Existence, Stability and Controllability Results for a Class of Switched Evolution System with Impulses over Arbitrary Time Domain

Vipin Kumar[1], Amar Debbouche[2], Juan J. Nieto[3]

[1]Max-Planck Institute for Dynamics of Complex Technical Systems, 39106 Magdeburg, Germany
vkumar@mpi-magdeburg.mpg.de

[2]Department of Mathematics, Guelma University, Guelma 24000, Algeria.
amar_debbouche@yahoo.fr

[3]Instituto de Matemáticas, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain.
juanjose.nieto.roig@usc.es

Abstract: We establish several qualitative properties of a neutral switched impulsive evolution system on an arbitrary time domain by using the theory of time scales. This is the first attempt for switched evolution systems with impulses in abstract spaces. Firstly, we investigate the existence of a unique solution and Ulam's type stability results. After that, we establish the total controllability results, i.e., controllability not only with respect to the endpoint of the interval but also on the impulsive points. We transform the controllability problem into a solvability problem of an operator equation. We used the Banach fixed point theory and evolution operator theory to establish these results. To illustrate the effectiveness and implications of the developed results, we provide theoretical and simulated numerical examples for different time domains.

Keywords: Time Scales; Stability; Controllability; Impulses; Switched System.

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1 Introduction

Controllability is one of the most important qualitative properties of the modern mathematical control theory. In general, controllability denotes the ability to steer the state of a dynamical control system from an initial state to the desired final state by using a suitable control function. Since the seminal work by Kalman (1963) for finite-dimensional linear systems, controllability has been a very active area of research. In the last few decades, many authors investigated the controllability results for nonlinear dynamic systems in both finite and infinite-dimensional spaces by using the fixed point technique, see for instance (Joshi and George, 1989; Agarwal et al., 2009; Fu, 2003; Zhou, 1984), and the cited references therein. Another important notion of mathematical analysis which is very useful in many fields of applied sciences and engineering is dedicated to stability analysis. In the existing literature, there are numerous concepts of stability like Mittag-Leffler stability, finite-time stability, h-stability, exponential and Lyapunov stability. A particular type of stability called Ulam-Hyers stability was introduced by Ulam (1940) and Hyers (1941). Obłoza (1993), was the first researcher who established the Ulam-Hyers type stability of the finite-dimensional linear differential equations. Thereafter, many authors investigated the Ulam-Hyers type stability results for different types of dynamic systems, please see (Miura et al., 2001; Jung, 2004; Popa and Raşa, 2011; Wang et al., 2012), and the references cited therein.

On the other side, in many real-world problems, systems have some sudden changes in their state, such sudden changes are known as the impulsive effect in the system and the corresponding differential equations are known as the impulsive differential equations. It is observed that these types of equations have many applications in various areas of science and engineering, for instance, in control systems with communication constraints, sampled-data systems, mechanical systems, and networked control systems with scheduling protocol (Liu and Willms, 1995; Yang and Chua, 1997, etc.). In the existing literature, there are mainly two kinds of impulses, one is the instantaneous impulses, where the duration of these sudden changes is very small in comparison with the duration of the entire evolution process, for instance in shocks and natural disasters. The models in such cases are modelled by using the instantaneous impulsive differential equations. The second one is the non-instantaneous impulses, where the duration of these sudden changes starts impulsively at some points and continues over a finite-time interval. For example, in some real biological medical problems, the introduction of a drug or a vaccine in the bloodstream is a gradual process, then one is forced to consider the drug or vaccine as a non-instantaneous impulse since it starts abruptly and remains active for a finite time interval. The models in such cases are modelled by using the non-instantaneous impulsive differential equations. Since, in practicality, there is no impulse that occurs instantaneously rather it is non-instantaneous howsoever time of occurrence of impulse is small. Therefore, it is advantageous to study a class of differential equations with non-instantaneous impulses. Recently, few authors established the different types of results such as the existence of solutions, stability, and controllability for non-instantaneous impulsive systems by using the theory of analytic semigroup, fixed-point methods, and variational method, see for instance (Hernández and O'Regan, 2013; Kumar et al., 2021b; Liu et al., 2018; Luo and Luo, 2020; Wang and Fečkan, 2018; Wang and Fečkan, 2015; Wang et al., 2017; Kavitha et al., 2020; Abbas and Benchohra, 2015). However, these results cannot be easily extended to the case of switched impulsive systems on an arbitrary time domain.

Switched systems consist of a family of subsystems and a switching rule that orchestrates the switching between them. This class of systems has received significant attention because of their broad applications in many useful engineering systems, for example, the following phenomena evolve switching behavior: the dynamics of a vehicle changing unexpectedly because of wheels bolting and opening on ice, airplane entering, intersection and leaving an air traffic control area, biological cells developing and separating, a thermostat turning the heat on and off, a valve or a power switch opening and closing (Sun et al., 2011; Yu et al., 2008; Zhang et al., 2016). Stability and controllability are the important studied problems for this class of systems, and in recent years, many researchers have focused on stability and controllability results of the switched dynamic systems (Babiarz et al., 2016; Liberzon et al., 1999; Zhou et al., 2020). Furthermore, in many switched systems, at the time of switching there arise some impulse effects and hence it is very beneficial to investigate the switched systems with impulsive conditions. Recently, few authors studied the switched systems with instantaneous impulses, see for instance (Wang et al., 2004; Xie and Wang, 2004; Zhao and Sun, 2010).

However, all the above-mentioned results on existence, stability, and controllability are true for either discrete-time systems or continuous-time systems. Apart from the discrete and continuous-time systems, a wide class of systems exists, wherein the time domain is neither discrete nor continuous. For example, to study the dynamic behavior of some special species like Magicicada septendecim, Magicicada cassinii, and Pharaoh cicada, we need a particular time domain of form (Bohner and Peterson, 2001)

$$T = \bigcup_{i=1} [i(a+b), i(a+b)+b], \ \ a, b \in \mathbb{R}^+.$$

Also, to study the behavior of a simple electric circuit with resistance R, inductance L and capacitance C, if at every time unit we discharge the capacitor periodically and assume that the discharging takes a small $\delta > 0$ time unit, we need the time domain of the form (Bohner and Peterson, 2001)

$$T = \bigcup_{i \in \mathbb{N}_0} [i, i + 1 - \delta].$$

Since these models cannot be studied by using discrete or continuous dynamic systems but time scale theory can be used to address such systems. This theory was first introduced by Hilger (1988) in his Ph.D. thesis to unify and extend the discrete and continuous analysis into a general framework. A time scale, denoted by $\mathbb{T}$, is an arbitrary non-empty closed subset of real numbers. The most common examples of time scales are $\mathbb{R}, \mathbb{Z}$ and $h\mathbb{Z}$ (where $h > 0$). The results obtained on time scales are valid for the continuous-time systems (by setting the time scales to be real numbers $\mathbb{T} = \mathbb{R}$), the discrete dynamic systems (by setting the time scales to be integers $\mathbb{T} = \mathbb{Z}$) and as well as for any non-uniform time domains (a discrete non-uniform domain or the combination of discrete points with continuous intervals) which are very useful in the study of many complex dynamic systems. For further study on time scales, we refer to the books (Bohner and Peterson, 2001, 2003).

Nowadays, the notion of time scales theory has been attracting a lot of interest from many researchers. In recent years, few authors studied the finite-dimensional dynamic systems on time scales and investigated the existence of solutions, Ulam-Hyers type stability, and controllability results, see for instance (András and Mészáros, 2013; Bohner and Wintz, 2012; Davis et al., 2009; Lupulescu and Younus, 2011; Malik and Kumar, 2020; Shen and Li, 2019; Zada et al., 2017; Shah and Zada, 2019, 2022; Ben Nasser et al., 2021; Yasmin et al., 2020; Pervaiz et al., 2021). Particularly, in (András and Mészáros, 2013), the authors studied the Ulam-Hyers stability of linear and nonlinear dynamic equations and integral equations on time scales by using the theory of Picard operators. The work in (Davis et al., 2009), focused on the stability, controllability, and observability of the linear dynamic systems on time scales while in Lupulescu and Younus (2011), the authors extend the controllability and observability results of (Davis et al., 2009) to the time-varying systems with instantaneous impulses on time scales. In (Malik and Kumar, 2020), the authors established the existence of a unique solution, Ulam-Hyers stability, and controllability results for a Volterra integro-dynamic system with non-instantaneous impulses on time scales. In (Pervaiz et al., 2021), the authors considered the finite-dimensional fractional delay dynamical systems with instantaneous and non-instantaneous impulses on time scales and studied the stability and controllability results. Further, some results related to stability and controllability for finite-dimensional switched dynamic systems on time scales were addressed in the literature (Kumar et al., 2020; Kumar et al., 2021a; Lu and Zhang, 2019; Taousser et al., 2019). In particular, the work in (Kumar et al., 2020), focused on the total controllability of a class of switched impulsive dynamic systems with non-instantaneous impulses on time scales by using the parameter variation method and Gramian types matrices. In Kumar et al. (2021a), the authors examined the stability results for switched dynamic systems on arbitrary time domain by using the Lyapunov function and time scales theory. However, only a few researchers studied the stability, existence of almost periodic, periodic solutions, and controllability results of abstract equations on arbitrary time domain by using the time scale theory (Dhama and Abbas, 2019; Kumar and Malik, 2019; Kumar et al., 2021; Wang and Agarwal, 2014). In (Kumar and Malik, 2019), the authors studied the Ulam-Hyers stability and controllability results of non-linear evolution systems with instantaneous impulses on time scales by using the Banach fixed point theorem while in (Kumar et al., 2021), the authors investigated the existence of a unique solution, stability and controllability results for an abstract integro-hybrid evolution system with non-instantaneous impulses on time scales.

Nevertheless, all the above-mentioned works cannot be easily extended for the switched dynamic system with non-instantaneous impulses on time scales in infinite-dimensional spaces. To the best of our knowledge, there is no work reported that discussed the switched dynamic evolution system with non-instantaneous impulses over the arbitrary time domain. The inclusion of impulses, infinite-dimensional states, switching, and time scales requires a high level of abstraction. Therefore, in this manuscript, we investigate the existence of unique solution, stability, and controllability results for a class of switched dynamic evolution systems with non-instantaneous impulses on an arbitrary time scale in infinite-dimensional spaces by applying the fixed point technique with evolution operator theory.

- We consider a new class of neutral switched evolution systems with non-instantaneous impulses in the abstract spaces over the arbitrary time domain and formulated by the time scales theory.
- We use the concept of piecewise continuous mild solution to construct a suitable operator and with the help of this operator, we derived the existence of solution and Ulam-Hyers stability.
- We also define a new piecewise control function and study the total controllability result.
- We apply the fixed point technique with evolution operator and time scales theory to study these results.
- We provide some theoretical and simulated numerical examples with different time domains to illustrate the obtained analytical results.

The remainder of the manuscript is structured as follows: In Section 2, we give the problem of the statement. In Section 3, we review some preliminaries, important definitions, and significant lemmas. In Sections 4 and 5, we examine the existence and stability results for switched dynamic evolution systems with impulses on time scales, respectively. Section 6 is dedicated to study the controllability issue for the considered systems. In the last Section 7, we present some theoretical and numerical examples to illustrate the effectiveness of the obtained analytical outcomes.

Notations: Throughout this manuscript, $\mathbb{T}$ denotes the time scales and $I = [0,T]_{\mathbb{T}}$ for $T > 0$. $(X, \|\cdot\|)$ denotes the Banach space X under the induced norm $\|\cdot\|$ and Id denotes the identity operator in X. $C(I,X)$ denotes the set of all continuous functions from I into X. The set of all linear bounded operators from X into X is denoted by $\mathbb{B}(X)$. Also, we denote the set of all square Lebesgue integrable functions from I to X by $L^2(I,X)$.

2 Problem Formulation

In this section, we will introduce our statement of the problem.

We consider the following neutral evolution dynamic system in a Banach space X

$$\begin{cases} [x(t)-\Upsilon_{r(t)}(t,x_a(t))]^{\Delta} = \mathcal{A}_{r(t)}(t)[x(t)-\Upsilon_{r(t)}(t,x_a(t))] + \Psi_{r(t)}(t,x_b(t)),\ t\in \cup_{i=0}^{\vartheta}(s_i,t_{i+1}]_{\mathbb{T}}, \\ x(t) = \dfrac{1}{\Gamma(\gamma)}\int_{t_i}^{t}(t-\zeta)^{\gamma-1}\aleph_{r(t)}(\zeta, x(t_i^-))\Delta\zeta,\ t\in(t_i,s_i]_{\mathbb{T}},\ i=1,2,\dots,\vartheta, \\ x(0)=x_0 \end{cases} \tag{2.1}$$

and investigate the existence, uniqueness and stability results. Also, we establish the controllability results, of the following control system

$$\begin{cases} [x(t)-\Upsilon_{r(t)}(t,x_a(t))]^{\Delta} = \mathcal{A}_{r(t)}(t)[x(t)-\Upsilon_{r(t)}(t,x_a(t))] + \mathcal{B}_{r(t)}u(t) + \Psi_{r(t)}(t,x_b(t)),\ t\in \cup_{i=0}^{\vartheta}(s_i,t_{i+1}]_{\mathbb{T}}, \\ x(t) = \dfrac{1}{\Gamma(\gamma)}\int_{t_i}^{t}(t-\zeta)^{\gamma-1}\aleph_{r(t)}(\zeta, x(t_i^-))\Delta\zeta,\ t\in(t_i,s_i]_{\mathbb{T}},\ i=1,2,\dots,\vartheta, \\ x(0)=x_0, \end{cases} \tag{2.2}$$

where x is the state function; $\mathbb{T}$ is a time scale; $x_a(t)=x(a(t)), x_b(t)=x(b(t))$, where $a,b:I\to I$ are the delay functions with $a(t),b(t)\le t$ for all $t\in I$; $\Gamma(\cdot)$ denotes the usual gamma function; $\gamma\in(0,1)$; t_i and $s_i\in\mathbb{T}$ are some points which satisfy the relation $0=s_0=t_0<t_1<s_1<t_2<\ldots<s_\vartheta<t_{\vartheta+1}=T$; $x(t_i^-)=\lim_{k\to 0^+}x(t_i-k)$ and $x(t_i^+)=\lim_{k\to 0^+}x(t_i+k)$, $i=1,2,\ldots,\vartheta$, denote the left and right limit of $x(t)$ at $t=t_i$, respectively; $r(t)$ is the switching law to be defined later; the family of bounded linear operators $\mathcal{A}_{r(t)}(t)$ generate the evolution operators $\{\mathcal{T}_{\mathcal{A}_{r(t)}(t)}(t,s):(t,s)\in I\times I:0\le s\le t\le T\}$; $\mathcal{B}_{r(t)}$ are linear bounded operators from a Banach space U to X; $u\in L^2(I,U)$ is a control function, U is called the control space; the functions $\Upsilon_{r(t)}$, $\Psi_{r(t)}$, and $\aleph_{r(t)}$ are satisfying some suitable conditions which will be specified later.

The switching signal $r:I\mapsto\{0,1,\ldots,\vartheta\}$ is assumed to be known and satisfies the minimal dwell time condition. It only changes its values at switching times t_i. The discrete state $r(t)\in\{0,1,\ldots,\vartheta\}$ determines the actual system dynamics among the possible operating modes which corresponds to a specific instance of $\mathcal{A}_i(t),\mathcal{B}_i(t),\Upsilon_i,\Psi_i$, and $\aleph_i$. That is to say,

$$r(t)=i,\quad t_i\le t<t_{i+1},\ i=0,1,\ldots,\vartheta.$$

Subsequently, using the above switching law in systems (2.1) and (2.2), we get the following class of evolution switched systems

$$[x(t)-\Upsilon_i(t,x_a(t))]^\Delta=\mathcal{A}_i(t)[x(t)-\Upsilon_i(t,x_a(t))]+\Psi_i(t,x_b(t)),\ t\in\cup_{i=0}^{\vartheta}(s_i,t_{i+1}]_{\mathbb{T}}, \tag{2.3a}$$

$$x(t)=\frac{1}{\Gamma(\gamma)}\int_{t_i}^{t}(t-\zeta)^{\gamma-1}\aleph_i(\zeta,x(t_i^-))\Delta\zeta,\ t\in(t_i,s_i]_{\mathbb{T}},\ i=1,2,\ldots,\vartheta, \tag{2.3b}$$

$$x(0)=x_0 \tag{2.3c}$$

and

$$[x(t)-\Upsilon_i(t,x_a(t))]^\Delta=\mathcal{A}_i(t)[x(t)-\Upsilon_i(t,x_a(t))]+\mathcal{B}_iu(t)+\Psi_i(t,x_b(t)),\ t\in\cup_{i=0}^{\vartheta}(s_i,t_{i+1}]_{\mathbb{T}}, \tag{2.4a}$$

$$x(t)=\frac{1}{\Gamma(\gamma)}\int_{t_i}^{t}(t-\zeta)^{\gamma-1}\aleph_i(\zeta,x(t_i^-))\Delta\zeta,\ t\in(t_i,s_i]_{\mathbb{T}},\ i=1,2,\ldots,\vartheta, \tag{2.4b}$$

$$x(0)=x_0, \tag{2.4c}$$

respectively. Now onwards, we study the systems (2.3a)-(2.3c) and (2.4a)-(2.4b).

Here, we are giving a brief description of the problem (2.3a)-(2.3c)(or (2.4a)-(2.4c)).

- $x(t)$ satisfies the dynamic equation (2.3a)(or 2.4a) when $t\in(0,t_1]_{\mathbb{T}}$.
- $x(t)$ is given by the equation (2.3b)(or (2.4b)) when $t\in(t_1,s_1]_{\mathbb{T}}$.
- $x(t)$ satisfies the dynamic equation (2.3a)(or (2.4a)) when $t\in(s_1,t_2]_{\mathbb{T}}$.
- After repeating this process, on the interval $(s_\vartheta,t_{\vartheta+1}]_{\mathbb{T}}$, $x(t)$ satisfies the dynamic equation (2.3a)(or (2.4a)) and on the interval $(s_\vartheta,t_{\vartheta+1}]_{\mathbb{T}}$, $x(t)$ is given by the equation (2.3b)(or (2.4b)).

Graphically, this means that the solution $x(t)$ satisfies the dynamic equation (2.3a) on the black intervals $(s_i,t_{i+1}]_{\mathbb{T}},\ i=0,1,\ldots,\vartheta$ and the equation (2.3b) on the red intervals $(t_i,s_i]_{\mathbb{T}},\ i=1,2,\ldots,\vartheta$.

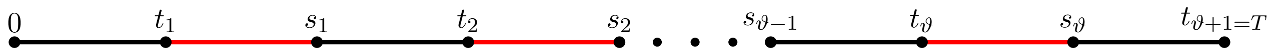

Figure 1: Description of statement of problem (2.3a)-(2.3c) (or (2.4a)-(2.4c))

3 Preliminaries and Definitions

In this section, we introduce the basic concept of the time scales theory as it is laid out in all detail in the textbook (Bohner and Peterson, 2001).

We define a time scales interval by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$. Similarly, we can define some other time scales intervals like $(a, b)_{\mathbb{T}}, [a, b)_{\mathbb{T}}$ and so on.

Next, we define some fundamental operators which are frequently used throughout the manuscript.

Definition 3.1 ((Bohner and Peterson, 2001), Def. 1.1)**.** *The forward jump operator* $\sigma(\mathbb{T}, \mathbb{T})$ *is defined by*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

with the substitution $\inf \emptyset = \sup \mathbb{T}$*.*

Definition 3.2 ((Bohner and Peterson, 2001), Def. 1.1)**.** *The backward jump operator* $\rho(\mathbb{T}, \mathbb{T})$ *is defined by*

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$

with the substitution $\sup \emptyset = \inf \mathbb{T}$*.*

Remark 3.3. *The graininess operator* $\mu(\mathbb{T}, [0, \infty))$ *is defined by* $\mu(t) = \sigma(t) - t$ *for all* $t \in \mathbb{T}$*. Now onwards, we set* $\bar{\mu} = \sup_{t \in I} \mu(t)$*.*

A point $t \in \mathbb{T}$ is called left dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, dense if it is left and right dense at the same time, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$.

We define the set $\mathbb{T}^\kappa$ as follows:

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T}^\kappa \setminus (\rho(\sup(\mathbb{T})), \sup(\mathbb{T})] & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty. \end{cases}$$

In the next definition, we define the delta-derivative which generalized the concept of differentiation to time scales.

Definition 3.4 ((Bohner and Peterson, 2001), Def. 1.10)**.** *A function* $\phi(\mathbb{T}, \mathbb{R})$ *is called delta differentiable at the point* $t \in \mathbb{T}^\kappa$*, if there exists a number* $\phi^\Delta(t)$ *such that for any* $\epsilon > 0$*, there exists a neighborhood* $\mathcal{U}$ *of* t *such that*

$$|[\phi(\sigma(t)) - \phi(s)] - \phi^\Delta(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|,$$

holds for all $s \in \mathcal{U}$*. We call* $\phi^\Delta(t)$ *the delta derivative of* ϕ *at* t*.*

Remark 3.5. *In the above definition, if we set* $\mathbb{T} = \mathbb{Z}$*, then* $\phi^\Delta(t) = \phi(t+1) - \phi(t)$*, which is the forward difference of* $\phi(t)$ *while if we set* $\mathbb{T} = \mathbb{R}$*, then* $\phi^\Delta(t) = \phi'(t)$*, which is the usual derivative of* $\phi(t)$*.*

Next, we define the delta-integral which generalized the ordinary integral to time scales.

Definition 3.6 ((Bohner and Peterson, 2001), Def. 1.71)**.** *A function* $\Phi(\mathbb{T}, \mathbb{R})$ *is called an antiderivative of a function* $\phi(\mathbb{T}, \mathbb{R})$ *provided* $\Phi^\Delta(t) = \phi(t)$ *holds for all* $t \in \mathbb{T}^\kappa$*. We define the Cauchy integral by*

$$\int_{t_0}^{t} \phi(\zeta)\Delta\zeta = \Phi(t) - \Phi(t_0) \text{ for all } t, t_0 \in \mathbb{T}.$$

Definition 3.7 ((Bohner and Peterson, 2001), Def. 1.57). *A function $f(\mathbb{T},\mathbb{R})$ is called regulated if its right-hand limit exist (finite) at all right-dense points in T and its left-hand limit exist (finite) at all left-dense points in $\mathbb{T}$.*

A function $f(\mathbb{T},\mathbb{R})$ is called rd-continuous, if it is regulated and it is continuous at all right-dense points. Moreover, function f is called piecewise rd-continuous if it is regulated and rd-continuous at all, except possibly at finitely many, right-dense points in $\mathbb{T}$.

All other concepts related to time scales used in this paper can be found in (Bohner and Peterson, 2001).

To define the exponential function on time scales, we first define the regressive functions as follows.

Definition 3.8 ((Bohner and Peterson, 2001), Def. 2.25). *We say that a function $q:\mathbb{T}\to\mathbb{R}$ is regressive provided*

$$1+\mu(t)q(t)\neq 0 \text{ for all } t\in\mathbb{T}^{\kappa}$$

holds

In the next definition, we define the generalized exponential function which generalized the concept of ordinary exponential function to time scales.

Definition 3.9 ((Bohner and Peterson, 2001), Def. 2.30). *Let q be regressive, then we define the exponential function is defined as*

$$e_q(t,s)=\exp\left(\int_s^t \zeta_{\mu(\zeta)}(q(\zeta))\Delta\zeta\right) \text{ for } t,s\in\mathbb{T},$$

where

$$\zeta_{\mu(s)}(q(s))=\begin{cases}\dfrac{1}{\mu(s)}Log(1+q(s)\mu(s)), & \text{if } \mu(s)\neq 0,\\ q(s), & \text{if } \mu(s)=0.\end{cases}$$

In the above definition, exp *and Log are the usual exponential and logarithmic functions, respectively.*

The next properties of exponential function on time scales are often used in the main results.

Theorem 3.10 ((Bohner and Peterson, 2001), Theorem 2.36). *Let q be regressive, then*

(i) $e_0(t,s)=1$ *and* $e_q(t,t)=1$. *(ii)* $e_q(\sigma(t),s)=(1+\mu(t)q(t))e_q(t,s)$.

(iii) $e_q(t,s)e_q(s,\zeta)=e_q(t,\zeta)$. *(iv)* $e_q(t,s)=\dfrac{1}{e_q(s,t)}=e_{\ominus q}(s,t)$.

Lemma 3.11 ((Dhama and Abbas, 2019), Lemma 2.12). *Let $\nu>0$ and $t,s\in\mathbb{T}$, then $e_{\ominus\nu}(t,s)\leq 1$.*

Next, we give some basic definitions related to evolution operator family which are often used throughout the manuscript.

Definition 3.12 ((Wang and Agarwal, 2014), Def. 4.1). *A two-parameter family $\mathcal{T}(t,s): I\times I\to\mathbb{B}(X)$ is called a linear evolution operator if the following hold:*

(i) $\mathcal{T}(t,t)=\mathrm{Id}$.

(ii) $\mathcal{T}(t,r)\mathcal{T}(r,s)=\mathcal{T}(t,s)$ *for* $0\leq s\leq r\leq t\leq T$.

(iii) For any fixed $x\in X$, $(t,s)\to\mathcal{T}(t,s)x$ is continuous mapping.

Definition 3.13 ((Wang and Agarwal, 2014), Def. 4.2)**.** *Let $\mathcal{T}(t,s)$ be an evolution operator, then it is called exponentially stable if there exist $\mathsf{C} \geq 1$ and $\nu > 0$ such that*

$$\|\mathcal{T}(t,s)\| \leq \mathsf{C}e_{\ominus\nu}(t,s),\ t \geq s.$$

Before defining the mild solution of the considered class of switched impulsive evolution system (2.3a)-(2.3c), we first provide the mild solution of the corresponding semilinear evolution system.

Let us consider the following semillinear evolution system

$$x^{\Delta}(t) = \mathcal{A}_0(t)x(t) + \Psi_0(t, x(t)),\ x(0) = x_0,\ t \in (0, t_1]_{\mathbb{T}}. \tag{3.5}$$

Definition 3.14 ((Wang and Agarwal, 2014), Def. 4.3)**.** *A function $x \in C(I, X)$ is called a mild solution of (3.5) if $x(t)$ satisfies the following integral equation*

$$x(t) = \mathcal{T}_{\mathcal{A}_0(t)}(t,0)x_0 + \int_0^t \mathcal{T}_{\mathcal{A}_0(t)}(t, \sigma(\zeta))\Psi_0(\zeta, x(\zeta))\Delta\zeta,\ t \in (0, t_1]_{\mathbb{T}},$$

where $\mathcal{T}_{\mathcal{A}_0(t)}(t,s)$ denotes the linear evolution operator generated by $\mathcal{A}_0(t)$ on $(0, t_1]_{\mathbb{T}}$.

For the notational convenience, now onwards we set $\mathcal{T}_i(t,s)$ for $\mathcal{T}_{\mathcal{A}_i(t)}(t,s)$.

Now to define the solution of the considered problems (2.3a)-(2.3c) and (2.4a)-(2.4c), we define the space of piecewise continuous functions $PC(I,X) = \{x : I \to X : x \in C((t_i, t_{i+1}]_{\mathbb{T}}, X),\ i = 0, 1, \ldots, \vartheta$ and there exist $x(t_i^-)$ and $x(t_i^+)$, $i = 1, 2, \ldots, \vartheta$, with $x(t_i^-) = x(t_i)\}$. It can be seen easily that $PC(I,X)$ is a Banach space endowed with the sup norm $\|x\|_P = \sup_{t\in I}\|x(t)\|$. Further, we define $PC^1(I,X) = \{x \in PC(I,X) : x^{\Delta} \in PC(I,X)\}$. Clearly, $PC^1(I,X)$ forms a Banach space endowed with the norm $\|x\|_{PC^1} = \max\{\|x\|_P, \|x^{\Delta}\|_P\}$.

Next, by Definition 3.14 and definition 2.2 of (Kavitha et al., 2020), we can define the solution of the system (2.3a)-(2.3c) as follows.

Definition 3.15 ((Kavitha et al., 2020), Def. 2.2)**.** *A function $x \in PC(I,X)$ is called a mild solution f the system (2.3a)-(2.3c), if $x(t)$ satisfies the following*

(i) $x(0) = x_0$;

(ii) $x(t) = \frac{1}{\Gamma(\gamma)} \int_{t_i}^t (t-\zeta)^{\gamma-1}\aleph_i(\zeta, x(t_i^-))\Delta\zeta,\ t \in (t_i, s_i]_{\mathbb{T}},\ i = 1, 2, \ldots, \vartheta$

and the following integral equations

$$x(t) = \mathcal{T}_0(t,0)[x_0 - \Upsilon_0(0, x_0)] + \Upsilon_0(t, x_a(t)) + \int_0^t \mathcal{T}_0(t, \sigma(\zeta))\Psi_0(\zeta, x_b(\zeta))\Delta\zeta \tag{3.6}$$

for all $t \in (0, t_1]_{\mathbb{T}}$ and

$$\begin{aligned} x(t) = {} & \mathcal{T}_i(t, s_i)\left[\frac{1}{\Gamma(\gamma)}\int_{t_i}^{s_i}(s_i - \zeta)^{\gamma-1}\aleph_i(\zeta, x(t_i^-))\Delta\zeta - \Upsilon_i(s_i, x_a(s_i))\right] + \Upsilon_i(t, x_a(t)) \\ & + \int_{s_i}^t \mathcal{T}_i(t, \sigma(\zeta)\Psi_i(\zeta, x_b(\zeta))\Delta\zeta \end{aligned} \tag{3.7}$$

for all $t \in (s_i, t_{i+1}]_{\mathbb{T}},\ i = 1, 2, \ldots, \vartheta$.

Now, we set the following standard and feasible assumptions on the non-linear functions Υ_i, Ψ_i and $\aleph_i$, which are often used to establish the existence of a unique solution.

(A1) (Wang and Fečkan, 2015; Kumar et al., 2021b): The functions $\Upsilon_i, \Psi_i : T_0 \times X \to X, T_0 = \cup_{i=0}^{\vartheta}[s_i, t_{i+1}]_{\mathbb{T}}$ are continuous. Also, there exist positive constants L_{Υ_i} and $L_{\Psi_i}, i = 0, 1, \dots, \vartheta$, such that

(a) $\|\Upsilon_i(t, x_1) - \Upsilon_i(t, x_2)\| \leq L_{\Upsilon_i}\|x_1 - x_2\|$ for all $x_1, x_2 \in X$ and $t \in T_0$.

(b) $\|\Psi_i(t, x_1) - \Psi_i(t, x_2)\| \leq L_{\Psi_i}\|x_1 - x_2\|$ for all $x_1, x_2 \in X$ and $t \in T_0$.

(A2) (Wang and Fečkan, 2015; Kumar et al., 2021b): The functions $\aleph_i : T_i \times X \to X, T_i = [t_i, s_i]_{\mathbb{T}},\ i = 1, 2, \dots, \vartheta$, are continuous. Also, there exist positive constants $L_{\aleph_i}, i = 1, 2, \dots, \vartheta$, such that

$$\|\aleph_i(t, x_1) - \aleph_i(t, x_2)\| \leq L_{\aleph_i}\|x_1 - x_2\| \text{ for all } x_1, x_2 \in X \text{ and } t \in T_i.$$

(A3) (Wang and Agarwal, 2014): $\{\mathcal{A}_i(t) : t \in T_0\}$ generate the exponentially stable evolution operators $\{\mathcal{T}_i(t, s) : t \geq s\}$, i.e., there exist $\mathsf{C} \geq 1$ and $\nu > 0$ such that $\|\mathcal{T}_i(t, s)\| \leq \mathsf{C}e_{\ominus\nu}(t, s)$ for all $i = 0, 1, \dots, \vartheta$ and $t \geq s$.

For the notational convenience, we set

$L_\Upsilon = \max_{i=0,1,\dots,\vartheta}\{L_{\Upsilon_i}\},\ L_\Psi = \max_{i=0,1,\dots,\vartheta}\{L_{\Psi_i}\},\ L_\aleph = \max_{i=1,2,\dots,\vartheta}\{L_{\aleph_i}\};$

$\sup_{t\in I,\ i=0,1,\dots,\vartheta}\|\Upsilon_i(t, 0)\| \leq M_\Upsilon,\ \sup_{t\in I,\ i=0,1,\dots,\vartheta}\|\Psi_i(t, 0)\| \leq M_\Psi,\ \sup_{t\in I,\ i=1,2,\dots,\vartheta}\|\aleph_i(t, 0)\| \leq M_\aleph;$

$$\mathsf{N}_0 = \mathsf{C}[\|x_0\| + \|\Upsilon_0(0, x_0)\|] + M_\Upsilon + \frac{\mathsf{C}M_\Psi(1 + \bar{\mu}\nu)}{\nu};\ \ \mathsf{N}_1 = L_\Upsilon + \frac{\mathsf{C}L_\Psi(1 + \bar{\mu}\nu)}{\nu};$$

$$\mathsf{N}_2 = \mathsf{C}\left(\frac{M_\aleph T^\gamma}{\Gamma(\gamma + 1)} + M_\Upsilon\right) + M_\Upsilon + \frac{\mathsf{C}M_\Psi(1 + \bar{\mu}\nu)}{\nu};\ \ \mathsf{N}_3 = \mathsf{C}\left(\frac{L_\aleph T^\gamma}{\Gamma(\gamma + 1)} + L_\Upsilon\right) + L_\Upsilon + \frac{\mathsf{C}L_\Psi(1 + \bar{\mu}\nu)}{\nu}.$$

4 Existence and Uniqueness Result

Here, we establish the existence of a unique solution by using the Banach contraction principle with evolution operator theory.

Theorem 4.1. *If the assumptions (A1)-(A3) hold, then the switched system (2.3a)-(2.3c) has a unique solution, provided* $\mathsf{N}_3 < 1$.

Proof. Let us define a subset $\mathsf{D}_1 \subset PC(I, X)$ such that

$$\mathsf{D}_1 = \{x \in PC(I, X) : \|x\|_P \leq \delta_1\},$$

where

$$\delta_1 = \max\left\{\frac{\mathsf{N}_0}{1 - \mathsf{N}_3},\ \frac{M_\aleph T^\gamma}{\Gamma(\gamma + 1)(1 - \mathsf{N}_3)},\ \frac{\mathsf{N}_2}{1 - \mathsf{N}_3}\right\}.$$

Now, we define an operator $\digamma_1 : \mathsf{D}_1 \to \mathsf{D}_1$ such that

$$(\digamma_1 x)t = \begin{cases} \mathcal{T}_0(t, 0)[x_0 - \Upsilon_0(0, x_0)] + \Upsilon_0(t, x_a(t)) + \int_0^t \mathcal{T}_0(t, \sigma(\zeta))\Psi_0(\zeta, x_b(\zeta))\Delta\zeta, & t \in (0, t_1]_{\mathbb{T}}, \\ \dfrac{1}{\Gamma(\gamma)} \int_{t_i}^t (t - \zeta)^{\gamma-1}\aleph_i(\zeta, x(t_i^-))\Delta\zeta, & t \in (t_i, s_i]_{\mathbb{T}},\ i = 1, 2, \dots, \vartheta, \\ \mathcal{T}_i(t, s_i)\left[\dfrac{1}{\Gamma(\gamma)} \int_{t_i}^{s_i} (s_i - \zeta)^{\gamma-1}\aleph_i(\zeta, x(t_i^-))\Delta\zeta - \Upsilon_i(s_i, x_a(s_i))\right] & \\ +\Upsilon_i(t, x_a(t)) + \int_{s_i}^t \mathcal{T}_i(t, \sigma(\zeta)\Psi_i(\zeta, x_b(\zeta))\Delta\zeta, & t \in (s_i, t_{i+1}]_{\mathbb{T}},\ i = 1, 2, \dots, \vartheta. \end{cases}$$

Now, we dived the proof into the following two steps:
Step 1: Here, we will show that $\digamma_1$ maps D_1 into D_1. Let for any $t \in (0, t_1]_{\mathbb{T}}$ and $x \in \mathsf{D}_1$, we have

$$\begin{aligned}
\|(\digamma_1 x)t\| &\leq \mathsf{C}e_{\ominus\nu}(t,0)[\|x_0\| + \|\Upsilon_0(0,x_0)\|] + \|\Upsilon_0(t,x_a(t))\| + \mathsf{C}\int_0^t e_{\ominus\nu}(t,\sigma(\zeta))\|\Psi_0(\zeta,x_b(\zeta))\|\Delta\zeta \\
&\leq \mathsf{C}e_{\ominus\nu}(t,0)[\|x_0\| + \|\Upsilon_0(0,x_0)\|] + L_{\Upsilon_0}\|x_a(t)\| + M_\Upsilon + \mathsf{C}\int_0^t e_{\ominus\nu}(t,\sigma(\zeta))(L_{\Psi_0}\|x_b(\zeta)\| + M_\Psi)\Delta\zeta \\
&\leq \mathsf{C}e_{\ominus\nu}(t,0)[\|x_0\| + \|\Upsilon_0(0,x_0)\|] + L_{\Upsilon_0}\|x_a(t)\| + M_\Upsilon + \frac{\mathsf{C}(L_{\Psi_0}\sup_{t\in[0,t_1]_{\mathbb{T}}}\|x_b(t)\| + M_\Psi)(1+\bar{\mu}\nu)}{\nu} \\
&\leq \mathsf{N}_0 + L_\Upsilon\delta_1 + \frac{\mathsf{C}L_{\Psi_0}\delta_1(1+\bar{\mu}\nu)}{\nu} \\
&\leq \mathsf{N}_0 + \mathsf{N}_1\delta_1 \leq \mathsf{N}_0 + \mathsf{N}_3\delta_1 \\
&\leq \delta_1. \qquad (4.8)
\end{aligned}$$

Now, for any $t \in (t_i, s_i]_{\mathbb{T}},\ i = 1, 2, \ldots, \vartheta$, and $x \in \mathsf{D}_1$,

$$\begin{aligned}
\|(\digamma_1 x)t\| &\leq \frac{1}{\Gamma(\gamma)}\int_{t_i}^t (t-\zeta)^{\gamma-1}\|\aleph_i(\zeta, x(t_i^-))\|\Delta\zeta \\
&\leq \frac{1}{\Gamma(\gamma)}\int_{t_i}^t (t-\zeta)^{\gamma-1}\|\aleph_i(\zeta, x(t_i^-)) - \aleph_i(\zeta,0) + \aleph_i(\zeta,0)\|\Delta\zeta \\
&\leq \frac{1}{\Gamma(\gamma)}\int_{t_i}^t (t-\zeta)^{\gamma-1}(L_{\aleph_i}\|x(t_i^-)\| + M_\aleph)\Delta\zeta \\
&\leq \frac{(L_{\aleph_i}\delta_1 + M_\aleph)(t-t_i)^\gamma}{\Gamma(\gamma+1)} \\
&\leq \frac{M_\aleph T^\gamma}{\Gamma(\gamma+1)} + \mathsf{N}_3\delta_1 \\
&\leq \delta_1. \qquad (4.9)
\end{aligned}$$

Similarly, for any $t \in (s_i, t_{i+1}]_{\mathbb{T}},\ i = 1, 2, \ldots, \vartheta$ and $x \in \mathsf{D}_1$,

$$\begin{aligned}
\|(\digamma_1 x)t\| &\leq \mathsf{C}e_{\ominus\nu}(t,s_i)\left[\frac{1}{\Gamma(\gamma)}\int_{t_i}^{s_i}(s_i-\zeta)^{\gamma-1}\|\aleph_i(\zeta,x(t_i^-))\|\Delta\zeta + \|\Upsilon_i(s_i,x_a(s_i))\|\right] + \|\Upsilon_i(t,x_a(t))\| \\
&\quad + \mathsf{C}\int_{s_i}^t e_{\ominus\nu}(t,\sigma(\zeta)\|\Psi_i(\zeta,x_b(\zeta))\|\Delta\zeta \\
&\leq \mathsf{C}e_{\ominus\nu}(t,s_i)\left[\frac{1}{\Gamma(\gamma)}\int_{t_i}^{s_i}(s_i-\zeta)^{\gamma-1}(L_{\aleph_i}\|x(t_i^-)\| + M_\aleph)\Delta\zeta + L_{\Upsilon_i}\|x_a(s_i)\| + M_\Upsilon\right] + L_{\Upsilon_i}\|x_a(t)\| \\
&\quad + M_\Upsilon + \mathsf{C}\int_{s_i}^t e_{\ominus\nu}(t,\sigma(\zeta)(L_{\Psi_i}\|x_b(\zeta)\| + M_\Psi)\Delta\zeta \\
&\leq \mathsf{C}e_{\ominus\nu}(t,s_i)\left[\frac{(L_{\aleph_i}\|x(t_i^-)\| + M_\aleph)t_{i+1}^\gamma}{\Gamma(\gamma+1)} + L_{\Upsilon_i}\|x_a(s_i)\| + M_\Upsilon\right] + L_{\Upsilon_i}\|x_a(t)\| + M_\Upsilon \\
&\quad + \frac{\mathsf{C}(L_{\Psi_i}\sup_{t\in[s_i,t_{i+1}]_{\mathbb{T}}}\|x_b(t)\| + M_\Psi)(1+\bar{\mu}\nu)}{\nu} \\
&\leq \mathsf{N}_2 + \mathsf{N}_3\delta_1 \\
&\leq \delta_1. \qquad (4.10)
\end{aligned}$$

From the above equations (4.8), (4.9) and (4.10), for all $t \in I$, we have

$$\|\mathcal{F}_1 x\|_P \leq \delta_1.$$

Hence, $\mathcal{F}_1$ maps D_1 into D_1.
Step 2: Here, we shall show that the operator $\mathcal{F}_1$ is a contracting operator. Let for any $t \in (0, t_1]_{\mathbb{T}}$ and $x, y \in \mathsf{D}_1$, we have

$$\begin{aligned}
\|(\mathcal{F}_1 x)t - (\mathcal{F}_1 y)t\| &\leq \|\Upsilon_0(t, x_a(t)) - \Upsilon_0(t, y_a(t))\| + \mathsf{C}\int_0^t e_{\ominus\nu}(t, \sigma(\zeta))\|\Psi_0(\zeta, x_b(\zeta)) - \Psi_0(\zeta, y_b(\zeta))\|\Delta\zeta \\
&\leq L_{\Upsilon_0}\|x_a(t) - y_a(t)\| + \mathsf{C}\int_0^t e_{\ominus\nu}(t, \sigma(\zeta))L_{\Psi_0}\|x_b(\zeta) - y_b(\zeta)\|\Delta\zeta \\
&\leq L_{\Upsilon_0}\|x_a(t) - y_a(t)\| + \frac{\mathsf{C}L_{\Psi_0}\sup_{t\in[0,t_1]_{\mathbb{T}}}\|x_b(t) - y_b(t)\|(1 + \bar{\mu}\nu)}{\nu} \\
&\leq \left(L_{\Upsilon_0} + \frac{\mathsf{C}L_{\Psi_0}(1 + \bar{\mu}\nu)}{\nu}\right)\|x - y\|_P \\
&\leq \mathsf{N}_3\|x - y\|_P. \qquad (4.11)
\end{aligned}$$

Now, for any $t \in (t_i, s_i]_{\mathbb{T}},\ i = 1, 2, \ldots, \vartheta$ and $x, y \in \mathsf{D}_1$,

$$\begin{aligned}
\|(\mathcal{F}_1 x)t - (\mathcal{F}_1 y)t\| &\leq \frac{1}{\Gamma(\gamma)}\int_{t_i}^t (t - \zeta)^{\gamma-1}\|\aleph_i(\zeta, x(t_i^-)) - \aleph_i(\zeta, y(t_i^-))\|\Delta\zeta \\
&\leq \frac{1}{\Gamma(\gamma)}\int_{t_i}^t (t - \zeta)^{\gamma-1}L_{\aleph_i}\|x(t_i^-) - y(t_i^-)\|\Delta\zeta \\
&\leq \frac{L_{\aleph_i}(t - t_i)^\gamma}{\Gamma(\gamma + 1)}\|x - y\|_P \\
&\leq \mathsf{N}_3\|x - y\|_P \qquad (4.12)
\end{aligned}$$

Similarly, for any $t \in (s_i, t_{i+1}]_{\mathbb{T}},\ i = 1, 2, \ldots, \vartheta$ and $x, y \in \mathsf{D}_1$,

$$\begin{aligned}
\|(\mathcal{F}_1 x)t - (\mathcal{F}_1 y)t\| &\leq \mathsf{C}e_{\ominus\nu}(t, s_i)\Bigg[\frac{1}{\Gamma(\gamma)}\int_{t_i}^{s_i}(s_i - \zeta)^{\gamma-1}\|\aleph_i(\zeta, x(t_i^-)) - \aleph_i(\zeta, y(t_i^-))\|\Delta\zeta \\
&\quad + \|\Upsilon_i(s_i, x_a(s_i)) - \Upsilon_i(s_i, y_a(s_i))\|\Bigg] + \|\Upsilon_i(t, x_a(t)) - \Upsilon_i(t, y_a(t))\| \\
&\quad + \mathsf{C}\int_{s_i}^t e_{\ominus\nu}(t, \sigma(\zeta)\|\Psi_i(\zeta, x_b(\zeta)) - \Psi_i(\zeta, y_b(\zeta))\|\Delta\zeta \\
&\leq \mathsf{C}e_{\ominus\nu}(t, s_i)\left[\frac{1}{\Gamma(\gamma)}\int_{t_i}^{s_i}(s_i - \zeta)^{\gamma-1}L_{\aleph_i}\|x(t_i^-) - y(t_i^-)\|\Delta\zeta + L_{\Upsilon_i}\|x_a(s_i) - y_a(s_i)\|\right] \\
&\quad + L_{\Upsilon_i}\|x_a(t) - y_a(t)\| + \mathsf{C}\int_{s_i}^t e_{\ominus\nu}(t, \sigma(\zeta)L_{\Psi_i}\|x_b(\zeta) - y_b(\zeta)\|\Delta\zeta \\
&\leq \mathsf{C}e_{\ominus\nu}(t, s_i)\left[\frac{(L_{\aleph_i}\|x(t_i^-) - y(t_i^-)\|t_{i+1}^\gamma}{\Gamma(\gamma + 1)} + L_{\Upsilon_i}\|x_a(s_i) - y_a(s_i)\|\right] + L_{\Upsilon_i}\|x_a(t) - y_a(t)\| \\
&\quad + \frac{\mathsf{C}(L_{\Psi_i}\sup_{t\in[s_i,t_{i+1}]_{\mathbb{T}}}\|x_b(t) - y_b(t)\| + M_\Psi)(1 + \bar{\mu}\nu)}{\nu} \\
&\leq \mathsf{N}_3\|x - y\|_P. \qquad (4.13)
\end{aligned}$$

From the above equations (4.11), (4.12) and (4.13), for all $t \in I$, we have

$$\|\digamma_1 x - \digamma_1 y\|_P \le \mathsf{N}_3 \|x - y\|_P.$$

Hence, $\digamma_1$ is a contracting operator.

Now collecting the step 1 and step 2 along with the Banach contraction principle, we can conclude that the operator $\digamma_1$ has a unique fixed point which is the solution of the systems (2.3a)-(2.3c). □

Remark 4.2. *In the existing literature, many authors established the existence of solutions for different types of dynamic systems with non-instantaneous impulses by using the Banach contraction principle. Particularly. in (Hernández and O'Regan, 2013), the authors investigated the existence of mild solutions for a new class of differential equations with non-instantaneous impulses while in (Abbas and Benchohra, 2015), the authors studied the existence of a unique solution for partial fractional differential equations with non-instantaneous impulses. Further, in (Zada et al., 2017) the authors considered a nonlinear impulsive Volterra integro-delay dynamic system on time scales and investigated the existence of a unique solution. The works in (Shah and Zada, 2019) mainly focused on the existence and uniqueness of solutions for the mixed integral dynamic systems with both instantaneous and non-instantaneous impulses on time scales. In (Shah and Zada, 2022), the authors studied the existence of a unique solution of the nonlinear Volterra integro-delay dynamic system with fractional integrable impulses on time scales in the finite-dimensional spaces. However, all these results are either for the continuous-time domain or for the finite-dimensional spaces, and cannot be directly applied to the case of the switched dynamic systems on an arbitrary time domain in the infinite-dimensional spaces. Therefore, the existence and uniqueness result of this paper is new which extends and generalizes the existing results.*

5 Ulam-Hyers Stability Result

Stability analysis is the fundamental property of the mathematical analysis which is very important in many fields of engineering and science. Ulam and Hyer introduced an interesting type of stability called Ulam-Hyer's stability and since then it has been picked up a great deal of attention due to its wide range of applications in many fields of science, especially in optimization and mathematical modelling. Therefore, in this segment of the paper, we will investigate the Ulam-Hyers type stability for the system (2.3a)-(2.3c).

Let us consider the following inequalities

$$\begin{cases} \left\| [y(t) - \Upsilon_i(t, y_a(t))]^\Delta - \mathcal{A}_i(t)[y(t) - \Upsilon_i(t, y_a(t))] - \Psi_i(t, y_b(t)) \right\| \le \epsilon, \ t \in \cup_{i=0}^{\vartheta} (s_i, t_{i+1}]_{\mathbb{T}}, \\ \left\| y(t) - \dfrac{1}{\Gamma(\gamma)} \int_{t_i}^{t} (t-\zeta)^{\gamma-1} \aleph_i(\zeta, y(t_i^-)) \Delta\zeta \right\| \le \epsilon, \ t \in (t_i, s_i]_{\mathbb{T}}, \ i = 1, 2, \ldots, \vartheta, \end{cases} \tag{5.14}$$

where $\epsilon > 0$ is a constant.

Now, Before giving the main result of Ulam-Hyers type stability, we introduce the following important definitions.

Definition 5.1 ((Wang et al., 2012), Def. 3.1)**.** *Evolution system (2.3a)-(2.3c) is Ulam-Hyers stable if there exists a positive constant $H_{(L_\Upsilon, L_\Psi, L_\aleph, \vartheta)}$ such that for $\epsilon > 0$ and for each solution y of inequality (5.14), there exist a unique solution x of the system (2.3a)-(2.3c) such that*

$$\|y(t) - x(t)\| \le H_{(L_\Upsilon, L_\Psi, L_\aleph, \vartheta)} \epsilon \text{ for all } t \in I.$$

Remark 5.2. *A function $y \in PC^1(I,X)$ is a solution of the inequality (5.14) if and only if there is $\mathtt{G}, \mathtt{G}_i \in PC(I,X),\ i=1,2,\dots,\vartheta$, such that*

(i) $\|\mathtt{G}(t)\| \le \epsilon$ *for all* $t \in \cup_{i=0}^{\vartheta}(s_i,t_{i+1}]_{\mathbb{T}}$ *and* $\|\mathtt{G}_i(t)\| \le \epsilon$ *for all* $t \in (t_i,s_i]_{\mathbb{T}},\ i=1,2,\dots,\vartheta$;

(ii) $[y(t) - \Upsilon_i(t,y_a(t))]^{\Delta} = \mathcal{A}_i(t)[y(t)-\Upsilon_i(t,y_a(t))] + \Psi_i(t,y_b(t)) + \mathtt{G}(t),\ t \in \cup_{i=0}^{\vartheta}(s_i,t_{i+1}]_{\mathbb{T}}$;

(iii) $y(t) = \dfrac{1}{\Gamma(\gamma)} \int_{t_i}^{t} (t-\zeta)^{\gamma-1} \aleph_i(\zeta, y(t_i^-))\Delta\zeta + \mathtt{G}_i(t),\ t \in (t_i,s_i]_{\mathbb{T}},\ i=1,2,\dots,\vartheta.$

Lemma 5.3. *If $y \in PC^1(I,X)$ satisfies inequality (5.14), then for $y(0)=x_0$, the following inequalities*

$$\begin{cases} \left\| y(t) - \mathcal{T}_0(t,0)[x_0 - \Upsilon_0(0,x_0)] - \Upsilon_0(t,y_a(t)) - \int_0^t \mathcal{T}_0(t,\sigma(\zeta))\Psi_0(\zeta,y_b(\zeta))\Delta\zeta \right\| \le \dfrac{\mathsf{C}\epsilon(1+\bar{\mu}\nu)}{\nu},\ t \in (0,t_1]_{\mathbb{T}}, \\ \Bigg\| y(t) - \mathcal{T}_i(t,s_i)\left[\dfrac{1}{\Gamma(\gamma)} \int_{t_i}^{s_i}(s_i-\zeta)^{\gamma-1}\aleph_i(\zeta,y(t_i^-))\Delta\zeta - \Upsilon_i(s_i,y_a(s_i))\right] - \Upsilon_i(t,y_a(t)) \\ \quad - \int_{s_i}^{t} \mathcal{T}_i(t,\sigma(\zeta)\Psi_i(\zeta,y_b(\zeta))\Delta\zeta \Bigg\| \le \dfrac{\mathsf{C}\epsilon(1+\bar{\mu}(1+\nu))}{\nu},\ t \in (s_i,t_{i+1}]_{\mathbb{T}},\ i=1,2,\dots,\vartheta, \\ \left\| y(t) - \dfrac{1}{\Gamma(\gamma)} \int_{t_i}^{t}(t-\zeta)^{\gamma-1}\aleph_i(\zeta,y(t_i^-))\Delta\zeta \right\| \le \epsilon,\ t \in (t_i,s_i]_{\mathbb{T}},\ i=1,2,\dots,\vartheta, \end{cases}$$

hold.

Proof. If $y \in PC^1(I,X)$ satisfies inequality (5.14), then by Remark 5.2, we have

$$\begin{cases} [y(t)-\Upsilon_i(t,y_a(t))]^{\Delta} = \mathcal{A}_i(t)[y(t)-\Upsilon_i(t,y_a(t))] + \Psi_i(t,y_b(t)) + \mathtt{G}(t),\ t \in \cup_{i=0}^{\vartheta}(s_i,t_{i+1}]_{\mathbb{T}}, \\ y(t) = \dfrac{1}{\Gamma(\gamma)}\int_{t_i}^{t}(t-\zeta)^{\gamma-1}\aleph_i(\zeta,y(t_i^-))\Delta\zeta + \mathtt{G}_i(t),\ t \in (t_i,s_i]_{\mathbb{T}},\ i=1,2,\dots,\vartheta. \end{cases} \tag{5.15}$$

Clearly, from Definition 3.15, the solution of the equation (5.15) with $y(0)=x_0$ is given as

$$y(t) = \begin{cases} \mathcal{T}_0(t,0)[x_0-\Upsilon_0(0,x_0)] + \Upsilon_0(t,y_a(t)) + \int_0^t \mathcal{T}_0(t,\sigma(\zeta))(\Psi_0(\zeta,y_b(\zeta)) + \mathtt{G}(\zeta))\Delta\zeta,\ t \in (0,t_1]_{\mathbb{T}}, \\ \dfrac{1}{\Gamma(\gamma)}\int_{t_i}^{t}(t-\zeta)^{\gamma-1}\aleph_i(\zeta,y(t_i^-))\Delta\zeta + \mathtt{G}_i(t),\ t\in(t_i,s_i]_{\mathbb{T}},\ i=1,2,\dots,\vartheta, \\ \mathcal{T}_i(t,s_i)\left[\dfrac{1}{\Gamma(\gamma)}\int_{t_i}^{s_i}(s_i-\zeta)^{\gamma-1}\aleph_i(\zeta,y(t_i^-))\Delta\zeta + \mathtt{G}_i(s_i) - \Upsilon_i(s_i,y_a(s_i))\right] \\ +\Upsilon_i(t,y_a(t)) + \int_{s_i}^{t}\mathcal{T}_i(t,\sigma(\zeta)(\Psi_i(\zeta,y_b(\zeta)) + \mathtt{G}(\zeta))\Delta\zeta,\ t\in(s_i,t_{i+1}]_{\mathbb{T}},\ i=1,2,\dots,\vartheta. \end{cases}$$

Now, for any $t \in (0,t_1]_{\mathbb{T}}$, we have

$$\begin{aligned} \left\| y(t) - \mathcal{T}_0(t,0)[x_0-\Upsilon_0(0,x_0)] - \Upsilon_0(t,y_a(t)) - \int_0^t \mathcal{T}_0(t,\sigma(\zeta))\Psi_0(\zeta,y_b(\zeta))\Delta\zeta \right\| &\le \left\| \int_0^t \mathcal{T}_0(t,\sigma(\zeta))\mathtt{G}(\zeta)\Delta\zeta \right\| \\ &\le \mathsf{C}\epsilon \int_0^t e_{\ominus\nu}(t,\sigma(\zeta))\Delta\zeta \\ &\le \frac{\mathsf{C}\epsilon(1+\bar{\mu}\nu)}{\nu}. \end{aligned}$$

Also, for any $t \in (s_i,t_{i+1}]_{\mathbb{T}},\ i=1,2,\dots,\vartheta$,

$$\left\| y(t) - \mathcal{T}_i(t,s_i)\left[\frac{1}{\Gamma(\gamma)}\int_{t_i}^{s_i}(s_i-\zeta)^{\gamma-1}\aleph_i(\zeta,y(t_i^-))\Delta\zeta - \Upsilon_i(s_i,y_a(s_i))\right] - \Upsilon_i(t,y_a(t)) \right.$$

$$-\int_{s_i}^{t}\mathcal{T}_i(t,\sigma(\zeta)\Psi_i(\zeta,y_b(\zeta))\Delta\zeta\Big\| \leq \Big\|\mathcal{T}_i(t,s_i)\mathtt{G}_i(s_i)+\int_{s_i}^{t}\mathcal{T}_i(t,\sigma(\zeta)\mathtt{G}(\zeta)\Delta\zeta\Big\|$$
$$\leq \frac{\mathsf{C}\epsilon(1+\bar{\mu}(1+\nu))}{\nu}.$$

Similarly, for any $t\in(t_i,s_i]_{\mathbb{T}},\ i=1,2,\ldots,\vartheta$,

$$\left\|y(t)-\frac{1}{\Gamma(\gamma)}\int_{t_i}^{t}(t-\zeta)^{\gamma-1}\aleph_i(\zeta,y(t_i^-))\Delta\zeta\right\|\leq\epsilon.$$

Hence, the result follows. □

Theorem 5.4. *If the assumptions (A1)-(A3) and* $\mathsf{N}_3<1$ *hold, then the system* (2.3a)-(2.3c) *is Ulam-Hyers stable.*

Proof. Let $y(t)$ be a solution of the inequality (5.14) and $x(t)$ is a unique mild solution of the system (2.3a)-(2.3c) which is given by

$$x(t)=\begin{cases}\mathcal{T}_0(t,0)[x_0-\Upsilon_0(0,x_0)]+\Upsilon_0(t,x_a(t))+\int_0^t\mathcal{T}_0(t,\sigma(\zeta))\Psi_0(\zeta,x_b(\zeta))\Delta\zeta, & t\in(0,t_1]_{\mathbb{T}},\\ \dfrac{1}{\Gamma(\gamma)}\int_{t_i}^{t}(t-\zeta)^{\gamma-1}\aleph_i(\zeta,x(t_i^-))\Delta\zeta, & t\in(t_i,s_i]_{\mathbb{T}},\ i=1,2,\ldots,\vartheta,\\ \mathcal{T}_i(t,s_i)\left[\dfrac{1}{\Gamma(\gamma)}\int_{t_i}^{s_i}(s_i-\zeta)^{\gamma-1}\aleph_i(\zeta,x(t_i^-))\Delta\zeta-\Upsilon_i(s_i,x_a(s_i))\right] & \\ +\Upsilon_i(t,x_a(t))+\int_{s_i}^{t}\mathcal{T}_i(t,\sigma(\zeta)\Psi_i(\zeta,x_b(\zeta))\Delta\zeta, & t\in(s_i,t_{i+1}]_{\mathbb{T}},\ i=1,2,\ldots,\vartheta.\end{cases}$$

Now, for any $t\in(0,t_1]_{\mathbb{T}}$,

$$\begin{aligned}\|y(t)-x(t)\|&=\left\|y(t)-\mathcal{T}_0(t,0)[x_0-\Upsilon_0(0,x_0)]-\Upsilon_0(t,x_a(t))-\int_0^t\mathcal{T}_0(t,\sigma(\zeta))\Psi_0(\zeta,x_b(\zeta))\Delta\zeta\right\|\\&\leq\left\|y(t)-\mathcal{T}_0(t,0)[x_0-\Upsilon_0(0,x_0)]-\Upsilon_0(t,y_a(t))-\int_0^t\mathcal{T}_0(t,\sigma(\zeta))\Psi_0(\zeta,y_b(\zeta))\Delta\zeta\right\|\\&\quad+\|\Upsilon_0(t,y_a(t))-\Upsilon_0(t,x_a(t))\|+\int_0^t\mathcal{T}_0(t,\sigma(\zeta))\|\Psi_0(\zeta,y_b(\zeta))-\Psi_0(\zeta,x_b(\zeta))\|\Delta\zeta\\&\leq\frac{\mathsf{C}\epsilon(1+\bar{\mu}\nu)}{\nu}+L_{\Upsilon_0}\|y_a(t)-x_a(t)\|+\frac{\mathsf{C}L_{\Psi_0}(1+\bar{\mu}\nu)\sup_{t\in[0,t_1]_{\mathbb{T}}}\|y_b(t)-x_b(t)\|}{\nu}\\&\leq\frac{\mathsf{C}\epsilon(1+\bar{\mu}\nu)}{\nu}+\mathsf{N}_1\|y-x\|_P.\end{aligned}\tag{5.16}$$

Also, for any $t\in(s_i,t_{i+1}]_{\mathbb{T}},\ i=1,2,\ldots,\vartheta$,

$$\begin{aligned}\|y(t)-x(t)\|&=\Bigg\|y(t)-\mathcal{T}_i(t,s_i)\left[\frac{1}{\Gamma(\gamma)}\int_{t_i}^{s_i}(s_i-\zeta)^{\gamma-1}\aleph_i(\zeta,x(t_i^-))\Delta\zeta-\Upsilon_i(s_i,x_a(s_i))\right]\\&\quad-\Upsilon_i(t,x_a(t))-\int_{s_i}^{t}\mathcal{T}_i(t,\sigma(\zeta)\Psi_i(\zeta,x_b(\zeta))\Delta\zeta\Bigg\|\\&\leq\Bigg\|y(t)-\mathcal{T}_i(t,s_i)\left[\frac{1}{\Gamma(\gamma)}\int_{t_i}^{s_i}(s_i-\zeta)^{\gamma-1}\aleph_i(\zeta,y(t_i^-))\Delta\zeta-\Upsilon_i(s_i,y_a(s_i))\right]-\Upsilon_i(t,y_a(t))\end{aligned}$$

$$
\begin{aligned}
&- \int_{s_i}^{t} \mathcal{T}_i(t,\sigma(\zeta)\Psi_i(\zeta, y_b(\zeta))\Delta\zeta\Bigg\| + \Bigg\|\frac{\mathcal{T}_i(t,s_i)}{\Gamma(\gamma)}\int_{t_i}^{s_i}(s_i-\zeta)^{\gamma-1}\aleph_i(\zeta, y(t_i^-)) - \aleph_i(\zeta, x(t_i^-))\Delta\zeta\Bigg\| \\
&+ \|\mathcal{T}_i(t,s_i)\|\|\Upsilon_i(s_i, y_a(s_i)) - \Upsilon_i(s_i, x_a(s_i))\| + \|\Upsilon_i(t, y_a(t)) - \Upsilon_i(t, x_a(t))\| \\
&+ \int_{s_i}^{t} \|\mathcal{T}_i(t,\sigma(\zeta)\|\|\Psi_i(\zeta, y_b(\zeta)) - \Psi_i(\zeta, x_b(\zeta))\|\Delta\zeta\Bigg\| \\
\le\ & \frac{\mathsf{C}\epsilon(1+\bar{\mu}(1+\nu))}{\nu} + \frac{\mathsf{C}e_{\ominus\nu}(t,s_i)t_{i+1}^{\gamma}L_{\aleph_i}\|y(t_i^-) - x(t_i^-)\|}{\Gamma(\gamma+1)} + \mathsf{C}e_{\ominus\nu}(t,s_i)L_{\Upsilon_i}\|y_a(s_i) - x_a(s_i)\| \\
&+ L_{\Upsilon_i}\|y_a(t) - x_a(t)\| + \frac{\mathsf{C}L_{\Psi_i}(1+\bar{\mu}\nu)\sup_{t\in[s_i,t_{i+1}]_{\mathbb{T}}}\|y_b(t) - x_b(t)\|}{\nu} \\
\le\ & \frac{\mathsf{C}\epsilon(1+\bar{\mu}(1+\nu))}{\nu} + \mathsf{N}_3\|y-x\|_P. \qquad (5.17)
\end{aligned}
$$

Similarly, for any $t \in (t_i, s_i]_{\mathbb{T}},\ i = 1, 2, \ldots, \vartheta$, we have

$$
\|y(t) - x(t)\| \le \epsilon + \frac{T^{\gamma}L_J\|y-x\|}{\Gamma(\gamma+1)}. \qquad (5.18)
$$

From the above inequalities (5.16), (5.17) and (5.18), we have

$$
\|y-x\|_P \le \frac{\mathsf{C}\epsilon(1+\bar{\mu}(1+\nu))}{\nu} + \mathsf{N}_3\|y-x\|_P \text{ for all } t \in I,
$$

which immediately gives

$$
\|y-x\|_P \le H_{(L_{\Upsilon}, L_G, L_J, \vartheta)}\epsilon,
$$

where $H_{(L_{\Upsilon}, L_G, L_J, \vartheta)} = \dfrac{\mathsf{C}(1+\bar{\mu}(1+\nu))}{\nu(1-\mathsf{N}_3)} > 0$. Thus, the system (2.3a)-(2.3c) is Ulam-Hyers stable. □

Remark 5.5. *Many researchers studied the Ulam-Hyers type stability results for different classes of systems. In (Wang et al., 2017), the authors established the stability results for the non-instantaneous impulsive differential equations while in (Abbas and Benchohra, 2015), the authors considered the partial fractional differential equations with non-instantaneous impulses and established the different types of Ulam-Hyers stability results. Further, in (Zada et al., 2017), the authors investigated the Ulam-Hyers stability results for the nonlinear impulsive Volterra integro-delay dynamic systems on time scales. In (Shah and Zada, 2019), the authors mainly focused on the problem of Ulam-Hyers stability of the mixed integral dynamic systems with both instantaneous and non-instantaneous impulses on time scales. In (Shah and Zada, 2022), the authors considered the non–linear Volterra integro-delay dynamic system with fractional integrable impulses on time scales in the finite-dimensional spaces and studied the Ulam-Hyers stability results. Nevertheless, these results cannot be directly applied to the case of the switched dynamic systems in the infinite-dimensional spaces on an arbitrary time domain. In this regard, the stability results of this manuscript are completely new even in the case of the continuous-time domain.*

6 Controllability Result

In the previous sections 4 and 5, we established the existence of a unique solution and stability results of the switched impulsive system (2.3a)-(2.3c), respectively. However, among all the qualitative properties of a dynamic system, controllability is one of the most important ones. It has many applications in engineering including biological networks, filter design, optimal control, pole assignment problem, and safety checking. Therefore, in this segment, we establish the controllability result for the switched impulsive control system (2.4a)-(2.4c) by applying the Banach contraction principle.

To establish the controllability results for the system (2.4a)-(2.4c), we define the linear operators $\mathsf{W}_{s_i}^{t_{i+1}} : L^2(I, U) \to X$ given by

$$\mathsf{W}_{s_i}^{t_{i+1}} u = \int_{s_i}^{t_{i+1}} \mathcal{T}_i(t_{i+1}, \sigma(\zeta)) B u(\zeta) \Delta\zeta, \ i = 0, 1, \ldots, \vartheta.$$

Before giving the main results of this section, we give some important definitions.

Definition 6.1. *Switched control system (2.4a)-(2.4c) is exact controllable on I, if for any initial state $x_0 \in X$ and arbitrary final state $x_T \in X$, there exists a function $u \in L^2(I, X)$ such that the mild solution of (2.4a)-(2.4c) satisfies $x(0) = x_0$ and $x(T) = x_T$.*

Definition 6.2. *Switched control system (2.4a)-(2.4c) is totally controllable on I, if it is exact controllable on $(0, t_1]_{\mathbb{T}}$ and $(s_i, t_{i+1}]_{\mathbb{T}}$, $i = 1, 2, \ldots, \vartheta$, i.e., for any initial state $x_0 \in X$ and arbitrary final states $x_{t_{i+1}} \in X$, $i = 0, 1, \ldots, \vartheta$, there exists a function $u \in L^2(I, X)$ such that the mild solution of (2.4a)-(2.4c) satisfies $x(0) = x_0$ and $x(t_{i+1}) = x_{t_{i+1}}$, $i = 0, 1, \ldots, \vartheta$.*

Next, by using the Definition 3.15, we give the solution of the system (2.4a)-(2.4c) in the next definition.

Definition 6.3. *A function $x \in PC(I, X)$ is said to be a mild solution of the system (2.4a)-(2.4c), if $x(t)$ satisfies the following*

(i) $x(0) = x_0$;

(ii) $x(t) = \frac{1}{\Gamma(\gamma)} \int_{t_i}^{t} (t-\zeta)^{\gamma-1} \aleph_i(\zeta, x(t_i^-)) \Delta\zeta, \ t \in (t_i, s_i]_{\mathbb{T}}, \ i = 1, 2, \ldots, \vartheta$

and the following integral equations

$$x(t) = \mathcal{T}_0(t, 0)[x_0 - \Upsilon_0(0, x_0)] + \Upsilon_0(t, x_a(t)) + \int_0^t \mathcal{T}_0(t, \sigma(\zeta))(\Psi_0(\zeta, x_b(\zeta)) + \mathcal{B}_0 u(\zeta)) \Delta\zeta \tag{6.19}$$

for all $t \in (0, t_1]_{\mathbb{T}}$ and

$$\begin{aligned} x(t) = \mathcal{T}_i(t, s_i) &\left[\frac{1}{\Gamma(\gamma)} \int_{t_i}^{s_i} (s_i - \zeta)^{\gamma-1} \aleph_i(\zeta, x(t_i^-)) \Delta\zeta - \Upsilon_i(s_i, x_a(s_i)) \right] + \Upsilon_i(t, x_a(t)) \\ &+ \int_{s_i}^{t} \mathcal{T}_i(t, \sigma(\zeta)(\Psi_i(\zeta, x_b(\zeta)) + \mathcal{B}_i u(\zeta)) \Delta\zeta \end{aligned} \tag{6.20}$$

for all $t \in (s_i, t_{i+1}]_{\mathbb{T}}$, $i = 1, 2, \ldots, \vartheta$.

We need the following condition to establish the controllability results.

(A4): (Kumar et al., 2021; Malik et al., 2019): The linear operators $\mathsf{W}_{s_i}^{t_{i+1}}$ have the bounded invertible operators $(\mathsf{W}_{s_i}^{t_{i+1}})^{-1}$, $i = 0, 1, \ldots, \vartheta$, which take values in $L^2(I, U) \setminus \ker \mathsf{W}_{s_i}^{t_{i+1}}$. Further, there exist positive constants M_{W_i}, $i = 0, 1, \ldots, \vartheta$, such that $\|(\mathsf{W}_{s_i}^{t_{i+1}})^{-1}\| \leq M_{\mathsf{W}_i}$.
Also, $\mathcal{B}_i$ are continuous operators from U to X and there exists a positive constant $M_{\mathcal{B}}$ such that $\|\mathcal{B}_i\| \leq M_{\mathcal{B}}$, $i = 0, 1, \ldots, m$.

Now, we are in position to give the important lemmas which require to examine the controllability.

Lemma 6.4. *If the assumptions **(A1)-(A4)** hold, then the control function*

$$u(t) = (\mathsf{W}_0^{t_1})^{-1}\left[x_{t_1} - \mathcal{T}_0(t_1,0)[x_0 - \Upsilon_0(0,x_0)] - \Upsilon_0(t_1,x_a(t_1)) - \int_0^{t_1} \mathcal{T}_0(t_1,\sigma(\zeta))\Psi_0(\zeta,x_b(\zeta))\Delta\zeta\right](t),\ t\in(0,t_1]_{\mathbb{T}}, \tag{6.21}$$

transfers the state $x(t)$ of the system (2.4a)-(2.4c) from x_0 to x_{t_1} at the time $t = t_1$. Further, the control function $u(t)$ is bounded on $t \in (0,t_1]_{\mathbb{T}}$, i.e., $\|u(t)\| \le M_{u_0}$ for all $t \in (0,t_1]_{\mathbb{T}}$, where

$$M_{u_0} = M_{\mathsf{W}_0}\left[\|x_{t_1}\| + \mathsf{N}_0 + \mathsf{N}_1 \sup_{t\in[0,t_1]_{\mathbb{T}}} \|x(t)\|\right].$$

Proof. By using the control function $u(t)$ given by the equation (6.21) in the mild solution $x(t)$ of the system (2.4a)-(2.4c) at $t = t_1$, we get

$$\begin{aligned} x(t_1) &= \mathcal{T}_0(t_1,0)[x_0 - \Upsilon_0(0,x_0)] + \Upsilon_0(t_1,x_a(t)) + \int_0^{t_1} \mathcal{T}_0(t_1,\sigma(\zeta))(\Psi_0(\zeta,x_b(\zeta)) + \mathcal{B}_0u(\zeta))\Delta\zeta \\ &= \mathcal{T}_0(t_1,0)[x_0 - \Upsilon_0(0,x_0)] + \Upsilon_0(t_1,x_a(t)) + \int_0^{t_1} \mathcal{T}_0(t_1,\sigma(\zeta))\Psi_0(\zeta,x_b(\zeta))\Delta\zeta \\ &\quad + \mathsf{W}_0^{t_1}(\mathsf{W}_0^{t_1})^{-1}\left[x_{t_1} - \mathcal{T}_0(t_1,0)[x_0 - \Upsilon_0(0,x_0)] - \Upsilon_0(t_1,x_a(t_1)) - \int_0^{t_1} \mathcal{T}_0(t_1,\sigma(\zeta))\Psi_0(\zeta,x_b(\zeta))\Delta\zeta\right] \\ &= x_{t_1} \end{aligned}$$

Further for all $t \in (0,t_1]_{\mathbb{T}}$, the estimate for the control function $u(t)$ is calculated as

$$\begin{aligned} \|u(t)\| &\le M_{\mathsf{W}_0}\left[\|x_{t_1}\| + \|\mathcal{T}_0(t_1,0)\|[\|x_0\| + \|\Upsilon_0(0,x_0)\|] + \|\Upsilon_0(t_1,x_a(t_1))\| + \int_0^{t_1} \|\mathcal{T}_0(t_1,\sigma(\zeta))\|\|\Psi_0(\zeta,x_b(\zeta))\|\Delta\zeta\right] \\ &\le M_{\mathsf{W}_0}\Bigg[\|x_{t_1}\| + \mathsf{C}e_{\ominus\nu}(t_1,0)\|[\|x_0\| + \|\Upsilon_0(0,x_0)\|] + L_{\Upsilon_0}\|x_a(t_1)\| + M_\Upsilon \\ &\quad + \mathsf{C}\int_0^{t_1} e_{\ominus\nu}(t_1,\sigma(\zeta))L_{\Psi_0}\|x_b(\zeta)\|\Delta\zeta\Bigg] \\ &\le M_{\mathsf{W}_0}\Bigg[\|x_{t_1}\| + \mathsf{C}e_{\ominus\nu}(t_1,0)\|[\|x_0\| + \|\Upsilon_0(0,x_0)\|] + L_{\Upsilon_0}\|x_a(t_1)\| + M_\Upsilon \\ &\quad + \frac{\mathsf{C}(L_{\Psi_0}\sup_{t\in[0,t_1]_{\mathbb{T}}}\|x(t)\| + M_\Psi)(1+\bar{\mu}\bar{\nu})}{\nu}\Bigg] \\ &\le M_{\mathsf{W}_0}\left[\|x_{t_1}\| + \mathsf{N}_0 + \mathsf{N}_1 \sup_{t\in[0,t_1]_{\mathbb{T}}} \|x(t)\|\right] \\ &= M_{u_0}. \end{aligned}$$

□

Lemma 6.5. *If the assumptions **(A1)-(A4)** hold, then the control function*

$$u(t) = (\mathsf{W}_i^{t_{i+1}})^{-1}\left[x_{t_{i+1}} - \mathcal{T}_i(t_{i+1},s_i)\left(\frac{1}{\Gamma(\gamma)}\int_{t_i}^{s_i}(s_i-\zeta)^{\gamma-1}\aleph_i(\zeta,x(t_i^-))\Delta\zeta - \Upsilon_i(s_i,x_a(s_i))\right)\right.$$

$$- \Upsilon_i(t_{i+1}, x_a(t_{i+1})) - \int_{s_i}^{t_{i+1}} \mathcal{T}_i(t_{i+1}, \sigma(\zeta)\Psi_i(\zeta, x_b(\zeta))\Delta\zeta\Bigg](t),\ t \in (s_i, t_{i+1}]_{\mathbb{T}},\ i = 1, 2, \dots, \vartheta, \qquad (6.22)$$

transfers the state $x(t)$ of the system (2.4a)-(2.4c) from x_0 to $x_{t_{t+1}}$ at the time $t = t_{i+1}$. Further, the control function $u(t)$ is bounded on $t \in (s_i, t_{i+1}]_{\mathbb{T}},\ i = 1, 2, \dots, \vartheta,$, i.e., $\|u(t)\| \leq M_{u_i}$ for all $t \in (s_i, t_{i+1}]_{\mathbb{T}},\ i = 1, 2, \dots, \vartheta,$ where

$$M_{u_i} = M_{\mathsf{W}_i}\left[\|x_{t_{i+1}}\| + \mathsf{N}_2 + \mathsf{N}_3 \sup_{t\in[s_i,t_{i+1}]_{\mathbb{T}}} \|x(t)\|\right].$$

Proof. By using the control function $u(t)$ given by the equation (6.22) in the mild solution $x(t)$ of the system (2.4a)-(2.4c) at $t = t_{i+1}$, we get

$$\begin{aligned}
x(t_{i+1}) &= \mathcal{T}_i(t_{i+1}, s_i)\left[\frac{1}{\Gamma(\gamma)}\int_{t_i}^{s_i}(s_i-\zeta)^{\gamma-1}\aleph_i(\zeta, x(t_i^-))\Delta\zeta - \Upsilon_i(s_i, x_a(s_i))\right] + \Upsilon_i(t_{i+1}, x_a(t)) \\
&\quad + \int_{s_i}^{t_{i+1}} \mathcal{T}_i(t_{i+1}, \sigma(\zeta)(\Psi_i(\zeta, x_b(\zeta)) + \mathcal{B}_i u(\zeta))\Delta\zeta \\
&= \mathcal{T}_i(t_{i+1}, s_i)\left[\frac{1}{\Gamma(\gamma)}\int_{t_i}^{s_i}(s_i-\zeta)^{\gamma-1}\aleph_i(\zeta, x(t_i^-))\Delta\zeta - \Upsilon_i(s_i, x_a(s_i))\right] + \int_{s_i}^{t_{i+1}} \mathcal{T}_i(t_{i+1}, \sigma(\zeta)\Psi_i(\zeta, x_b(\zeta))\Delta\zeta \\
&\quad + \Upsilon_i(t_{i+1}, x_a(t)) + \mathsf{W}_i^{t_{i+1}}(\mathsf{W}_i^{t_{i+1}})^{-1}\Bigg[x_{t_{i+1}} - \mathcal{T}_i(t_{i+1}, s_i)\Bigg(\frac{1}{\Gamma(\gamma)}\int_{t_i}^{s_i}(s_i-\zeta)^{\gamma-1}\aleph_i(\zeta, x(t_i^-))\Delta\zeta \\
&\quad - \Upsilon_i(s_i, x_a(s_i))\Bigg) - \Upsilon_i(t_{i+1}, x_a(t_{i+1})) - \int_{s_i}^{t_{i+1}} \mathcal{T}_i(t_{i+1}, \sigma(\zeta)\Psi_i(\zeta, x_b(\zeta))\Delta\zeta\Bigg] \\
&= x_{t_1}
\end{aligned}$$

Further, for all $t \in (s_i, t_{i+1}]_{\mathbb{T}},\ i = 1, 2, \dots, \vartheta,$, the estimate for the control function $u(t)$ is calculated as

$$\begin{aligned}
\|u(t)\| &\leq M_{\mathsf{W}_i}\Bigg[\|x_{t_{i+1}}\| + \|\mathcal{T}_i(t_{i+1}, s_i)\|\left(\frac{1}{\Gamma(\gamma)}\int_{t_i}^{s_i}(s_i-\zeta)^{\gamma-1}\|\aleph_i(\zeta, x(t_i^-))\|\Delta\zeta + \|\Upsilon_i(s_i, x_a(s_i))\|\right) \\
&\quad + \|\Upsilon_i(t_{i+1}, x_a(t))\| + \int_{s_i}^{t_{i+1}} \|\mathcal{T}_i(t_{i+1}, \sigma(\zeta)\|\|\Psi_i(\zeta, x_b(\zeta))\|\Delta\zeta\Bigg] \\
&\leq M_{\mathsf{W}_i}\Bigg[\|x_{t_{i+1}}\| + \mathsf{C}e_{\ominus\nu}(t_{i+1}, s_i)\left(\frac{1}{\Gamma(\gamma)}\int_{t_i}^{s_i}(s_i-\zeta)^{\gamma-1}(L_{\aleph_i}\|x(t_i^-)\| + M_{\aleph})\Delta\zeta + L_{\Upsilon_i}\|x_a(s_i)\| + M_{\Upsilon}\right) \\
&\quad + L_{\Upsilon_i}\|x_a(t)\| + M_{\Upsilon} + \mathsf{C}\int_{s_i}^{t_{i+1}} e_{\ominus\nu}(t_{i+1}, \sigma(\zeta)(L_{\Psi_i}\|x_b(\zeta)\| + M_{\Psi})\Delta\zeta\Bigg] \\
&\leq M_{\mathsf{W}_i}\Bigg[\|x_{t_{i+1}}\| + \frac{\mathsf{C}e_{\ominus\nu}(t_{i+1}, s_i)(L_{\aleph_i}\sup_{t\in[s_i,t_{i+1}]_{\mathbb{T}}}\|x(t)\| + M_{\aleph})T^{\gamma}}{\Gamma(\gamma+1)} + \mathsf{C}e_{\ominus\nu}(t_{i+1}, s_i)L_{\Upsilon_i}\sup_{t\in[s_i,t_{i+1}]_{\mathbb{T}}}\|x(t)\| \\
&\quad + \mathsf{C}e_{\ominus\nu}(t_{i+1}, s_i)M_{\Upsilon} + L_{\Upsilon_i}\sup_{t\in[s_i,t_{i+1}]_{\mathbb{T}}}\|x(t)\| + M_{\Upsilon} + \frac{\mathsf{C}(L_{\Psi_i}\sup_{t\in[s_i,t_{i+1}]_{\mathbb{T}}}\|x(t)\| + M_{\Psi})(1+\bar{\mu}\nu)}{\nu}\Bigg] \\
&\leq M_{\mathsf{W}_i}\left[\|x_{t_{i+1}}\| + \mathsf{N}_2 + \mathsf{N}_3\sup_{t\in[s_i,t_{i+1}]_{\mathbb{T}}}\|x(t)\|\right] \\
&= M_{u_i}.
\end{aligned}$$

□

We set

$$M_{\mathsf{W}} = \max_{i=0,1,\dots,\vartheta}\{M_{\mathsf{W}_i}\};\ \ \mathsf{Q}_1 = \frac{\mathsf{C}M_{\mathcal{B}}M_{\mathsf{W}}(1+\bar{\mu}\nu)}{\nu};\ \ \mathsf{N}_4 = \mathsf{N}_0(1+\mathsf{Q}_1)+\mathsf{Q}_1\|x_{t_1}\|;\ \ \mathsf{N}_5 = \mathsf{N}_1(1+\mathsf{Q}_1);$$

$$\mathsf{N}_6 = \mathsf{N}_2(1+\mathsf{Q}_1)+\mathsf{Q}_1\|x_{t_{i+1}}\|;\ \ \mathsf{N}_7 = \mathsf{N}_3(1+\mathsf{Q}_1);$$

Now, we give the main theorem of controllability.

Theorem 6.6. *If the assumptions (A1)-(A4) hold, then the system (2.4a)-(2.4c) is totally controllable on I, provided $\mathsf{N}_7 < 1$.*

Proof. Consider a subset $\mathsf{D}_2 \subset PC(I,X)$ such that

$$\mathsf{D}_2 = \{x \in PC(I,X) : \|x\|_P \le \delta_2\},$$

where

$$\delta = \max\left\{\frac{\mathsf{N}_4}{1-\mathsf{N}_7},\ \frac{M_{\aleph}T^{\gamma}}{\Gamma(\gamma+1)(1-\mathsf{N}_7)},\ \frac{\mathsf{N}_6}{1-\mathsf{N}_7}\right\}.$$

Now, we define an operator $\digamma_2 : \mathsf{D}_2 \to \mathsf{D}_2$ such that

$$(\digamma_2 x)t = \begin{cases} \mathcal{T}_0(t,0)[x_0 - \Upsilon_0(0,x_0)] + \Upsilon_0(t,x_a(t)) + \int_0^t \mathcal{T}_0(t,\sigma(\zeta))(\Psi_0(\zeta,x_b(\zeta)) + \mathcal{B}_0 u(\zeta))\Delta\zeta, & t \in (0,t_1]_{\mathbb{T}}, \\ \dfrac{1}{\Gamma(\gamma)}\int_{t_i}^t (t-\zeta)^{\gamma-1}\aleph_i(\zeta, x(t_i^-))\Delta\zeta, & t \in (t_i,s_i]_{\mathbb{T}},\ i=1,2,\dots,\vartheta, \\ \mathcal{T}_i(t,s_i)\left[\dfrac{1}{\Gamma(\gamma)}\int_{t_i}^{s_i}(s_i-\zeta)^{\gamma-1}\aleph_i(\zeta,x(t_i^-))\Delta\zeta - \Upsilon_i(s_i,x_a(s_i))\right] & \\ +\Upsilon_i(t,x_a(t)) + \int_{s_i}^t \mathcal{T}_i(t,\sigma(\zeta)(\Psi_i(\zeta,x_b(\zeta)) + \mathcal{B}_i u(\zeta))\Delta\zeta, & t \in (s_i,t_{i+1}]_{\mathbb{T}},\ i=1,2,\dots,\vartheta, \end{cases}$$

where $u(t)$ is given by the equations (6.21) and (6.22) in the intervals $(0,t_1]_{\mathbb{T}}$ and $(s_i,t_{i+1}]_{\mathbb{T}},\ i=1,2,\dots,\vartheta$, respectively. It is clear from the Lemma 6.4 and 6.5, $x(t)$ satisfies $x(t_1) = x_{t_1}$ and $x(t_{i+1}) = x_{t_{i+1}},\ i = 1,2,\dots,\vartheta$. Thus, to demonstrate the controllability of the switched control systems (6.26), we need to show that the operator $\digamma_2$ has a fixed point. For the simplicity, we split the proof into the following two main steps:

Step 1: We shall show that $\digamma_2$ maps D_2 into D_2. Let for any $t \in (0,t_1]_{\mathbb{T}}$ and $x \in \mathsf{D}_2$, we have

$$\begin{aligned}
\|(\digamma_2 x)t\| &\le \mathsf{C}e_{\ominus\nu}(t,0)[\|x_0\| + \|\Upsilon_0(0,x_0)\|] + \|\Upsilon_0(t,x_a(t))\| + \mathsf{C}\int_0^t e_{\ominus\nu}(t,\sigma(\zeta))\|\Psi_0(\zeta,x_b(\zeta))\|\Delta\zeta \\
&\quad + \mathsf{C}\int_0^t e_{\ominus\nu}(t,\sigma(\zeta))\|\mathcal{B}_0 u(\zeta)\|\Delta\zeta \\
&\le \mathsf{C}e_{\ominus\nu}(t,0)[\|x_0\| + \|\Upsilon_0(0,x_0)\|] + L_{\Upsilon_0}\|x_a(t)\| + M_{\Upsilon} + \frac{\mathsf{C}(L_{\Psi_0}\sup_{t\in[0,t_1]_{\mathbb{T}}}\|x_b(t)\| + M_{\Psi})(1+\bar{\mu}\nu)}{\nu} \\
&\quad + \frac{\mathsf{C}M_{\mathcal{B}}M_{u_0}(1+\bar{\mu}\nu)}{\nu} \\
&\le \mathsf{N}_0 + L_{\Upsilon}\delta_2 + \frac{\mathsf{C}L_{\Psi_0}\delta_2(1+\bar{\mu}\nu)}{\nu} + \frac{\mathsf{C}M_{\mathcal{B}}M_{\mathsf{W}_0}(1+\bar{\mu}\nu)}{\nu}\left[\|x_{t_1}\| + \mathsf{N}_0 + \mathsf{N}_1\sup_{t\in[0,t_1]_{\mathbb{T}}}\|x(t)\|\right] \\
&\le \mathsf{N}_0 + \mathsf{N}_1\delta_2 + \mathsf{Q}_1(\|x_{t_1}\| + \mathsf{N}_0 + \mathsf{N}_1\delta_2) \\
&\le \mathsf{N}_4 + \mathsf{N}_5\delta_2 \le \delta_2.
\end{aligned} \tag{6.23}$$

Now, for any $t \in (s_i, t_{i+1}]_{\mathbb{T}},\ i = 1, 2, \ldots, \vartheta$ and $x \in \mathsf{D}_2$,

$$\begin{aligned}
\|(F_2x)t\| &\leq \mathsf{C}e_{\ominus\nu}(t, s_i)\left[\frac{1}{\Gamma(\gamma)}\int_{t_i}^{s_i}(s_i - \zeta)^{\gamma-1}\|\aleph_i(\zeta, x(t_i^-))\|\Delta\zeta + \|\Upsilon_i(s_i, x_a(s_i))\|\right] + \|\Upsilon_i(t, x_a(t))\| \\
&\quad + \mathsf{C}\int_{s_i}^{t} e_{\ominus\nu}(t, \sigma(\zeta)\|\Psi_i(\zeta, x_b(\zeta))\|\Delta\zeta + \mathsf{C}\int_{s_i}^{t} e_{\ominus\nu}(t, \sigma(\zeta)\|\mathcal{B}u(\zeta))\|\Delta\zeta \\
&\leq \mathsf{C}e_{\ominus\nu}(t, s_i)\left[\frac{(L_{\aleph_i}\|x(t_i^-)\| + M_{\aleph})t_{i+1}^{\gamma}}{\Gamma(\gamma+1)} + L_{\Upsilon_i}\|x_a(s_i)\| + M_{\Upsilon}\right] + L_{\Upsilon_i}\|x_a(t)\| + M_{\Upsilon} \\
&\quad + \frac{\mathsf{C}(L_{\Psi_i}\sup_{t\in[s_i,t_{i+1}]_{\mathbb{T}}}\|x_b(t)\| + M_{\Psi})(1 + \bar{\mu}\nu)}{\nu} + \frac{\mathsf{C}M_{\mathcal{B}}M_{u_i}(1 + \bar{\mu}\nu)}{\nu} \\
&\leq \mathsf{N}_2 + \mathsf{N}_3\delta_2 + \mathsf{Q}_1(\|x_{t_{i+1}}\| + \mathsf{N}_2 + \mathsf{N}_3\delta_2) \\
&\leq \mathsf{N}_6 + \mathsf{N}_7\delta_2 \leq \delta_2.
\end{aligned} \tag{6.24}$$

Similarly, for any $t \in (t_i, s_i]_{\mathbb{T}},\ i = 1, 2, \ldots, \vartheta$ and $x \in \mathsf{D}_2$,

$$\begin{aligned}
\|(F_2x)t\| &\leq \frac{1}{\Gamma(\gamma)}\int_{t_i}^{t}(t - \zeta)^{\gamma-1}\|\aleph_i(\zeta, x(t_i^-))\|\Delta\zeta \\
&\leq \frac{M_{\aleph}t^{\gamma}}{\Gamma(\gamma+1)} + \mathsf{N}_7\delta_2 \leq \delta_2.
\end{aligned} \tag{6.25}$$

From the above equations (6.23), (6.24) and (6.25), for all $t \in I$, we have

$$\|F_2x\|_P \leq \delta_2.$$

Hence, F_2 maps D_2 into D_2.

Step 2: Here, we shall show that the operator F_2 is a contracting operator. Let for any $t \in (0, t_1]_{\mathbb{T}}$ and $x, y \in \mathsf{D}_2$, we have

$$\begin{aligned}
&\|(F_2x)t - (F_2y)t\| \\
&\leq \|\Upsilon_0(t, x_a(t)) - \Upsilon_0(t, y_a(t))\| + \mathsf{C}\int_0^t e_{\ominus\nu}(t, \sigma(\zeta))\|\Psi_0(\zeta, x_b(\zeta)) - \Psi_0(\zeta, y_b(\zeta))\|\Delta\zeta \\
&\quad + \mathsf{C}\int_0^t e_{\ominus\nu}(t, \sigma(\zeta))\|M_{\mathcal{B}}\|\|u_x(\zeta) - u_y(\zeta)\|\Delta\zeta \\
&\leq L_{\Upsilon_0}\|x_a(t) - y_a(t)\| + \mathsf{C}\int_0^t e_{\ominus\nu}(t, \sigma(\zeta))L_{\Psi_0}\|x_b(\zeta) - y_b(\zeta)\|\Delta\zeta + \mathsf{C}M_{\mathcal{B}}M_{\mathsf{W}_0}\int_0^t e_{\ominus\nu}(t, \sigma(\tau)) \\
&\quad \times\left[\|\Upsilon_0(t_1, x_a(t_1)) - \Upsilon_0(t_1, y_a(t_1))\| + \mathsf{C}\int_0^{t_1} e_{\ominus\nu}(t_1, \sigma(\zeta))\|\Psi_0(\zeta, x_b(\zeta)) - \Psi_0(\zeta, y_b(\zeta))\|\Delta\zeta\right]\Delta\tau \\
&\leq L_{\Upsilon_0}\|x_a(t) - y_a(t)\| + \frac{\mathsf{C}L_{\Psi_0}\sup_{t\in[0,t_1]_{\mathbb{T}}}\|x_b(t) - y_b(t)\|(1 + \bar{\mu}\nu)}{\nu} \\
&\quad + \frac{\mathsf{C}M_{\mathcal{B}}M_{\mathsf{W}_0}(1 + \bar{\mu}\nu)}{\nu}\left[L_{\Upsilon_0}\|x_a(t_1) - y_a(t_1)\| + \frac{\mathsf{C}L_{\Psi_0}\sup_{t\in[0,t_1]_{\mathbb{T}}}\|x_b(t) - y_b(t)\|(1 + \bar{\mu}\nu)}{\nu}\right] \\
&\leq \left(L_{\Upsilon_0} + \frac{\mathsf{C}L_{\Psi_0}(1 + \bar{\mu}\nu)}{\nu} + \mathsf{Q}_1L_{\Upsilon_0} + + \frac{\mathsf{C}\mathsf{Q}_1L_{\Psi_0}(1 + \bar{\mu}\nu)}{\nu}\right)\|x - y\|_P \\
&\leq \mathsf{N}_1(1 + \mathsf{Q}_1)\|x - y\|_P
\end{aligned}$$

$$\leq \mathsf{N}_7\|x-y\|_P. \tag{6.26}$$

Similarly, for any $t \in (s_i, t_{i+1}]_{\mathbb{T}},\ i = 1, 2, \dots, \vartheta$ and $x, y \in \mathsf{D}_2$,

$$\begin{aligned}
\|(F_2x)t-(F_2y)t\| \leq{} & \mathsf{C}e_{\ominus\nu}(t,s_i)\Big[\frac{1}{\Gamma(\gamma)}\int_{t_i}^{s_i}(s_i-\zeta)^{\gamma-1}\|\aleph_i(\zeta,x(t_i^-))-\aleph_i(\zeta,y(t_i^-))\|\Delta\zeta \\
& + \|\Upsilon_i(s_i,x_a(s_i))-\Upsilon_i(s_i,y_a(s_i))\|\Big] + \|\Upsilon_i(t,x_a(t))-\Upsilon_i(t,y_a(t))\| \\
& + \mathsf{C}\int_{s_i}^{t} e_{\ominus\nu}(t,\sigma(\zeta)\|\Psi_i(\zeta,x_b(\zeta))-\Psi_i(\zeta,y_b(\zeta))\|\Delta\zeta \\
& + \mathsf{C}\int_{s_i}^{t} e_{\ominus\nu}(t,\sigma(\zeta)\|\mathcal{B}\|\|u_x(\zeta)-u_xy(\zeta)\|\Delta\zeta \\
\leq{} & \mathsf{C}e_{\ominus\nu}(t,s_i)\left[\frac{1}{\Gamma(\gamma)}\int_{t_i}^{s_i}(s_i-\zeta)^{\gamma-1}L_{\aleph_i}\|x(t_i^-)-y(t_i^-)\|\Delta\zeta + L_{\Upsilon_i}\|x_a(s_i)-y_a(s_i)\|\right] \\
& + L_{\Upsilon_i}\|x_a(t)-y_a(t)\| + \mathsf{C}\int_{s_i}^{t} e_{\ominus\nu}(t,\sigma(\zeta)L_{\Psi_i}\|x_b(\zeta)-y_b(\zeta)\|\Delta\zeta \\
& + \mathsf{C}M_{\mathcal{B}}M_{\mathsf{W}_i}\int_{s_i}^{t} e_{\ominus\nu}(t,\sigma(\tau)\Big[\mathsf{C}e_{\ominus\nu}(t_{i+1},s_i)\Big(\frac{1}{\Gamma(\gamma)}\int_{t_i}^{s_i}(s_i-\zeta)^{\gamma-1}L_{\aleph_i}\|x(t_i^-)-y(t_i^-)\|\Delta\zeta \\
& + L_{\Upsilon_i}\|x_a(s_i)-y_a(s_i)\|\Big) + L_{\Upsilon_i}\|x_a(t_{i+1})-y_a(t_{i+1})\| \\
& + \mathsf{C}\int_{s_i}^{t_{i+1}} e_{\ominus\nu}(t_{i+1},\sigma(\zeta)L_{\Psi_i}\|x_b(\zeta)-y_b(\zeta)\|\Delta\zeta\Big]\Delta\tau \\
\leq{} & \mathsf{N}_3(1+\mathsf{Q}_1)\|x-y\|_P \\
\leq{} & \mathsf{N}_7\|x-y\|_P.
\end{aligned} \tag{6.27}$$

Similarly, for any $t \in (t_i, s_i]_{\mathbb{T}},\ i = 1, 2, \dots, \vartheta$ and $x, y \in \mathsf{D}_1$,

$$\begin{aligned}
\|(F_2x)t-(F_2y)t\| &\leq \frac{1}{\Gamma(\gamma)}\int_{t_i}^{t}(t-\zeta)^{\gamma-1}\|\aleph_i(\zeta,x(t_i^-))-\aleph_i(\zeta,y(t_i^-))\|\Delta\zeta \\
&\leq \mathsf{N}_7\|x-y\|_P
\end{aligned} \tag{6.28}$$

From the above equations (6.26), (6.27) and (6.28), for all $t \in I$, we have

$$\|F_2x-F_2y\|_P \leq \mathsf{N}_7\|x-y\|_P.$$

Hence, F_2 is a contracting operator.

Now collecting the step 1 and step 2 along with the Banach contraction principle, we can conclude that the operator F_2 has a unique fixed point which is the solution of the systems (2.4a)-(2.4c) and hence the system (2.4a)-(2.4c) is totally controllable on I. □

Remark 6.7. *In the existing literature, many authors established the controllability results for the different types of dynamic systems by using different techniques for the continuous and discrete-time domain, but they are studied separately. Particularly. in (Agarwal et al., 2009), the authors studied the controllability of two classes of first-order semilinear functional and neutral functional differential evolution equations with infinite delay by using the fixed point theory. The work in (Malik et al., 2019), focused on the controllability* *of non-autonomous nonlinear differential systems with non-instantaneous impulses by using Rothe's fixed point theorem. Very recently, few authors studied the controllability problems for the impulsive dynamic systems on time scales in finite-dimensional spaces. In (Lupulescu and Younus, 2011), the authors studied the controllability and observability results of the time-varying dynamic systems with instantaneous impulses on time scales. In (Ben Nasser et al., 2021), the authors studied the reachability and controllability results for the time-varying linear systems evolving on time scales while in (Yasmin et al., 2020), the authors investigated the controllability results for the linear impulsive adjoint dynamic system on time scale. Furthermore, in (Pervaiz et al., 2021), the authors studied the controllability and stability analysis of fractional delay dynamical systems with both instantaneous and non-instantaneous impulses on time scales. However, for the considered class of systems of this paper, this is the first attempt to deal with the controllability results on the arbitrary time domain. Since the problem is formulated in terms of time scales, and thus the obtained results can be applied to the continuous-time domain, discrete-time domain as well as any combination of these two; henceforth the results of this manuscript are completely new which extends and generalizes the existing results.*

7 Examples

In this section, we will give some examples to illustrate the obtained analytical results obtained in previous sections.

Example 7.1. *Consider the following partial dynamic equation on time scale in $X = L^2[0,\pi]_{\mathbb{T}}$.*

$$\begin{aligned}
\frac{\partial}{\Delta_1 t}\left[\mathfrak{X}(t,\xi) - \frac{it+\cos(\mathfrak{X}(a(t),\xi))}{15e^{t+2i}}\right] &= \beta_i(t,\xi)\frac{\partial^2}{\Delta_2\xi^2}\left[\mathfrak{X}(t,\xi) - \frac{it+\cos(\mathfrak{X}(a(t),\xi))}{15e^{t+2i}}\right] \\
&\quad + \frac{t\sin(\mathfrak{X}(b(t),\xi))}{(1+i)e^{(i+t^2)^2}} + d_i(\xi)S(t,\xi),\ t\in\cup_{i=0}^{\vartheta}(s_i,t_{i+1}]_{\mathbb{T}}, \xi\in[0,\pi]_{\mathbb{T}}, \\
\mathfrak{X}(t,0) = \mathfrak{X}(t,\pi) &= 0,\ t\in I=[0,T]_{\mathbb{T}}, \qquad (7.29)\\
\mathfrak{X}(t,\xi) &= \frac{1}{\Gamma(\gamma)}\int_{t_i}^{t}(t-\zeta)^{\gamma-1}\frac{1+\cos(i\mathfrak{X}(t_i^-,\xi))}{(it+1)^2e^{t+3}},\ t\in(t_i,s_i]_{\mathbb{T}},\ i=1,2,\ldots,\vartheta, \\
\mathfrak{X}(0,\xi) &= x_0,\ \xi\in[0,\pi]_{\mathbb{T}},
\end{aligned}$$

where Δ_1 and Δ_2 denote the partial derivative of order one and two, respectively. $\mathbb{T}$ is a time scale with $t_i, s_i \in \mathbb{T}$ are some points which satisfy the relation $0 = s_0 = t_0 < t_1 < s_1 < t_2 < \ldots s_\vartheta < t_{\vartheta+1} = T$. The functions $a, b : I \to I$ satisfies $a(t), b(t) \le t$. $\mathfrak{X}, S, \beta_i : T_0 \times [0,\pi]_{\mathbb{T}} \to \mathbb{R}$, are the real valued functions where $T_0 = \cup_{i=0}^{\vartheta}[s_i, t_{i+1}]_{\mathbb{T}}$.

Now, we define the operators $\mathcal{A}_i(t)$ by $\mathcal{A}_i(t)x = \beta_i(t,\xi)\frac{\partial^2}{\Delta_2\xi^2}x$ for all $x \in D(A_i) = \{x \in H_0^1[0,\pi]_{\mathbb{T}} \cap H^2[0,\pi]_{\mathbb{T}}\}$, where $H_0^1[0,\pi]_{\mathbb{T}}$ and $H^2[0,\pi]_{\mathbb{T}}$ are the Sobolev spaces (Wang and Agarwal, 2014; Edmunds and Evans, 2018). Clearly, it is well known that $\mathcal{A}_i(t)$ generate the evolution operators $\{\mathcal{T}_i(t,s) : (t,s) \in I \times I : t \ge s\}$ such that $\|\mathcal{T}_i(t,s)\| \le \mathsf{C}e_{\ominus\nu}(t,s)$ for all (t,s) $(t \ge s)$ with $\mathsf{C} = 1$ and $\nu = \frac{1}{2}$ (please see (Dhama and Abbas, 2019; Wang and Agarwal, 2014)).

Now, for $(t,\xi) \in I \times [0,\pi], \mathcal{B}_i \in \mathbb{B}(U,X)$, we set $x(t) = \mathfrak{X}(t,\cdot)$, i.e., $x(t)(\xi) = \mathfrak{X}(t,\xi)$,

$$\Upsilon_i(t,x_a(t))(\xi) = \frac{it+\cos(\mathfrak{X}(a(t),\xi))}{15e^{t+2i}},\ \Psi_i(t,x_b(t))(\xi) = \frac{t\sin(\mathfrak{X}(b(t),\xi))}{(1+i)e^{(i+t^2)^2}},\ i=0,1,\ldots,\vartheta,$$

$$\aleph_i(t,x(t_i^-))(\xi) = \frac{1+\cos(i\mathfrak{X}(t_i^-,\xi))}{(it+1)^2e^{t+3}},\ i=1,2,\ldots,\vartheta,\ \mathcal{B}_iu(t)(\xi) = d_iS(t,\xi),\ i=0,1,\ldots,\vartheta.$$

With this formulation, the equation (7.29) can be rewritten in the abstract form (2.4a)-(2.4c). Clearly, we can see the functions $\Upsilon_i, \Psi_i,\ i = 0, 1, \ldots, \vartheta$ and $\aleph_i,\ i = 1, 2, \ldots, \vartheta$, satisfy all the assumptions of Theorem 6.6, and hence the system (7.29) is totally controllable on I.

Example 7.2. *Consider the following impulsive system when $X = \mathbb{R}$*

$$\begin{aligned}
\left[x(t) - \frac{t\sin(x_a(t))}{e^{t^2+3}}\right]^{\Delta} &= \frac{-3}{2+3\mu(t)}\left[x(t) - \frac{t\sin(x_a(t))}{e^{t^2+3}}\right] + \frac{3+\cos(x_b(t))}{e^{(t+3)^2}} + \frac{t^2}{e^{1+t^2}},\ t \in (0, t_1]_{\mathbb{T}},\\
\left[x(t) - \frac{t\sin(x_a(t))}{2e^{t^2+3}}\right]^{\Delta} &= \frac{-3}{2+3\mu(t)}\left[x(t) - \frac{t\sin(x_a(t))}{2e^{t^2+3}}\right] + \frac{3+\cos(x_b(t))}{e^{(t+3)^2+1}} + \frac{t^2}{e^{1+t^2}},\ t \in (s_1, T]_{\mathbb{T}},\\
x(t) &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{t_1}^{t} \frac{(5+\cos(x(t_1^-)))}{15e^{\zeta^2+1}(t-\zeta)^{\frac{1}{2}}}\Delta\zeta,\ t \in (t_1, s_1]_{\mathbb{T}},\\
x(0) &= 1.
\end{aligned} \tag{7.30}$$

The system (7.30) can be written in the form of (2.1), where $r(t) = i,\ t_i \le t < t_{i+1},\ i = 0, 1,$ $\gamma = 0.5, \vartheta = 1, x_0 = 1,$

$$\mathcal{A}_0(t) = \frac{-3}{2+3\mu(t)},\ \mathcal{A}_1(t) = \frac{-2}{1+2\mu(t)},\ \Upsilon_i(t, x_a(t))) = \frac{t\sin(x_a(t))}{(1+i)e^{t^2+3}},\quad i = 0, 1,$$

$$\Psi_i(t, x_b(t))) = \frac{3+\cos(x_b(t))}{e^{(t+3)^2+i}} + \frac{t^2}{e^{1+t^2}},\ i = 0, 1,\ \aleph_i(t, x(t_i^-)) = \frac{(5+\cos(ix(t_i^-)))}{15e^{it^2+1}},\ i = 1.$$

Here $\mathcal{T}_0(t,s) = e_{\ominus\frac{3}{2}}(t,s)$ and $\mathcal{T}_{\mathcal{A}_1}(t,s) = e_{\ominus 2}(t,s)$ and hence $\|\mathcal{T}_i(t,s)\| \le e_{\ominus\frac{3}{2}}(t,s),\ i = 0, 1,$ therefore $\mathcal{T}_i(t,s),\ i = 0, 1,$ are exponentially stable where $\mathsf{C} = 1$ and $\nu = \frac{3}{2}$. Next, we consider the two cases for different time scale as follows.

Case 1: When $\mathbb{T} = \mathbb{R}$. We choose $t_0 = 0, t_1 = 0.4, s_1 = 0.5, T = 1, a(t) = b(t) = t^2/4$. Also, we choose the desire points as $x(t_1) = 2$ and $x(T) = 1$. Now, from the Figure 2, it is clear that the trajectory of the system (7.30) does not passes throw the desire points $x(t_1) = 2$ and $x(T) = 1$. But after adding a control function

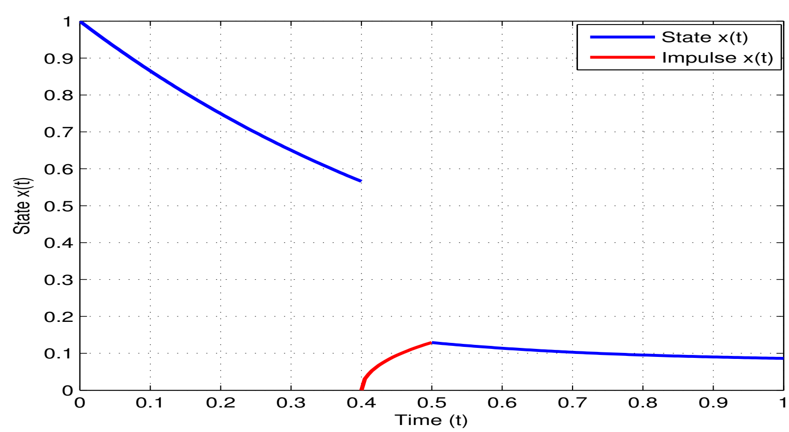

Figure 2: State trajectory of the system (7.30) when $\mathbb{T} = \mathbb{R}$.

$u(t) =$

$$
\begin{cases}
(\mathsf{W}_0^{t_1})^{(-1)}\Bigg(2 - e_{\ominus\frac{3}{2}}(t_1, 0) - \dfrac{t_1 \sin(x_a(t_1))}{e^{t_1^2+3}} \\
\quad - \int_0^{t_1} e_{\ominus\frac{3}{2}}(t_1, \sigma(\zeta))\Bigg(\dfrac{3+\cos(x_b(\zeta))}{e^{(\zeta+3)^2}} + \dfrac{\zeta^2}{e^{1+\zeta^2}}\Bigg)\Delta\zeta\Bigg)(t),\ t \in (0, t_1]_{\mathbb{T}}, \\
(\mathsf{W}_{s_1}^{T})^{(-1)}\Bigg(1 - e_{\ominus 2}(T, s_1)\Bigg(\dfrac{1}{\Gamma\left(\frac{1}{2}\right)} \int_{t_1}^{s_1} \dfrac{(5+\cos(x(t_1^-)))}{15e^{s_1^2+1}(s_1-\zeta)^{\frac{1}{2}}}\Delta\zeta - \dfrac{s_1 \sin(x_a(s_1))}{2e^{s_1^2+3}}\Bigg) - \dfrac{T\sin(x_a(T))}{2e^{T^2+3}} \\
\quad - \int_{s_1}^{T} e_{\ominus 2}(T, \sigma(\zeta))\Bigg(\dfrac{3+\cos(x_b(\zeta))}{e^{(\zeta+3)^2+1}} + \dfrac{\zeta^2}{e^{1+\zeta^2}}\Bigg)\Delta\zeta\Bigg)(t),\ t \in (s_1, T]_{\mathbb{T}},
\end{cases}
\tag{7.31}
$$

where

$$
\mathsf{W}_0^{t_1} = \int_0^{t_1} e_{\ominus\frac{3}{2}}(t_1, \sigma(\zeta))\Delta\zeta \text{ and } \mathsf{W}_{s_1}^{T} = \int_{s_1}^{T} e_{\ominus 2}(T, \sigma(\zeta))\Delta\zeta,
$$

with $\mathcal{B}_0 = \mathcal{B}_1 = 1$, in the system (7.30), one can easily calculate

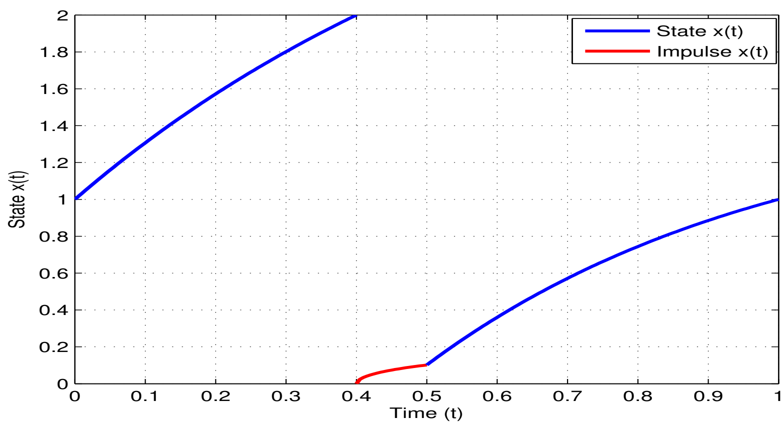

Figure 3: Totally controlled trajectory of the system (7.30) when $\mathbb{T} = \mathbb{R}, x(t_1) = 2$ and $x(T) = 1$.

$$
\begin{aligned}
&\mathcal{W}_0^{t_1} = 3.3246,\ \mathcal{W}_{s_1}^{T} = 3.1640,\ \mathsf{Q}_1 = \frac{\mathsf{C}M_{\mathcal{B}}M_{\mathsf{W}}(1+\bar{\mu}\nu)}{\nu} = 2.2164, \\
&\mathsf{N}_3 = \mathsf{C}\left(\frac{L_{\aleph}T^{\gamma}}{\Gamma(\gamma+1)} + L_{\Upsilon}\right) + L_{\Upsilon} + \frac{\mathsf{C}L_{\Psi}(1+\bar{\mu}\nu)}{\nu} = 0.1771, \\
&\mathsf{N}_7 = \mathsf{N}_3(1+\mathsf{Q}_1) = 0.5697.
\end{aligned}
$$

Thus, the assumptions of Theorem 6.6 are fulfilled. Therefore, the system (7.30) is totally controllable and the totally controlled state trajectory is shown in Figure 3.
Case 2: When $\mathbb{T} = [0,1]_{\mathbb{R}} \cup [2,3]_{\mathbb{R}} = \mathbb{J}$(say). We choose $t_0 = 0, t_1 = 0.4, s_1 = 0.5, T = 3, a(t) = b(t) = t/3$. Also, we choose the desire points as $x(t_1) = 3$ and $x(T) = 2$. Now, from the Figure 4, it is clear that the trajectory of the system (7.30) does not passes throw the desire points $x(t_1) = 3$ and $x(T) = 2$. But if we add a control function $u(t)$ given by the equation (7.31) with $\mathcal{B}_0 = \mathcal{B}_1 = 1$, in the system (7.30), we can find

$$
\mathcal{W}_0^{t_1} = 2.8429,\ \mathcal{W}_{s_1}^{T} = 1.9155,\ \mathsf{Q}_1 = 1.8953,\ \mathsf{N}_3 = 0.1975,\ \mathsf{N}_7 = 0.5718.
$$

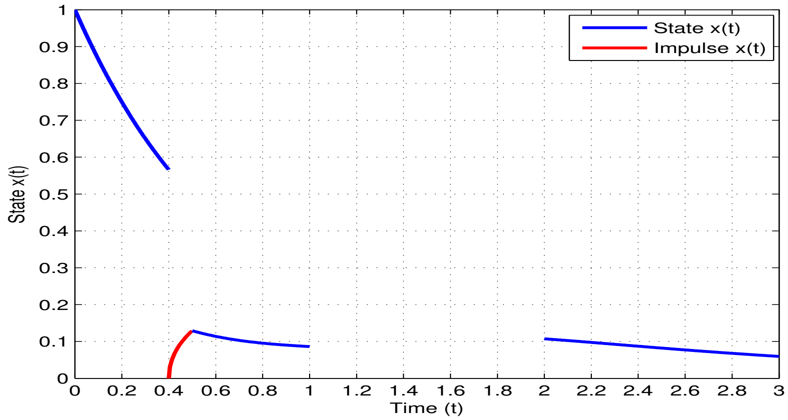

Figure 4: State trajectory of the system (7.30), when $\mathbb{T} = \mathbb{J}$.

Thus, the assumptions of Theorem 6.6 are fulfilled. Therefore, the system (7.30) is totally controllable and the totally controlled state trajectory is shown in Figure 5.

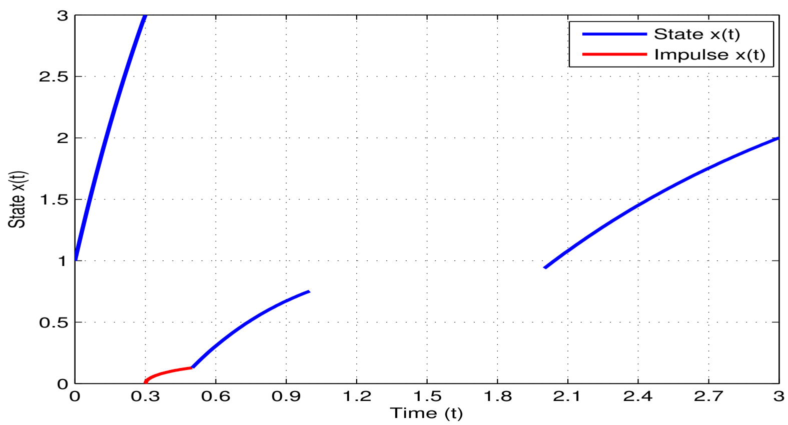

Figure 5: Totally controlled trajectory of the system (7.30), when $\mathbb{T} = \mathbb{J}, x(t_1) = 3, x(T) = 2$.

Conclusion

We have successfully established some qualitative properties for a class of switched evolution system with impulses over a arbitrary time domain. More precisely, first we established the existence and Ulam-Hyers type stability results and then we established the total controllability results for the considered systems. We applied the time scales theory, functional analysis, evolution operator theory, and fixed point theory to established these results. Furthermore, we have given two examples for different time domains to illustrate the obtained analytical results. As further directions, the developed methodology can be used to control an epidemic such as COVID-19 by different measures (confinement, vaccination,... etc.) (Noeiaghdam et al., 2021; Silva et al., 2021; Tyagi et al., 2021).

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