Second-order topology and supersymmetry in two-dimensional topological insulators: Supplemental material

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I. WEYL PHASE FOR SMALL MAGNETIC FIELDS

To substantiate the analytical result that topological hole states will not exist for a Corbino disc in the Weyl phase by numerical results, we show in Fig. 1 the four wave functions of lowest absolute value of the energy for a disc in the Weyl phase for increasing value of the radius of the disc. For small radius it seems as if a center state is present, but this state moves to the outer surface if the square root of the radius of the disc exceeds significantly the magnetic length. This is consistent with the analytical theory which predicts an additional topological state at the outer surface in the regime of strong localization $\tilde{l}_B \ll \sqrt{R_>}$.

II. TOPOLOGICAL STATES FOR CORBINO DISC: COMPARISON BETWEEN NUMERICS AND ANALYTICS

Here we compare the analytical and numerical results for the topological states of a Corbino disc in the topological phase. There are four topological states $\bar{\psi}_{su}$ in the transformed basis, labelled by chiral symmetry $s = \pm 1$ and SUSY $u = \pm 1$, two at the outer surface ($s = -u = \pm 1$) and two at the inner surface ($s = u = \pm 1$). Since $\bar{\psi}_{su}$ and $\bar{\psi}_{-s,-u}$ are related by the inversion symmetry, we concentrate on the two states with chirality $s = +1$, localized around the polar angle $\varphi = 0$.

The analytical formulas for $\bar{\psi}_{1,\pm 1}(\tilde{r}, \varphi; \sigma_z, s_z)$ have been provided in Section IV.E of the main text as function of the radial coordinate $\tilde{r} = r/\lambda_{\rm so}$, the polar angle φ, and the spinors σ_z and s_z. Omitting the normalization factor, we obtained

$$\bar{\psi}_{1,\pm 1}(\tilde{r}, \varphi; \sigma_z, s_z) \sim \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\sigma_z} \begin{pmatrix} 1 \\ \mp 1 \end{pmatrix}_{s_z} \hat{\chi}^{(\pm 1)}_{+,n}(\tilde{r})\, e^{-\frac{1}{4}(\varphi/\Delta\varphi_{\lessgtr})^2} , \quad (1)$$

with $\Delta\varphi_{\gtrless} = \tilde{l}_B/\sqrt{\tilde{R}_{\gtrless}}$,

$$\hat{\chi}^{(\pm 1)}_{+,n}(\tilde{r}) \sim e^{i\tilde{k}_1^{(\pm 1)}(\tilde{r}-\tilde{R}_{\lessgtr})} - e^{i\tilde{k}_2^{(\pm 1)}(\tilde{r}-\tilde{R}_{\lessgtr})} , \quad (2)$$

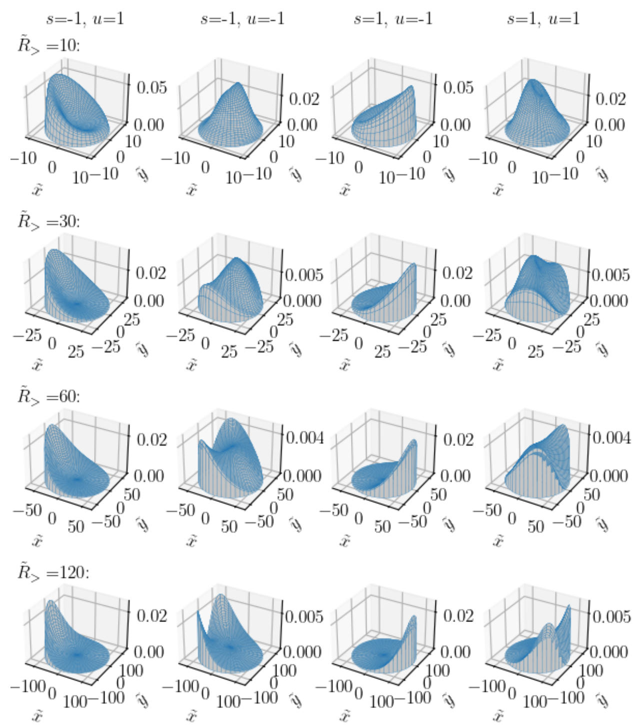

FIG. 1. The wave functions $\sum_{\sigma_z, s_z} |\bar{\psi}_{su}(x, y; \sigma_z, s_z)|^2$ as function of $\tilde{x} = x/\lambda_{\rm so}$ and $\tilde{y} = y/\lambda_{\rm so}$ of the four states with lowest absolute value of the energy for a disc with various radius $\tilde{R}_>$ and zero hole radius $\tilde{R}_< = 0$ at half-integer flux $f = 1/2$ in the Weyl regime $\tilde{\delta} = 0$ and $\tilde{l}_B = 5$. As one can see two of the center states with $s = u = \pm 1$ move to the outer surface when $\tilde{R}_>$ is sufficiently large such that $\sqrt{\tilde{R}_>}$ exceeds significantly $\tilde{l}_B$.

and

$$\tilde{k}^{(u)}_{1/2} = iu \pm \sqrt{|\tilde{\delta} - 1 - u/\tilde{l}_B^2|} \begin{cases} 1 & \text{for } \tilde{\delta} > 1 + u/\tilde{l}_B^2 \\ iu & \text{for } \tilde{\delta} < 1 + u/\tilde{l}_B^2 \end{cases} . \quad (3)$$

Using the tight-binding approach for a Corbino disc with $\tilde{R}_> = 45$ and $\tilde{R}_< = 20$, we show in the left/right parts of Fig. 2(a) the absolute value squared of the two topological states averaged over the spinor degrees of

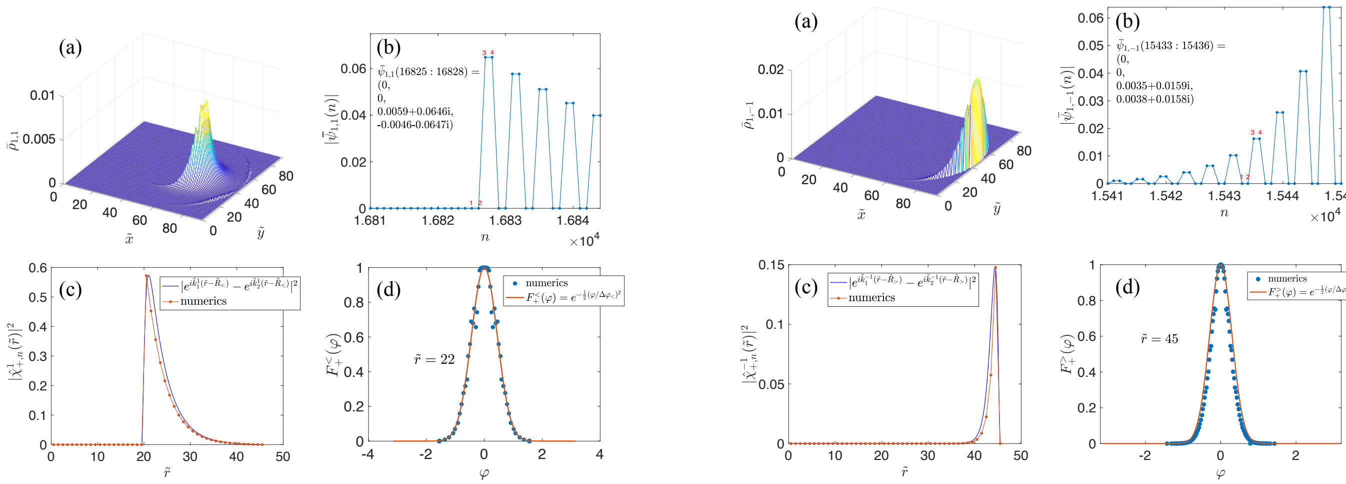

FIG. 2. Left/right part: The comparison between the analytical form (1) and the tight-binding numerics for the inner/outer topological state $\bar{\psi}_{1,\pm 1}$. (a) $\bar{\rho}_{1,\pm 1}$ as defined in (4) as function of the lattice sites labelled by $(\tilde{x}, \tilde{y})$. (b) The absolute value of the wave function $|\bar{\psi}_{1,\pm 1}(n)|$ with the index n accounting for both the position and spinor component. Here, the labelling $(1, 2, 3, 4)$ corresponds to $(\sigma_z, s_z) = (1, 1), (1, -1), (-1, 1), (-1, -1)$. (c) The fitting for the radial dependence $|\hat{\chi}^{(\pm 1)}_{+,n}(\tilde{r})|$ between the analytical form (2) and the tight-binding result, where the polar angle is fixed to $\varphi = 0$. (d) The fitting for the angular dependence $F^{<}_{+}(\varphi)$ between the analytical result (6) and the tight-binding result, where the radial coordinate is fixed to $\tilde{r} = 22/45$. The other parameters in the tight-binding calculation are: $\tilde{R}_{>} = 45$, $\tilde{R}_{<} = 20$, $\tilde{\delta} = 0.5$, $\tilde{l}_B = 2$, and $f = 1/2$.

freedom

$$\bar{\rho}_{1,\pm 1}(\tilde{x}, \tilde{y}) = \sum_{\sigma_z, s_z} |\bar{\psi}_{1,\pm 1}(\tilde{x}, \tilde{y}; \sigma_z, s_z)|^2 \tag{4}$$

$$\sim |\hat{\chi}^{(\pm 1)}_{+,n}(\tilde{r})|^2 \; F^{\lessgtr}_{+}(\varphi)\,, \tag{5}$$

where $(\tilde{x}, \tilde{y})$ are the coordinates of the lattice sites, and

$$F^{\lessgtr}_{+}(\varphi) \sim e^{-\frac{1}{2}(\varphi/\Delta\varphi_{\lessgtr})^2}\,. \tag{6}$$

Fixing either the polar angle to $\varphi = 0$ or the radial coordinate to $\tilde{r} = 22/45$ for the inner/outer state, we find in Fig. 2(c,d) a very good agreement between the numerical and analytical results for both the radial and angular dependence of the wave functions. To check the spinor dependence we show in Fig. 2(b) also the four spinor components of the wave functions for each fixed lattice site. The results are consistent with the analytical predictions.

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