Analytical approach to the Bose polaron via a *q*-deformed Lie algebra

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We present an approach to the Bose polaron based on the notion of quantum groups, also known as *q*-deformed Lie algebras. In this approach, a mobile impurity can be depicted as a deformation of the Lie algebra of the bosonic creation and annihilation operators of the bath, in which the impurity is immersed. Accordingly, the Bose polaron can be described as a bath of noninteracting *q*-deformed bosons, which allows us to provide an analytical formulation of the Bose polaron at arbitrary couplings. We derive its ground state energy in the phonon branch of the Bogoliubov dispersion and demonstrate that the previously observed transition from a repulsive to an attractive polaron occurs at the vicinity where the quantum group symmetry is broken. Furthermore, our approach has the potential to open up avenues in polaron physics by connecting it with seemingly unrelated research topics where quantum groups play an essential role, such as anyons.

The polaron model, which was initially introduced for a quasiparticle consisting of an electron dressed by lattice excitations in a crystal [1–3], has been adapted to several different contexts and nowadays is used to describe more general classes of impurities interacting with a quantum many-particle bath. Within recent advances in ultracold atomic physics, Bose polarons, which correspond to quantum impurities interacting with a surrounding Bose-Einstein condensate (BEC), broke ground in the study of polaron physics. Although, Bose polarons have been investigated intensively in several experimental [4–9] and theoretical [10–17] studies, there remain many open questions, which shape the current research directions in the field. One is whether it is possible to provide a complete analytical description of Bose polarons at arbitrary couplings. In this paper, inspired by the phenomenon of how a single impurity drastically changes its surrounding bath [18,19], we address the aforementioned question within the concept of quantum groups.

Quantum groups, also known as *q*-deformed Lie algebras [20–23], are Hopf algebras possessing a coproduct, a counit, and an antipode, in addition to the regular structures of an algebra [24]. In physics, quantum groups have been applied to solve the quantum Yang-Baxter equation [22]. Although quantum group symmetries are not believed to be common in physical problems, over the years, they have found several applications to spin chains [25], anyons [26,27], quantum optics [28,29], nuclear physics [30], rotational and vibrational molecular spectra [31,32], and molecular impurities [33].

Now we demonstrate that the Bose polaron features quantum group symmetries within the framework of the extended Bogoliubov-Fröhlich Hamiltonian, which provides the most accurate model for the Bose polaron if the Bogoliubov description of the Bose gas is valid [13]. We show that the Lie algebra of the bosonic creation and annihilation operators of the bath, called the Heisenberg-Weyl algebra, can be deformed due to the presence of an impurity. Such an approach, first, allows us to introduce a simple parent Hamiltonian in terms of *q*-deformed bosons, whose series expansion in the usual bosons corresponds to the extended Bogoliubov-Fröhlich Hamiltonian in the branch where the Bogoliubov dispersion takes the phononlike linear form. Furthermore, as an insightful physical picture, the parent Hamiltonian describes the gas of interacting Bogoliubov bosons in terms of a gas of noninteracting *q*-deformed bosons. Within this picture, we derive the ground state energy of the Bose polaron analytically in the phonon branch and demonstrate that, at the vicinity of a critical value of the *q* parameter, where the quantum group symmetry is broken, the Bose polaron changes from repulsive to attractive.

We start by considering a single impurity immersed in a weakly interacting Bose gas. Although our approach is independent of dimension, for the sake of simplicity, we will consider the impurity-bath system in one dimension with the quantization length L, which we set to unity. Within the Bogoliubov approximation, the most accurate Hamiltonian describing the Bose polaron is given by

$$H = \frac{P^2}{2} + \sum_k \omega_k b_k^\dagger b_k + \sum_k \lambda_k \exp(-ikX)(b_k^\dagger + b_{-k})$$
$$+ \sum_{k,k'} \lambda_{k,k'}^+ \exp[-i(k-k')X]b_k^\dagger b_{k'}$$
$$+ \frac{1}{2} \sum_{k,k'} \lambda_{k,k'}^- \exp[-i(k+k')X](b_k^\dagger b_{k'}^\dagger + b_{-k}b_{-k'}), \quad (1)$$

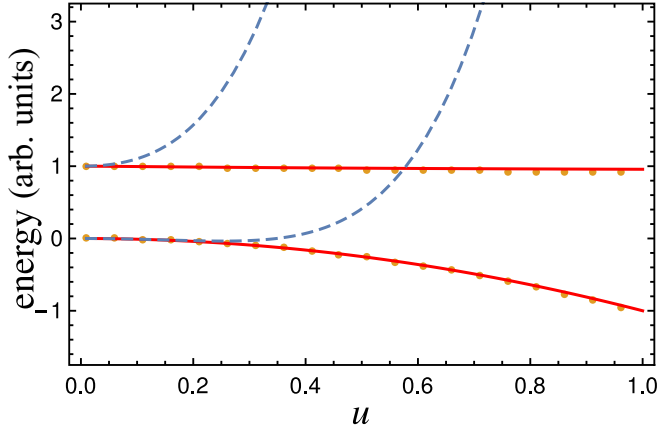---

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FIG. 1. Comparison of the first two energy eigenvalues obtained from a numerical diagonalization of the Hamiltonian in Eq. (5) (dotted yellow) and the analytical result in Eq. (11) (solid red) as a function of the effective coupling u. We further plot the perturbative result (dashed blue). The parameters are chosen as $v = 1$ and $\alpha = -0.2$.

which might be called the extended Bogoliubov-Fröhlich Hamiltonian, see Refs. [11–13,15] for its detailed derivation. Here, the first line corresponds to the well-known Fröhlich Hamiltonian, where $P^2/2$ is the kinetic energy operator of the impurity with mass $M = 1$, and X is the position operator. Also, $b_k^\dagger$ and b_k are the Bogoliubov creation and annihilation operators, ω_k is the Bogoliubov dispersion relation, $\lambda_k = g_{IB}\sqrt{n_B}\chi_k$ is the linear coupling, g_{IB} is the impurity-boson coupling constant, n_B is the condensate density, and $\chi_k = \sqrt{\varepsilon_k/\omega_k}$, with the energy $\varepsilon_k = k^2/(2m_B)$ and mass m_B of the bare bosons. The second and third lines, which are beyond the Fröhlich paradigm [13], describe two-phonon scattering processes with the quadratic couplings $\lambda_{k,k'}^\pm = g_{IB}(\chi_k\chi_{k'} \pm \chi_k^{-1}\chi_{k'}^{-1})/2$.

As the total linear momentum of the system is conserved, one can decouple the degree of freedom of the impurity from the bath by applying the Lee-Low-Pines (LLP) transformation [34] $U_{LLP} = \exp(-iX\sum_k b_k^\dagger b_k)$, which translates the creation and annihilation operators $U_{LLP}^{-1} b_k^\dagger U_{LLP} = \exp(ikX)b_k^\dagger$. The transformed Hamiltonian $H_B = U_{LLP}^{-1} H U_{LLP}$ yields

$$H_B = \sum_k \tilde\omega_k b_k^\dagger b_k + \sum_k \lambda_k(b_k^\dagger + b_k) + \sum_{k,k'} \lambda_{k,k'}^+ b_k^\dagger b_{k'}$$
$$+ \frac{1}{2}\sum_{k,k'} \lambda_{k,k'}^-(b_k^\dagger b_{k'}^\dagger + b_k b_{k'}) + \frac{1}{2}\sum_{k,k'} kk' b_k^\dagger b_{k'}^\dagger b_{k'} b_k, \tag{2}$$

where we replaced the momentum operator P with its eigenvalue p, defined $\tilde\omega_k = \omega_k - pk + k^2/2$, and omitted the constant term $p^2/2$.

The Hamiltonian in Eq. (2) is the main focus of this paper. We underline that, as it is purely bosonic, the Hamiltonian in Eq. (2) describes a bath of interacting Bogoliubov bosons. Furthermore, its spectrum, in principle, can be identified solely by using the Heisenberg-Weyl algebra of the bosonic
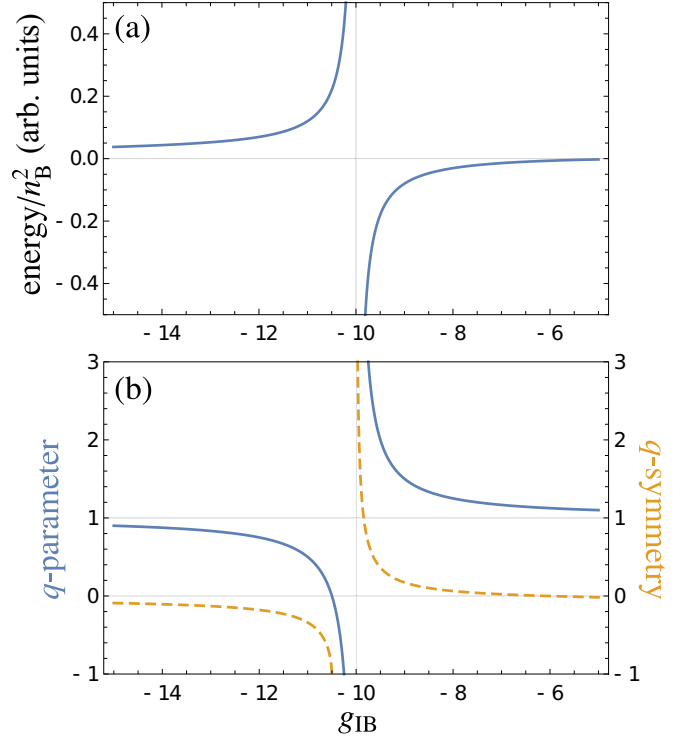

FIG. 2. (a) The ground state energy of the Bose polaron, given by Eq. (17), as a function of g_{BI}. The Bose polaron undergoes a transition from repulsive to attractive as g_{BI} increases. (b) The deformation parameter $q(k, k')$ (solid blue) and the parameter expressing the quantum group symmetry $\rho(k, k') - kk'\alpha_k/u_k$ (dashed yellow), which are calculated at $k = k' = \Lambda/10$. The transition is located at the vicinity where the quantum group symmetry is broken. The applied parameters are $\Lambda = c$, $g_{BB} = 1$, $c = 10$, $m_B = 1$, and $p = 0$.

creation and annihilation operators:

$$[b_k, b_k^\dagger] = 1, \quad [B_k, b_k^\dagger] = b_k^\dagger, \quad [B_k, b_k] = -b_k, \tag{3}$$

with $B_k = b_k^\dagger b_k$ being the number operator. We will show that the Hamiltonian in Eq. (2) can be written elegantly in terms of the elements of the deformed Heisenberg-Weyl algebra. This allows us to approach the Bose polaron within quantum groups.

To present the method in a transparent way, we first focus on the case of a single-phonon mode and discuss the full Hamiltonian later. In the former case, the Hamiltonian in Eq. (2) can be given by

$$H_{sm} = \bar\omega b^\dagger b + \lambda(b^\dagger + b) + \frac{\lambda^-}{2}(b^\dagger b^\dagger + bb) + \frac{k^2}{2}b^\dagger b^\dagger bb, \tag{4}$$

where we define $\bar\omega = \tilde\omega + \lambda^+$, and for a shorthand notation, we omit the index of the mode k from the relevant expressions, i.e., $b_k^\dagger \to b^\dagger$, $\bar\omega_k \to \bar\omega$, and so on. This Hamiltonian, by summing over k, can also be used to approximate the full Hamiltonian, when the elastic scattering ($k' = k$) is dominant.

We now apply the displacement operator $U_{disp} = \exp(\alpha b^\dagger - \alpha^* b)$, which shifts the creation and annihilation operators $U_{disp}^{-1} b^\dagger U_{disp} = b^\dagger + \alpha^*$ with a complex parameter α. The transformed Hamiltonian $H'_{sm} = U_{disp}^{-1} H_{sm} U_{disp}$ can be

written as

$$H'_{\text{sm}} = v\, b^{\dagger}\left(1 + \frac{k^2}{2v}b^{\dagger}b\right)b + u\, b^{\dagger}\left(1 + \frac{\alpha k^2}{u}b^{\dagger}b\right) + \text{H.c.},$$

(5)

where $v = \bar{\omega} + 2|\alpha|^2 k^2$, $u = \lambda + \bar{\omega}\alpha$, H.c. stands for the Hermitian conjugate of the preceding terms, and we omitted the constant term $\bar{\omega}|\alpha|^2 + \lambda(\alpha + \alpha^*) + \lambda^-(\alpha^2 + \alpha^{*2})/2 + k^2|\alpha|^4/2$. In the derivation of the Hamiltonian in Eq. (5), we eliminated the quadratic term $(\lambda^- + k^2\alpha^2)b^{\dagger}b^{\dagger}/2 + \text{H.c.}$ by choosing $\alpha^2 = \alpha^{*2} = -\lambda^-/k^2$. For the sake of simplicity, we further assume $\lambda^- < 0$, and hence, $\alpha \in \mathbb{R}$.

The structure of the Hamiltonian in Eq. (5) inspires us to introduce deformed creation and annihilation operators that can simplify the Hamiltonian further. From the first term, we define a deformed operator, whose series expansion is in the form of

$$a^{\dagger} = b^{\dagger}(1 + \rho\, b^{\dagger}b + \cdots),$$

(6)

with $\rho = \sqrt{1 + k^2/(2v)} - 1$, such that the series expansion of the term $v\, a^{\dagger}a$, truncated at the quartic order in $b^{(\dagger)}$, corresponds to the first term of the Hamiltonian in Eq. (5). The second term of the Hamiltonian also matches with the expansion of the term $u\,(a^{\dagger} + a)$ if there exists the relation $\sqrt{1 + k^2/(2v)} - 1 = \alpha k^2/u$. This relation, which can be interpreted as an expression of the quantum group symmetry, imposes certain restrictions on the values of the dispersion relation as well as the coupling. However, as we will discuss later, the physical parameters of the BEC, indeed, meet such a relation for the full Hamiltonian. Therefore, for the single-phonon-mode case, without loss of physical generality, we assume the existence of this symmetry. As a matter of fact, one can always consider the certain phonon mode k, which satisfies the relation.

The Hamiltonian in Eq. (5) then can be identified as the quartic order series expansion in the usual creation and annihilation operators of the following deformed parent Hamiltonian:

$$H_{\text{sm}}^{\text{def}} = v\, a^{\dagger}a + u(a^{\dagger} + a),$$

(7)

where v can be interpreted as the dispersion relation of the deformed bosons. As we discuss below, the expansion can be truncated, if the higher-order terms can be neglected. This allows us to derive the spectrum of the Hamiltonian in Eq. (5) by using the parent Hamiltonian in Eq. (7).

The Hamiltonian in Eq. (7) is simply in the form of a shifted harmonic oscillator Hamiltonian describing noninteracting deformed bosons. Therefore, the system of interacting regular bosons, which is originally governed by the Hamiltonian in Eq. (4), can be identified as a system of free deformed bosons. This follows from the fact that the introduced deformed bosons accumulate certain correlations which describe implicit interactions between the ordinary bosons. This is in analogy to the Bogoliubov transformation, which describes an interacting Bose gas in terms of a gas of noninteracting quasiparticles [35]. However, there is a price we pay. Despite its simple form, finding the spectrum of the Hamiltonian in Eq. (7) is not trivial due to the deformed commutation relations, which we define now.

The form of the deformed operator in Eq. (6) is known in the context of q deformation, see, for instance, Ref. [32]. It is an element of the called q-deformed Heisenberg-Weyl algebra [36]. This deformed algebra is defined by the group elements $a^{\dagger}$, a, and N, which satisfy the following commutation relations:

$$aa^{\dagger} - qa^{\dagger}a = 1, \quad [N, a^{\dagger}] = a^{\dagger}, \quad [N, a] = -a, \quad (8)$$

with q being the deformation parameter. We note that the number operator N does not equal $a^{\dagger}a$. It is also worth noticing that the q-commutation relation in Eq. (8) can be used to build anyonic algebras [26,27].

One realization of this algebra can be fulfilled with the following expressions:

$$a^{\dagger} = \sqrt{\frac{[N]}{N}}b^{\dagger}, \quad a = b\sqrt{\frac{[N]}{N}}, \quad N = B = b^{\dagger}b. \quad (9)$$

Here, we recall that b and $b^{\dagger}$ are the usual bosonic operators, and $[N] = (q^N - 1)/(q - 1)$. In the limit of $q \to 1$, one recovers the usual Heisenberg-Weyl algebra in Eq. (3). One can obtain the expressions for $a^{\dagger}$ and a in the form of an infinite series of B; $a^{\dagger} = \sum_{n=0} c_n B^n b^{\dagger}$, which allows us to define the q-deformed bosons in terms of the usual ones. Specifically, by using the identity $(b^{\dagger}b)^l = \sum_{k=1}^{l} S(l, k)(b^{\dagger})^k b^k$, with $S(l, k)$ being Stirling numbers of the second kind, the q-deformed creation operator can be written as

$$a^{\dagger} = b^{\dagger}\left\{1 + \left(\sqrt{\frac{[2]}{2}} - 1\right)b^{\dagger}b + \cdots\right\}. \quad (10)$$

This series is in the form of Eq. (6) with $\rho = \sqrt{[2]/2} - 1$, which identifies the deformation parameter as $q = 1 + k^2/v$. As the deformation part k^2/v results from the LLP transformation, we interpret that the Lie algebra of the bath is deformed by the impurity. Moreover, we deduce from the series expansion in Eq. (10) that the Hamiltonian in Eq. (7) can be truncated at the quartic order under the condition of $\zeta \ll 1$, see Eq. (A4) in Appendix A for the definition of ζ.

To find the spectrum, as an initial guess, one might think of applying the displacement operator to shift the q-deformed creation and annihilation operators. However, as we mentioned earlier, due to the nontrivial commutation relation, this is not possible. In fact, it is known that a unitary displacement operator for the q-deformed Weyl-Heisenberg algebra does not exist [36–38]. Nevertheless, we find the eigenvalues exactly:

$$E_{\text{sm}}^{\text{def}(n)} = v[n] - \frac{1}{q^n}\frac{u^2}{v}, \quad (11)$$

see Appendix B for the derivation. In the limit of $q \to 1$, one recovers the well-known eigenvalues $E_{\text{sm}}^{\text{def}(n)} \to vn - u^2/v$. We would like to emphasize that solving the eigenvalue equation of the Hamiltonian in Eq. (7) is, in itself, a very important result for the field of quantum groups, as it corresponds to the resolution of the problem of the q-displacement operator.

In summary, under the condition of $\zeta \ll 1$, the ground state energy of the single-phonon-mode Hamiltonian in Eq. (5) coincides with that of the deformed parent Hamiltonian, hence given by Eq. (11). This correspondence turns the picture of

interacting regular bosons into the one of noninteracting q-deformed bosons. To illustrate the ability of the analytical result in Eq. (11), in Fig. 1, we compare it with the exact numerical result, which we calculated by diagonalizing the Hamiltonian in Eq. (5) in the basis of the eigenstates of the number operator $|m\rangle$. Instead of the original parameters which appeared in the Hamiltonian in Eq. (4), the eigenvalues are calculated in terms of the parameters v, u, and α of the Hamiltonian in Eq. (5), which is unitarily equivalent to the former. In the figure, the parameter α is chosen to satisfy the condition $\zeta \ll 1$. In the same figure, we also present the result of the first-order perturbation theory, where the term $k^2 b^\dagger b^\dagger bb/2 + \alpha k^2 b^\dagger b^\dagger b + $ H.c. is treated perturbatively. It is clear that the analytical result agrees with the numerical one even beyond the perturbative regime.

Now we will focus on the full Hamiltonian in Eq. (2). In a similar way, to eliminate the quadratic terms, we apply the displacement operator $U_{\text{disp}} = \exp[\sum_k (\alpha_k b_k^\dagger - \alpha_k^* b_k)]$. The transformed Bogoliubov-Fröhlich Hamiltonian can be written as

$$H'_{\text{B}} = \sum_k v_k b_k^\dagger \left(1 + \sum_{k'} \frac{kk'}{2v_k} b_{k'}^\dagger b_{k'}\right) b_k$$
$$+ \sum_k u_k b_k^\dagger \left(1 + \sum_{k'} \frac{kk'\alpha_k}{u_k} b_{k'}^\dagger b_{k'}\right) + \text{H.c.,} \quad (12)$$

where $v_k = \tilde{\omega}_k + k \sum_{k'} k'|\alpha_{k'}|^2$, $u_k = \lambda_k + \alpha_k(\tilde{\omega}_k - k \sum_{k'} k' |\alpha_{k'}|^2)$, and the constant term $\sum_k \tilde{\omega}_k |\alpha_k|^2 - 3 \sum_{k,k'} kk' |\alpha_k|^2 |\alpha_{k'}|^2/2$ was omitted. In the derivation of Eq. (12), the quadratic terms $b_k^\dagger b_{k'}^\dagger$, $b_k b_{k'}$, and $b_k^\dagger b_{k'}$ are canceled out by specifying $\alpha_k \alpha_{k'} = -\lambda_{k,k'}^-/(kk')$ and $\alpha_k \alpha_{k'}^* = -\lambda_{k,k'}^+/(kk')$. This is possible only if the couplings satisfy the relation $|\lambda_{k,k'}^+| = |\lambda_{k,k'}^-|$. Strikingly, the relation is satisfied via $\lambda_{k,k'}^+ = -\lambda_{k,k'}^- \approx g_{\text{IB}} \chi_k^{-1} \chi_{k'}^{-1}/2$ in the phonon branch of the Bogoliubov dispersion, i.e., when $\omega_k \approx ck$, with $c = \sqrt{g_{\text{BB}} n_{\text{B}}/m_{\text{B}}}$ being the sound velocity and g_{BB} the interaction strength between bosons, see Ref. [35]. The relation $\lambda_{k,k'}^+ = -\lambda_{k,k'}^-$ further implies that $\alpha_k = i|\alpha_k|$ and hence allows us to consider the nontrivial phases of $g_{\text{IB}} < 0$, where the polaron changes from repulsive to attractive as g_{IB} increases across a critical value [15,39].

The form of the Hamiltonian in Eq. (12) now allows us to introduce deformed operators. For a later purpose, we first symmetrize the term $kk'/(2v_k)$ by replacing it with $kk'/(v_k + v_{k'})$ in the first line of the Hamiltonian in Eq. (12). This replacement has no effect, as the quartic term $\sum_{k,k'} kk' b_k^\dagger b_{k'}^\dagger b_{k'} b_k/2$ is symmetric with respect to the exchange of k and k', but the remainder term is antisymmetric. Then we introduce the following deformed operator:

$$a_k^\dagger = b_k^\dagger \prod_{k'} q_{kk'}^{B_{k'}/2}$$
$$= b_k^\dagger \left[1 + \sum_{k'} \rho(k, k') b_{k'}^\dagger b_{k'} + \cdots \right]. \quad (13)$$

Here, the deformation parameter $q_{kk'} = [1 + \rho(k, k')]^2$ is a complex number, as the coupling u_k is—and we define $\rho(k, k')$ via the relation $\rho(k, k') + \rho^*(k, k') + |\rho(k, k')|^2 =$

$kk'/(v_k + v_{k'})$—such that it corresponds to the first line of the Hamiltonian in Eq. (12). We emphasize that, in the phonon branch of the Bogoliubov dispersion, where $k \ll m_{\text{B}} c$, the expansion in Eq. (13) naturally truncates at the quartic order, as the higher orders are proportional to $k/(m_{\text{B}} c)$. Furthermore, the relation $\rho(k, k') \approx kk'\alpha_k/u_k$ also holds true, see Fig. 2(b); thereby, the deformed operator in Eq. (13) matches with the second line of the Hamiltonian in Eq. (12), as well. This is basically the manifestation of the quantum group symmetry of the Bose polaron. Thus, the full Bogoliubov-Fröhlich Hamiltonian in Eq. (12) in the phonon branch can be written as the following deformed parent Hamiltonian:

$$H_{\text{B}}^{\text{def}} = \sum_k v_k a_k^\dagger a_k + \sum_k (u_k a_k^\dagger + u_k^* a_k). \quad (14)$$

The Hamiltonian in Eq. (14) is elegantly in the form of a collection of uncoupled shifted harmonic oscillator so that it describes a gas of noninteracting deformed bosons. However, as a price to pay, one has to deal with the complicated deformed Lie algebra, which can be written as

$$a_k a_k^\dagger - |q_{kk}| \, a_k^\dagger a_k = \prod_{k'} |q_{kk'}|^{B_{k'}},$$
$$[N_k, a_k^\dagger] = a_k^\dagger, \quad [N_k, a_k] = -a_k, \quad (15)$$

with $N_k = B_k$. Moreover, in contrast to the case of the usual algebra, the commutation relation between different modes ($k \neq k'$) do not vanish but are given by

$$a_k a_{k'}^\dagger - \sqrt{q_{kk'}^*} \sqrt{q_{k'k}} \, a_{k'}^\dagger a_k = 0, \quad (16a)$$
$$\sqrt{q_{k'k}^*} \, a_k a_{k'} - \sqrt{q_{kk'}^*} \, a_{k'} a_k = 0. \quad (16b)$$

These commutators lead to further complications for finding the energy eigenvalues. Nevertheless, since the deformation parameter is symmetric, i.e., $q_{kk'} = q_{k'k}$, the commutator in Eq. (16b) simply reads $[a_k, a_{k'}] = 0$. This simplification allows us to construct the ground state in terms of the eigenstate of the annihilation operator a_k and hence to derive the ground state energy of the deformed parent Hamiltonian in Eq. (14) as

$$E_{\text{B}}^{\text{def}\,(0)} = -\sum_k \frac{|u_k|^2}{v_k}, \quad (17)$$

which, consequently, also identifies the ground state energy of the full Bogoliubov-Fröhlich Hamiltonian in the phonon branch.

Before presenting a numerical result, we would like to further note the following. It is known that the Bose polaron shows an infrared (IR) divergent behavior in the phonon branch, which requires certain regularization techniques, see, for instance, Ref. [13]. The explicit form of the ground state energy in Eq. (17) enables us to regularize the IR divergence straightforwardly by means of the Hadamard fine part method. For instance, by setting $p = 0$ and $m_{\text{B}} = 1$, we present a simple and explicit formula for the ground state energy of the Bose polaron:

$$E_{\text{B}}^{\text{def}\,(0)} = -\frac{g_{\text{IB}} c^2}{\Lambda^2} \frac{(\Lambda - g_{\text{IB}})^2}{\Lambda + g_{\text{IB}}} - \frac{g_{\text{IB}}^2 n_{\text{B}} \Lambda^2}{2c^2 (\Lambda + g_{\text{IB}})}, \quad (18)$$

where Λ is the momentum cutoff, until which the Bogoliubov dispersion is linear, and it can be identified via the healing length $\Lambda \sim 1/\xi$.

The formula in Eq. (18) shows explicitly that there exists a transition from a repulsive to an attractive polaron at the critical value of $g_{IB} = -\Lambda$, which is demonstrated in Fig. 2(a). This is qualitatively consistent with the previously reported results [11–13,15]. The divergent nature of the energy observed in the figure is a direct consequence of the broken quantum group symmetry at the vicinity of the critical value of g_{IB}, where the value of the parameter $\rho(k, k')$ deviates from the value of $kk'\alpha_k/u_k$, see the dashed yellow curve in Fig. 2(b). In fact, in this regime, $\rho(k, k') \propto (1 + g_{IB}/\Lambda)^{-1/2}$, and the bosonic expansion of the Hamiltonian in Eq. (14) cannot be truncated at the quartic order. Thereby, this is the regime where the picture of the q-deformed bosons breaks down. This is analogous to the so-called *quasiparticle instability*, at which the quasiparticle picture fails [40]. Moreover, as shown in Fig. 2(b), the behavior of the deformation parameter $q(k, k')$ changes as the coupling increases across the critical value. While the deformation is negative for the repulsive polaron, it becomes positive for the attractive one (solid blue curve). We also observe in the same figure that, far from the critical value, the deformed bosons coincide with the usual Bogoliubov bosons, as $q(k, k') \to 1$, so that the quantum group symmetry becomes trivial.

In conclusion, in the phonon branch of the Bogoliubov dispersion, where $k \ll m_B c$, the extended Bogoliubov-Fröhlich Hamiltonian in Eq. (2) can be written in terms of noninteracting q-deformed bosons, which allow us to derive the ground state energy of the Bose polaron analytically. Our result can be straightforwardly extended to three dimensions, and by including the omitted constant terms, the analytical result can be compared more quantitatively with an *ab initio* calculation. Lastly, as our approach has connected seemingly unrelated branches of many-body physics and mathematics, we anticipate that the Bose polaron and, in general, quantum impurities can be explored in different contexts where quantum groups play crucial roles. The deformation of the bosonic commutation relations due to the presence of an impurity might open horizons for anyons, the basis of topological quantum computing.

APPENDIX A: TRUNCATION OF THE DEFORMED HAMILTONIAN IN EQ. (7)

The deformed operator in Eq. (10) can be expanded as

$$a^\dagger = b^\dagger (1 + \rho\, b^\dagger b + \delta\, b^\dagger b^\dagger bb + \cdots), \qquad (A1)$$

where $\rho = \sqrt{[2]/2} - 1$ and $\delta = \sqrt{[3]/3}/2 - \sqrt{[2]/2} + 1/2$. Furthermore,

$$a^\dagger a^\dagger a^\dagger aa = \frac{[2]}{2}\sqrt{\frac{[3]}{3}}\, b^\dagger b^\dagger b^\dagger bb + \cdots. \qquad (A2)$$

Then without any approximation, the single-phonon-mode Hamiltonian in Eq. (5) can be written in terms of the deformed operators as

$$H'_{sm} = v\, a^\dagger a + u(a^\dagger + a)$$
$$- \frac{u\delta}{\sqrt{\frac{[3]}{3}\frac{[2]}{2}}}(a^\dagger a^\dagger a^\dagger aa + \text{H.c.}) + \cdots. \qquad (A3)$$

Now we define the first line as $H_{sm}^{def} = v\, a^\dagger a + u(a^\dagger + a)$ and the second line as H_1. The latter is negligible if

$$\zeta = \left|\frac{\langle H_1 \rangle}{\langle H_{sm}^{def} \rangle}\right| = \frac{2\delta}{\sqrt{\frac{[3]}{3}\frac{[2]}{2}}}\left(\frac{u}{v}\right)^4 + \cdots \ll 1, \qquad (A4)$$

where the expectation value is taken in the ground state of H_{sm}^{def}, which is given by the q-deformed coherent state $|-u/v\rangle$, see Eq. (B2).

APPENDIX B: DERIVATION OF THE EIGENVALUES IN EQ. (11)

To find the eigenvalues of the deformed Hamiltonian in Eq. (7), in the view of the form of the eigenstate of the corresponding nondeformed Hamiltonian, which is $vb^\dagger b + u(b^\dagger + b)$, we introduce the following state:

$$|\psi_n\rangle = \mathcal{N}_\psi^n \sum_{j=0}^{n} d_j^n (a^\dagger)^j |\beta_n\rangle, \qquad (B1)$$

where $\mathcal{N}_\psi^n$ is the normalization factor, $|\beta_n\rangle$ is the q-deformed coherent state, i.e., the eigenstate of the q-deformed annihilation operator:

$$|\beta_n\rangle = \mathcal{N}_\beta^n \sum_{m=0}^{\infty} \frac{\beta_n^m}{\sqrt{[m]!}} |m\rangle, \qquad (B2)$$

with $\mathcal{N}_\beta^n$ being the normalization factor. Here, $[m]! = [1], [2], ..., [m]$, where $[0]! = 1$ is the q factorial, and $|m\rangle$ is the eigenstate of the number operator N such that $N|m\rangle = [m]|m\rangle$. Then using the q-deformed algebra in Eq. (8), we find $\beta_n = -uq^{-n}/v$ and

$$d_j^n = \binom{n}{j}_q \left(\frac{u}{v}\right)^{n-j} q^{j(j-1)/2 - n(n-1)/2}, \qquad (B3)$$

where $\binom{n}{j}_q$ is the q-binomial coefficient given by $[n]!/([n-j]![j]!)$. Thus, the eigenstate in Eq. (B1) can be written as

$$|\psi_n\rangle = \mathcal{N}_\psi^n q^{-n(n-1)/2}\left(a^\dagger + \frac{u}{v}\right)_q^n |\beta_n\rangle, \qquad (B4)$$

where $(x + y)_q^n = \sum_{j=0}^{n} \binom{n}{j}_q x^j y^{n-j} q^{j(j-1)/2}$ is identified as the q-binomial theorem [41,42]. Finally, the nth eigenvalue can be derived to be

$$E_{sm}^{def\,(n)} = v[n] - \frac{1}{q^n}\frac{u^2}{v}. \qquad (B5)$$

In the limit of $q \to 1$, one recovers the well-known results for the corresponding nondeformed Hamiltonian $E_{sm}^{def\,(n)} \to vn - u^2/v$ and $|\psi_n\rangle \to \mathcal{N}_\psi^n (b^\dagger + u/v)^n |-u/v\rangle$.

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