak{h}$, $\mathfrak{v}$, and $\mathfrak{i}$ (and hence respect E_Γ and Λ_Γ). We denote $\mathrm{Aut}\,\Gamma$ the automorphism group of Γ.

We denote by $\mathbf{G}_{g,n}$ the set of stable graphs of type (g,n). It parametrises the topological types of simple multicurves without multiplicities on a bordered surface Σ of genus g with n labelled boundaries

$$\mathbf{G}_{g,n} = M'_\Sigma \,/\, \mathrm{Mod}^\partial_\Sigma.$$

The stable graph with a single vertex of genus g corresponds to the empty multicurve. The other stable graphs are in bijective correspondence with the boundary strata of $\overline{\mathfrak{M}}_{g,n}$; more precisely $\Gamma \in \mathbf{G}_{g,n}$ refers to a boundary stratum of complex codimension $|E_\Gamma|$ that contains the union over $v \in V_\Gamma$ of smooth complex curves of genus $h(v)$ with $k(v)$ punctures, glued in a nodal way along punctures that correspond to the two ends of the same edge.

By direct integration, we have that

Theorem 2.10. [3] Assume $V\Omega_{g,n}$ is $\mathfrak{S}_n$-invariant for any $g \geqslant 0$ and $n \geqslant 1$ such that $2g-2+n>0$. Then,

$$V\Omega_{g,n}(f;L_1,\ldots,L_n) = \sum_{\Gamma\in\mathbf{G}_{g,n}} \frac{1}{|\mathrm{Aut}\,\Gamma|}\int_{\mathbb{R}_+^{E_\Gamma}} \prod_{v\in V_\Gamma} V\Omega_{h(v),k(v)}\big((\ell_e)_{e\in E(v)},(L_\lambda)_{\lambda\in\Lambda(v)}\big)\prod_{e\in E_\Gamma}\ell_e\, f(\ell_e)d\ell_e.$$

If $\Omega_{g,0}$ is defined, this formula is also valid for $n=0$. ■

We record two useful combinatorial identities, valid for any $\Gamma \in \mathbf{G}_{g,n}$.

Lemma 2.11.

$$\chi_\Gamma \coloneqq \sum_{v\in V_\Gamma}\big(2-2h(v)-k(v)\big) = 2-2g-n,$$
$$d_\Gamma \coloneqq \sum_{v\in V_\Gamma}\big(3h(v)-3+k(v)\big) = 3g-3+n-|E_\Gamma|.$$

Proof. The claim follows by combining edge counting with the definition of the first Betti number $b_1(\Gamma)$, namely

$$\sum_{v\in V_\Gamma} k(v) = 2|E_\Gamma|+n, \qquad 1-|V_\Gamma|+|E_\Gamma|+\sum_{v\in V_\Gamma} h(v) = g.$$

□

2.7 Equivalent forms of the topological recursion

In this section, we describe equivalent form of the topological recursion (2.9), which can be convenient for either carrying out calculations or for exploiting properties proved in the context of Eynard–Orantin topological recursion.

2.7.1 Polynomial cases

Let ϕ_i be a measurable function on $\mathbb{R}_+^i$. The operators

$$\hat{B}[\phi_1](L_1,L_2) = \int_{\mathbb{R}_+} B(L_1,L_2,\ell)\,\phi_1(\ell)\,\ell\,d\ell, \qquad \hat{C}[\phi_2](L_1) = \int_{\mathbb{R}_+^2} C(L_1,\ell,\ell')\,\phi_2(\ell,\ell')\,\ell\,\ell'\,d\ell\,d\ell' \tag{2.10}$$

play an essential role in the topological recursion (2.9). It turns out that for Mirzakhani or Kontsevich initial data, these operators preserve the space of polynomials in one (for $\hat{B}$) or two (for $\hat{C}$) variables that are even with respect of each variable (we call them even polynomials). Since in both examples the base cases $(g,n)=(0,3)$ and $(1,1)$ are even polynomials in the length variables, it implies that $V\Omega^{M}_{g,n}$ and $V\Omega^{K}_{g,n}$ are even polynomials.

Definition 2.12. We say that an initial data (A, B, C, D) is polynomial if (B, C) are such that (2.10) preserve spaces of even polynomials and A and VD are themselves even polynomials.

For polynomial initial data, it is sometimes more efficient for computations to decompose $V\Omega_{g,n}$ on a basis of monomials and write the effect of $\hat{B}$ and $\hat{C}$ on these monomials. For instance, let us decompose

$$V\Omega_{g,n}(L_1, \ldots, L_n) = \sum_{d_1,\ldots,d_n \geqslant 0} F_{g,n}[d_1, \ldots, d_n] \prod_{i=1}^{n} e_{d_i}(L_i), \qquad e_d(\ell) = \frac{\ell^{2d}}{(2d+1)!},$$

and

$$\hat{B}[e_{d_3}](L_1, L_2) = \sum_{d_1,d_2 \geqslant 0} B^{d_1}_{d_2,d_3}\, e_{d_1}(L_1) e_{d_2}(L_2), \qquad \hat{C}[e_{d_2} \otimes e_{d_3}](L_1) = \sum_{d_1 \geqslant 0} C^{d_1}_{d_2,d_3}\, e_{d_1}(L_1).$$

The topological recursion (2.9) then takes the form, if $2g + n - 2 \geqslant 2$

$$\begin{aligned} &F_{g,n}[d_1, \ldots, d_n] \\ &= \sum_{m=2}^{n} \sum_{a \geqslant 0} B^{d_1}_{d_m,a}\, F_{g,n-1}[a, d_2, \ldots, \widehat{d_m}, \ldots, d_n] \\ &\quad + \frac{1}{2} \sum_{a,b \geqslant 0} C^{d_1}_{a,b} \Bigg(F_{g-1,n+1}[a, b, d_2, \ldots, d_n] + \sum_{\substack{h+h'=g \\ J \sqcup J' = \{d_2,\ldots,d_n\}}} F_{h,1+|J|}[a, J]\, F_{h',1+|J'|}[b, J'] \Bigg). \end{aligned} \tag{2.11}$$

For the sake of uniformity, we introduce a similar notation for the base cases of the recursion

$$F_{0,3}[d_1, d_2, d_3] = A^{d_1}_{d_2,d_3}, \qquad F_{1,1}[d_1] = D^{d_1}.$$

For the Kontsevich initial data, we have in the chosen basis

$$F_{g,n}[d_1, \ldots, d_n] = \prod_{i=1}^{n} (2d_i + 1)!! \int_{\overline{\mathfrak{M}}_{g,n}} \prod_{i=1}^{n} \psi_i^{d_i},$$

which vanishes unless $d_1 + \cdots + d_n = 3g - 3 + n$. Translating the Virasoro constraints of [41] – or computing directly with (2.4) – we find

$$F_{0,3}[d_1, d_2, d_3] = \delta_{d_1,d_2,d_3,0}, \qquad F_{1,1}[d] = \frac{\delta_{d,1}}{8}. \tag{2.12}$$

and

$$B^{d_1}_{d_2,d_3} = (2d_2 + 1)\delta_{d_1+d_2,d_3}, \qquad C^{d_1}_{d_2,d_3} = \delta_{d_1,d_2+d_3+2}. \tag{2.13}$$

A similar computation for Mirzakhani initial data can be found in [30] and is reviewed in [10] with notations closer to ours.

Other bases of the space of even polynomials are sometimes useful to consider. For instance, the linear isomorphism given by the Laplace transform

$$\mathcal{L}\colon \begin{array}{ccc} \mathbb{C}[L^2] & \longrightarrow & \mathbb{C}[p^{-2}]dp \\ \phi & \longrightarrow & \Big(\int_{\mathbb{R}_+} e^{-p\ell} \phi(\ell)\, \ell\, d\ell\Big) dp \end{array}$$

makes the bridge towards the Eynard–Orantin form of the topological recursion (see Section 2.7.3).

2.7.2 Twisting

The operation of twisting (2.8) preserves the polynomiality of initial data. Indeed, the condition (2.6) guarantees that all moments of the test function f exist and if we set

$$u_{d_1,d_2} = \int_{\mathbb{R}_+} \frac{\ell^{2d_1+2d_2+1}}{(2d_1+1)!(2d_2+1)!} f(\ell)\, d\ell, \tag{2.14}$$

we obtain

$$\begin{aligned}
A[f]^{d_1}_{d_2,d_3} &= A^{d_1}_{d_2,d_3}, \\
B[f]^{d_1}_{d_2,d_3} &= B^{d_1}_{d_2,d_3} + \sum_{a\geqslant 0} A^{d_1}_{d_2,a}\, u_{a,d_3}, \\
C[f]^{d_1}_{d_2,d_3} &= C^{d_1}_{d_2,d_3} + \sum_{a\geqslant 0} \left(B^{d_1}_{a,d_3}\, u_{a,d_2} + B^{d_1}_{a,d_2}\, u_{a,d_3}\right) + \sum_{a,b\geqslant 0} A^{d_1}_{a,b} u_{a,d_2} u_{b,d_3}, \\
D[f]^{d_1} &= D^{d_1} + \frac{1}{2}\sum_{a,b\geqslant 0} A^{d_1}_{a,b} u_{a,b}.
\end{aligned} \tag{2.15}$$

Let us denote by $F_{g,n}[f;\cdot]$ the coefficients of decomposition of the twisted TR amplitudes. According to Theorem 2.10, it can be expressed as a sum over decorated stable graphs

$$F_{g,n}[f;d_1,\ldots,d_n] = \sum_{\substack{\Gamma\in \mathbf{G}_{g,n} \\ d\colon H_\Gamma\to\mathbb{N}}} \frac{1}{|\mathrm{Aut}\,\Gamma|} \prod_{e\in E_\Gamma} u_{v(e),v'(e)} \prod_{v\in V_\Gamma} F_{h(v),k(v)}\big[(d_e)_{e\in \overline{E}(v)}\big], \tag{2.16}$$

where $H_\Gamma = \bigsqcup_{v\in V_\Gamma} \overline{E}(v)$ is the set of half-edges and $\{v(e),v'(e)\}$ are the endpoints of an edge e. In this sum we impose that the decoration of the i-th leaf is d_i. Similar twisting operations appear in the context of Givental group action on cohomological field theories, see [3] for the comparison.

2.7.3 Eynard–Orantin form

Originally, the topological recursion was formulated by Eynard and Orantin as a residue computation on spectral curves [18]. We present it in a restricted setting adapted to our needs. A local spectral curve is a triple $(x,y,\omega_{0,2})$ where

- x and y are holomorphic functions on a smooth complex curve $\mathcal{C}$;
- dx has a unique zero at a point $\alpha\in\mathcal{C}$, which is simple, and $dy(\alpha)\neq 0$;
- $\omega_{0,2}$ is a meromorphic symmetric bidifferential on $\mathcal{C}^2$ with a double pole on the diagonal with biresidue 1. The latter means that for any choice of local coordinate p on $\mathcal{C}$, the bidifferential $\omega_{0,2}(z_1,z_2) - \frac{dp(z_1)dp(z_2)}{(p(z_1)-p(z_2))^2}$ is holomorphic near the diagonal in $\mathcal{C}^2$.

We consider $x\colon \mathcal{C}\to\mathbb{C}$ as a double branched cover in a neighbourhood of α; it admits a non-trivial holomorphic automorphism τ exchanging the two sheets, i.e. $x\circ\tau = x$ but $\tau\neq \mathrm{id}$ and $\tau(\alpha)=\alpha$. We introduce the recursion kernel

$$K(z_1,z) = \frac{1}{2}\,\frac{\int_{\tau(z)}^{z}\omega_{0,2}(\cdot,z_1)}{[y(z)-y(\tau(z))]dx(z)},$$

and proceed to define multidifferentials $\omega_{g,n}(z_1,\ldots,z_n)$ for $g\geqslant 0$ and $n\geqslant 1$ as follows. We set $\omega_{0,1} = ydx$, further $\omega_{0,2}$ is part of the data of the local spectral curve, and for $2g-2+n>0$ we define inductively

$$\begin{aligned}
\omega_{g,n}(z_1,z_2,\ldots,z_n) = \operatorname*{Res}_{z\to\alpha} K(z_1,z)\bigg\{&\omega_{g-1,n+1}(z,\tau(z),z_2,\ldots,z_n) \\
&+ \sum_{\substack{h+h'=g \\ J\sqcup J'=\{z_2,\ldots,z_n\}}}^{\text{no }(0,1)} \omega_{h,1+|J|}(z,J)\otimes\omega_{h',1+|J'|}(\tau(z),J')\bigg\},
\end{aligned} \tag{2.17}$$

where $\sum^{\text{no }(0,1)}$ means that the sum excludes the cases where $(h,1+|J|)=(0,1)$ or $(h',1+|J'|)=(0,1)$. For $n=0$ and $g\geqslant 2$, we also define the numbers

$$\omega_{g,0} = \frac{1}{2-2g}\operatorname*{Res}_{z\to\alpha}\left(\int_\alpha^z ydx\right)\omega_{g,1}(z). \tag{2.18}$$

2.7.4 Equivalences

The correspondence with Section 2.7.1 appears if we decompose the $\omega_{g,n}$ on a suitable basis of 1-forms. Let us choose a coordinate p near α such that $x = p^2/2 + x(\alpha)$. We introduce the 1-form globally defined on $\mathcal{C}$

$$\xi_d(z_0) = \operatorname*{Res}_{z\to\alpha} \frac{dp(z)}{p(z)^{2d+2}} \left(\int_{\alpha}^{z} \omega_{0,2}(\cdot, z_0) \right). \tag{2.19}$$

We also introduce

$$\begin{aligned}
\xi_d^*(z) &= (2d+1)p(z)^{2d+1}, \\
\theta(z) &= \frac{-2}{(y(z) - y(\tau(z)))dx(z)} \underset{z\to\alpha}{\sim} \sum_{k\geqslant -1} \theta_k \frac{p(z)^{2k}}{dp(z)}, \\
u_{0,0} &= \lim_{z_1\to z_2} \left(\frac{\omega_{0,2}(z_1,z_2)}{dp(z_1)dp(z_2)} - \frac{1}{(p(z_1)-p(z_2))^2} \right).
\end{aligned}$$

Theorem 2.13. [2] For $2g - 2 + n > 0$, we have

$$\omega_{g,n}(z_1,\ldots,z_n) = \sum_{\substack{d_1,\ldots,d_n \geqslant 0 \\ d_1+\cdots+d_n \leqslant 3g-3+n}} F_{g,n}[d_1,\ldots,d_n] \bigotimes_{i=1}^{n} \xi_{d_i}(z_i),$$

where the $F_{g,n}$'s are given by the recursion (2.11) with

$$\begin{aligned}
A^{d_1}_{d_2,d_3} &= \operatorname*{Res}_{z\to\alpha} \xi^*_{d_1}(z) d\xi^*_{d_2}(z) d\xi^*_{d_3}(z)\, \theta(z), \\
B^{d_1}_{d_2,d_3} &= \operatorname*{Res}_{z\to\alpha} \xi^*_{d_1}(z) d\xi^*_{d_2}(z) \xi_{d_3}(z)\, \theta(z), \\
C^{d_1}_{d_2,d_3} &= \operatorname*{Res}_{z\to\alpha} \xi^*_{d_1}(z) \xi_{d_2}(z) \xi_{d_3}(z)\, \theta(z), \\
VD^{d} &= \frac{\theta_0 + u_{0,0}\theta_{-1}}{8}\, \delta_{d,0} + \frac{\theta_{-1}}{24}\, \delta_{d,1}.
\end{aligned}$$

■

The $F_{g,n}$'s associated with Kontsevich or Mirzakhani initial data are described by Theorem 2.13 for the spectral curve $\mathcal{C} = \mathbb{C}$, $x(z) = z^2/2$ and $\omega^K_{0,2}(z_1,z_2) = \frac{dz_1 \otimes dz_2}{(z_1-z_2)^2}$ for which $\tau(z) = -z$, and

$$y^K(z) = -z, \qquad y^M(z) = -\frac{\sin(2\pi z)}{2\pi}. \tag{2.20}$$

In other words

$$\theta^K(z) = \frac{1}{z^2 dz}, \qquad \theta^M(z) = \frac{2\pi}{z \sin(2\pi z) dz}. \tag{2.21}$$

More generally, if we assume that a polynomial GR initial data (A, B, C, D) leads to TR amplitudes described by Theorem 2.13 for a certain spectral curve, then $\omega_{g,n}(z_1,\ldots,z_n)$ and $V\Omega_{g,n}(L_1,\ldots,L_n)$ are two equivalent ways of collecting the numbers $F_{g,n}$'s, which are related by the Laplace transform. Indeed, we notice that $\xi_d = p^{-(2d+2)}dp + O(dp)$ and $p^{-(2d+2)}dp = \mathcal{L}[e_d]$. Let us introduce the projection operator

$$\mathcal{P}[\phi](p_0) = \operatorname*{Res}_{z\to\alpha} \frac{\phi(z)}{p(z) - p_0},$$

which takes as input a meromorphic 1-form on $\mathcal{C}$ and outputs the element of $\mathbb{C}[p_0^{-1}]dp_0$ such that $\phi(z_0) - \mathcal{P}[\phi](p_0)$ is holomorphic when $z_0 \to \alpha$. Hence $\mathcal{P}[\xi_d] = \mathcal{L}[e_d]$ and

$$\mathcal{L}^{\otimes n}[V\Omega_{g,n}](p_1,\ldots,p_n) = \mathcal{P}^{\otimes n}[\omega_{g,n}](p_1,\ldots,p_n). \tag{2.22}$$

Furthermore, twisting the GR initial data amounts to shifting [3]

$$\omega_{0,2}(z_1,z_2) \longrightarrow \omega_{0,2}(z_1,z_2) + \mathcal{L}[f]\big(\pm p(z_1) \pm p(z_2)\big)\, dp(z_1)dp(z_2), \tag{2.23}$$

where the two choice of signs $\pm$ are independent and arbitrary – they do not affect the right-hand side of (2.22).

3 Asymptotic growth of multicurves

3.1 Preliminaries

We review some aspects of the space of measured foliations which play a key role in this article. For a more complete description we refer to [20].

Let Σ be a closed or punctured surface. A measured foliation is an ordered pair $\lambda = (\mathcal{F}, \nu)$ where: $\mathcal{F}$ is a foliation of Σ whose leaves are 1-dimensional submanifolds except for the possible existence of isolated singular points, of valency $p \geqslant 3$ away from the punctures and univalent at the punctures; ν is a transverse measure invariant along $\mathcal{F}$. Two measured foliations are Whitehead equivalent if they are related by a sequence of isotopies (relatively to the punctures), and contraction or expansion of edges between two singularities (that should not be both punctures). We denote MF_Σ the set of Whitehead equivalence classes of measured foliations. For each $\sigma \in \mathfrak{T}_\Sigma$, MF_Σ is equipped with a hyperbolic length function which we denote by $\ell_\sigma \colon MF_\Sigma \to \mathbb{R}_+$.

The space MF_Σ is endowed with an integral piecewise linear structure, and the set of simple multicurves M_Σ is in (length-preserving) bijection with the set of integral points of MF_Σ. One can then define a measure μ_{Th} by lattice point counting, which is called the Thurston measure in this context; we normalise μ_{Th} such that M_Σ has covolume one in MF_Σ. Let us emphasise that our normalisation differs from the Thurston symplectic volume form by a constant factor, see [4, 34].

The space $Q\mathfrak{T}_\Sigma$ is intimately linked to MF_Σ by considering the horizontal and vertical foliations associated to a quadratic differential. More precisely we have a homeomorphism

$$\begin{array}{ccc} Q\mathfrak{T}_\Sigma & \longrightarrow & MF_\Sigma \times MF_\Sigma \setminus \Delta_\Sigma \\ q & \longmapsto & \left(\left[\sqrt{|\mathrm{Im}(q)|}\right], \left[\sqrt{|\mathrm{Re}(q)|}\right]\right) \end{array} \tag{3.1}$$

where

$$\Delta_\Sigma = \left\{ (\lambda_1, \lambda_2) \in MF_\Sigma^2 \;\middle|\; \exists \eta \in MF_\Sigma, \quad \iota(\eta, \lambda_1) + \iota(\eta, \lambda_2) = 0 \right\},$$

and $\iota \colon MF_\Sigma \times MF_\Sigma \to \mathbb{R}$ is the geometric intersection pairing, which extends continuously the topological intersection of (formal $\mathbb{Q}$-linear combinations of) simple multicurves, see e.g. [8].

The space $Q\mathfrak{T}_\Sigma$ has an integral piecewise linear structure defined in terms of holonomy coordinates. The Masur–Veech measure μ_{MV} is defined from this structure by lattice point counting [28, 38]. We define the Masur–Veech measure on the bundle $Q^1\mathcal{T}_\Sigma$ of quadratic differentials of unit area as follows. If $Y \subseteq Q^1\mathfrak{T}_\Sigma$, we put

$$\mu^1_{\mathrm{MV}}(Y) = (12g - 12 + 4n)\mu_{\mathrm{MV}}(\widetilde{Y}), \qquad \widetilde{Y} = \left\{ tq \;\middle|\; t \in (0, \tfrac{1}{2}) \;\text{ and }\; q \in Y \right\}$$

when $\tilde{Y}$ is measurable. This normalisation follows the one chosen in [5, 11, 22]. Then the Masur–Veech volume is by definition the total mass $MV_{g,n} = \mu^1_{\mathrm{MV}}(Q^1\mathfrak{M}_{g,n}) < \infty$.

Finally, we need to discuss Teichmüller spaces with zero boundary lengths. We introduce the space

$$\widehat{\mathcal{T}}_\Sigma = \bigcup_{L_1,\ldots,L_n \geqslant 0} \mathcal{T}_\Sigma(L_1, \ldots, L_n),$$

which is a stratified manifold. Its top-dimensional stratum is $\mathcal{T}_\Sigma$ and lower-dimensional strata correspond to some of the boundary length L_i are equal to zero. The lowest-dimensional stratum $\mathfrak{T}_\Sigma = \mathcal{T}_\Sigma(0, \ldots, 0)$ is identified with the Teichmüller space of punctured Riemann surfaces on Σ. The quotient of the action of $\mathrm{Mod}^\partial_\Sigma$ on $\widehat{\mathcal{T}}_\Sigma$ obviously respects the stratification, and is denoted by $\widehat{\mathcal{M}}_{g,n}$. Inside this moduli space, the lowest-dimensional stratum $\mathfrak{M}_{g,n} = \mathcal{M}_{g,n}(0, \ldots, 0)$ is identified with the usual moduli space of complex curves with punctures.

Following Thurston [36], we consider an asymmetric pseudo-distance on $\widehat{\mathcal{T}}_\Sigma$ defined for $\sigma, \sigma' \in \widehat{\mathcal{T}}_\Sigma$ as

$$d_{\mathrm{Th}}(\sigma, \sigma') = \sup_{\gamma \in S_\Sigma} \ln\left(\frac{\ell_{\sigma'}(\gamma)}{\ell_\sigma(\gamma)}\right).$$

The fact that this quantity is finite follows from the compactness of the space of measured foliations. We emphasize that S_Σ does not include boundary curves and hence d_{Th} is constant equal to zero on the Teichmüller space $\widehat{\mathfrak{T}}_P$ of the pair of pants P. It is expected that, on any other stable surface, d_{Th} is actually an asymmetric distance, but this is irrelevant for our purposes. We will simply use the facts that d_{Th} is non-negative, continuous and vanishes on the diagonal, i.e. $d_{\mathrm{Th}}(\sigma, \sigma) = 0$.

3.2 Masur–Veech volumes

In this paragraph, g, n are non-negative integers and $2g - 2 + n > 0$. Let $\phi\colon \mathbb{R}_+ \to \mathbb{C}$ be an admissible test function and Σ a surface of type (g, n). We introduce the additive statistics for $\sigma \in \widehat{\mathfrak{T}}_\Sigma$

$$N_\Sigma^+(\phi; \sigma) = \sum_{c \in M_\Sigma} \phi(\ell_\sigma(c)).$$

We are interested in some scaling limit of the additive statistics $N_\Sigma^+(\phi; \sigma)$. Namely, we define for $\beta > 0$ the scaling operator

$$\rho_\beta^* \phi(x) = \phi(x/\beta).$$

and we want to understand the behaviour of $N_\Sigma^+(\rho_\beta^* \phi; \sigma)$ and its integrals over the moduli spaces with fixed boundary lengths.

The result (Lemma 3.2 below) will be governed by two ingredients. First, the dependence on the test function will involve the following linear forms, for $k \geqslant 0$

$$c_k[\phi] = \int_{\mathbb{R}_+} \frac{\ell^{k-1}}{(k-1)!} \phi(\ell)\, d\ell. \tag{3.2}$$

Note that $c_0[\phi]$ is not always well-defined; we will assume it is only when necessary. Second, the dependence on the metric will be governed by the function

$$X_\Sigma\colon \begin{array}{ccc} \widehat{\mathfrak{T}}_\Sigma & \longrightarrow & \mathbb{R}_+ \\ \sigma & \longmapsto & (6g-6+2n)!\, \mu_{\mathrm{Th}}\big(\{\lambda \in MF_\Sigma \mid \ell_\sigma(\lambda) \leqslant 1\}\big) \end{array}$$

The function X_Σ is an important ingredient in [30] where most of its properties are proven. In particular, its integral over moduli space is proportional to the Masur–Veech volumes.

Lemma 3.1. The function X_Σ descends to a function $X_{g,n}$ on the moduli space $\widehat{\mathcal{M}}_{g,n}$ and

- The logarithm $\ln(X_\Sigma)$ is Lipschitz with respect to d_{Th}, namely

$$\frac{X_\Sigma(\sigma)}{X_\Sigma(\sigma')} \leqslant e^{(6g-6+2n)d_{\mathrm{Th}}(\sigma,\sigma')}.$$

- The average $VX_{g,n}(L_1, \ldots, L_n)$ exists and is a continuous function of $(L_1, \ldots, L_n) \in (\mathbb{R}_{\geqslant 0})^n$.
- We have that

$$MV_{g,n} = \frac{2^{4g-2+n}(4g-4+n)!}{(6g-7+2n)!} VX_{g,n}(0, \ldots, 0).$$

⋆

Lemma 3.2. Let $\sigma \in \widehat{\mathfrak{T}}_\Sigma$ and $\phi\colon \mathbb{R}_+ \to \mathbb{C}$ be an admissible test function. Then

$$\lim_{\beta\to\infty} \beta^{-(6g-6+2n)} N_\Sigma^+(\rho_\beta^*\phi; \sigma) = c_{6g-6+2n}[\phi]\, X_\Sigma(\sigma),$$

and further, the following limit exist and it equals

$$\lim_{\beta\to\infty} \beta^{-(6g-6+2n)} VN_{g,n}^+(\rho_\beta^*\phi; L_1, \ldots, L_n) = c_{6g-6+2n}[\phi]\, VX_{g,n}(L_1, \ldots, L_n).$$

for all $(L_1, \ldots, L_n) \in \mathbb{R}_{\geqslant 0}^n$. ⋆

Proof of Lemma 3.1. The first property follows from the inclusion of the unit ℓ_σ-ball in a $\ell_{\sigma'}$-ball:

$$\{\,\lambda \in \mathrm{MF}_\Sigma \mid \ell_\sigma(\lambda) \leqslant 1\,\} \subseteq \{\,\lambda \in \mathrm{MF}_\Sigma \mid \ell_{\sigma'}(\lambda) \leqslant \exp(d_{\mathrm{Th}}(\sigma,\sigma'))\,\}.$$

The integrability of X_Σ is proven in Theorem 3.3 of [33, p. 106]. Namely, the function X_Σ is bounded by the function

$$K_\Sigma(\sigma) = \kappa \prod_{\substack{\gamma \in S_\Sigma \\ \ell_\sigma(\gamma) \leqslant \epsilon}} \frac{1}{\ell_\sigma(\gamma)}$$

for appropriate constants $\kappa, \epsilon > 0$ that depend only on g and n. The function K_Σ is invariant under the action of the mapping class group and we denote $K_{g,n}$ the function it induces on moduli space. Mirzakhani showed that $K_{g,n}$ is integrable with respect to μ_{WP} over $\mathcal{M}_{g,n}(L)$ for any $L \in \mathbb{R}^n_{\geqslant 0}$ (see her proof of Theorem 3.3 in [33, pp. 111-112]).

We now prove that the integral $VX_{g,n}(L)$ is a continuous function of L. Let us choose a pair of pants decomposition of Σ and consider the corresponding Fenchel–Nielsen coordinates $(\ell_i,\tau_i)_{i=1}^{3g-3+n}$ realising $\mathcal{T}_\Sigma(L) \simeq (\mathbb{R}_+ \times \mathbb{R})^{3g-3+n}$. By continuity of $X_\Sigma(\sigma)$, for any compact set $Z \subset (\mathbb{R}_+ \times \mathbb{R})^{3g-3+n}$ the following function is continuous

$$L \longmapsto \int_{\{L\}\times Z} X_\Sigma(\sigma) d\mu_{\mathrm{WP}}(\sigma).$$

In order to show the continuity of $VX_{g,n}$, it remains to show that the contribution coming from the set $\mathcal{M}^{<\epsilon'}_{g,n}(L) \subset \mathcal{M}_{g,n}(L)$ of surfaces with a non-peripheral curve of length smaller than ϵ' is uniformly small in ϵ'. We use again the function $K_{g,n}$ for which

$$\int_{\mathcal{M}^{<\epsilon'}_{g,n}(L)} X_{g,n}(\sigma) d\mu_{\mathrm{WP}} \leqslant \int_{\mathcal{M}^{<\epsilon'}_{g,n}(L)} K_{g,n}(\sigma) d\mu_{\mathrm{WP}}.$$

The set $\mathcal{M}^{<\epsilon'}_{g,n}(L)$ is covered by the $(3g-3+n)2^{3g-4+n}$ sets

$$Y^{<\epsilon'}_{i_0,J}(L) = \pi\Big(\{L\} \times \Big\{\, (\ell_i,\tau_i)_{i=1}^{3g-3+n} \;\Big|\; \ell_{i_0} \leqslant \epsilon',\ \ell_j \leqslant \epsilon\ \ \forall j \in J,\ \ell_i \leqslant b_{g,n}(L)\ \ \forall i,\ 0 \leqslant \tau_i \leqslant \ell_i \,\Big\}\Big),$$

where J is a subset of $\{1,2,\ldots,3g-3+n\}$, i_0 an integer in the complement of J, $\pi\colon \mathcal{T}_\Sigma \to \mathcal{M}_\Sigma$ is the projection map, and $b_{g,n}(L)$ is the Bers constant of $\mathcal{T}_\Sigma(L)$. It is shown in [6] that $b_{g,n}(L)$ is uniformly bounded for L in compact subsets of $\mathbb{R}^n_{\geqslant 0}$. Now, given a point in $\mathcal{M}^{<\epsilon'}_{g,n}$, one can always choose a hyperbolic structure in its π-preimage so that all curves shorter than ϵ are contained in the pants decomposition. Hence

$$\begin{aligned}
\int_{\mathcal{M}^{<\epsilon'}_{g,n}(L)} K_{g,n}(\sigma)\, d\mu_{\mathrm{WP}}(\sigma) &\leqslant \kappa \sum_{i_0,J} \int_{Y^{<\epsilon'}_{i_0,J}} \prod_{j \in J} \frac{1}{\ell_j} \prod_{i=1}^{3g-3+n} d\ell_i d\tau_i \\
&\leqslant \kappa \sum_{i_0,J} \epsilon'\, \epsilon^{|J|} \, \big(b_{g,n}(L)\big)^{2(3g-4+n-|J|)} \\
&\leqslant \kappa(3g-3+n)\, 2^{3g-4+n} \big(b_{g,n}(L)\big)^{2(3g-4+n)} \epsilon'.
\end{aligned}$$

This concludes the proof of the continuity.

The proportionality with the Masur–Veech volume is derived in [32] for closed surfaces and extended to punctured surfaces in [11]. We only sketch the idea. Associated to any maximal measured foliation λ, Thurston [36] and Bonahon [9] constructed an analytic embedding

$$G_\lambda\colon \mathfrak{T}_\Sigma \longrightarrow H_\Sigma(\lambda)$$

where $H_\Sigma(\lambda)$ are the transverse Hölder distributions on (the support of) λ. The transverse Hölder distributions form a vector space of dimension $6g-6+2n$ which plays the role of the tangent space at λ in MF_Σ, see [9]. Mirzakhani then proved that G_λ factors through the space of measured laminations as $G_\lambda = I_\lambda \circ F_\lambda$ where

$$F_\lambda\colon \mathfrak{T}_\Sigma \longrightarrow \mathrm{MF}_\Sigma(\lambda) \qquad I_\lambda\colon \mathrm{MF}_\Sigma(\lambda) \longrightarrow H_\Sigma(\lambda)$$

are respectively the *horocyclic foliation* and *shearing coordinates* and where

$$\mathrm{MF}_\Sigma(\lambda) = \{\, \eta \in \mathrm{MF}_\Sigma \ \mid \ \forall \gamma \in S_\Sigma, \quad \iota(\lambda,\gamma) + \iota(\eta,\gamma) > 0 \,\}.$$

It is shown in [9, 32] that these maps are symplectomorphisms with respect to the Weil–Petersson symplectic form on $\mathfrak{T}_\Sigma$ and the Thurston symplectic forms on $H_\Sigma(\lambda)$ and MF_Σ. As a consequence, on the subset $\mathrm{MF}_\Sigma^{\max}$ of maximal foliations – which has full measure in MF_Σ – we obtain a map

$$\begin{array}{ccc} \mathfrak{T}_\Sigma \times \mathrm{MF}_\Sigma^{\max} & \longrightarrow & \mathrm{MF}_\Sigma \times \mathrm{MF}_\Sigma \\ (\sigma,\lambda) & \longmapsto & \big(\lambda, F_\lambda(\sigma)\big) \end{array}.$$

which is again a symplectomorphism.

On the other hand, the homeomorphism (3.1) $Q\mathfrak{T}_\Sigma \to \mathrm{MF}_\Sigma \times \mathrm{MF}_\Sigma \setminus \Delta_\Sigma$ maps μ_{MV} to $\mu_{\mathrm{Th}} \otimes \mu_{\mathrm{Th}}$, up to a constant factor. In order to match our normalisation of μ_{MV} one has to include the factor that corresponds to the ratio between the Thurston symplectic volume form and the measure obtained via integral points in MF_Σ, see [4, 34]. ☐

Proof of Lemma 3.2. Since ϕ is Riemann integrable, we have

$$\lim_{\beta\to\infty} \beta^{-6g-6+2n} N_\Sigma^{+}(\rho_\beta^* \phi;\sigma) = \int_{\mathrm{MF}_\Sigma} \phi \circ \ell_\sigma(\lambda)\, d\mu_{\mathrm{Th}}(\lambda). \tag{3.3}$$

Now, we can desintegrate the Thurston measure with respect to the function ℓ_σ. We denote by $\overline{\mu}$ the projectivised measure on $\mathbb{P}\mathrm{MF}_\Sigma$ defined by

$$\overline{\mu}(A) = \mu_{\mathrm{Th}}\big(\{\, \lambda \in \mathrm{MF}_\Sigma \mid [\lambda] \in A \ \text{ and } \ \ell_\sigma(\lambda) \leqslant 1 \,\}\big),$$

where $[\lambda]$ denotes the projective class of λ. Then we have the "polar form" of the Thurston measure

$$\mu_{\mathrm{Th}} = (6g-6+2n)\, t^{6g-7+2n}\, dt\, d\overline{\mu}.$$

The right hand side in (3.3) hence can be rewritten

$$(6g-6+2n)\left(\int_{\mathbb{R}_+} t^{6g-7+2n}\, \phi(t)\, dt\right) \mu_{\mathrm{Th}}\big(\{\, \lambda \in \mathrm{MF}_\Sigma \mid \ell_\sigma(\lambda) = 1 \,\}\big).$$

The above is equivalent to the first part of the Lemma.

To complete the proof, we should justify that the limit $\beta \to \infty$ and the integral over the moduli spaces can be exchanged. We will do so by dominated convergence. Let us denote

$$\mathscr{N}_\Sigma(R;\sigma) = \{\, c \in M_\Sigma \mid \ell_\sigma(c) \leqslant R \,\}.$$

By [33, Proposition 3.6 p. 110 and Theorem 3.3 p. 106] we have

$$|\mathscr{N}_\Sigma(R;\sigma)| \leqslant K_\Sigma(\sigma)\, R^{6g-6+2n}.$$

Now we have

$$\begin{aligned} \beta^{-(6g-6+2n)}\, N_\Sigma^{+}(\rho_\beta^*\phi;\sigma) &= \beta^{-(6g-6+2n)} \sum_{k\geqslant 0} \sum_{\substack{c\in M_\Sigma \\ \beta k \leqslant \ell_\sigma(c) < \beta(k+1)}} \phi\left(\frac{\ell_\sigma(c)}{\beta}\right) \\ &\leqslant K_\Sigma(\sigma)\left(\sum_{k\geqslant 0} (k+1)^{6g-6+2n} \sup_{k\leqslant \ell < k+1} |\phi(\ell)|\right) \\ &\leqslant K_\Sigma(\sigma)\left(\sum_{k\geqslant 0} (k+1)^{-2}\right) \sup_{\ell\geqslant 0}\, (\ell+2)^{6g-4+2n} |\phi(\ell)|. \end{aligned}$$

The right hand side is bounded by the decay assumption (2.6). By Lemma 3.1, for any $L \in \mathbb{R}^n_{\geqslant 0}$ the right-hand side is integrable against the Weil-Petersson measure over $\mathcal{M}_{g,n}(L)$. It is independent of β, so the conclusion follows by dominated convergence. ☐

3.3 Definition of the Masur–Veech polynomials

We introduce the Masur–Veech polynomials, for any $g, n \geqslant 0$ such that $2g - 2 + n > 0$

$$V\Omega^{MV}_{g,n}(L_1, \dots, L_n) = \sum_{\Gamma \in \mathbf{G}_{g,n}} \frac{1}{|\mathrm{Aut}\,\Gamma|} \int_{\mathbb{R}_+^{E_\Gamma}} \prod_{v \in V_\Gamma} V\Omega^K_{g(v),k(v)}\big((\ell_e)_{e \in E(v)}, (L_\lambda)_{\lambda \in \Lambda(v)}\big) \prod_{e \in E_\Gamma} \frac{\ell_e d\ell_e}{e^{\ell_e} - 1}. \qquad (3.4)$$

These are polynomials in the variable $(L_i^2)_{i=1}^n$ of total degree $3g-3+n$. Its terms of maximal total degree come from the stable graph with a single vertex of genus g with n leaves; therefore coincide with the Kontsevich volumes of the combinatorial moduli space $V\Omega^K_{g,n}(L_1, \dots, L_n)$. In order to evaluate the sum over stable graphs, we need the following integral.

Lemma 3.3. The function $f^{MV}(\ell) = \frac{1}{e^\ell - 1}$ is such that, for any $k \geqslant 0$,

$$\int_{\mathbb{R}_+} f^{MV}(\ell)\ell^{2k+1}\, d\ell = (2k+1)!\zeta(2k+2).$$

Proof. We compute

$$\begin{aligned}(2k+1)!\zeta(2k+2) &= \sum_{n \geqslant 1} \frac{1}{n^{2k+2}} \int_{\mathbb{R}_+} e^{-t}\, t^{2k+1}\, dt = \sum_{n \geqslant 1} \int_{\mathbb{R}_+} e^{-n\ell}\, \ell^{2k+1}\, d\ell \\ &= \int_{\mathbb{R}_+} \frac{e^{-\ell}}{1 - e^{-\ell}}\, \ell^{2k+1}\, d\ell = \int_{\mathbb{R}_+} \frac{1}{e^\ell - 1}\, \ell^{2k+1}\, d\ell.\end{aligned}$$

□

For $n = 0$ and $g \geqslant 2$, $V\Omega^{MV}_{g,0}$ is a number, which can also be extracted from $V\Omega^{MV}_{g,1}(L_1)$, as a particular case $n = 0$ of the following formula.

Lemma 3.4. For any $g, n \geqslant 0$ such that $2g - 2 + n > 0$, we have the dilaton equation

$$\big[\tfrac{L_{n+1}^2}{2}\big]\, V\Omega^{MV}_{g,n+1}(L_1, \dots, L_{n+1}) = (2g - 2 + n)\, V\Omega^{MV}_{g,n}(L_1, \dots, L_n),$$

where $\big[\frac{\ell^2}{2}\big]$ extracts the coefficient of $\frac{\ell^2}{2}$ in the polynomial to its right. In particular, for $g \geqslant 2$ we have that

$$\big[\tfrac{L^2}{2}\big]\, V\Omega^{MV}_{g,1}(L) = (2g-2)V\Omega^{MV}_{g,0}.$$

Proof. We introduce

$$\mathbf{G}^\bullet_{g,n} = \{\, (\Gamma, v) \mid \Gamma \in \mathbf{G}_{g,n} \text{ and } v \in V_\Gamma \,\}$$

and the surjective map $\pi \colon \mathbf{G}_{g,n+1} \to \mathbf{G}^\bullet_{g,n}$ which erase the $(n+1)$-th leaf from the stable graph but records the information of the vertex v to which this leaf was incident. In general π is not injective, but one can check that for any $\Gamma \in \mathbf{G}_{g,n}$, $v \in V_\Gamma$ and $\tilde{\Gamma} \in \pi^{-1}(\Gamma, v)$, we have

$$|\mathrm{Aut}\,\Gamma| = |\pi^{-1}(\Gamma, v)|\, |\mathrm{Aut}\,\tilde{\Gamma}|. \qquad (3.5)$$

The dilaton equation for the ψ classes intersections yields, for $2h - 2 + (k+1) > 0$

$$\big[\tfrac{\ell_{k+1}^2}{2}\big]\, V\Omega^K_{h,k+1}(\ell_1, \dots, \ell_{k+1}) = (2h - 2 + k)\, V\Omega^K_{h,k}(\ell_1, \dots, \ell_k),$$

and this expression vanishes when $2h - 2 + k = 0$. Therefore

$$\begin{aligned}&\big[\tfrac{L_{n+1}^2}{2}\big]\, V\Omega^{MV}_{g,n+1}(L_1, \dots, L_{n+1}) \\ &= \sum_{(\Gamma,v) \in \mathbf{G}^\bullet_{g,n}} \frac{2h(v) - 2 + k(v)}{|\mathrm{Aut}\,\tilde{\Gamma}|} \int_{\mathbb{R}_+^{E_\Gamma}} \prod_{w \in V_\Gamma} V\Omega^K_{h(w),k(w)}\big((\ell_e)_{e \in E(w)}, (L_\lambda)_{\lambda \in \Lambda(w)}\big) \prod_{e \in E_\Gamma} \frac{\ell_e\, d\ell_e}{e^{\ell_e} - 1},\end{aligned}$$

where $\tilde{\Gamma}$ is any element of $\pi^{-1}(\Gamma, v)$, $E(w)$ and $\Lambda(w)$ are the sets of edges and leaves incident to w in the graph Γ (and not in $\tilde{\Gamma}$). Using (3.5) we deduce that

$$\left[\tfrac{L_{n+1}^2}{2}\right] V\Omega^{\mathrm{MV}}_{g,n+1}(L_1,\ldots,L_{n+1}) = \sum_{\Gamma \in \mathbf{G}_{g,n}} \bigg(\sum_{v \in V_\Gamma} 2h(v) - 2 + k(v) \bigg) \\ \times \frac{1}{|\mathrm{Aut}\,\Gamma|} \int_{\mathbb{R}_+^{E_\Gamma}} \prod_{w \in V_\Gamma} V\Omega^{K}_{h(w),k(w)}\big((\ell_e)_{e \in E(w)}, (L_\lambda)_{\lambda \in \Lambda(v)}\big) \prod_{e \in E_\Gamma} \frac{\ell_e\, d\ell_e}{e^{\ell_e} - 1}.$$

By Lemma 2.11 the sum of Euler characteristics at the vertices is $2g - 2 + n$, hence the claim. □

3.4 Main result

In [11], the fourth author and his collaborators obtained by combinatorial methods a formula for the Masur–Veech volumes as a sum over stable graphs, exploiting the relation between Masur–Veech volumes and lattice point counting in the moduli space of quadratic differentials. Our proof is different and relies on ideas of the geometric recursion developed in [3]. Our method gives access to more general quantities, which we introduced under the name of Masur–Veech polynomials. We now prove that they record the asymptotic growth of the number of multicurves on surfaces with large boundaries, after integration against the Weil–Petersson measure. As a consequence of Lemma 3.2, we then show that the Masur–Veech volumes arise as the constant term of the Masur–Veech polynomials, up to normalisation.

Let us denote $\hat{c}[\phi]\colon \mathbb{C}[t^{-1}] \to \mathbb{C}$ the linear operator sending t^{-k} to $c_k[\phi]$ for $k > 0$, and t^0 to $\phi(0)$ when it exists.

Theorem 3.5. Let ϕ be an admissible test function admitting a Laplace representation

$$\phi(\ell) = \int_{\mathbb{R}_+} \Phi(t)\, e^{-t\ell}\, dt$$

for a measurable function Φ such that $t \mapsto |\Phi(t)|$ is integrable on $\mathbb{R}_+$. In particular, $\phi(0) = \lim_{\ell \to 0} \phi(\ell)$ exists. Then, for any $g, n \geqslant 0$ such that $2g - 2 + n > 0$, we have that

$$\lim_{\beta \to \infty} \beta^{-(6g-6+2n)}\, VN^{+}_{g,n}\big(\rho_\beta^* \phi; L_1, \ldots, L_n\big) = c_{6g-6+2n}[\phi]\, V\Omega^{\mathrm{MV}}_{g,n}(L_1,\ldots,L_n),$$
$$\lim_{\beta \to \infty} \beta^{-(6g-6+2n)}\, VN^{+}_{g,n}\big(\rho_\beta^* \phi; \beta L_1, \ldots, \beta L_n\big) = \hat{c}[\phi]\big(t^{-(6g-6+2n)}\, V\Omega^{\mathrm{MV}}_{g,n}(tL_1,\ldots,tL_n)\big),$$

and the convergence is uniform for L_i in any compact of $\mathbb{R}_{\geqslant 0}$. ⋆

Notice that the contribution of the test function factors out for finite boundary lengths. The assumption that ϕ has a Laplace representation is not essential. It could be waived by an approximation argument, if we had an integrable upper bound for the number of simple closed multicurves of lengths between βL_1 and βL_2. This is not currently available in the literature and we do not address this question here.

In particular, comparing with the second formula in Lemma 3.2 and the last formula of Lemma 3.1, we obtain

Corollary 3.6. For any $g, n \geqslant 0$ such that $2g - 2 + n > 0$, $VX_{g,n}(L_1,\ldots,L_n) = V\Omega^{\mathrm{MV}}_{g,n}(0,\ldots,0)$ is independent of $L_1,\ldots,L_n \in \mathbb{R}_{\geqslant 0}$, and the Masur-Veech volumes are

$$MV_{g,n} = \frac{2^{4g-2+n}(4g-4+n)!}{(6g-7+2n)!}\, V\Omega^{\mathrm{MV}}_{g,n}(0,\ldots,0).$$

■

It would be interesting to provide an a priori explanation why $VX_{g,n}$ is independent of the boundary lengths $L_1,\ldots,L_n$; for us it is merely the consequence of a computation.

Proof of Theorem 3.5. We fix once and for all g and n such that $2g-2+n>0$. In the admissibility assumption, we will only use a weaker form of decay

$$\sup_{\ell>0}(1+\ell)^{6g-6+2n+\delta}|\phi(\ell)|<+\infty, \tag{3.6}$$

with $\delta=1$.

The Laplace representation of ϕ allows us to convert additive statistics into multiplicative statistics. We are going to apply many times Fubini–Tonelli and dominated convergence theorem.

Admissibility implies convergence of the series

$$N_{\Sigma}^{+}(\rho_{\beta}^{*}\phi;\sigma)=\sum_{c\in M_{\Sigma}}\phi\left(\frac{\ell_{\sigma}(c)}{\beta}\right)=\sum_{c\in M_{\Sigma}}\int_{\mathbb{R}_{+}}\Phi(t)\prod_{\gamma\in\pi_{0}(c)}e^{-t\ell_{\sigma}(c)/\beta}dt.$$

By Fubini–Tonelli theorem applied twice, we have

$$N_{\Sigma}^{+}(\rho_{\beta}^{*}\phi;\sigma)=\int_{\mathbb{R}_{+}}\Phi(t)\,N_{\Sigma}^{t/\beta}(\sigma)\,dt, \tag{3.7}$$

$$VN_{g,n}^{+}(\rho_{\beta}^{*}\phi;\beta L_{1},\ldots,L_{n})=\int_{\mathbb{R}_{+}}\Phi(t)\,VN_{g,n}^{t/\beta}(\beta L_{1},\ldots,\beta L_{n})\,dt, \tag{3.8}$$

where

$$N_{\Sigma}^{t}(\sigma)=\sum_{c\in M_{\Sigma}}\prod_{\gamma\in\pi_{0}(c)}e^{-t\ell_{\sigma}(\gamma)}=\sum_{c\in M_{\Sigma}'}\prod_{\gamma\in\pi_{0}(c)}\frac{1}{e^{t\ell_{\sigma}(\gamma)}-1},$$

$$VN_{\Sigma}^{t}(L_{1},\ldots,L_{n})=\int_{\mathcal{M}_{g,n}(L_{1},\ldots,L_{n})}N_{g,n}^{t}(\sigma)\,d\mu_{WP}(\sigma),$$

are now multiplicative statistics, to which we can apply the theory reviewed in Section 2.

For $t>0$, by Theorem 2.10

$$\begin{aligned}
&VN_{g,n}^{t/\beta}(\beta L_{1},\ldots,\beta L_{n})\\
&:=\int_{\mathcal{M}_{g,n}(\beta L_{1},\ldots,\beta L_{n})}N_{g,n}^{t/\beta}(\sigma)\,d\mu_{WP}(\sigma)\\
&=\sum_{\Gamma\in\mathbf{G}_{g,n}}\frac{1}{|\mathrm{Aut}\,\Gamma|}\int_{\mathbb{R}_{+}^{E_{\Gamma}}}\prod_{v\in V_{\Gamma}}V\Omega^{M}_{h(v),k(v)}\big((\ell_{e})_{e\in E(v)},(\beta L_{\lambda})_{\lambda\in\Lambda(v)}\big)\prod_{e\in E_{\Gamma}}\frac{\ell_{e}\,d\ell_{e}}{e^{t\ell_{e}/\beta}-1}\\
&=\sum_{\Gamma\in\mathbf{G}_{g,n}}\frac{(\beta/t)^{2|E_{\Gamma}|}}{|\mathrm{Aut}\,\Gamma|}\int_{\mathbb{R}_{+}^{E_{\Gamma}}}\prod_{v\in V_{\Gamma}}V\Omega^{M}_{h(v),k(v)}\big((\beta\ell_{e}/t)_{e\in E(v)},(t\cdot\beta L_{\lambda}/t)_{\lambda\in\Lambda(v)}\big)\prod_{e\in E_{\Gamma}}\frac{\ell_{e}\,d\ell_{e}}{e^{\ell_{e}}-1}.
\end{aligned} \tag{3.9}$$

We remark that $\beta^{-(6g-6+2n)}VN_{g,n}^{t/\beta}(\beta L_{1},\ldots,\beta L_{n})$ is a polynomial in t^{-1} and β^{-1} of bounded degree. We observe that

$$\int_{\mathbb{R}_{+}}\Phi(t)\,dt\,=\phi(0),$$

which here is assumed to exist, while for $k\geqslant 1$

$$\int_{\mathbb{R}_{+}}\frac{1}{t^{k}}\,\Phi(t)\,dt=\int_{\mathbb{R}_{+}}\Phi(t)\int_{\mathbb{R}_{+}}\frac{\ell^{k-1}}{(k-1)!}\,e^{-t\ell}\,d\ell\,dt\,=c_{k}[\phi].$$

The assumptions on ϕ guarantee that $c_{k}[\phi]$ are finite for all $k\geqslant 0$. Hence (3.8) is finite for a fixed $\beta>0$.

We now study the $\beta\to\infty$ limit. For $\beta\geqslant 1$, we can bound the aforementioned polynomial by a β-independent polynomial in t^{-1} and integrating the latter against $\Phi(t)dt$ gives a finite result. Therefore, by dominated convergence, we have

$$\lim_{\beta\to\infty}\beta^{-(6g-6+2n)}\,VN_{g,n}^{+}(\rho_{\beta}^{*}\phi;\beta L_{1},\ldots,\beta L_{n})=\int_{\mathbb{R}_{+}}\Phi(t)\left(\lim_{\beta\to\infty}\beta^{-(6g-6+2n)}\,VN_{g,n}^{t/\beta}(\beta L_{1},\ldots,\beta L_{n})\right)dt.$$

Comparing Theorems 2.7 and 2.8 yields

$$\lim_{\beta\to\infty} \beta^{-2(3h-3+k)}\, V\Omega^{M}_{h,k}(\beta\ell_1/t,\dots,\beta\ell_k/t) = t^{-2(3h-3+k)}\, V\Omega^{K}_{h,k}(\ell_1,\dots,\ell_k),$$

and the limit is uniform for $(\ell_1,\dots,\ell_k,t^{-1})$ in any compact of $\mathbb{R}^{k+1}_{\geqslant 0}$. Thus, uniformly for $(L_1,\dots,L_n,t^{-1})$ in any compact of $\mathbb{R}^{n+1}_{\geqslant 0}$, we have that

$$\begin{aligned}&\lim_{\beta\to\infty} \beta^{-(6g-6+2n)} VN^{t/\beta}_{g,n}(\beta L_1,\dots,\beta L_n)\\ &= \frac{1}{t^{6g-6+2n}} \sum_{\Gamma\in \mathbf{G}_{g,n}} \int_{\mathbb{R}^{E_\Gamma}_+} \prod_{v\in V_\Gamma} V\Omega^{K}_{h(v),k(v)}\big((\ell_e)_{e\in E(v)},(tL_\lambda)_{\lambda\in\Lambda(v)}\big) \prod_{e\in E_\Gamma} \frac{\ell_e\, d\ell_e}{e^{\ell_e}-1},\end{aligned} \tag{3.10}$$

where we recognise the Masur–Veech polynomials introduced in Section 3.3. We arrive at

$$\begin{aligned}\lim_{\beta\to\infty} \beta^{-(6g-6+2n)}\, VN^{+}_{g,n}(\rho^*_\beta;\beta L_1,\dots,\beta L_n) &= \int_{\mathbb{R}_+} \Phi(t)\, t^{-(6g-6+2n)}\, V\Omega^{MV}(tL_1,\dots,tL_n)\, dt\\ &= \hat{c}[\phi]\big(t^{-(6g-6+2n)}\, V\Omega^{MV}(tL_1,\dots,tL_n)\big).\end{aligned} \tag{3.11}$$

Using finite boundary lengths L_i instead of rescaling them by β amounts to replacing L_i by L_i/β in (3.9), and by the aforementioned uniformity we then have

$$\begin{aligned}\lim_{\beta\to\infty} \beta^{-(6g-6+2n)}\, VN^{+}_{g,n}(\rho^*_\beta\phi;L_1,\dots,L_n) &= \hat{c}[\phi]\big(t^{-(6g-6+2n)}\, V\Omega^{MV}_{g,n}(0,\dots,0)\big)\\ &= c_{6g-6+2n}[\phi]\, V\Omega^{MV}_{g,n}(0,\dots,0).\end{aligned} \tag{3.12}$$

This concludes the proof of the theorem. □

Proof of Theorem 1.1. The expression of the Masur–Veech polynomials in terms of stable graphs is actually our Definition 3.4. Note that this is not a circular argument: in the beginning of the paper we stated that Masur–Veech polynomials have four different equivalent formulations, we then chose the formulation in terms of stable graphs to be their definition, and we show in the rest of the paper that the same polynomials are expressed in the remaining three formulations. Therefore, the only non-trivial statement left to prove is the second part of the theorem, i.e. formula (1.4), which follows immediately from Corollary 3.6. □

3.5 Expression via geometric and topological recursion

By comparison with Theorem 2.10, the structure of this formula implies that the Masur–Veech polynomials satisfy the topological recursion.

Proposition 3.7 (Geometric recursion for Masur–Veech volumes)**.** Let Ω^{MV} be the GR amplitudes produced by the initial data

$$\begin{aligned}
A^{MV}(L_1,L_2,L_3) &= 1,\\
B^{MV}(L_1,L_2,\ell) &= \frac{1}{(e^{\ell}-1)} + \frac{1}{2L_1}\big([L_1-L_2-\ell]_+ - [-L_1+L_2-\ell]_+ + [L_1+L_2-\ell]_+\big),\\
C^{MV}(L_1,\ell,\ell') &= \frac{1}{(e^{\ell}-1)(e^{\ell'}-1)} + \frac{1}{L_1}[L_1-\ell-\ell']_+\\
&\quad + \frac{1}{2L_1}\bigg\{\frac{1}{e^{\ell}-1}\Big([L_1-\ell-\ell']_+ - [-L_1+\ell-\ell']_+ + [L_1+\ell-\ell']_+\Big)\\
&\qquad + \frac{1}{e^{\ell'}-1}\Big([L_1-\ell-\ell']_+ - [-L_1-\ell+\ell']_+ + [L_1-\ell+\ell']_+\Big)\bigg\},\\
D^{MV}_T(\sigma) &= \sum_{\gamma\in S_T} e^{-\ell_\sigma(\gamma)}.
\end{aligned} \tag{3.13}$$

Then, for any $g \geqslant 0$ and $n \geqslant 1$ such that $2g-2+n>0$, the Masur–Veech polynomials satisfy

$$V\Omega^{\mathrm{MV}}_{g,n}(L_1,\ldots,L_n) = \int_{\mathcal{M}_{g,n}(L_1,\ldots,L_n)} \Omega^{\mathrm{MV}}_{g,n}\, d\mu_{\mathrm{WP}}.$$

In particular, they are computed by the topological recursion (2.9). ■

Notice that the notation $V\Omega^{\mathrm{MV}}$ is consistent with its use in (1.1). The above initial data is obtained by twisting the Kontsevich initial data (2.4) by the function $f^{\mathrm{MV}}(\ell) = \frac{1}{e^{\ell}-1}$ – it is admissible according to Definition 2.1 with $\eta = 1$. The function $\Omega^{\mathrm{MV}}_{g,n}$ is a non-trivial function on $\mathcal{T}_{\Sigma}$, which is not equal to the function $X_{g,n}$ from Lemma 3.1. For instance, we saw in Corollary 3.6 that $VX_{g,n}(L_1,\ldots,L_n)$ does not depend on $L_1,\ldots,L_n$, while $V\Omega^{\mathrm{MV}}_{g,n}(L_1,\ldots,L_n)$ are non-trivial polynomials whose constant term is $VX_{g,n}$. The relation between $X_{g,n}$ and $\Omega^{\mathrm{MV}}_{g,n}$ will be discussed in a broader context in [1].

Recall the decomposition

$$V\Omega^{\mathrm{MV}}_{g,n}(L_1,\ldots,L_n) = \sum_{d_1+\cdots+d_n \leqslant 3g-3+n} F^{\mathrm{MV}}_{g,n}[d_1,\ldots,d_n] \prod_{i=1}^{n} \frac{L_i^{2d_i}}{(2d_i+1)!}.$$

By Section 2.7 we can give two equivalent forms of Proposition 3.7, in terms of the $F_{g,n}$. The first one is the recursion of Theorem 1.2, of which we give a proof in the following. This recursion is spelled out explicitly in Section 5.2.

Proof of Theorem 1.2. From the topological recursion in Proposition 3.7, it follows that the $F_{g,n}$'s are computed by the recursion (2.11), by twisting the Kontsevich initial data (2.12)-(2.13) with $f^{\mathrm{MV}}(\ell) = \frac{1}{e^{\ell}-1}$, that is, by

$$u_{d_1,d_2} = \int_{\mathbb{R}_+} \frac{\ell^{2d_1+2d_2+1}}{(2d_1+1)!(2d_2+1)!}\,\frac{d\ell}{e^{\ell}-1} = \frac{(2d_1+2d_2+1)!}{(2d_1+1)!(2d_2+1)!}\zeta(2d_1+2d_2+2)$$

according to Lemma 3.3. □

The second equivalent form is the topological recursion à la Eynard–Orantin. Let us introduce the Hurwitz zeta function, for $k \geqslant 1$

$$\zeta_H(2k;z) = \frac{1}{z^{2k}} + \frac{1}{2}\sum_{m\in\mathbb{Z}^*}\frac{1}{(z+m)^{2k}},$$

and define the multidifferentials

$$\omega^{\mathrm{MV}}_{g,n}(z_1,\ldots,z_n) = \sum_{d_1+\cdots+d_n\leqslant 3g-3+n} F_{g,n}[d_1,\ldots,d_n] \bigotimes_{i=1}^{n} \zeta_H(2d_i+2;z_i)\,dz_i.$$

Proposition 3.8. For $g,n \geqslant 0$ such that $2g-2+n>0$, the $\omega^{\mathrm{MV}}_{g,n}(z_1,\ldots,z_n)$ are computed by Eynard–Orantin topological recursion (2.17) for the spectral curve

$$\mathcal{C} = \mathbb{C}, \qquad x(z) = \frac{z^2}{2}, \qquad y(z) = -z, \qquad \omega^{\mathrm{MV}}_{0,2}(z_1,z_2) = \left(\frac{1}{(z_1-z_2)^2} + \frac{\pi^2}{\sin^2\pi(z_1-z_2)}\right)\frac{dz_1\otimes dz_2}{2}.$$

Proof. Recall the spectral curve (2.20) associated with the Kontsevich initial data. The effect of twisting amounts to shifting $\omega^{K}_{0,2}(z_1,z_2) = \frac{dz_1\otimes dz_2}{(z_1-z_2)^2}$ according to (2.23). We compute, for $\mathrm{Re}\, z > 0$

$$\int_{\mathbb{R}_+} \frac{1}{e^{\ell}-1}\, e^{-\ell z}\,\ell\, d\ell = \sum_{m\geqslant 1}\int_{\mathbb{R}_+} e^{-\ell(z+m)}\,\ell\, d\ell = \sum_{m\geqslant 1}\frac{1}{(z+m)^2}. \tag{3.14}$$

As the choice of signs in (2.23) is arbitrary, we can also take

$$\omega^{\mathrm{MV}}_{0,2}(z_1,z_2) = \left(\frac{1}{(z_1-z_2)^2} + \frac{1}{2}\sum_{m\geqslant 1}\frac{1}{(z_1-z_2+m)^2} + \frac{1}{(z_1-z_2-m)^2}\right) dz_1\otimes dz_2$$
$$= \left(\frac{1}{(z_1-z_2)^2} + \frac{\pi^2}{\sin^2\pi(z_1-z_2)}\right)\frac{dz_1\otimes dz_2}{2}.$$

The sector of convergence for the integral (3.14) is irrelevant as we only need the (well-defined) Taylor expansion when $z_i \to 0$ to compute the $\omega_{g,n}$. Finally, we compute the differential forms ξ_d defined in (2.19) and which are used in Theorem 2.13 to decompose the $\omega^{\mathrm{MV}}_{g,n}$

$$\frac{\xi_d(z_0)}{dz_0} = \operatorname*{Res}_{z\to 0}\frac{dz}{z^{2d+2}}\left(\frac{1}{z_0-z} + \frac{1}{2}\sum_{m\geqslant 1}\frac{1}{z_0-z-m} + \frac{1}{z_0-z+m}\right)$$
$$= \frac{1}{z_0^{2d+2}} + \frac{1}{2}\sum_{m\geqslant 1}\frac{1}{(z_0+m)^{2d+2}} + \frac{1}{(z_0-m)^{2d+2}}$$
$$= \zeta_H(2d+2;z_0).$$

For $n=0$ and $g\geqslant 2$, Lemma 3.4 gives

$$V\Omega^{\mathrm{MV}}_{g,0} = \frac{1}{2g-2}\frac{F_{g,1}[1]}{3}.$$

This agrees with the definition (2.18) of $\omega^{\mathrm{MV}}_{g,0}$ by the following computation

$$\omega^{\mathrm{MV}}_{g,0} = \frac{1}{2-2g}\operatorname*{Res}_{z\to 0}\left(\int_0^z y\,dx\right)\omega_{g,1}(z) = \frac{1}{2g-2}\operatorname*{Res}_{z\to 0}\frac{z^3}{3}\,\omega^{\mathrm{MV}}_{g,1}(z)$$
$$= \frac{1}{2g-2}\sum_{d\geqslant 0}\left(\operatorname*{Res}_{z\to 0}\frac{z^3}{3}\,\zeta_H(2d+2;z)\,dz\right)F_{g,1}[d]$$
$$= \frac{1}{2g-2}\frac{F_{g,1}[1]}{3},$$

where we used that $\zeta_H(2d+2;z) = z^{-(2d+2)}dz + O(1)$ when $z\to 0$, which implies that only the $d=1$ term contributes to the residue. □

3.6 Equivalent expression in intersection theory

We can express Masur–Veech polynomials as a single integral over moduli space of curves of a certain class, which involves boundary divisors. This is just another way to express the sum over stable graphs (i.e. boundary strata of $\overline{\mathfrak{M}}_{g,n}$) by pushing-forward systematically the class to $\overline{\mathfrak{M}}_{g,n}$, that will be useful in section 5.1. We first recall that the even zeta values are related to Bernoulli numbers by

$$\zeta(2m+2) = (-1)^m\frac{B_{2m+2}(2\pi)^{2m+2}}{2(2m+2)!}, \tag{3.15}$$

with $B_2=\frac{1}{6}$, $B_4=-\frac{1}{30}$, etc.

Corollary 3.9. For $2g-2+n>0$, let us define the cohomology class in $H^\bullet(\overline{\mathfrak{M}}_{g,n})$

$$\Xi_{g,n} = \exp\left(\sum_{\delta}\frac{1}{2^{\jmath_\delta}}\sum_{m\geqslant 0}\frac{|B_{2m+2}|}{(2m+2)}\frac{(\psi_\delta+\psi'_\delta)^m}{m!}J_\delta\right).\mathbf{1},$$

where the sum runs over boundary divisors δ of $\overline{\mathfrak{M}}_{g,n}$, ψ_δ and ψ'_δ are the psi-classes on the nodes of δ,

$$\jmath_\delta = \begin{cases}1 & \text{if } \delta \text{ is irreducible},\\ 0 & \text{otherwise},\end{cases}$$

J_δ is the operator pinching the loop corresponding to the divisor δ (whose application is performed after the formal expansion of the exponential), and $\mathbf{1}$ is the class represented by a smooth curve. Note in particular that the operators J_δ commute and that $J_\delta^2 = 0$. The Masur-Veech polynomials can be expressed as follows

$$V\Omega^{\mathrm{MV}}_{g,n}(L_1,\ldots,L_n) = (2\pi^2)^{(3g-3+n)} \int_{\overline{\mathfrak{M}}_{g,n}} \Xi_{g,n} \exp\left(\sum_{i=1}^{n} \frac{L_i^2}{4\pi^2}\,\psi_i\right).$$

⋆

Once the spectral curve for a certain enumerative geometric problem satisfying topological recursion is known (here Proposition 3.8), one could apply Eynard's formula [17, Theorem 3.1] to obtain such a representation for $\omega^{\mathrm{MV}}_{g,n}(z_1,\ldots,z_n)$, and thus the Masur–Veech polynomials. To be self-contained, we prove the result by direct computation.

It would be interesting to obtain this formula by algebro-geometric methods. A first hint in this direction would be to express $\Xi_{g,n}$ in a more intrinsic way, as a characteristic class of a bundle over $\overline{\mathfrak{M}}_{g,n}$, maybe obtained by pushforward from the moduli space of quadratic differentials.

Proof. We shall examine the contribution in Theorem 3.5 of a given $\Gamma \in \mathbf{G}_{g,n}$ before integration over the product of moduli spaces at the vertices. Given a decoration $d\colon H_\Gamma \to \mathbb{N}$, an edge e receives a weight $(2d_{v(e)}+2d_{v'(e)}+1)!\zeta(2d_{v(e)}+2d_{v'(e)}+2)$ where $\{v(e), v'(e)\}$ are the vertices connected by e. We remark that it only depends on the total degree $D_e := d_{v(e)} + d_{v'(e)}$ associated to this edge. On the other hand, the contribution of the ψ classes at the ends of the edge is

$$\frac{\psi^{d_{v(e)}}(\psi')^{d_{v(e)}}}{2^{D_e}\, d_{v(e)}!\, d_{v'(e)}!}.$$

Therefore, we can replace the sum over decorations of half-edges $d\colon H_\Gamma \to \mathbb{N}$ by the sum over decorations of edges $D\colon E_\Gamma \to \mathbb{N}$, and attach to each edge a contribution of

$$\frac{(2D_e+1)!\,\zeta(2D_e+2)}{2^{D_e}\,D_e!}\big(\psi_{v(e)}+\psi_{v'(e)}\big)^{D_e} = (2\pi^2)^{D_e+1}\frac{|B_{2D_e+2}|}{2D_e+2}\big(\psi_{v(e)}+\psi_{v'(e)}\big)^{D_e}.$$

If we are interested in the coefficient of $\prod_{i=1}^n L_1^{2m_i+2}$ in $V\Omega^{\mathrm{MV}}_{g,n}(L_1,\ldots,L_n)$, the integration over the product of moduli spaces at each vertex will select the terms of cohomological degree $2d_\Gamma = \sum_{i=1}^n 2m_i + \sum_{e\in E_\Gamma} 2D_e$. According to Lemma 2.11 this implies

$$\sum_{e\in E_\Gamma}(D_e+1) = 3g-3+n-\sum_{i=1}^n m_i.$$

Therefore, factoring out $(2\pi^2)^{3g-3+n}$ while replacing $L_i^2/2$ with $L_i^2/(4\pi^2)$ in the sum over (Γ, D) leaves the sum invariant, and brings it in the desired form. □

4 Statistics of hyperbolic lengths for Siegel–Veech constants

4.1 Preliminaries

The area Siegel–Veech constant $SV_{g,n}$ of $Q\mathfrak{M}_{g,n}$ is a positive real number related to the asymptotic number of flat cylinders of a generic quadratic differential. Given a quadratic differential $q \in Q\mathfrak{M}_{g,n}$, we define

$$\mathcal{N}_{\mathrm{area}}(q,L) = \frac{1}{\mathrm{Area}(q)} \sum_{\substack{\mathbf{c}\subset q\\ w(\mathbf{c})\leqslant L}} \mathrm{Area}(\mathbf{c})$$

where the sum is over flat cylinders $\mathbf{c}$ of q whose width $w(\mathbf{c})$ (or circumference) is less or equal to L and Area refers to the total mass of the measure induced by the flat metric of q. By a theorem of Veech [39] and Vorobets [40], the number

$$SV_{g,n} = \frac{1}{MV_{g,n}} \frac{1}{\pi L^2} \int_{Q^1\mathfrak{M}_{g,n}} \mathcal{N}_{\text{area}}(q, L)\, d\mu^1_{MV}(q)$$

exists and is independent of $L > 0$. It is called the (area) Siegel–Veech constant of $Q\mathfrak{M}_{g,n}$.

4.2 Goujard's formula

Goujard showed in [22, Section 4.2, Corollary 1] how to compute $SV_{g,n}$ in terms of the Masur–Veech volumes. Her result is in fact more general as it deals with all strata of the moduli space of quadratic differentials, while the present article is only concerned with the principal stratum.

Theorem 4.1. [22] For $g, n \geqslant 0$ such that $2g + n - 2 \geqslant 2$, we have

$$\begin{aligned} SV_{g,n} \cdot MV_{g,n} &= \tfrac{(4g-4+n)(4g-5+n)}{(6g-7+2n)(6g-8+2n)}\, MV_{g-1,n+2} \\ &\quad + \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n}} \tfrac{n!}{n_1!n_2!}\, \tfrac{(4g-4+n)!}{(4g_1-3+n_1)!(4g_2-3+n_2)!}\, \tfrac{(6g_1-5+2n_1)!(6g_2-5+2n_1)!}{(6g-7+2n)!}\, MV_{g_1,1+n_1} MV_{g_2,1+n_2}. \end{aligned} \tag{4.1}$$

■

In [22] the contribution of $MV_{0,3} \cdot MV_{g,n-1}$ was written separately, but this term can be included in the sum if we remark that $MV_{0,3} = 4$ (see Section 5.3) and

$$\frac{(2n-5)!}{(n-3)!}\bigg|_{n=2} \coloneqq \lim_{n\to 2} \frac{\Gamma(2n-4)}{\Gamma(n-2)} = \frac{1}{2}.$$

The structure of this formula becomes more transparent if we rewrite it in terms of the rescaled Masur–Veech volumes that are sums over stable graphs

$$MV_{g,n} = \frac{2^{4g-2+n}(4g-4+n)!}{(6g-7+2n)!}\, V\Omega^{MV}_{g,n}(0,\ldots,0).$$

Corollary 4.2. For $g, n \geqslant 0$ such that $2g + n - 2 \geqslant 2$, we have

$$SV_{g,n} \cdot V\Omega^{MV}_{g,n}(0) = \frac{1}{4}\left(V\Omega^{MV}_{g-1,2+n}(0) + \frac{1}{2} \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n}} \frac{n!}{n_1!n_2!}\, V\Omega^{MV}_{g_1,1+n_1}(0)\, V\Omega^{MV}_{g_2,1+n_2}(0)\right). \tag{4.2}$$

■

We can give an even more compact form to this relation, in terms of generating series. If we introduce

$$\mathcal{Z}_\hbar(x) = \exp\left(\sum_{g\geqslant 0} \sum_{\substack{n\geqslant 1 \\ 2g-2+n>0}} \frac{\hbar^{g-1}x^n}{n!} \frac{V\Omega^{MV}_{g,n}(0)}{\pi^{6g-6+2n}}\right), \tag{4.3}$$

then Corollary (4.2) is equivalent to

Corollary 4.3. We have that

$$\sum_{g\geqslant 0} \sum_{\substack{n\geqslant 0 \\ 2g+n-2\geqslant 2}} \frac{\hbar^g}{n!} \frac{SV_{g,n} \cdot V\Omega^{MV}_{g,n}(0)}{\pi^{6g-4+2n}} = \frac{1}{2} \frac{\hbar^2 \partial_x^2 \sqrt{\mathcal{Z}_\hbar(x)}}{\sqrt{\mathcal{Z}_\hbar(x)}}. \tag{4.4}$$

Proof. Let us write $\mathscr{Z}_\hbar(x) = \exp\big(\sum_{g \geqslant 0} \hbar^{g-1} \mathscr{F}_g(x)\big)$. For $\alpha \in \mathbb{C}$ we compute

$$\begin{aligned}
&\frac{\hbar^2 \partial_x \mathscr{Z}_\hbar^\alpha(x)}{\mathscr{Z}_\hbar^\alpha(x)} \\
&= \sum_{g \geqslant 0} \hbar^g \Bigg(\alpha \partial_x^2 \mathscr{F}_{g-1}(x) + \alpha^2 \sum_{g_1+g_2=g} \partial_x \mathscr{F}_{g_1}(x) \cdot \partial_x \mathscr{F}_{g_2}(x)\Bigg) \\
&= \sum_{g \geqslant 0} \sum_{\substack{n \geqslant 0 \\ 2g+n>0}} \frac{\hbar^g}{n!} \Bigg(\alpha \, V\Omega^{MV}_{g-1,2+n}(0) + \alpha^2 \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n}} \frac{n!}{n_1! n_2!} V\Omega^{MV}_{g_1,1+n_1}(0)\, V\Omega^{MV}_{g_2,1+n_2}(0)\Bigg),
\end{aligned} \tag{4.5}$$

where we noticed that the restriction $2g-2+n > 0$ in (4.3) implies that there are no terms for $2g+n \leqslant 0$ in (4.5). The relative factor of $\frac{1}{2}$ between the two types of terms in (4.2) is reproduced by choosing $\alpha = \frac{1}{2}$, and we need to multiply (4.5) by an overall factor of a $\frac{1}{2}$ to reproduce the prefactor $\frac{1}{4}$ in (4.2). The factors of π also match since

$$6(g-1) - 6 + 2(2+n) = 6g_1 - 6 + 2(1+n_1) + 6g_2 - 6 + 2(1+n_2) = 6g - 4 + 2n. \tag{4.6}$$

They have been included so that the coefficients of $\hbar^g x^n$ in the generating series are rational numbers. □

The contributions in (4.2) correspond to the topology of surfaces obtained from Σ of genus g with n boundaries after cutting along a simple closed curve. It is important to note that the (somewhat unusual) feature that separating curves receive an extra factor of a $\frac{1}{2}$, which is reflected in the squareroot in the right-hand side of (4.4). Such sums (without this relative factor of a $\frac{1}{2}$) can be obtained by differentiating a sum over stable graphs with respect to the edge weight. Therefore, they also arise by integrating over the moduli space derivatives of the statistics of hyperbolic lengths of simple multicurves with respect to the test function. We make this precise in the next paragraphs.

4.3 Derivatives of hyperbolic length statistics

We define two natural derivative statistics for which we are going to study the scaling limit. First, if $\gamma_0 \in S_\Sigma$, we denote

$$\jmath(\gamma_0) = \begin{cases} 1 & \text{if } \gamma_0 \text{ is separating,} \\ 0 & \text{otherwise}. \end{cases}$$

Let ψ, ϕ be admissible test functions, and consider

$$N_\Sigma^{\pm}(\phi;\psi;\sigma) = \sum_{c \in M_\Sigma} \sum_{\gamma_0 \in \pi_0(c)} 2^{-\jmath(\gamma_0)}\, \psi(\ell_\sigma(\gamma_0)) \cdot \phi(\ell_\sigma(c)), \tag{4.7}$$

$$\widetilde{N}_\Sigma^{\pm}(\phi;\psi;\sigma) = \sum_{c \in M_\Sigma} \sum_{\gamma_0 \in \pi_0(c)} 2^{-\jmath(\gamma_0)}\, \psi\big(\ell_\sigma(\gamma_0)\big) \cdot \phi\big(\ell_\sigma(c-\gamma_0)\big). \tag{4.8}$$

Theorem 4.4. Assume that ψ is bounded, $\ell \mapsto \ell^{-1}\psi(\ell)$ is integrable over $\mathbb{R}_+$ and recall that $c_0[\psi] = \int_{\mathbb{R}_+} \frac{d\ell}{\ell}\, \psi(\ell)$. Assume that ϕ has a Laplace representation

$$\phi(\ell) = \int_{\mathbb{R}_+} \Phi(t)\, e^{-t\ell}\, dt$$

for some measurable function Φ such that $t \mapsto |\Phi(t)|$ is integrable over $\mathbb{R}_+$. For $g, n \geqslant 0$ such that $2g+n-2 \geqslant 2$ and fixed $L_1, \ldots, L_n \geqslant 0$, we have

$$\begin{aligned}
&\lim_{\beta \to \infty} \beta^{-(6g-6+2n)}\, VN_\Sigma^{\pm}(\rho_\beta^* \phi; \psi; \beta L_1, \ldots, \beta L_n) \\
&= \frac{1}{2} c_0[\psi]\, \hat{c}[\phi] \Bigg[t^{-(6g-6+2n)} \\
&\quad \cdot \Bigg(V\Omega^{MV}_{g-1,2+n}(0,0,L_1,\ldots,L_n) + \frac{1}{2} \sum_{\substack{g_1+g_2=g \\ J_1 \sqcup J_2 = \{L_1,\ldots,L_n\}}} V\Omega^{MV}_{g_1,1+|J_1|}(0,J_1)\, V\Omega^{MV}_{g_2,1+|J_2|}(0,J_2) \Bigg) \Bigg],
\end{aligned}$$

and

$$\lim_{\beta\to\infty} \beta^{-(6g-6+2n)}\, VN^{+}_{\Sigma}(\rho_\beta^*\phi;\psi;L_1,\dots,L_n)$$
$$= \frac{1}{2}\, c_0[\psi]\, c_{6g-6+2n}[\phi] \bigg(V\Omega^{MV}_{g-1,2+n}(0) + \frac{1}{2} \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n}} \frac{n!}{n_1!n_2!}\, V\Omega^{MV}_{g_1,1+n_1}(0)\, V\Omega^{MV}_{g_2,1+n_2}(0) \bigg).$$

In particular, this last expression is independent of $L_1,\dots,L_n$. Furthermore, replacing N with $\widetilde{N}$ gives the same limits. ⋆

By comparison with Goujard's formula (Corollary 4.2), we deduce that

Corollary 4.5. Under the assumptions of Theorem 4.4, for any fixed $L_1,\dots,L_n \in \mathbb{R}^n_+$, we have

$$2c_0[\psi]\, SV_{g,n} = \lim_{\beta\to\infty} \frac{VN^{+}_{g,n}(\rho^*_\beta\phi;\psi;L_1,\dots,L_n)}{VN^{+}_{g,n}(\rho^*_\beta\phi;L_1,\dots,L_n)}.$$

The same equality holds if N in the numerator is replaced with $\widetilde{N}$. ■

The corollary gives a hyperbolic geometric interpretation of the area Siegel–Veech constant. However, our proof is done by comparison of values from Goujard's formula. It would be desirable to find a direct and geometric proof of this identity, which would give a new proof of Goujard's formula. The derivative statistics of hyperbolic lengths $N^{+}_{g,n}(\phi;\psi;L_1,\dots,L_n)$ is indeed reminiscent of the Siegel–Veech transform. Via the Hubbard-Masur correspondence [25], the multicurve $c \in M_\Sigma$ is associated to a holomorphic quadratic differential q and the component γ_0 is the core curve of a cylinder of q. The difficulty, however, lies in comparing hyperbolic and flat lengths.

Proof of Theorem 4.4. The assumption $2g+n-2 \geqslant 2$ is made so that M_Σ does not only consist of the empty multicurve. If we encode multicurves with multiplicities as a pair consisting of a multicurve without multiplicity and integers k remembering the multiplicity for each of its component, we have that

$$N^{+}_{\Sigma}(\rho^*_\beta\phi;\psi;\sigma) = \sum_{\substack{c\in M'_\Sigma \\ m\colon \pi_0(c)\to\mathbb{N}^*}} \sum_{\gamma_0\in\pi_0(c)} 2^{-\jmath(\gamma_0)} m(\gamma_0)\, \psi(\ell_\sigma(\gamma_0))\, \phi\bigg(\beta^{-1}\sum_{\gamma\in\pi_0(c)} m(\gamma)\,\ell_\sigma(\gamma)\bigg) \tag{4.9}$$

since γ_0 in (4.7) can be any of the $m(\gamma_0)$ component of the multicurve with multiplicity.

As in the proof of Theorem 3.5, we rely on the Laplace representation for ϕ

$$\phi(\ell) = \int_{\mathbb{R}_+} \Phi(t)\, e^{-t\ell}\, dt \tag{4.10}$$

to convert additive statistics into multiplicative statistics. As their application is similar to the proof of Theorem 3.5, we will silently use the Fubini–Tonelli and dominated convergence theorems at many places – the estimates necessary for their application use that ψ is bounded and $c_0[\psi]$ exists.

The Laplace representation allows us to convert (4.9) into derivatives (with respect to the test function) of multiplicative statistics, namely

$$\begin{aligned} N^{+}_{\Sigma}(\rho^*_\beta\phi;\psi;\sigma) &= \int_{\mathbb{R}_+} \Phi(t) \bigg(\sum_{c\in M'_\Sigma} \sum_{\gamma_0\in\pi_0(c)} \frac{\psi(\ell_\sigma(\gamma_0))}{2^{\jmath(\gamma_0)}(1-e^{-t\ell_\sigma(\gamma_0)/\beta})} \prod_{\gamma\in\pi_0(c)} \frac{e^{-t\ell_\sigma(c)/\beta}}{1-e^{-t\ell_\sigma(\gamma)/\beta}} \bigg) dt \\ &= \int_{\mathbb{R}_+} \Phi(t)\, \partial_{z=0}\big(N^{t/\beta,z}_{\Sigma}(\psi;\sigma)\big) dt, \end{aligned} \tag{4.11}$$

where

$$N^{t,z}_{\Sigma}(\psi;\sigma) = \sum_{c\in M'_\Sigma} \prod_{\gamma\in\pi_0(c)} \frac{1}{e^{t\ell_\sigma(\gamma)}-1}\bigg(1+\frac{z\,\psi(\ell_\sigma(\gamma))}{2^{\jmath(\gamma)}(1-e^{-t\ell_\sigma(\gamma)})}\bigg)$$

is a polynomial of degree $3g-3+n$ in z. Integrating over the moduli space, we obtain a sum over the topological types of simple multicurves, that is, over stable graphs

$$
\begin{aligned}
&VN^{t,z}_{g,n}(\psi; L_1,\ldots,L_n) \\
&= \sum_{\Gamma\in\mathbf{G}_{g,n}} \frac{1}{|\mathrm{Aut}\,\Gamma|}\int_{\mathbb{R}_+^{E_\Gamma}} \prod_{v\in V_\Gamma} V\Omega^{M}_{h(v),k(v)}\big((\ell_e)_{e\in E(v)},(L_\lambda)_{\lambda\in\Lambda(v)}\big)\prod_{e\in E_\Gamma}\frac{1}{e^{t\ell_e}-1}\left(1+\frac{z\,\psi(\ell_e)}{2^{\jmath_e}(1-e^{-t\ell_e})}\right)\ell_e\,d\ell_e,
\end{aligned}
\tag{4.12}
$$

where $\jmath_e=1$ if e is separating and $\jmath_e=0$ otherwise. The coefficient of z in this sum reads

$$
\begin{aligned}
&\partial_{z=0}\big(VN^{t,z}_{g,n}(\psi; L_1,\ldots,L_n)\big) \\
&= \sum_{\substack{\Gamma\in\mathbf{G}_{g,n}\\ e_0\in E_\Gamma}} \frac{1}{|\mathrm{Aut}\,\Gamma|}\int_{\mathbb{R}_+^{E_\Gamma}} \prod_{v\in V_\Gamma} V\Omega^{M}_{h(v),k(v)}\big((\ell_e)_{e\in E(v)},(L_\lambda)_{\lambda\in\Lambda(v)}\big)\frac{\psi(\ell_{e_0})\,e^{-t\ell_{e_0}}}{2^{\jmath_{e_0}}(1-e^{-t\ell_{e_0}})^2}\,\ell_{e_0}d\ell_{e_0}\prod_{e\neq e_0}\frac{\ell_e\,d\ell_e}{e^{t\ell_e}-1}.
\end{aligned}
\tag{4.13}
$$

Let $\mathbf{G}'_{g,n}$ be the set of ordered pairs (Γ,e_0) where $\Gamma\in\mathbf{G}_{g,n}$ and $e_0\in E_\Gamma$. We introduce the map

$$
\mathbf{glu}\colon \Bigg(\mathbf{G}_{g-1,2+n}\sqcup \bigsqcup_{\substack{\{(g_1,J_1),(g_2,J_2)\}\\ g_1+g_2=g\\ J_1\sqcup J_2=\{1,\ldots,n\}}} \mathbf{G}_{g_1,1+n_1}\times\mathbf{G}_{g_2,1+n_2}\Bigg)\longrightarrow \mathbf{G}'_{g,n},
$$

which consists in adding an edge between the two special leaves – the two first leaves in the connected situation and the first leaf of each graph in the disconnected situation. This map is surjective but not necessarily injective. More precisely, if $(\Gamma,e_0)\in\mathbf{G}'_{g,n}$, let us cut e_0 to create the stable graph Γ' with n labelled leaves and 2 unlabelled leaves. Let $a_{\Gamma'}$ be equal to 2 if Γ' is invariant under the permutation of the two unlabelled leaves, and $a_{\Gamma'}=1$ otherwise. If Γ' is disconnected, we must have $a_{\Gamma'}=1$ because the two connected components can be distinguished by the subsets J_1 and J_2 of leaves of the initial graph that they contain. Furthermore, the number of automorphisms of Γ is the product of the number of automorphism of its connected components. If Γ' is connected, it must have genus $g-1$. If $a_{\Gamma'}=1$, there are two distinct graphs in $\mathbf{G}_{g-1,2+n}$, that differ by the labels of the two first leaves, which lead to (Γ,e_0) after application of $\mathbf{glu}$. If $a_{\Gamma'}=2$, these two graphs are actually isomorphic. So, when Γ' is connected, we always have

$$
\big|\mathbf{glu}^{-1}(\Gamma,e_0)\big|=\frac{2}{a_{\Gamma'}}.
$$

Finally, we notice that for $(\Gamma,e_0)\in\mathbf{G}'_{g,n}$ and $\tilde\Gamma\in\mathbf{glu}^{-1}(\Gamma,e_0)$, we always have

$$
|\mathrm{Aut}\,\Gamma| = a_{\Gamma'}\,|\mathrm{Aut}\,\tilde\Gamma|.
$$

Partitioning the sum (4.13) according to the fibers of $\mathbf{glu}$ we obtain

$$
\begin{aligned}
&\partial_{z=0}\big(VN^{t,z}_{g,n}(\psi; L_1,\ldots,L_n)\big) \\
&= \int_{\mathbb{R}_+}\frac{\psi(\ell_0)\,e^{-t\ell_0}}{(1-e^{-t\ell_0})^2}\,\ell_0\,d\ell_0 \\
&\times\Bigg\{\frac{1}{2}\sum_{\Gamma\in\mathbf{G}_{g-1,2+n}}\frac{1}{|\mathrm{Aut}\,\Gamma|}\int_{\mathbb{R}_+^{E_\Gamma}}\prod_{v\in V_\Gamma}V\Omega^{M}_{h(v),k(v)}\big((\ell_e)_{e\in E(v)},(L_\lambda)_{\lambda\in\Lambda(v)}\big)\prod_{e\in E_\Gamma}\frac{\ell_e\,d\ell_e}{e^{t\ell_e}-1} \\
&+\frac{1}{2}\sum_{\substack{\{(g_1,J_1),(g_2,J_2)\}\\ g_1+g_2=g\\ J_1\sqcup J_2=\{1,\ldots,n\}}}\prod_{i=1}^{2}\Bigg(\sum_{\substack{\Gamma_i\\ \in\mathbf{G}_{g_i,1+|J_i|}}}\frac{1}{|\mathrm{Aut}\,\Gamma_i|}\int_{\mathbb{R}_+^{E_{\Gamma_i}}}\prod_{v\in V_{\Gamma_i}}V\Omega^{M}_{h(v),k(v)}\big((\ell_e)_{e\in E_i(v)},(L_\lambda)_{\lambda\in\Lambda_i(v)}\big)\prod_{e\in E_{\Gamma_i}}\frac{\ell_e\,d\ell_e}{e^{t\ell_e}-1}\Bigg)\Bigg\},
\end{aligned}
$$

where $E_i(v)$ and $\Lambda_i(v)$ are the set of edges and leaves of Γ_i, and if λ is a special leaf we set $L_\lambda = \ell_0$. We stress that $\frac{1}{2}$ in the last line comes from $\jmath_{e_0} = 1$. We recognise the sums over stable graphs

$$VN^{t}_{h,k}(\widetilde{L}_1, \ldots, \widetilde{L}_k) = \sum_{\Gamma \in \mathbf{G}_{\tilde{g},\tilde{n}}} \frac{1}{|\mathrm{Aut}\,\Gamma|} \int_{\mathbb{R}_+^{E_\Gamma}} \prod_{v \in V_\Gamma} V\Omega^{M}_{h(v),k(v)}\big((\ell_e)_{e\in E(v)}, (\widetilde{L}_\lambda)_{\lambda \in \Lambda(v)}\big) \prod_{e \in E_\Gamma} \frac{\ell_e \, d\ell_e}{e^{t\ell_e} - 1},$$

which already appeared in the proof of Theorem 3.5. We can replace the last sum over pairs with a sum over ordered pairs up to multiplication by an extra factor of $\frac{1}{2}$. All in all,

$$\begin{aligned} \partial_{z=0}\big(VN^{t,z}_{g,n}(\psi; L_1, \ldots, L_n)\big) = \frac{1}{2} \int_{\mathbb{R}_+} \frac{\psi(\ell)\, e^{-t\ell}}{(1 - e^{-t\ell})^2} \Bigg(& VN^{t}_{g-1,2+n}(\ell, \ell, L_1, \ldots, L_n) \\ & + \frac{1}{2} \sum_{\substack{g_1 + g_2 = g \\ J_1 \sqcup J_2 = \{L_1, \ldots, L_n\}}} VN^{t}_{g_1, 1+|J_1|}(\ell, J_1) VN^{t}_{g_2,1+|J_2|}(\ell, J_2) \Bigg) \ell d\ell. \end{aligned} \tag{4.14}$$

We multiply the boundary lengths by β and divide t by β in order to insert this formula in (4.12). Notice that the quantity in parenthesis in (4.14) now contributes to an even polynomial in ℓ, such that the monomial ℓ^{2m} is a polynomial in (β/t), of top degree $6g - 6 + 2n - (2m + 2)$. We recall that the $\beta \to \infty$ leading behaviour of $VN^{t/\beta}_{h,k}$ from (3.10) is expressed via the Masur–Veech polynomials $V\Omega^{MV}_{h,k}$. Since

$$\lim_{\beta \to \infty} \beta^2 \int_{\mathbb{R}_+} \frac{\psi(\ell)\, e^{-t\ell/\beta}}{(1 - e^{-t\ell/\beta})^2} \ell^{2m+1} d\ell = \frac{1}{t^2} \int_{\mathbb{R}_+} \psi(\ell) \ell^{2m-1} d\ell \tag{4.15}$$

is finite for any $m \geqslant 0$, only the $m = 0$ terms will contribute in the leading $\beta \to \infty$ behaviour of (4.12), in which case (4.15) is equal to $t^{-2}\, c_0[\psi]$ which exist since $\ell \mapsto \ell^{-1}\psi(\ell)$ is integrable. We arrive at the formula

$$\begin{aligned} & \lim_{\beta \to \infty} \beta^{-(6g-6+2n)} VN^{+}_{g,n}(\rho^*_\beta \phi; \psi; \beta L_1, \ldots, \beta L_n) \\ & = \frac{1}{2}\, c_0[\psi]\, \hat{c}[\phi] \Bigg[t^{-(6g-6+2n)} \\ & \quad \cdot \Bigg(V\Omega^{MV}_{g-1,2+n}(0, 0, tL_1, \ldots, tL_n) + \frac{1}{2} \sum_{\substack{g_1 + g_2 = g \\ J_1 \sqcup J_2 = \{L_1, \ldots, L_n\}}} V\Omega^{MV}_{g_1,1+|J_1|}(0, J_1) V\Omega^{MV}_{g_2,1+|J_2|}(0, J_2) \Bigg) \Bigg], \end{aligned}$$

which is the first desired formula. To obtain the second formula, we remark that all $\beta \to \infty$ limits used in the previous arguments are uniform for $L_1, \ldots, L_n$ in any compact of $\mathbb{R}_{\geqslant 0}$. Hence

$$\begin{aligned} & \lim_{\beta \to \infty} \beta^{-(6g-6+2n)} VN^{+}_{g,n}(\rho^*_\beta \phi; \psi; L_1, \ldots, L_n) \\ & = \frac{1}{2}\, c_0[\psi]\, \hat{c}[\phi] \Bigg[t^{-(6g-6+2n)} \\ & \quad \cdot \Bigg(V\Omega^{MV}_{g-1,2+n}(0) + \frac{1}{2} \sum_{\substack{g_1 + g_2 = g \\ J_1 \sqcup J_2 = \{L_1, \ldots, L_n\}}} V\Omega^{MV}_{g_1,1+|J_1|}(0) V\Omega^{MV}_{g_2,1+|J_2|}(0) \Bigg) \Bigg]. \end{aligned}$$

The effect of $\hat{c}[\phi]$ factors out to give $c_{6g-6+2n}[\phi]$ and the sum over the partition $J_1 \sqcup J_2 = \{L_1, \ldots, L_n\}$ yields binomial coefficients, hence the formula we sought for.

The statistics $\widetilde{N}$ are perhaps more natural. Their expression slightly differs from (4.11) by one factor $e^{-t\ell/\beta}$ less in front of ψ – this factor was previously due to the contribution of γ_0 to the total length that was included before evaluating ϕ. Namely, we have

$$\widetilde{N}_\Sigma(\rho^*_\beta \phi; \psi; \sigma) = \int_{\mathbb{R}_+} \Phi(t)\, \partial_{z=0}\big(\widetilde{N}^{t/\beta, z}_\Sigma(\psi; \sigma)\big)\, dt,$$

with

$$\widetilde{N}_{\Sigma}^{t,z}(\psi;\sigma) = \sum_{c \in M'_{\Sigma}} \prod_{\gamma \in \pi_0(c)} \frac{1}{e^{t\ell_\sigma(\gamma)} - 1}\left(1 + \frac{z\,\psi(\ell_\sigma(\gamma))\,e^{t\ell_\sigma(\gamma)}}{2^{\jmath(\gamma)}(1 - e^{-t\ell_\sigma(\gamma)})}\right).$$

All the previous argument can be carried over, except that we use instead of (4.15) the limit

$$\lim_{\beta\to\infty} \beta^2 \int_{\mathbb{R}_+} \frac{\psi(\ell)}{(1 - e^{-t\ell/\beta})^2} \ell^{2m+1}\, d\ell = \frac{1}{t^2} \int_{\mathbb{R}_+} \psi(\ell)\ell^{2m-1} d\ell,$$

which yields the same result. □

5 Computing Masur–Veech polynomials

The Masur–Veech polynomial $V\Omega^{MV}_{g,n}$ has degree $6g - 6 + 2n$. As explained in Section 2.7.1, we decompose it as follows

$$V\Omega^{MV}_{g,n}(L_1, \ldots, L_n) = \sum_{\substack{d_1,\ldots,d_n \geqslant 0 \\ d_1+\cdots+d_n \leqslant 3g-3+n}} F_{g,n}[d_1, \ldots, d_n] \prod_{j=1}^{n} \frac{L_j^{2d_j}}{(2d_j + 1)!}, \tag{5.1}$$

In this section we drop the superscript ^{MV} on the $F_{g,n}$'s as it will always refer to the coefficients of (5.1). By symmetry, $F_{g,n}$ can be considered as a function on the set of partitions of size less or equal to $3g - 3 + n$. It is convenient to give a name to the value of $F_{g,n}$ on partitions with a single row

$$H_{g,n}[d] = F_{g,n}[d, 0, \ldots, 0] \qquad \text{and} \qquad H_n[d] = H_{0,n}[d].$$

By convention, if some d_i is negative or if $2g - 2 + n \leqslant 0$, we declare $F_{g,n}[d_1, \ldots, d_n] = 0$. We are particularly interested in the Masur–Veech volumes which – up to normalisation – are the values of this function on the empty partition

$$MV_{g,n} = \frac{2^{4g-2+n}(4g - 4 + n)!}{(6g - 7 + 2n)!} H_{g,n}[0]. \tag{5.2}$$

5.1 Leading coefficients via stable graphs

We denote $H^*_{g,n}[d] = H_{g,n}[3g - 3 + n - d]$ and consider low values of d. In other words, these are the coefficients appearing in front of the terms of high(est) degrees in the Masur–Veech polynomial $V\Omega^{MV}_{g,n}(L, 0, \ldots, 0)$. They can be computed efficiently with the stable graph formula, or equivalently with Proposition 3.9, since for degree reasons only stable graphs with less than d edges will contribute. We give a few examples of such computations, starting from the expression

$$H^*_{g,n}[d] = \frac{(6g - 5 + 2n - 2d)!}{(3g - 3 + n - d)!}\, 2^{-(3g-3+n)+2d}\, \pi^{2d} \int_{\overline{\mathfrak{M}}_{g,n}} \left[\Xi_{g,n}\right]_d \psi_1^{3g-3+n-d}$$

where $[\cdots]_d$ extracts the component of cohomological degree $2d$.

5.1.1 Genus zero.

To compute the vertex weights, we will use the formula [41]

$$\int_{\overline{\mathfrak{M}}_{0,n}} \prod_{i=1}^{n} \psi_i^{m_i} = \delta_{\sum_i m_i, n-3} \frac{(n-3)!}{d_1! \cdots d_n!},$$

which is a consequence of the string equation for the ψ classes.

• $\mathbf{d = 0}$. The computation of the integral is trivial and we have

$$H_n^*[0] = \frac{(2n-5)!}{2^{n-3}(n-3)!}\int_{\overline{\mathfrak{M}}_{0,n}} \psi_1^{n-3} = \frac{(2n-5)!}{2^{n-3}(n-3)!}.$$

• $\mathbf{d = 1}$. The only contribution comes from the stable graph with one edge joining two genus zero vertices (Figure 1). As the ψ^{n-4} carried by the first leaf saturates the dimension of the moduli space at its incident vertex v_1, this edge must have degree 0 and receives a weight $\frac{\pi^2 B_2}{2} = \frac{\pi^2}{12}$. Then, the contribution from each vertex after integration is equal to 1. It remains to distribute the leaves labelled $2, \ldots, n$ between a first group of $n-3$ which will be incident to v_1, and a second group of 2 which will be incident to the second vertex. Hence

$$H_n^*[1] = \frac{(2n-7)!}{2^{n-2}(n-4)!}\frac{(n-1)(n-2)}{3}\pi^2.$$

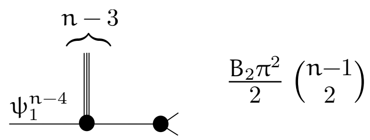

Figure 1 – Stable graph contributing to $H_n^*[1]$.

• $\mathbf{d = 2}$. We have to consider stable graphs with vertices of genus zero with 1 or 2 edges (for cohomological degree reasons the graph with no edges does not contribute). There are four cases (Figure 2).

1a– Two vertices are connected by an edge, the extra ψ class lies on the same vertex as ψ_1^{n-5}. There are $\binom{n-1}{2}$ ways to pick two leaves carried by the second vertex. The contribution of the first vertex is $\int_{\overline{\mathfrak{M}}_{0,n-1}} \psi_1^{n-5}\psi' = n-4$, and the contribution of the second vertex is $\int_{\overline{\mathfrak{M}}_{0,3}} 1 = 1$. The edge contribution is $\frac{\pi^4|B_4|}{4}$.

1b– Two vertices are connected by an edge, the first vertex carries ψ_1^{n-5} and the extra ψ class lies on the second vertex. There are $\binom{n-1}{3}$ ways to pick three leaves to the second vertex. Both vertex contributions are equal to 1, and the edge contribution is $\frac{\pi^4|B_4|}{4}$.

2a– A central vertex carrying ψ_1^{n-5} is connected to two other vertices carrying no ψ class. There are $\binom{n-1}{2,2,n-5}$ ways to pick two leaves for each of the two non-central vertices. The contribution of each vertex is 1, each edge contributes to a factor $\frac{\pi^2 B_2}{2}$ and we get an extra factor of a $\frac{1}{2}$ from the automorphism of the graph (exchange of the two non-central vertices).

2b– There are three vertices connected by two edges and ψ_1^{n-5} is carried by an extremal vertex. There are $n-1$ choices for the leaf on the central vertex, and $\binom{n-2}{2}$ ways to pick the two leaves for the second extremal vertex. The contribution of each vertex is 1, each edge contributes to a factor of $\frac{\pi^2 B_2}{2}$, and there are no automorphisms.

Summing up all contributions we obtain:

$$\begin{aligned} H_n^*[2] &= \frac{(2n-9)!}{2^{n-7}(n-5)!}\left\{\frac{|B_4|\pi^4}{4}\left((n-4)\binom{n-1}{2} + \binom{n-1}{3}\right)\right. \\ &\qquad\left. + \left(\frac{B_2\pi^2}{2}\right)^2\left(\frac{1}{2}\binom{n-1}{2,2,n-5} + (n-1)\binom{n-2}{2}\right)\right\} \\ &= \frac{(2n-9)!}{2^{n-7}(n-5)!}\frac{(n-1)(n-2)(5n^2+17n-120)}{5760}\pi^4. \end{aligned}$$

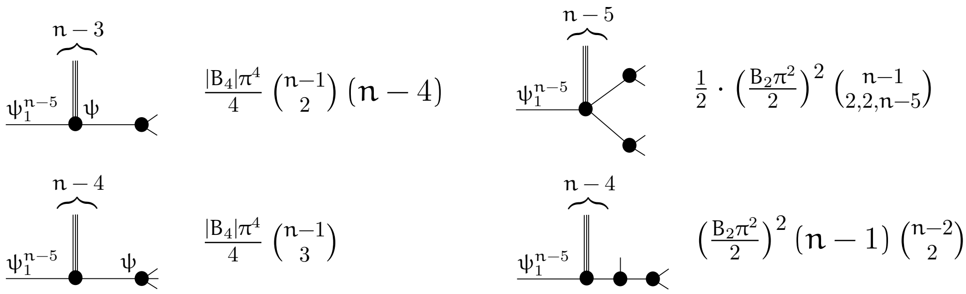

Figure 2 – Stable graphs contributing to $H_n^*[2]$.

5.1.2 Genus one

Here we will need the classical formula

$$\int_{\overline{\mathfrak{M}}_{1,n}} \psi_1^n = \frac{1}{24} \tag{5.3}$$

which is sufficient to compute $H_{1,n}^*[d]$ for $d = 0, 1$. More general ψ classes intersections in genus one would be necessary to push the stable graphs computations further. For instance, we will compute below $H_{1,n}^*[2]$ using

$$\int_{\overline{\mathfrak{M}}_{1,n}} \psi_1^{n-1}\psi_2 = \frac{n-1}{24}. \tag{5.4}$$

We present in Appendix A a closed formula for arbitrary genus one ψ classes intersections (including (5.3) and (5.4)), which we prove in an elementary way using well-known facts, but for which we could not find a reference.

• $\mathbf{d = 0}$. The only stable graph contributing has a single vertex, and with (5.3) we obtain

$$H_{1,n}^*[0] = \frac{1}{24}\,\frac{(2n+1)!}{2^n n!}.$$

• $\mathbf{d = 1}$. We have to consider the stable graphs with a single edge, which is either separating or non-separating (Figure 3). This edge cannot carry ψ classes and its contribution is $\frac{B_2\pi^2}{2}$. In the separating case, there is a vertex of genus one which carries the ψ_1^{n-1}, which is connected to a second vertex of genus zero. There are $\binom{n-1}{2}$ ways to distribute the two leaves on the genus zero vertex. The contribution of the genus zero vertex is 1 and the contribution of the genus one vertex is $\int_{\overline{\mathfrak{M}}_{1,n-1}} \psi_1^{n-1} = \frac{1}{24}$. In the non-separating case, there is a single vertex, which has genus zero; its contribution is 1, and we have an automorphism factor of $\frac{1}{2}$ (exchange of the two ends of the edge). Hence

$$\begin{aligned} H_{1,n}^*[1] &= \frac{(2n-2)!}{2^{n-2}(n-1)!}\,\frac{B_2\pi^2}{2}\left(\frac{1}{2} + \frac{1}{24}\binom{n-1}{2}\right) \\ &= \frac{(2n-2)!}{2^{n-2}(n-1)!}\,\frac{(n^2-3n+26)}{576}\pi^2. \end{aligned}$$

• $\mathbf{d = 2}$. We have to consider stable graphs with one or two edges (Figure 4). When there is a single edge, its contribution is $\frac{|B_4|\pi^4}{4}$ as we have an extra ψ class to distribute at one of its ends. Four cases appear.

1a– There is one non-separating edge on a single vertex of genus zero. The extra ψ class is carried by one extremity of the edge, forbidding non-trivial automorphisms. The contribution of the vertex is $\int_{\overline{\mathfrak{M}}_{0,n+2}} \psi_1^{n-2}\psi_2 = (n-1)$, and the contribution of the edge is $\frac{\pi^4|B_4|}{4}$.

Figure 3 – Stable graphs contributing to $H^*_{g=1,n}[1]$. The black vertices have genus zero and the green ones have genus one.

1b– The vertex carrying ψ_1^{n-2} has genus one and also carries the extra ψ class. It is connected to a genus zero vertex, which has two leaves and there are $\binom{n-1}{2}$ ways to choose them. The contribution of the genus one vertex is $\int_{\overline{\mathfrak{M}}_{1,n-1}} \psi_1^{n-2}\psi_2 = \frac{n-2}{24}$ using (5.4), the contribution of the genus zero vertex is 1.

1c– The vertex carrying ψ_1^{n-2} has genus one and is connected to a genus zero vertex carrying the extra ψ class. We can pick the 3 leaves incident to the genus zero vertex in $\binom{n-1}{3}$ ways. The contribution of the genus one vertex is $\int_{\overline{\mathfrak{M}}_{1,n-2}} \psi_1^{n-2} = \frac{1}{24}$ while the contribution of the genus zero vertex is 1.

1d – The vertex carrying ψ_1^{n-2} has genus zero and is connected to a genus one vertex carrying the extra ψ class. The contribution of the genus zero vertex is 1 and the contribution of the genus one vertex is $\int_{\overline{\mathfrak{M}}_{1,1}} \psi = \frac{1}{24}$.

When there are two edges, each of them contributes by a factor of $\frac{B_2\pi^2}{2}$ and there is no extra ψ class.

2a– The vertex carrying ψ_1^{n-2} has genus zero, is incident to a non-separating edge forming a loop, and the second edge connects it to another vertex of genus zero. There are $\binom{n-1}{2}$ ways to choose the two leaves on the second vertex. The loop is responsible for a symmetry factor of a $\frac{1}{2}$, and the contribution of both vertices is 1.

2b– The vertex carrying ψ_1^{n-2} has genus zero and is connected to another vertex of genus zero which carries a loop. The latter yields a symmetry factor of a $\frac{1}{2}$ and the contribution of both vertices is 1.

2c– The vertex carrying ψ_1^{n-2} has genus zero and is connected to another vertex of genus zero by two edges. To the second vertex should be assigned a leaf and this can be done in $(n-1)$ ways. There is a symmetry factor of a $\frac{1}{2}$ for the exchange of the two edges, and the contribution of both vertices is 1.

2d– The vertex carrying ψ_1^{n-2} has genus one, it is connected to a vertex of genus zero with one leaf, which itself is connected to another vertex of genus zero with 2 leaves. There are $(n-1)\binom{n-2}{2}$ ways to assign the leaves. The contribution from the genus one vertex is $\int_{\overline{\mathfrak{M}}_{1,n-2}} \psi_1^{n-2} = \frac{1}{24}$ and the contribution of the genus zero vertices is 1.

2e– There are three vertices, the central one has genus one and carries ψ_1^{n-2}, the extremal ones have genus zero and carry two leaves each. There are $\binom{n-1}{2,2,n-5}$ ways to assign the leaves but there is a symmetry factor of a $\frac{1}{2}$ for the exchange of the two extremal vertices. The contribution of the genus one vertex is $\int_{\overline{\mathfrak{M}}_{1,n-2}} \psi_1^{n-2} = \frac{1}{24}$ and the contribution of the genus zero vertices is 1.

Summing all contributions, we obtain

$$\begin{aligned} H^*_{1,n}[2] = \frac{(2n-3)!}{2^{n-4}(n-2)!}\Bigg\{ &\left(\frac{B_2\pi^2}{2}\right)^2 \left(\frac{1}{2}\binom{n-1}{2} + \frac{1}{2} + \frac{n-1}{2} + \frac{n-1}{24}\binom{n-2}{2} + \frac{1}{24}\frac{1}{2}\binom{n-1}{2,2,n-5}\right) \\ &+ \frac{|B_4|\pi^4}{4}\left(n-1+\frac{n-2}{24}\binom{n-1}{2} + \frac{1}{24}\binom{n-1}{3} + \frac{1}{24}\right)\Bigg\} \\ = \frac{5n^4+2n^3+127n^2+1162n-768}{138240}\pi^4. \end{aligned}$$

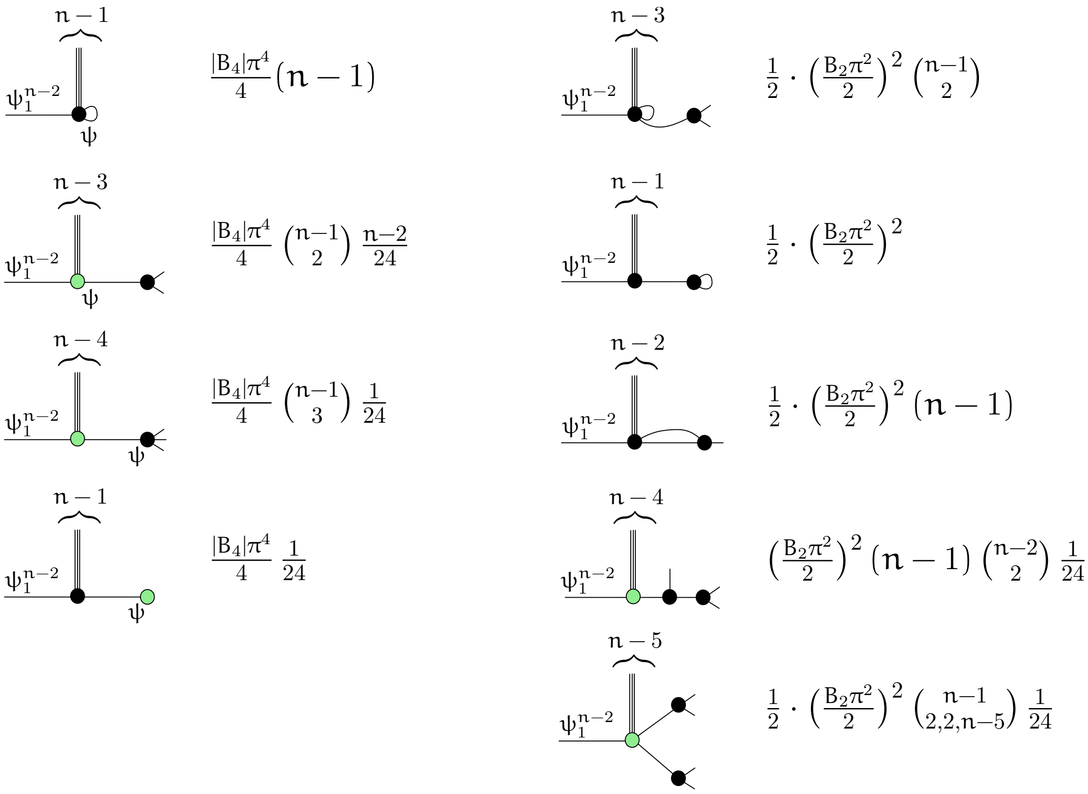

Figure 4 – Stable graphs contributing to $H^*_{g=1,n}[2]$. The black vertices have genus zero and the green ones have genus one.

5.2 Virasoro constraints

In this paragraph, we write down explicitly the recursion of Theorem 1.2 for the coefficients of the Masur–Veech polynomials. It is obtained by inserting the Kontsevich initial data (2.12)-(2.13) and the twist $u_{a,b}$ given by Theorem 1.2 into the general formula (2.15) and recursion (2.11). It is equivalent to Virasoro constraints, obtained (see e.g. [10]) by conjugation of the Virasoro constraints for ψ classes intersections, with the operator

$$\mathcal{U} = \exp\bigg(\sum_{a,b\geqslant 0} \frac{\hbar}{2}\, u_{a,b}\, \partial_{x_a}\partial_{x_b}\bigg), \qquad u_{a,b} = \frac{(2a+2b+1)!\zeta(2a+2b+2)}{(2a+1)!(2b+1)!}.$$

Base cases – When $2g-2+n=1$ we have

$$F_{0,3}[d_1,d_2,d_3] = \delta_{d_1,d_2,d_3,0}, \qquad F_{1,1}[d] = \delta_{d,0}\,\frac{\zeta(2)}{2} + \delta_{d,1}\,\frac{1}{8}.$$

We assume $2g+n-2 \geqslant 2$ in what follows.

String equation –

$$\begin{aligned} F_{g,n}[0,d_2,\ldots,d_n] = \sum_{i=2}^{n}\bigg(& F_{g,n-1}[d_2,\ldots,d_i-1,\ldots,d_n] + \delta_{d_i,0}\sum_{a\geqslant 0}\zeta(2a+2)F_{g,n-1}(a,d_2,\ldots,\widehat{d_i},\ldots,d_n)\bigg) \\ &+ \frac{1}{2}\sum_{a,b\geqslant 0}\bigg(\frac{(2a+2b+4)!}{(2a+2)!(2b+2)!}\zeta(2a+2b+4) + \zeta(2a+2)\zeta(2b+2)\bigg) \\ &\qquad \times\bigg(F_{g-1,n+1}[a,b,d_2,\ldots,d_n] + \sum_{\substack{h+h'=g \\ J\sqcup J'=\{d_2,\ldots,d_n\}}} F_{h,1+|J|}[a,J]F_{h',1+|J'|}[b,J']\bigg) \end{aligned}$$

DILATON EQUATION –

$$F_{g,n}[1,d_2,\dots,d_n] = \left(\sum_{i=2}^{n}(2d_i+1)\right)F_{g,n-1}[d_2,\dots,d_n] + \frac{1}{2}\sum_{a,b\geqslant 0}\frac{(2a+2b+2)!\zeta(2a+2b+2)}{(2a+1)!(2b+1)!}$$
$$\times\left(F_{g-1,n+1}[a,b,d_2,\dots,d_n] + \sum_{\substack{h+h'=g \\ J\sqcup J'=\{d_2,\dots,d_n\}}} F_{h,1+|J|}[a,J]F_{h',1+|J'|}[b,J']\right)$$

FOR $d_1 \geqslant 2$

$$F_{g,n}[d_1,\dots,d_n] = \sum_{i=2}^{n}(2d_i+1)F_{g,n-1}[d_1+d_i-1,d_2,\dots,\widehat{d_i},\dots,d_n]$$
$$+\sum_{a,b\geqslant 0}\left(\frac{1}{2}\delta_{a+b,d_1-2} + \delta_{a\geqslant d_1-1}\frac{(2a+2b+3-2d_1)!\zeta(2a+2b+4-2d_1)}{(2b+1)!(2a+2-2d_1)!}\right)$$
$$\times\left(F_{g-1,n+1}[a,b,d_2,\dots,d_n] + \sum_{\substack{h+h'=g \\ J\sqcup J'=\{d_2,\dots,d_n\}}} F_{h,1+|J|}[a,J]F_{h',1+|J'|}[b,J']\right)$$

In genus zero, the string equation (i.e. the first member of the Virasoro constraints) gives a recursion which uniquely determines all $F_{0,n}(d_1,\dots,d_n)$. Indeed, this number could be non-zero only when $d_1+\cdots+d_n \leqslant n-3$, which implies that at least 3 of the d_i's are zero. By symmetry we can take one of these zeroes to be d_1, and apply the string equation.

5.3 Recursion for genus zero, one row

If we specialise the Virasoro constraints to $g=0$ and $d_2=\cdots=d_n=0$, we obtain a recursion for the $H_n[d]=F_{0,n}[d,0,\dots,0]$.

Corollary 5.1. We have that

$$H_n[0] = \delta_{n,3} + (n-1)\sum_{a\geqslant 0}\zeta(2a+2)H_{n-1}[a]$$
$$+\frac{1}{2}\sum_{\substack{2\leqslant j\leqslant n-3 \\ a,b\geqslant 0}}\frac{(n-1)!}{j!(n-1-j)!}\left(\frac{(2a+2b+4)!\zeta(2a+2b+4)}{(2a+2)!(2b+2)!} + \zeta(2a+2)\zeta(2b+2)\right)H_{1+j}[a]H_{n-j}[b],$$
$$H_n[1] = (n-1)H_{n-1}[0] + \frac{1}{2}\sum_{\substack{2\leqslant j\leqslant n-3 \\ a,b\geqslant 0}}\frac{(n-1)!}{j!(n-1-j)!}\frac{(2a+2b+2)!\zeta(2a+2b+2)}{(2a+1)!(2b+1)!}H_{1+j}[a]H_{n-j}[b]$$

and for $d \geqslant 2$

$$H_n[d] = (n-1)H_{n-1}[d-1] + \sum_{\substack{2\leqslant j\leqslant n-3 \\ a,b\geqslant 0}}\frac{(n-1)!}{j!(n-1-j)!}$$
$$\times\left(\frac{1}{2}\delta_{a+b,d-2} + \delta_{a\geqslant d-1}\frac{(2a+2b+3-2d)!\zeta(2a+2b+4-2d)}{(2b+1)!(2a+2-2d)!}\right)H_{1+j}[a]H_{n-j}[b].$$

■

The last equation could also be written so as to give symmetric roles to a and b in the last term, and it is then easy to see that it is also valid for $d=1$. This recursion determines uniquely the $H_n[d]$, and a fortiori the genus zero Masur–Veech volumes

$$MV_{0,n} = \frac{2^{n-2}(n-4)!}{(2n-7)!}H_n[0].$$

We have not been able – even using generating series – to solve this recursion. It can however be used to generate efficiently the numbers $H_n[d]$.
From intersection theory on the moduli space of quadratic differentials, a closed formula is known for area Siegel–Veech constants in genus zero [15] and then the Masur–Veech volumes in genus zero [5].

Theorem 5.2. We have that

$$SV_{0,n} = \frac{n+5}{6\pi^2}, \qquad MV_{0,n} = \pi^{2(n-3)}\, 2^{5-n}.$$

■

In fact, using Goujard's formula[2] (Theorem 4.1), it is easy to see that the formula for $SV_{0,n}$ and the formula for $MV_{0,n}$ are equivalent. If one could guess a closed formula for the $H_n[d]$, it should be possible to check that it satisfies the recursion of Corollary 5.1, and by uniqueness deduce a new proof of Theorem 5.2.

Based on numerical data, we can guess the shape of a formula for fixed d but all n.

Conjecture 5.3. For each $d \geqslant 0$, there exists a polynomial P_d of degree d with rational coefficients such that

$$H_n[d] = \frac{(2d+1)}{(2d-1)!!}\,\frac{P_d(n)}{2^{(n-3-d)}}\,\frac{(2(n-3-d))!}{(n-3-d)!}\,\pi^{2(n-3-d)}. \tag{5.5}$$

The formula for $d = 0$ uses the convention $(-1)!! = 1$. Equivalently, there exists $\tilde{P}_d \in \mathbb{Q}[x]$ such that

$$\mathscr{H}(x;d) = \sum_{n \geqslant d+3} \frac{H_n[d]}{\pi^{2(n-3-d)}}\,\frac{x^{n-1}}{(n-1)!} = \Big[\tilde{P}_d(x)(1-x)^{3/2}\Big]_{\geqslant d+2}$$

where $[\cdots]_{\geqslant d+2}$ means that we only keep monomials of degree greater than $d+2$.

The formula is true for $d = 0$ with $P_0(n) = 1$ according to Theorem 5.2. For low values of d, we can find polynomials $P_d(n)$ interpolating the values $H_{d+3}[d], \ldots, H_{2d+3}[d]$ (see Table 5). Formula (5.5) then gives the correct values $H_n[d]$ for the $n > 2d+3$ that appear in Table 11.

5.4 Conjectures for Masur-Veech volumes with fixed g

For fixed g, the number of a priori non-zero coefficients $F_{g,n}[d_1, \ldots, d_n]$ grows faster than any polynomial in n, and the Virasoro constraint determines them by induction on $2g-2+n$. If one is interested primarily in obtaining $F_{g,n}[0, \ldots, 0]$, there is a more efficient way to use the Virasoro constraints.

In genus zero, we already saw that it implies recursion for the values of $F_{0,n}$ on partitions with one row. More generally, if $k > 0$ and we specialise $d_{k+1} = \cdots = d_n = 0$, we also get a recursion expressing the values of $F_{0,n}$ on partitions with at most k rows, in terms of the values of $F_{0,n'}$ for $n' < n$ on partitions with at most k rows. The same specialisation in genus $g > 0$ expresses the values of $F_{g,n}$ on partitions with at most k rows in terms of the values of $F_{g',n'}$ on partitions with at most k rows for $2g'-2+n' < 2g-2+n$, and the values of $F_{g-1,n+1}$ on partitions with at most $k+1$ rows. In this way, reaching $F_{g,n}[0, \ldots, 0]$ only requires the computation of a number of values of the $F_{g,n}$'s which grows polynomially with n.

Based on numerical data, we could guess general formulas for $MV_{g,n}$ for low values of g but all n. We start by defining the generating series

$$\mathscr{H}_g(x) = \sum_{n \geqslant 1} \frac{H_{g,n}[0]}{\pi^{6g-6+2n}}\,\frac{x^n}{n!} + \delta_{g,0}\,\mathscr{A}(x) \tag{5.6}$$

[2] The reader looking at Theorem 4.1 may think that to derive a formula for $MV_{0,n}$ from the knowledge of $SV_{0,n}$ one also needs the data of $MV_{0,3}$. Actually, in the literature $MV_{0,3}$ is ill-defined, while for us, in the context of statistics of length of multicurves, it makes perfect sense and is equal to 4. In the formulation of her result [22], Goujard wrote separately the terms that we included as contributions of $(0,3)$ in Theorem 4.1. Therefore the extra value of $MV_{0,3} = 4$ can be seen as a convention and the two formulas for $SV_{0,n}$ and $MV_{0,n}$ are indeed equivalent.

d	$P_d(n)$
0	1
1	$n-3$
2	$5n^2-34n+52$
3	$\frac{3}{2}(32n^3-367n^2+1307n-1392)$
4	$\frac{1}{6}(4138n^4-70496n^3+419969n^2-1002721n+751506)$
5	$\frac{15}{2}(1766n^5-41536n^4+365383n^3-1459754n^2+2493951n-1221210)$
6	$\frac{1}{20}(6377776n^6-197270496n^5+2385358645n^4-14079371820n^3+40768140229n^2$ $-48501218874n+9190581840)$
7	$\frac{7}{40}(52783968n^7-2073237920n^6+32861488488n^5-266767548125n^4+1152274787382n^3$ $-2422330473875n^2+1627352271762n+713960984880)$
8	$\frac{5}{56}(3504015400n^8-170178415232n^7+3416784683368n^6-36378043869776n^5+217683482202865n^4$ $-701967732545618n^3+976060154881647n^2+86564417888466n-937368548035920)$

Table 5 – Polynomials appearing in Conjecture 5.3 for $H_n[d]$.

where we allow for a conventional choice of a quadratic polynomial $\mathscr{A}(x)$. Theorem 5.2 implies that we can take

$$\mathscr{H}_0(x) = -\frac{8}{15}(1-x)^{5/2}, \qquad \mathscr{A}(x) = \frac{8}{15} - \frac{4}{3}x + x^2$$

where the role of $\mathscr{A}(x)$ is to cancel the coefficients of x^0, x^1, x^2 in the expansion of $\mathscr{H}_0(x)$, since they do not correspond to Masur-Veech volumes. The MAPLE command guessgf recognises that the values of $H_{1,n}[0]$ that we have computed for $n = 1, \ldots, 20$ match with the expansion of

$$\mathscr{H}_1(x) = -\frac{\ln(1-x)}{24} - \frac{\sqrt{1-x}}{12} + \frac{1}{12}.$$

It suggests that, for $g \geqslant 2$, $\mathscr{H}_g(x)$ could be a polynomial of degree $5(g-1)$ in the variable $(1-x)^{-1/2}$ with rational coefficients. Although the command guessgf fails for $g \geqslant 2$, we are on good tracks. If we attempt to match this ansatz for $g = 2$ and 3 with the data of Table 13, we discover that this polynomial has valuation $4(g-1)$. This leads us to guess that the generating series we look for may have the form

$$\mathscr{H}_g(x) = -\frac{\ln y}{12}\,\delta_{g,1} + y^{5(1-g)}\,Q_g(y) \qquad \text{with} \qquad y = \sqrt{1-x} \tag{5.7}$$

where Q_g is a polynomial of degree g with rational coefficients. We then determine the polynomials P_g such that (5.7) reproduces correctly the values $H_{g,n}[0]$ for $n \leqslant g+1$, and checked that they predict the correct values $H_{g,n}[0]$ for higher n that we computed in Table 11 with the recursion of Section 5.2.

Empirically, we recognise the top coefficients of these polynomials

$$\text{coeff of } y^g \text{ in } Q_g(y) = 2^{3-2g}\,(4g-7)!!\,b_g \tag{5.8}$$

where $b_g = (2^{1-2g}-1)(-1)^g\,\frac{B_{2g}}{2g!}$ are the coefficients of expansion in

$$\frac{z/2}{\sin(z/2)} = \sum_{g \geqslant 0} b_g\,z^{2g}.$$

g	$Q_g(y)$
0	$-\frac{8}{15}$
1	$-\frac{1}{12}y + \frac{1}{12}$
2	$\frac{7}{11520}y^2 + \frac{5}{2304}y + \frac{7}{2880}$
3	$\frac{31}{516096}y^3 + \frac{17}{36864}y^2 + \frac{223}{165888}y + \frac{245}{165888}$
4	$\frac{127}{5242880}y^4 + \frac{2521}{8847360}y^3 + \frac{24551}{17694720}y^2 + \frac{8785}{2654208}y + \frac{259553}{79626240}$
5	$\frac{6643}{301989888}y^5 + \frac{10949}{31457280}y^4 + \frac{1352317}{566231040}y^3 + \frac{1132327}{127401984}y^2 + \frac{9147257}{509607936}y + \frac{1337455}{84934656}$
6	$\frac{24046109}{676457349120}y^6 + \frac{1332533}{1887436800}y^5 + \frac{9522007931}{1522029035520}y^4 + \frac{26920481}{849346560}y^3$ $+ \frac{15810556787}{163074539520}y^2 + \frac{2079231455}{12230590464}y + \frac{245229441961}{1834588569600}$

Table 6 – Conjectural generating series for Masur–Veech polynomials.

Equation (5.8) is also valid for $g = 0$ and $g = 1$, if we use the values $(-7)!! = -\frac{7}{15}$ and $(-3)!! = -1$ given by the analytic continuation of the double factorial via the Gamma function, and if $g \geqslant 2$ we discard the coefficients of x^0, x^1 and x^2.

Returning to the coefficients of the generating series and then to the Masur–Veech volumes (5.2), Equation (5.7) is equivalent to the following structure for the Masur–Veech volumes. Let us first define

$$\gamma_k = \frac{1}{4^k}\binom{2k}{k}.$$

Conjecture 5.4. For any $g \geqslant 0$, there exist polynomials $p_g, q_g \in \mathbb{Q}[n]$ of degrees

$$\deg p_g = \begin{cases} \lfloor (g-1)/2 \rfloor & \text{if } g > 0 \\ -\infty & \text{if } g = 0 \end{cases} \qquad \text{and} \qquad \deg q_g = \lfloor g/2 \rfloor$$

such that, for any $n \geqslant 0$,

$$\frac{MV_{g,n}}{\pi^{6g-6+2n}} = 2^n \frac{(2g-3+n)!(4g-4+n)!}{(6g-7+2n)!}\big(p_g(n) + \gamma_{2g-3+n}\, q_g(n)\big). \tag{5.9}$$

For $g = 0$, the formula (5.9) agrees with Theorem 5.2 if we choose $p_0(n) = 0$ and $q_0(n) = \frac{1}{4}$. Up to genus 5, the conjecture is numerically true in the range of Table 11 for the following choice of polynomials (which can be deduced from Table 6).

5.5 Conjectures for area Siegel–Veech with fixed g

Area Siegel–Veech constants $SV_{g,n}$ can be computed from Masur–Veech volumes thanks to Goujard's formula, see Section 4.2. The correspondence between the notations of Section 4.2 and the present one is $\mathscr{F}_g(x) = \mathscr{H}_g(x) - \delta_{g,0}\mathscr{A}(x)$. If we insert the conjectural formulas for the Masur–Veech volumes, we can obtain conjectural formulas for the area Siegel–Veech constants.

Corollary 5.5. Assuming Conjecture 5.4, for any $g \geqslant 0$, there exist polynomials $p_g^*, q_g^* \in \mathbb{Q}[n]$ with degrees

$$\deg p_g^* = \begin{cases} \lfloor (g+3)/2 \rfloor & \text{if } g > 0 \\ -\infty & \text{if } g = 0 \end{cases} \qquad \text{and} \qquad \deg q_g^* = 1 + \lfloor g/2 \rfloor$$

g	$p_g(n)$	$q_g(n)$
0	0	$\frac{1}{4}$
1	$\frac{1}{6}$	$\frac{1}{6}$
2	$\frac{5}{36}$	$\frac{28}{135}n + \frac{7}{18}$
3	$\frac{245}{3888}n + \frac{643}{1944}$	$\frac{1784}{8505}n + \frac{6523}{8505}$
4	$\frac{1757}{23328}n + \frac{95413}{194400}$	$\frac{1186528}{23455575}n^2 + \frac{40882696}{54729675}n + \frac{5951381}{2296350}$
5	$\frac{38213}{3359232}n^2 + \frac{4218671}{16796160}n + \frac{63657059}{48988800}$	$\frac{83632064}{1196234325}n^2 + \frac{50144427856}{41868201375}n + \frac{63849553}{12629925}$
6	$\frac{59406613}{3325639680}n^2 + \frac{11411443987}{27713664000}n + \frac{61888029881}{26453952000}$	$\frac{2562397434368}{352859220016875}n^3 + \frac{185272285982144}{640374140030625}n^2 + \frac{9008283258227896}{2470014540118125}n + \frac{1636294928657}{110827591875}$

Table 7 – Polynomials conjecturally appearing in the Masur–Veech volumes.

such that, for any $n \geqslant 0$ such that $2g-2+n \geqslant 2$

$$\frac{SV_{g,n} \cdot MV_{g,n}}{\pi^{6g-8+2n}} = 2^n \frac{(2g-3+n)!(4g-4+n)!}{(6g-7+2n)!} \left(\frac{p_g^*(n)}{2g-3+n} + \gamma_{2g-3+n}\, q_g^*(n) \right),$$

or equivalently

$$SV_{g,n} = \frac{1}{\pi^2} \frac{\frac{p_g^*(n)}{2g-3+n} + \gamma_{2g-3+n}\, q_g^*(n)}{p_g(n) + \gamma_{2g-3+n}\, q_g(n)}.$$

The expression of the polynomials is displayed in Table 8: it is deduced, after computation of the sums (4.1), from Table 7. For $g = 0$ the conjecture matches with Theorem 5.2 with $p_0^* = 0$ and $q_1^* = \frac{n+5}{24}$.

g	$p_g^*(n)$	$q_g^*(n)$
0	0	$\frac{n+5}{24}$
1	$\frac{1}{36}n^2 - \frac{1}{36}n$	$\frac{1}{36}n + 1$
2	$\frac{5}{216}n^2 + \frac{20}{27}n + \frac{811}{1080}$	$\frac{14}{405}n^2 - \frac{35}{324}n + \frac{329}{540}$
3	$\frac{245}{23328}n^3 - \frac{143}{7776}n^2 + \frac{355}{1458}n + \frac{11861}{9720}$	$\frac{892}{25515}n^2 + \frac{52907}{51030}n + \frac{69617}{18255}$
4	$\frac{1757}{139968}n^3 + \frac{1428289}{3499200}n^2 + \frac{514241}{129600}n + \frac{4368611}{388800}$	$\frac{593264}{70366725}n^3 - \frac{322892}{164189025}n^2 + \frac{480686827}{1970268300}n + \frac{14820167}{4592700}$
5	$\frac{38213}{20155392}n^4 + \frac{867413}{50388480}n^3 + \frac{353997223}{3527193600}n^2$ $+ \frac{124054303}{55112400}n + \frac{128194553}{10497600}$	$\frac{41816032}{3588702975}n^3 + \frac{48489191848}{125604604125}n^2 + \frac{1269997838947}{251209208250}n + \frac{957632944}{44778825}$
6	$\frac{59406613}{19953838080}n^4 + \frac{63937638461}{498845952000}n^3 + \frac{8797861897271}{3491921664000}n^2$ $\frac{7511464839971}{317447424000}n + \frac{10221213098113}{123451776000}$	$\frac{1281198717184}{1058577660050625}n^4 + \frac{931707432208544}{51870305342480625}n^3 + \frac{67020810213757 16}{51870305342480625}n^2$ $+ \frac{302389725584289713}{103740610684961250}n + \frac{1719710639461433}{79130900598750}$

Table 8 – Polynomials conjecturally appearing in the numerator of $SV_{g,n}$.

Proof. We already mentioned that an equivalent form of Conjecture 5.4 is

$$\mathscr{H}_g(x) = \sum_{n \geqslant 0} \frac{x^n}{n!} \frac{H_{g,n}[0]}{\pi^{6g-6+2n}} = -\frac{\ln y}{12} \delta_{g,1} + y^{5(1-g)}\, Q_g(y), \qquad y = \sqrt{1-x}.$$

We recall from Table 6 that $Q_0(y) = -\frac{8}{15}$ and $Q_1(y) = \frac{1-y}{12}$. Therefore

$$\partial_x \mathscr{H}_g(x) = y^{3-5g}\, Q_{g;1}(y), \qquad \partial_x^2 \mathscr{H}_g(x) = y^{1-5g}\, Q_{g;2}(y), \qquad \partial_x \mathscr{A}(x) \partial_x \mathscr{H}_g(x) = y^{3-5g}\, Q_{g+2;3}(y) \tag{5.10}$$

where $Q_{g;i}$ are polynomials of degree g with rational coefficients

$$Q_{g;1}(y) = \begin{cases} \frac{16}{15} & \text{if } g = 0 \\ \frac{1+y}{24} & \text{if } g = 1 \\ \frac{5}{2}(g-1)Q_g(y) - \frac{1}{2}Q_g'(y) & \text{if } g \geqslant 2 \end{cases}$$

$$Q_{g;2}(y) = \begin{cases} -\frac{16}{10} & \text{if } g = 0 \\ \frac{1+2y}{48} & \text{if } g = 1 \\ \frac{5}{4}(g-1)(5g-3)Q_g(y) + \frac{1}{4}(9-10g)Q_g'(y) + \frac{1}{4}y^2 Q_g''(y) & \text{if } g \geqslant 2 \end{cases}$$

$$Q_{g+2;3}(y) = 2\big(\tfrac{1}{3} - y^2\big)\, Q_{g;1}(y)$$

With $\mathscr{F}_g(x) = \mathscr{H}_g(x) - \delta_{g,0}\mathscr{A}(x)$, we recall from the proof of Corollary 4.3 that

$$\begin{aligned} \mathscr{S}_g(x) &:= \sum_{\substack{n \geqslant 1 \\ 2g+n>0}} \frac{SV_{g,n} \cdot H_{g,n}[0]}{\pi^{6g-4+2n}} \frac{x^n}{n!} \\ &= \frac{1}{4}\bigg(\partial_x^2 \mathscr{F}_{g-1}(x) + \frac{1}{2} \sum_{g_1+g_2=g} \partial_x \mathscr{F}_{g_1}(x) \cdot \partial_x \mathscr{F}_{g_2}(x)\bigg). \end{aligned}$$

From (5.10) we deduce for any $g \geqslant 0$ the existence of $R_{g+3} \in \mathbb{Q}_g[y]$ such that $\mathscr{S}_g(x) = y^{3-5g}\, R_{g+3}(y)$, that is

$$S_g(x) = \sum_{k=0}^{g+3} \frac{r_{g,k}}{(1-x)^{2g-3+k/2}} \tag{5.11}$$

for some rational numbers $r_{g,k} \in \mathbb{Q}$. For $g \geqslant 2$, (5.11) contains only negative powers of $y = \sqrt{1-x}$. From the expansions

$$\begin{aligned} \frac{1}{(1-x)^{b+1}} &= \sum_{n \geqslant 0} \frac{(b+n)!}{b!} \frac{x^n}{n!}, \\ \frac{1}{(1-x)^{b+1/2}} &= \sum_{n \geqslant 0} \frac{b!}{2b!} \frac{(2b+2n)!}{(b+n)!} \frac{(x/4)^n}{n!}, \end{aligned}$$

it easily follows that

$$SV_{g,n}\, H_{g,n}[0] = (2g-4+n)!\, \tilde{p}_g^*(n) + \frac{(4g-6+2n)!}{4^{2g-3+n}(2g-3+n)!} \tilde{q}_g^*(n)$$

for some polynomials $\tilde{p}_g^*$ and $\tilde{q}_g^*$ with rational coefficients and degrees as announced. Multiplying by the prefactor of Equation (5.2) yields the claim, with polynomials p_g^* and q_g^* differing from $\tilde{p}_g^*$ and $\tilde{q}_g^*$ by prefactors that only depend on g. The cases $g = 0$ and $g = 1$ can be treated separately, with the same conclusion. □

5.6 Conjectural asymptotics for fixed g and large n

Let us examine the asymptotics when $n \to \infty$ assuming the conjectural formulas for Masur–Veech volumes and area Siegel-Veech constants. Since $\gamma_k \sim (\pi k)^{-1/2}$ when $k \to \infty$, we obtain when $n \to \infty$

$$MV_{g,n} \sim 2^{-n}\, \pi^{6g-6+2n+\epsilon(g)/2}\, n^{g/2}\, m_g, \qquad \epsilon(g) = \begin{cases} 0 & \text{if } g \text{ is even} \\ 1 & \text{if } g \text{ is odd} \end{cases}, \tag{5.12}$$

g	m_g	s_g
0	32	0
1	$\frac{1}{3}$	6
2	$\frac{7}{1080}$	$\frac{225}{56}$
3	$\frac{245}{7962624}$	$\frac{171264}{8575}$
4	$\frac{37079}{96074035200}$	$\frac{24227775}{2712064}$
5	$\frac{38213}{28179280429056}$	$\frac{85639233536}{2322395075}$
6	$\frac{5004682489}{369999709488414720000}$	$\frac{19363429564990875}{1311947486396416}$

Table 9 – Constants in the conjectural asymptotics of $MV_{g,n}$ and $SV_{g,n}$.

where $2^{6g-7}m_g \in \mathbb{Q}$ is the top coefficient of q_g if g is even and the top coefficient of p_g if g is odd, see Table 9. We observe that the coefficients which we could recognise in (5.8) are not relevant in this leading asymptotics, as they rather appear proportional to the constant term in p_g or q_g. For the area Siegel–Veech constants we find when $n \to \infty$

$$SV_{g,n} = \frac{n+5-5g}{6\pi^2} + \frac{s_g}{\pi^{3/2+\epsilon(g)} n^{1/2}} + O(n^{-1}), \tag{5.13}$$

where $s_g \in \mathbb{Q}$ are given in Table 9 for $g \leqslant 5$.

By [15, Theorem 2] we have that

$$\frac{\pi^2}{3} SV_{g,n} = \frac{n+5-5g}{18} + \Lambda^+_{g,n}, \tag{5.14}$$

where $\Lambda^+_{g,n}$ are the sum of the g Lyapunov exponents of the Hodge bundle along the Teichmüller flow on the moduli space of area one quadratic differentials $Q^1\mathfrak{M}_{g,n}$. In particular $\Lambda^+_{g,n} \in [0, g]$ and we can observe the coincidence of the main term in (5.13) and (5.14). Based on extensive numerical experiments, Fougeron [21] conjectured that for each g we have $\Lambda^+_{g,n} = O(n^{-1/2})$ as $n \to \infty$. The conjectural asymptotics (5.13) provides a refined version of Fougeron's conjecture.

We notice that the power of π appearing in the asymptotics depends on the parity of g. Both for $MV_{g,n}$ and $SV_{g,n}$, we have an all-order asymptotic expansion in powers of $n^{-1/2}$ beyond the leading terms (5.12)-(5.13).

5.7 Conjectures for $H_{1,n}[d]$

We can generate the numbers $H_{1,n}[d]$ in the following way (see Tables 15-16).

(i) We record the $H_n[d] = F_{0,n}[d, 0, \ldots, 0]$ computed in Section 5.3.

(ii) The specialisation of the Virasoro constraints to genus zero and $d_3 = \cdots = d_n = 0$ gives a recursion (on the variable n) for $F_{0,n}[d_1, d_2, 0, \ldots, 0]$ using (i) as input.

(iii) The specialisation of the Virasoro constraints to genus one and $d_2 = \cdots = d_n = 0$ gives a recursion (on the variable n) for $H_{1,n}[d] = F_{1,n}[d, 0, \ldots, 0]$ using (i) and (ii) as input.

Notice that obtaining $H_{1,n}[d]$ requires from (ii) the knowledge of $F_{0,m}[d_1, d_2, 0, \ldots, 0]$ for arbitrary d_1, d_2 (they can be non-zero only for $d_1 + d_2 \leqslant m - 3$) and $m \leqslant n + 1$, and from (i) the knowledge of $H_{n'}[d]$ for arbitrary $d \leqslant m - 3$ and $n' \leqslant n$.

d	ρ_d	$R_d(n)$
0	$\frac{1}{6}$	$\frac{1}{6}$
1	$\frac{1}{4}$	$\frac{1}{4}(n-1)$
2	$\frac{25}{72}$	$\frac{25}{72}n^2 - \frac{65}{72}n + \frac{35}{144}$
3	$\frac{7}{15}$	$\frac{7}{15}n^3 - \frac{217}{96}n^2 + \frac{973}{480}n + \frac{413}{480}$
4	$\frac{2069}{3360}$	$\frac{2069}{3360}n^4 - \frac{4009}{840}n^3 + \frac{60479}{6720}n^2 + \frac{1189}{840}n - \frac{12549}{2240}$
5	$\frac{9713}{12096}$	$\frac{9713}{12096}n^5 - \frac{55033}{6048}n^4 + \frac{710501}{24192}n^3 - \frac{90299}{6048}n^2 - \frac{61105}{4032}n - \frac{259919}{4032}$

Table 10 – Parameters of the conjectural formula (5.15) for $H_{1,n}[d]$ with $d \leqslant 5$.

The data we have generated leads us to propose the ansatz, for $n \geqslant d+1$

$$H_{1,n}[d] = 2^{d-2} \cdot (n-1-d)!\big(\rho_d (n-1) \cdots (n-d) + \gamma_{n-1-d}\, r_d(n)\big) \tag{5.15}$$

for some rational constant ρ_d and some polynomial $r_d(n)$ of degree d with rational coefficients. This formula makes sense even though the arguments of the factorials can be negative. Indeed, the first term yields $(n-1)!\rho_d$, while for the second term, if k is a negative integer, we use

$$\lim_{N\to k} N!\,\gamma_N = \lim_{N\to k} \frac{\Gamma(2N+1)}{\Gamma(N+1)} = \frac{1}{2}\frac{(k+1)!}{(2k+1)!}.$$

We wrote (5.15) in this form to stress the analogy with (5.9). Equation (5.15) for $d=0$ indeed matches (5.9) with the values $r_0 = q_1 = \frac{1}{6}$ and $\rho_0 = p_1 = \frac{1}{6}$ already found in Table 7.

For a fixed value of $d \in \{0,1,2,3,4,5\}$, we have determined R_d and ρ_d (Table 10) by matching the values of $H_{1,d+1}[d]$, $H_{1,d+2}[d]$, …, $H_{2d+2}[d]$ and we checked that (5.15) predicts the correct values for $2d+2 \leqslant n \leqslant 14$. We observed that the formula (5.15) does not give the correct value for $n = d$. However, for this particular case we prove in Section 5.1.2 that $H_{1,n}[n] = \frac{1}{24}\frac{(2n+1)!}{2^n n!}$ using stable graphs.

A Closed formulae for the intersection of ψ classes in genus one

Lemma A.1. For a fixed integer $n \geqslant 1$, we have

$$\int_{\overline{\mathfrak{M}}_{1,n}} \psi_1^{a_1}\cdots\psi_n^{a_n} = \frac{1}{24}\left\{\binom{n}{a_1,\ldots,a_n} - \sum_{\substack{b_1,\ldots,b_n\\ b_i\in\{0,1\}}} \binom{n-\sum_i b_i}{a_1-b_1,\ldots,a_n-b_n}(b_1+\cdots+b_n-2)!\right\}, \tag{A.1}$$

and the sum of all such integrals is

$$\int_{\overline{\mathfrak{M}}_{1,n}} \frac{1}{\prod_{i=1}^n (1-\psi_i)} = \frac{1}{24}\left(n^n - \sum_{j=1}^{n-1} \frac{n^{n-j}}{j(j+1)}\frac{(n-1)!}{(n-j-1)!}\right). \tag{A.2}$$

We use the convention that summands involving negative factorials are excluded from the summation. In particular we retrieve (5.3) and (5.4) used in the text.

Proof. Let us recall the following result.

Theorem A.2 (The conjecture of Goulden–Jackson–Vainshtein conjecture [24], the theorem of Vakil [37]). Let μ be a partition of d of length $n = \ell(\mu)$. The simple connected Hurwitz numbers $h_{g=1,\mu}$ of genus one and ramification profile μ over zero are given by

$$\frac{h_{g=1,\mu}}{(n+d)!} = \frac{1}{24}\prod_{i=1}^n \frac{\mu_i^{\mu_i}}{\mu_i!}\left(d^n - d^{n-1} - \sum_{j=2}^n (j-2)!\,d^{n-j} s_j(\mu_1,\ldots,\mu_n)\right), \tag{A.3}$$

where s_j is the j-th elementary symmetric polynomial. ■

On the other hand the ELSV formula [14] in genus one gives

$$\frac{h_{g=1,\mu}}{(n+d)!} = \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{1,n}} \frac{1-\lambda_1}{\prod_{i=1}^{n}(1-\mu_i\psi_i)}. \tag{A.4}$$

Combining the two we obtain

$$\int_{\overline{\mathfrak{M}}_{1,n}} \frac{1-\lambda_1}{\prod_{i=1}^{n}(1-\mu_i\psi_i)} = \frac{1}{24}\Big(d^n - d^{n-1} - \sum_{j=2}^{n}(j-2)!d^{n-j}s_j(\mu_1,\ldots,\mu_n)\Big). \tag{A.5}$$

The contribution corresponding to the intersections of λ_1 is easily removed by erasing the second summand d^{n-1}: indeed, considering d as $\mu_1+\ldots+\mu_n$, the right hand side is a polynomial in the μ_i involving only two degrees, which are n and $n-1$. On the other hand, the summands of degree $n-1$ must correspond to all and only the monomials in ψ classes intersecting λ_1. Now, to prove (A.2) substitute $\mu_i = 1$ for each i and observe that $s_j(1,\ldots,1) = \binom{n}{j}$. To prove (A.1), collect the coefficient of $\mu_1^{a_1}\cdots\mu_n^{a_n}$ in the right hand side after the substitution $d = \mu_1+\ldots+\mu_n$. This concludes the proof of the lemma. □

Remark. A longer but more detailed way to remove the contribution of λ_1 to prove (A.2) is by using λ_g theorem.

Theorem A.3 (The λ_g-theorem [19])**.**

$$\int_{\overline{\mathfrak{M}}_{g,n}} \psi_1^{a_1}\cdots\psi_n^{a_n}\lambda_g = \binom{2g-3+n}{a_1,\ldots,a_n}\int_{\overline{\mathfrak{M}}_{g,1}} \psi_1^{2g-2}\lambda_g. \tag{A.6}$$

■

Its specialisation in genus one reads

$$\int_{\overline{\mathfrak{M}}_{1,n}} \psi_1^{a_1}\cdots\psi_d^{a_n}\lambda_1 = \binom{n-1}{a_1,\ldots,a_n}\int_{\overline{\mathfrak{M}}_{1,1}} \lambda_1 = \binom{n-1}{a_1,\ldots,a_n}\frac{1}{24}. \tag{A.7}$$

This can also be seen for instance using that λ_1 is represented by the Poincaré dual of the divisor of curves with at least one non-separating node times $\frac{1}{24}$, then pulling back the class via the attaching map and integrating over $\overline{\mathfrak{M}}_{0,n+2}$ gives the same result. In any case, summing over all tuples of non-negative integers a_i such that $\sum_i a_i = d-1$ gives, using the multinomial theorem

$$\int_{\overline{\mathfrak{M}}_{1,n}} \frac{\lambda_1}{\prod_{i=1}^{n}(1-\psi_i)} = \sum_{\substack{a_1,\ldots,a_n\geqslant 0\\ \sum a_i = n-1}} \int_{\overline{\mathfrak{M}}_{1,d}} \psi_1^{a_1}\cdots\psi_n^{a_n}\lambda_1 = \frac{1}{24}\sum_{\substack{a_1,\ldots,a_n\geqslant 0\\ \sum a_i = n-1}} \binom{n-1}{a_1,\ldots,a_n} = \frac{1}{24}n^{n-1}, \tag{A.8}$$

which equals the second summand in equation (A.5) after the substitution $\mu_i = 1$ for all i, and therefore $n = d$. Removing it from (A.5) and simplifying the expression proves again (A.2).

B Computing Masur–Veech polynomials with Eynard–Orantin topological recursion

For readers who are unfamiliar with the topological recursion à la Eynard–Orantin, we compute a few Masur–Veech polynomials via $\omega_{g,n}^{MV}$ (Proposition 3.8), i.e. applying the residue formula (2.9) to the spectral curve given by $\mathcal{C} = \mathbb{C}$ and

$$x(z) = \frac{z^2}{2}, \qquad y(z) = -z, \qquad \omega_{0,2}^{MV}(z_1,z_2) = \frac{dz_1\otimes dz_2}{(z_1-z_2)^2} + \frac{1}{2}\sum_{m\in\mathbb{Z}^*}\frac{dz_1\otimes dz_2}{(z_1-z_2+m)^2}. \tag{B.1}$$

Let us first compute the recursion kernel

$$
\begin{aligned}
K(z,z_0) &= -\frac{1}{2}\frac{1}{(y(z)-y(-z))\,z\,dz}\int_{-z}^{z}\omega_{0,2}^{\mathrm{MV}}(z',z_0) \\
&= \frac{dz_0}{4z^2dz}\int_{-z}^{z}\left(\frac{dz'}{(z_0-z')^2}+\frac{1}{2}\sum_{m\in\mathbb{Z}^*}\frac{dz'}{(z_0-z'+m)^2}\right) \\
&= \frac{dz_0}{2zdz}\left(\frac{1}{z_0^2-z^2}+\frac{1}{2}\sum_{m\in\mathbb{Z}^*}\frac{1}{(z_0+m)^2-z^2}\right).
\end{aligned}
$$

It is handy, in order to compute residues at $z=0$, to write down the expansion in power series near $z=0$ of the recursion kernel.

$$
\begin{aligned}
\frac{1}{z_0^2-z^2}+\frac{1}{2}\sum_{m\in\mathbb{Z}^*}\frac{1}{(z_0+m)^2-z^2} &= \sum_{d\geqslant 0}\left(\frac{1}{z_0^{2d+2}}+\frac{1}{2}\sum_{m\in\mathbb{Z}^*}\frac{1}{(z_0+m)^{2d+2}}\right)z^{2d} \\
&= \sum_{d\geqslant 0}\zeta_H(2d+2;z_0)z^{2d}.
\end{aligned}
$$

In the same way we have that

$$
\begin{aligned}
\zeta_H(2;z-z_i) &= \sum_{d\geqslant 0}(2d+1)\,\zeta(2d+2;z_i)\,z^{2d}+\text{odd part in } z, \\
\zeta_H(2k;z) &= \frac{1}{z^{2k}}+\sum_{d\geqslant 0}\binom{2k-1+2d}{2d}\zeta(2k+2d)\,z^{2d}.
\end{aligned}
$$

The topological recursion formula, specialised to our case and expressed in terms of

$$
\mathcal{W}_{g,n}^{\mathrm{MV}}(z_1,\ldots,z_n)=\frac{\omega_{g,n}^{\mathrm{MV}}(z_1,\ldots,z_n)}{dz_1\otimes\cdots\otimes dz_n},
$$

reads

$$
\begin{aligned}
\mathcal{W}_{g,n}^{\mathrm{MV}}(z_1,z_2,\ldots,z_n) = \frac{1}{2}[z^0]\sum_{d\geqslant 0}\zeta_H(2d+2,z_1)\,z^{2d}\Bigg\{&\mathcal{W}_{g-1,n+1}^{\mathrm{MV}}(z,-z,z_2,\ldots,z_n) \\
&+\sum_{\substack{h+h'=g \\ J\sqcup J'=\{z_2,\ldots,z_n\}}}\mathcal{W}_{h,1+|J|}^{\mathrm{MV}}(z,J)\,\mathcal{W}_{h',1+|J'|}^{\mathrm{MV}}(-z,J')\Bigg\}.
\end{aligned}
$$

• $(\mathbf{g},\mathbf{n})=(\mathbf{0},\mathbf{3})$.

$$
\begin{aligned}
\mathcal{W}_{0,3}^{\mathrm{MV}}(z_0,z_1,z_2) &= \frac{1}{2}[z^0]\sum_{d\geqslant 0}\zeta_H(2d+2;z_0)z^{2d}\cdot\left(\mathcal{W}_{0,2}^{\mathrm{MV}}(z,z_1)\mathcal{W}_{0,2}^{\mathrm{MV}}(-z,z_2)+\mathcal{W}_{0,2}^{\mathrm{MV}}(z,z_2)\mathcal{W}_{0,2}^{\mathrm{MV}}(-z,z_1)\right) \\
&= [z^0]\sum_{d\geqslant 0}\zeta_H(2d+2;z_0)\,z^{2d}\cdot\mathcal{W}_{0,2}^{\mathrm{MV}}(z,z_1)\mathcal{W}_{0,2}^{\mathrm{MV}}(z,z_2) \\
&= \zeta_H(2;z_0)\zeta_H(2;z_1)\zeta_H(2;z_2).
\end{aligned}
$$

The inverse Laplace transform of the principal part near $z_0=z_1=z_2=0$ then reads

$$
V\Omega_{0,3}^{\mathrm{MV}}(L_1,L_2,L_3)=1.
$$

Multiplying by the combinatorial factor $2^{4g-2+n}\,\frac{(4g-4+n)!}{(6g-7+2n)!}$ whose value for $g=0$ and $n\to 3$ is 4, we get $MV_{0,3}=4$.

• $(\mathbf{g},\mathbf{n})=(\mathbf{0},\mathbf{4})$

$$\begin{aligned}
\mathcal{W}^{\mathrm{MV}}_{0,4}(z_0,z_1,z_2,z_3) &= [z^0]\sum_{d\geqslant 0}\zeta_H(2d+2;z_0)\,z^{2d}\cdot\Big(\mathcal{W}^{\mathrm{MV}}_{0,2}(z,z_1)\mathcal{W}^{\mathrm{MV}}_{0,3}(z,z_2,z_3)\\
&\quad+\mathcal{W}^{\mathrm{MV}}_{0,2}(z,z_2)\mathcal{W}^{\mathrm{MV}}_{0,3}(z,z_1,z_3)+\mathcal{W}^{\mathrm{MV}}_{0,2}(z,z_3)\mathcal{W}^{\mathrm{MV}}_{0,3}(z,z_1,z_2)\Big)\\
&= [z^0]\sum_{d\geqslant 0}\zeta_H(2d+2;z_0)\,z^{2d}\cdot\Big(\zeta_H(2;z-z_1)\zeta_H(2;z)\zeta_H(2;z_2)\zeta_H(2;z_3)\\
&\quad+\zeta_H(2;z-z_2)\zeta_H(2;z)\zeta_H(2;z_1)\zeta_H(2;z_3)+\zeta_H(2;z-z_3)\zeta_H(2;z)\zeta_H(2;z_1)\zeta_H(2;z_2)\Big)\\
&= 3\sum_{i=0}^{3}\zeta_H(4;z_i)\prod_{j\in\{0,1,2,3\}\setminus\{i\}}\zeta_H(2;z_j)+3\zeta(2)\zeta_H(2;z_0)\zeta_H(2;z_1)\zeta_H(2;z_2)\zeta_H(2;z_3).
\end{aligned}$$

The inverse Laplace transform of the principal part near $z_0=z_1=z_2=z_3=0$ then reads

$$V\Omega^{\mathrm{MV}}_{0,4}(L_1,\ldots,L_4)=\frac{1}{2}\Big(\pi^2+\sum_{i=1}^{4}L_i^2\Big)$$

from which we deduce $MV_{0,4}=2\pi^2$.

• $(\mathbf{g},\mathbf{n})=(\mathbf{1},\mathbf{1})$

$$\begin{aligned}
\mathcal{W}^{\mathrm{MV}}_{1,1}(z_0) &= \frac{1}{2}[z^0]\sum_{d\geqslant 0}\zeta_H(2d+2;z_0)\,z^{2d}\,\mathcal{W}^{\mathrm{MV}}_{0,2}(z,-z)\\
&= \frac{1}{2}\sum_{d\geqslant 0}\zeta_H(2d+2;z_0)\,z^{2d}\left(\frac{1}{4z^2}+\sum_{m\geqslant 1}\sum_{d'\geqslant 0}(2d'+1)\frac{(2z)^{2d'}}{m^{2d'+2}}\right)\\
&= \frac{1}{8}\zeta_H(4;z_0)+\frac{\pi^2}{12}\zeta_H(2;z_0).
\end{aligned}$$

The inverse Laplace transform of the principal part near $z_0=0$ then reads

$$V\Omega^{\mathrm{MV}}_{1,1}=\frac{\pi^2}{12}+\frac{L^2}{48}.$$

Multiplying the constant term by the combinatorial factor $\frac{2^{4g-2+n}(4g-4+n)!}{(6g-7+2n)!}=8$ we deduce $MV_{1,1}=\frac{2\pi^2}{3}$.

• $(\mathbf{g},\mathbf{n})=(\mathbf{1},\mathbf{2})$

$$\begin{aligned}
&\mathcal{W}^{\mathrm{MV}}_{1,2}(z_1,z_2)\\
&= [z^0]\sum_{d\geqslant 0}^{\infty}\zeta_H(2d+2;z_1)\,z^{2d}\Big(\frac{1}{2}\,\mathcal{W}^{\mathrm{MV}}_{0,3}(z,z,z_2)+\mathcal{W}^{\mathrm{MV}}_{0,2}(z,z_2)\mathcal{W}^{\mathrm{MV}}_{1,1}(z)\Big)\\
&= \frac{1}{2}[z^0]\sum_{d\geqslant 0}\zeta_H(2d+2;z_1)\,z^{2d}\cdot\zeta_H(2;z)^2\zeta_H(2;z_2)+\frac{1}{8}[z^0]\sum_{d\geqslant 0}\zeta_H(2d+2;z_1)\,z^{2d}\cdot\zeta_H(2;z-z_1)\zeta_H(4;z)\\
&\qquad+\frac{1}{2}[z^0]\sum_{d\geqslant 0}\zeta_H(2d+2;z_1)\,z^{2d}\cdot\zeta_H(2;z-z_2)\zeta_H(2;z)\zeta(2)\\
&= \frac{\zeta_H(2;z_2)}{2}\Big\{\zeta_H(6;z_1)+2\zeta_H(4;z_1)\zeta(2)+6\zeta_H(2;z_1)\zeta(4)+\zeta_H(2;z_0)\zeta(2)^2\Big\}\\
&\qquad+\frac{1}{8}\Big\{\zeta_H(2;z_1)\zeta_H(2;z_2)\zeta(4)+\zeta_H(6;z_1)\zeta_H(2;z_2)+3\zeta_H(4;z_1)\zeta_H(4;z_2)+5\zeta_H(2;z_1)\zeta_H(6;z_2)\Big\}\\
&\qquad+\frac{1}{2}\Big\{\zeta_H(2;z_1)\zeta_h(2;z_2)\,\zeta(2)^2+\zeta_H(4;z_1)\zeta_H(2;z_2)\,\zeta(2)+3\zeta_H(2;z_1)\zeta_H(4;z_2)\,\zeta(2)\Big\}.
\end{aligned}$$

Rearranging the terms, we obtain

$$\mathcal{W}_{1,2}^{\mathrm{MV}}(z_1,z_2) = \frac{5}{8}\Big(\zeta_{\mathrm{H}}(6;z_1)\zeta_{\mathrm{H}}(2;z_2) + \zeta_{|\mathrm{H}}(2;z_1)\zeta_{\mathrm{H}}(6;z_2)\Big) + \frac{3}{8}\zeta_{\mathrm{H}}(4;z_1)\zeta_{\mathrm{H}}(4;z_2)$$
$$+ \frac{\pi^2}{4}\Big(\zeta_{\mathrm{H}}(4;z_1)\zeta_{\mathrm{H}}(2;z_2) + \zeta_{\mathrm{H}}(2;z_1)\zeta_{\mathrm{H}}(4;z_2)\Big) + \frac{\pi^4}{16}\zeta_{\mathrm{H}}(2;z_1)\zeta_{\mathrm{H}}(2;z_2).$$

The inverse Laplace transform of the principal part near $z_0 = z_1 = 0$ then reads

$$V\Omega_{1,2}^{\mathrm{MV}}(L_1,L_2) = \frac{1}{192}(L_1^4 + L_2^4) + \frac{1}{96}L_1^2L_2^2 + \frac{\pi^2}{24}(L_1^2 + L_2^2) + \frac{\pi^4}{16}.$$

Multiplying the constant term by the combinatorial factor $\frac{2^{4g-2+n}(4g-4+n)!}{(6g-7+2n)!} = \frac{16}{3}$ we obtain $MV_{1,2} = \frac{\pi^4}{3}$.

C Numerical data

$n \setminus g$	0	1	2	3	4	5	6
0	-	-	$\frac{1}{15}$	$\frac{115}{33264}$	$\frac{2106241}{11548293120}$	$\frac{7607231}{790778419200}$	$\frac{51582017261473}{1017356012351078400000}$
1	-	**$\frac{2}{3}$**	**$\frac{29}{840}$**	**$\frac{4111}{2223936}$**	**$\frac{58091}{592220160}$**	**$\frac{35161328707}{6782087854080000}$**	**$\frac{1725192578138153}{630760727657668608000}$**
2	-	**$\frac{1}{3}$**	**$\frac{337}{18144}$**	**$\frac{77633}{77837760}$**	**$\frac{160909109}{3038089420800}$**	**$\frac{27431847097}{9796349122560000}$**	**$\frac{236687293214441}{1601932006749634560000}$**
3	**4**	$\frac{11}{60}$	**$\frac{29}{2880}$**	**$\frac{207719}{384943104}$**	**$\frac{14674841399}{512424415641600}$**	**$\frac{5703709895459}{3767985230929920000}$**	**$\frac{37679857842043}{471817281090355200000}$**
4	**2**	$\frac{1}{10}$	$\frac{919}{168480}$	**$\frac{16011391}{54854392320}$**	**$\frac{9016171639}{582300472320000}$**	**$\frac{143368101519407}{175211313238241280000}$**	**$\frac{13237209152580169}{306665505466027868160000}$**
5	**1**	$\frac{163}{3024}$	$\frac{653}{221760}$	$\frac{6208093}{39382640640}$	**$\frac{442442475179}{52900285261824000}$**	**$\frac{259645860580231}{587375069141532672000}$**	**$\frac{6359219722433607397}{2726869674603919803678720000}$**
6	**$\frac{1}{2}$**	$\frac{29}{1008}$	$\frac{88663}{56010240}$	$\frac{5757089}{67781007360}$	$\frac{1537940628689}{340912949465088000}$	**$\frac{229686916047007}{962777317187911680000}$**	**$\frac{43310941179948284069}{34400509741157142138716160000}$**
7	**$\frac{1}{4}$**	$\frac{1255}{82368}$	$\frac{295133}{348281856}$	$\frac{2598992519}{56936046182400}$	$\frac{643391778377}{264869710110720000}$	$\frac{11267167909498433}{876187158474365337600000}$	**$\frac{744084879305048387 27}{10957199399035237866405888000}$**
8	**$\frac{1}{8}$**	$\frac{2477}{308880}$	$\frac{1835863}{4063288320}$	$\frac{1769539}{720943441920}$	$\frac{127802659622551}{97895844856922112000}$	$\frac{2762333771707}{39907473380632166400}$	$\frac{760349474493855607 73}{207808954119633821604249600000}$
9	**$\frac{1}{16}$**	$\frac{39203}{9335040}$	$\frac{12653167}{52718561280}$	$\frac{6756335603}{516534771916800}$	$\frac{76170641989903}{108773160952135680000}$	$\frac{46331482996262911}{1245354014578231266508800}$	$\frac{7583038108310022233611}{3850996789771271334071894016000}$
10	**$\frac{1}{32}$**	$\frac{1363}{622336}$	$\frac{5219989}{41079398400}$	$\frac{2863703603}{410578921267200}$	$\frac{364975959330977}{973541287193739264000}$	$\frac{110488317513510709}{5533939090421837306265600}$	$\frac{15977883277628053521622 51}{1509590741590338362956182454272000}$
11	**$\frac{1}{64}$**	$\frac{308333}{270885888}$	$\frac{644710519}{9612579225600}$	$\frac{28221517763}{7606514751897600}$	$\frac{2627412792296 1227}{1311625625110110535680000}$	$\frac{390740937497025565 51}{365239979967841262213529 6000}$	$\frac{328937919726664092191 89}{5789091497188304121420054528 0000}$

Table 11 – Masur–Veech volumes $\pi^{-(6g-6+2n)} MV_{g,n}$. We display in black the values that were honestly computed from the recursion, in bold the values used to determine the polynomials appearing in Conjecture 5.4, and in grey the values that the conjecture predicts. The first column reproduces Theorem 5.2. The first row is computed by the relation $MV_{g,0} = \frac{2^{4g-2}(4g-4)!}{(6g-g)!} H_{g,1}$[1] proved in Lemma 3.4.

$n \setminus g$	0	1	2	3	4	5	6
0	-	-	$\frac{19}{6}$	$\frac{24199}{8625}$	$\frac{283794163}{105312050}$	$\frac{180693680}{68465079}$	$\frac{806379495590975}{309492103568838}$
1	-	-	$\frac{230}{87}$	$\frac{529239}{205550}$	$\frac{14053063}{5518645}$	$\frac{533759417507}{210967972242}$	$\frac{4346055982466800}{1725192578138153}$
2	-	$\frac{7}{3}$	$\frac{8131}{3370}$	$\frac{2843354}{1164495}$	$\frac{11842209371}{4827273270}$	$\frac{606925117339}{246886623873}$	$\frac{122318875814791931}{49704331575032610}$
3	-	$\frac{47}{22}$	$\frac{11041}{4785}$	$\frac{73870699}{31157850}$	$\frac{35221419482}{14674841399}$	$\frac{82681229028041}{34222259372754}$	$\frac{5057811587495459887}{2085014933689449405}$
4	$\frac{3}{2}$	$\frac{44}{21}$	$\frac{688823}{303270}$	$\frac{187549387}{80056955}$	$\frac{1414826039249}{595067328174}$	$\frac{1031120131654286}{430104304558221}$	$\frac{1339844245835171101}{555962784408367098}$
5	$\frac{5}{3}$	$\frac{2075}{978}$	$\frac{96716}{42445}$	$\frac{87365995}{37248558}$	$\frac{15788133716389}{6636637127685}$	$\frac{1245335246460801}{519291721160462}$	$\frac{321899861240823487478}{133543614171105755337}$
6	$\frac{11}{6}$	$\frac{697}{319}$	$\frac{8622217}{3723846}$	$\frac{1433623484}{604494345}$	$\frac{7380284015613}{3075881257378}$	$\frac{18305424406953487}{7579668229551231}$	$\frac{3150765025310943712637}{1299328235398448522070}$
7	2	$\frac{17101}{7530}$	$\frac{10506949}{4426995}$	$\frac{12557689333}{5197985038}$	$\frac{32906433038620}{13511227345917}$	$\frac{165332043184123111}{67603007456990598}$	$\frac{1276869600669686371105}{520859415513533871089}$
8	$\frac{13}{6}$	$\frac{17630}{7431}$	$\frac{44927707}{18358630}$	$\frac{3273823127}{1322965425}$	$\frac{1905176709014543}{766815957735306}$	$\frac{931701551880070892}{374503401099176525}$	$\frac{32923598627691820002839}{13230080856193087574502}$
9	$\frac{7}{3}$	$\frac{194829}{78406}$	$\frac{480821458}{189797505}$	$\frac{515867741141}{202690068090}$	$\frac{3294839869674121}{1294900913828351}$	$\frac{13416096198217292533}{5281789061573971854}$	$\frac{403660475951758341605956}{159243800274510466905831}$
10	$\frac{5}{2}$	$\frac{202415}{77691}$	$\frac{905804827}{344519274}$	$\frac{488680850166}{186140734195}$	$\frac{658216299971112017}{251833411938374130}$	$\frac{2586449275763662283}{994394857621596381}$	$\frac{57921215793035879725637191}{22369036588679274930271514}$
11	$\frac{8}{3}$	$\frac{5054467}{1849998}$	$\frac{1761936475}{644710519}$	$\frac{2297552653219}{846645532890}$	$\frac{212103557000574050}{78822383768883681}$	$\frac{208627514502680586639}{78148187499405113102}$	$\frac{9156519282251402538004459}{3453848157129972968014845}$

Table 12 – Area Siegel–Veech constants $\pi^2 SV_{g,n}$. They are computed from Table 11 thanks to Theorem 4.1. Theorem 5.2 gives the first column.

$n \setminus d$	0	1	2	3	4	5	6
3	**1**						
4	$\mathbf{\frac{1}{2}}$	**3**					
5	$\mathbf{\frac{3}{4}}$	**3**	**15**				
6	$\mathbf{\frac{15}{8}}$	$\frac{27}{4}$	**25**	**105**			
7	$\mathbf{\frac{105}{16}}$	$\frac{45}{2}$	$\mathbf{\frac{305}{4}}$	$\mathbf{\frac{525}{2}}$	**945**		
8	$\mathbf{\frac{945}{32}}$	$\frac{1575}{16}$	$\frac{1275}{4}$	**1029**	$\mathbf{\frac{6615}{2}}$	**10395**	
9	$\mathbf{\frac{10395}{64}}$	$\frac{8505}{16}$	$\frac{26775}{16}$	$\mathbf{\frac{20853}{4}}$	$\mathbf{\frac{32193}{2}}$	**48510**	**135135**
10	$\mathbf{\frac{135135}{128}}$	$\frac{218295}{64}$	$\frac{168525}{16}$	$\frac{512883}{16}$	$\mathbf{\frac{386271}{4}}$	$\mathbf{\frac{571725}{2}}$	**810810**
11	$\mathbf{\frac{2027025}{256}}$	$\frac{405405}{16}$	$\frac{4937625}{64}$	$\frac{7388955}{32}$	$\mathbf{\frac{10938159}{16}}$	$\mathbf{\frac{3992175}{2}}$	**5675670**
12	$\frac{54729675}{256}$	$\frac{41216175}{64}$	$\frac{60904305}{32}$	$\frac{177968745}{32}$	$\mathbf{\frac{256944105}{16}}$	$\mathbf{\frac{182035425}{4}}$	$\mathbf{\frac{497972475}{4}}$
13	$\frac{516891375}{256}$	$\frac{1543917375}{256}$	$\frac{564729165}{32}$	$\frac{816623775}{16}$	**146029950**	$\mathbf{\frac{6588578139}{16}}$	$\mathbf{\frac{9070035975}{8}}$
14	$\frac{21606059475}{1024}$	$\frac{16023632625}{256}$	$\frac{46514953485}{256}$	$\frac{16675523865}{32}$	$\frac{23672927025}{16}$	$\mathbf{\frac{66411007767}{16}}$	$\mathbf{\frac{45759368655}{4}}$
15	$\frac{123743795175}{512}$	$\frac{730022918625}{1024}$	$\frac{1052618271705}{512}$	$\frac{1499012960445}{256}$	$\frac{264272735725}{16}$	**46109143548**	$\mathbf{\frac{2030915622645}{16}}$
16	$\frac{12333131585775}{4096}$	$\frac{9051629461875}{1024}$	$\frac{811209323925}{32}$	$\frac{36751659059115}{512}$	$\frac{51544870752375}{256}$	$\frac{17905106587095}{32}$	$\mathbf{\frac{49185509458365}{32}}$
17	$\frac{166022925193125}{4096}$	$\frac{485419409850375}{4096}$	$\frac{346386381315975}{1024}$	$\frac{487957139745075}{512}$	$\frac{340526280419325}{128}$	$\frac{1884818339393535}{256}$	$\mathbf{\frac{1291176578473425}{64}}$
18	$\frac{9605612100459375}{16384}$	$\frac{6996680418853125}{4096}$	$\frac{19889546438136375}{4096}$	$\frac{13948724683018125}{1024}$	$\frac{19385928833646375}{512}$	$\frac{1670327729904015}{16}$	$\frac{73041994202191875}{256}$
19	$\frac{18570850060888125}{2048}$	$\frac{431541017698415625}{16384}$	$\frac{611196796805970375}{8192}$	$\frac{854034802440694875}{4096}$	$\frac{295622640130442625}{512}$	$\frac{50770632935200815}{32}$	$\frac{2214495362565119025}{512}$
20	$\frac{9786837982088041875}{65536}$	$\frac{7087874439905634375}{16384}$	$\frac{10007297452695918375}{8192}$	$\frac{27873406624278340125}{8192}$	$\frac{38464769339580709125}{4096}$	$\frac{26342984670839932095}{1024}$	$\frac{71637824483699384925}{1024}$

Table 13 – Values of $\pi^{-2(n-3-d)} H_n[d]$ computed from the recursion of Section 5.3, for $n \leqslant 20$. We display in bold the values that were used to determine the polynomials $P_d(n)$ in Table 5.

$n \setminus d$	7	8	9	10	11	12	13
10	**2027025**						
11	**$\frac{30405375}{2}$**	**34459425**					
12	**$\frac{34459425}{512}$**	**$\frac{631756125}{2}$**	654729075				
13	**$\frac{654729075}{1024}$**	**$\frac{11952825885}{4}$**	7202019825	13749310575			
14	**$\frac{13749310575}{2048}$**	**$\frac{245021061285}{8}$**	$\frac{311520093885}{4}$	178741037475	316234143225		
15	**$\frac{316234143225}{4096}$**	**$\frac{2737188448995}{8}$**	891435459915	$\frac{8758310836275}{4}$	$\frac{9592435677825}{2}$	7905853580625	
16	**$\frac{7905853580625}{8192}$**	**$\frac{66469370924775}{16}$**	$\frac{87660410042205}{8}$	$\frac{111209007034125}{4}$	$\frac{132080460486975}{2}$	$\frac{276704875321875}{2}$	213458046676875
17	**$\frac{213458046676875}{16384}$**	**$\frac{1746012465049635}{32}$**	$\frac{1158795684753525}{8}$	$\frac{2992184547677325}{8}$	$\frac{1849593268648125}{2}$	2126674613188125	4269160933537500
18	**$\frac{6190283353629375}{32768}$**	**$\frac{49363995431765475}{64}$**	$\frac{65744175999773079}{32}$	$\frac{42871352128175025}{8}$	$\frac{108418387292179965}{8}$	$\frac{65350162166664375}{2}$	72817654989704625
19	**$\frac{191898783962510625}{65536}$**	**$\frac{2990161669353791775}{256}$**	$\frac{498588953042078271}{16}$	$\frac{2616690164139006165}{32}$	$\frac{3354058884543031665}{16}$	$\frac{4155321837251710125}{8}$	1221898913059324500
20	**$\frac{6332659870762850625}{131072}$**	$\frac{9660174300790922587 5}{512}$	$\frac{128930498895120139989}{256}$	$\frac{42442944415051043115}{32}$	$\frac{27436266153939996045}{8}$	$\frac{138292854856604823375}{16}$	$\frac{167929854347224344375}{8}$

$n \setminus d$	14	15	16	17
17	6190283353629375			
18	140313089348932500	191898783962510625		
19	$\frac{5282787813987308625}{2}$	$\frac{9786837982088041875}{2}$	6332659870762850625	
20	$\frac{19283434212002247787 5}{4}$	$\frac{40471453537693490812 5}{4}$	$\frac{36096161263348248562 5}{2}$	221643095476699771875

Table 14 – (Continued) Values of $\pi^{-2(n-3-d)} H_n[d]$ computed from the recursion of Section 5.3, for $n \leqslant 20$.

$n \setminus d$	0	1	2	3	4	5	6	7	8
1	$\mathbf{\frac{1}{12}}$	$\frac{1}{8}$							
2	$\mathbf{\frac{1}{16}}$	$\mathbf{\frac{1}{4}}$	$\frac{5}{8}$						
3	$\frac{11}{96}$	$\mathbf{\frac{3}{8}}$	$\mathbf{\frac{65}{48}}$	$\frac{35}{8}$					
4	$\frac{21}{64}$	$\mathbf{\frac{33}{32}}$	$\mathbf{\frac{305}{96}}$	$\mathbf{\frac{175}{16}}$	$\frac{315}{8}$				
5	$\frac{163}{128}$	$\frac{63}{16}$	$\mathbf{\frac{745}{64}}$	$\mathbf{\frac{1127}{32}}$	$\frac{945}{8}$	$\frac{3465}{8}$			
6	$\frac{1595}{256}$	$\frac{2445}{128}$	$\mathbf{\frac{21275}{384}}$	$\mathbf{\frac{10283}{64}}$	$\mathbf{\frac{15477}{32}}$	$\mathbf{\frac{12705}{8}}$	$\frac{45045}{8}$		
7	$\frac{18825}{512}$	$\frac{14355}{128}$	$\frac{82375}{256}$	$\mathbf{\frac{116907}{128}}$	$\mathbf{\frac{84279}{32}}$	$\mathbf{\frac{252945}{32}}$	$\frac{405405}{16}$	$\frac{675675}{8}$	
8	$\frac{260085}{1024}$	$\frac{395325}{512}$	$\frac{1126475}{512}$	$\mathbf{\frac{1578339}{256}}$	$\mathbf{\frac{2222919}{128}}$	$\mathbf{\frac{1600005}{32}}$	$\frac{2387385}{16}$	$\frac{7432425}{16}$	$\frac{11486475}{8}$
9	$\frac{4116315}{2048}$	$\frac{780255}{128}$	$\frac{17702825}{1024}$	$\frac{24596187}{512}$	$\mathbf{\frac{17069781}{128}}$	$\mathbf{\frac{47966325}{128}}$	$\frac{34462285}{32}$	$\frac{25450425}{8}$	$\frac{19144125}{2}$
10	$\frac{73417995}{4096}$	$\frac{11140505}{2048}$	$\frac{314129025}{2048}$	$\frac{433887111}{1024}$	$\mathbf{\frac{596350053}{512}}$	$\mathbf{\frac{412393245}{128}}$	$\frac{1157510783}{128}$	$\frac{103528425}{4}$	$\frac{602657055}{8}$
11	$\frac{1456873425}{8192}$	$\frac{1101269925}{2048}$	$\frac{6209382375}{4096}$	$\frac{8539707015}{2048}$	$\frac{5827734675}{512}$	$\mathbf{\frac{15945353325}{512}}$	$\frac{22022084487}{256}$	$\frac{30846831015}{128}$	$\frac{21914407515}{32}$
12	$\frac{31832972325}{16384}$	$\frac{48076823025}{8192}$	$\frac{135269701875}{8192}$	$\frac{185426394615}{4096}$	$\frac{251769660615}{2048}$	$\mathbf{\frac{170859190425}{512}}$	$\frac{7290122177}{8}$	$\frac{643590359685}{256}$	$\frac{898172490215}{128}$
13	$\frac{759408232275}{32768}$	$\frac{286496750925}{4096}$	$\frac{3219307441625}{16384}$	$\frac{4401709330095}{8192}$	$\frac{2976421242465}{2048}$	$\frac{8031549267225}{2048}$	$\frac{339777246393}{32}$	$\frac{3707081523495}{128}$	$\frac{10205899285855}{128}$
14	$\frac{19639202658075}{65536}$	$\frac{29616921058725}{32768}$	$\frac{83094183797625}{32768}$	$\frac{113381114321475}{16384}$	$\frac{152856379904085}{8192}$	$\frac{102641114315025}{2048}$	$\frac{276095298454833}{2048}$	$\frac{466586730530555}{128}$	$\frac{127090011696915}{128}$

Table 15 – Values of $\pi^{-2(n-d)}H_{1,n}[d]$ computed from the Virasoro constraints. We display in bold the values that were used to determine ρ_d and $R_d(n)$ in Table 10.

$n \setminus d$	9	10	11	12	13	14
9	$\frac{218243025}{8}$					
10	218243025	$\frac{4583103525}{8}$				
11	$\frac{31296049785}{16}$	$\frac{87078966975}{16}$	$\frac{105411381075}{8}$			
12	$\frac{631580764815}{32}$	$\frac{1764494857125}{32}$	$\frac{2354187510675}{16}$	$\frac{2635284526875}{8}$		
13	$\frac{28319986049355}{128}$	$\frac{39312334336275}{64}$	$\frac{53584118713125}{32}$	$\frac{34258698849375}{8}$	$\frac{71152682225625}{8}$	
14	$\frac{348671089506285}{128}$	$\frac{479892186999225}{64}$	$\frac{1311975146807325}{64}$	$\frac{1741923072264575}{32}$	$\frac{1067290233384375}{8}$	$\frac{2063427784543125}{8}$

Table 16 – Values of $\pi^{-2(n-d)}H_{1,n}[d]$ computed from the Virasoro constraints (continued).

$n \setminus d$	0	1	2	3	4	5	6	7	8	9	10	11
1	$\frac{29}{2560}$	$\frac{1}{32}$	$\frac{119}{1152}$	$\frac{35}{96}$	$\frac{105}{128}$							
2	$\frac{337}{9216}$	$\frac{261}{2560}$	$\frac{75}{256}$	$\frac{119}{128}$	$\frac{105}{32}$	$\frac{1155}{128}$						
3	$\frac{319}{2048}$	$\frac{337}{768}$	$\frac{1399}{1152}$	$\frac{2695}{768}$	$\frac{2779}{256}$	$\frac{9625}{256}$	$\frac{15015}{128}$					
4	$\frac{10109}{12288}$	$\frac{4785}{2048}$	$\frac{19583}{3072}$	$\frac{567}{32}$	$\frac{26161}{512}$	$\frac{79695}{512}$	$\frac{135135}{256}$	$\frac{225225}{128}$				
5	$\frac{42445}{8192}$	$\frac{30327}{2048}$	$\frac{185063}{4608}$	$\frac{105007}{960}$	$\frac{1559847}{5120}$	$\frac{897039}{1024}$	$\frac{1354353}{512}$	$\frac{1126125}{128}$	$\frac{3828825}{128}$			
6	$\frac{620641}{16384}$	$\frac{891345}{8192}$	$\frac{1205735}{4096}$	$\frac{24343627}{30720}$	$\frac{4426961}{2048}$	$\frac{12338073}{2048}$	$\frac{53113775}{3072}$	$\frac{26591565}{512}$	$\frac{21696675}{128}$	$\frac{72747675}{128}$		
7	$\frac{10329655}{32768}$	$\frac{1861923}{2048}$	$\frac{15099635}{6144}$	$\frac{26907839}{4096}$	$\frac{362089239}{20480}$	$\frac{197828785}{4096}$	$\frac{827349809}{6144}$	$\frac{49424375}{128}$	$\frac{591425835}{512}$	$\frac{945719775}{256}$	$\frac{1527701175}{128}$	
8	$\frac{192765615}{65536}$	$\frac{278900685}{32768}$	$\frac{1130917375}{49152}$	$\frac{250801845}{4096}$	$\frac{1337766877}{8192}$	$\frac{3604505311}{8192}$	$\frac{4929397655}{4096}$	$\frac{6875894025}{2048}$	$\frac{4926846925}{512}$	$\frac{916620705}{32}$	$\frac{22915517625}{256}$	$\frac{35137127025}{128}$

Table 17 – Values of $\pi^{-2(n+3-d)}H_{2,n}[d]$ computed from the Virasoro constraints.

n \ d	0	1	2	3	4	5	6	7	8	9	10	11	12
1	$\frac{20555}{1327104}$	$\frac{575}{14336}$	$\frac{8099}{73728}$	$\frac{56749}{184320}$	$\frac{8203}{9218}$	$\frac{17479}{6144}$	$\frac{5005}{512}$	$\frac{25025}{1024}$					
2	$\frac{77633}{884736}$	$\frac{102775}{442368}$	$\frac{1920563}{3096576}$	$\frac{624463}{368640}$	$\frac{48189}{10240}$	$\frac{500489}{36864}$	$\frac{87087}{2048}$	$\frac{75075}{512}$	$\frac{425425}{1024}$				
3	$\frac{1038595}{1769472}$	$\frac{77633}{49152}$	$\frac{11069909}{2654208}$	$\frac{8245679}{737280}$	$\frac{3737107}{122880}$	$\frac{6218927}{73728}$	$\frac{26863837}{110592}$	$\frac{575575}{768}$	$\frac{15740725}{6144}$	$\frac{8083075}{1024}$			
4	$\frac{16011391}{3538944}$	$\frac{7270165}{589824}$	$\frac{172014797}{5308416}$	$\frac{379718161}{4423680}$	$\frac{2354953}{10240}$	$\frac{92087039}{147456}$	$\frac{383277895}{221184}$	$\frac{183688505}{36864}$	$\frac{186931745}{12288}$	$\frac{105079975}{2048}$	$\frac{169744575}{1024}$		
5	$\frac{31040465}{786432}$	$\frac{16011391}{147456}$	$\frac{1008891097}{3538944}$	$\frac{6627865109}{8847360}$	$\frac{2277007409}{1146880}$	$\frac{3658297225}{688128}$	$\frac{2129633311}{147456}$	$\frac{739407955}{18432}$	$\frac{8496138365}{73728}$	$\frac{1430704275}{4096}$	$\frac{1188212025}{1024}$	$\frac{3904125225}{1024}$	
6	$\frac{201498115}{524288}$	$\frac{279364185}{262144}$	$\frac{2200898005}{786432}$	$\frac{8629787107}{1179648}$	$\frac{9925860397}{516096}$	$\frac{210466671811}{4128768}$	$\frac{156612960491}{1146880}$	$\frac{18256755985}{49152}$	$\frac{456931637591}{442368}$	$\frac{72883701605}{24576}$	$\frac{36630879285}{4096}$	$\frac{29931626725}{1024}$	$\frac{97603130625}{1024}$

Table 18 – Values of $\pi^{-2(n+6-d)} H_{3,n}[d]$ computed from the Virasoro constraints.

n \ d	0	1	2	3	4	5	6	7
1	$\frac{1103729}{18874368}$	$\frac{2106241}{14155776}$	$\frac{249375997}{637009920}$	$\frac{55140311}{53084160}$	$\frac{7636607}{2752512}$	$\frac{26723675}{3538944}$	$\frac{111772375}{5308416}$	$\frac{3379805}{55296}$
2	$\frac{160909109}{339738624}$	$\frac{7726103}{6291456}$	$\frac{209435}{65536}$	$\frac{8920693349}{1061683200}$	$\frac{1834236191}{82575360}$	$\frac{4903282417}{82575360}$	$\frac{571236653}{3538944}$	$\frac{793954733}{1769472}$
3	$\frac{14674841399}{3397386240}$	$\frac{160909109}{14155776}$	$\frac{14996838119}{509607936}$	$\frac{2173908251}{28311552}$	$\frac{62374893397}{309657600}$	$\frac{17585088257}{33030144}$	$\frac{1058028040511}{743178240}$	$\frac{1140294935}{294912}$
4	$\frac{99177888029}{2264924160}$	$\frac{14674841399}{125829120}$	$\frac{399887731}{1327104}$	$\frac{110651102527}{141557760}$	$\frac{2524091828537}{1238630400}$	$\frac{756362323453}{141557760}$	$\frac{466656238555}{33030144}$	$\frac{29716909709}{786432}$
5	$\frac{4424424475179}{905969664}$	$\frac{99177888029}{75497472}$	$\frac{2305616474459}{679477248}$	$\frac{1240791326569}{141557760}$	$\frac{3575909389403}{157286400}$	$\frac{117418444025993}{1981808640}$	$\frac{153980456785597}{990904320}$	$\frac{969929385535}{2359296}$

Table 19 – Values of $\pi^{-2n+9-d)} H_{4,n}[d]$ computed from the Virasoro constraints.

$n \setminus d$	8	9	10	11	12	13	14
1	$\frac{28503475}{147456}$	$\frac{8083075}{12288}$	$\frac{56581525}{32768}$				
2	$\frac{1147322605}{884736}$	$\frac{66281215}{16384}$	$\frac{56581525}{4096}$	$\frac{1301375075}{32768}$			
3	$\frac{56993061697}{5308416}$	$\frac{4563242255}{147456}$	$\frac{6257916665}{65536}$	$\frac{63767378675}{196608}$	$\frac{32534376875}{32768}$		
4	$\frac{272361575941}{2654208}$	$\frac{42015685283}{147456}$	$\frac{322445178055}{393216}$	$\frac{986702581865}{393216}$	$\frac{553084406875}{65536}$	$\frac{878428175625}{32768}$	
5	$\frac{7785790643045}{7077888}$	$\frac{3525813704023}{1179648}$	$\frac{2177058238355}{262144}$	$\frac{18781333535965}{786432}$	$\frac{9519558673625}{131072}$	$\frac{7905853580625}{32768}$	$\frac{25474417093125}{32768}$

Table 20 – Values of $\pi^{-2(n+9-d)} H_{4,n}[d]$ computed from the Virasoro constraints (continued).

$n \setminus d$	0	1	2	3	4	5	6	7	8
1	$\frac{35161328707}{81537269760}$	$\frac{22821693}{20971520}$	$\frac{7644175835}{2717908992}$	$\frac{109134615047}{14948499456}$	$\frac{35307225403}{1857945600}$	$\frac{296352439757}{5945425920}$	$\frac{87114140087}{660602880}$	$\frac{4992748537}{14155776}$	$\frac{244417472273}{254803968}$
2	$\frac{27431847097}{6039797760}$	$\frac{35161328707}{3019898880}$	$\frac{1463192937811}{48922361856}$	$\frac{210033343633}{2717908992}$	$\frac{174554274549551}{871995801600}$	$\frac{12396787304077}{23781703680}$	$\frac{2438015036977}{1783627776}$	$\frac{68208841817}{18874368}$	$\frac{547141537559}{56623104}$
3	$\frac{5703709895459}{108716359680}$	$\frac{27431847097}{201326592}$	$\frac{3793692043765}{10871635968}$	$\frac{14642582778043}{16307453952}$	$\frac{68867984889821}{29727129600}$	$\frac{8918977503973}{1486356480}$	$\frac{2787426440647199}{178362777600}$	$\frac{23198438807111}{566231040}$	$\frac{551941078807889}{5096079360}$
4	$\frac{143368101519407}{217432719360}$	$\frac{62740808850049}{36238786560}$	$\frac{288900738269347}{65229815808}$	$\frac{617405851484437}{54358179840}$	$\frac{3086856384144917}{105696460800}$	$\frac{7165942204049803}{95126814720}$	$\frac{19890284967197827}{101921587200}$	$\frac{1499002879859}{2949120}$	$\frac{4527331166325601}{3397386240}$

Table 21 – Values of $\pi^{-2n+12-d)} H_{5,n}[d]$ computed from the Virasoro constraints.

$n \setminus d$	9	10	11	12	13	14	15	16
1	$\frac{4730354057}{1769472}$	$\frac{18324331025}{2359296}$	$\frac{115301831645}{4718592}$	$\frac{32534376875}{393216}$	$\frac{58561878375}{262144}$			
2	$\frac{247779861745}{9437184}$	$\frac{689116092665}{9437184}$	$\frac{1993595068465}{9437184}$	$\frac{344864394875}{524288}$	$\frac{292809391875}{131072}$	$\frac{1698294472875}{262144}$		
3	$\frac{81986849479019}{283115520}$	$\frac{44535267213073}{56623104}$	$\frac{123749786026867}{56623104}$	$\frac{59509093518875}{9437184}$	$\frac{5102691669075}{262144}$	$\frac{34531987615125}{524288}$	$\frac{52647128659125}{262144}$	
4	$\frac{221681265259901}{62914560}$	$\frac{1067126014204999}{113246208}$	$\frac{966187786983419}{37748736}$	$\frac{447428553196625}{6291456}$	$\frac{214834529684775}{1048576}$	$\frac{658598596580925}{1048576}$	$\frac{1105589701841625}{524288}$	$\frac{1737355245751125}{262144}$

Table 22 – Values of $\pi^{-2(n+12-d)} H_{5,n}[d]$ computed from the Virasoro constraints (continued).

$n \backslash d$	0	1	2	3	4	5	6
1	$\frac{1725192578138153}{328758271672320}$	$\frac{51582017261473}{3913788948480}$	$\frac{118778241943205}{3522410053632}$	$\frac{28213838434057}{326149079040}$	$\frac{769565168808731549}{3462616055808000}$	$\frac{7853909947536887}{13698261319680}$	$\frac{16788873181356453 97}{1130106558873600}$
2	$\frac{236687293214441}{3478923509760}$	$\frac{18977118359519683}{109586090557440}$	$\frac{2072797311487703}{4696546738176}$	$\frac{13256652596923381}{11741366845440}$	$\frac{18867920053695301}{6522981580800}$	$\frac{102951894998214038 71}{1385046422323200}$	$\frac{144451655411576760 91}{753404372582400}$
3	$\frac{19857285082756661}{20873541058560}$	$\frac{236687293214441}{96636764160}$	$\frac{341353246695400481}{54793045278720}$	$\frac{4608517054909483}{289910292480}$	$\frac{11130008225501778383}{273965226393600}$	$\frac{271549970945534501}{2609192632320}$	$\frac{94035520375500582436 9}{3515887072051200}$

$n \backslash d$	7	8	9	10	11	12
1	$\frac{2208394994566309}{570760888320}$	$\frac{18384224416373}{1811939328}$	$\frac{15195072225949}{566231040}$	$\frac{585329319725327}{8153726976}$	$\frac{530536293549353}{2717908992}$	$\frac{246412953611825}{452984832}$
2	$\frac{534554804193487589}{10762919608320}$	$\frac{221467745692382411}{171228266496}$	$\frac{9216122383558207}{27179089920}$	$\frac{1624463784480019}{1811939328}$	$\frac{39095700824091245}{16307453952}$	$\frac{1967183255694475}{301989888}$
3	$\frac{112491864394018121}{163074539520}$	$\frac{5769459355627848601 7}{32288758824960}$	$\frac{196729681979823383}{42278584320}$	$\frac{132613775029696513}{10871635968}$	$\frac{1051799750173130861}{32614907904}$	$\frac{4218609560944475125}{48922361856}$

$n \backslash d$	13	14	15	16	17	18
1	$\frac{49725541615 75}{3145728}$	$\frac{20794672323425}{4194304}$	$\frac{17549042886375}{1048576}$	$\frac{193039471750125}{4194304}$		
2	$\frac{304054573898075}{16777216}$	$\frac{1320243340390975}{25165824}$	$\frac{684412672568625}{4194304}$	$\frac{579118415250375}{1048576}$	$\frac{6756381511254375}{4194304}$	
3	$\frac{17683899568619375}{75497472}$	$\frac{24580850021252525}{37748736}$	$\frac{141960893257558375}{75497472}$	$\frac{48710293371614875}{8388608}$	$\frac{164405283440523125}{8388608}$	$\frac{249986115916411875}{4194304}$

Table 23 – Values of $\pi^{-2(n+15-d)}\, H_{6,n}[d]$ computed from the Virasoro constraints.

Table 24 – Values of $\pi^{-2(3g-3+n)+2d_1+\cdots+2d_k} F_{g,n}[d_1,\ldots,d_k,0,\ldots,0]$ for $k \geqslant 2$, computed from the Virasoro constraints.

(g,n)	$(d_1,\ldots,d_k)$	$*$
$(0,5)$	$(1,1)$	18
$(0,6)$	$(1,1)$	27
	$(2,1)$	135
	$(1,1,1)$	162
$(0,7)$	$(1,1)$	81
	$(2,1)$	300
	$(3,1)$	1260
	$(2,2)$	1350
	$(1,1,1)$	324
	$(2,1,1)$	1620
	$(1,1,1,1)$	1944
$(0,8)$	$(1,1)$	$\frac{675}{2}$
	$(2,1)$	$\frac{4575}{4}$
	$(3,1)$	$\frac{7875}{2}$
	$(4,1)$	14175
	$(2,2)$	4125
	$(3,2)$	15750
	$(1,1,1)$	1215
	$(2,1,1)$	4500
	$(3,1,1)$	18900
	$(2,2,1)$	20250
	$(1,1,1,1)$	4860
	$(2,1,1,1)$	24300
	$(1,1,1,1,1)$	29160

(g,n)	$(d_1,\ldots,d_k)$	$*$
$(1,2)$	$(1,1)$	$\frac{3}{8}$
$(1,3)$	$(1,1)$	$\frac{3}{2}$
	$(2,1)$	$\frac{15}{4}$
	$(1,1,1)$	$\frac{9}{4}$
$(1,4)$	$(1,1)$	$\frac{27}{8}$
	$(2,1)$	$\frac{195}{16}$
	$(3,1)$	$\frac{315}{8}$
	$(2,2)$	$\frac{75}{2}$
	$(1,1,1)$	$\frac{27}{2}$
	$(2,1,1)$	$\frac{135}{4}$
	$(1,1,1,1)$	$\frac{81}{4}$
$(1,5)$	$(1,1)$	$\frac{99}{8}$
	$(2,1)$	$\frac{305}{8}$
	$(3,1)$	$\frac{525}{4}$
	$(4,1)$	$\frac{945}{2}$
	$(2,2)$	$\frac{1075}{8}$
	$(3,2)$	$\frac{3675}{8}$
	$(1,1,1)$	$\frac{81}{2}$
	$(2,1,1)$	$\frac{585}{4}$
	$(3,1,1)$	$\frac{945}{2}$
	$(2,2,1)$	450
	$(1,1,1,1)$	162
	$(2,1,1,1)$	405
	$(1,1,1,1,1)$	243

(g,n)	$(d_1,\ldots,d_k)$	$*$
$(1,6)$	$(1,1)$	$\frac{945}{16}$
	$(2,1)$	$\frac{11175}{64}$
	$(3,1)$	$\frac{16905}{32}$
	$(4,1)$	$\frac{14175}{8}$
	$(5,1)$	$\frac{51975}{8}$
	$(2,2)$	$\frac{6475}{12}$
	$(3,2)$	$\frac{29225}{16}$
	$(4,2)$	$\frac{51975}{8}$
	$(3,3)$	$\frac{25725}{4}$
	$(1,1,1)$	$\frac{1485}{8}$
	$(2,1,1)$	$\frac{4575}{8}$
	$(3,1,1)$	$\frac{7875}{4}$
	$(4,1,1)$	$\frac{14175}{2}$
	$(2,2,1)$	$\frac{16125}{8}$
	$(3,2,1)$	$\frac{55125}{8}$
	$(2,2,2)$	6750
	$(1,1,1,1)$	$\frac{1215}{2}$
	$(2,1,1,1)$	$\frac{8775}{4}$
	$(3,1,1,1)$	$\frac{14175}{2}$
	$(2,2,1,1)$	6750
	$(1,1,1,1,1)$	2430
	$(2,1,1,1,1)$	6075
	$(1,1,1,1,1,1)$	3645

Table 25 – Values of $\pi^{-2(3g-3+n)+2d_1+\cdots+2d_k} F_{g,n}[d_1,\ldots,d_k,0,\ldots,0]$ for $k \geqslant 2$ (continued).

(g,n)	$(d_1,\ldots,d_k)$	$*$
$(2,2)$	$(1,1)$	$\frac{9}{32}$
	$(2,1)$	$\frac{119}{128}$
	$(3,1)$	$\frac{105}{32}$
	$(4,1)$	$\frac{945}{128}$
	$(2,2)$	$\frac{1225}{384}$
	$(3,2)$	$\frac{1015}{128}$
$(2,3)$	$(1,1)$	$\frac{783}{640}$
	$(2,1)$	$\frac{225}{64}$
	$(3,1)$	$\frac{357}{32}$
	$(4,1)$	$\frac{315}{8}$
	$(5,1)$	$\frac{3465}{32}$
	$(2,2)$	$\frac{25585}{2304}$
	$(3,2)$	$\frac{14875}{384}$
	$(4,2)$	$\frac{3465}{32}$
	$(3,3)$	$\frac{7105}{64}$
	$(1,1,1)$	$\frac{27}{8}$
	$(2,1,1)$	$\frac{357}{32}$
	$(3,1,1)$	$\frac{315}{8}$
	$(4,1,1)$	$\frac{2835}{32}$
	$(2,2,1)$	$\frac{1225}{32}$
	$(3,2,1)$	$\frac{3045}{32}$
	$(2,2,2)$	$\frac{1575}{16}$

(g,n)	$(d_1,\ldots,d_k)$	$*$
$(2,4)$	$(1,1)$	$\frac{1685}{256}$
	$(2,1)$	$\frac{6995}{384}$
	$(3,1)$	$\frac{13475}{256}$
	$(4,1)$	$\frac{41685}{256}$
	$(5,1)$	$\frac{144375}{256}$
	$(6,1)$	$\frac{225225}{128}$
	$(2,2)$	$\frac{241825}{4608}$
	$(3,2)$	$\frac{125167}{768}$
	$(4,2)$	$\frac{72135}{128}$
	$(5,2)$	$\frac{3465}{2}$
	$(3,3)$	$\frac{71785}{128}$
	$(4,3)$	$\frac{112455}{64}$
	$(1,1,1)$	$\frac{2349}{128}$
	$(2,1,1)$	$\frac{3375}{64}$
	$(3,1,1)$	$\frac{5355}{32}$
	$(4,1,1)$	$\frac{4725}{8}$
	$(5,1,1)$	$\frac{51975}{32}$
	$(2,2,1)$	$\frac{127925}{768}$
	$(3,2,1)$	$\frac{74375}{128}$
	$(3,3,1)$	$\frac{106575}{64}$
	$(2,2,2)$	$\frac{18375}{32}$
	$(3,2,2)$	$\frac{13125}{8}$
	$(1,1,1,1)$	$\frac{405}{8}$
	$(2,1,1,1)$	$\frac{5355}{32}$
	$(3,1,1,1)$	$\frac{4725}{8}$
	$(4,1,1,1)$	$\frac{42525}{32}$
	$(2,2,1,1)$	$\frac{18375}{32}$
	$(3,2,1,1)$	$\frac{45675}{32}$
	$(2,2,2,1)$	$\frac{23625}{16}$

(g,n)	$(d_1,\ldots,d_k)$	$*$
$(3,2)$	$(1,1)$	$\frac{8625}{14336}$
	$(2,1)$	$\frac{40495}{24576}$
	$(3,1)$	$\frac{56749}{12288}$
	$(4,1)$	$\frac{41015}{3072}$
	$(5,1)$	$\frac{87395}{2048}$
	$(6,1)$	$\frac{75075}{512}$
	$(7,1)$	$\frac{375375}{1024}$
	$(2,2)$	$\frac{56765}{12288}$
	$(3,2)$	$\frac{743917}{55296}$
	$(4,2)$	$\frac{87353}{2048}$
	$(5,2)$	$\frac{297605}{2048}$
	$(6,2)$	$\frac{385385}{1024}$
	$(3,3)$	$\frac{131467}{3072}$
	$(4,3)$	$\frac{37485}{256}$
	$(5,3)$	$\frac{193655}{512}$
	$(4,4)$	$\frac{191205}{512}$

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