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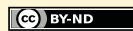
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# ON REDUCTION OF MODULI SCHEMES OF ABELIAN VARIETIES WITH DEFINITE QUATERNION MULTIPLICATIONS

by Chia-Fu YU (\*)

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ABSTRACT. — In this paper we make an initial study on type D moduli spaces in positive characteristic  $p \neq 2$ , where we allow the prime  $p$  to ramify in the defining datum. We classify explicitly the isogeny classes of  $p$ -divisible groups with additional structures in question. We also study the reduction of the type D moduli spaces of minimal rank.

RÉSUMÉ. — Dans cet article, nous faisons une étude initiale sur les espaces de modules de type D en caractéristique positive  $p \neq 2$ , où le nombre premier  $p$  peut être ramifié dans la donnée définissant l'espace de modules. Nous classifions explicitement les classes d'isogénie des groupes  $p$ -divisibles avec structures supplémentaires en question. Nous étudions également la réduction des espaces de modules de type D de rang minimal.

## 1. Introduction

### 1.1.

PEL moduli spaces parametrize abelian varieties with additional structures of polarizations, endomorphisms and level structures. When the adjoint group  $G^{\text{ad}}$  of the defining algebraic group  $G$  is  $\mathbb{Q}$ -simple, these moduli spaces are divided into types A, C and D according to the Dynkin diagram of  $G^{\text{ad}}$ . Previous studies of these moduli spaces and their integral models have mainly focused on the spaces of types A and C in the case of good reduction. There is comparatively less known about type D moduli spaces in the literature. Certain important results on all smooth PEL-type moduli

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spaces, which of course include the case of type D, have been obtained by Wedhorn [36, 37] and Moonen [15, 16, 17], where they consider the density of the  $\mu$ -ordinary locus and the Ekedahl–Oort (EO) strata. In this paper we study the type D moduli spaces in mainly positive characteristic and certain basic classification problems for abelian varieties and associated  $p$ -divisible groups with additional structures in question. A main point here is that we allow the prime  $p$  to be ramified in the definite quaternion algebra concerned. In the minimal rank case, we also exhibit a method for studying the case with arbitrary polarization degree.

Throughout this section let  $p$  denote an odd prime number. Let  $F$  be a totally real algebraic number field and  $O_F$  the ring of integers. Let  $B$  be a totally definite quaternion algebra over  $F$  and let  $*$  be the canonical involution on  $B$ , which is the unique positive involution on  $B$ . Let  $O_B$  be an  $O_F$ -order in  $B$  which is stable under the involution  $*$  and maximal at  $p$ , that is, the completion  $O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$  at  $p$  is a maximal order in the algebra  $B_p := B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ . A *polarized  $O_B$ -abelian variety*<sup>(1)</sup> is a tuple  $\underline{A} = (A, \lambda, \iota)$ , where  $(A, \lambda)$  is a polarized abelian variety and  $\iota : O_B \rightarrow \text{End}(A)$  is a ring monomorphism such that  $\lambda \circ \iota(b^*) = \iota(b)^t \circ \lambda$ , i.e. the map  $\iota$  is compatible with the involution  $*$  and the Rosati involution induced by the polarization  $\lambda$ . Clearly, this notion can be defined over any base scheme and one can study families of such objects.

Let  $m \geq 1$  be a positive integer, and let  $\mathcal{M} = \mathcal{M}_{m, O_B}$  be the coarse moduli scheme over  $\text{Spec } \mathbb{Z}_{(p)}$  of  $2m[F : \mathbb{Q}]$ -dimensional polarized  $O_B$ -abelian varieties  $\underline{A} = (A, \lambda, \iota)$ . Let  $\mathcal{M}^{(p)} \subset \mathcal{M}$  be the open and closed subscheme consisting of objects  $\underline{A}$  with prime-to- $p$  polarization degree. Both moduli spaces  $\mathcal{M}$  and  $\mathcal{M}^{(p)}$  are schemes locally of finite type. Let  $\mathcal{M}_K \subset \mathcal{M}$  (resp.  $\mathcal{M}_K^{(p)} \subset \mathcal{M}^{(p)}$ ) denote the closed subscheme parametrizing the objects  $\underline{A}$  in  $\mathcal{M}$  (resp.  $\mathcal{M}^{(p)}$ ) that satisfy the determinant condition; see Section 2.3. We shall call  $\mathcal{M}$  or one of its variants a moduli scheme (or moduli space) of type  $D_m$ . The fibers of these moduli schemes are non-empty; see Lemmas 2.2 and 2.3. The goal of the paper is to investigate the geometry of these moduli spaces. We make a detailed study on the moduli spaces of type  $D_1$  because this is the most basic and most accessible case, and some speculation indicates that this family behaves quite differently from the higher rank cases.

We explain the relation of the usual Shimura varieties of PEL-type D and the moduli spaces of type  $D_m$ . Let  $(V, \psi)$  be a  $\mathbb{Q}$ -valued non-degenerate

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<sup>(1)</sup>The author also used the terminology “abelian  $O_B$ -variety” for the same object in his earlier papers [39, 40, 41, 42, 44]. He apologizes for the inconsistency.

skew-Hermitian  $B$ -module of  $B$ -rank  $m$ . Let  $G := GU_B(V, \psi)$  denote the  $\mathbb{Q}$ -group of  $B$ -linear similitudes on  $(V, \psi)$ , and let  $X$  be the  $G(\mathbb{R})$ -conjugacy class of an  $\mathbb{R}$ -homomorphism  $h : \mathbb{C}^\times \rightarrow G_{\mathbb{R}}$  such that  $\psi(x, h(i)y)$  is definite on  $V_{\mathbb{R}}$ . Note that up to a central torus, the  $\overline{\mathbb{Q}}$ -group  $G_{\overline{\mathbb{Q}}} := G \otimes \overline{\mathbb{Q}}$  is isogenous to the product of copies of the even orthogonal group  $O_{2m}$ ; see Section 2.2. Let  $U \subset G(\mathbb{A}_f)$  be the open compact subgroup that stabilizes a lattice  $\Lambda_0$ . Then the associated Shimura variety  $\text{Sh}_U(G, X)_{\overline{\mathbb{Q}}}$  is an open and closed subscheme of  $\mathcal{M}_{\overline{\mathbb{Q}}}$ . Conversely, any irreducible component of  $\mathcal{M}_{\overline{\mathbb{Q}}}$  is isomorphic to a component of  $\text{Sh}_U(G, X)_{\overline{\mathbb{Q}}}$  for some  $(V, \psi)$  and  $U$  as above; see Section 2.5.

The main contents of this paper handle the following two basic problems:

- (a) Classify explicitly the isogeny classes of quasi-polarized  $p$ -divisible groups with additional structures (of arbitrary rank  $m$ ) in question.
- (b) Study the reduction of the moduli spaces of type  $D_1$ .

As the reader can see from known results on classical moduli spaces like Siegel or Hilbert moduli spaces, the results obtained so far for moduli spaces of type D are comparably much less. Wishfully we could obtain more results based on the present work. Below we illustrate our main results.

### 1.2. Part (a)

For any polarized  $O_B$ -abelian variety  $\underline{A} = (A, \lambda_A, \iota_A)$  over an algebraically closed field  $k$  of characteristic  $p$ , the associated  $p$ -divisible group  $(H, \lambda_H, \iota_H) := (A, \lambda_A, \iota_A)[p^\infty]$  with additional structures is a quasi-polarized  $p$ -divisible  $O_B \otimes \mathbb{Z}_p$ -module (see Section 5.1). We would like to determine the slope sequences and isogeny classes of these  $p$ -divisible groups with additional structures. As a first standard step, we decompose these  $p$ -divisible groups and study the same problem for each component independently. Write

$$(1.1) \quad F \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{v|p} F_v, \quad B \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{v|p} B_v, \quad O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p = \prod_{v|p} O_{B_v}.$$

Then we get a decomposition  $(H, \lambda_H, \iota_H) = \prod_{v|p} (H_v, \lambda_{H_v}, \iota_{H_v})$ . The slope sequence  $\underline{\nu}(\underline{A})$  of  $\underline{A}$  is then defined to be the collection  $(\nu(H_v))_{v|p}$  with slope sequences  $\nu(H_v)$  indexed by the set of places  $v|p$  of  $F$ . So we may study quasi-polarized  $p$ -divisible  $O_{B_v}$ -modules for each place  $v|p$  separately. We shall write  $\mathbf{B}$ ,  $\mathbf{F}$  and  $\mathcal{O}_{\mathbf{B}}$  for  $B_v$ ,  $F_v$  and  $O_{B_v}$ , respectively for brevity. In this part we do the following:

- (1) Study the structure of skew-Hermitian  $\mathcal{O}_{\mathbf{B}} \otimes W$ -modules and quasi-polarized Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -modules (see Section 5.1), where  $W$  denotes the ring of Witt vectors over  $k$ . See Sections 4 and 5.
- (2) Determine all possible slope sequences of quasi-polarized Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -modules of rank  $4dm$ , where  $d = [\mathbf{F} : \mathbb{Q}_p]$ . Moreover, we show that these slope sequences can be also realized by *separably* quasi-polarized Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -modules. See Theorems 6.4 and 7.3 for the precise statements; also see Corollaries 7.7 and 7.8 for the list of all possible slope sequences in the cases  $m = 1$  and  $m = 2$ .
- (3) Classify the isogeny classes of quasi-polarized Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -modules of rank  $4dm$ ; see Section 8.

The method of finding possible slope sequences in (2) uses a criterion for embeddings of a simple algebra into another one over a local field (see [45] and Section 6.3). This gives a description for possible slope sequences. Then we construct a separably quasi-polarized Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module realizing each possible slope sequence. The construction is divided into the supersingular part and non-supersingular part. For the supersingular part we can even write down a separably quasi-polarized superspecial Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module that also satisfies the determinant condition. For the non-supersingular part we use the “double construction”; see Lemma 7.1. The “double construction” easily produces a separable  $\mathcal{O}_{\mathbf{B}}$ -linear polarization. However, a Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module obtained in this manner rarely satisfies the determinant condition (see Remark 7.4). In fact, given a possible slope sequence  $\nu$  as in (2) (see Theorem 7.3 for a precise description), it is not always possible to construct a Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module  $M$  with slope sequence  $\nu$  which *both* admits a separable  $\mathcal{O}_{\mathbf{B}}$ -linear quasi-polarization and satisfies the determinant condition. We will discuss this in more detail in the minimal case later (cf. Theorem 1.2).

To classify the isogeny classes of the  $p$ -divisible groups with additional structures, it suffices to classify those with a fixed slope sequence  $\nu$ . Rapoport and Richartz [27] gave a cohomological description of this finite set  $I(\nu)$ . In fact, they treated a general case of connected groups while the groups here are not connected. (For more general classification by Galois cohomology which also includes non-connected groups, see Kottwitz [12].) Here we carry out a more elementary approach which involves only quadratic forms and Hermitian forms, and obtain more explicit results in terms of invariants (rather than cohomology classes). One can first reduce the case to where  $\nu$  is supersingular; see Lemma 8.1. Then we establish a bijection between the set  $I(\nu)$  ( $\nu$  is supersingular) and the set of isomorphism classes

of skew-Hermitian  $\mathbf{B}'$ -modules for a certain twisted quaternion  $\mathbf{F}$ -algebra  $\mathbf{B}'$ ; see Theorem 8.2. When  $\mathbf{B}'$  is the matrix algebra, one reduces to classifying quadratic forms over  $\mathbf{F}$ , and we apply the classical theory of quadratic forms over local fields (cf. O'Meara [20]). When  $\mathbf{B}'$  is the quaternion division algebra, we adopt the work of Tsukamoto [32].

### 1.3. Part (b)

In this part we restrict ourselves to the case where  $m = 1$ . Part (b) consists of Sections 10–13. A main result of this part states as follows.

**THEOREM 1.1.** — *Assume that  $m = 1$ .*

- (1) *Suppose that  $p$  is unramified in  $B$ , that is, the algebra  $B \otimes \mathbb{Q}_p$  is a product of matrix algebras over unramified field extensions of  $\mathbb{Q}_p$ . Then the moduli scheme  $\mathcal{M}_K \rightarrow \mathrm{Spec} \mathbb{Z}_{(p)}$  is flat and every connected component is projective of relative dimension zero.*
- (2) *The moduli scheme  $\mathcal{M}_K^{(p)} \rightarrow \mathrm{Spec} \mathbb{Z}_{(p)}$  is flat and every connected component is projective of relative dimension zero.*

Theorem 1.1 confirms a special case of the Rapoport–Zink conjecture on integral models of Shimura varieties; see [29]. Theorem 1.1(1) is proved in [38, Theorem 4.5]. The proof of Theorem 1.1(2), given in Section 10, uses local models (see Rapoport–Zink [29]). More precisely, let  $\mathbf{M}_\Lambda$  be the local model over  $\mathrm{Spec} \mathbb{Z}_p$  associated to a unimodular skew-Hermitian free  $\mathcal{O}_B \otimes \mathbb{Z}_p$ -module  $\Lambda$  (cf. Section 9.1). Using the Rapoport–Zink local model diagram, one can prove Theorem 1.1(2) by proving the flatness of  $\mathbf{M}_\Lambda$ . To prove the latter, we show that any point in  $\mathbf{M}_\Lambda(k)$ , where  $k$  is an algebraically closed field of characteristic  $p$ , can be lifted to characteristic zero and that all its geometric fibers are zero-dimensional.

We remark that the analog of the flatness result in Theorem 1.1(1) does not hold for higher  $m$ ; cf. [25, Example 8.3], which shows that flatness fails without the “spin condition” for  $m > 1$ . Example 8.2 of loc. cit. shows that the parahoric analog of the flatness result in Theorem 1.1(1) for  $m = 1$  does not hold without the “spin condition”, either.

For the possible slope sequences of objects in the case  $m = 1$ , we have the following result, which is a refinement of Theorem 7.3.

**THEOREM 1.2** (Theorem 11.3). — *Let  $M$  be a separably quasi-polarized Dieudonné  $\mathcal{O}_B$ -module of rank  $4d$  satisfying the determinant condition, where  $d = [\mathbf{F} : \mathbb{Q}_p]$ .*

(1) If  $\mathbf{B}$  is the matrix algebra, then

$$\nu(M) = \left\{ \left( \frac{a}{d} \right)^{2d}, \left( \frac{d-a}{d} \right)^{2d} \right\},$$

where  $a$  can be any integer with  $0 \leq a < d/2$ , or

$$\nu(M) = \left\{ \left( \frac{1}{2} \right)^{4d} \right\}.$$

(2) If  $\mathbf{B}$  is the division algebra, then

$$\nu(M) = \left\{ \left( \frac{a}{2d} \right)^{2d}, \left( \frac{2d-a}{2d} \right)^{2d} \right\},$$

where  $a$  can be any odd integer with  $2 \lfloor e/2 \rfloor f \leq a < d$ , or

$$\nu(M) = \left\{ \left( \frac{1}{2} \right)^{4d} \right\}.$$

Here  $e$  and  $f$  are the ramification index and the inertia degree of  $\mathbf{F}$  over  $\mathbb{Q}_p$ , respectively.

In the remainder of this part (Sections 12 and 13) we limit ourselves to the case  $F = \mathbb{Q}$ . We determine the dimensions of the special fibers of various moduli schemes as above.

**THEOREM 1.3** (Theorem 13.1 and Proposition 13.7). — Assume that  $m = 1$  and  $F = \mathbb{Q}$ . Let  $\mathcal{M}_{\overline{\mathbb{F}}_p}, \mathcal{M}_{\overline{\mathbb{F}}_p}^{(p)}$ , and  $\mathcal{M}_{K, \overline{\mathbb{F}}_p}$  be the geometric special fibers of  $\mathcal{M}, \mathcal{M}^{(p)}$  and  $\mathcal{M}_K$ , respectively.

- (1) If  $p$  is unramified in  $B$ , then  $\dim \mathcal{M}_{\overline{\mathbb{F}}_p} = 0$ .
- (2) If  $p$  is ramified in  $B$ , then  $\dim \mathcal{M}_{\overline{\mathbb{F}}_p} = 1$ .
- (3) We have  $\dim \mathcal{M}_{\overline{\mathbb{F}}_p}^{(p)} = 0$ .
- (4) If  $p$  is ramified in  $B$ , then  $\dim \mathcal{M}_{K, \overline{\mathbb{F}}_p} = 1$ .

We explain the ideas of the proof. For (1) any object  $\underline{A} = (A, \lambda, \iota) \in \mathcal{M}(k)$  is either ordinary or superspecial. For the first case we use the canonical lifting for ordinary abelian varieties and the fact that the generic fiber has dimension zero. For the superspecial case, we use the fact that the superspecial locus has dimension zero. For (2) and (4), we construct a  $\mathbf{P}^1$ -family of supersingular polarized  $O_B$ -abelian surfaces. This produces a one-dimensional family in the moduli space  $\mathcal{M}_{\overline{\mathbb{F}}_p}$ . A close examination shows that this family actually lands in the locus  $\mathcal{M}_{K, \overline{\mathbb{F}}_p} \subset \mathcal{M}_{\overline{\mathbb{F}}_p}$ ; see

Lemma 13.6. This gives a lower bound for the dimensions of our moduli spaces:

$$1 \leq \dim \mathcal{M}_{K, \overline{\mathbb{F}}_p} \leq \dim \mathcal{M}_{\overline{\mathbb{F}}_p}.$$

For the other bound, we consider the finite morphism  $f : \mathcal{M}_{\overline{\mathbb{F}}_p} \rightarrow \mathcal{A}_{2, \overline{\mathbb{F}}_p}$  to the moduli space  $\mathcal{A}_{2, \overline{\mathbb{F}}_p}$  of polarized abelian surfaces, through forgetting the endomorphism structure. As  $p$  is ramified, every object in  $\mathcal{M}(k)$  is supersingular (cf. Corollary 7.7), and hence the whole space  $\mathcal{M}_{\overline{\mathbb{F}}_p}$  is supersingular. The image of  $\mathcal{M}_{\overline{\mathbb{F}}_p}$  in  $\mathcal{A}_{2, \overline{\mathbb{F}}_p}$  then lands in the supersingular locus  $\mathcal{S}_2$  of  $\mathcal{A}_{2, \overline{\mathbb{F}}_p}$ . Then we use a result of Norman–Oort [19] (also cf. Katsura and Oort [9] for the principally polarized case) that  $\dim \mathcal{S}_2 = 1$  and get the other bound  $\dim \mathcal{M}_{\overline{\mathbb{F}}_p} \leq 1$ . This proves (2) and (4). Note that the result

$$\dim \mathcal{S}_2 = \dim \mathcal{A}_{2, \overline{\mathbb{F}}_p}^{(0)} = 1$$

we use is a special case of a theorem of Norman and Oort stating that the  $p$ -rank zero locus  $\mathcal{A}_{g, \overline{\mathbb{F}}_p}^{(0)}$  of the Siegel moduli space  $\mathcal{A}_{g, \overline{\mathbb{F}}_p}$  has co-dimension  $g$ . For (3) we only need to treat the case where  $p$  is ramified. In this case we show that any separably quasi-polarized  $O_B \otimes \mathbb{Z}_p$ -Dieudonné module (of rank  $4d$ ) is superspecial, and again use the dimension of the superspecial locus.

*Remark 1.4.* — We end this part with a few remarks about Theorem 1.3.

- (1) Theorem 1.3(3) yields another proof of the result  $\dim \mathcal{M}_{K, \overline{\mathbb{F}}_p}^{(p)} = 0$ , which follows from Theorem 1.1.
- (2) When  $p$  is ramified in  $B$ , both  $\mathcal{M}_{\overline{\mathbb{F}}_p}$  and  $\mathcal{M}_{K, \overline{\mathbb{F}}_p}$  contain components of dimension zero and one. This follows from the result that both  $\mathcal{M}_{\overline{\mathbb{F}}_p}^{(p)}$  and  $\mathcal{M}_{K, \overline{\mathbb{F}}_p}^{(p)}$  are zero-dimensional and non-empty (by Theorem 1.3(3) and Lemma 2.3).
- (3) Suppose  $p$  is ramified in  $B$ . By Theorem 1.3(2) and (4), we conclude that the moduli schemes  $\mathcal{M}$  and  $\mathcal{M}_K$  are not flat over  $\text{Spec } \mathbb{Z}_{(p)}$ . Moreover, there is a (non-separably) polarized  $O_B$ -abelian surface satisfying the determinant condition which can not be lifted to characteristic zero.
- (4) When  $p$  is unramified in  $B$ , we have  $\mathcal{M}^{(p)} = \mathcal{M}_K^{(p)}$  and hence the moduli scheme  $\mathcal{M}^{(p)}$  is flat over  $\text{Spec } \mathbb{Z}_{(p)}$ . When  $p$  is ramified in  $B$ , we construct a superspecial prime-to- $p$  degree polarized  $O_B$ -abelian surface which does not satisfy the determinant condition; see Lemma 13.3. In particular, this point can not be lifted to characteristic zero. This shows that the inclusion  $\mathcal{M}_K^{(p)}(k) \subset \mathcal{M}^{(p)}(k)$  is strict. This phenomenon is different from the reduction modulo



$p$  of Hilbert moduli schemes or Hilbert–Siegel moduli schemes. In the Hilbert–Siegel case, any separably polarized abelian varieties with RM by  $O_F$  of a totally real algebraic number field  $F$  satisfy the determinant condition automatically; see Yu [39], Görtz [3] and Vollaard [35].

- (5) To generalize Theorem 1.3 to the case where  $B$  is a quaternion algebra over any totally real field  $F$ , one can construct Moret–Bailly families to get a lower bound for the dimensions. However, we do not know how to produce a good upper bound.

Local models for orthogonal groups of general rank have been studied in Smithling [30, 31] in the case when  $p$  is unramified in  $B$ . Vasiu [33, 34] shows the existence of the canonical integral model of certain Shimura varieties of PEL-type D over  $\mathbb{Z}_{(2)}$ . He and Rapoport [7] show non-emptiness of various stratifications (NP, KR and EO) in the reduction of Shimura varieties under certain axioms. Rapoport and Viehmann [28] studies local analogues of Shimura varieties (which are RZ spaces in the PEL-type cases) and explore non-emptiness of these spaces and the Kottwitz set  $B(G, \{\mu\})$  (see [12, the definition]).

We make a remark on the assumption of  $p$ . Our primary goal was to establish as many results as we can only limited to the assumption  $p \neq 2$ . However, as far as we can see, several pointwise statements (in terms of Dieudonné modules or isocrystals), as long as which do not rely on the classification up to isomorphism, do not require this assumption. After a close examination, we do not need to assume this in Sections 2–8, except for Section 4.4, Lemma 5.1, Corollary 5.7 and Proposition 8.8. The assumption  $p \neq 2$  is needed for the construction of the local model diagram by Rapoport and Zink (see [29, p. 75 and Theorem 3.16]). Thus, we make this assumption starting from Section 9.

## 2. Moduli spaces

### 2.1. Moduli spaces

Let  $p$  be a prime number. Let  $F$  be a totally real field of degree  $d = [F : \mathbb{Q}]$  and  $O_F$  the ring of integers. Let  $B$  be a totally definite quaternion algebra over  $F$  and let  $*$  be the canonical involution on  $B$ , which is the unique positive involution on  $B$ . Let  $O_B \subset B$  be an  $O_F$ -order in  $B$  which is stable under the involution  $*$  and maximal at  $p$ , that is,  $O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$  is a maximal

order in the algebra  $B_p := B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ . A polarized  $O_B$ -abelian scheme over a base scheme  $S$  is a tuple  $(A, \lambda, \iota)$ , where

- $(A, \lambda)$  is a polarized abelian scheme over  $S$ , and
- $\iota : O_B \rightarrow \text{End}_S(A)$  is a ring monomorphism that satisfies the compatibility condition  $\lambda \circ \iota(b^*) = \iota(b)^t \circ \lambda$  for all  $b \in O_B$ .

The pair  $(A, \iota)$ , where  $A$  and  $\iota$  are as above, is called an  $O_B$ -abelian scheme. A polarization  $\lambda$  on an  $O_B$ -abelian scheme  $(A, \iota)$  satisfying the above compatibility condition is said to be  $O_B$ -linear. Similar objects can be also defined when the algebra  $B$  is replaced by an arbitrary semi-simple  $\mathbb{Q}$ -algebra together with a positive involution.

Let  $m \geq 1$  be a positive integer, and let  $\mathcal{M} = \mathcal{M}_{m, O_B}$  be the coarse moduli scheme over  $\text{Spec } \mathbb{Z}_{(p)}$  of  $2dm$ -dimensional polarized  $O_B$ -abelian varieties  $(A, \lambda, \iota)$ . Let  $\mathcal{M}^{(p)} \subset \mathcal{M}$  be the subscheme parametrizing the objects in  $\mathcal{M}$  which have a prime-to- $p$  degree polarization. The moduli spaces  $\mathcal{M}$  and  $\mathcal{M}^{(p)}$  are schemes both locally of finite type. Each of them is an union of infinitely many open and closed subschemes which are of finite type:

$$(2.1) \quad \mathcal{M} = \coprod_{D \geq 1} \mathcal{M}_D, \quad \mathcal{M}^{(p)} = \coprod_{D \geq 1, p \nmid D} \mathcal{M}_D,$$

where  $D$  runs through all positive integers and  $\mathcal{M}_D \subset \mathcal{M}$  is the subscheme parametrizing the objects  $(A, \lambda, \iota)$  with polarization degree  $\deg \lambda = D^2$ .

### 2.2. Study of $\mathcal{M}_{\mathbb{C}}$ .

Let  $(V, \psi)$  be a  $\mathbb{Q}$ -valued non-degenerate skew-Hermitian  $B$ -module of  $B$ -rank  $m$ . That is,  $\psi : V \times V \rightarrow \mathbb{Q}$  is a non-degenerate alternating pairing such that  $\psi(ax, y) = \psi(x, a^*y)$  for all  $a \in B$  and  $x, y \in V$ . Let  $G := GU_B(V, \psi)$  denote the group of  $B$ -linear similitudes on  $(V, \psi)$  over  $\mathbb{Q}$ . For any commutative  $\mathbb{Q}$ -algebra  $R$ , the group of  $R$ -valued points is given by

$$(2.2) \quad G(R) = \{g \in \text{GL}_{B \otimes_{\mathbb{Q}} R}(V \otimes_{\mathbb{Q}} R) \mid c(g) = g'g \in R^\times\},$$

where  $g \mapsto g'$  is the adjoint involution with respect to  $\psi$ . Let  $G_1 = U_B(V, \psi)$  be the kernel of the multiplier homomorphism  $c : G \rightarrow \mathbb{G}_m$ ; one has a short exact sequence of algebraic  $\mathbb{Q}$ -groups

$$(2.3) \quad 1 \longrightarrow G_1 \longrightarrow G \xrightarrow{c} \mathbb{G}_m \longrightarrow 1.$$

One can easily show that

$$(2.4) \quad G_{\overline{\mathbb{Q}}} := G \otimes \overline{\mathbb{Q}} \\ \simeq \left\{ (A_i) \in \mathrm{GL}_{2m, \overline{\mathbb{Q}}}^d \mid A_i \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix} A_i^t = c \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix} \text{ for all } i \right\}.$$

A simple calculation shows that for  $t \in \mathbb{G}_m$ , one has

$$\left( \begin{bmatrix} I_m & 0 \\ 0 & tI_m \end{bmatrix}, \dots, \begin{bmatrix} I_m & 0 \\ 0 & tI_m \end{bmatrix} \right) \in G_{\overline{\mathbb{Q}}}.$$

This gives a section of  $c$  over  $\overline{\mathbb{Q}}$ . Thus, one has  $G_{1, \overline{\mathbb{Q}}} \simeq O_{2m}^d$  and  $G_{\overline{\mathbb{Q}}} \simeq O_{2m}^d \times \mathbb{G}_m$  over  $\overline{\mathbb{Q}}$ . In particular, both  $G_1$  and  $G$  have  $2^d$  connected components. One can also easily show that

$$(2.5) \quad Z(G)(\overline{\mathbb{Q}}) = \left\{ x \in (F \otimes \overline{\mathbb{Q}})^\times \mid x^2 \in \overline{\mathbb{Q}}^\times \right\}, \\ Z(G) = Z(G^0), \quad \text{if } m > 1,$$

and that for  $m = 1$ , the group  $G^0$  is a torus of dimension  $d + 1$ .

There is a unique  $B$ -valued skew-Hermitian pairing  $\psi_B : V \times V \rightarrow B$ , i.e.

$$\psi_B(a_1x, a_2y) = a_1\psi_B(x, y)a_2^*, \quad \forall a_1, a_2 \in B, \quad x, y \in V,$$

such that  $\psi(x, y) = \mathrm{Trd} \psi_B(x, y)$ , where  $\mathrm{Trd}$  is the reduced trace from  $B$  to  $\mathbb{Q}$ . Note that the property  $\psi_B(ax, y) = \psi_B(x, a^*y)$  for all  $a \in B$  and  $x, y \in V$  does not hold anymore as  $B$  is not commutative. We can choose an orthogonal basis  $\{e_i\}$  for  $\psi_B$  and put  $b_i := \psi_B(e_i, e_i)$ . Then  $b_i^* = -b_i$  for  $i = 1, \dots, m$  and

$$\psi \left( \sum_{i=1}^m x_i e_i, \sum_{i=1}^m y_i e_i \right) = \sum_{i=1}^m \mathrm{Trd}(x_i b_i y_i^*).$$

For any anti-symmetric element  $b \in B^\times$ , i.e.  $b^* = -b$ , we define a ( $\mathbb{Q}$ -valued) rank-one skew-Hermitian  $B$ -module  $(B, \psi_b)$ , where  $\psi_b(x, y) := \mathrm{Trd}(xby^*)$ . Then we have a decomposition of skew-Hermitian  $B$ -modules

$$(2.6) \quad (V, \psi) = \bigoplus_{i=1}^m (B, \psi_{b_i}).$$

LEMMA 2.1. — *There is a  $B \otimes \mathbb{R}$ -linear complex structure  $J_0$  on  $V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R}$  such that  $\psi(J_0x, J_0y) = \psi(x, y)$  for  $x, y \in V_{\mathbb{R}}$  and the symmetric bilinear form  $(x, y) := \psi(x, J_0y)$  is negative definite.*

*Proof.* — By (2.6) we may assume that  $m = 1$  and  $(V, \psi) = (B, \psi_b)$ . Let  $J_0$  be the right multiplication of the element  $b/\sqrt{\mathrm{Nr}_{B/F}(b)}$  in  $B \otimes \mathbb{R}$ , where  $\mathrm{Nr}_{B/F}$  is the reduced norm from  $B$  to  $F$ . Then one obtains  $\psi_b(x, J_0y) = -\mathrm{Trd}(xy^*)$ , which is negative definite. □

We call a complex structure  $J_0$  as in Lemma 2.1 an *admissible* complex structure on  $(V_{\mathbb{R}}, \psi)$ . The group  $G_1(\mathbb{R})$  of real points acts transitively on the set of all admissible complex structures on  $(V_{\mathbb{R}}, \psi)$  by conjugation (see [11, Lemma 4.3]). It is well known that the Hermitian symmetric space

$$X_1 := G_1(\mathbb{R})/K_\infty$$

has dimension  $dm(m-1)/2$ , where  $K_\infty$  is the stabilizer of a fixed admissible complex structure  $J_0$ . Fix an  $O_B$ -lattice  $\Lambda$  such that  $\psi(\Lambda, \Lambda) \subset \mathbb{Z}$  and let  $\Gamma_\Lambda \subset G_1(\mathbb{Q})$  be the arithmetic subgroup which stabilizes the lattice  $\Lambda$ . The natural map  $g \mapsto (V_{\mathbb{R}}/\Lambda, \text{Int}(g)J_0, \psi)$  induces an open and closed immersion of analytic spaces

$$\Phi_{(\Lambda, \psi)} : \Gamma_\Lambda \backslash X_1 \hookrightarrow \mathcal{M}(\mathbb{C}).$$

Let  $\mathcal{M}_{(\Lambda, \psi)}$  denote the open and closed subscheme of  $\mathcal{M}_{\mathbb{C}}$  over  $\mathbb{C}$  whose underlying space is the image of  $\Phi_{(\Lambda, \psi)}$ . Then we have a decomposition of  $\mathcal{M}_{\mathbb{C}}$  into open and closed subschemes

$$(2.7) \quad \mathcal{M}_{\mathbb{C}} = \coprod_{(\Lambda, \psi)} \mathcal{M}_{(\Lambda, \psi)},$$

where  $(\Lambda, \psi)$  runs through the isomorphism classes of all  $\mathbb{Z}$ -valued non-degenerate skew-Hermitian  $O_B$ -lattices of rank  $m$ . We will see that each subscheme  $\mathcal{M}_{(\Lambda, \psi)}$  is irreducible (Proposition 2.7).

LEMMA 2.2.

- (1) *There is an anti-symmetric element  $b \in B^\times$  such that (a)  $\psi_b(O_B, O_B) \subset \mathbb{Z}$  and (b)  $O_B \otimes \mathbb{Z}_p$  is a self-dual lattice with respect to  $\psi_b$ , where  $\psi_b(x, y) := \text{Trd}(xy^*)$ .*
- (2) *The moduli space  $\mathcal{M}_{\mathbb{C}}^{(p)}$  is non-empty.*

*Proof.* — (1) We have the decomposition  $O_B \otimes \mathbb{Z}_p = \bigoplus_{v|p} \mathcal{O}_{B_v}$  with respect to  $O_F \otimes \mathbb{Z}_p = \prod_{v|p} \mathcal{O}_v$ . We first show that for each place  $v$  of  $F$  over  $p$ , one can choose an anti-symmetric element  $b_v \in B_v^\times$  such that  $\psi_{b_v}$  is a  $\mathbb{Z}_p$ -valued perfect paring on  $\mathcal{O}_{B_v}$ . When  $v$  is unramified in  $B$ , we are reduced to finding a  $\mathbb{Z}_p$ -valued perfect symmetric pairing on  $\mathcal{O}_v^2$  which clearly exists, and let  $b_v$  be the element corresponding to this perfect pairing. When  $v$  is ramified in  $B$ , one may choose a prime element  $\Pi_v$  of  $B_v$  such that  $\Pi_v^2$  is a uniformizer of  $F_v$ , and let  $b_v := \delta_v \Pi_v^{-1}$ , where  $\delta_v$  is a generator of the inverse different  $\mathcal{D}_{F_v/\mathbb{Q}_p}^{-1}$ .

Using weak approximation, there is an anti-symmetric element  $b \in B^\times$  close to  $b_v$  for each place  $v|p$ . Replacing  $b$  by a prime-to- $p$  multiple  $bM$  of  $b$ , one gets a pairing  $\psi_b$  that satisfies the desired properties.

(2) Choose a pairing  $\psi = \psi_b$  on  $B$  as in (1). Then the triple  $(V_{\mathbb{R}}/O_B, J_0, \psi)$  defines a  $2d$ -dimensional polarized complex  $O_B$ -abelian variety  $(A_0, \lambda_0, \iota_0)$ . Put  $\underline{A} = (A_0^m, \lambda_0^m, \iota)$ , where  $\iota : O_B \rightarrow \text{End}(A_0^m)$  is the diagonal embedding. This gives an object in  $\mathcal{M}^{(p)}(\mathbb{C})$  and hence that  $\mathcal{M}_{\mathbb{C}}^{(p)}$  is non-empty. □

### 2.3. Moduli spaces with the determinant condition

Let  $(V, \psi)$  be any skew-Hermitian  $B$ -module of  $B$ -rank  $m$ . Let  $J_0$  be an admissible complex structure on  $V_{\mathbb{R}}$ . Let  $V_{\mathbb{C}} = V^{-1,0} \oplus V^{0,-1}$  be the eigenspace decomposition where  $J_0$  acts respectively by  $i$  and  $-i$  on  $V^{-1,0}$  and  $V^{0,-1}$ . Let  $\Sigma := \text{Hom}(F, \mathbb{R}) = \text{Hom}(F, \mathbb{C})$  be the set of real embeddings of  $F$ . We have

$$B_{\mathbb{C}} := B \otimes_{\mathbb{Q}} \mathbb{C} = \prod_{\sigma \in \Sigma} B_{\sigma}, \quad B_{\sigma} := B \otimes_{F, \sigma} \mathbb{C} \simeq \text{Mat}_2(\mathbb{C}).$$

The action of  $B_{\mathbb{C}}$  on  $V^{-1,0}$  gives the decomposition

$$V^{-1,0} = \sum_{\sigma \in \Sigma} V_{\sigma},$$

where each subspace  $V_{\sigma}$  is a  $B_{\sigma}$ -module of  $\mathbb{C}$ -dimension  $2m$  and it is isomorphic to the direct sum of  $m$ -copies of the simple  $B_{\sigma}$ -module  $\mathbb{C}^2$ . If  $a \in B$ , then the characteristic polynomial of  $a$  on  $V_{\sigma}$  is equal to  $\sigma(\text{char}_F(a)^m)$ , where  $\text{char}_F(a) \in F[T]$  is the reduced characteristic polynomial of  $a$ . Therefore, the characteristic polynomial of  $a \in B$  on  $V^{-1,0}$  is given by

$$(2.8) \quad \text{char}(a|V^{-1,0}) = \prod_{\sigma \in \Sigma} \sigma(\text{char}_F(a))^m = \text{char}(a)^m \in \mathbb{Q}[T],$$

where  $\text{char}(a) = N_{F/\mathbb{Q}} \text{char}_F(a) \in \mathbb{Q}[T]$  is the reduced characteristic polynomial of  $a$  from  $B$  to  $\mathbb{Q}$ .

Let  $\underline{A} = (A, \lambda, \iota)_S \in \mathcal{M}(S)$  be a polarized  $O_B$ -abelian scheme over  $S$ , where  $S$  is a connected locally Noetherian  $\mathbb{Z}_{(p)}$ -scheme. Since the Lie algebra  $\text{Lie}(A)$  is a locally free  $\mathcal{O}_S$ -module, the characteristic polynomial  $\text{char}(\iota(a)|\text{Lie}(A))$ , for any element  $a \in O_B$ , is defined and it is a polynomial in  $\mathcal{O}_S[T]$  of degree  $2dm$ . The *determinant condition* for  $\underline{A}$  is the equality of the following two polynomials

$$(2.9) \quad (\text{K}) \quad \text{char}(\iota(a)|\text{Lie}(A)) = \text{char}(a)^m \in \mathcal{O}_S[T], \quad \forall a \in O_B.$$

This is a closed condition and it depends on  $m$ , but not on the choice of  $(V, \psi)$ .

Let  $\mathcal{M}_K \subset \mathcal{M}$  (resp.  $\mathcal{M}_K^{(p)} \subset \mathcal{M}^{(p)}$ ) denote the closed subscheme parametrizing the objects  $\underline{A}$  in  $\mathcal{M}$  (resp.  $\mathcal{M}^{(p)}$ ) that satisfy the determinant condition (K).

Let  $L \subset B$  be a maximal subfield such that any place  $v$  of  $F$  lying over  $p$  is unramified in  $L$  and that the order  $L \cap O_B$  is maximal at  $p$ . We can construct such a maximal subfield  $L$  by

- (a) constructing a maximal commutative semi-simple subalgebra  $\mathbf{L} \subset B \otimes \mathbb{Q}_p$  such that  $\mathbf{L}$  is the unramified quadratic extension of  $F \otimes \mathbb{Q}_p$  and the maximal order  $\mathcal{O}_{\mathbf{L}}$  of  $\mathbf{L}$  is contained in  $O_B \otimes \mathbb{Z}_p$ , and
- (b) applying weak approximation.

LEMMA 2.3.

- (1) The moduli space  $\mathcal{M}_{K, \mathbb{Q}}^{(p)} := \mathcal{M}_K^{(p)} \otimes \mathbb{Q}$  is non-empty.
- (2) The special fiber  $\mathcal{M}_{K, \overline{\mathbb{F}}_p}^{(p)} := \mathcal{M}_K^{(p)} \otimes \overline{\mathbb{F}}_p$  is non-empty.

*Proof.*

(1) This follows from the fact that the determinant condition for any polarized  $O_B$ -abelian variety in characteristic zero holds automatically, and Lemma 2.2(2).

(2) We may assume that  $m = 1$  by the same reduction step as we show Lemma 2.2(2). In this case,  $\mathcal{M}(\mathbb{C}) = \mathcal{M}(\overline{\mathbb{Q}})$  as each subscheme  $\mathcal{M}_D \otimes \mathbb{Q}$  is zero-dimensional and of finite type. By Grothendieck’s semi-stable reduction theorem [5, Chapter IX, Theorem 3.6] <sup>(2)</sup>, any object  $(A, \lambda, \iota) \in \mathcal{M}(\overline{\mathbb{Q}}_p)$  has good reduction, due to the fact that the  $\mathbb{Z}$ -rank of  $O_B$  is larger than  $\dim A$ . Since  $\mathcal{M}^{(p)}(\overline{\mathbb{Q}}_p) = \mathcal{M}_K^{(p)}(\overline{\mathbb{Q}}_p)$  is non-empty by (1), the reduction modulo  $p$  of some point  $y \in \mathcal{M}_K^{(p)}(\overline{\mathbb{Q}}_p)$  gives a point  $x$  which is contained in  $\mathcal{M}_K^{(p)}(\overline{\mathbb{F}}_p)$ , because  $x \in \overline{\{y\}}$  and (K) is a closed condition. Thus,  $\mathcal{M}_{K, \overline{\mathbb{F}}_p}^{(p)}$  is non-empty. □

### 2.4. Study of $\mathcal{M}_{(\Lambda, \psi)}$ .

Let  $\mathbb{H}$  denote the real Hamilton quaternion algebra. One has

$$(2.10) \quad \mathbb{H} = \mathbb{C} + \mathbb{C}\mathbf{j}, \quad \mathbf{j}a = \bar{a}\mathbf{j}, \quad a \in \mathbb{C}.$$

---

<sup>(2)</sup> We refer to the Math. Review of Artin–Winters [1] for a historic background of this theorem.

It is a standard fact that any non-degenerate skew-Hermitian  $\mathbb{H}$ -module of rank  $m$  is isomorphic to  $(\mathbb{H}^m, \psi_0)$ , where  $\psi_0(e_i, e_j) = \mathbf{j} \delta_{i,j}$ . Put  $\mathbf{J}_m := \text{diag}(\mathbf{j}, \dots, \mathbf{j}) \in \text{Mat}_m(\mathbb{H})$ . We extend the canonical involution  $*$  on  $\text{Mat}_m(\mathbb{H})$  by  $(a_{ij})^* = (a'_{ij})$ , where  $a_{ij} \in \mathbb{H}$  and  $a'_{ij} := a_{ji}^*$ . Let  $\text{O}_{2m}^*$  denote the algebraic  $\mathbb{R}$ -group of isometries of  $(\mathbb{H}^m, \psi_0)$ ; one has

$$(2.11) \quad \text{O}_{2m}^*(\mathbb{R}) = \{A \in \text{GL}_m(\mathbb{H}) \mid A\mathbf{J}_m A^* = \mathbf{J}_m\}.$$

The group  $G_1 \otimes \mathbb{R}$  is isomorphic to  $\prod_{\sigma \in \Sigma} \text{O}_{2m}^*$ .

LEMMA 2.4. — *One has  $G_1(\mathbb{R}) = G_1^0(\mathbb{R})$ ,  $G(\mathbb{R}) = G^0(\mathbb{R})$  and  $c(G(\mathbb{R})) = \mathbb{R}^\times$ .*

*Proof.* — We first show that for any element  $g \in \text{GL}_m(\mathbb{H})$  one has  $\text{Nrd}(g) > 0$ . Since the set  $\text{GL}_m(\mathbb{H})^{\text{ss}} \subset \text{GL}_m(\mathbb{H})$  of semi-simple elements is open and dense in the classical topology, it suffices to show  $\text{Nrd}(g) > 0$  for  $g \in \text{GL}_m(\mathbb{H})^{\text{ss}}$ . As such  $g$  is contained in a maximal commutative semi-simple subalgebra of  $\text{Mat}_m(\mathbb{H})$ , which is isomorphic to  $\mathbb{C}^m$ , one has  $\text{Nrd}(g) > 0$ .

It follows that

$$(2.12) \quad \text{O}_{2m}^*(\mathbb{R}) = \{A \in \text{GL}_m(\mathbb{H}) \mid A\mathbf{J}_m A^* = \mathbf{J}_m, \text{Nrd}(A) = 1\}.$$

Let  $\text{SO}_{2m}^* = \{A \in \text{O}_{2m}^* \mid \text{Nrd}(A) = 1\}$ . The group  $\text{SO}_{2m}^*$  is a form of  $\text{SO}_{2m}$  and hence it is the neutral component of  $\text{O}_{2m}^*$ . Therefore,

$$(2.13) \quad G_1(\mathbb{R}) = \prod_{\sigma} \text{O}_{2m}^*(\mathbb{R}) = \prod_{\sigma} \text{SO}_{2m}^*(\mathbb{R}) = G_1^0(\mathbb{R}).$$

For the second statement, if  $A\mathbf{J}_m A^* = c\mathbf{J}_m$  for some  $c \in \mathbb{R}^\times$ , then  $\text{Nrd}(A/c^m) > 0$  and one has  $G(\mathbb{R}) = G^0(\mathbb{R})$ . For the last statement we just need to find an element  $g$  such that  $c(g) < 0$ . Consider diagonal elements  $x = \text{diag}(y, \dots, y), y \in \mathbb{H}^\times$  and we are reduced to showing this in the case where  $m = 1$ . In this case one has  $\mathbf{j}\mathbf{i}^* = -\mathbf{j}$  and hence  $c(\mathbf{i}) = -1$ . This proves the Lemma 2.4. □

LEMMA 2.5.

- (1) *The Lie group  $G_1(\mathbb{R})$  is connected.*
- (2) *The Lie group  $G(\mathbb{R})$  has two connected components with the neutral component*

$$G(\mathbb{R})^+ = \{g \in G(\mathbb{R}) \mid c(g) > 0\}.$$

*Proof.*

(1) It suffices to show that the Lie group  $O_{2m}^*(\mathbb{R}) = SO_{2m}^*(\mathbb{R})$  is connected. We embed  $\mathbb{H} \hookrightarrow \text{Mat}_2(\mathbb{C})$  as

$$\mathbb{H} \ni a + b\mathbf{j} \mapsto \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \text{Mat}_2(\mathbb{C}),$$

and have  $\text{Mat}_m(\mathbb{H}) \subset \text{Mat}_{2m}(\mathbb{C})$ . Let  $J$  and  $J_m$  be the image of  $\mathbf{j}$  and  $\mathbf{J}_m$  in  $\text{Mat}_2(\mathbb{C})$  and  $\text{Mat}_{2m}(\mathbb{C})$ , respectively. Clearly,  $J^t = -J$  and  $J_m^{-1} = -J_m$ . The complex conjugation on  $\text{Mat}_2(\mathbb{C})$  coming from the  $\mathbb{R}$ -structure of  $\mathbb{H}$  is given by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto J\bar{A}J^{-1} = \begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix}$$

where  $A \mapsto \bar{A}$  is the usual complex conjugation. Thus, we can recover  $\text{Mat}_m(\mathbb{H})$  from  $\text{Mat}_{2m}(\mathbb{C})$  by

$$(2.14) \quad \text{Mat}_m(\mathbb{H}) = \{A \in \text{Mat}_{2m}(\mathbb{C}) \mid J_m \bar{A} J_m^{-1} = A\}.$$

We have  $A^* = \bar{A}^t$  for  $A \in \text{Mat}_m(\mathbb{H})$ . By (2.11) and (2.14), one gets

$$(2.15) \quad O_{2m}^*(\mathbb{R}) = \left\{ A \in \text{Mat}_{2m}(\mathbb{C}) \mid AJ_m \bar{A}^t = J_m, J_m \bar{A} J_m^{-1} = A, \det(A) = 1 \right\}.$$

The first condition in (2.15) gives

$$(2.16) \quad J_m \bar{A}^t J_m^{-1} = A^{-1}.$$

Taking the transpose, the second condition in (2.15) becomes the condition

$$(2.17) \quad A^t = (J_m^{-1})^t \bar{A}^t J_m^t = J_m^{-1} \bar{A}^t J_m = J_m \bar{A}^t J_m^{-1}.$$

With (2.16), the condition (2.17) becomes  $A^t A = I_{2m}$ . Therefore,

$$(2.18) \quad O_{2m}^*(\mathbb{R}) = \left\{ A \in \text{Mat}_{2m}(\mathbb{C}) \mid AJ_m \bar{A}^t = J_m, A^t A = I_{2m}, \det(A) = 1 \right\}.$$

The group in the RHS of (2.18) is denoted as  $SO^*(2m)$  in [8, Chapter X, Section 2] and this group is connected [8, Chapter X, Lemma 2.4]. Thus,  $O_{2m}^*(\mathbb{R})$  is connected.

(2) This follows from (1) and Lemma 2.4. □

*Remark 2.6.* — In Section 2.2, we constructed a section for  $c : G \rightarrow \mathbb{G}_m$  over  $\mathbb{C}$ . Now we show that there is no section of  $c$  over  $\mathbb{R}$  when  $m$  is odd. Suppose that there is a section  $s : t \mapsto (A_1(t), \dots, A_d(t))$  over  $\mathbb{R}$ . The composition of the first component with the reduced norm map  $\text{Nrd}$  gives a character  $\mathbb{G}_m \rightarrow \mathbb{G}_m, t \mapsto \text{Nrd}(A_1(t))$ , which is  $t^n$  for some  $n \in \mathbb{Z}$ . By  $A_1(t)\mathbf{J}_m A_1(t)^* = t\mathbf{J}_m$ , one has  $\text{Nrd}(A_1(t))^2 = t^{2m}$  and hence  $\text{Nrd}(A_1(t)) = t^m$ . Since  $t^m = \text{Nrd}(A_1(t)) > 0$  for all  $t \in \mathbb{R}^\times$ ,  $m$  must be even.



When  $m$  is even, the map

$$\mathbb{G}_m \rightarrow G_{\mathbb{R}}, \quad t \mapsto \left( \begin{pmatrix} 0 & t \cdot I_{m/2} \\ I_{m/2} & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & t \cdot I_{m/2} \\ I_{m/2} & 0 \end{pmatrix} \right)$$

gives a section of  $c$  over  $\mathbb{R}$ .

PROPOSITION 2.7. — *Let  $(\Lambda, \psi)$  be a non-degenerate skew-Hermitian  $O_B$ -lattice. The subscheme  $\mathcal{M}_{(\Lambda, \psi)}$  of  $\mathcal{M}_{\mathbb{C}}$  is irreducible.*

*Proof.* — This follows immediately from Lemma 2.5(1). □

COROLLARY 2.8. — *If  $m = 1$ , then the group  $G_1(\mathbb{R})$  is isomorphic to  $(\mathbb{C}^{\times})^d$  and the quotient space  $X_1 = G_1(\mathbb{R})/K_{\infty}$  consists of one point, where  $\mathbb{C}_1^{\times} = \{z \in \mathbb{C}^{\times} \mid z\bar{z} = 1\}$ .* □

The following result is an immediate consequence of Proposition 2.7.

PROPOSITION 2.9. — *Suppose that  $m = 1$ . The map that sends each object  $(A, \lambda, \iota) \in \mathcal{M}(\overline{\mathbb{Q}}) = \mathcal{M}(\mathbb{C})$  to its first homology group  $(H_1(A(\mathbb{C}), \mathbb{Z}), \psi_{\lambda})$ , together with the Riemann form and the  $O_B$ -action, induces a bijection between the space  $\mathcal{M}(\overline{\mathbb{Q}})$  and the discrete set of isomorphism classes of  $\mathbb{Z}$ -valued rank one skew-Hermitian  $O_B$ -modules  $(V_{\mathbb{Z}}, \psi)$ .*

*Moreover, the subspace  $\mathcal{M}^{(p)}(\overline{\mathbb{Q}})$  corresponds to the subset of classes with the property that  $V_{\mathbb{Z}} \otimes \mathbb{Z}_p$  is self-dual with respect to the pairing  $\psi$ .* □

### 2.5. Connection with the adelic description

Let  $(V, \psi)$ ,  $G_1$ ,  $G$ ,  $J_0$  be as before. Let  $h_0 : \mathbb{C} \rightarrow \text{End}_{B \otimes \mathbb{R}}(V_{\mathbb{R}})$  be the  $\mathbb{R}$ -algebra homomorphism defined by  $h_0(i) = J_0$  and denote again by  $h_0 : \mathbb{C}^{\times} \rightarrow G_{\mathbb{R}}$  the homomorphism of  $\mathbb{R}$ -groups. Let  $X$  be the  $G(\mathbb{R})$ -conjugacy class of  $h_0$ . Fix an  $O_B$ -lattice  $\Lambda_0$  in  $V$  and let  $U \subset G(\mathbb{A}_f)$  be the open and compact subgroup that stabilizes the lattice  $\Lambda_0 \otimes \widehat{\mathbb{Z}}$ . Here  $\widehat{\mathbb{Z}}$  is the profinite completion of  $\mathbb{Z}$  and  $\mathbb{A}_f = \widehat{\mathbb{Z}} \otimes \mathbb{Q}$  is the finite adèle ring of  $\mathbb{Q}$ . One forms a complex Shimura variety (even though  $G$  is not connected)

$$\text{Sh}_U(G, X)_{\mathbb{C}} = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / U.$$

LEMMA 2.10. — *The Hermitian symmetric space  $X$  is  $G(\mathbb{R})/\mathbb{R}^{\times}K_{\infty}$  and it has two connected components.*

*Proof.* — Since  $c(G(\mathbb{R})) = \mathbb{R}^{\times}$  (Lemma 2.4), the closed immersion  $G \rightarrow \text{GSp}(V, \psi)$  induces a surjective map  $\pi_0(X) \rightarrow \pi_0(\mathbb{H}_g^{\pm})$ , where  $\mathbb{H}_g^{\pm}$  is the

Siegel double space. This shows that  $X$  has at least two connected components. On the other hand, since the group  $G(\mathbb{R})$  has two connected components (Lemma 2.5),  $X$  can only have two connected components. From the short exact sequence

$$(2.19) \quad G_1(\mathbb{R}) \longrightarrow G(\mathbb{R})/Z(G(\mathbb{R})) \xrightarrow{c} \{\pm 1\} \longrightarrow 1,$$

we conclude that  $X = G(\mathbb{R})/Z(G(\mathbb{R}))K_\infty = G(\mathbb{R})/\mathbb{R}^\times K_\infty$ . □

By Lemma 2.4, the group  $G(\mathbb{Q})$  is dense in  $G(\mathbb{R})$ . Thus,

$$(2.20) \quad \begin{aligned} \text{Sh}_U(G, X)_{\mathbb{C}} &= G(\mathbb{Q})^+ \backslash X_1 \times G(\mathbb{A}_f)/U \\ &= \prod_{i=1}^h \Gamma_i \backslash X_1, \quad \Gamma_i = G(\mathbb{Q})^+ \cap c_i U c_i^{-1}, \end{aligned}$$

where  $G(\mathbb{Q})^+ = G(\mathbb{Q}) \cap G(\mathbb{R})^+$  and  $c_1, \dots, c_h$  are coset representatives for the finite set  $G(\mathbb{Q})^+ \backslash G(\mathbb{A}_f)/U$ .

We now describe (2.20) in terms of lattices. We say two  $O_B$ -lattices  $\Lambda$  and  $\Lambda'$  in  $V$  are *similar* (resp. *strictly similar*), denoted  $\Lambda \sim \Lambda'$  (resp.  $\Lambda \sim_s \Lambda'$ ), if there is an element  $g \in G(\mathbb{Q})$  (resp.  $g \in G(\mathbb{Q})^+$ ) such that  $\Lambda' = g\Lambda$ . Similarly, for each prime  $\ell$ , write  $\Lambda_\ell \sim \Lambda'_\ell$  if  $\Lambda'_\ell = g\Lambda_\ell$  for some  $g \in G(\mathbb{Q}_\ell)$ , where  $\Lambda_\ell := \Lambda \otimes \mathbb{Z}_\ell$  denotes the completion of  $\Lambda$  at  $\ell$ . We say  $\Lambda$  and  $\Lambda'$  are in the same *idealcomplex* if  $\Lambda_\ell \sim \Lambda'_\ell$  for all primes  $\ell$ . Let

$$\mathfrak{J} := \{ \Lambda \subset V \mid \Lambda_\ell \sim \Lambda_{0,\ell} \ \forall \ell \}$$

be the ideal complex containing the  $O_B$ -lattice  $\Lambda_0$ . The map  $c \mapsto c\Lambda_0$ , where  $c \in G(\mathbb{A}_f)$ , induces a bijection between the double coset space  $G(\mathbb{Q})^+ \backslash G(\mathbb{A}_f)/U$  and the set of strict similitude classes in  $\mathfrak{J}$ . In particular, the complex Shimura variety  $\text{Sh}_U(G, X)_{\mathbb{C}}$  has  $h(\mathfrak{J})$  connected components, where  $h(\mathfrak{J})$  is the strict class number of  $\mathfrak{J}$ .

Put  $\Lambda_i = c_i\Lambda_0$ , then  $\Lambda_1, \dots, \Lambda_h$  are representatives of the strict similitude classes of  $\mathfrak{J}$ . After rescaling we may assume that  $\psi$  takes  $\mathbb{Z}$ -values on  $\Lambda_i$  for all  $i$ . It is easy to verify that  $\Gamma_i = \text{Aut}(\Lambda_i, \psi) = \Gamma_{\Lambda_i}$ . Thus, by (2.20) we get an isomorphism

$$(2.21) \quad \text{Sh}_U(G, X)_{\mathbb{C}} \simeq \prod_{i=1}^h \mathcal{M}_{(\Lambda_i, \psi)}.$$

We now describe  $\text{Sh}_U(G, X)_{\mathbb{C}}$  in terms of the Shimura variety for  $(G^0, X)$ . By Lemma 2.4, the  $G^0(\mathbb{R})$ -conjugacy class of  $h_0$  is also  $X$ . We also have  $G(\mathbb{Q}) = G^0(\mathbb{Q})$ . Let  $g_1, \dots, g_r$  be coset representatives for the double coset space  $G^0(\mathbb{A}_f) \backslash G(\mathbb{A}_f)/U$ . Then elements  $(1, g_1), \dots, (1, g_r)$  are also coset

representatives for  $G^0(\mathbb{A}) \backslash G(\mathbb{A}) / \mathbb{R}^\times K_\infty U$ . The map  $x \mapsto G(\mathbb{Q})x(1, g_i) \mathbb{R}^\times K_\infty U$ , for  $x \in G^0(\mathbb{A})$ , induces an isomorphism

$$(2.22) \quad G^0(\mathbb{Q}) \backslash G^0(\mathbb{A}) / \mathbb{R}^\times K_\infty U_i^0 \simeq G(\mathbb{Q}) \backslash G^0(\mathbb{A})(1, g_i) \mathbb{R}^\times K_\infty U / \mathbb{R}^\times K_\infty U,$$

where  $U_i^0 = g_i U g_i^{-1} \cap G^0(\mathbb{A}_f)$ . The left hand side of (2.22) is equal to  $\text{Sh}_{U_i^0}(G^0, X)_{\mathbb{C}}$ . The union of RHS of (2.22) for  $i = 1, \dots, r$  is equal to  $G(\mathbb{Q}) \backslash G(\mathbb{A}) / \mathbb{R}^\times K_\infty U$ . Thus, we have the decomposition

$$(2.23) \quad \text{Sh}_U(G, X)_{\mathbb{C}} = \coprod_{i=1}^r \text{Sh}_{U_i^0}(G^0, X)_{\mathbb{C}}.$$

It would be interesting to determine the number  $r$  in (2.23) explicitly.

*Remark 2.11.* — In [12], Kottwitz studied the set  $B(G)$  of  $(F)$ -isocrystals with  $G$ -structure for (not necessarily connected) reductive  $\mathbb{Q}_p$ -groups  $G$ . However, the study of the Kottwitz set  $B(G, \{\mu\})$  is limited to connected ones ([12, Section 6]). In the case of good reduction and of PEL-type D, Wedhorn [36] introduced the union of the sets  $B(G^0, \{\mu_i\})$ , where  $\{\mu_1\}, \dots, \{\mu_s\}$  are the  $G^0(\overline{\mathbb{Q}_p})$ -orbits of the  $G(\overline{\mathbb{Q}_p})$ -conjugacy class of the cocharacter  $\mu$  over  $\mathbb{Q}_p$  arising from  $X$ . He showed that the  $\mu$ -ordinary locus, which is the union of points in the special fiber whose associated isocrystals are the maximal element in  $B(G^0, \{\mu_i\})$  (the  $\mu_i$ -ordinary locus), is dense in the special fiber. The description (2.23) shows that the set of the isocrystals of the points in the special fiber are the same as those of the subscheme defined by  $(G^0, X)$ , which is expected to be  $B(G^0, \{\mu\})$ .

### 3. Arithmetic properties

#### 3.1.

Let  $\underline{A} = (A, \lambda, \iota)$  be a  $2dm$ -dimensional polarized  $O_B$ -abelian variety over any field  $K$ . Denote by  $\Gamma_K = \text{Gal}(K_s/K)$  the Galois group of  $K$ , where  $K_s$  is a separable closure of  $K$ . For any prime  $\ell \neq \text{char } K$ , let  $T_\ell = T_\ell(A)$  denote the Tate module of  $A$ , and put  $V_\ell := T_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ . Let

$$\rho_\ell : \Gamma_K \rightarrow \text{Aut}(T_\ell)$$

denote the associated  $\ell$ -adic representation of the Galois group  $\Gamma_K$ .

Let  $\langle, \rangle_A : T_\ell(A) \times T_\ell(A^t) \rightarrow \mathbb{Z}_\ell(1)$  be the canonical perfect pairing, where  $A^t$  denotes the dual abelian variety of  $A$ . The Galois group  $\Gamma_K$  acts on  $\mathbb{Z}_\ell(1)$  by the  $\ell$ -adic cyclotomic character  $\chi_\ell$  and this pairing is  $\Gamma_K$ -equivariant. We may identify  $T_\ell(A^t)$   $\Gamma_K$ -equivariantly with the linear dual

$T_\ell(A)^* := \text{Hom}(T_\ell(A), \mathbb{Z}_\ell(1))$ . The polarization  $\lambda$  induces an alternating non-degenerate pairing

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_\lambda : T_\ell(A) \times T_\ell(A) \rightarrow \mathbb{Z}_\ell(1), \quad \langle x, y \rangle_\lambda := \langle x, \lambda y \rangle_A.$$

Denote by  $g \mapsto g'$  ( $g \in \text{End}(V_\ell)$ ) the adjoint with respect to  $\langle \cdot, \cdot \rangle$ . Clearly  $g' = \lambda^{-1}g^t\lambda$ , where  $g^t \in \text{End}(V_\ell^*)$  is the pull-back map. It follows from

$$(3.1) \quad \langle \rho_\ell(\sigma)x, \rho_\ell(\sigma)y \rangle = \rho_\ell(\sigma)(\langle x, y \rangle) = \chi_\ell(\sigma)\langle x, y \rangle, \quad x, y \in V_\ell, \sigma \in \Gamma_K$$

that  $\rho_\ell(\sigma)'\rho_\ell(\sigma) = \chi_\ell(\sigma)$ . Thus, if we put  $B_\ell := B \otimes \mathbb{Q}_\ell$  and let  $G_\ell = GU_{B_\ell}(V_\ell)$  be the group of  $B_\ell$ -linear similitudes on  $V_\ell$ , then the  $\ell$ -adic representation  $\rho_\ell$  factors through this subgroup

$$\rho_\ell : \Gamma_K \rightarrow G_\ell(\mathbb{Q}_\ell).$$

We have the following basic properties:

LEMMA 3.1. — *Keep the notation as above.*

- (1)  $T_\ell$  is a free  $O_F \otimes \mathbb{Z}_\ell$ -module of rank  $4m$ .
- (2)  $V_\ell$  is a free  $O_B \otimes \mathbb{Q}_\ell$ -module of rank  $m$ .
- (3) If  $O_B$  is maximal at  $\ell$ , i.e.  $O_B \otimes \mathbb{Z}_\ell$  is a maximal order, then  $T_\ell$  is an  $O_B \otimes \mathbb{Z}_\ell$ -module of rank  $m$ .
- (4) The center of  $G_\ell(\mathbb{Q}_\ell)$  is given by

$$Z(G_\ell(\mathbb{Q}_\ell)) = \{x \in (F \otimes \mathbb{Q}_\ell)^\times \mid x^2 \in \mathbb{Q}_\ell^\times\}.$$

- (5) If  $m = 1$ , then the connected component  $G_\ell^0$  of  $G_\ell$  is a torus and  $G_\ell(\mathbb{Q}_\ell)/G_\ell^0(\mathbb{Q}_\ell)$  is a finite elementary 2-group.

*Proof.* — Statement (1) follows from the fact that  $\text{Tr}(a; V_\ell; \mathbb{Q}_\ell) = 4m \text{Tr}_{F/\mathbb{Q}}(a)$  for all  $a \in O_F$  and that  $O_F \otimes \mathbb{Z}_\ell$  is a maximal order. Statements (2) and (3) are obvious. Statement (4) follows from a direct computation. Statement (5) is proved in [44, Lemma 3.4]. □

*Remark 3.2.* — (1) The group  $G_\ell(\mathbb{Q}_\ell)$  does not need to be Zariski dense in  $G_\ell$ . For example, if  $F_\ell = F \otimes \mathbb{Q}_\ell$  remains a field and  $B$  splits at the prime over  $\ell$ , then  $[G_\ell(\mathbb{Q}_\ell) : G_\ell^0(\mathbb{Q}_\ell)] = 1$  or  $2$ . However, we always have  $[G_\ell : G_\ell^0] = 2^d$ .

Recall that an abelian variety  $A$  over any field  $K$  is said to have *sufficiently many complex multiplications* (smCM) or be of *CM-type* over  $K$  if there exists a semi-simple commutative  $\mathbb{Q}$ -subalgebra  $L \subset \text{End}_K^0(A) = \text{End}_K(A) \otimes_{\mathbb{Z}} \mathbb{Q}$  such that  $[L : \mathbb{Q}] = 2 \dim A$ . It is said to have *potentially smCM* or be *potentially of CM-type* if there exists a finite field extension  $K_1$  over  $K$  such that the base change  $A_{K_1}$  is of CM type over  $K_1$ .

PROPOSITION 3.3. — *For any  $2d$ -dimensional polarized  $O_B$ -abelian variety or  $O_B$ -abelian variety  $\underline{A}$  over  $K$ , the underlying abelian variety  $A$  is potentially of CM-type.*

*Proof.* — We prove the polarized version since any  $O_B$ -abelian variety admits an  $O_B$ -linear polarization. We may assume that  $K$  is finitely generated over its prime field. Let  $\mathbb{Q}_\ell[\Gamma_\ell]$  be the subalgebra of  $\text{End}(V_\ell)$  generated by the image  $\Gamma_\ell := \rho_\ell(\Gamma_K)$ . Replacing  $K$  by a finite extension of  $K$ , we may assume that  $\Gamma_\ell \subset G_\ell^0(\mathbb{Q}_\ell)$  is abelian. By the semi-simplicity of Tate modules due to Faltings and Zarhin (see [2] and [46]),  $\mathbb{Q}_\ell[\Gamma_\ell]$  is a commutative and semi-simple subalgebra. Let  $L$  be a maximal semi-simple commutative subalgebra in  $\text{End}^0(A)$ , then so is  $L \otimes \mathbb{Q}_\ell \subset \text{End}^0(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ . By the theorem of Faltings and Zarhin on Tate's conjecture (see [2] and [46]), we have  $\text{End}^0(A) \otimes \mathbb{Q}_\ell = \text{End}_{\mathbb{Q}_\ell[\Gamma_\ell]}(V_\ell)$ . As  $\mathbb{Q}_\ell[\Gamma_\ell]$  is commutative and semi-simple, any maximal semi-simple commutative subalgebra in  $\text{End}_{\mathbb{Q}_\ell[\Gamma_\ell]}(V_\ell)$  has degree  $2 \dim A$  over  $\mathbb{Q}_\ell$ . This shows  $[L : \mathbb{Q}] = 2 \dim A$  and finishes the proof of the proposition.  $\square$   $\square$

COROLLARY 3.4. — *Let  $\underline{A}$  be as in Proposition 3.3.*

- (1) *If  $\text{char } K = 0$ , then for a suitable finite extension  $K_1/K$ , the base change  $\underline{A} \otimes K_1$  admits a model  $\underline{A}_0$  defined over a number field.*
- (2) *If  $\text{char } K = p > 0$ , then for a suitable finite extension  $K_1/K$ , the base change  $\underline{A} \otimes K_1$  is isogenous to  $\underline{A}'$  which admits a model  $\underline{A}'_0$  defined over a finite field.*

*Proof.* — Statement (1) follows from Proposition 3.3 and the fact that any complex abelian variety of CM type is defined over a number field. Statement (2) follows from Proposition 3.3 and the Grothendieck theorem, stating that any isogeny class of abelian varieties of CM type in positive characteristic is defined over a finite field (see [22] and [40]).  $\square$

## 4. Skew-Hermitian $\mathcal{O}_B \otimes_{\mathbb{Z}_p} W$ -modules

### 4.1.

In this section we investigate basic properties of related local modules with additional structures. We use the following notation.

Let  $k$  be an algebraically closed field of characteristic  $p > 0$ . Let  $W = W(k)$  be the ring of Witt vectors over  $k$ , and  $B(k) := \text{Frac } W$  the fraction field of  $W$ . Let  $\sigma$  be the Frobenius map on  $W$  and on  $B(k)$ .

Let  $\mathbf{F}$  be a finite field extension of  $\mathbb{Q}_p$  and  $\mathcal{O}$  the ring of integers. Let  $e$  and  $f$  be the ramification index and inertial degree of  $\mathbf{F}/\mathbb{Q}_p$ , respectively, and  $\pi$  a uniformizer of  $\mathcal{O}$ . Let  $\mathbf{F}^{\text{nr}}$  denote the maximal unramified subfield extension of  $\mathbb{Q}_p$  in  $\mathbf{F}$ , and put  $\mathcal{O}^{\text{nr}} := \mathcal{O}_{\mathbf{F}^{\text{nr}}}$  the ring of integers.

Let  $\mathbf{B}$  be a quaternion algebra over  $\mathbf{F}$  and  $\mathcal{O}_{\mathbf{B}}$  be a maximal order. As before, we denote by  $*$  the canonical involution. If  $\mathbf{B}$  is the matrix algebra, then we fix an isomorphism  $\mathbf{B} = \text{Mat}_2(\mathbf{F})$  with  $\mathcal{O}_{\mathbf{B}} = \text{Mat}_2(\mathcal{O})$ .

Choose an unramified maximal subfield  $\mathbf{L} \subset \mathbf{B}$  such that the integral ring  $\mathcal{O}_{\mathbf{L}}$  is contained in  $\mathcal{O}_{\mathbf{B}}$ . If  $\mathbf{B}$  is a division algebra, then  $\mathcal{O}_{\mathbf{L}}$  is contained in  $\mathcal{O}_{\mathbf{B}}$  always. In this case we choose a presentation

$$(4.1) \quad \mathcal{O}_{\mathbf{B}} = \mathcal{O}_{\mathbf{L}}[\Pi] = \{a + b\Pi; a, b \in \mathcal{O}_{\mathbf{L}}\}$$

with the relations

$$(4.2) \quad \Pi^2 = -\pi \quad \text{and} \quad \Pi a = \bar{a}\Pi, \quad \forall a \in \mathcal{O}_{\mathbf{L}},$$

where  $a \mapsto \bar{a}$  is the non-trivial automorphism of  $\mathbf{L}/\mathbf{F}$ .

Indeed we first choose a presentation of  $\mathcal{O}_{\mathbf{B}}$  as (4.1) with relations  $\Pi a = \bar{a}\Pi$  and  $\Pi^2 = -\pi u$  for some element  $u \in \mathcal{O}^\times$ . Then replacing  $\Pi$  by  $\alpha\Pi$  for some element  $\alpha \in \mathcal{O}_{\mathbf{L}}^\times$ , one gets the relation  $\Pi^2 = -\pi$ . Similarly, one could also choose a presentation but with the relation  $\Pi^2 = \pi$  instead. Nevertheless we simply fix the presentation of  $\mathbf{B}$  as in (4.1) and (4.2).

We may regard  $\mathcal{O}_{\mathbf{B}}$  as an  $\mathcal{O}$ -subalgebra of  $\text{Mat}_2(\mathcal{O}_{\mathbf{L}})$  by sending

$$\Pi \mapsto \begin{pmatrix} 0 & -1 \\ \pi & 0 \end{pmatrix}, \quad \text{and} \quad a \mapsto \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}, \quad \forall a \in \mathcal{O}_{\mathbf{L}}.$$

Thus,

$$\mathcal{O}_{\mathbf{B}} = \left\{ a + b\Pi = \begin{pmatrix} a & -b \\ \pi\bar{b} & \bar{a} \end{pmatrix} \mid a, b \in \mathcal{O}_{\mathbf{L}} \right\} \subset \text{Mat}_2(\mathcal{O}_{\mathbf{L}}).$$

We have the following properties

$$\Pi^* = -\Pi, \quad \text{and} \quad (a + b\Pi)^* = \bar{a} - b\Pi.$$

### 4.2.

Let  $\Sigma_0 := \text{Hom}_{\mathbb{Z}_p}(\mathcal{O}^{\text{nr}}, W)$  be the set of embeddings of  $\mathcal{O}^{\text{nr}}$  into  $W$ . Write  $\Sigma_0 = \{\sigma_i\}_{i \in \mathbb{Z}/f\mathbb{Z}}$  in the way that  $\sigma\sigma_i = \sigma_{i+1}$  for all  $i \in \mathbb{Z}/f\mathbb{Z}$ . For any  $W$ -module  $M$  together with a  $W$ -linear action of  $\mathcal{O}^{\text{nr}}$ , write

$$(4.3) \quad M^i := \{x \in M \mid ax = \sigma_i(a)x, \forall a \in \mathcal{O}^{\text{nr}}\}$$

for the  $\sigma_i$ -component, and we have the decomposition

$$(4.4) \quad M = \bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} M^i.$$

If  $V$  is a finite-dimensional  $k$ -vector space with a  $k$ -linear action of  $\mathbb{F}_{p^f}$ , we write

$$(4.5) \quad V = k^{m_0} \oplus \dots \oplus k^{m_{f-1}}$$

for the decomposition  $V = \bigoplus V^i$  as in (4.4) with  $m_i = \dim_k V^i$  for all  $i \in \mathbb{Z}/f\mathbb{Z}$ .

Let  $P(T) \in \mathcal{O}^{\text{nr}}[T]$  be the minimal polynomial of  $\pi$ ; one has  $\mathcal{O} = \mathcal{O}^{\text{nr}}[\pi] = \mathcal{O}^{\text{nr}}[T]/P(T)$ . For any  $i \in \mathbb{Z}/f\mathbb{Z}$ , set  $W^i := W[T]/(\sigma_i(P(T)))$  and denote again by  $\pi$  the image of  $T$  in  $W^i$ . Each  $W^i$  is a complete discrete valuation ring and one has the decomposition

$$(4.6) \quad \mathcal{O} \otimes_{\mathbb{Z}_p} W = \prod_{i \in \mathbb{Z}/f\mathbb{Z}} W^i.$$

The action of the Frobenius map  $\sigma$  on  $\mathcal{O} \otimes_{\mathbb{Z}_p} W$  through the right factor gives a map  $\sigma : W^i \rightarrow W^{i+1}$  which sends  $a$  to  $\sigma(a)$  for  $a \in W$  and  $\sigma(\pi) = \pi$ . If  $M$  is an  $\mathcal{O} \otimes_{\mathbb{Z}_p} W$ -module, then we have the decomposition (4.4) with each component  $M^i$  a  $W^i$ -module. Note that the structure of  $M$  as an  $\mathcal{O} \otimes_{\mathbb{Z}_p} W$ -module is determined by the structure of each  $M^i$  as a  $W^i$ -module for all  $i \in \mathbb{Z}/f\mathbb{Z}$ .

Let  $\mathbf{L}^{\text{nr}}$  be the maximal unramified extension of  $\mathbb{Q}_p$  in  $\mathbf{L}$ , and let  $\mathcal{O}_{\mathbf{L}^{\text{nr}}}$  be the ring of integers. Let  $\Sigma := \text{Hom}_{\mathbb{Z}_p}(\mathcal{O}_{\mathbf{L}^{\text{nr}}}, W)$  be the set of embeddings of  $\mathcal{O}_{\mathbf{L}^{\text{nr}}}$  into  $W$ . Write  $\Sigma = \{\tau_j\}_{j \in \mathbb{Z}/2f\mathbb{Z}}$  in the way that  $\sigma\tau_j = \tau_{j+1}$  and  $\tau_j|_{\mathcal{O}^{\text{nr}}} = \sigma_i$  where  $i = j \bmod f$  for all  $j$ . The Galois group  $\text{Gal}(\mathbf{L}/\mathbf{F})$  acts on the set  $\Sigma$  by composing with the conjugate:  $\bar{\tau}_i(x) := \tau_i(\bar{x})$ . One has  $\bar{\tau}_i = \sigma^f \circ \tau_i = \tau_{i+f}$ . For any  $W$ -module  $M$  together with a  $W$ -linear action of  $\mathcal{O}_{\mathbf{L}}$ , write

$$(4.7) \quad M^j := \{x \in M \mid ax = \tau_j(a)x, \forall a \in \mathcal{O}_{\mathbf{L}^{\text{nr}}}\}$$

for the  $\tau_j$ -component, and we have the decomposition

$$(4.8) \quad M = \bigoplus_{j \in \mathbb{Z}/2f\mathbb{Z}} M^j.$$

Similarly, each  $M^j$  is a  $W^i$ -module where  $i = j \bmod f$  and the structure of  $M$  as an  $\mathcal{O}_{\mathbf{L}} \otimes_{\mathbb{Z}_p} W$  is determined by the structure of each  $M^j$  as a  $W^i$ -module for all  $j \in \mathbb{Z}/2f\mathbb{Z}$ .

**4.3. Finite  $\mathcal{O}_{\mathbf{B}} \otimes_{\mathbb{Z}_p} W$ -modules**

Suppose  $M$  is a finite  $W$ -module together with a  $W$ -linear action by  $\mathcal{O}_{\mathbf{B}}$ . Suppose  $\mathbf{B}$  is the matrix algebra. One has the decomposition

$$(4.9) \quad M = e_{11}M \oplus e_{22}M \quad M =: M_1 \oplus M_2,$$

where  $e_{11}$  and  $e_{22}$  are standard idempotents of  $\text{Mat}_2(\mathcal{O})$ , and  $M_1$  and  $M_2$  are finite  $W$ -modules with a  $W$ -linear action by  $\mathcal{O}$  with  $\text{rank}_W M_1 = \text{rank}_W M_2$ . The Morita equivalence states that the module  $M$  is uniquely determined by the  $\mathcal{O} \otimes_{\mathbb{Z}_p} W$ -module  $M_1$ . Furthermore, the structure of  $M_1$  as an  $\mathcal{O} \otimes_{\mathbb{Z}_p} W$ -module is given by its decomposition  $M_1 = \bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} M_1^i$  as  $W^i$ -submodules and the  $W^i$ -module structure of each component  $M_1^i$ . This describes finite  $\mathcal{O}_{\mathbf{B}} \otimes W$ -modules when  $\mathbf{B}$  is the matrix algebra. In particular, if  $M$  is free as a  $W$ -module, then  $M$  is uniquely determined by the numbers  $\text{rank}_{W^i} M^i$ , which equals  $2 \text{rank}_{W^i} M_1^i$ , up to isomorphism (and these ranks can be arbitrary even non-negative integers). The module  $M$  is a free  $\mathcal{O}_{\mathbf{B}} \otimes W$ -module if and only if the ranks  $\text{rank}_{W^i} M^i$  are constant.

Suppose  $\mathbf{B}$  is the division algebra. Write  $\mathcal{O}_{\mathbf{B}} = \mathcal{O}_{\mathbf{L}}[\Pi]$  as in Section 4.1. The action by  $\mathcal{O}_{\mathbf{L}^{\text{nr}}}$  gives the decomposition

$$(4.10) \quad M = \bigoplus_{j \in \mathbb{Z}/2f\mathbb{Z}} M^j$$

with each component  $M^j$  a finite  $W^i$ -module where  $i = j \pmod f$ . Moreover, one has

$$\Pi : M^j \rightarrow M^{j+f}, \quad \Pi^2 = -\pi \quad \text{on } M^j$$

for all  $j \in \mathbb{Z}/2f\mathbb{Z}$ . To see this, if  $x \in M^j$ , then for  $a \in \mathcal{O}_{\mathbf{L}^{\text{nr}}}$ ,

$$a \cdot \Pi x = \Pi \bar{a} \cdot x = \Pi (\tau_j(\bar{a}))x = \Pi \tau_{j+f}(a)x = \tau_{j+f}(a)\Pi x.$$

The map  $\Pi$  induces an isomorphism  $M^j \otimes \text{Frac } W^i \simeq M^{j+f} \otimes \text{Frac } W^i$ . Thus,

$$(4.11) \quad \text{rank}_{W^i} M^j = \text{rank}_{W^i} M^{j+f}, \quad \forall j \in \mathbb{Z}/2f\mathbb{Z}.$$

This is the only constraint for  $M$  to be an  $\mathcal{O}_{\mathbf{B}} \otimes W$ -module. Put

$$(4.12) \quad a_j := \dim_k M^j / \Pi M^{j-f}.$$

If  $M$  is free as a  $W$ -module, then  $M$  is uniquely determined by the numbers  $\{a_j\}_j$  up to isomorphism. The number  $a_j$  can be arbitrary between 0 and  $\text{rank}_{W^i} M^j$  subject to the condition  $a_j + a_{j+f} = \text{rank}_{W^i} M^j$ . Furthermore,  $M$  is a free  $\mathcal{O}_{\mathbf{B}} \otimes W$ -module if and only if the numbers  $a_j$  are constant.



### 4.4. Skew-Hermitian $\mathcal{O}_{\mathbf{B}} \otimes_{\mathbb{Z}_p} W$ -modules

In this subsection we assume that  $p \neq 2$ . Let  $M$  be a finite non-degenerate skew-Hermitian  $\mathcal{O}_{\mathbf{B}}$ -module over  $W$ , that is, it is a finite and free  $W$ -module with a  $W$ -linear action of  $\mathcal{O}_{\mathbf{B}}$  and together with an alternating non-degenerate bilinear pairing

$$\psi : M \times M \rightarrow W$$

satisfying the condition

$$(4.13) \quad \psi(bx, y) = \psi(x, b^*y), \quad \forall x, y \in M, b \in \mathcal{O}_{\mathbf{B}}.$$

(non-degeneracy here means that the induced map  $M \rightarrow M^t := \text{Hom}_W(M, W)$  is injective). If the pairing  $\psi$  is perfect, we call  $M$  *self-dual*.

Suppose  $\mathbf{B}$  is the matrix algebra. Let  $C := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  be the Weil element. Put  $\varphi(x, y) := \psi(x, Cy)$ . Then the pairing  $\varphi : M \times M \rightarrow W$  is symmetric; this follows from the property  $C^* = -C$ . One also has  $C^{-1}a^*C = a^t$  for all  $a \in \mathbf{B} = \text{Mat}_2(\mathbf{F})$  and the idempotents  $e_{11}$  and  $e_{22}$  are fixed by the adjoint with respect to  $\varphi$ . Thus, the decomposition  $M = M_1 \oplus M_2$  in (4.9) respects the pairing  $\varphi$ . By the Morita equivalence, we may describe the symmetric  $\mathcal{O} \otimes W$ -module  $M_1$ . Note that the condition (4.13) becomes equivalent to the condition  $\varphi(ax, y) = \varphi(x, ay)$  for  $x, y \in M_1$  and  $a \in \mathcal{O}$ . The latter implies that  $\varphi(M_1^i, M_1^{i'}) = 0$  for  $i \neq i'$  in  $\mathbb{Z}/f\mathbb{Z}$ , where the  $M_1^i$ 's are the components in the decomposition (4.4). Consider the restriction of  $\varphi$  to each component  $M_1^i$ . Then there is a unique  $W^i$ -bilinear pairing

$$\varphi_i : M_1^i \times M_1^i \rightarrow \mathcal{D}_{W^i/W}^{-1}$$

such that  $\varphi = \text{Tr}_{W^i/W} \varphi_i$  on each  $M_1^i$ , where  $\mathcal{D}_{W^i/W}^{-1}$  is the inverse different of  $W^i$  over  $W$ . Thus, one can describe the symmetric  $W^i$ -modules  $M_1^i$  instead. The classification of symmetric local lattices is well known; see O'Meara [20, Section 92]. Since  $W^i$  is non-dyadic, each  $M_1^i$  has an orthogonal basis for  $\varphi_i$  (see loc. cit.). As our ground field  $k$  is algebraically closed of characteristic  $p \neq 2$ , one has  $W^{i \times} = (W^{i \times})^2$ . Thus, if  $M_1^i$  is self-dual (with respect to the values in  $\mathcal{D}_{W^i/W}^{-1}$ ), then the isomorphism class of  $M_1^i$  is uniquely determined by its rank  $\text{rank}_{W^i} M_1^i$ . Note that  $M$  is self-dual with respect to the pairing  $\psi$  if and only if each submodule  $M_1^i$  is self-dual with respect to the pairing  $\varphi_i$  (with values in  $\mathcal{D}_{W^i/W}^{-1}$ ). Following from this, we conclude the following result.

LEMMA 4.1. — Assume  $\mathbf{B}$  is the matrix algebra. Any two self-dual skew-Hermitian  $\mathcal{O}_{\mathbf{B}} \otimes W$ -modules  $M$  and  $N$  are isomorphic if and only if  $\text{rank}_{W^i} M^i = \text{rank}_{W^i} N^i$  for all  $i \in \mathbb{Z}/f\mathbb{Z}$ . Moreover, for any given non-negative even integers  $n_i$  for  $i \in \mathbb{Z}/f\mathbb{Z}$ , there is a unique self-dual skew-Hermitian  $\mathcal{O}_{\mathbf{B}} \otimes W$ -module  $M$  up to isomorphism such that  $\dim_{W^i} M^i = n_i$  for all  $i \in \mathbb{Z}/f\mathbb{Z}$ .  $\square$

Suppose  $\mathbf{B}$  is the division algebra. Let  $M = \bigoplus_j M^j$  be the decomposition by the action of  $\mathcal{O}_{\mathbf{L}^{\text{nr}}}$  as in (4.8). It is easy to see using (4.13) that  $\psi(M^{j_1}, M^{j_2}) = 0$  if  $j_1 - j_2 \neq f$  in  $\mathbb{Z}/2f\mathbb{Z}$  and hence  $\psi$  is determined by its restriction

$$\psi : M^j \times M^{j+f} \rightarrow W$$

for  $0 \leq j < f$ . Note that  $\text{rank}_{W^i} M^j = \text{rank}_{W^i} M^{j+f}$  (4.11), where  $i := j \bmod f$ . Let  $\mathcal{D}_{W^i/W}^{-1} = W^i \delta_i$ , where  $\delta_i$  is a generator. Then there is a unique  $W^i$ -bilinear pairing

$$\psi_i : M^j \times M^{j+f} \rightarrow W^i$$

such that  $\psi = \text{Tr}_{W^i/W}(\delta_i \psi_i)$ . Clearly, the module  $M$  is self-dual with respect to the pairing  $\psi$  if and only if each  $\psi_i$  is a perfect pairing.

Now we only consider the case where  $M$  is self-dual. For any  $x, y \in M^j$ , one easily sees

$$\psi_i(x, \Pi y) = \psi_i(\Pi^* x, y) = \psi_i(y, \Pi x),$$

so the pairing

$$\varphi_i : M^j \times M^j \rightarrow W^i, \quad \varphi_i(x, y) := \psi_i(x, \Pi y),$$

is symmetric. Put  $\overline{M^j} := M^j / \pi M^j$ . Then  $\psi_i$  induces the perfect pairing<sup>(3)</sup>  $\psi_i : \overline{M^j} \times \overline{M^{j+f}} \rightarrow k$ . Recall  $a_j := \dim_k M^j / \Pi M^{j+f}$ . From the isomorphism

$$\Pi : M^{j+f} / \Pi M^j \simeq \Pi M^{j+f} / \pi M^j,$$

we get

$$a_j + a_{j+f} = \dim_k \overline{M^j} = \dim_k \overline{M^{j+f}}.$$

LEMMA 4.2. — Assume that  $\mathbf{B}$  is a division algebra. Let  $M = \bigoplus_{j \in \mathbb{Z}/2f\mathbb{Z}} M^j$  be a self-dual skew-Hermitian  $\mathcal{O}_{\mathbf{B}} \otimes_{\mathbb{Z}_p} W$ -module. For each  $j \in \mathbb{Z}/2f\mathbb{Z}$ , there are  $W^i$ -bases

$$\{x_1^j, \dots, x_{a_j+a_{j+f}}^j\}, \quad \{x_1^{j+f}, \dots, x_{a_j+a_{j+f}}^{j+f}\}$$

<sup>(3)</sup> To ease our notation, we still denote the induced pairing by  $\psi_i$ .

for  $M^j$  and  $M^{j+f}$ , respectively, where the positive integers  $a_j$  and  $a_{j+f}$  are as above, such that

$$(4.14) \quad \begin{cases} \Pi(x_r^j) = x_r^{j+f}, & \forall 1 \leq r \leq a_j, \\ \Pi(x_{a_j+r}^{j+f}) = x_{a_j+r}^j, & \forall 1 \leq r \leq a_{j+f}, \end{cases}$$

and for  $1 \leq r, s \leq a_j + a_{j+f}$ , one has

$$(4.15) \quad \psi_i(x_r^j, x_s^{j+f}) = \delta_{r,s}.$$

*Proof.* — Consider the induced symmetric pairing  $\varphi_i : \overline{M^j} \times \overline{M^j} \rightarrow k$ . Since  $\overline{\Pi M^j}$  and  $\overline{\Pi M^{j+f}}$  are mutually orthogonally complemented with respect to the pairing  $\psi_i$ , one obtains a non-degenerate symmetric pairing

$$(4.16) \quad \varphi_i : M^j / \Pi M^{j+f} \times M^j / \Pi M^{j+f} \rightarrow k.$$

We prove the statement by induction on the rank of  $M^j$  (or  $M^{j+f}$ ). Suppose  $a_j > 0$ , using (4.16) we may choose an element  $x_1^j \in M^j$  such that  $\varphi_i(x_1^j, x_1^j) = 1$ , because  $W^{i \times} = (W^{i \times})^2$ . Put  $x_1^{j+f} := \Pi x_1^j \in M^{j+f}$ . Then we have

$$M^j \oplus M^{j+f} = N \oplus N^\perp,$$

where  $N$  is the  $W^i$ -submodule generated by  $x_1^j$  and  $x_1^{j+f}$  and  $N^\perp$  is the orthogonal complement of  $N$ . Clearly  $N$  is stable under the  $\mathcal{O}_{\mathbf{B}}$ -action and hence so is  $N^\perp$ . If  $a_j = 0$  then  $a_{j+f} > 0$  and we do the same for  $M^{j+f}$ . As  $N^\perp$  has lower rank, by induction we can choose bases  $\{x_1^j, \dots, x_{a_j+a_{j+f}}^j\}$  and  $\{x_1^{j+f}, \dots, x_{a_j+a_{j+f}}^{j+f}\}$  for  $M^j$  and  $M^{j+f}$ , respectively, satisfying (4.14) and (4.15). This completes the proof of the Lemma 4.2.  $\square$

We obtain the following classification result.

**COROLLARY 4.3.** — *Assume that  $\mathbf{B}$  is the division algebra. Any two self-dual skew-Hermitian  $\mathcal{O}_{\mathbf{B}} \otimes W$ -modules  $M$  and  $N$  are isomorphic if and only if*

$$\dim_k M^j / \Pi M^{j+f} = \dim_k N^j / \Pi N^{j+f}, \quad \text{for all } j \in \mathbb{Z}/2f\mathbb{Z}.$$

Moreover, for any given non-negative integers  $a_j$  for  $j \in \mathbb{Z}/2f\mathbb{Z}$ , there is a unique up to isomorphism self-dual skew-Hermitian  $\mathcal{O}_{\mathbf{B}} \otimes W$ -modules  $M$  such that  $a_j = \dim_k M^j / \Pi M^{j+f}$  for all  $j \in \mathbb{Z}/2f\mathbb{Z}$ .  $\square$

### 5. Quasi-polarized Dieudonné $\mathcal{O}_{\mathbf{B}}$ -modules

#### 5.1.

We keep the notation of Section 4. In particular, we have the decomposition  $\mathcal{O} \otimes_{\mathbb{Z}_p} W = \bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} W^i$  as in (4.6). All Dieudonné modules over  $k$  in this paper are assumed to be finite and free as  $W$ -modules. For basic theory of Dieudonné modules, we refer to Manin [13] and Zink [47]. The Frobenius and Verschiebung operators of a Dieudonné module are denoted by  $F$  and  $V$ , respectively.

By a *Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module* over  $k$  we mean a Dieudonné module  $M$  over  $k$  together with a ring monomorphism  $\mathcal{O}_{\mathbf{B}} \rightarrow \text{End}_{\text{DM}}(M)$  of  $\mathbb{Z}_p$ -algebras. Recall that a *quasi-polarization*<sup>(4)</sup> on a Dieudonné module  $M$  is a non-degenerate alternating  $W$ -bilinear form

$$\langle \cdot, \cdot \rangle : M \times M \rightarrow W$$

such that  $\langle Fx, y \rangle = \langle x, Vy \rangle^\sigma$  for all  $x, y \in M$ . When the pairing  $\langle \cdot, \cdot \rangle$  is perfect, we call it *separable*<sup>(5)</sup>. A quasi-polarization is called  *$\mathcal{O}_{\mathbf{B}}$ -linear* if it satisfies the condition

$$(5.1) \quad \langle bx, y \rangle = \langle x, b^*y \rangle, \quad \forall x, y \in M, b \in \mathcal{O}_{\mathbf{B}}.$$

A (resp. *separably*) *quasi-polarized Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module* is a Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module  $M$  together with an (resp. separable)  $\mathcal{O}_{\mathbf{B}}$ -linear quasi-polarization.

We also recall that a *quasi-polarized  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -module* is a triple  $(H, \lambda, \iota)$  where  $H$  is a  $p$ -divisible group,  $\iota : \mathcal{O}_{\mathbf{B}} \rightarrow \text{End}(H)$  is a ring monomorphism of  $\mathbb{Z}_p$ -algebras and  $\lambda : H \rightarrow H^t$  is a quasi-polarization (i.e. an isogeny  $\lambda$  satisfying  $\lambda^t = -\lambda$ ) such that  $\lambda \circ \iota(b^*) = \iota(b)^t \circ \lambda$  for all  $b \in \mathcal{O}_{\mathbf{B}}$ , where  $H^t$  is the Serre dual of  $H$ . For a quasi-polarized  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -module  $(H, \lambda, \iota)$  over  $k$ , the associated (covariant) Dieudonné module  $M = M(H)$  together with the additional structures is a quasi-polarized Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module over  $k$ .

Clearly these notions can be defined for orders in general semi-simple  $\mathbb{Q}_p$ -algebras with involutions (i.e. for general PEL data); cf. [40, Section 2.1].

<sup>(4)</sup> The terminology “quasi-polarization” has been adopted in several papers of Oort e.g. in [19]. A reason for calling it quasi-polarization is that there is no notion of positivity for a biextension on a  $p$ -divisible group, unlike abelian varieties. Here we simply use the same terminology. So a quasi-polarization on a  $p$ -divisible group is really an *isogeny* to its Serre dual, not a quasi-isogeny.

<sup>(5)</sup> Some authors also call this *principal*.

Let  $(M, \langle, \rangle)$  be a (not necessarily separably) quasi-polarized Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module.

Suppose  $\mathbf{B}$  is the matrix algebra. As in Section 4.4, we define the symmetric pairing  $(x, y) := \langle x, Cy \rangle$ , where  $C$  is the Weil element. Then we obtain the decomposition  $M = e_{11}M \oplus e_{22}M =: M_1 \oplus M_2$ , which respects the pairing  $(, )$ . By the Morita equivalence, one may consider the anti-quasi-polarized Dieudonné  $\mathcal{O}$ -module  $M_1$  instead, i.e. for all  $a \in \mathcal{O}$  and  $x, y \in M_1$ , one has the properties

$$(5.2) \quad (y, x) = (x, y), \quad (ax, y) = (x, ay), \quad \text{and} \quad (Fx, y) = (x, Vy)^\sigma.$$

Let  $M_1 = \bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} M_1^i$  be the decomposition by the action of  $\mathcal{O}^{\text{nr}}$  as in (4.4). Each component  $M_1^i$  is then a  $W^i$ -module. Then one has

$$F : M_1^i \rightarrow M_1^{i+1}, \quad V : M_1^{i+1} \rightarrow M_1^i.$$

It follows that the ranks  $\text{rank}_{W^i} M_1^i$  (or equivalently  $\text{rank}_{W^i} M^i$ ) for  $i \in \mathbb{Z}/f\mathbb{Z}$  are constant. As a result, the module  $M_1$  (or  $M$ ) is free as an  $\mathcal{O} \otimes_{\mathbb{Z}_p} W$ -module. (This only uses the property that  $M$  is a Dieudonné  $\mathcal{O}$ -module.) One also has the orthogonal property  $(M_1^{i_1}, M_1^{i_2}) = 0$  if  $i_1 \neq i_2$  in  $\mathbb{Z}/f\mathbb{Z}$ .

LEMMA 5.1. — Assume that  $\mathbf{B}$  is the matrix algebra and  $p \neq 2$ . If  $M$  and  $N$  are two separably quasi-polarized Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -modules of the same rank, then  $M$  and  $N$  are isomorphic as skew-Hermitian  $\mathcal{O}_{\mathbf{B}} \otimes W$ -modules.

*Proof.* — This follows from the fact that  $M_1$  (or  $M$ ) is free as an  $\mathcal{O} \otimes_{\mathbb{Z}_p} W$ -module and Lemma 4.1. □

Suppose  $\mathbf{B}$  is the division algebra. Let  $M = \bigoplus_{j \in \mathbb{Z}/2f\mathbb{Z}} M^j$  be the decomposition as in (4.10). Then we have

$$(5.3) \quad F : M^j \rightarrow M^{j+1}, \quad V : M^{j+1} \rightarrow M^j,$$

and  $\langle M^{j_1}, M^{j_2} \rangle = 0$  if  $j_1 - j_2 \neq f$  in  $\mathbb{Z}/2f\mathbb{Z}$ . Similarly, it follows from (5.3) that the ranks  $\text{rank}_{W^i} M^j$  for  $j \in \mathbb{Z}/2f\mathbb{Z}$  are constant, and that  $M$  is free as an  $\mathcal{O}_{\mathbf{L}} \otimes_{\mathbb{Z}_p} W$ -module. (This uses only the property that  $M$  is a Dieudonné  $\mathcal{O}_{\mathbf{L}}$ -module.)

### 5.2.

From now on until Section 8 we assume that  $\text{rank}_W M = m[\mathbf{B} : \mathbb{Q}_p]$  for some positive integer  $m$ . As in (2.8), let  $\text{char}(a) \in \mathbb{Z}_p[T]$  be the reduced characteristic polynomial of  $a$  from  $\mathbf{B}$  to  $\mathbb{Q}_p$ , which is of degree  $[\mathbf{B} : \mathbb{Q}_p]/2$ .

We say a Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module  $M$  satisfies the *determinant condition* if one has the equality of the following characteristic polynomials, taken in  $k[T]$ , cf. (2.9)

$$(5.4) \quad (\text{K}) \quad \text{char}(\iota(a)|M/\mathcal{V}M) = \text{char}(a)^m, \quad \forall a \in \mathcal{O}_{\mathbf{B}}.$$

If we set  $d = [\mathbf{F} : \mathbb{Q}_p]$ , then the above polynomials are of degree  $2dm$ .

LEMMA 5.2. — *Let  $M$  be a Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module.*

- (1) *If  $\mathbf{B}$  is the matrix algebra, then  $M$  satisfies the determinant condition (K) if and only if for all  $i \in \mathbb{Z}/f\mathbb{Z}$ , one has  $\dim_k(M_1/\mathcal{V}M_1)^i = em$ .*
- (2) *If  $\mathbf{B}$  is the division algebra, then  $M$  satisfies the determinant condition (K) if and only if for all  $j \in \mathbb{Z}/2f\mathbb{Z}$ , one has  $\dim_k(M/\mathcal{V}M)^j = em$ .*

*Proof.*

(1) Using the Morita equivalence,  $M$  satisfies the condition (K) if and only if  $M_1$  satisfies (K) for all  $a \in \mathcal{O}$ . Choose an algebraic closure  $B(k)^{\text{alg}}$  of  $B(k) := W(k)[1/p]$  and put  $\Sigma_{\mathbf{F}} := \text{Hom}(\mathbf{F}, B(k)^{\text{alg}})$ . For  $a \in \mathcal{O}$ , the left hand side of the equation (5.4) is

$$\prod_{i \in \mathbb{Z}/f\mathbb{Z}} (T - \tilde{\sigma}_i(a))^{\dim_k(M_1/\mathcal{V}M_1)^i},$$

where  $\tilde{\sigma}_i \in \Sigma_{\mathbf{F}}$  is any lift of  $\sigma_i \in \Sigma_0$ . The right hand side of the equation (5.4) is equal to

$$\prod_{\sigma \in \Sigma_{\mathbf{F}}} (T - \sigma'(a))^m = \prod_{i \in \mathbb{Z}/f\mathbb{Z}} (T - \tilde{\sigma}_i(a))^{em}.$$

Therefore, the condition (K) is satisfied if and only if  $\dim_k(M_1/\mathcal{V}M_1)^i = em$  for all  $i \in \mathbb{Z}/f\mathbb{Z}$ .

(2) As  $\mathbf{B}$  is generated by the element  $\Pi$  over  $\mathbf{L}$ , by [6, Proposition 2.1.1] one may check the equality (5.4) for  $a = \Pi$  and all  $a \in \mathcal{O}_{\mathbf{L}}$ . Since  $\iota(\Pi)$  on  $M/\mathcal{V}M$  is nilpotent, its characteristic polynomial is  $T^{2dm}$ , where  $d = [\mathbf{F} : \mathbb{Q}_p]$ . Since  $\Pi$  satisfies  $\Pi^2 + \pi = 0$ , the (reduced) characteristic polynomial of  $\Pi$  is  $(T^2 + \pi)^d$ , whose image in  $k[T]$  is equal to  $T^{2d}$ . This verifies (K) for  $a = \Pi$ . We then must show that (K) holds for all  $a \in \mathcal{O}_{\mathbf{L}}$  if and only if  $\dim_k(M/\mathcal{V}M)^j = em$  for all  $j \in \mathbb{Z}/2f\mathbb{Z}$ . But the same proof of (1) also proves this. □

LEMMA 5.3. — Suppose  $M$  is a separably quasi-polarized Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module.

(1) If  $\mathbf{B}$  is the matrix algebra, then for any  $i \in \mathbb{Z}/f\mathbb{Z}$ , one has

$$\dim_k(M_1/\mathcal{V}M_1)^i = em.$$

(2) If  $\mathbf{B}$  is the division algebra, then for any  $i \in \mathbb{Z}/f\mathbb{Z}$ , one has  $\dim_k(M/\mathcal{V}M)^i = 2em$  (recall that  $(M/\mathcal{V}M)^i$  denotes the  $\sigma_i$ -component of  $M/\mathcal{V}M$ ).

*Proof.*

(1) The proof is similar to that of [39, Lemma 2.6(2)]. Consider  $M_1$  as a separably anti-quasi-polarized Dieudonné  $\mathcal{O}$ -module. Since the restriction of  $(, )$  to each component  $M_1^i$  is perfect, for any  $W$ -basis  $\{x_r\}$  of  $M_1^i$ , the discriminant  $\det(x_r, x_s)$  is a unit. Consider the map  $\mathcal{V} : M_1^{i+1} \rightarrow M_1^i$  of modules of  $W$ -rank  $2em$ ; one has  $M_1^i/\mathcal{V}M_1^{i+1} = (M_1/\mathcal{V}M_1)^i$ . By the elementary divisor theorem, there exist bases  $\{x_1, \dots, x_{2em}\}$  and  $\{y_1, \dots, y_{2em}\}$  of  $M_1^{i+1}$  and  $M_1^i$ , respectively, such that for all  $r$ , one has  $\mathcal{V}x_r = p^{n_r}y_r$  for some integer  $n_r \geq 0$ . Then

$$\begin{aligned} p^{2em} \det(x_r, x_s)^{\sigma^{-1}} &= \det(\mathcal{V}x_r, \mathcal{V}x_s) \\ &= \det p^{n_r+n_s}(y_r, y_s) = p^2 \sum n_r \det(y_r, y_s). \end{aligned}$$

It follows that  $\sum n_r = em$  and  $\dim_k(M_1/\mathcal{V}M_1)^i = em$ .

(2) Consider  $M$  as a separably quasi-polarized Dieudonné  $\mathcal{O}$ -module (the Hilbert–Siegel analogue). The same proof of (1) with  $(, )$  and  $M_1$  replaced by  $\langle , \rangle$  and  $M$ , respectively, will prove the result.  $\square$

*Remark 5.4.* — According to Lemmas 5.2 and 5.3, if  $\mathbf{B}$  is the matrix algebra, then any separably quasi-polarized Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module satisfies the determinant condition. However, in the case where  $\mathbf{B}$  is the division algebra, there are a few possibilities for the numbers  $\dim_k(M/\mathcal{V}M)^j$ . Therefore, the determinant condition would impose a further condition for separably quasi-polarized Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -modules.

### 5.3.

We discuss the relationship between the numbers  $a_j := \dim_k(M/\Pi M)^j$  and the numbers  $\dim_k(M/\mathcal{V}M)^j$  for a Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module  $M$  when  $\mathbf{B}$  is the division algebra. Put

$$(5.5) \quad c_j := \dim_k(M/\mathcal{V}M)^j \quad \text{for } j \in \mathbb{Z}/2f\mathbb{Z}.$$

Write  $V_j : M^{j+1} \rightarrow M^j$  for the restriction of  $V$  on  $M^{j+1}$ , and  $\Pi_j : M^j \rightarrow M^{j+f}$  for that of  $\Pi$  on  $M^j$ . We have the commutative diagram

$$(5.6) \quad \begin{array}{ccc} M^j & \xleftarrow{V_j} & M^{j+1} \\ \Pi_j \downarrow & & \Pi_{j+1} \downarrow \\ M^{j+f} & \xleftarrow{V_{j+f}} & M^{j+f+1}. \end{array}$$

Let  $\text{ord}$  be the normalized valuation on  $W^i$ , that is, one has  $\text{ord}(\pi) = 1$ . Let  $\text{ord det } \Pi_j$  denote the valuation of  $\det(A_j)$ , where  $A_j$  is the matrix representing the map  $\Pi_j$  with respect to a set of  $W^i$ -bases for  $M^j$  and  $M^{j+f}$  respectively; this is well-defined. Similarly we define  $\text{ord det } V_j$  for suitable bases of  $M^{j+1}$  and  $M^j$ . It is easy to see that

$$(5.7) \quad \text{ord det } V_j = c_j, \text{ and } \text{ord det } \Pi_j = a_{j+f}, \quad \forall j \in \mathbb{Z}/2f\mathbb{Z}.$$

As  $\Pi_{j+1} = V_{j+f}^{-1} \circ \Pi_j \circ V_j$ , one has the relation

$$a_{j+f+1} = a_{j+f} + c_j - c_{j+f},$$

or equivalently

$$(5.8) \quad a_{j+1} = a_j + c_{j+f} - c_j, \quad \forall j \in \mathbb{Z}/2f\mathbb{Z}.$$

Since  $\text{rank}_{W^i} M^j = 2m$ , it follows from  $\Pi_j \circ \Pi_{j+f} = -\pi$  that

$$(5.9) \quad a_j + a_{j+f} = 2m.$$

Since  $a_j$ 's are integers between 0 and  $2m$ , it follows from (5.8) that

$$(5.10) \quad \left| \sum_{i=j}^{j+r} (c_{i+f} - c_i) \right| \leq 2m, \quad \forall j \in \mathbb{Z}/2f\mathbb{Z}, 0 \leq r \leq f - 1.$$

On the other hand, the collection  $\{c_j\}$  satisfies the condition

$$(5.11) \quad \sum_{j \in \mathbb{Z}/2f\mathbb{Z}} c_j = 2dm.$$

We say  $M$  is *separably quasi-polarizable* if it admits a separable  $\mathcal{O}_{\mathbf{B}}$ -linear quasi-polarization. If  $M$  is separably quasi-polarizable, then by Lemma 5.3 (2), one has

$$(5.12) \quad c_j + c_{j+f} = 2em, \quad \forall j \in \mathbb{Z}/2f\mathbb{Z},$$

because  $(M/\mathbf{V}M)^i = (M/\mathbf{V}M)^j \oplus (M/\mathbf{V}M)^{j+f}$  and  $\dim_k (M/\mathbf{V}M)^i = 2em$ , with  $i = j \pmod f$ .



LEMMA 5.5. — Suppose that  $\mathbf{B}$  is the division algebra and let the notation be as above. The sets of numbers  $\{a_j\}$  and  $\{c_j\}$  satisfy the conditions (5.8)–(5.11). Moreover, if  $M$  is separably quasi-polarizable, then one has in addition the condition (5.12).

It is interesting to know whether the necessary conditions in Lemma 5.5 are also sufficient, in the sense that these invariants can be realized by a (separably quasi-polarizable) Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module.

PROPOSITION 5.6. — Let  $M$  be a Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module of  $W$ -rank  $4dm$ , where  $d = [\mathbf{F} : \mathbb{Q}_p]$ .

- (1) Suppose  $\mathbf{B}$  is the matrix algebra.
  - (a) The module  $M$  is free as an  $\mathcal{O}_{\mathbf{B}} \otimes W$ -module.
  - (b) The module  $M$  is separably quasi-polarizable if and only if it satisfies the determinant condition.
- (2) Suppose  $\mathbf{B}$  is the division algebra.
  - (a) If  $M$  satisfies the determinant condition, then it is free as an  $\mathcal{O}_{\mathbf{B}} \otimes W$ -module.
  - (b) If  $M$  is separably quasi-polarizable and free as an  $\mathcal{O}_{\mathbf{B}} \otimes W$ -module, then it satisfies the determinant condition.

*Proof.*

(1) Part (1a) is proved in the paragraph before Lemma 5.1. For part (1b), the only if part follows from Lemmas 5.2 and 5.3. For the other direction, we refer to the discussion in [39, Lemma 2.6].

(2) (2a) Since  $M$  satisfies the determinant condition, by Lemma 5.2 each  $c_j$  is equal to  $em$ . By (5.8) and (5.9), each  $a_j$  is equal to  $m$ . Therefore,  $M$  is free as an  $\mathcal{O}_{\mathbf{B}} \otimes W$ -module. (2b) Note that  $M$  is free as an  $\mathcal{O}_{\mathbf{B}} \otimes W$ -module if and only if  $a_j = m$  for all  $j$ . By (5.8), one has  $c_j = c_{j+f}$  for all  $j$ . Since  $M$  is separably quasi-polarizable, by (5.12) each  $c_j$  is equal to  $em$ . By Lemma 5.2, the Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module  $M$  satisfies the determinant condition.  $\square$

COROLLARY 5.7. — Assume that  $p \neq 2$ . Let  $\Lambda$  be a  $\mathbb{Z}_p$ -valued unimodular skew-Hermitian  $\mathcal{O}_{\mathbf{B}}$ -module of  $\mathbb{Z}_p$ -rank  $4dm$ . For any separably quasi-polarized Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module  $M$  of rank  $4dm$  that satisfies the determinant condition, one has an isomorphism  $M \simeq \Lambda \otimes_{\mathbb{Z}_p} W$  as skew-Hermitian  $\mathcal{O}_{\mathbf{B}} \otimes W$ -modules.

*Proof.* — By Proposition 5.6,  $M$  is free as an  $\mathcal{O}_{\mathbf{B}} \otimes W$ -module. When  $\mathbf{B}$  is the matrix algebra, the assertion follows from Lemma 4.1. When  $\mathbf{B}$  is the division algebra, the assertion then follows from Lemma 4.2.  $\square$

*Remark 5.8.* — The converse of Proposition 5.6(2a) does not hold. We give a counterexample. Let  $\mathcal{O} = \mathbb{Z}_{p^2}$ . Then  $\mathcal{O}_{\mathbf{B}} = \mathbb{Z}_{p^4}[\Pi]$  with  $\Pi^2 = -p$  and  $\Pi a = \sigma^2(a)\Pi$  for  $a \in \mathbb{Z}_{p^4}$ . Put  $M = (\mathcal{O}_{\mathbf{B}} \otimes W)^{\oplus 2}$ , which is obviously a free  $\mathcal{O}_{\mathbf{B}} \otimes W$ -module. We have the decomposition  $M = M^0 \oplus M^1 \oplus M^2 \oplus M^3$  by the action of  $\mathbb{Z}_{p^4}$ . Choose a basis  $\{e_1^j, e_2^j\}$  of  $M^j$  for each  $i$  such that the representative matrix for each  $\Pi_i : M^i \rightarrow M^{j+2}$  is given by  $\begin{bmatrix} 0 & -p \\ 1 & 0 \end{bmatrix}$ . With respect to these bases, we define the map  $V_j : M^{j+1} \rightarrow M^j$  by  $V_j = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  for  $j = 0, 2$ , and  $V_j = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}$  for  $j = 1, 3$ . Then we have the following commutative diagram

$$\begin{array}{ccccc}
 M^0 & \xleftarrow{V_0} & M^1 & \xleftarrow{V_1} & M^2 \\
 \downarrow \Pi_0 & & \downarrow \Pi_1 & & \downarrow \Pi_2 \\
 M^2 & \xleftarrow{V_2} & M^3 & \xleftarrow{V_3} & M^0 \\
 \downarrow \Pi_2 & & \downarrow \Pi_3 & & \downarrow \Pi_0 \\
 M^0 & \xleftarrow{V_0} & M^1 & \xleftarrow{V_1} & M^2.
 \end{array}$$

This defines a Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module  $M$  and one has  $c_0 = c_2 = 0$  and  $c_1 = c_3 = 2$ . Thus,  $M$  does not satisfy the determinant condition by Lemma 5.2(2).

## 6. Isogeny classes of $p$ -divisible $\mathcal{O}_{\mathbf{B}}$ -modules

### 6.1.

Keep the notation of Sections 4 and 5. Our goal is to classify the isogeny classes of quasi-polarized  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -modules  $\underline{H} = (H, \lambda, \iota)$  over an algebraically closed field  $k$  of characteristic  $p$ . Let  $(M, \langle \cdot, \cdot \rangle, \iota)$  be the associated Dieudonné module with the additional structures. Assume that  $\text{rank}_W M = m[\mathbf{B} : \mathbb{Q}_p]$  for some integer  $m \geq 1$ . Note that this is the general type D case (in the local situation). Let  $d := [\mathbf{F} : \mathbb{Q}_p]$ . So the  $p$ -divisible group  $H$  has height  $4dm$ .

The slope sequence (or Newton polygon) of a  $p$ -divisible group  $H$  is denoted by  $\nu(H)$ . Write

$$\{\beta_i^{m_i}\}_{1 \leq i \leq t}$$

for the slope sequence with each slope  $\beta_i$  of multiplicity  $m_i$ .

Two quasi-polarized  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -modules  $\underline{H} = (H, \lambda, \iota)$  and  $\underline{H}' = (H', \lambda', \iota')$  are said to be *isogenous* if there is an  $\mathcal{O}_{\mathbf{B}}$ -linear quasi-isogeny

$\varphi : H \rightarrow H'$  such that  $\varphi^* \lambda' = \lambda$ . This is equivalently saying that the associated  $F$ -isocrystals  $\underline{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \underline{M}' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$  are isomorphic compatible with the additional structures. Similarly, one defines the isogeny relation for  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -modules  $(H, \iota)$ . Clearly, the slope sequence of a (quasi-polarized)  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -module is determined by its isogeny class. The relationship between the Newton polygon and isogeny class of  $p$ -divisible groups with additional structures (in the general setting of  $F$ -isocrystals with  $G$ -structure) is known due to the works of Kottwitz [10, 12] and Rapoport–Richartz [27]. Here we describe the image of the Newton map  $\nu$  for  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -modules.

### 6.2.

We describe the isogeny classes of  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -modules.

LEMMA 6.1. — *Let  $(H, \iota)$  and  $(H', \iota')$  be two  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -modules of the same height. Then  $(H, \iota)$  is isogenous to  $(H', \iota')$  if and only if  $\nu(H) = \nu(H')$ .*

*Proof.* — The direction  $\implies$  is obvious and we show the other direction. Since  $\nu(H) = \nu(H')$ , we may choose an isogeny  $\varphi : H \rightarrow H'$ . Then we have an isomorphism  $j : \text{End}^0(H') \simeq \text{End}^0(H)$  of  $\mathbb{Q}_p$ -algebras, sending  $a \mapsto \varphi^{-1} a \varphi$ , where  $\text{End}^0(H) := \text{End}(H) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ . Put  $\iota_1 := j \circ \iota' : \mathbf{B} \rightarrow \text{End}^0(H)$ . Since the center of  $\text{End}^0(H)$  is a product of copies of  $\mathbb{Q}_p$ , by the Noether–Skolem theorem, there is an element  $\alpha \in \text{End}^0(H)^\times$  such that  $\iota_1 = \text{Int}(\alpha) \circ \iota$ . For  $b \in B$ , one has

$$\varphi^{-1} \iota'(b) \varphi = \iota_1(b) = \alpha \iota(b) \alpha^{-1}.$$

□

### 6.3. The isoclinic case.

Let  $H$  be an isoclinic  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -module of height  $h > 0$  and of slope  $\beta$ , and let  $M$  be the associated Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module. Let  $E := \text{End}^0(H)$  be the endomorphism algebra. Write  $E = \text{Mat}_n(\Delta)$ , where  $\Delta$  is a central division  $\mathbb{Q}_p$ -algebra with  $\text{inv}(\Delta) = \beta$ .

6.3.1.

Suppose  $\mathbf{B}$  is the matrix algebra. Then  $H = H_1 \oplus H_2$ ,  $H_1$  has height  $h_1 := h/2$  and  $\deg(\text{End}^0(H_1)) = h_1$ . It is known that there is a monomorphism  $\mathbf{F} \rightarrow \text{End}^0(H_1)$  if and only if  $d|h_1$ , or equivalently  $2d|h$ . As  $\deg(\text{End}^0(H_1)) = h_1$ , one has  $\beta = a/h_1$  for some integer  $0 \leq a \leq h_1$ . Therefore,

$$\nu(H) = \left\{ \left( \frac{a}{h_1} \right)^h \right\}$$

for some integer  $0 \leq a \leq h_1$ . Conversely, suppose  $2d|h$  and we are given a slope sequence  $\nu = \{(2a/h)^h\}$  for some integer  $0 \leq a \leq h/2$ . Then there is a  $p$ -divisible group  $H$  of height  $h$  together with an algebra monomorphism  $\mathbf{B} \rightarrow \text{End}^0(H)$  such that  $\nu(H) = \nu$ . Replacing  $H$  by another  $p$ -divisible group in its isogeny class, we have a ring monomorphism  $\iota : \mathcal{O}_{\mathbf{B}} \rightarrow \text{End}(H)$ .

6.3.2.

Suppose that  $\mathbf{B}$  is the division algebra. Write  $\beta = a/h$  for some integer  $0 \leq a \leq h$ . As the field  $\mathbf{L}$  can be embedded into  $E$ , one has  $2d|h$ . Therefore, we can write  $h = 2dh'$  for some integer  $h'$ .

LEMMA 6.2. — *Notations and assumptions are as above. There is an embedding of  $\mathbf{B}$  into  $E$  if and only if*

$$(6.1) \quad a \equiv h' \pmod{2}.$$

*Proof.* — We write

$$\Delta \otimes_{\mathbb{Q}_p} \mathbf{B}^{\text{op}} = \Delta_{\mathbf{F}} \otimes_{\mathbf{F}} \mathbf{B}^{\text{op}} = \text{Mat}_c(\Delta'),$$

where  $\Delta_{\mathbf{F}} := \Delta \otimes_{\mathbb{Q}_p} \mathbf{F}$ ,  $\mathbf{B}^{\text{op}}$  is the opposite algebra of  $\mathbf{B}$ ,  $\Delta'$  is a central division  $\mathbf{F}$ -algebra.

By an embedding theorem for general simple algebras [45, Theorem 2.7], there is an embedding of  $\mathbf{B}$  into  $E = \text{Mat}_n(\Delta)$  if and only if the following condition holds

$$(6.2) \quad [\mathbf{B} : \mathbb{Q}_p] \mid nc.$$

Write  $\delta := \deg(\Delta)$  and  $\delta' := \deg(\Delta')$ . We have  $c\delta' = 2\delta$  and  $h = n\delta$ . Then one has

$$4d|nc \iff 4d\delta'|nc\delta' \iff 2d\delta'|n\delta = h,$$

and this is equivalent to the condition

$$(6.3) \quad \delta'|h'.$$

As  $\delta'$  is the denominator of  $\text{inv}(\Delta')$ , the condition (6.3) holds if and only if  $h' \cdot \text{inv}(\Delta') \in \mathbb{Z}$ . We compute

$$\text{inv}(\Delta') = d \cdot \frac{a}{h} - \frac{1}{2} = \frac{a - h'}{2h'} \quad \text{and hence} \quad h' \cdot \text{inv}(\Delta') = \frac{a - h'}{2}.$$

Therefore, the condition (6.3) holds if and only if the condition (6.1) holds. This proves the Lemma 6.2.  $\square$

By Lemma 6.2 one has

$$\nu(H) = \left\{ \left( \frac{a}{h} \right)^h \right\}$$

for some integer  $0 \leq a \leq h$  with  $a \equiv h' \pmod{2}$ . Conversely, suppose  $h = 2dh'$  for some  $h' \in \mathbb{Z}_{\geq 1}$  and we are given a slope sequence  $\nu = \{(a/h)^h\}$  for some integer  $0 \leq a \leq h$  with  $a \equiv h' \pmod{2}$ . Then by adjusting  $H$  in its isogeny class if necessary, there is a  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -module  $H$  of height  $h$  such that  $\nu(H) = \nu$ .

We conclude our discussion by the following Proposition 6.3.

**PROPOSITION 6.3.** — *Let  $h = 2dh'$  with  $h' \in \mathbb{Z}_{\geq 1}$ , and let  $(H, \iota)$  be an isoclinic  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -module of height  $h$  over  $k$ .*

(1) *If  $\mathbf{B}$  is the matrix algebra, then*

$$\nu(H) = \left\{ \left( \frac{a}{dh'} \right)^h \right\},$$

*where  $a$  is an integer with  $0 \leq a \leq dh'$ . Conversely, any slope sequence of this form is realized by a  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -module of height  $h$  over  $k$ .*

(2) *If  $\mathbf{B}$  is the division algebra, then*

$$\nu(H) = \left\{ \left( \frac{a}{h} \right)^h \right\},$$

*where  $a$  can be any integer with  $0 \leq a \leq h$  and  $a \equiv h' \pmod{2}$ . Conversely, any slope sequence of this form is realized by a  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -module of height  $h$  over  $k$ .  $\square$*

#### 6.4. The general case

It is not hard to state the general case for possible slope sequences of  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -modules based on the isoclinic case. One simply considers the decomposition  $H \sim H_1 \times H_2 \times \cdots \times H_t$  into the isoclinic components in the isogeny class.

THEOREM 6.4. — Let  $h = 2dh'$  with  $h' \in \mathbb{Z}_{\geq 1}$ , and let  $(H, \iota)$  be a  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -module of height  $h$  over  $k$ .

(1) If  $\mathbf{B}$  is the matrix algebra, then

$$(6.4) \quad \nu(H) = \left\{ \left( \frac{a_i}{dh'_i} \right)^{2dh'_i} \right\}_{1 \leq i \leq t},$$

where  $h'_1 + \dots + h'_t = h'$  is a partition of the integer  $h'$  and  $a_i$  is an integer with  $0 \leq a_i \leq dh'_i$ . Moreover, after combining the indices  $i$  with same slope  $a_i/dh'_i$  together and rearranging the indices, we may assume that

$$\frac{a_i}{dh'_i} < \frac{a_{i+1}}{dh'_{i+1}}, \quad i = 1, \dots, t - 1.$$

Conversely, any slope sequence of the form (6.4) subject to the condition above arises from a  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -module of height  $h$  over  $k$ .

(2) If  $\mathbf{B}$  is the division algebra, then

$$(6.5) \quad \nu(H) = \left\{ \left( \frac{a_i}{2dh'_i} \right)^{2dh'_i} \right\}_{1 \leq i \leq t},$$

where  $h'_1 + \dots + h'_t = h'$  is a partition of the integer  $h'$  and  $a_i$  is an integer with  $0 \leq a_i \leq 2dh'_i$  and  $a_i \equiv h'_i \pmod{2}$ . Similarly, we may rearrange the indices such that

$$\frac{a_i}{2dh'_i} < \frac{a_{i+1}}{2dh'_{i+1}}, \quad i = 1, \dots, t - 1.$$

Conversely, any slope sequence of this form arises from a  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -module of height  $h$  over  $k$ . □

### 7. Slope sequences of quasi-polarized Dieudonné $\mathcal{O}_{\mathbf{B}}$ -modules

Keep the notation of Section 6. Let  $h = 4dm$  with  $m \in \mathbb{Z}_{\geq 1}$ , and let  $(H, \lambda, \iota)$  be a quasi-polarized  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -module of height  $h$  over  $k$ . Then the slope sequence  $\nu(H)$  of  $H$  is given as Theorem 6.4 together with the symmetric condition: for all  $0 \leq i, j \leq t$  with  $i + j = t + 1$ , one has  $h'_i = h'_j$  and

$$(7.1) \quad \begin{aligned} \frac{a_i}{dh'_i} + \frac{a_j}{dh'_j} &= 1 \text{ in the matrix algebra case,} \\ \frac{a_i}{2dh'_i} + \frac{a_j}{2dh'_j} &= 1 \text{ in the division algebra case,} \end{aligned}$$

where  $h'_1 + \dots + h'_t = 2m$  is a partition of  $2m$ .

Note that the integer  $t$  is even if and only if  $H$  has no supersingular component. For any symmetric slope sequence  $\nu$  which has the form in Theorem 6.4, we write

$$(7.2) \quad \nu = \nu_n \cup \nu_s,$$

where  $\nu_n$ , the *non-supersingular part*, consists of all slopes in  $\nu$  which are not  $1/2$ , and  $\nu_s$ , the *supersingular part*, consists of all slopes  $1/2$  in  $\nu$ .

LEMMA 7.1. — *Let  $h$  be a positive integer and  $\beta$  a positive rational number such that there exists an isoclinic  $\mathcal{O}_{\mathbf{B}}$ -module of height  $h$  and with slope  $\beta$ . Then there exists a separably quasi-polarized  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -module  $(H, \lambda, \iota)$  of height  $2h$  such that  $\nu(H) = \{\beta^h, (1 - \beta)^h\}$ .*

*Proof.* — Choose an isoclinic  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -module  $(H_1, \iota_1)$  of height  $h$  and with the slope  $\beta$ . Put  $H_2 := H_1^t$ , which is an isoclinic  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -module of height  $h$  and with the slope  $1 - \beta$ . The monomorphism  $\iota_2 : \mathcal{O}_{\mathbf{B}} \rightarrow \text{End}(H_2)$  is given by  $\iota_2(a) := \iota_1(a^*)^t$  for  $a \in \mathcal{O}_{\mathbf{B}}$ . Put  $H := H_1 \times H_2$ , and then  $H^t = H_1^t \times H_2^t$ . Let  $\lambda = \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} : H \rightarrow H^t$  be an  $\mathcal{O}_{\mathbf{B}}$ -linear isogeny, where  $\lambda_1 : H_1 \rightarrow H_1^t$  and  $\lambda_2 : H_2 \rightarrow H_2^t$  are  $\mathcal{O}_{\mathbf{B}}$ -linear isogenies. Then  $\lambda^t = \begin{bmatrix} \lambda_1^t & \lambda_2^t \end{bmatrix}$ , so  $\lambda^t = -\lambda$  if and only if  $\lambda_2 = -\lambda_1^t$ . Choose  $\lambda_1$  an  $\mathcal{O}_{\mathbf{B}}$ -linear isomorphism and put

$$\lambda = \begin{bmatrix} & -\lambda_1^t \\ \lambda_1 & \end{bmatrix}.$$

Then  $\lambda$  is a separably  $\mathcal{O}_{\mathbf{B}}$ -linear quasi-polarization. □

Note that the construction in Lemma 7.1 works for any finite-dimensional simple  $\mathbb{Q}_p$ -algebra  $\mathbf{B}'$  with involution, not just for the quaternion algebra treated in this paper. This method of construction appears quite often in dealing with symmetric slope sequences with two slopes and we call this *the double construction*.

For the supersingular case, we have the following result.

THEOREM 7.2. — *For any positive integer  $m$ , there exists a superspecial separably quasi-polarized Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module  $M$  of rank  $4dm$  that satisfies the determinant condition  $(K)$ .*

The proof of this theorem is given by construction. Namely, we directly write down such a Dieudonné module. The construction is lengthy and we omit the details of the construction<sup>(6)</sup>. We conclude the main result of this section.

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<sup>(6)</sup>The reader can find the detailed construction in older versions of <https://arxiv.org/abs/1306.1400v1>.

**THEOREM 7.3.** — *Let  $h = 4dm$  with any  $m \in \mathbb{Z}_{\geq 1}$ . Let  $\nu$  be a slope sequence of the form in Theorem 6.4 that satisfies the symmetric condition (7.1), where  $h'_1 + \dots + h'_t = 2m$  is any partition of  $2m$ . Then there exists a separably quasi-polarized Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module  $M$  of rank  $h$  and with  $\nu(M) = \nu$ .*

*Proof.* — Write  $\nu = \nu_n + \nu_s$  into non-supersingular part and supersingular part, say of length  $4dm_n$  and  $4dm_s$ , respectively. As each isoclinic component  $\{\beta_i^{m_i}\}$  of  $\nu_n$  can be realized by a  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -module (Theorem 6.4), by Lemma 7.1 there is a separably quasi-polarized  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -module  $(H_n, \lambda_n, \iota_n)$  of height  $4dm_n$  such that  $\nu(H_n) = \nu_n$ . On the other hand, by Theorem 7.2, there is a superspecial separably quasi-polarized  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -module  $(H_s, \lambda_s, \iota_s)$  of height  $4dm_s$  that satisfies the determinant condition. The product  $(H_n, \lambda_n, \iota_n) \times (H_s, \lambda_s, \iota_s)$  satisfies the desired properties.  $\square$

*Remark 7.4.*

- (1) In Lemma 7.1 one may choose  $H$  such that  $H$  is a *minimal*  $p$ -divisible group in the sense of Oort, that is, the endomorphism ring  $\text{End}(H)$  of  $H$  is a maximal  $\mathbb{Z}_p$ -order in the semi-simple  $\mathbb{Q}_p$ -algebra  $\text{End}^0(H)$ . This follows from the construction of the minimal isogeny; see [43, Section 4 (particularly Proposition 4.8)]. Therefore, the  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -module in Theorem 7.3 can be chosen to be minimal.
- (2) We shall see that when  $\mathbf{B}$  is the division algebra, the determinant condition (K) will rule out some possibilities of the slope sequences that are realized by separably quasi-polarized Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module in Theorem 7.3. That is, not all symmetric slope sequences in Theorem 7.3 occurring as those of separably quasi-polarized Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -modules that satisfy *the determinant condition*. We refer to Section 11 for more details in the case of rank  $4d$ .
- (3) When  $\mathbf{B}$  is a division algebra and  $d$  is odd, the double construction as in Lemma 7.1 provides an alternative way to produce a separably quasi-polarized superspecial Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module  $M$ . However, we checked that such a Dieudonné module  $M$  rarely satisfies the determinant condition.
- (4) We refer to [41] for a classification of superspecial quasi-polarized Dieudonné  $\mathcal{O}_F \otimes \mathbb{Z}_p$ -modules of HB type.



COROLLARY 7.5. — *There is an ordinary separably quasi-polarized  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -module of height  $4dm$  if and only if one of the following holds:*

- (1)  $\mathbf{B}$  is the matrix algebra;
- (2)  $\mathbf{B}$  is the division algebra and  $m$  is even.

*Proof.* — Any ordinary  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -module always admits an  $\mathcal{O}_{\mathbf{B}}$ -linear separable quasi-polarization.

If  $\mathbf{B}$  is the matrix algebra, then the ordinary slope sequence appears in Theorem 6.4 (or in Proposition 6.3). Therefore, by Theorem 7.3 there is an ordinary separably quasi-polarized  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -module of height  $4dm$ .

Suppose  $\mathbf{B}$  is the division algebra. Then a slope sequence with 2 slopes  $\{(a_1/2dh'_1)^{2dh'_1}, (a_2/2dh'_2)^{2dh'_2}\}$  which is of the form (6.5) can be the ordinary slope sequence if and only if  $h'_1 = h'_2 = m$  and  $m \equiv 2dm \pmod{2}$  (taking  $a_2 = 2dh'_2$  and  $h'_2 = m$ ). That is,  $m$  is even.  $\square$

Remark 7.6. — For a smooth PEL-type moduli space  $\mathcal{M}'$ , the ordinary locus of  $\mathcal{M}' \otimes \overline{k}(v)$  is non-empty if and only if  $E_v = \mathbb{Q}_p$ , where  $E$  is the reflex field and  $v$  is a finite place of  $E$  lying over  $p$ ; see Wedhorn [36]. Corollary 7.5 shows that the latter condition is not sufficient for the non-emptiness of the ordinary locus in some ramified cases.

We give a few examples of possible slope sequences of quasi-polarized Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -modules.

COROLLARY 7.7 ( $m = 1$ ). — *Let  $(H, \lambda, \iota)$  be a quasi-polarized  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -module of height  $4d$ .*

- (1) *If  $\mathbf{B}$  is the matrix algebra, then*

$$\nu(H) = \left\{ \left(\frac{a}{d}\right)^{2d}, \left(\frac{d-a}{d}\right)^{2d} \right\},$$

where  $a$  can be any integer with  $0 \leq a < d/2$ , or

$$\nu(H) = \left\{ \left(\frac{1}{2}\right)^{4d} \right\}.$$

- (2) *If  $\mathbf{B}$  is the division algebra, then*

$$\nu(H) = \left\{ \left(\frac{a}{2d}\right)^{2d}, \left(\frac{2d-a}{2d}\right)^{2d} \right\},$$

where  $a$  can be any integer with  $0 \leq a < d$  with  $a \equiv 1 \pmod{2}$ , or

$$\nu(H) = \left\{ \left(\frac{1}{2}\right)^{4d} \right\}. \quad \square$$

COROLLARY 7.8 ( $m = 2$ ). — Let  $(H, \lambda, \iota)$  be a quasi-polarized  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -module of height  $8d$ .

(1) If  $\mathbf{B}$  is the matrix algebra, then we have the following possibilities of  $\nu(H)$ :

(a) one slope case:

$$\nu(H) = \left\{ \left( \frac{1}{2} \right)^{8d} \right\}.$$

(b) two slopes case:

$$\nu(H) = \left\{ \left( \frac{a}{2d} \right)^{4d}, \left( \frac{2d-a}{2d} \right)^{4d} \right\}, \quad 0 \leq a < d, \quad a \in \mathbb{Z}.$$

(c) three slopes case:

$$\nu(H) = \left\{ \left( \frac{a}{d} \right)^{2d}, \left( \frac{1}{2} \right)^{4d}, \left( \frac{d-a}{d} \right)^{2d} \right\}, \quad 0 \leq a < \frac{d}{2}, \quad a \in \mathbb{Z}.$$

(d) four slopes case:

$$\nu(H) = \left\{ \left( \frac{a}{d} \right)^{2d}, \left( \frac{b}{d} \right)^{2d}, \left( \frac{d-b}{d} \right)^{2d}, \left( \frac{d-a}{d} \right)^{2d} \right\},$$

where  $a$  and  $b$  can be any integers with  $0 \leq a < b < d/2$ .

(2) If  $\mathbf{B}$  is the division algebra, then we have the following possibilities of  $\nu(H)$ :

(a) one slope case:

$$\nu(H) = \left\{ \left( \frac{1}{2} \right)^{8d} \right\}.$$

(b) two slopes case:

$$\nu(H) = \left\{ \left( \frac{a}{4d} \right)^{4d}, \left( \frac{4d-a}{4d} \right)^{4d} \right\}, \quad 0 \leq a < 2d, \quad a \in \mathbb{Z}, \quad a \equiv 0 \pmod{2}.$$

(c) three slopes case:

$$\nu(H) = \left\{ \left( \frac{a}{2d} \right)^{2d}, \left( \frac{1}{2} \right)^{4d}, \left( \frac{2d-a}{2d} \right)^{2d} \right\},$$

where  $a$  can be any integer with  $0 \leq a < d$  and  $a \equiv 1 \pmod{2}$ .

(d) four slopes case:

$$\nu(H) = \left\{ \left( \frac{a}{2d} \right)^{2d}, \left( \frac{b}{2d} \right)^{2d}, \left( \frac{2d-b}{2d} \right)^{2d}, \left( \frac{2d-a}{2d} \right)^{2d} \right\},$$

where  $a$  and  $b$  can be any integers with  $0 \leq a < b < d$  and  $a, b \equiv 1 \pmod{2}$ .  $\square$

## 8. Isogeny classes of quasi-polarized $p$ -divisible $\mathcal{O}_{\mathbf{B}}$ -modules

In this section we would like to classify isogeny classes of quasi-polarized  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -modules of height  $4dm$  over an algebraically closed field  $k$  of characteristic  $p > 0$ . Consider rational quasi-polarized Dieudonné  $\mathbf{B}$ -modules  $N$  with  $B(k)$ -rank  $= 4dm$ , or quasi-polarized  $\mathbf{B}$ -linear  $F$ -isocrystals. When all slopes of  $N$  are between 0 and 1, there is a Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -lattice  $M$  in  $N$ , so  $N = M \otimes_{W(k)} B(k)$  for some quasi-polarized Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module. Thus, classifying isogeny classes in question may be reduced to classifying rational quasi-polarized Dieudonné  $\mathbf{B}$ -modules  $N$ . Let  $\nu$  be a symmetric slope sequence as in Theorem 7.3. Let  $I(\nu)$  denote the set of isogeny classes of quasi-polarized  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -modules  $(H, \lambda, \iota)$  of height  $4dm$  such that  $\nu(H) = \nu$ . By Dieudonné theory, the set  $I(\nu)$  is isomorphic to the set of isomorphism classes of rational quasi-polarized Dieudonné  $\mathbf{B}$ -modules  $N$  of  $B(k)$ -rank  $= 4dm$  and with slope sequence  $\nu$ .

Rapoport and Richartz [27] have classified the isomorphism classes of isocrystals with  $G$ -structure with a fixed Newton vector by the Galois cohomology  $H^1(\mathbb{Q}_p, J)$  for a certain reductive group  $J$  over  $\mathbb{Q}_p$  when  $G$  is connected. For the present case one needs to work a bit more, because our group  $G$  is not connected. The description by Galois cohomology helps us to understand the classification problem. However, in order to obtain more explicit results, one also needs to compute the set  $H^1(\mathbb{Q}_p, J)$ .

We shall work along with Dieudonné modules and translate the classification problem into the theory of (skew-)Hermitian forms over local fields; see Theorem 8.2.

Write  $\nu = \nu_n + \nu_s$  into the non-supersingular and supersingular parts.

LEMMA 8.1. — We have  $I(\nu) = I(\nu_n) \times I(\nu_s)$ , and  $I(\nu_n)$  consists of one isogeny class.

*Proof.* — Suppose  $N_1$  and  $N_2$  are rational quasi-polarized Dieudonné  $\mathbf{B}$ -modules with  $\nu(N_1) = \nu(N_2) = \nu$ . Write

$$N_1 = N_1^{ns} \oplus N_1^{ss}, \quad N_2 = N_2^{ns} \oplus N_2^{ss}$$

for the decomposition of  $N_1$  and  $N_2$  into the non-supersingular component and supersingular component, respectively. Clearly, one has  $N_1 \simeq N_2$  if and only if  $N_1^{ns} \simeq N_2^{ns}$  and  $N_1^{ss} \simeq N_2^{ss}$ . This shows the first part.

For the second part, we decompose the rational Dieudonné modules

$$N_1^{ns} = \oplus_{\beta < 1/2} (N_{1,\beta} \oplus N_{1,\beta}^t), \quad N_2^{ns} = \oplus_{\beta < 1/2} (N_{2,\beta} \oplus N_{2,\beta}^t)$$

into isotypic components. By Lemma 6.1, we have an isomorphism  $N_{1,\beta} \simeq N_{2,\beta}$  as rational Dieudonné  $\mathbf{B}$ -modules for each  $\beta < 1/2$ . It follows that  $N_1^{ns} \simeq N_2^{ns}$  as rational quasi-polarized Dieudonné  $\mathbf{B}$ -modules.  $\square$

By Lemma 8.1, one reduces the classification to classifying the isomorphism classes of supersingular rational quasi-polarized Dieudonné  $\mathbf{B}$ -modules of  $B(k)$ -rank  $4dm_s$ , where  $4dm_s$  is the length of the supersingular part  $\nu_s$ .

Let  $N$  be a supersingular rational quasi-polarized Dieudonné  $\mathbf{B}$ -module of  $B(k)$ -rank  $4dm_s$ . Put

$$\tilde{N} := \{x \in N \mid F^2x = px\}.$$

This is a  $B(\mathbb{F}_{p^2})$ -vector space of dimension  $4dm_s$  such that

- $W(k) \otimes_{B(\mathbb{F}_{p^2})} \tilde{N} = N$ ,
- $F = V$  on  $\tilde{N}$ , and
- the action of  $\mathbf{B}$  leaves  $\tilde{N}$  invariant.

Let  $\mathbf{D}$  be the quaternion division algebra over  $\mathbb{Q}_p$ . We can write  $\mathbf{D} = B(\mathbb{F}_{p^2})[F]$  with relations  $F^2 = p$  and  $Fa = \sigma(a)F$  for all  $a \in B(\mathbb{F}_{p^2})$ . Then  $\tilde{N}$  naturally becomes a left  $\mathbf{D}$ -module of  $\mathbb{Q}_p$ -rank  $8dm_s$ . Define the involution  $*_{\mathbf{D}}$  on  $\mathbf{D}$  by

$$(a + bF)^{*_{\mathbf{D}}} := \sigma(a) + bF.$$

This is an orthogonal involution as the fixed subspace is 3-dimensional. As the actions of  $\mathbf{B}$  and  $\mathbf{D}$  commute,  $\tilde{N}$  becomes a left  $\mathbf{B} \otimes_{\mathbb{Q}_p} \mathbf{D}$ -module. Write

$$\mathbf{B} \otimes_{\mathbb{Q}_p} \mathbf{D} = \mathbf{B} \otimes_{\mathbf{F}} (\mathbf{F} \otimes_{\mathbb{Q}_p} \mathbf{D}) \simeq \text{Mat}_2(\mathbf{B}'),$$

where  $\mathbf{B}'$  is a quaternion algebra over  $\mathbf{F}$ , which is determined by the relation

$$\text{inv}(\mathbf{B}') = 1/2[F : \mathbb{Q}_p] - \text{inv}(\mathbf{B}).$$

The alternating pairing

$$\langle \cdot, \cdot \rangle : \tilde{N} \times \tilde{N} \rightarrow B(\mathbb{F}_{p^2})$$

has values in  $B(\mathbb{F}_{p^2})$  satisfying  $\langle Fx, y \rangle = \langle x, Vy \rangle^\sigma$ . Define

$$\psi(x, y) := \text{Tr}_{B(\mathbb{F}_{p^2})/\mathbb{Q}_p} \langle x, Fy \rangle.$$

For all  $x, y \in \tilde{N}$ ,  $a \in \mathbf{D}$ , and  $b \in \mathbf{B}$ , one has

- (i)  $\psi(y, x) = -\psi(x, y)$ ,
- (ii)  $\psi(ax, y) = \psi(x, a^{*_{\mathbf{D}}}y)$ , and

(iii)  $\psi(bx, y) = \psi(x, b^*y)$ .

That is,  $\tilde{N}$  is a  $\mathbb{Q}_p$ -valued skew-Hermitian  $\mathbf{B} \otimes_{\mathbb{Q}_p} \mathbf{D}$ -module with respect to the product involution  $* \otimes *_{\mathbf{D}}$ . We check (i)–(iii). For (i),

$$\psi(y, x) = \text{Tr}\langle y, \mathbf{F}x \rangle = \text{Tr}\langle \mathbf{F}y, x \rangle^\sigma = -\text{Tr}\langle x, \mathbf{F}y \rangle = -\psi(x, y).$$

For (ii), one has for  $a \in B(\mathbb{F}_{p^2}) \subset \mathbf{D}$

$$\begin{aligned} \psi(ax, y) &= \text{Tr}\langle ax, \mathbf{F}y \rangle = \text{Tr}\langle x, \mathbf{F}a^\sigma y \rangle = \psi(x, a^\sigma y), \\ \psi(\mathbf{F}x, y) &= \text{Tr}\langle \mathbf{F}x, \mathbf{F}y \rangle = \text{Tr}p\langle x, y \rangle^\sigma = \text{Tr}\langle x, \mathbf{F}^2y \rangle = \psi(x, \mathbf{F}y). \end{aligned}$$

For (iii),

$$\psi(bx, y) = \text{Tr}\langle bx, \mathbf{F}y \rangle = \text{Tr}\langle x, \mathbf{F}b^*y \rangle = \psi(x, b^*y).$$

Note that if we replace  $\tilde{N}$  by  $\tilde{N}' := \{x \in N \mid \mathbf{F}^2x + px = 0\}$ , then  $\mathbf{F} = -\mathbf{V}$  on  $\tilde{N}'$  and the pairing  $\psi'(x, y) := \text{Tr}\langle x, \mathbf{F}y \rangle$  becomes Hermitian instead of skew-Hermitian. Moreover, the adjoint involution  $*'_{\mathbf{D}}$  on  $\mathbf{D}$  is the canonical involution.

Since the canonical involution  $*$  is symplectic and  $*_{\mathbf{D}}$  is orthogonal, the product involution  $* \otimes *_{\mathbf{D}}$  is symplectic. Therefore, we can choose an  $\mathbf{F}$ -algebra isomorphism  $\mathbf{B} \otimes_{\mathbb{Q}_p} \mathbf{D} \simeq \text{Mat}_2(\mathbf{B}')$  such that the induced involution is the map  $(b_{ij}) \mapsto (b'_{ji})$ , where  $*'$  is the canonical involution on  $\mathbf{B}'$ .

Let  $e_{11}$  and  $e_{22}$  be the standard idempotents of  $\text{Mat}_2(\mathbf{B}')$  and let  $\tilde{N} = \tilde{N}_1 \oplus \tilde{N}_2$  be the corresponding decomposition. We have proven

**THEOREM 8.2.** — *The association  $(N, \langle \cdot, \cdot \rangle) \mapsto (\tilde{N}_1, \psi)$  gives rise to a bijection between the set  $I(\nu_s)$  and the set of isomorphism classes of  $\mathbb{Q}_p$ -valued skew-Hermitian free  $\mathbf{B}'$ -modules of  $\mathbf{B}'$ -rank  $m_s$ , where  $\mathbf{B}'$  is the quaternion algebra (unique up to isomorphism) over  $\mathbf{F}$  with  $\text{inv}(\mathbf{B}') = 1/2[\mathbf{F} : \mathbb{Q}_p] - \text{inv}(\mathbf{B})$ .  $\square$*

**COROLLARY 8.3.**

- (1) *If  $\mathbf{B}'$  is the matrix algebra, then there is a natural bijection between the set  $I(\nu_s)$  and the set of isomorphism classes of non-degenerate symmetric spaces over  $\mathbf{F}$  of dimension  $2m_s$ .*
- (2) *If  $\mathbf{B}'$  is the quaternion division algebra, then there is a natural bijection between the set  $I(\nu_s)$  and the set of isomorphism classes of non-degenerate skew-Hermitian  $\mathbf{B}'$ -modules of  $\mathbf{B}'$ -rank  $m_s$ .*

*Proof.* — For the matrix algebra case, we do the Morita equivalence again as before. The corollary follows from Theorem 8.2.  $\square$

In the following we use the theory of quadratic forms and the skew-Hermitian quaternionic forms over local fields; see O’Meara [20, Chapter IV] and Tsukamoto [32].

Consider non-degenerate symmetric spaces  $V$  of dimension  $n_0$  over a non-Archimedean local field  $k_0$  of characteristic different from 2. Recall the discriminant  $\delta V \in k_0^\times/k_0^{\times 2}$  of  $V$  is defined by

$$\delta V := (-1)^{\lfloor n_0/2 \rfloor} \det V.$$

Note that we have  $\delta V = [1]$  when  $V$  is the hyperbolic plane. Let  $SV \in \{\pm 1\}$  denote the Hasse symbol of  $V$  (see [20, p. 167]). Denote by  $Q(n_0)$  the set of isomorphism classes of non-degenerate symmetric spaces  $V$  of dimension  $n_0$  over  $k_0$ .

**THEOREM 8.4.**

- (1) For any  $n_0 \geq 1$ , the map  $(\delta, S) : Q(n_0) \rightarrow k_0^\times/k_0^{\times 2} \times \{\pm 1\}$  is injective. This map is also surjective for any  $n_0 \geq 3$ .
- (2) For  $n_0 = 1$ , the map  $\delta : Q(1) \simeq k_0^\times/k_0^{\times 2}$  is a bijection.
- (3) For  $n_0 = 2$ , the image of the map  $(\delta, S)$  is

$$\{([a], \pm 1); [a] \neq [1]\} \cup \left\{ \left( [1], \left( \frac{-1, -1}{k_0} \right) \right) \right\},$$

where  $(-1, -1/k_0)$  is a Hilbert symbol.

*Proof.* — See [20, Theorems 63:20, 63:22 and 63:23 p. 170–171]. □

**COROLLARY 8.5.** — *If  $k_0$  is non-dyadic, then one has*

$$|Q(1)| = 4, \quad |Q(2)| = 7, \quad \text{and} \quad |Q(n_0)| = 8, \quad \forall n_0 \geq 3. \quad \square$$

Let  $B_0$  be the quaternion division algebra over  $k_0$  together with the canonical involution  $*$ . Denote by  $SQ(n_0)$  the set of isomorphism classes of skew-Hermitian  $B_0$ -modules  $(V, \psi)$  of rank  $n_0$  for  $n_0 \geq 1$ . The discriminant  $\delta V \in k_0^\times/k_0^{\times 2}$  is defined by

$$\delta V := (-1)^{\lfloor n_0/2 \rfloor} \text{Nr}(\psi(e_i, e_j)),$$

where  $\{e_i\}$  is a basis for  $V$  over  $B_0$  and  $\text{Nr} : \text{Mat}_{n_0}(B_0) \rightarrow k_0$  is the reduced norm.

**THEOREM 8.6.**

- (1) For  $n_0 \geq 2$ , the map  $\delta : SQ(n_0) \rightarrow k_0^\times/k_0^{\times 2}$  is a bijection.
- (2) For  $n_0 = 1$ , the map  $\delta : SQ(n_0) \rightarrow k_0^\times/k_0^{\times 2}$  is injective and its image is equal to  $\{[a]; [a] \neq [1]\}$ .

*Proof.* — This is [32, Theorem 3]. □

**COROLLARY 8.7.** — *If  $k_0$  is non-dyadic, then one has*

$$|SQ(1)| = 3, \quad \text{and} \quad |SQ(n_0)| = 4, \quad \forall n_0 \geq 2. \quad \square$$

PROPOSITION 8.8. — *Let the notation be as above and assume that  $p \neq 2$ .*

(1) *If  $\mathbf{B}'$  is the matrix algebra, then we have*

$$(8.1) \quad |I(\nu_s)| = \begin{cases} 7 & \text{if } m_s = 1, \\ 8 & \text{if } m_s \geq 2. \end{cases}$$

(2) *If  $\mathbf{B}'$  is the quaternion division algebra, then we have*

$$(8.2) \quad |I(\nu_s)| = \begin{cases} 3 & \text{if } m_s = 1, \\ 4 & \text{if } m_s \geq 2. \end{cases}$$

*Proof.* — These follow from Corollaries 8.3, 8.5 and 8.7. □

Combining Lemma 8.1, Theorems 8.2, 8.4 and 8.6, we obtain an explicit classification of isogeny classes of quasi-polarized  $p$ -divisible  $\mathcal{O}_{\mathbf{B}}$ -modules over  $k$ .

### 9. Local model for $\mathcal{M}_K^{(p)}$

We shall use the notation of Section 2 and Section 4.1. For the remainder of this paper, we assume  $p \neq 2$ .

#### 9.1. Local models

Let  $\Lambda$  be a free  $O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$ -module of rank  $m$  together with a perfect  $\mathbb{Z}_p$ -valued skew-Hermitian pairing

$$\psi : \Lambda \times \Lambda \rightarrow \mathbb{Z}_p.$$

For such a lattice  $\Lambda$ , we define, following Rapoport and Zink [29], a projective  $\mathbb{Z}_p$ -scheme  $\mathbf{M}_{\Lambda}$ , called the *local model associated to  $\Lambda$*  (and  $\psi$ ), which represents the following functor. For any  $\mathbb{Z}_p$ -scheme  $S$ ,  $\mathbf{M}_{\Lambda}(S)$  is the set of locally free  $\mathcal{O}_S$ -submodules  $\mathcal{F} \subset \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S$  of rank  $m[B : \mathbb{Q}]/2 = 2dm$  such that

- (i)  $\mathcal{F}$  is isotropic with respect to the pairing  $\psi$ ;
- (ii) locally for Zariski topology on  $S$ ,  $\mathcal{F}$  is a direct summand of  $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S$ ;
- (iii)  $\mathcal{F}$  is invariant under the  $O_B$ -action;
- (iv)  $\mathcal{F}$  satisfies the determinant condition (cf. Section 2.3):

$$(K) \quad \text{char}(a|\Lambda \otimes \mathcal{O}_S/\mathcal{F}) = \text{char}(a)^m \in \mathcal{O}_S[T], \quad \forall a \in O_B.$$

Recall that for an abelian scheme  $A$  over a base scheme  $S$ , we have the Hodge filtration

$$0 \rightarrow \omega_{A/S} \rightarrow H_{\text{DR}}^1(A/S) \rightarrow \text{Lie}(A^t/S) \rightarrow 0.$$

Taking the dual one obtains the short exact sequence

$$0 \rightarrow \omega_{A^t/S} \rightarrow H_1^{\text{DR}}(A/S) \rightarrow \text{Lie}(A/S) \rightarrow 0.$$

If  $M$  is the covariant Dieudonné module of an abelian variety  $A$  over a perfect field  $k_0$ , then there is a canonical isomorphism  $M/pM \simeq H_1^{\text{DR}}(A/k_0)$  with the Hodge filtration  $\mathbb{V}M/pM$  corresponding to  $\omega_{A^t}$ . This justifies the definition of the determinant condition for objects in the local model  $\mathbf{M}_\Lambda$  in (iv).

By an automorphism of the lattice  $\Lambda \otimes \mathcal{O}_S$ , where  $S$  is a  $\mathbb{Z}_p$ -scheme, we mean an  $\mathcal{O}_B \otimes \mathbb{Z}_p$ -linear automorphism of the  $\mathcal{O}_S$ -module  $\Lambda \otimes \mathcal{O}_S$  that preserves the pairing  $\psi$ . We denote by  $\text{Aut}_{\mathcal{O}_B \otimes \mathcal{O}_S}(\Lambda \otimes \mathcal{O}_S, \psi)$  the group of automorphisms of  $\Lambda \otimes \mathcal{O}_S$ .

Let  $\mathcal{G} = \text{Aut}_{\mathcal{O}_B \otimes \mathbb{Z}_p}(\Lambda, \psi)$  be the group scheme over  $\mathbb{Z}_p$  that represents the group functor

$$S \mapsto \text{Aut}_{\mathcal{O}_B \otimes \mathcal{O}_S}(\Lambda \otimes \mathcal{O}_S, \psi).$$

By [29, Theorem 3.16] and [23, Theorem 2.2(a)],  $\mathcal{G}$  is an affine smooth group scheme over  $\mathbb{Z}_p$ . The generic fiber  $\mathcal{G}_{\mathbb{Q}_p}$  of  $\mathcal{G}$  is a  $\mathbb{Q}_p$ -form of  $(\text{Res}_{F/\mathbb{Q}} \mathcal{O}_{2m, F}) \otimes_{\mathbb{Q}} \mathbb{Q}_p$ ; see (2.4). The group scheme  $\mathcal{G}$  acts naturally on  $\mathbf{M}_\Lambda$  on the left.

### 9.2. Local model diagrams

We impose a level structure away from  $p$  on  $\mathcal{M}_K^{(p)}$  and consider a fine moduli scheme  $\mathcal{M}_{K,*}^{(p)}$  over  $\mathbb{Z}_{(p)}$ . More precisely, write  $\mathcal{M}_K^{(p)} = \coprod_{p \nmid D} \mathcal{M}_{K,D}$ , where  $\mathcal{M}_{K,D} = \mathcal{M}_K \cap \mathcal{M}_D$  and  $\mathcal{M}_D$  is defined in (2.1). For each  $D$ , we fix a prime-to- $pD$  integer  $N_D \geq 3$ . Set  $\mathcal{M}_{K,*}^{(p)} := \coprod_{p \nmid D} \mathcal{M}_{K,D,N_D}$ , where  $\mathcal{M}_{K,D,N_D}$  is the moduli scheme over  $\mathbb{Z}_{(p)}$  which parameterizes objects in  $\mathcal{M}_{K,D}$  with a level  $N_D$  structure.

Let  $S$  be a  $\mathbb{Z}_p$ -scheme and  $\underline{A} = (A, \lambda, \iota, \eta)$  be an object in  $\mathcal{M}_{K,*}^{(p)}(S)$ . A *trivialization*  $\gamma$  of the de Rham homology  $H_1^{\text{DR}}(A/S)$  by  $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S$  is an  $\mathcal{O}_B \otimes \mathbb{Z}_p$ -linear isomorphism  $\gamma : H_1^{\text{DR}}(A/S) \rightarrow \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S$  of  $\mathcal{O}_S$ -modules such that  $\psi(\gamma(x), \gamma(y)) = \langle x, y \rangle_\lambda$  for  $x, y \in H_1^{\text{DR}}(A/S)$ , where

$$\langle \cdot, \cdot \rangle_\lambda : H_1^{\text{DR}}(A/S) \times H_1^{\text{DR}}(A/S) \rightarrow \mathcal{O}_S$$

is the perfect alternating pairing induced by  $\lambda$ .



Let  $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}_{K,*}^{(p)}}$  denote the moduli space over  $\mathbb{Z}_p$  that parametrizes equivalence classes of objects  $(\underline{A}, \gamma)_S$ , where

- $\underline{A} = (A, \lambda, \iota, \eta)$  is an object over a  $\mathbb{Z}_p$ -scheme  $S$  in  $\mathcal{M}_{K,*}^{(p)} \otimes \mathbb{Z}_p$ , and
- $\gamma$  is a trivialization of  $H_1^{\text{DR}}(A/S)$  by  $\Lambda \otimes \mathcal{O}_S$ .

The moduli scheme  $\widetilde{\mathcal{M}}$  has two natural projections  $\varphi^{\text{mod}}$  and  $\varphi^{\text{loc}}$ . The morphism

$$\varphi^{\text{mod}} : \widetilde{\mathcal{M}} \rightarrow \mathcal{M}_{K,*}^{(p)} \otimes \mathbb{Z}_p$$

forgets the trivialization. The morphism

$$\varphi^{\text{loc}} : \widetilde{\mathcal{M}} \rightarrow \mathbf{M}_\Lambda$$

sends any object  $(\underline{A}, \gamma)$  to  $\gamma(\omega_{A^t/S})$ , where  $\omega_{A^t/S} \subset H_1^{\text{DR}}(A/S)$  is the  $\mathcal{O}_S$ -submodule in the Hodge filtration. Thus, we have the so called *local model diagram*:

$$(9.1) \quad \mathcal{M}_{K,*}^{(p)} \otimes \mathbb{Z}_p \xleftarrow{\varphi^{\text{mod}}} \widetilde{\mathcal{M}} \xrightarrow{\varphi^{\text{loc}}} \mathbf{M}_\Lambda.$$

The local model diagram above was introduced by Rapoport and Zink [29] in a more general setting. The moduli scheme  $\widetilde{\mathcal{M}}$  also admits a left action by the group scheme  $\mathcal{G}$ . Recall that  $k$  denotes an algebraically closed field of characteristic  $p > 0$  and  $W = W(k)$  the ring of Witt vectors over  $k$ . Using Corollary 5.7, for any  $k$ -valued point  $\underline{A}$  in  $\mathcal{M}_{K,*}^{(p)}$ , there is an  $\mathcal{O}_B \otimes \mathbb{Z}_p$ -linear isomorphism of  $W$ -modules

$$M(A) \simeq \Lambda \otimes W$$

which is compatible with the alternating pairings. This shows that the morphism  $\varphi^{\text{mod}}$  is surjective. It follows that  $\varphi^{\text{mod}}$  is a left  $\mathcal{G}$ -torsor, and hence this morphism is affine and smooth.

By the Grothendieck–Messing deformation theory for abelian schemes ([4, 14] and [29, 3.29]), for any  $k$ -valued point  $x$  of  $\mathcal{M}_{K,*}^{(p)}$ , there is a  $k$ -valued point  $y$  in  $\mathbf{M}_\Lambda$  such that there is a (non-canonical) isomorphism

$$(9.2) \quad \mathcal{M}_{K,*}^{(p)}|_x^\wedge \simeq \mathbf{M}_\Lambda|_y^\wedge$$

of formal local moduli spaces. This shows, in particular, that if the local model  $\mathbf{M}_\Lambda$  is flat over  $\text{Spec } \mathbb{Z}_p$ , then the integral model  $\mathcal{M}_{K,*}^{(p)} \otimes \mathbb{Z}_p$  is flat over  $\text{Spec } \mathbb{Z}_p$  [29, p. 95].

The morphism  $\varphi^{\text{loc}}$  is smooth,  $\mathcal{G}$ -equivariant, and of relative dimension the same as  $\varphi^{\text{mod}}$ . However, at this moment we do not know whether the morphism  $\varphi^{\text{loc}}$  is surjective. If this is so, then the integral model  $\mathcal{M}_{K,*}^{(p)} \otimes \mathbb{Z}_p$  is flat over  $\text{Spec } \mathbb{Z}_p$  if and only if the local model  $\mathbf{M}_\Lambda$  is flat over  $\text{Spec } \mathbb{Z}_p$ .

**9.3. A reduction step**

Let  $\Lambda$  and  $\mathbf{M}_\Lambda$  be as above. Let

$$\Lambda = \bigoplus_{v|p} \Lambda_v$$

be the decomposition of  $\Lambda$  obtained from the decomposition  $O_F \otimes \mathbb{Z}_p = \prod_{v|p} \mathcal{O}_v$ , where  $\mathcal{O}_v$  is the ring of integers in the local field  $F_v$  of  $F$  at  $v$ . Then we have

$$\mathbf{M}_\Lambda = \prod_{v|p} \mathbf{M}_{\Lambda_v},$$

where the product  $\prod$  means the fiber product of the schemes  $\mathbf{M}_{\Lambda_v}$ 's over  $\text{Spec } \mathbb{Z}_p$  and  $\mathbf{M}_{\Lambda_v}$  is the local model defined by the lattice  $\Lambda_v$  in the same way as  $\mathbf{M}_\Lambda$ ; see Section 9.1.

Write  $O_B \otimes \mathbb{Z}_p = \prod_{v|p} \mathcal{O}_{B_v}$  for the decomposition with respect to  $O_F \otimes \mathbb{Z}_p = \prod_{v|p} \mathcal{O}_v$ . Then  $\mathcal{O}_{B_v}$  is a maximal order in  $B_v$ . Similarly we have the automorphism group scheme  $\mathcal{G}_v = \text{Aut}_{\mathcal{O}_{B_v}}(\Lambda_v, \psi_v)$  associated to the local lattice  $(\Lambda_v, \psi_v)$ , and we have the fiber product decomposition

$$\mathcal{G} = \prod_{v|p} \mathcal{G}_v.$$

Now we fix a place  $v$  of  $F$  over  $p$ . Let  $\mathcal{O}_v^{\text{nr}} \subset \mathcal{O}_v$  be the maximal étale extension of  $\mathbb{Z}_p$  in  $\mathcal{O}_v$  and put

$$I_v := \text{Hom}_{\mathbb{Z}_p}(\mathcal{O}_v^{\text{nr}}, W).$$

Let  $e = e_v$  be the ramification index and  $f = f_v$  be the inertia degree. Let  $\pi$  be a uniformizer of  $\mathcal{O}_v$  and let  $P(T)$  be the minimal polynomial of  $\pi$  over  $\mathcal{O}_v^{\text{nr}}$ . For any  $\sigma \in I_v$  put  $W_\sigma := W[T]/(\sigma(P(T)))$  and denote by  $\pi$  again the image of  $T$  in  $W_\sigma$ . One has  $W_\sigma = W[\pi]$  and the element  $\pi$  satisfies the equation  $\sigma(P(T)) = 0$ . We have the decomposition

$$\Lambda_v \otimes_{\mathbb{Z}_p} W = \bigoplus_{\sigma \in I_v} \Lambda_\sigma, \quad \Lambda_\sigma := \Lambda_v \otimes_{\mathcal{O}_v^{\text{nr}}, \sigma} W.$$

Write

$$\psi_\sigma : \Lambda_\sigma \times \Lambda_\sigma \rightarrow W$$

for the induced alternating pairing.

Similarly we define the local model  $\mathbf{M}_{\Lambda_\sigma}$  over  $\text{Spec } W$  attached to each skew-Hermitian lattice  $(\Lambda_\sigma, \psi_\sigma)$ . If  $\mathcal{F}_v \subset \Lambda_v \otimes \mathcal{O}_S$  is an object in  $\mathbf{M}_{\Lambda_v}$  and let  $\mathcal{F}_v = \bigoplus_{\sigma \in I_v} \mathcal{F}_\sigma$  be the natural decomposition, then every factor  $\mathcal{F}_\sigma$  is a locally free  $\mathcal{O}_S$ -module of rank  $2me$ ; this follows from the determinant condition (K). Therefore we have a natural isomorphism

$$\mathbf{M}_{\Lambda_v} \otimes W \simeq \prod_{\sigma \in I_v} \mathbf{M}_{\Lambda_\sigma}, \quad \mathcal{F}_v \mapsto (\mathcal{F}_\sigma)_{\sigma \in I_v},$$

where the product  $\amalg$  means the fiber product of the schemes  $\mathbf{M}_{\Lambda_\sigma}$ 's over  $\text{Spec } W$ .

### 9.4. Maps to the local model

A basic question in the local model diagram asks whether the morphism  $\varphi^{\text{loc}}$  is surjective. The local model diagram gives rise to a morphism of Artin stacks

$$(9.3) \quad \theta : \mathcal{M}_{K,*}^{(p)} \otimes \mathbb{Z}_p \rightarrow [\mathcal{G} \backslash \mathbf{M}_\Lambda],$$

and this amounts to asking about the surjectivity of the map of the sets of geometric points

$$(9.4) \quad \theta_k : \mathcal{M}_{K,*}^{(p)}(k) \rightarrow \mathcal{G}(k) \backslash \mathbf{M}_\Lambda(k).$$

Let  $\text{Dieu}_m^{O_B \otimes \mathbb{Z}_p}(k)$  denote the set of isomorphism classes of separably quasi-polarized Dieudonné  $O_B \otimes \mathbb{Z}_p$ -modules of  $W$ -rank  $4md$  satisfying the determinant condition. The map  $\theta_k$  factors through the natural map  $\mathcal{M}_{K,*}^{(p)}(k) \rightarrow \text{Dieu}_m^{O_B \otimes \mathbb{Z}_p}(k)$ ,  $\underline{A} \mapsto M(\underline{A})$ , and let

$$(9.5) \quad \alpha : \text{Dieu}_m^{O_B \otimes \mathbb{Z}_p}(k) \rightarrow \mathcal{G}(k) \backslash \mathbf{M}_\Lambda(k)$$

be the induced map. We will show in Section 11 that  $\alpha$  is surjective when  $m = 1$ .

## 10. Computation of local models for $m = 1$

Keep the notation of the previous section. For the remainder of this paper, assume that  $m = 1$ . We shall compute the geometric special fiber  $\mathbf{M}_{\Lambda_\sigma} \otimes k$  of  $\mathbf{M}_{\Lambda_\sigma}$ . Put  $\mathcal{O}_{B_\sigma} := \mathcal{O}_{B_v} \otimes_{\mathcal{O}_{v^{\text{nr}},\sigma}} W$  for  $\sigma \in I_v$ .

### 10.1. Unramified case.

Suppose  $v$  is unramified in  $B$ . Then  $\mathcal{O}_{B_\sigma} = \text{Mat}_2(W_\sigma)$ . By the Morita equivalence reduction as before, we have  $\Lambda_\sigma = \Lambda_{\sigma,1} \oplus \Lambda_{\sigma,2}$  and a unimodular Hermitian pairing

$$\varphi_\sigma : \Lambda_{\sigma,1} \times \Lambda_{\sigma,1} \rightarrow W.$$

Recall that  $\varphi_\sigma(x, y)$  is the restriction of the symmetric pairing  $\psi(x, Cy)$  on the first factor  $\Lambda_{\sigma,1}$ , where  $C$  is the Weil element. The local model  $\mathbf{M}_{\varphi_\sigma}$  associated to the symmetric lattice  $(\Lambda_{\sigma,1}, \psi_\sigma)$  is defined by parametrizing the  $W_\sigma \otimes \mathcal{O}_S$ -submodules  $\mathcal{F}$  of  $\Lambda_{\sigma,1} \otimes \mathcal{O}_S$  with the following properties:

- (i)  $\mathcal{F}$  is a locally free  $\mathcal{O}_S$ -module of rank  $e$  and locally for Zariski topology on  $S$  is a direct summand of  $\Lambda_{\sigma,1} \otimes \mathcal{O}_S$ ;
- (ii)  $\mathcal{F}$  is isotropic with respect to the pairing  $\varphi_\sigma$ .

Every object  $\mathcal{F}_\sigma$  in  $\mathbf{M}_{\Lambda_\sigma}$  has the decomposition  $\mathcal{F}_\sigma = \mathcal{F}_{\sigma,1} \oplus \mathcal{F}_{\sigma,2}$ .

LEMMA 10.1. — *The map which sends any object  $\mathcal{F}_\sigma$  in  $\mathbf{M}_{\Lambda_\sigma}$  to its first factor  $\mathcal{F}_{\sigma,1}$  induces an isomorphism of schemes*

$$\mathbf{M}_{\Lambda_\sigma} \simeq \mathbf{M}_{\varphi_\sigma}.$$

*Proof.* — It suffices to check that  $\mathcal{F}_\sigma$  is isotropic with respect to the pairing  $\psi_\sigma$  if and only if  $\mathcal{F}_{\sigma,1}$  is isotropic with respect to the pairing  $\varphi_\sigma(x, y) = \psi_\sigma(x, Cy)$ . Using  $e_{11}^* = e_{22}$ , we get  $\psi_\sigma(\mathcal{F}_\sigma, \mathcal{F}_\sigma) = 0$  if and only if  $\psi_\sigma(\mathcal{F}_{\sigma,1}, \mathcal{F}_{\sigma,2}) = 0$ . On the other hand, the isomorphism  $C : \Lambda_\sigma \xrightarrow{\sim} \Lambda_\sigma$  induces the isomorphism  $C : \Lambda_{\sigma,1} \xrightarrow{\sim} \Lambda_{\sigma,2}$ . Therefore,  $\varphi_\sigma(\mathcal{F}_{\sigma,1}, \mathcal{F}_{\sigma,1}) = 0$  if and only if  $\psi_\sigma(\mathcal{F}_{\sigma,1}, \mathcal{F}_{\sigma,2}) = 0$ . This shows the lemma.  $\square$

Put  $\bar{\Lambda}_\sigma := \Lambda_\sigma/p\Lambda_\sigma$  and  $\bar{\Lambda}_{\sigma,1} := \Lambda_{\sigma,1}/p\Lambda_{\sigma,1}$ . Let  $\mathcal{D}_{W_\sigma/W}^{-1}$  be the inverse different of the extension  $W_\sigma/W$  and choose a generator  $\delta_\sigma$  of this fractional ideal. Then there is a unique  $W_\sigma$ -valued  $W_\sigma$ -bilinear symmetric pairing

$$\varphi'_\sigma : \Lambda_{\sigma,1} \times \Lambda_{\sigma,1} \rightarrow W_\sigma$$

such that  $\varphi_\sigma(x, y) = \text{Tr}[\delta_\sigma \cdot \varphi'_\sigma(x, y)]$ . One can show that a  $k[\pi]/(\pi^e)$ -submodule  $\mathcal{F}_{\sigma,1} \subset \bar{\Lambda}_{\sigma,1}$  is isotropic with respect to the pairing  $\varphi_\sigma$  if and only if so is it for the pairing  $\varphi'_\sigma$ .

Since  $\Lambda_{\sigma,1}$  is a self-dual lattice and  $k$  is algebraically closed, we can choose a  $W_\sigma$ -basis  $x_1, x_2$  for  $\Lambda_{\sigma,1}$  such that

$$\varphi'_\sigma(x_1, x_1) = \varphi'_\sigma(x_2, x_2) = 0 \quad \text{and} \quad \varphi'_\sigma(x_1, x_2) = \varphi'_\sigma(x_2, x_1) = 1.$$

Denote by  $\bar{x}_i$ , for  $i = 1, 2$ , the image of  $x_i$  in  $\bar{\Lambda}_{\sigma,1}$ . Let  $\mathcal{F} \subset \bar{\Lambda}_{\sigma,1}$  be an object in  $\mathbf{M}_{\varphi_\sigma}(k)$ . As  $\bar{\Lambda}_{\sigma,1}$  is a free  $k[\pi]/(\pi^e)$ -module of rank two, one has

$$\bar{\Lambda}_{\sigma,1}/\mathcal{F} \simeq k[\pi]/(\pi^{e_1}) \oplus k[\pi]/(\pi^{e_2})$$

for some integers  $e_1, e_2$  with  $0 \leq e_1 \leq e_2 \leq e$  and  $e_1 + e_2 = e$ . The pair  $(e_1, e_2)$  will be called the *Lie type* of the object  $\mathcal{F}$ . We can write

$$\mathcal{F} = \text{Span}\{\pi^{e_1}\bar{y}_1, \pi^{e_2}\bar{y}_2\},$$

where  $\bar{y}_1$  and  $\bar{y}_2$  generate  $\bar{\Lambda}_{\sigma,1}$  over  $k[\pi]/(\pi^e)$ . Moreover, we can write either

- (a)  $\bar{y}_1 = \bar{x}_1 + t\bar{x}_2$  and  $\bar{y}_2 = \bar{x}_2$ , or
- (b)  $\bar{y}_1 = t\bar{x}_1 + \bar{x}_2$  and  $\bar{y}_2 = \bar{x}_1$ ,

where  $t \in k[\pi]/(\pi^e)$ . We can represent  $t$  as

$$t = t_0 + t_1\pi + \dots + t_{e-2e_1-1}\pi^{e-2e_1-1}, \quad t_i \in k$$

because if  $\text{ord}_\pi(t) \geq e - 2e_1$  then one can replace  $\bar{x}_1 + t\bar{x}_2$  by  $\bar{x}_1$  in the case (a) (and the same for the case (b)). Now one easily computes that

$$\varphi'_\sigma(\mathcal{F}, \mathcal{F}) = 0 \iff 2t\pi^{2e_1} = 0.$$

This condition gives  $t_0\pi^{2e_1} + \dots + t_{e-2e_1-1}\pi^{e-1} = 0$  and hence

$$t_0 = \dots = t_{e-2e_1-1} = 0.$$

Therefore, we get two objects.

$$\mathcal{F} = \text{Span} \{ \pi^{e_1}\bar{x}_1, \pi^{e_2}\bar{x}_2 \}, \quad \text{or} \quad \mathcal{F} = \text{Span} \{ \pi^{e_1}\bar{x}_2, \pi^{e_2}\bar{x}_1 \}.$$

Notice that these two members are in the same orbit under the action of the group  $\mathcal{G}_\sigma(k)$  as the automorphism of  $\bar{\Lambda}_{\sigma,1}$  switching  $\bar{x}_1$  and  $\bar{x}_2$  lies in  $\mathcal{G}_\sigma(k)$ , where

$$\mathcal{G}_\sigma = \text{Aut}_{W_\sigma}(\Lambda_{\sigma,1}, \varphi_\sigma)$$

is the automorphism group scheme of the symmetric lattice  $(\Lambda_{\sigma,1}, \varphi_\sigma)$  over  $W$ . We obtain the following the result.

PROPOSITION 10.2. — *Assume that  $v$  is unramified in  $B$  and let  $\sigma \in I_v$ . Then  $\mathbf{M}_{\varphi_\sigma}(k)$  consists of the  $k[\pi]/(\pi^e)$ -submodules*

$$\mathcal{F}_{e_1} = \text{Span} \{ \pi^{e_1}\bar{x}_1, \pi^{e-e_1}\bar{x}_2 \}, \quad \text{for } 0 \leq e_1 \leq e.$$

Moreover, two objects  $\mathcal{F}_{e_1}$  and  $\mathcal{F}_{e'_1}$  are in the same orbit under the action of  $\mathcal{G}_\sigma(k)$  if and only if  $e_1 = e'_1$  or  $e_1 + e'_1 = e$ . □

PROPOSITION 10.3. — *Assume that  $v$  is unramified in  $B$ , and let  $\sigma \in I_v$ .*

- (1) *The special fiber  $\mathbf{M}_{\varphi_\sigma} \otimes_W k$  is zero-dimensional and two objects  $\mathcal{F}$  and  $\mathcal{F}'$  in  $\mathbf{M}_{\varphi_\sigma}(k)$  are in the same orbit under  $\mathcal{G}_\sigma(k)$  if and only if they have the same Lie type.*
- (2) *The structure morphism  $\mathbf{M}_{\varphi_\sigma} \rightarrow \text{Spec } W$  is finite and flat.*

*Proof.*

(1) This follows immediately from Proposition 10.2.

(2) Since the structure morphism is quasi-finite and projective, it is finite. We now show that any object  $\mathcal{F}_0$  in  $\mathbf{M}_{\varphi_\sigma}(k)$  can be lifted to an object  $\mathcal{F}_R$  over an integral domain  $R$  with residue field  $k$  and fraction field  $K$  of characteristic zero. Then the coordinate ring of  $\mathbf{M}_{\varphi_\sigma}$  is torsion-free as a  $W$ -module and hence is flat over  $W$ .

By Proposition 10.2, write  $\mathcal{F}_0 = \text{Span}\{\pi^{e_1}\bar{x}_1, \pi^{e_2}\bar{x}_2\}$  for two integers  $e_1, e_2$  with  $0 \leq e_1, e_2 \leq e$  and  $e_1 + e_2 = e$ . Write  $W_\sigma = W[T]/(\sigma P(T))$ .

Let  $R$  be the ring of integers in a finite separable field extension  $K$  of  $B(k) = \text{Frac}(W)$  such that the polynomial  $\sigma P(T)$  decomposes completely over  $R$ :

$$\sigma P(T) = (T - \pi_1) \cdots (T - \pi_e) \in R[T].$$

Let  $\pi_R$  be a uniformizer of  $R$ . We have  $W_\sigma \otimes_W R = R[T]/(\sigma P(T))$ . As  $W_\sigma$  is a free  $W$ -module, we have an exact sequence:

$$0 \longrightarrow W_\sigma \otimes_W (\pi_R) \longrightarrow W_\sigma \otimes_W R \longrightarrow W_\sigma \otimes_W k \longrightarrow 0.$$

So an element  $f(T)$  in  $R[T]/(\sigma P(T))$  specializes to zero in  $W_\sigma \otimes k = k[T]/(T^e)$  if and only if  $f(T) \in \pi_R \cdot R[T]/(\sigma P(T))$ . We shall construct a  $W_\sigma \otimes_W R$ -submodule  $\mathcal{F}_R \subset \Lambda_{\sigma,1} \otimes_W R$  such that

- (i)  $\mathcal{F}_R \otimes_R k = \mathcal{F}_0$ ;
- (ii)  $\mathcal{F}_R$  and  $(\Lambda_{\sigma,1} \otimes_W R)/\mathcal{F}_R$  are both free of rank  $e$  over  $R$ ;
- (iii)  $\mathcal{F}_R$  is isotropic with respect to the pairing  $\psi'_\sigma$ .

Now we let  $\mathcal{F}_R$  be the submodule generated by the elements  $(T - \pi_1) \cdots (T - \pi_{e_1})x_1$  and  $(T - \pi_{e_1+1}) \cdots (T - \pi_e)x_2$ . Clearly  $\pi_i \in \pi_R R$  for all  $i$  so one has (i). The statement (ii) follows from (i) by the right exactness of the tensor product. To check (iii), as  $\mathcal{F}_R \subset \mathcal{F}_K := \mathcal{F}_R \otimes K$ , it suffices to check (iii) for  $\mathcal{F}_K$ . Now we have

$$W_\sigma \otimes_W K = \prod_{i=1}^e K \quad \text{and} \quad \mathcal{F}_K = (\mathcal{F}_{K,i})_{1 \leq i \leq e}.$$

It is easy to see that each component  $\mathcal{F}_{K,i}$  is one-dimensional  $K$ -subspace generated by either  $x_1$  or  $x_2$  and hence  $\mathcal{F}_K$  satisfies (iii). □

Let  $\mathcal{F}_v$  be an object in  $\mathbf{M}_{\Lambda_v}(k)$  and let  $\mathcal{F}_v = \bigoplus_{\sigma \in I_v} \mathcal{F}_\sigma$  be the natural decomposition. The *reduced Lie type* of  $\mathcal{F}_v$  is defined to the system of pairs  $(e_{\sigma,1}, e_{\sigma,2})$  indexed by  $I_v$ , where  $(e_{\sigma,1}, e_{\sigma,2})$  is the Lie type of  $\mathcal{F}_{\sigma,1}$ . Proposition 10.3 immediately gives the following result.

**THEOREM 10.4.** — *Suppose that  $v$  is unramified in  $B$ .*

- (1) *The special fiber  $\mathbf{M}_{\Lambda_v} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$  is zero-dimensional and two objects  $\mathcal{F}_v$  and  $\mathcal{F}'_v$  in  $\mathbf{M}_{\Lambda_v}(k)$  are in the same orbit under the  $\mathcal{G}_v(k)$  if and only if they have the same reduced Lie type.*
- (2) *The structure morphism  $\mathbf{M}_{\Lambda_v} \rightarrow \text{Spec } \mathbb{Z}_p$  is flat and finite.* □

### 10.2. Ramified case.

Now we compute the local model  $\mathbf{M}_{\Lambda_v}$  for the case where  $v$  is ramified in  $B$ . Recall that  $\Lambda_v$  is a free  $\mathcal{O}_{B_v}$ -module of rank one together with a perfect

$\mathbb{Z}_p$ -valued skew-Hermitian pairing  $\psi_v : \Lambda_v \times \Lambda_v \rightarrow \mathbb{Z}_p$ . We fix an unramified quadratic field extension  $L_v \subset B_v$  as in Section 4.1. Notice that the ring  $\mathcal{O}_{L_v}$  of integers is contained in the unique maximal order  $\mathcal{O}_{B_v}$ . We choose a presentation  $\mathcal{O}_{B_v} = \mathcal{O}_{L_v}[\Pi]$  as in (4.1) and (4.2).

Let  $\mathcal{O}_{L_v^{\text{nr}}}$  denote the maximal étale extension over  $\mathbb{Z}_p$  in  $L_v$ , and put  $J_v := \text{Hom}_{\mathbb{Z}_p}(\mathcal{O}_{L_v^{\text{nr}}}, W)$ . Let  $\text{pr} : J_v \rightarrow I_v$  be the restriction map from  $\mathcal{O}_{L_v^{\text{nr}}}$  to  $\mathcal{O}_v^{\text{nr}}$ ; this is a two-to-one map. We have the decomposition

$$\Lambda_v \otimes_{\mathbb{Z}_p} W = \bigoplus_{\sigma \in I_v} \Lambda_\sigma, \quad \Lambda_\sigma = \Lambda_\tau \oplus \Lambda_{\tau'}$$

where  $\{\tau, \tau'\} = \text{pr}^{-1}(\sigma)$  and  $\Lambda_\tau$  (resp.  $\Lambda_{\tau'}$ ) is the  $\tau$ -component (resp.  $\tau'$ -component) of  $\Lambda_v$ . Notice that the pairing  $\psi_\sigma$  induces a perfect pairing

$$\psi_\sigma : \Lambda_\tau \times \Lambda_{\tau'} \rightarrow W.$$

Let

$$\psi'_\sigma : \Lambda_\tau \times \Lambda_{\tau'} \rightarrow W_\sigma$$

be the unique  $W_\sigma$ -valued  $W_\sigma$ -bilinear pairing such that  $\psi_\sigma(x, y) = \text{Tr}[\delta_\sigma \cdot \psi'_\sigma(x, y)]$ , where  $\delta_\sigma$  is a generator of the inverse different  $\mathcal{D}_{W_\sigma/W}^{-1}$ .

The local model  $\mathbf{M}_{\Lambda_\sigma}$  over  $\text{Spec } W$  parametrizes the  $W_\sigma \otimes_W \mathcal{O}_S$ -submodules

$$\mathcal{F}_v = \mathcal{F}_\tau \oplus \mathcal{F}_{\tau'} \subset (\Lambda_\tau \oplus \Lambda_{\tau'}) \otimes \mathcal{O}_S$$

such that

- (i)  $\mathcal{F}_\tau$  and  $\mathcal{F}_{\tau'}$  are locally free  $\mathcal{O}_S$ -modules of rank  $e$  and they are locally direct summands of  $\Lambda_\tau \otimes \mathcal{O}_S$  and  $\Lambda_{\tau'} \otimes \mathcal{O}_S$ , respectively;
- (ii)  $\Pi(\mathcal{F}_\tau) \subset \mathcal{F}_{\tau'}$  and  $\Pi(\mathcal{F}_{\tau'}) \subset \mathcal{F}_\tau$ ;
- (iii)  $\psi_\sigma(\mathcal{F}_\tau, \mathcal{F}_{\tau'}) = 0$ .

As  $\mathcal{F}_\tau$  and  $\mathcal{F}_{\tau'}$  are of rank  $e$ , condition (iii) says that one is the orthogonal complement of the other and hence one submodule determines the other.

We check that  $\psi_\sigma(\mathcal{F}_\tau, \mathcal{F}_{\tau'}) = 0$  if and only if  $\psi'_\sigma(\mathcal{F}_\tau, \mathcal{F}_{\tau'}) = 0$ . As  $\mathcal{D}^{-1} = \mathcal{D}_{W_\sigma/W}^{-1}$  is the largest  $W_\sigma$ -submodule in  $W_\sigma[1/p]$  such that  $\text{tr}(\mathcal{D}^{-1}) \subset W$ , one has  $\text{tr}(\pi^{-1}\mathcal{D}^{-1}) = p^{-1}W$ . So  $\text{tr}(\pi^{e-1}\mathcal{D}^{-1}) = W$ . Consider the structure map  $\phi : W \rightarrow \mathcal{O}_S$ . If  $\ker \phi = 0$ , then  $\psi'_\sigma(\mathcal{F}_\tau, \mathcal{F}_{\tau'}) \neq 0$  implies  $\psi_\sigma(\mathcal{F}_\tau, \mathcal{F}_{\tau'}) \neq 0$ . Suppose  $\ker \phi = p^r W$ . If  $\psi'_\sigma(\mathcal{F}_\tau, \mathcal{F}_{\tau'}) \neq 0$ , then

$$\delta_\sigma \psi'_\sigma(\mathcal{F}_\tau, \mathcal{F}_{\tau'}) \supset p^{r-1} \pi^{e-1} \mathcal{D}^{-1} \otimes_W \mathcal{O}_S$$

Taking the trace one gets

$$\psi_\sigma(\mathcal{F}_\tau, \mathcal{F}_{\tau'}) \supset p^{r-1} W \otimes_W \mathcal{O}_S \neq 0.$$

This verifies the assertion.

By Lemma 4.2 we can choose a  $W_\sigma$ -basis  $x_1, x_2$  for  $\Lambda_\tau$  and a  $W_\sigma$ -basis  $x'_1, x'_2$  for  $\Lambda_{\tau'}$  such that

$$(10.1) \quad \psi'_\sigma(x_i, x'_j) = \delta_{i,j}, \quad \text{for } 1 \leq i, j \leq 2$$

and

$$(10.2) \quad \Pi(x_1) = x'_1, \quad \Pi(x_2) = -\pi x'_2, \quad \Pi(x'_1) = -\pi x_1, \quad \Pi(x'_2) = x_2.$$

Put  $\bar{\Lambda}_\tau := \Lambda_\tau/p\Lambda_\tau$  and  $\bar{\Lambda}_{\tau'} := \Lambda_{\tau'}/p\Lambda_{\tau'}$ . Write  $\bar{x}_i$  or  $\bar{x}'_i$  for the image of  $x_i$  or  $x'_i$  in  $\bar{\Lambda}_\tau$  or  $\bar{\Lambda}_{\tau'}$ , respectively. Let  $\mathcal{F}_\sigma = \mathcal{F}_\tau \oplus \mathcal{F}_{\tau'}$  be an object in  $\mathbf{M}_{\Lambda_\sigma}(k)$ . One has

$$\bar{\Lambda}_\tau/\mathcal{F}_\tau \simeq k[\pi]/(\pi^{e_1}) \oplus k[\pi]/(\pi^{e_2})$$

as  $k[\pi]/(\pi^e)$ -modules for some integers  $e_1, e_2$  with  $0 \leq e_1 \leq e_2 \leq e$  and  $e_1 + e_2 = e$ ; the pair  $(e_1, e_2)$  is called the *Lie type* of  $\mathcal{F}_\tau$ . It is easy to see that  $\mathcal{F}_{\tau'}$  has the same Lie type as  $\mathcal{F}_\tau$ . The *reduced Lie type* of  $\mathcal{F}_\sigma$  is defined to be the Lie type of  $\mathcal{F}_\tau$ . We call a reduced Lie type  $(e_1, e_2)$  of an object  $\mathcal{F}_\sigma$  *minimal* if  $e_2 - e_1 \in \{0, 1\}$ .

Similar to the unramified case, we can write

$$\mathcal{F}_\tau = \text{Span} \{ \pi^{e_1} \bar{y}_1, \pi^{e_2} \bar{y}_2 \},$$

where  $\bar{y}_1$  and  $\bar{y}_2$  are in one of the following cases

- (a)  $\bar{y}_1 = \bar{x}_1 + t\bar{x}_2$  and  $\bar{y}_2 = \bar{x}_2$ , or
- (b)  $\bar{y}_1 = t\bar{x}_1 + \bar{x}_2$  and  $\bar{y}_2 = \bar{x}_1$ ,

where  $t \in k[\pi]/(\pi^e)$ .

In case (a), we compute

$$\mathcal{F}_{\tau'} = \text{Span} \{ \pi^{e_1} (t\bar{x}'_1 - \bar{x}'_2), \pi^{e_2} \bar{x}'_1 \}.$$

As  $\mathcal{F}_\tau$  and  $\mathcal{F}_{\tau'}$  are orthogonal to each other, condition (ii) is equivalent to

$$(10.3) \quad \psi'_\sigma(\mathcal{F}_\tau, \Pi\mathcal{F}_\tau) = \psi'_\sigma(\Pi\mathcal{F}_{\tau'}, \mathcal{F}_{\tau'}) = 0.$$

This yields the equation

$$(10.4) \quad \pi^{2e_1}(1 - t^2\pi) = 0.$$

If  $e = 2c + 1$  is odd, then there is no solution to the equation (10.4). If  $e = 2c$  is even, then a solution to (10.4) exists only when  $(e_1, e_2) = (c, c)$ , and any solution gives rise to the same object

$$(10.5) \quad \mathcal{F}_\tau = \pi^c \bar{\Lambda}_\tau \quad \text{and} \quad \mathcal{F}_{\tau'} = \pi^c \bar{\Lambda}_{\tau'}.$$

In case (b), we compute

$$\mathcal{F}_{\tau'} = \text{Span} \{ \pi^{e_1} (\bar{x}'_1 - t\bar{x}'_2), \pi^{e_2} \bar{x}'_2 \}.$$



The condition (10.3) yields the following equation

$$(10.6) \quad \pi^{2e_1}(t^2 - \pi) = 0.$$

If  $e = 2c$  is even, then a solution to (10.6) exists only when  $(e_1, e_2) = (c, c)$ . In this case we only get the object  $\mathcal{F}_\sigma$  as in (10.5). If  $e = 2c + 1$  is odd, then to have a solution we must have  $(e_1, e_2) = (c, c + 1)$  and we only get the object

$$(10.7) \quad \mathcal{F}_\tau = \text{Span} \{ \pi^c \bar{x}_2, \pi^{c+1} \bar{x}_1 \} \quad \text{and} \quad \mathcal{F}_{\tau'} = \text{Span} \{ \pi^c \bar{x}'_1, \pi^{c+1} \bar{x}'_2 \}.$$

PROPOSITION 10.5. — *Notations being as above, assume that  $v$  is ramified in  $B$ , and let  $\sigma \in I_v$ .*

- (1) *If  $e = 2c$  is even, then  $\mathbf{M}_{\Lambda_\sigma}(k)$  consists of the single  $k[\pi]/(\pi^e)$ -submodule  $\mathcal{F}_\sigma = \mathcal{F}_\tau \oplus \mathcal{F}_{\tau'}$  with*

$$\mathcal{F}_\tau = \pi^c \bar{\Lambda}_\tau \quad \text{and} \quad \mathcal{F}_{\tau'} = \pi^c \bar{\Lambda}_{\tau'}.$$

- (2) *If  $e = 2c + 1$  is odd, then  $\mathbf{M}_{\Lambda_\sigma}(k)$  consists of the single  $k[\pi]/(\pi^e)$ -submodule  $\mathcal{F}_\sigma = \mathcal{F}_\tau \oplus \mathcal{F}_{\tau'}$  with*

$$\mathcal{F}_\tau = \text{Span} \{ \pi^c \bar{x}_2, \pi^{c+1} \bar{x}_1 \} \quad \text{and} \quad \mathcal{F}_{\tau'} = \text{Span} \{ \pi^c \bar{x}'_1, \pi^{c+1} \bar{x}'_2 \},$$

where the bases  $\{x_i\}$  and  $\{x'_i\}$  are chosen as in (10.1) and (10.2). □

In particular, only the minimal reduced Lie type can occur in the space  $\mathbf{M}_{\Lambda_\sigma}(k)$ .

PROPOSITION 10.6. — *Assume that  $v$  is ramified in  $B$ , and let  $\sigma \in I_v$ . The structure morphism  $f : \mathbf{M}_{\Lambda_\sigma} \rightarrow \text{Spec } W$  is finite and flat.*

*Proof.* — As  $f$  is projective and quasi-finite (Proposition 10.5), the morphism  $f$  is finite. Let  $B(k)^{\text{alg}}$  be an algebraic closure of the fraction field  $B(k) = \text{Frac}(W)$ . Since  $\mathbf{M}_{\Lambda_\sigma}(k)$  consists of one element, the specialization map

$$\mathbf{M}_{\Lambda_\sigma}(B(k)^{\text{alg}}) \rightarrow \mathbf{M}_{\Lambda_\sigma}(k)$$

is surjective. Therefore, any (the unique) object in  $\mathbf{M}_{\Lambda_\sigma}(k)$  can be lifted to characteristic zero. This shows that the coordinate ring of  $\mathbf{M}_{\Lambda_\sigma}$  is torsion free and hence  $f$  is flat. □

THEOREM 10.7. — *Suppose  $v$  is ramified in  $B$ . The structure morphism  $\mathbf{M}_{\Lambda_v} \rightarrow \text{Spec } \mathbb{Z}_p$  is finite and flat.*

*Proof.* — This follows from Proposition 10.6 immediately. □

**10.3. Flatness of  $M_\Lambda$ .**

**THEOREM 10.8.** — *Let  $\Lambda$  be a free unimodular skew-Hermitian  $O_B \otimes \mathbb{Z}_p$ -module of rank one and let  $M_\Lambda$  be the associated local model. The structure morphism  $M_\Lambda \rightarrow \text{Spec } \mathbb{Z}_p$  is finite and flat.*

*Proof.* — This follows from Theorems 10.4 and 10.7. □

**THEOREM 10.9.** — *The moduli scheme  $\mathcal{M}_K^{(p)} \rightarrow \text{Spec } \mathbb{Z}_{(p)}$  is flat and every connected component is projective and of relative dimension zero.* □

**11. More constructions of Dieudonné modules**

In this section we handle two technical problems that arise from the results of previous sections. Keep the notation and assumptions of Section 10.

**11.1. Dieudonné  $O_B$ -modules with given Lie type**

In (9.5) we introduced the map  $\alpha : \text{Dieu}_m^{O_B \otimes \mathbb{Z}_p}(k) \rightarrow \mathcal{G}(k) \backslash M_\Lambda(k)$ . For each place  $v|p$  of  $F$ , let  $\text{Dieu}_1^{O_{B_v}}(k)$  denote the set of isomorphism classes of separably quasi-polarized Dieudonné  $O_{B_v}$ -modules of  $W$ -rank  $4d_v$  satisfying the determinant condition, where  $d_v = [F_v : \mathbb{Q}_p]$ .

Suppose  $M = \bigoplus_{v|p} M_v$  is a Dieudonné  $O_F \otimes \mathbb{Z}_p$ -module of ( $W$ -)rank  $4d$  such that  $\text{rank}_W M_v = 4d_v$ . Put  $I = \coprod_{v|p} I_v$ . For each place  $v|p$  and  $i \in I_v$ , let  $e_{i,1} \leq e_{i,2} \leq e_{i,3} \leq e_{i,4}$  be the integers such that

$$(M_v/\mathcal{V}M_v)^i \simeq k[\pi]/(\pi^{e_{i,1}}) \oplus k[\pi]/(\pi^{e_{i,2}}) \oplus k[\pi]/(\pi^{e_{i,3}}) \oplus k[\pi]/(\pi^{e_{i,4}}),$$

where  $(M_v/\mathcal{V}M_v)^i$  denotes the  $i$ -component of  $M_v/\mathcal{V}M_v$ . We define the *Lie type* of  $M_v$  by

$$(11.1) \quad \underline{e}(M_v) := (\underline{e}_i)_{i \in I_v}, \quad \underline{e}_i := (e_{i,1}, e_{i,2}, e_{i,3}, e_{i,4}).$$

The *Lie type* of  $M$  is defined by

$$(11.2) \quad \underline{e}(M) := (\underline{e}(M_v))_{v|p}.$$

For two Dieudonné  $O_F \otimes \mathbb{Z}_p$ -modules  $M_1$  and  $M_2$ , one has  $M_1/\mathcal{V}M_1 \simeq M_2/\mathcal{V}M_2$  as  $O_F \otimes k$ -modules if and only if  $\underline{e}(M_1) = \underline{e}(M_2)$ . If  $(M_v, \langle \cdot, \cdot \rangle) \in \text{Dieu}_1^{O_{B_v}}(k)$ , then  $\underline{e}_i = (e_{i,1}, e_{i,1}, e_{i,2}, e_{i,2})$  for two integers  $e_{i,1}$  and  $e_{i,2}$  with  $0 \leq e_{i,1} \leq e_{i,2} \leq e_v$  and  $e_{i,1} + e_{i,2} = e_v$  for each  $i \in I_v$ . In this case we define the *reduced Lie type* of  $M_v$  and that of  $M$ , respectively, by

$$(11.3) \quad \underline{e}^r(M_v) := ((e_{i,1}, e_{i,2}))_{i \in I_v} \quad \text{and} \quad \underline{e}^r(M) := (\underline{e}^r(M_v))_{v|p}.$$

PROPOSITION 11.1. — *The map  $\alpha : \text{Dieu}_1^{O_B \otimes \mathbb{Z}_p}(k) \rightarrow \mathcal{G}(k) \backslash \mathbf{M}_\Lambda(k)$  is surjective.*

*Proof.* — It suffices to show the surjectivity of the map

$$(11.4) \quad \alpha_v : \text{Dieu}_1^{\mathcal{O}_{B_v}}(k) \rightarrow \mathcal{G}_v(k) \backslash \mathbf{M}_{\Lambda_v}(k)$$

for each place  $v|p$ . The target orbit space in (11.4) is classified by the reduced Lie types of the objects (Theorem 10.4 and Proposition 10.5). When  $v$  is unramified in  $B$ , this consists of tuples  $(e_{i,1}, e_{i,2})_{i \in I_v}$  of pairs of integers with  $0 \leq e_{i,1} \leq e_{i,2} \leq e_v$  and  $e_{i,1} + e_{i,2} = e_v$ . When  $v$  is ramified in  $B$ , this consists of the single tuple  $((c, e_v - c))_{i \in I_v}$ , where  $c := \lfloor e_v/2 \rfloor$ .

In the ramified case, by Theorem 7.2 there is a separably quasi-polarized Dieudonné  $\mathcal{O}_{B_v}$ -module  $M$  with the determinant condition whose Lie type is the minimal one, that is,  $(M/\mathbf{V}M)^j \simeq k[\pi]/(\pi^c) \oplus k[\pi]/(\pi^{e_v-c})$  for all  $j \in \mathbb{Z}/2f_v\mathbb{Z}$ . So one has the surjectivity of  $\alpha_v$ .

It remains to treat the unramified case. We need to write down a separably anti-quasi-polarized Dieudonné  $\mathcal{O}_v$ -module  $M_1$  of rank  $2d_v$  such that the Lie type  $\underline{e}(M_1)$  of  $M_1$  is equal to the given one  $((e_{i,1}, e_{i,2}))_{i \in I_v}$ . Fix an identification  $I_v \simeq \mathbb{Z}/f_v\mathbb{Z}$ . Let  $M_1 = \bigoplus_{i \in \mathbb{Z}/f_v\mathbb{Z}} M_1^i$ , where each  $M_1^i$  is a free rank two  $W^i$ -module generated by two elements  $X_i$  and  $Y_i$ . For each  $i \in \mathbb{Z}/f_v\mathbb{Z}$ , let  $(, ) : M_1^i \times M_1^i \rightarrow W$  be the pairing that satisfies

$$(11.5) \quad (X_i, \pi^{e-1}Y_i) = 1 \quad \text{and} \quad (X_i, \pi^b Y_i) = 0, \quad \forall 0 \leq b \leq e-2$$

and  $(ax, y) = (x, ay)$  for  $a \in \mathcal{O}$  and  $x, y \in M_1^i$ . Define the Verschiebung map  $\mathbf{V} : M_1^{i+1} \rightarrow M_1^i$  by

$$(11.6) \quad \mathbf{V}X_{i+1} = \pi^{e_{i,1}}X_i, \quad \mathbf{V}Y_{i+1} = p\pi^{e_v - e_{i,2}}Y_i.$$

It is easy to show that  $(\mathbf{V}X, \mathbf{V}Y) = p(X, Y)^{\sigma^{-1}}$  for  $X, Y \in M_1$  and that the Lie type  $\underline{e}(M_1)$  of  $M_1$  is equal to  $(e_{i,1}, e_{i,2})_{i \in I_v}$ . Therefore, one has the surjectivity of  $\alpha_v$ . □

*Remark 11.2.* — The Dieudonné module  $M_1$  constructed in the proof of Proposition 11.1 has the slope sequence

$$(11.7) \quad \nu(M_1) = \left\{ \left( \frac{\sum_i e_{i,1}}{d_v} \right)^{d_v}, \left( \frac{\sum_i e_{i,2}}{d_v} \right)^{d_v} \right\}.$$

This exhausts all possible slope sequences that can occur in Corollary 7.7 in the case where  $v$  is unramified in  $B$ .

**11.2. Slope sequences of Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -modules: a refinement**

Our goal is to determine precisely slope sequences that arise from separably quasi-polarized Dieudonné  $\mathcal{O}_{\mathbf{B}} \otimes \mathbb{Z}_p$ -modules satisfying the determinant condition (for  $m = 1$ ). This problem is local. So it suffices to treat Dieudonné  $\mathcal{O}_{B_v}$ -modules for each place  $v$  lying over  $p$ . To simplify the notation as we did in Sections 4–8, we write  $\mathbf{B}, \mathbf{F}$  etc. for  $B_v, F_v$  etc. and drop the subscript  $v$  from our notation.

Theorem 7.3 determines exactly all possible slope sequences for separably polarized Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -modules of rank  $4dm$ , in particular for the present case of rank  $4d$  (Corollary 7.7). Recall that  $d, e, f$  denote the degree, ramification index and inertia degree of  $\mathbf{F}$ , respectively. The following result refines Corollary 7.7 for those Dieudonné modules in addition satisfying the determinant condition.

**THEOREM 11.3.**

- (1) *Suppose that  $\mathbf{B}$  is the  $2 \times 2$  matrix algebra. Let  $\nu$  be a slope sequence as follows:*

$$(11.8) \quad \nu = \left\{ \left( \frac{1}{2} \right)^{4d} \right\}, \quad \text{or} \quad \nu = \left\{ \left( \frac{a}{d} \right)^{2d}, \left( \frac{d-a}{d} \right)^{2d} \right\}$$

*for an integer  $a$  with  $0 \leq a < d/2$ . Then there exists a separably quasi-polarized Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module  $M$  of rank  $4d$  satisfying the determinant condition such that  $\nu(M) = \nu$ .*

- (2) *Suppose that  $\mathbf{B}$  is the quaternion division algebra. Suppose  $M$  is a separably quasi-polarized Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module of rank  $4d$  satisfying the determinant condition. Then*

$$(11.9) \quad \nu(M) = \left\{ \left( \frac{1}{2} \right)^{4d} \right\}, \quad \text{or} \quad \nu(M) = \left\{ \left( \frac{a}{2d} \right)^{2d}, \left( \frac{2d-a}{2d} \right)^{2d} \right\},$$

*for an odd integer  $a$  with  $2\lfloor e/2 \rfloor f \leq a < d$ . Conversely, if  $\nu$  is a slope sequence as in (11.9), then there exists a separably quasi-polarized Dieudonné  $\mathcal{O}_{\mathbf{B}}$ -module  $M$  of rank  $4d$  satisfying the determinant condition such that  $\nu(M) = \nu$ .*

*Proof.*

- (1) This is proved in Proposition 11.1 and Remark 11.2.

(2) Proposition 10.5 asserts that the reduced Lie type  $\underline{e}^r(M)$  of  $M$  is the minimal one  $\{(c, e - c); i \in \mathbb{Z}/f\mathbb{Z}\}$ , where  $c := \lfloor e/2 \rfloor$ . This yields  $F^{2f}(M) \subset \pi^{2fc}M$  and hence that smallest slope  $\beta \geq 2cf/2d$ . Then the first assertion follows from Corollary 7.7. We now prove the second statement.

Suppose that  $\nu$  is a slope sequence as in (11.9). Suppose  $e = 2c$  is even. Then  $\nu$  is supersingular and this follows from Theorem 7.2. It remains to treat the case where  $e = 2c + 1$  is odd. We may also assume that  $\nu$  is non-supersingular as the supersingular case has been done by Theorem 7.2. Write  $a = 2cf + 2r + 1$ , where  $0 < 2r + 1 < f$ . Let

$$M := \bigoplus_{j \in \mathbb{Z}/2f\mathbb{Z}} M^j,$$

where each  $M^j$  is a free rank two  $W^i$ -module generated by elements  $X_j$  and  $Y_j$  (with  $i = j \bmod f$ ). As before, we fix a presentation  $\mathcal{O}_{\mathbf{B}} = \mathcal{O}_{\mathbf{L}}[\Pi]$  as in (4.1) and (4.2). We shall describe the Frobenius map  $F$  and the map  $\Pi$  by their representative matrices with respect to the bases  $\{X_i, Y_i\}$ :

$$F_j : M^j \rightarrow M^{j+1}, \quad \Pi_j : M^j \rightarrow M^{j+f}, \quad \forall j \in \mathbb{Z}/2f\mathbb{Z}$$

in the sense that  $F_j = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$  if the latter matrix presents  $F_j$  for the bases  $\{X_j, Y_j\}$  and  $\{X_{j+1}, Y_{j+1}\}$ .

Put

$$(11.10) \quad \Pi_j = \begin{pmatrix} 0 & -\pi \\ 1 & 0 \end{pmatrix}, \quad \forall j \in \mathbb{Z}/2f\mathbb{Z}.$$

For each  $j \in \mathbb{Z}/2f\mathbb{Z}$ , define a  $W^i$ -bilinear pairing

$$\langle \cdot, \cdot \rangle_{\mathbf{F}} : M^j \times M^{j+f} \rightarrow W^i$$

by

$$(11.11) \quad \begin{aligned} \langle X_j, X_{j+f} \rangle_{\mathbf{F}} &= \langle Y_j, Y_{j+f} \rangle_{\mathbf{F}} = 0, \\ \langle X_j, Y_{j+f} \rangle_{\mathbf{F}} &= 1 \quad \text{and} \quad \langle Y_j, X_{j+f} \rangle_{\mathbf{F}} = -1. \end{aligned}$$

It is easy to show that  $\langle \Pi X, \Pi Y \rangle_{\mathbf{F}} = \pi \langle X, Y \rangle_{\mathbf{F}}$  for  $X \in M^j$  and  $Y \in M^{j+f}$ . So  $\langle \cdot, \cdot \rangle_{\mathbf{F}}$  is a unimodular skew-Hermitian form on  $M$  over  $W \otimes \mathcal{O}$ . Put

$$(11.12) \quad F_j = \begin{cases} \begin{pmatrix} 0 & -p\pi^{-c} \\ \pi^c & 0 \end{pmatrix}, & j = 0, \\ \begin{pmatrix} \pi^c & 0 \\ 0 & p\pi^{-c} \end{pmatrix}, & 1 \leq j \leq r, \\ \begin{pmatrix} p\pi^{-c} & 0 \\ 0 & \pi^c \end{pmatrix}, & r < j < f. \end{cases}$$

Using the commutative relation  $\Pi_{j+1}F_j = F_{j+1}\Pi_j$ , we compute

$$(11.13) \quad F_j = \begin{cases} \begin{pmatrix} 0 & -\pi^{c+1} \\ p\pi^{-c}\pi^{-1} & 0 \end{pmatrix}, & j = f; \\ \begin{pmatrix} \pi^c & 0 \\ 0 & p\pi^{-c} \end{pmatrix}, & f + 1 \leq j \leq f + r; \\ \begin{pmatrix} p\pi^{-c} & 0 \\ 0 & \pi^c \end{pmatrix}, & f + r < j < 2f. \end{cases}$$

As the matrix coefficients of  $F_j$  lie in the image of  $\mathbb{Z}_p[\pi]$  and  $\det F_j = p$ , one has  $\langle FX, FY \rangle_{\mathbf{F}} = p\langle X, Y \rangle_{\mathbf{F}}^\sigma$  for  $X \in M^j$  and  $Y \in M^{j+f}$ .

Putting  $\langle x, y \rangle := \text{Tr}_{\mathbf{F}/\mathbb{Q}_p}(\delta\langle x, y \rangle_{\mathbf{F}})$ , where  $\delta$  is a generator of  $\mathcal{D}_{\mathbf{F}/\mathbb{Q}_p}^{-1}$ , we obtain a separable  $\mathcal{O}_{\mathbf{B}}$ -linear quasi-polarization  $\langle, \rangle : M \times M \rightarrow W$ . It is easy to see from our construction that  $\dim_k(M/\mathcal{V}M)^j = e$  for all  $j \in \mathbb{Z}/2f\mathbb{Z}$ . Thus, by Lemma 5.2,  $M$  satisfies the determinant condition.

We compute

$$(11.14) \quad F^f = p^r \begin{pmatrix} 0 & -(p\pi^{-c})^{f-2r} \\ (\pi^c)^{f-2r} & 0 \end{pmatrix} : M^0 \rightarrow M^f;$$

$$(11.15) \quad F^f = p^r \begin{pmatrix} 0 & -(\pi^c)^{f-2r}\pi \\ (p\pi^{-c})^{f-2r}\pi^{-1} & 0 \end{pmatrix} : M^f \rightarrow M^0;$$

and

$$(11.16) \quad F^{2f} = p^{2r} \begin{pmatrix} -(\pi^c)^{2(f-2r)}\pi & 0 \\ 0 & -(p\pi^{-c})^{2(f-2r)}\pi^{-1} \end{pmatrix} : M^0 \rightarrow M^0.$$

The valuation ( $\text{ord}_p$ ) of the first diagonal entry of the last matrix is

$$[2er + 2c(f - 2r) + 1]/e = (2cf + 2r + 1)/e = a/e.$$

This shows that the slope sequence of the Dieudonné module  $M$  is equal to  $\nu$ . This completes the construction of a desired Dieudonné module  $M$  and hence completes the proof of the Theorem 11.3.  $\square$

## 12. Construction of Moret–Bailly families with $O_B$ -action

Keeping our assumption  $m = 1$ , we assume in addition that  $F = \mathbb{Q}$  in the remainder of this paper (Sections 12 and 13). In this section we assume in addition that  $p$  is ramified in  $B$ . Recall that  $k$  denotes an algebraically closed field of characteristic  $p$ . We shall prove

**THEOREM 12.1.** — *There is a non-constant family of supersingular polarized  $O_B$ -abelian surfaces over  $\mathbf{P}_k^1$ .*

**12.1. Case  $B = B_{p,\infty}$**

We begin with the construction of a Moret–Bailly family with  $O_B$ -action for the case  $B = B_{p,\infty}$ , where  $B_{p,\infty}$  is the definite quaternion  $\mathbb{Q}$ -algebra ramified exactly at  $\{p, \infty\}$ . Choose a supersingular elliptic curve  $E$  over  $k$ . There is an isomorphism  $B \simeq \text{End}^0(E) := \text{End}(E) \otimes \mathbb{Q}$  of  $\mathbb{Q}$ -algebras and we fix one. Then the endomorphism ring  $\text{End}(E)$  is a maximal order  $O_B$  of  $B$ . The subgroup scheme  $E[\mathbb{F}] := \ker \mathbb{F} = \alpha_p$  is  $O_B$ -stable as the Frobenius morphism is functorial. This induces a ring homomorphism

$$(12.1) \quad \phi : O_B/(p) = \mathbb{F}_{p^2}[\Pi]/(\Pi^2) \rightarrow \text{End}_k(\alpha_p) = k.$$

Since  $k$  is commutative, this map factors through the maximal commutative quotient  $(\mathbb{F}_{p^2}[\Pi]/(\Pi^2))^{\text{ab}} = \mathbb{F}_{p^2}[\Pi]/(\Pi^2, I)$ , where  $I$  is the two-sided ideal of  $\mathbb{F}_{p^2}[\Pi]/(\Pi^2)$  generated by elements of the form  $ab - ba$  for all  $a, b \in \mathbb{F}_{p^2}[\Pi]/(\Pi^2)$ . Since  $\Pi a - a\Pi = (a^p - a)\Pi$  and  $a^p - a$  is invertible if  $a \notin \mathbb{F}_p$ , the element  $\Pi$  lies in  $I$ . This shows that the action of  $O_B$  on  $E[\mathbb{F}] = \alpha_p$  factors through the quotient  $O_B \twoheadrightarrow \mathbb{F}_{p^2}$ . Put  $\text{Hom}_{\mathbb{F}_p}(\mathbb{F}_{p^2}, k) = \{\sigma_1, \sigma_2\}$ . We may assume this action is given by the embedding  $\sigma_1 : \mathbb{F}_{p^2} \rightarrow k$ .

Let  $A_0 := E \times E$  and let  $\iota_0 : O_B \rightarrow \text{End}(A_0)$  be the diagonal action. Let  $M$  be the Dieudonné module of  $A_0$ . Then the Lie algebra  $\text{Lie}(A_0) = M/\mathbb{V}M$  has the decomposition of  $\sigma_i$ -components (4.5):

$$(12.2) \quad \text{Lie}(A_0) = \text{Lie}(A_0)^1 \oplus \text{Lie}(A_0)^2 = k^2 \oplus 0,$$

and satisfies the condition

$$(12.3) \quad \Pi(\text{Lie}(A_0)) = 0.$$

Consider the functor  $\mathcal{X}$  over  $k$  which sends each  $k$ -scheme  $S$  to the set of  $O_B$ -stable locally free finite subgroup schemes over  $S$  of order  $p$  in  $A_0[\mathbb{F}]_S = \alpha_{p,S} \times \alpha_{p,S}$ . Clearly this functor is representable by a projective scheme (again denoted)  $\mathcal{X} \subset \mathbf{P}^1$  over  $k$ , as the condition of being  $O_B$ -stable is closed. Each locally free finite subgroup  $S$ -scheme  $H$  of order  $p$  corresponds to a rank one locally free  $\mathcal{O}_S$ -submodule  $\mathcal{L}$  in  $\mathcal{O}_S^2 = \text{Lie}(A_0[\mathbb{F}]_S)$  which locally for Zariski topology is a direct summand. As the action of  $O_B$  on  $\text{Lie}(E[\mathbb{F}]_S)$  is given by the scalar multiplication through the map  $\sigma_1 : \mathbb{F}_{p^2} \rightarrow k \rightarrow \mathcal{O}_S$ , the condition that  $\mathcal{L}$  is  $O_B$ -stable is automatically satisfied. This shows that  $\mathcal{X} = \mathbf{P}^1$ .

Let  $\mathbf{H} \subset A_0 \times \mathbf{P}^1$  be the universal family, and let  $\mathbf{A} := (A_0 \times \mathbf{P}^1)/\mathbf{H}$ . As  $\mathbf{H}$  is  $O_B$ -stable, we have a supersingular  $O_B$ -abelian scheme  $(\mathbf{A}, \iota_{\mathbf{A}})$  over  $\mathbf{P}^1$ . Ignoring the structure of the  $O_B$ -action, this family is constructed by Moret–Bailly and it is non-constant [18, p. 131]. By [11, Lemma 9.2], one

can choose an  $O_B$ -linear polarization  $\lambda_0$  on  $(A_0, \iota_0)$ . Replacing  $\lambda_0$  by  $p\lambda_0$  if necessary one may assume that  $\ker \lambda_0 \supset A_0[\mathbf{F}]$ . Since  $\mathbf{H}$  is isotropic with respect to the Weil pairing defined by  $\lambda_0$  (every finite flat subgroup scheme of order  $p$  has this property), the polarization  $\lambda_0 \times \mathbf{P}^1$  on  $A_0 \times \mathbf{P}^1$  descends to a polarization  $\lambda_{\mathbf{A}}$  on  $\mathbf{A}$ , which is also  $O_B$ -linear. Therefore, we have constructed a non-constant family of supersingular polarized  $O_B$ -abelian surfaces  $(\mathbf{A}, \lambda_{\mathbf{A}}, \iota_{\mathbf{A}})$  over  $\mathbf{P}^1$  for  $B = B_{p,\infty}$ .

## 12.2. General case

Now let  $B$  be an arbitrary definite quaternion algebra over  $\mathbb{Q}$  ramified at  $p$ . By the construction above, we only need to construct a superspecial  $O_B$ -abelian surface  $(A_0, \iota_0)$  that satisfies the conditions (12.2) and (12.3).

We first find a superspecial  $p$ -divisible  $O_B \otimes \mathbb{Z}_p$ -module  $(H_2, \iota_2)$  (of height 4) over  $k$  such that the conditions (12.2) and (12.3) for  $\text{Lie}(H_2)$  are satisfied. One can directly write down a superspecial Dieudonné  $O_B \otimes \mathbb{Z}_p$ -module of rank 4 with such conditions (see an example in Section 12.3) and let  $(H_2, \iota_2)$  be the corresponding  $p$ -divisible  $O_B \otimes \mathbb{Z}_p$ -module. Alternatively, let  $O_{B_{p,\infty}}$  be the maximal order in Section 12.1 and  $(A_0, \lambda_0)$  be the superspecial  $O_{B_{p,\infty}}$ -abelian surface used there. After identifying  $O_B \otimes \mathbb{Z}_p$  with  $O_{B_{p,\infty}} \otimes \mathbb{Z}_p$ , the attached  $p$ -divisible  $O_B \otimes \mathbb{Z}_p$ -module  $(H_2, \iota_2) := (A_0, \lambda_0)[p^\infty]$  shares the desired property.

Choose a supersingular  $O_B$ -abelian surface  $(A_1, \iota_1)$ . It exists by the non-emptiness of moduli spaces and Corollary 7.7(2). Indeed, there exists a polarized  $O_B$ -abelian surface  $(A_1, \lambda_1, \iota_1)$ . Since  $p$  is ramified in  $B$ ,  $A_1$  must be supersingular by Corollary 7.7(2). Alternatively, one can first construct an embedding  $B \rightarrow \text{Mat}_2(B_{p,\infty})$  of  $\mathbb{Q}$ -algebras. One does this by choosing an appropriate quaternion algebra  $B'$  such that  $B \otimes B' \simeq M_2(B_{p,\infty})$ . So we obtain a supersingular  $B$ -abelian surface  $A'$  up to isogeny. Replacing  $A'$  by an abelian surface in the isogeny class, we obtain a supersingular  $O_B$ -abelian surface. Let  $(H_1, \iota_1) := (A_1, \lambda_1)[p^\infty]$  be the associated  $p$ -divisible  $O_B \otimes \mathbb{Z}_p$ -module.

LEMMA 12.2. — *There is an  $O_B \otimes \mathbb{Z}_p$ -linear isogeny  $\varphi : (H_1, \iota_1) \rightarrow (H_2, \iota_2)$  over  $k$ .*

*Proof.* — It suffices to find a  $B_p$ -linear quasi-isogeny  $(H_1, \iota_1) \rightarrow (H_2, \iota_2)$  in the category of  $p$ -divisible groups over  $k$  up to isogeny, where  $B_p := B \otimes \mathbb{Q}_p$ . Since  $H_1$  and  $H_2$  are supersingular, one chooses an isogeny  $\varphi : H_1 \rightarrow H_2$ . Define the map  $\iota'_2 : B_p \rightarrow \text{End}^0(H_1)$  by  $\iota'_2(a) := \varphi^{-1}\iota_2(a)\varphi$



for all  $a \in B_p$ . We have two  $\mathbb{Q}_p$ -algebra homomorphisms  $\iota_1, \iota'_2 : B_p \rightarrow \text{End}^0(H_1)$ , and  $\text{End}^0(H_1)$  is a central simple  $\mathbb{Q}_p$ -algebra, because  $H_1$  is supersingular. By the Noether–Skolem theorem, there exists an element  $g \in \text{End}^0(H_1)^\times$  such that  $\iota'_2 = g \circ \iota_1 \circ g^{-1}$ . That is, we have the following commutative diagram

$$\begin{array}{ccccc} H_1 & \xrightarrow{g} & H_1 & \xrightarrow{\varphi} & H_2 \\ \downarrow \iota_1(a) & & \downarrow \iota'_2(a) & & \downarrow \iota_2(a) \\ H_1 & \xrightarrow{g} & H_1 & \xrightarrow{\varphi} & H_2. \end{array}$$

Replacing  $\varphi$  by  $\varphi \circ g$ , we get a  $B_p$ -linear quasi-isogeny  $\varphi : (H_1, \iota_1) \rightarrow (H_2, \iota_2)$ . This proves the Lemma 12.2. □

By Lemma 12.2, we choose an  $O_B \otimes \mathbb{Z}_p$ -linear isogeny  $\varphi : (H_1, \iota_1) \rightarrow (H_2, \iota_2)$ . Let  $K := \ker \varphi$ ; this is an  $O_B$ -stable subgroup scheme of  $A_1$ . Let  $A_0 := A_1/K$  and let  $\iota_0 : O_B \rightarrow \text{End}(A_0)$  the induced action. Then one has an isomorphism  $(A_0, \iota_0)[p^\infty] \simeq (H_2, \iota_2)$ . This yields an  $O_B$ -abelian surface satisfying the conditions (12.2) and (12.3). Applying the construction in Section 12.1 to this  $(A_0, \iota_0)$ , we construct a non-constant family of polarized  $O_B$ -abelian surfaces  $(\mathbf{A}, \lambda_{\mathbf{A}}, \iota_{\mathbf{A}})$  over  $\mathbf{P}^1$ . This finishes the proof of Theorem 12.1. □

### 12.3. An example.

We write down a superspecial Dieudonné  $O_B \otimes \mathbb{Z}_p$ -module  $M$  such that the conditions (12.2) and (12.3) for  $M/\mathcal{V}M$  are satisfied. Write  $M = M^1 \oplus M^2$  as a free  $W$ -module of rank 4 with a  $\mathbb{Z}_{p^2}$ -action, where  $M^1$  and  $M^2$  are free module generated by elements  $\{e_1^1, e_2^1\}$  and  $\{e_1^2, e_2^2\}$ , respectively. Define the Verschiebung map  $\mathcal{V}$  by

$$\mathcal{V}(e_1^2) = -pe_1^1, \quad \mathcal{V}(e_2^2) = -pe_2^1, \quad \mathcal{V}(e_1^1) = e_1^2, \quad \mathcal{V}(e_2^1) = e_2^2.$$

This determines the Frobenius map and defines a Dieudonné module with a  $\mathbb{Z}_{p^2}$ -action which satisfies the condition (12.2) for  $M/\mathcal{V}M$ . By the property  $\Pi a = \sigma(a)\Pi$  for  $a \in \mathbb{Z}_{p^2}$ , we must define  $\Pi : M^1 \rightarrow M^2$  and  $\Pi : M^2 \rightarrow M^1$  such that  $\Pi^2 = -p$  and  $\Pi\mathcal{V} = \mathcal{V}\Pi$ . Note that the map  $\Pi$  is determined by its restriction to  $M^1$ ,  $\Pi|_{M^1} : M^1 \rightarrow M^2$ , because  $\Pi^2 = -p$ . The condition (12.3) implies that  $\Pi(M^2) = pM^1$  and hence  $\Pi(M^1) = M^2$ .

Clearly,  $\Pi\mathcal{V} = \mathcal{V}\Pi$  if and only if  $\Pi\mathcal{V}(e_j^i) = \mathcal{V}\Pi(e_j^i)$  for all  $i, j$ . We define  $\Pi$  by setting  $\Pi(e_j^i) := \mathcal{V}(e_j^i)$  for all  $i, j$ . Then the conditions  $\Pi\mathcal{V} = \mathcal{V}\Pi$  and  $\Pi^2 = -p$  are satisfied. Therefore, this gives a superspecial Dieudonné  $O_B \otimes \mathbb{Z}_p$ -module for which the conditions (12.2) and (12.3) are fulfilled.

### 13. Dimensions of special fibers

Keep the setting of the previous section. We have proven that  $\dim \mathcal{M}_K^{(p)} \otimes_{\overline{\mathbb{F}}_p} = 0$ ; see Theorem 10.9. Our goal is to determine the dimensions of the special fibers  $\mathcal{M}_{\overline{\mathbb{F}}_p} := \mathcal{M} \otimes_{\overline{\mathbb{F}}_p}$ ,  $\mathcal{M}_{K, \overline{\mathbb{F}}_p} := \mathcal{M}_K \otimes_{\overline{\mathbb{F}}_p}$  and  $\mathcal{M}_{\overline{\mathbb{F}}_p}^{(p)} := \mathcal{M}^{(p)} \otimes_{\overline{\mathbb{F}}_p}$ .

**THEOREM 13.1.** — *Assume that  $m = 1$  and  $F = \mathbb{Q}$ .*

- (1) *If  $p$  is unramified in  $B$ , then  $\dim \mathcal{M}_{\overline{\mathbb{F}}_p} = 0$ .*
- (2) *If  $p$  is ramified in  $B$ , then  $\dim \mathcal{M}_{\overline{\mathbb{F}}_p} = 1$ .*
- (3) *We have  $\dim \mathcal{M}_{\overline{\mathbb{F}}_p}^{(p)} = 0$ .*

A close examination shows that when  $p$  is ramified in  $B$ , one has  $\dim \mathcal{M}_{K, \overline{\mathbb{F}}_p} = 1$ ; see Proposition 13.7. This refines Theorem 13.1(2).

#### 13.1. Unramified case

Suppose that  $p$  is unramified in  $B$ . Let  $(A, \lambda, \iota)$  be a polarized  $O_B$ -abelian surface over  $\overline{\mathbb{F}}_p$ . By Corollary 7.7,  $A$  is either ordinary or supersingular. If  $A$  is ordinary, then one has the canonical lifting  $(\mathbf{A}, \lambda_{\mathbf{A}}, \iota_{\mathbf{A}})$  over  $W(\overline{\mathbb{F}}_p)$  of  $(A, \lambda, \iota)$ . Since the generic fiber  $\mathcal{M}_{\overline{\mathbb{Q}}_p}$  has dimension zero, each subscheme  $\mathcal{M}_{D, \overline{\mathbb{Q}}_p}$  has finitely many points, if it is not empty. Recall that  $\mathcal{M}_D \subset \mathcal{M}$  denotes the subscheme parametrizing the objects  $(A, \lambda, \iota)$  in  $\mathcal{M}$  with polarization degree  $\deg \lambda = D^2$ . This implies that the ordinary locus  $\mathcal{M}_{D, \overline{\mathbb{F}}_p}^{\text{ord}}$  of  $\mathcal{M}_{D, \overline{\mathbb{F}}_p}$  has finitely many points and hence it has dimension zero. Therefore, the ordinary locus  $\mathcal{M}_{\overline{\mathbb{F}}_p}^{\text{ord}}$  has dimension zero.

Suppose now that  $A$  is supersingular. Then  $A$  must be superspecial. To see this, let  $H := A[p^\infty]$  be the associated  $p$ -divisible group. Since  $O_B \otimes \mathbb{Z}_p \simeq \text{Mat}_2(\mathbb{Z}_p)$ , the  $p$ -divisible group  $H$  is isomorphic to  $H_1 \times H_2$ , where  $H_1$  and  $H_2$  are supersingular  $p$ -divisible groups of height 2. Therefore,  $A$  is superspecial.

For any positive integers  $g$  and  $D$ , let  $\mathcal{A}_{g,D}$  denote the coarse moduli space over  $\overline{\mathbb{F}}_p$  of  $g$ -dimensional polarized abelian varieties  $(A, \lambda)$  with polarization degree  $\deg \lambda = D^2$ . Let  $\Lambda_{g,D} \subset \mathcal{A}_{g,D}$  be the superspecial locus. It is known that  $\Lambda_{g,D}$  is a finite closed subscheme. Let  $f : \mathcal{M}_{D, \overline{\mathbb{F}}_p} \rightarrow \mathcal{A}_{2,D}$  be the forgetful morphism:  $f(A, \lambda, \iota) = (A, \lambda)$ . The morphism  $f$  induces a map

$$f : \mathcal{M}_{D, \overline{\mathbb{F}}_p}^{\text{ss}} \rightarrow \Lambda_{g,D},$$

where  $\mathcal{M}_{D, \overline{\mathbb{F}}_p}^{\text{ss}}$  is the supersingular locus of  $\mathcal{M}_{D, \overline{\mathbb{F}}_p}$ . As  $\dim \Lambda_{g, D} = 0$  and the forgetful map  $f$  is finite (see [38]), the supersingular locus  $\mathcal{M}_{D, \overline{\mathbb{F}}_p}^{\text{ss}}$  also has dimension zero. We conclude that  $\mathcal{M}_{\overline{\mathbb{F}}_p}$  has dimension zero. This proves Theorem 13.1 (1). □

### 13.2. Ramified case

Suppose that the prime  $p$  is ramified in  $B$ . We know that any polarized  $O_B$ -abelian surface over  $k$  is supersingular (Corollary 7.7). By Theorem 12.1, there is a non-constant family  $\underline{\mathbf{A}} \rightarrow \mathbf{P}_{\overline{\mathbb{F}}_p}^1$  of supersingular polarized  $O_B$ -abelian surfaces. This gives rise to a non-constant moduli map

$$f' : \mathbf{P}_{\overline{\mathbb{F}}_p}^1 \rightarrow \mathcal{M}_{\overline{\mathbb{F}}_p}.$$

Therefore,  $\dim \mathcal{M}_{\overline{\mathbb{F}}_p} \geq \dim f'(\mathbf{P}^1) = 1$ .

On the other hand, the forgetful morphism  $f : \mathcal{M}_{D, \overline{\mathbb{F}}_p} \rightarrow \mathcal{A}_{2, D}$  factors through the supersingular locus  $\mathcal{A}_{2, D}^{\text{ss}} \subset \mathcal{A}_{2, D}$ . Since  $f$  is finite, one gets

$$(13.1) \quad \dim \mathcal{M}_{D, \overline{\mathbb{F}}_p} \leq \dim \mathcal{A}_{2, D}^{\text{ss}}.$$

For any integer  $i$  with  $0 \leq i \leq g$ , let  $\mathcal{A}_{g, D}^{(i)} \subset \mathcal{A}_{g, D}$  denote the reduced locally closed subscheme that consists of objects  $(A, \lambda)$  of  $p$ -rank equal to  $i$ . Norman and Oort [19] showed that the collection of  $p$ -strata forms a stratification and for each  $i$

$$\dim \mathcal{A}_{g, D}^{(i)} = g(g - 1)/2 + i.$$

When  $g = 2$ , one has  $\mathcal{A}_{2, D}^{(0)} = \mathcal{A}_{2, D}^{\text{ss}}$  and gets  $\dim \mathcal{A}_{2, D}^{\text{ss}} = 1$ . This proves the other direction

$$\dim \mathcal{M}_{D, \overline{\mathbb{F}}_p} \leq 1.$$

We conclude that  $\dim \mathcal{M}_{\overline{\mathbb{F}}_p} = 1$ . This proves Theorem 13.1 (2). □

### 13.3. Dimension of $\mathcal{M}_{\overline{\mathbb{F}}_p}^{(p)}$

We now prove Theorem 13.1 (3). For the case where the prime  $p$  is unramified in  $B$ , we have shown in Theorem 13.1 (1) that  $\mathcal{M}_{\overline{\mathbb{F}}_p}$  is zero-dimensional. Therefore,  $\dim \mathcal{M}_{\overline{\mathbb{F}}_p}^{(p)} = 0$  in the unramified case.

We now treat the case where  $p$  is ramified in  $B$ . Recall that  $k$  denotes an algebraically closed field of characteristic  $p$ , and that  $O_B \otimes \mathbb{Z}_p = \mathbb{Z}_{p^2}[\Pi]$ ,  $\Pi^2 = -p$ ,  $\Pi a = \sigma(a)\Pi$  for  $a \in \mathbb{Z}_{p^2}$ .

PROPOSITION 13.2. — Assume that  $p$  is ramified in  $B$ . Every prime-to- $p$  degree polarized  $O_B$ -abelian surface  $(A, \lambda, \iota)$  over  $k$  is superspecial.

*Proof.* — Let  $(M, \langle \cdot, \cdot \rangle)$  be the covariant Dieudonné  $O_B \otimes \mathbb{Z}_p$ -module associated to  $(A, \lambda, \iota)$ .

Suppose  $M/\mathbb{V}M = k^2 \oplus 0$  with respect to the action of  $\mathbb{F}_{p^2} \otimes_{\mathbb{F}_p} k = k \times k$ . Then  $\mathbb{V}M^2 = pM^1$  and  $\mathbb{V}M^1 = M^2$ , and hence  $\mathbb{F}M^1 = M^2$  and  $\mathbb{F}M^2 = pM^1$ . This proves that  $\mathbb{F}M = \mathbb{V}M$  and that  $M$  is superspecial

We show that if  $\Pi(M/\mathbb{V}M) = 0$ , then  $M$  is superspecial. Indeed, it follows that  $\Pi M \subset \mathbb{V}M$ . From  $\dim M/\Pi M = 2$  (as  $\Pi^2 = -p$ ) and  $\dim M/\mathbb{V}M = 2$  it follows that  $\Pi M = \mathbb{V}M$ . Then  $\mathbb{V}^2 M = \mathbb{V}\Pi M = \Pi \mathbb{V}M = \Pi^2 M = pM$  and hence  $M$  is superspecial. So far we have not used the separability of polarizations.

Suppose that  $M/\mathbb{V}M = k \oplus k$ . Consider the induced perfect pairing  $\langle \cdot, \cdot \rangle : \overline{M} \times \overline{M} \rightarrow k$ , where  $\overline{M} = M/pM$ . Since  $\Pi$  is nilpotent on  $M/\mathbb{V}M$  one may assume for example that  $\Pi \overline{M}^2 = \overline{\mathbb{V}M}^2$ . Taking the orthogonal complements of  $\Pi \overline{M}^2$  and  $\overline{\mathbb{V}M}^2$ , we have  $\Pi \overline{M}^1 = \overline{\mathbb{V}M}^1$ . This proves that  $\Pi(M/\mathbb{V}M) = 0$ . By the above argument,  $M$  is superspecial. □

Since the superspecial locus of the Siegel moduli space is zero-dimensional, the superspecial locus of any PEL-type moduli space is zero-dimensional, too. Proposition 13.2 implies that  $\dim \mathcal{M}_{\mathbb{F}_p}^{(p)} = 0$ . This proves Theorem 13.1(3) and hence completes the proof of Theorem 13.1. □

LEMMA 13.3. — Assume that  $p$  is ramified in  $B$ . There is a prime-to- $p$  degree polarized superspecial  $O_B$ -abelian surface over  $k$  that does not satisfy the determinant condition.

*Proof.* — Using the construction in Section 12.1, we have a superspecial  $p$ -divisible  $O_B \otimes \mathbb{Z}_p$ -module  $(H, \iota_H)$  of height 4 such that the conditions (12.2) and (12.3) for  $\text{Lie}(H)$  are satisfied. Thus,  $(H, \iota_H)$  does not satisfy the determinant condition. There is a superspecial abelian  $O_B$ -surface  $(A_0, \iota_0)$  such that  $(A_0, \iota_0)[p^\infty] \simeq (H, \iota_H)$ . We fix an identification  $(A_0, \iota_0)[p^\infty] = (H, \iota_H)$ .

We can choose a separable  $O_B \otimes \mathbb{Z}_p$ -linear quasi-polarization  $\lambda_H$ . Note that  $(H, \iota_H)$  is isomorphic to the  $p$ -divisible  $O_{B_{p,\infty}} \otimes \mathbb{Z}_p$ -module  $E_0[p^\infty]^2$  ( $E_0$  is a supersingular elliptic curve) through the identification  $O_{B_{p,\infty}} \otimes \mathbb{Z}_p = O_B \otimes \mathbb{Z}_p$ . We can pick the product principal polarization on  $E_0^2$  which yields such a quasi-polarization  $\lambda_H$ .

Choose an  $O_B$ -linear polarization  $\lambda$  on  $(A_0, \iota_0)$  and let  $*$  denote the Rosati involution induced by  $\lambda$ . Then  $\lambda a_p = \lambda_H$  for some element  $a_p \in \text{End}_{O_B \otimes \mathbb{Z}_p}^0(A_0[p^\infty])$  with  $a_p^* = a_p$ . Since

$$\text{End}_B^0(A_0) \otimes \mathbb{Q}_p = \text{End}_{B \otimes \mathbb{Q}_p}^0(A_0[p^\infty]),$$

by weak approximation we can choose a totally positive symmetric element  $a \in \text{End}_B^0(A_0)$  such that  $a$  is sufficiently close to  $a_p$ . Then we have  $(A, \lambda a, \iota)[p^\infty] \simeq (H, \lambda_H, \iota_H)$ . Replacing  $a$  by  $Na$  for a positive prime-to- $p$  integer  $N$  if necessary, we get a prime-to- $p$  degree  $O_B$ -linear polarization  $\lambda_0 = \lambda a$  on  $(A_0, \iota_0)$ . This proves the Lemma 13.3.  $\square$

We know that when  $p$  is ramified in  $B$ , the whole moduli space  $\mathcal{M}_{\overline{\mathbb{F}}_p}$  is supersingular. On the other hand when  $p$  is unramified in  $B$ , the moduli space may have both supersingular and ordinary points according to Corollary 7.7. The following Lemma 13.4 says that this is indeed the case.

LEMMA 13.4. — *When  $p$  is unramified in  $B$ , the moduli space  $\mathcal{M}_{K, \overline{\mathbb{F}}_p}^{(p)}$  consists of both ordinary and supersingular points.*

*Proof.* — As  $p$  is unramified in  $B$ , the determinant condition is automatically satisfied for objects in  $\mathcal{M}(k)$ . Let  $(H, \lambda_H, \iota_H)$  be a supersingular or ordinary separably quasi-polarized  $p$ -divisible  $O_B \otimes \mathbb{Z}_p$ -module. Since  $B$  can be embedded into  $\text{End}^0(A)$  for any supersingular abelian surface, we can find a supersingular  $O_B$ -abelian surface with  $(A_0, \iota_0)[p^\infty] \simeq (H, \iota_H)$ . We use the argument in the proof of Lemma 13.3 again to obtain a prime-to- $p$  degree  $O_B$ -linear polarization. For the ordinary case, we choose any imaginary quadratic field  $K$  such that  $K$  splits  $B$  and  $p$  splits in  $K$ . Then there is a  $\mathbb{Q}$ -algebra embedding of  $B$  into  $\text{Mat}_2(K)$ . Choose an ordinary elliptic curve  $E$  such that  $\text{End}^0(E) \simeq K$  and take the ordinary abelian surface  $A = E^2$ . As  $\text{End}_B^0(A) \otimes \mathbb{Q}_p = \text{End}_{B \otimes \mathbb{Q}_p}^0(A[p^\infty])$ , we can repeat the previous argument and get a prime-to- $p$  degree polarized ordinary  $O_B$ -abelian surface.  $\square$

*Remark 13.5.*

(1) The proof of Theorem 13.1 does not use local models. The method of local models we understand is developed in [29], where Rapoport and Zink formulate the moduli spaces of polarized self-dual (multi-)chains of  $O_B$ -abelian varieties together with the determinant condition. The moduli spaces considered in Theorem 13.1 parametrize polarized  $O_B$ -abelian varieties only and are not exactly the same as those in loc. cit. In order to apply the method of local models in our case, one needs to first extend the objects  $(A, \lambda, \iota)$  over a base scheme to a self-dual chain of  $O_B$ -abelian

schemes, and then apply the theorem “normal forms of lattice chains” [29, Theorem 3.16]. To do that the following closed condition must hold: there is an  $O_B$ -linear isogeny  $\varphi : A^t \rightarrow A$  such that  $\varphi \circ \lambda = p$ . In the cases of Theorem 13.1, this is the case where  $\deg \lambda$  is at most divisible by  $p^2$  and in this case we use local models to calculate the dimension; see Corollary 13.10 and Section 10. Note that the degree of the polarization of objects here can be divisible by a high power of  $p$  and we also consider the objects which do not satisfy the determinant condition.

We remark that there is a subtle difference between the moduli space of polarized self-dual chains of  $O_B$ -abelian varieties and that of polarized  $O_B$ -abelian varieties in the chain. This may be seen from the fact the morphism of deformation functors  $\text{Def}[A_0, \lambda_0] \rightarrow \text{Def}[A_0, p\lambda_0]$ , sending  $(A, \lambda)$  to  $(A, p\lambda)$ , is not an isomorphism (for the sake of simplicity, we disregard the  $O_B$ -structures). For example, if  $\lambda_0$  is principal, then the deformation functor  $\text{Def}[A_0, \lambda_0]$  is formally smooth while  $\text{Def}[A_0, p\lambda_0]$  is not. Furthermore, the universal deformation  $(\tilde{A}, \tilde{\lambda})$  of  $(A_0, p\lambda_0)$  does not give rise to a self-dual chain of  $O_B$ -abelian schemes because the multiplication by  $p$  is not divisible by the polarization  $\tilde{\lambda}$ .

The referee points out that one still can apply local models for computing some subvariety in which points may have polarization highly divisible by  $p$  in the following way. Consider the locus  $\mathcal{N}$  of  $\mathcal{M}_{\mathbb{F}_p}$  consisting of points with polarization degree divisible by at most  $p^2$  and let  $\mathcal{V} := \cup_{n=1}^{\infty} \phi_p^n(\mathcal{N}) \subset \mathcal{M}_{\mathbb{F}_p}$ , where  $\phi_p : \mathcal{M}_{\mathbb{F}_p} \rightarrow \mathcal{M}_{\mathbb{F}_p}$  sends  $(A, \lambda, \iota)$  to  $(A, p\lambda, \iota)$ . Then we have  $\dim \mathcal{V} = \dim \mathcal{N}$  and we can use local models to compute  $\dim \mathcal{N}$ . We thank the referee for this comment.

(2) We directly show that the method of local models does not work well for the previous example  $(A_0, p\lambda_0)$  over  $k$ , where  $\lambda_0$  is a principal polarization. Let  $(\Lambda, \psi)$  be a symplectic  $\mathbb{Z}_p$ -lattice of rank 4 with elementary divisor type  $(p, p)$ . We can choose a symplectic basis  $\mathbb{Z}_p$ -basis  $e_1, e_2, f_1, f_2$  with  $\psi(e_i, f_j) = p\delta_{ij}$  such that the Hodge filtration of  $H_1^{DR}(A_0)$  corresponds to the  $k$ -subspace generated by  $\{f_1, f_2\}$  via a trivialization  $(M(A_0), \langle, \rangle) = (\Lambda \otimes W(k), \psi)$ . The universal deformation of this point is then spanned by  $\tilde{f}_1 := f_1 + t_{11}e_1 + t_{12}e_2$  and  $\tilde{f}_2 := f_2 + t_{21}e_1 + t_{22}e_2$  subject to the condition  $\psi(\tilde{f}_1, \tilde{f}_2) = 0$ . Thus, the universal deformation ring is isomorphic to  $W(k)[[t_{11}, t_{12}, t_{21}, t_{22}]]/(p(t_{11} - t_{22}))$  and in particular the local model  $\mathbf{M}_\Lambda$  is not flat over  $\mathbb{Z}_p$ . However, by a theorem of Mumford [21, Theorem 2.3.3] (cf. [38, Theorem 4.5]) the moduli scheme  $\mathcal{A}_{2, p^2}$  is flat over  $\mathbb{Z}_p$ . Therefore, the local model does not control of the singularity of the moduli scheme  $\mathcal{A}_{2, p^2}$  at the point  $(A_0, p\lambda_0)$ .

(3) We used local models to prove  $\dim \mathcal{M}_K^{(p)} \otimes \overline{\mathbb{F}}_p = 0$  (Theorem 10.9). Proposition 13.2 gives a different proof of this result. Lemma 13.3 shows that the inclusion  $\mathcal{M}_K^{(p)}(k) \subset \mathcal{M}^{(p)}(k)$  is strict at least when  $B \otimes \mathbb{Q}_p$  is a division  $\mathbb{Q}_p$ -algebra. This phenomenon is different from the reduction modulo  $p$  of Hilbert moduli schemes or Hilbert–Siegel moduli schemes. In the Hilbert–Siegel case, any separably polarized abelian variety with RM by  $O_F$  of a totally real field  $F$  satisfies the determinant condition automatically; see Yu [39], Görtz [3] and Vollaard [35].

(4) We know that the moduli space  $\mathcal{M}_{\overline{\mathbb{F}}_p}^{(p)}$  is non-empty (Lemma 2.3). When  $p$  is ramified in  $B$ , the moduli space  $\mathcal{M}_{\overline{\mathbb{F}}_p}$  consists of both one-dimensional components (e.g. Moret–Bailly families) and zero-dimensional components (e.g. points in  $\mathcal{M}_{\overline{\mathbb{F}}_p}^{(p)}$ ).

### 13.4. Dimension of $\mathcal{M}_{K, \overline{\mathbb{F}}_p}$

Using Theorem 13.1(1), we only need to treat the ramified case.

LEMMA 13.6. — *Assume that  $p$  is ramified in  $B$ . Let  $M_0$  be a Dieudonné  $O_B \otimes \mathbb{Z}_p$ -module over  $k$  such that*

$$M_0/\mathbb{V}M_0 = k^2 \oplus 0,$$

*that is the Lie type of  $M_0$  is  $(2, 0)$  with respect to the action of  $\mathbb{Z}_{p^2}$ . Let  $M$  be any Dieudonné module such that  $\mathbb{V}M_0 \subset M \subset M_0$  and  $\dim_k(M_0/M) = 1$ . Then one has*

$$M/\mathbb{V}M = k \oplus k.$$

*Proof.* — Choose bases  $\{X_1^1, X_2^1\}$  and  $\{X_1^2, X_2^2\}$  for  $M_0^1$  and  $M_0^2$ , respectively. Since  $M_0/\mathbb{V}M_0 = k^2 \oplus 0$ ,  $M_0$  is superspecial (see the proof of Proposition 13.2). Since  $\mathbb{V}M \supset \mathbb{V}^2M_0 = pM_0$ , we can check the statement by passage to  $\overline{M}_0 := M_0/pM_0$ . Write  $x_j^i$  for the image of  $X_j^i$  in  $\overline{M}_0$ . One has

$$\mathbb{V}\overline{M}_0 = \text{Span}_k \{x_1^2, x_2^2\}, \quad \overline{M} := M/pM_0 = \text{Span}_k \{x_1^2, x_2^2, ax_1^1 + bx_2^1\},$$

for some  $(a, b) \neq (0, 0) \in k^2$ . Then

$$\mathbb{V}\overline{M} = \text{Span}_k \{\mathbb{V}(ax_1^1 + bx_2^1)\} \subset \overline{M}_0^2, \quad \text{and} \quad \dim \mathbb{V}\overline{M} = 1.$$

This gives  $M/\mathbb{V}M = k \oplus k$ . □

PROPOSITION 13.7. — *Assume that  $p$  is ramified in  $B$ . We have  $\dim \mathcal{M}_{K, \overline{\mathbb{F}}_p} = 1$ .*

*Proof.* — In the previous section, we construct a polarized  $O_B$ -abelian surface  $(\mathbf{A}, \lambda_{\mathbf{A}}, \iota_{\mathbf{A}})$  over  $\mathbf{P}^1$  starting from a superspecial abelian surface  $(A_0, \lambda_0, \iota_0)$  with additional structures and get a non-constant moduli map  $f' : \mathbf{P}^1 \rightarrow \mathcal{M}_{\overline{\mathbb{F}}_p}$ . The Dieudonné module  $M_0$  of  $A_0$  has the property  $M_0/\mathcal{V}M_0 = k^2 \oplus 0$ . By Lemma 13.6, every fiber of the family  $(\mathbf{A}, \lambda_{\mathbf{A}}, \iota_{\mathbf{A}}) \rightarrow \mathbf{P}^1$  has Lie type  $(1, 1)$ . Then the image  $f'(\mathbf{P}^1)$  is contained in  $\mathcal{M}_{K, \overline{\mathbb{F}}_p}$ . This shows that  $\dim \mathcal{M}_{K, \overline{\mathbb{F}}_p} = 1$ .  $\square$

*Remark 13.8.* — As a consequence of Proposition 13.7, we have the following result: When  $p$  is ramified in  $B$ , there is a polarized  $O_B$ -abelian surface over  $k$  with the determinant condition that can not be lifted to a polarized  $O_B$ -abelian surface in characteristic zero.

### 13.5. Dimension of $\mathcal{M}_{K, D, \overline{\mathbb{F}}_p}$

For an integer  $D \geq 1$ , let  $\mathcal{M}_{K, D, \overline{\mathbb{F}}_p} := \mathcal{M}_{K, D} \otimes_{\mathbb{Z}_{(p)}} \overline{\mathbb{F}}_p$ , where  $\mathcal{M}_{K, D} = \mathcal{M}_K \cap \mathcal{M}_D$ . We would like to determine the dimension of  $\mathcal{M}_{K, D, \overline{\mathbb{F}}_p}$  if it is non-empty. By Theorem 13.1,  $\dim \mathcal{M}_{K, D, \overline{\mathbb{F}}_p} = 0$  if  $p$  is unramified in  $B$  or  $p \nmid D$ . Thus, we shall assume that  $p$  is ramified in  $B$  and  $p \mid D$ . For the case where  $p \parallel D$ , we use local models to compute the dimension.

Let  $(\Lambda_1, \psi)$  be a skew-Hermitian  $O_B \otimes \mathbb{Z}_p$ -module of  $\mathbb{Z}_p$ -rank 4 of discriminant  $(p^2)$ . We may write  $\Lambda_1 = O_B \otimes \mathbb{Z}_p = \mathbb{Z}_{p^2}[\Pi]$  and  $\psi(x, y) = \text{tr}_{\mathbf{B}/\mathbb{Q}_p}(x\xi y^*)$  for a skew-Hermitian element  $\xi \in B_p^\times$ . The condition  $\text{disc } \psi = (p^2)$  implies that  $\xi \in (O_B \otimes \mathbb{Z}_p)^\times$ . Up to isomorphism the element  $\xi$  is uniquely determined up to a scalar in  $(\mathbb{Z}_p^\times)^2$ .

Let  $\mathbf{M}_{\Lambda_1}$  be the local model over  $\mathbb{Z}_p$  associated to the lattice  $\Lambda_1$  defined as in Section 9.1.

LEMMA 13.9. — *The special fiber  $\mathbf{M}_{\Lambda_1, \overline{\mathbb{F}}_p}$  is a union of two projective lines meeting at one point.*

*Proof.* — Write  $\Lambda_{1, W} := \Lambda_1 \otimes W = \bigoplus_{j \in \mathbb{Z}/2\mathbb{Z}} \Lambda_{1, W}^j$ . We can choose a  $W$ -base  $\{e_1^j, e_2^j\}$  for  $\Lambda_{1, W}^j$  for each  $j \in \mathbb{Z}/2\mathbb{Z}$  such that

$$(13.2) \quad \begin{aligned} \Pi e_1^j &= e_2^{j+1}, & \Pi e_2^j &= -p e_1^{j+1}, \\ \psi(e_1^1, e_1^2) &= 1, & \psi(e_2^1, e_2^2) &= -p, & \psi(e_r^i, e_s^j) &= 0 \quad \text{if } i = j \text{ or } r \neq s. \end{aligned}$$

To see this, write  $O_B \otimes_{\mathbb{Z}} W = (W \times W)[\Pi]$  and let  $e_1^1 = (1, 0)$ ,  $e_2^1 = (1, 0)\Pi$ ,  $e_1^2 = (0, 1)$ ,  $e_2^2 = (0, 1)\Pi$ . Up to  $(W^\times)^2 = W^\times$ , we may choose  $\xi$  to be  $(1, -1)$ . Then the conditions in (13.2) are satisfied.



Any point in  $\mathbf{M}_{\Lambda_1}(\overline{\mathbb{F}}_p)$  is given by an  $\overline{\mathbb{F}}_p$ -subspace  $\mathcal{F} = \mathcal{F}^1 \oplus \mathcal{F}^2 \subset \Lambda_{1, \overline{\mathbb{F}}_p}^1 \oplus \Lambda_{1, \overline{\mathbb{F}}_p}^2$  such that

$$(13.3) \quad \dim_{\overline{\mathbb{F}}_p} \mathcal{F}^j = 1, \quad \Pi(\mathcal{F}^j) \subset \mathcal{F}^{j+1}, \quad \psi(\mathcal{F}^1, \mathcal{F}^2) = 0$$

for  $j \in \mathbb{Z}/2\mathbb{Z}$ . Write  $\mathcal{F}^j = \langle t_{j1}e_1^j + t_{j2}e_2^j \rangle$  with  $[t_{j1} : t_{j2}] \in \mathbf{P}^1(\overline{\mathbb{F}}_p)$  for  $j \in \mathbb{Z}/2\mathbb{Z}$ . The conditions in (13.3) is then given by the condition  $t_{11}t_{21} = 0$ . Thus,  $\mathcal{M}_{\Lambda_1, \overline{\mathbb{F}}_p} = \{[0 : 1]\} \times \mathbf{P}^1 \cup \mathbf{P}^1 \times \{[0 : 1]\}$  with two  $\mathbf{P}^1$ 's meeting at the point  $([0 : 1], [0 : 1])$ .  $\square$

**COROLLARY 13.10.** — *If  $p$  is ramified in  $B$ ,  $p \parallel D$  and  $\mathcal{M}_{K, D, \overline{\mathbb{F}}_p}$  is non-empty, then  $\dim \mathcal{M}_{K, D, \overline{\mathbb{F}}_p} = 1$ .*

*Proof.* — Choosing an auxiliary prime to  $pD$  level structure, we have a finite surjective cover  $\mathcal{M}_{K, D, N} \rightarrow \mathcal{M}_{K, D}$ . Using the local model diagram, we have  $\dim \mathcal{M}_{K, D, N, \overline{\mathbb{F}}_p} = \dim \mathbf{M}_{\Lambda_1, \overline{\mathbb{F}}_p} = 1$  by Lemma 13.9. It follows that  $\dim \mathcal{M}_{K, D, \overline{\mathbb{F}}_p} = 1$ .  $\square$

We mention a few problems which are not solved in this paper.

(1) What is the dimension of  $\mathcal{M}_{K, D, \overline{\mathbb{F}}_p}$  if it is non-empty and  $p^2 \mid D$ ? We expect the answer to be one.

(2) We prove in Theorem 10.9 that  $\mathcal{M}_K^{(p)} \rightarrow \text{Spec } \mathbb{Z}_{(p)}$  is finite and flat. Which points in  $\mathcal{M}_K^{(p)}$  are étale over  $\text{Spec } \mathbb{Z}_{(p)}$ ? How about the same questions for  $\mathcal{M}_K$  and  $\mathcal{M}^{(p)}$  when  $F = \mathbb{Q}$  in Theorem 13.1?

(3) What are the dimensions of variant moduli spaces in Theorem 13.1 where  $F$  is an arbitrary totally real field and  $m = 1$ ?

(4) Is the map  $\varphi^{\text{loc}}$  (and its variant for the canonical local model considered in [24] or [26]) in Section 9.2 surjective?

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## BIBLIOGRAPHY

- [1] M. ARTIN & G. B. WINTERS, “Degenerate fibres and stable reduction of curves”, *Topology* **10** (1971), p. 373-383.
- [2] G. FALTINGS, “Endlichkeitssätze für abelsche Varietäten über Zahlkörpern”, *Invent. Math.* **73** (1983), no. 3, p. 349-366.
- [3] U. GÖRTZ, “Topological flatness of local models in the ramified case”, *Math. Z.* **250** (2005), no. 4, p. 775-790.
- [4] A. GROTHENDIECK, *Groupes de Barsotti–Tate et Cristaux de Dieudonné*, Séminaire de Mathématiques Supérieures, vol. 45, Les Presses de l’Université de Montréal, 1974.
- [5] A. GROTHENDIECK, M. RAYNAUD & D. S. RIM, *Groupes de monodromie en géométrie algébrique I*, Lecture Notes in Mathematics, vol. 288, Springer, 1972, 1967-1969 pages.
- [6] P. HARTWIG, “Kottwitz–Rapoport and p-rank strata in the reduction of Shimura varieties of PEL type”, *Ann. Inst. Fourier* **65** (2015), no. 3, p. 1031-1103.
- [7] X. HE & M. RAPOPORT, “Stratifications in the reduction of Shimura varieties”, *Manusc. Math.* **152** (2017), no. 3-4, p. 317-343.
- [8] S. HELGASON, *Differential geometry, Lie groups, and symmetric spaces*, Graduate Studies in Mathematics, vol. 34, American Mathematical Society, 2001.
- [9] T. KATSURA & F. OORT, “Families of supersingular abelian surfaces”, *Compos. Math.* **62** (1987), no. 2, p. 107-167.
- [10] R. E. KOTTWITZ, “Isocrystals with additional structure”, *Compos. Math.* **56** (1985), no. 3, p. 201-220.
- [11] ———, “Points on some Shimura varieties over finite fields”, *J. Am. Math. Soc.* **5** (1992), no. 2, p. 373-444.
- [12] ———, “Isocrystals with additional structure. II”, *Compos. Math.* **109** (1997), no. 3, p. 255-339.
- [13] Y. I. MANIN, “The theory of commutative formal groups over fields of finite characteristic”, *Russ. Math. Surv.* **18** (1963), no. 6, p. 1-83.
- [14] W. MESSING, *The crystals associated to Barsotti–Tate groups: with applications to abelian schemes*, Lecture Notes in Mathematics, vol. 264, Springer, 1972.
- [15] B. MOONEN, “Group schemes with additional structures and Weyl group cosets”, in *Moduli of Abelian Varieties* (C. Faber, G. van der Geer & F. Oort, eds.), Progress in Mathematics, vol. 195, Birkhäuser, 2001, p. 255-298.
- [16] ———, “A dimension formula for Ekedahl-Oort strata”, *Ann. Inst. Fourier* **54** (2004), no. 3, p. 666-698.
- [17] ———, “Serre-Tate Theory for Moduli Spaces of PEL Type”, *Ann. Sci. Éc. Norm. Supér.* **37** (2004), no. 2, p. 223-269.
- [18] L. MORET-BAILLY, “Familles de courbes et de variétés abéliennes sur  $\mathbf{P}^1$ . I. Descente des polarisations. II. Exemples”, in *Séminaire sur les pincesaux de courbes de genre au moins deux* (L. Szpiro, ed.), Astérisque, vol. 86, Société Mathématique de France, 1981, p. 109-140.
- [19] P. NORMAN & F. OORT, “Moduli of abelian varieties”, *Ann. Math.* **112** (1980), no. 3, p. 413-439.
- [20] O. T. O’MEARA, *Introduction to quadratic forms*, reprint of the 1973 ed., Classics in Mathematics, Springer, 2000.
- [21] F. OORT, “Finite group schemes, local moduli for abelian varieties, and lifting problems”, *Compos. Math.* **23** (1971), p. 265-296.
- [22] ———, “The isogeny class of a CM-type abelian variety is defined over a finite extension of the prime field”, *J. Pure Appl. Algebra* **3** (1973), p. 399-408.

- [23] G. PAPPAS, “On the arithmetic moduli schemes of PEL Shimura varieties”, *J. Algebr. Geom.* **9** (2000), no. 3, p. 577-605.
- [24] G. PAPPAS & M. RAPOPORT, “Local models in the ramified case I. The EL-case”, *J. Algebr. Geom.* **12** (2003), no. 1, p. 107-145.
- [25] ———, “Local models in the ramified case. III. Unitary groups”, *J. Inst. Math. Jussieu* **8** (2009), no. 3, p. 507-564.
- [26] G. PAPPAS & X. ZHU, “Local models of Shimura varieties and a conjecture of Kottwitz”, *Invent. Math.* **194** (2013), no. 1, p. 147-254.
- [27] M. RAPOPORT & M. RICHARTZ, “On the classification and specialization of  $F$ -isocrystals with additional structure”, *Compos. Math.* **103** (1996), no. 2, p. 153-181.
- [28] M. RAPOPORT & E. VIEHMANN, “Towards a theory of local Shimura varieties”, *Münster J. Math.* **7** (2014), no. 1, p. 273-326.
- [29] M. RAPOPORT & T. ZINK, *Period Spaces for  $p$ -divisible groups*, Annals of Mathematics Studies, vol. 141, Princeton University Press, 1996.
- [30] B. D. SMITHLING, “Admissibility and permissibility for minuscule cocharacters in orthogonal groups”, *Manuscr. Math.* **136** (2011), no. 3-4, p. 295-314.
- [31] ———, “Topological flatness of orthogonal local models in the split, even case. I”, *Math. Ann.* **350** (2011), no. 2, p. 381-416.
- [32] T. TSUKAMOTO, “On the local theory of quaternionic anti-hermitian forms”, *J. Math. Soc. Japan* **13** (1961), p. 387-400.
- [33] A. VASIU, “Integral models in unramified mixed characteristic  $(0,2)$  of Hermitian orthogonal Shimura varieties of PEL type. Part I”, *J. Ramanujan Math. Soc.* **27** (2012), no. 4, p. 425-477.
- [34] ———, “Integral models in unramified mixed characteristic  $(0,2)$  of hermitian orthogonal Shimura varieties of PEL type. Part II”, *Math. Nachr.* **287** (2014), no. 14-15, p. 1756-1773.
- [35] I. VOLLAARD, “On the Hilbert–Blumenthal moduli problem”, *J. Inst. Math. Jussieu* **4** (2005), no. 4, p. 653-683.
- [36] T. WEDHORN, “Ordinariness in good reductions of Shimura varieties of PEL-type”, *Ann. Sci. Éc. Norm. Supér.* **32** (1999), no. 5, p. 575-618.
- [37] ———, “The dimension of Oort strata of Shimura varieties of PEL-type”, in *Moduli of Abelian Varieties* (C. Faber, G. van der Geer & F. Oort, eds.), Progress in Mathematics, vol. 195, Birkhäuser, 2001, p. 441-471.
- [38] C.-F. YU, “Lifting abelian varieties with additional structures”, *Math. Z.* **242** (2002), no. 3, p. 427-441.
- [39] ———, “On reduction of Hilbert-Blumenthal varieties”, *Ann. Inst. Fourier* **53** (2003), no. 7, p. 2105-2154.
- [40] ———, “The isomorphism classes of abelian varieties of CM-type”, *J. Pure Appl. Algebra* **187** (2004), no. 1-3, p. 305-319.
- [41] ———, “On the mass formula of supersingular abelian varieties with real multiplications”, *J. Aust. Math. Soc.* **78** (2005), no. 3, p. 373-392.
- [42] ———, “An exact geometric mass formula”, *Int. Math. Res. Not.* **2008** (2008), no. 11, article no. rnn113 (11 pages).
- [43] ———, “On finiteness of endomorphism rings of abelian varieties”, *Math. Res. Lett.* **17** (2010), no. 2, p. 357-370.
- [44] ———, “Simple mass formulas on Shimura varieties of PEL-type”, *Forum Math.* **22** (2010), no. 3, p. 565-582.
- [45] ———, “Embeddings of fields into simple algebras: generalizations and applications”, *J. Algebra* **368** (2012), p. 1-20.

- [46] J. G. ZARHIN, “Isogenies of abelian varieties over fields of finite characteristics”, *Math. USSR, Sb.* **24** (1974), no. 3, p. 451-461.
- [47] T. ZINK, *Cartiertheorie kommutativer formaler Gruppen. Unter Mitarb. von Harry Reimann*, Teubner-Texte zur Mathematik [Teubner Texts in Mathematics], vol. 68, Teubner, 1984, 124 pages.

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