

Packed swarms on dirt: two-dimensional incompressible flocks with quenched and annealed disorder

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We show that incompressible polar active fluids can exhibit an ordered, coherently moving phase even in the presence of quenched disorder in two dimensions. Unlike such active fluids with *annealed* (i.e., time-dependent) disorder *only*, which behave like equilibrium ferromagnets with long-range interactions, this robustness against quenched disorder is a fundamentally non-equilibrium phenomenon. The ordered state belongs to a new universality class, whose scaling laws we calculate using three different renormalization group schemes, which all give scaling exponents within 0.02 of each other, indicating that our results are quite accurate. Our predictions can be quantitatively tested in readily available artificial active systems, and imply that biological systems such as cell layers can move coherently *in vivo*, where disorder is inevitable.

One of the most significant discoveries in statistical mechanics in the 20th century was the “Mermin-Wagner-Hohenberg” theorem [1], which says that continuous symmetries cannot be spontaneously broken in equilibrium two-dimensional systems at finite temperature. This implies, in particular, that magnetic systems with continuous rotational symmetry like the XY or Heisenberg models cannot exhibit long-range polar order characterized by a non-zero magnetization in two dimensions (2D).

It is precisely for this reason that it was so surprising to learn that 2D *active* systems *can* spontaneously develop long-range polar order even in the presence of noise [2–6]. In particular, polar self-propelled particles moving over a frictional substrate (a system often described as a “dry polar active fluid”) can “flock”; that is, form a state with a non-zero average velocity $\langle \mathbf{v} \rangle$, even when perturbed by noise.

In equilibrium systems, it is known that *quenched* disorder [7–10] – that is, disorder that is time-*independent* – is even more destructive of long-range order than thermal noise, which is, of course, time-*dependent*. Indeed, even arbitrarily small quenched disorder destroys long-ranged ferromagnetic and crystalline order in all spatial dimensions $d \leq 4$ [7–10].

The above discussion raises the obvious question: can *active* systems retain long-range order even in the presence of *quenched* disorder [11–20]? It has been shown [11, 12] that for three-dimensional dry polar active systems with quenched disorder, long-range polar order (i.e., a non-zero average velocity $\langle \mathbf{v} \rangle$) can survive such quenched disorder. But in 2D, only quasi-long-range polar order (i.e., $\langle \mathbf{v} \rangle$ vanishes as a power of system size L as $L \rightarrow \infty$) was found [11, 12, 21].

In this paper, we report that it is possible to achieve

true *long-range* polar order in 2D in dry polar active systems with quenched disorder, if those systems are *incompressible*.

There are many ways to experimentally realize incompressibility in active systems. One is to make them very dense so that the effective compressibility of the flocers vanishes [22, 23]. An even more realizable incompressible system is a suspension of swimmers in a fluid confined in a narrow channel. Here, the active agents “inherit” the incompressibility of the background fluid [23–25]. Strictly speaking, swimmer suspensions differ from the systems we consider here, due to the presence of an extra hydrodynamic variable in the former: the density of the swimmers. However, if the swimmers in the channel are constantly being born and dying [26] or can switch between an active, motile state and a passive, immotile one, so that the number of active swimmers is not globally conserved, the dynamics of the system *is* described by the theory presented here.

Finally, spin-systems interacting via dipolar interactions have the same long-time, large-distance properties as magnets with a divergence-free constraint on the magnetization [22, 27]. It has further been shown [28–30] that the hydrodynamics of spin-systems with nonequilibrium and non-reciprocal asymmetric interactions are described by the equations originally constructed for (Malthusian) flocks [26]. Therefore, the hydrodynamics of dipolar magnets with asymmetric exchange interactions [30] is equivalent to incompressible flocks [22, 31]. If such a magnetic system were subject to quenched disorder, it would again be described by the theory we develop here.

Motivated by all of the examples above, we consider here incompressible flocks in 2D [22] on a substrate with quenched disorder. We also include the effects of

“annealed” disorder (i.e., time-dependent noise), which proves to be subdominant for the equal-time correlations, but gives time-dependence to the fluctuations, which would otherwise be static. Considering the full nonlinear theory, we demonstrate that the moving phase of such a flock can sustain long-range polar order. A similar result was found, based on a purely *linearized* theory, for number-conserving active suspensions with quenched disorder [32] (which, as noted above, differ somewhat from our system).

Using a dynamic renormalization group (DRG) analysis, we calculate the long-time, large-distance scaling of the fluctuations $\mathbf{u}(\mathbf{r}, t)$ of the local active fluid velocity $\mathbf{v}(\mathbf{r}, t)$ about its mean value $\langle \mathbf{v} \rangle \equiv v_0 \hat{\mathbf{x}}$, where we’ve defined our coordinate system so that $\hat{\mathbf{x}}$ is along the mean velocity spontaneously chosen by the system. This analysis leads to scaling laws for the correlations of $\mathbf{u}(\mathbf{r}, t) \equiv \mathbf{v}(\mathbf{r}, t) - v_0 \hat{\mathbf{x}}$ which we obtain using three different DRG schemes: two different $d = (d_c - \epsilon)$ -expansions, which we term “hard continuation” and “soft continuation”, and an uncontrolled expansion in exactly 2D.

Specifically, we show that the u - u correlations are given by

$$\langle \mathbf{u}(\mathbf{r}, t) \cdot \mathbf{u}(\mathbf{0}, 0) \rangle = |y|^{2\chi} \mathcal{G}_Q \left(\frac{|x|}{|y|^\zeta} \right) + |y|^{2\chi'} \mathcal{G}_A \left(\frac{|x|}{|y|^\zeta}, \frac{|t|}{|y|^z} \right), \quad (1)$$

where $\mathcal{G}_{A,Q}$ are scaling functions that are universal up to an overall multiplicative factor, corresponding to the annealed (i.e., time-dependent) and quenched parts of the correlations, respectively. The exponents in (1) are

$$z = 0.49 \pm 0.01, \quad \zeta = 0.77 \pm 0.01, \quad (2a)$$

$$\chi = -0.23 \pm 0.01, \quad \chi' = -0.37 \pm 0.02, \quad (2b)$$

in the physical case $d = 2$, where the error bars correspond to the differences between the three aforementioned DRG schemes. The fact that both χ and χ' are negative implies long-range polar order.

The existence of a common anisotropy exponent ζ for the quenched and the annealed parts of the correlation function (1) and a simple scaling form for the latter is a highly non-trivial result, which is a direct consequence of the anomalous hydrodynamics of our system. Indeed, a *linearized* treatment of our hydrodynamic model predicts a much more complicated “double scaling” form, with different anisotropy exponents for the quenched and annealed correlations, and a more complicated form for the annealed part of the correlation function (what we mean by this will become clear below). The latter is reminiscent of, e.g., simple fluids, whose spatiotemporal correlation functions have a structure that reflects both the dispersionless propagation of sound (which is characterized by a dynamic exponent ($z = 1$) and the diffusive nature of viscous damping ($z = 2$)).

Model.—The hydrodynamic equation of motion (EOM) of incompressible polar active fluids moving on

a disordered substrate can be constructed based on symmetry considerations alone [3, 4, 22, 31, 33]. We defer its derivation to the Associated Long Paper (ALP) [34]. The EOM in the moving phase is identical to that studied in [22], except for the presence of the quenched noise. Specifically, keeping only “relevant” terms, by which we mean terms that can change the long-distance, long-time behavior of the system, the EOM governing \mathbf{u} is, in Einstein component notation,

$$\begin{aligned} \partial_t u_i &= -\partial_i \Pi - \gamma \partial_x u_i - b \delta_{ix} \partial_x u_x + \mu_\perp \partial_y^2 u_i + f_Q^i + f_A^i \\ &+ \mu_x \partial_x^2 u_i - \alpha \left(u_x + \frac{u_y^2}{2v_0} \right) (\delta_{ix} + \frac{u_y}{v_0} \delta_{iy}), \end{aligned} \quad (3)$$

where the indices i, j label the spatial coordinates. In (3), the “pressure” Π acts as a Lagrange multiplier to enforce the incompressibility condition: $\nabla \cdot \mathbf{u} = 0$, while γ and b are (positive or negative) constants, and α and $\mu_{x,\perp}$ are positive constants [34]. The quenched and annealed noises $\mathbf{f}_Q(\mathbf{r})$ and $\mathbf{f}_A(\mathbf{r}, t)$ respectively are zero mean Gaussian white noises with variances:

$$\langle f_Q^i(\mathbf{r}) f_Q^j(\mathbf{r}') \rangle = 2D_Q \delta_{ij} \delta^2(\mathbf{r} - \mathbf{r}'), \quad (4a)$$

$$\langle f_A^i(\mathbf{r}, t) f_A^j(\mathbf{r}', t') \rangle = 2D_A \delta_{ij} \delta^2(\mathbf{r} - \mathbf{r}') \delta(t - t'). \quad (4b)$$

Note the time-independence of the noise $\mathbf{f}_Q(\mathbf{r})$; this is what we mean by “quenched”. As mentioned earlier, Eq. (3) differs from the EOM in [22] only through the presence of the quenched noise $\mathbf{f}_Q(\mathbf{r})$.

Linear regime.—The linearized version of EOM (3) is solved in Fourier space by projecting (3) transverse to the wavevector using the projection operator $P_{\perp i}(\mathbf{q}) = \delta_{li} - \frac{q_l q_i}{q^2}$. Autocorrelating these solutions, and using the correlations (4a) and (4b) for the noises, we obtain the correlation functions

$$\langle u_i(\tilde{\mathbf{q}}) u_i(\tilde{\mathbf{q}}') \rangle = C_A^i(\tilde{\mathbf{q}}, \tilde{\mathbf{q}}') + C_Q^i(\tilde{\mathbf{q}}, \tilde{\mathbf{q}}'), \quad (5)$$

where $\tilde{\mathbf{q}} \equiv (\omega, \mathbf{q})$, with ω being the frequency and \mathbf{q} the wavevector of the perturbation,

$$C_A^{x,y}(\tilde{\mathbf{q}}, \tilde{\mathbf{q}}') = \frac{q_{y,x}^2}{q^2} \frac{2D_A \delta(\omega + \omega') \delta(\mathbf{q} + \mathbf{q}')}{\left[\omega - \left(\frac{bq_y^2}{q^2} + \gamma \right) q_x \right]^2 + \left[\frac{\alpha q_y^2}{q^2} + \Gamma(\mathbf{q}) \right]^2}, \quad (6a)$$

$$C_Q^{x,y}(\tilde{\mathbf{q}}, \tilde{\mathbf{q}}') = \frac{q_{y,x}^2}{q^2} \frac{4\pi D_Q \delta(\omega) \delta(\omega') \delta(\mathbf{q} + \mathbf{q}')}{\left(\frac{bq_y^2}{q^2} + \gamma \right)^2 q_x^2 + \left[\frac{\alpha q_y^2}{q^2} + \Gamma(\mathbf{q}) \right]^2}, \quad (6b)$$

where $\Gamma(\mathbf{q}) \equiv \mu_\perp q_y^2 + \mu_x q_x^2$, and the subscripts A and Q denote the annealed and quenched parts, respectively.

The real-time, real-space autocorrelation of $\mathbf{u}(\mathbf{r}, t)$ obtained via inverse Fourier-transforming (6), is

$$\begin{aligned} \langle \mathbf{u}(\mathbf{r}, t) \cdot \mathbf{u}(\mathbf{0}, 0) \rangle &= |y|^{-\frac{1}{3}} C_Q \left(\frac{|x|}{|y|^{\frac{2}{3}}} \right) \\ &+ |y|^{-\frac{1}{2}} C_A \left(\frac{|x - \gamma t|}{|y|^{\frac{1}{2}}}, \frac{|t|}{|y|} \right), \end{aligned} \quad (7)$$

where $C_{A/Q}$ are scaling functions displayed in [34]. Thus the linear theory predicts $\chi = -1/6$, $\chi' = -1/4$, and $z = 1$, and also predicts different anisotropy exponents for the quenched and annealed parts: $\zeta_{\text{quenched}} = 2/3$ and $\zeta_{\text{annealed}} = 1/2$. Note also that the first argument of the scaling function C_A in the linear theory involves a “boosted” x coordinate $x - \gamma t$, in contrast to the non-linear result (1), in which the first argument of the annealed scaling function only involves x , with no t -dependence. The negativity of the roughness exponents χ and χ' implies that the incompressible flock has long-range polar order within the linear theory [32]. We will show that these exponents are modified in the nonlinear theory, and, in particular, the two anisotropy exponents ζ_{quenched} and ζ_{annealed} become equal.

This persistence of two-dimensional long-range polar order in a disordered medium is a result of a combina-

tion of the effects of incompressibility and, crucially, active motility, as can be seen from (6). Indeed, in a 2D equilibrium divergence-free magnet, the equal-time correlator of the magnetization diverges as $1/\mu_x^2 q_x^4$ instead of as $1/\gamma^2 q_x^2$ for $q_y = 0$ in the presence of quenched disorder. This divergence is strong enough to destroy *even* quasi-long-range order. This implies that, while incompressible flocks perturbed by *annealed* noise are equivalent to *equilibrium* divergence-free magnets [22, 27], they are much more resistant to quenched disorder. Due to motility, the quenched disorder is effectively “annealed” for fluctuations propagating along the ordered direction $\hat{\mathbf{x}}$ at a speed γ : along $q_y = 0$, $C_A^y(\mathbf{q}, \mathbf{q}')$ and $C_Q^y(\mathbf{q}, \mathbf{q}')$ both diverge as $\sim 1/q_x^2$ as $\mathbf{q} \rightarrow \mathbf{0}$.

Nonlinear regime & DRG analysis.— We now turn to the full EOM of \mathbf{u} (3). Fourier transforming this, and operating on both sides with the transverse projection operator P_{ly} , we obtain

$$-i\omega u_y = P_{yx}(\mathbf{q}) \mathcal{F}_{\bar{\mathbf{q}}} \left[-\alpha \left(u_x + \frac{u_y^2}{2v_0} \right) \right] + \mathcal{F}_{\bar{\mathbf{q}}} \left[-\gamma \partial_x u_y - \lambda u_y \partial_y u_y - \frac{\alpha}{v_0} \left(u_x + \frac{u_y^2}{2v_0} \right) u_y + f_A^y + f_Q^y \right], \quad (8)$$

where we have neglected terms that are irrelevant in the dominant regime of wavevector $q_y \ll q_x$, as we discovered in our treatment of the linear theory. We will verify *a posteriori* that this continues to hold for the nonlinear theory. The symbol $\mathcal{F}_{\bar{\mathbf{q}}}$ represents the $\bar{\mathbf{q}}$ th Fourier component.

To evaluate the importance of the nonlinear terms in (8), we rescale time, lengths, and fields as

$$t \rightarrow t e^{z\ell}, \quad x \rightarrow x e^{\zeta\ell}, \quad y \rightarrow y e^\ell, \quad u_y \rightarrow u_y e^{\ell\chi}, \quad (9a)$$

implying $u_x \rightarrow u_x e^{(\chi+\zeta-1)\ell}$, and keep the form of the resultant EOM unchanged by absorbing the rescaling factors into the coefficients. The coefficients of the linear terms $P_{yx}u_x$, $\partial_x u_y$, and the noise strength are rescaled respectively as

$$\alpha \rightarrow \alpha e^{(z+2\zeta-2)\ell}, \quad \gamma \rightarrow \gamma e^{(z-\zeta)\ell}, \quad (10a)$$

$$D_A \rightarrow D_A e^{(z-2\chi-\zeta-1)\ell}, \quad D_Q \rightarrow D_Q e^{(2z-2\chi-\zeta-1)\ell}, \quad (10b)$$

and the coefficients of the nonlinear terms $P_{yx}u_y^2$, $u_y \partial_y u_y$, $u_x u_y$, and u_y^3 as

$$\frac{\alpha}{2v_0} \rightarrow \frac{\alpha}{2v_0} e^{(z+\chi+\zeta-1)\ell}, \quad \frac{\alpha}{v_0} \rightarrow \frac{\alpha}{v_0} e^{(z+\chi+\zeta-1)\ell}, \quad (11a)$$

$$\lambda \rightarrow \lambda e^{(z+\chi-1)\ell}, \quad \frac{\alpha}{2v_0^2} \rightarrow \frac{\alpha}{2v_0^2} e^{(z+2\chi)\ell}. \quad (11b)$$

We choose z , ζ , and χ to fix α , γ and D_Q , which control the size of the dominant fluctuations (i.e., those coming

from the quenched noise), which yields

$$z = \frac{2}{3}, \quad \zeta = \frac{2}{3}, \quad \chi = -\frac{1}{6}. \quad (12)$$

As expected, these values of ζ and χ are identical to those for the quenched correlations obtained from our linear theory [e.g., see (7)]. Note that while the choice of z is formally necessary to restore the EOM to its original form, it does not appear in the correlation function (7) because the leading order part of that correlation function is static since the quenched disorder that induces those leading-order fluctuations is. Its value is different from the z in (7), since the time-dependent part of the fluctuations comes from the subdominant annealed disorder.

Substituting these values into (11), we find the coefficients of the nonlinear terms $P_{yx}u_y^2$, $u_x u_y$, and u_y^3 rescale as

$$\frac{\alpha}{2v_0} \rightarrow \frac{\alpha}{2v_0} e^{\ell/6}, \quad \frac{\alpha}{v_0} \rightarrow \frac{\alpha}{v_0} e^{\ell/6}, \quad \frac{\alpha}{2v_0^2} \rightarrow \frac{\alpha}{2v_0^2} e^{\ell/3}, \quad (13)$$

which clearly all diverge as $\ell \rightarrow \infty$. This implies that these nonlinear terms are *relevant* in the hydrodynamic limit. In contrast, $\lambda \rightarrow \lambda e^{(z+\chi-1)\ell}$ vanishes, which implies $u_y \partial_y u_y$ is irrelevant and hence can be neglected.

To deal with the relevant nonlinearities, we will perform a one-loop DRG calculation using an ϵ -expansion method. Our approach is that of [35, 36] and is explained in detail in the ALP [34].

To employ the ϵ -expansion method, we rely on analytic continuity to generalize our calculation to dimensions $d >$

2. We have two distinct choices for doing this; we can treat the “soft” x direction as one-dimensional and the “hard” y direction as $d-1$ dimensional or we can take x to be $d-1$ dimensional while treating y as one-dimensional. These lead to two *distinct* ϵ -expansion schemes which we term the *hard* and *soft* continuations, respectively [37]. Alternatively, we perform an uncontrolled one-loop calculation in exactly $d=2$. The numerical values given at the beginning of the Letter for the exponents represent an average of the results from these three schemes.

Crucially, in all three schemes, we make use of an important simplification: due to the rotation invariance of our hydrodynamic EOM, it is convenient to choose the values of χ and ζ so that all three “ α ”s appearing in the EOM (8) remain identical upon the rescaling, i.e.,

$$\chi = \zeta - 1. \quad (14)$$

Using the above simplification to eliminate the roughness exponent χ , the DRG recursion relations, to one-loop order, for the *hard* continuation are [34]:

$$\frac{d \ln \alpha}{d\ell} = z + 2\zeta - 2 + \eta_\alpha, \quad (15a)$$

$$\frac{d \ln \gamma}{d\ell} = z - \zeta + \eta_\gamma, \quad (15b)$$

$$\frac{d \ln \mu_x}{d\ell} = z - 2\zeta + \eta_\mu, \quad (15c)$$

$$\frac{d \ln D_Q}{d\ell} = 2z - 3\zeta + 3 - d + \eta_Q, \quad (15d)$$

$$\frac{d \ln D_A}{d\ell} = z - 3\zeta + 3 - d + \eta_A, \quad (15e)$$

where the η ’s represent graphical corrections. To lowest order in perturbation theory [34],

$$\eta_\alpha = -\frac{1}{27}g, \quad \eta_\gamma = \frac{8}{27}g, \quad \eta_Q = \frac{10}{27}g, \quad \eta_A = \frac{16}{27}g, \quad (16a)$$

$$\eta_\mu = g_\mu + \frac{2}{27}g, \quad (16b)$$

where we’ve defined

$$g \equiv \frac{S_{d-1}}{(2\pi)^{d-1}} |\gamma|^{-\frac{7}{3}} \alpha^{\frac{1}{3}} \Lambda^{\frac{3d-7}{3}} D_Q, \quad (17a)$$

$$g_\mu = \frac{S_{d-1}}{(2\pi)^{d-1}} \Lambda^{d-3} |\gamma|^{-1} \mu_x^{-1} D_Q, \quad (17b)$$

with S_{d-1} the area of a $d-1$ -dimensional unit sphere, and Λ the inverse of the short wavelength cut-off in the soft direction. In writing the recursion relations (15), we have ignored some corrections arising from the annealed noise D_A which prove to be irrelevant (that is, to vanish as $\ell \rightarrow \infty$) [34].

From (15), we can construct closed recursion relations

for g and g_μ :

$$\frac{dg}{d\ell} = \frac{1}{3} (7 - 3d - g) g, \quad (18a)$$

$$\frac{dg_\mu}{d\ell} = (3 - d - g_\mu) g_\mu. \quad (18b)$$

For $d < 7/3$, the above flow equations indicate that the generic stable fixed point is at

$$g^* = 3\epsilon + \mathcal{O}(\epsilon^2), \quad g_\mu^* = \frac{2}{3} + \epsilon + \mathcal{O}(\epsilon^2), \quad (19)$$

where $\epsilon = 7/3 - d$ and the $\mathcal{O}(\epsilon^2)$ part can only be obtained through higher-order in perturbation theory calculations. For $d=2$, $\epsilon=1/3$, which is extremely small for ϵ expansions. Therefore, we expect our one-loop DRG results to be quantitatively *very* accurate.

The fact that both g and g_μ flow to non-zero stable fixed points also implies two *exact* relations between the η ’s. The definition of g (17a), together with (15), implies

$$\frac{d \ln g}{d\ell} = \frac{1}{3} (7 - 3d) - \frac{7}{3} \eta_\gamma + \frac{1}{3} \eta_\alpha + \eta_Q. \quad (20)$$

which leads to the exact relation, valid to *all* loop orders,

$$7 - 3d - 7\eta_\gamma + \eta_\alpha + 3\eta_Q = 0, \quad (21)$$

since $d \ln g / d\ell = 0$ at the fixed point. Similarly, (17b) and (15) lead to the second *exact* relation:

$$3 - d + \eta_Q - \eta_\gamma - \eta_\mu = 0. \quad (22)$$

Note that η_μ does not vanish as ϵ goes to zero since the upper critical dimension of g_μ is 3 instead of $7/3$ (see (17b)). Since μ_x controls the annealed part of the correlation, this implies that this part starts to depart from the linear prediction, as d is decreased, from $d=3$, in contrast to the quenched part, which first does so at $d=7/3$. As shown in detail in [34], while within the linear theory, the anisotropy exponent for the quenched and annealed parts of the correlator are *distinct*, the annealed anisotropy exponent assumes the same value as the quenched one for all $d \leq 7/3$; that is, in precisely the dimension at which the quenched anisotropy exponent first departs from its linear value.

Scaling exponents.—We now use the DRG procedure to calculate the real time-real space correlations $C_Q(\mathbf{r})$ and $C_A(t, \mathbf{r})$, which represent the quenched and annealed parts of $\langle \mathbf{u}(\mathbf{r}, t) \cdot \mathbf{u}(\mathbf{0}, 0) \rangle$, respectively. They are related to those of the rescaled system [38] via

$$\begin{aligned} & C_Q(\alpha_0, \gamma_0, D_{Q_0}, \mathbf{r}) \\ &= e^{2\chi\ell} C_Q \left[\alpha(\ell), \gamma(\ell), D_Q(\ell), \frac{|x|}{e^{\zeta\ell}}, \frac{|y|}{e^\ell} \right], \end{aligned} \quad (23a)$$

$$\begin{aligned} & C_A(\alpha_0, \mu_{x0}, D_{A_0}, t, \mathbf{r}) \\ &= e^{2\chi\ell} C_A \left[\alpha(\ell), \mu_x(\ell), D_A(\ell), \frac{|t|}{e^{\zeta\ell}}, \frac{|x|}{e^{\zeta\ell}}, \frac{|y|}{e^\ell} \right], \end{aligned} \quad (23b)$$

where α , γ , and D_Q control the magnitude of $C_Q(\mathbf{r})$, while α , μ_x and D_A control the magnitude of $C_A(\mathbf{r}, t)$ and the subscript “0” denotes the bare values of the parameters. We choose ζ and z such that α , γ , μ_x , and D_Q are all fixed [i.e., equate the RHS of (15a-15d) to 0], which is possible since $\eta_{\gamma, \alpha, Q, \mu}$ are constrained by the *exact* relations (21,22). Choosing $\ell = \ln(\Lambda|y|)$ and taking the renormalization factor of D_A to the front, we write the RHS of (23a,23b) in the form displayed by (1) with

$$\mathcal{G}_Q \left(\frac{|x|}{|y|^\zeta} \right) \equiv \Lambda^{2\chi} C_Q \left[\alpha_0, \gamma_0, D_{Q_0}, \frac{|x|}{(|y|\Lambda)^\zeta}, \frac{1}{\Lambda} \right], \quad (24)$$

and

$$\mathcal{G}_A \left(\frac{|t|}{|y|^z}, \frac{|x|}{|y|^\zeta} \right) \equiv \Lambda^{2\chi'} C_A \left[\alpha_0, \gamma_0, D_{A_0}, \frac{|t|}{(|y|\Lambda)^z}, \frac{|x|}{(|y|\Lambda)^\zeta}, \frac{1}{\Lambda} \right], \quad (25)$$

with the various exponents being

$$\zeta = \frac{2 + \eta_\gamma - \eta_\alpha}{3} = \frac{2}{3} + \frac{1}{3}\epsilon + O(\epsilon^2), \quad (26a)$$

$$z = \frac{2 - 2\eta_\gamma - \eta_\alpha}{3} = \frac{2}{3} - \frac{5}{9}\epsilon + O(\epsilon^2), \quad (26b)$$

$$\chi = \frac{-1 + \eta_\gamma - \eta_\alpha}{3} = -\frac{1}{3} + \frac{1}{3}\epsilon + O(\epsilon^2), \quad (26c)$$

$$\chi' = \frac{\eta_A - \eta_\gamma - 1}{2} = -\frac{1}{2} + \frac{4}{9}\epsilon + O(\epsilon^2), \quad (26d)$$

where in the second equalities we have used (16) and the results for g and g_μ to $O(\epsilon)$ given by (19). Note that this calculation shows explicitly that, beyond the linear theory, the anisotropy exponents for the quenched and the annealed correlations become equal, and the x -dependence of the annealed part of the correlation function involves an “unboosted” x , *not* a boosted variable $x - \gamma t$. This confirms our earlier claim.

The analysis for the *soft* continuation and the uncontrolled one-loop calculation in precisely $d = 2$ are very similar, and give comparable quantitative results [34]. The exponents quoted earlier are a weighted average of these three results; the quoted error bars more than cover the spread between the three different calculations.

Summary & Outlook.—We have demonstrated that a combination of active motility and incompressibility leads to the formation of two-dimensional long-range ordered flocks, even in the presence of random quenched disorder. In contrast, neither active motility nor incompressibility *alone* can overcome quenched disorder, although either can successfully compete with *annealed* (i.e., time-dependent) noise [5, 27].

One experimental realization of incompressibility in a two-dimensional system is motile particles moving through a narrow channel filled with an incompressible fluid [23, 24] or at high particle densities [22, 23]. Furthermore, some degree of disorder will always be present

in all experimental systems, especially biological ones. Therefore, our work should be valuable in interpreting numerous experiments both in artificial and biological active systems. Further, it demonstrates that confluent cell layers on substrates can move coherently despite the presence of static random impurities.

We also look forward to quantitative tests of our predictions in artificial active systems, for instance, Quincke rotors [20] or vibrated granular systems [39].

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