Spinors, Proper Time and Higher-Spin Fields

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Abstract

We present a Lagrangian formulation for $4d$ integer-spin relativistic fields in the $5d$ space spanned by two conjugate Weyl spinors and a Lorentz-invariant proper-time coordinate. We construct a manifestly Poincaré-invariant free classical action, find a general solution to equations of motion and a corresponding positive-definite inner product. Our formulation displays a separation of variables: equations of motion represent ODE in a proper time only, while spinor coordinates parameterize the Cauchy hypersurface. We also find momentum eigenstates solutions for massless arbitrary integer-spin fields and a massive scalar field.

Contents

1 Introduction

Higher-spin (HS) theories represent an important class of models of fundamental interactions. Covariant Lagrangian formulations for free higher-spin fields have been constructed in massive case by Singh and Hagen [1, 2], and in massless case by Fronsdal and Fang, both in Minkowski [3, 4] and (A)dS [5, 6] spaces. But it turned out that constructing consistent interactions for massless HS fields, which problem is of the most interest, gets very involved in the covariant setup. Therefore the main progress beyond the free level is due to other approaches.

In particular, cubic HS interactions have been found and studied in detail within the light-cone framework (see e.g. [7–11]). However, already beyond the cubic level the analysis becomes too complicated.

Self-dual HS models are conveniently formulated and analyzed by means of the methods of twistor theory [12–15].

The full all-order system of classical e.o.m. of interacting HS gauge fields has been constructed by Vasiliev [16, 17] in terms of the generating equations, written in the so-called unfolded form [18–20] (for a review of Vasiliev theory see [21, 22]). But extracting HS vertices from Vasiliev equations represents a very nontrivial task, because one must restrict somehow the degree of non-locality while solving for auxiliary generating variables, which problem is currently under the active study (see [23] and references therein).

More references and a partial review of the recent HS literature can be found in [24].

Thus, the availability of different implementations of HS fields significantly enriches our possibilities for constructing and studying HS theories. In this paper we propose a new realization for the integer-spin representations of the $4d$ Poincaré group. Instead of dealing with $4d$ Minkowski space, we consider a $5d$ space spanned by a pair of conjugate spinors and one Lorentz scalar. This set of coordinates appeared previously in the unfolded formulation of the $4d$ off-shell fields [25–28], where they have been playing the role of the auxiliary fiber coordinates, encoding unfolded descendants of the space-time fields under consideration. In this paper we use these coordinates to build a self-contained Lagrangian formulation for $4d$ integer-spin fields without any reference to a space-time.

To give a preliminary intuitive idea of how such $5d$ space can encode $4d$ fields, let us consider a simple example. An asymptotic one-particle state of a scalar field is determined by

4-momentum $p^a = (E, \overrightarrow{p})$, which is forced to lie on the mass-shell $p_a p^a = m^2$. Hence, the state is fixed by three independent parameters: four variables with one constraint. Alternatively, the same information can be encoded in a Lorentz-scalar $\pi = \sqrt{E^2 - \overrightarrow{p}^2}$ and a null vector $n^a = (|\overrightarrow{p}|, \overrightarrow{p})$, with the constraint being $\pi = m$. In its turn, a real null $4d$ vector can be represented in terms of spinors as $n^a = (\bar{\sigma})^{\dot{\alpha}\beta}\bar{\xi}_{\dot{\alpha}}\xi_\beta$. Thus, a set of 5 variables $\{\pi, \xi^\alpha, \bar{\xi}^{\dot{\alpha}}\}$ (effectively, 4 of them, as the global phase of ξ does not contribute) determines the 4-momentum, while the mass-shell equation becomes simply $\pi = m$, putting no restrictions on ξ.

In our consideration, however, we make use of a similar $5d$ space as a substitute not for the momentum p^a, but rather for the coordinate x^a, so that classical e.o.m. become ODE in a scalar coordinate. We find expressions for Poincaré generators and identify appropriate modules supplied with a positive-definite inner product.We also construct simple Poincaré-invariant actions which lead to the appropriate e.o.m. and find their general solutions. In addition, we find solutions for momentum eigenstates for the cases of an arbitrary-mass scalar field and of massless arbitrary spin fields.

The paper is organized as follows. In Section 2 we introduce our conventions for Poincaré generators and give a brief reminder on how covariant quantum fields are constructed in the standard approach, to be later compared with our construction. In Section 3 we build a $4d$ integer-spin representation on a certain $5d$ space. In Section 4 we present a Poincaré-invariant action for a free field, give a general solution to e.o.m. and propose an inner product for solutions. In Section 5 we find solutions of e.o.m. corresponding to momentum eigenstates for a scalar field and massless fields. In Section 6 we sum up our results.

2 $4d$ Poincaré algebra and relativistic fields

Elementary particles are associated with unitary irreducible representations (UIRs) of the Poincaré group (or an isometry group of the spacetime in question, more generally) [29].

In the paper we consider $4d$ Poincaré algebra with generators $P_{\alpha\dot{\alpha}}$, $M_{\alpha\beta} = M_{\beta\alpha}$ and $\bar{M}_{\dot{\alpha}\dot{\beta}} = \bar{M}_{\dot{\beta}\dot{\alpha}}$, which correspond to translations, anti-selfdual and selfdual rotations of Minkowski space, respectively. Here indices belong to two conjugate spinor representations of the Lorentz algebra $sl(2, \mathbb{C})$. Commutation relations are

$$[M_{\alpha\beta}, M_{\gamma\delta}] = \epsilon_{\alpha\gamma}M_{\beta\delta} + \epsilon_{\alpha\delta}M_{\beta\gamma} + \epsilon_{\beta\gamma}M_{\alpha\delta} + \epsilon_{\beta\delta}M_{\alpha\gamma}, \tag{2.1}$$
$$[\bar{M}_{\dot{\alpha}\dot{\beta}}, \bar{M}_{\dot{\gamma}\dot{\delta}}] = \epsilon_{\dot{\alpha}\dot{\gamma}}\bar{M}_{\dot{\beta}\dot{\delta}} + \epsilon_{\dot{\alpha}\dot{\delta}}\bar{M}_{\dot{\beta}\dot{\gamma}} + \epsilon_{\dot{\beta}\dot{\gamma}}\bar{M}_{\dot{\alpha}\dot{\delta}} + \epsilon_{\dot{\beta}\dot{\delta}}\bar{M}_{\dot{\alpha}\dot{\gamma}}, \tag{2.2}$$
$$[M_{\alpha\beta}, \bar{M}_{\dot{\gamma}\dot{\delta}}] = 0, \tag{2.3}$$
$$[M_{\alpha\beta}, P_{\gamma\dot{\gamma}}] = \epsilon_{\alpha\gamma}P_{\beta\dot{\gamma}} + \epsilon_{\beta\gamma}P_{\alpha\dot{\gamma}}, \tag{2.4}$$
$$[\bar{M}_{\dot{\alpha}\dot{\beta}}, P_{\gamma\dot{\gamma}}] = \epsilon_{\dot{\alpha}\dot{\gamma}}P_{\gamma\dot{\beta}} + \epsilon_{\dot{\beta}\dot{\gamma}}P_{\gamma\dot{\alpha}}, \tag{2.5}$$
$$[P_{\alpha\dot{\alpha}}, P_{\beta\dot{\beta}}] = 0, \tag{2.6}$$

where $\epsilon_{\alpha\beta}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$ are Lorentz-invariant spinor metrics

$$\epsilon_{\alpha\beta} = \epsilon^{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.7}$$

which raise and lower spinor indices according to

$$v_\alpha = \epsilon_{\beta\alpha}v^\beta, \quad v^\alpha = \epsilon^{\alpha\beta}v_\beta, \quad \bar{v}_{\dot{\alpha}} = \epsilon_{\dot{\beta}\dot{\alpha}}\bar{v}^{\dot{\beta}}, \quad \bar{v}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{v}_{\dot{\beta}}. \tag{2.8}$$

UIRs are determined by the values of two Casimir operators: a square of the momentum, associated with the mass,

$$P^2 = m^2 \tag{2.9}$$

and, introducing the Pauli–Lubanski pseudovector as

$$W_{\alpha\dot\alpha} = \frac{1}{2}M_{\alpha\beta}P^{\beta}{}_{\dot\alpha} - \frac{1}{2}\bar M_{\dot\alpha\dot\beta}P_{\alpha}{}^{\dot\beta}, \tag{2.10}$$

either its square, associated with the spin s when $m^2 > 0$

$$W^2 = -m^2 s(s+1), \tag{2.11}$$

or the helicity λ when $m = 0$

$$W_{\alpha\dot\beta} = \lambda P_{\alpha\dot\beta}. \tag{2.12}$$

In (2.9), (2.11) and throughout the paper the square v^2 of a vector $v_{\alpha\dot\beta}$ is defined as

$$v^2 = \frac{1}{2}v_{\alpha\dot\beta}v^{\alpha\dot\beta}. \tag{2.13}$$

The standard covariant QFT approach is to implement momentum generators as coordinate derivatives

$$P_a = -i\frac{\partial}{\partial x^a} \tag{2.14}$$

on the Minkowski space with coordinates x^a. Then quantum fields look as $\phi^I(x)$, where index I belongs to some finite-dimensional representation of the Lorentz group (spin), so that rotations are realized as

$$M_{a,b} = i(x_a\frac{\partial}{\partial x^b} - x_b\frac{\partial}{\partial x^a}) + (S_{a,b})^I{}_J \tag{2.15}$$

with S being x-independent spin generators. In general, however, the resulting representation of the Poincaré algebra is neither irreducible nor unitary, and one has to remove undesirable subrepresentations by imposing additional constraints besides the Klein–Gordon equation (2.9). In order to represent all of them as following from some Lagrangian equations of motion, one has to introduce auxiliary fields (for massive fields with $s > 1$) and/or to provide certain gauge symmetry (for massless fields with $s \geq 1$). Corresponding Lagrangian formulations for arbitrary spin fields have been constructed by Sing and Hagen for massive fields [1, 2] and by Fronsdal and Fang for massless fields [3–6].

3 Spin-s representation

In the paper we construct a realization of bosonic UIRs on a $5d$ linear space spanned by a pair of conjugate commuting $sl(2,\mathbb{C})$ spinors $Y^A = (y^\alpha, \bar y^{\dot\alpha})$ and a Lorentz-invariant 'proper time' τ. This set of variables (Y,τ) was previously used in formulating off-shell unfolded equations for various $4d$ field systems [25–28]. And spinors Y were initially used in the unfolded Vasiliev equations [16, 17], where they play the crucial role of the generators of an associative HS gauge algebra. Here we propose to use (Y,τ)-space instead of a space-time and build a corresponding Lagrangian formulation for bosonic fields. All fields are 'scalar' (i.e. without non-contracted Lorentz indices) functions $F(Y,\tau)$ on this space.

For the rotation generators we take

$$M_{\alpha\beta} = y_\alpha \partial_\beta + y_\beta \partial_\alpha, \tag{3.1}$$

$$\bar{M}_{\dot\alpha\dot\beta} = \bar{y}_{\dot\alpha} \bar\partial_{\dot\beta} + \bar{y}_{\dot\beta} \bar\partial_{\dot\alpha}, \tag{3.2}$$

where Y-derivatives are defined as

$$\partial_\alpha y^\beta = \delta_\alpha{}^\beta, \quad \bar\partial_{\dot\alpha} \bar{y}^{\dot\beta} = \delta_{\dot\alpha}{}^{\dot\beta}. \tag{3.3}$$

It is easy to check that (3.1)-(3.2) satisfy (2.1)-(2.3). From here it also directly follows that the proper-time coordinate τ is Lorentz-invariant (but not translation-invariant, as we will see). The expressions (3.1)-(3.2) for rotations operators are universal: we demand that they look the same for all fields of arbitrary masses and spins, like it is the case for the translation operator in the standard construction (2.14). The price to pay for this is that the translation operator now depends on a spin, as we will see.

As Y commute with themselves, they have zero norm

$$y_\alpha y^\alpha = 0, \quad \bar{y}_{\dot\alpha} \bar{y}^{\dot\alpha} = 0, \tag{3.4}$$

and the only independent Lorentz-invariant Y-combinations one can form are Euler operators

$$N = y^\alpha \partial_\alpha, \quad \bar{N} = \bar{y}^{\dot\alpha} \bar\partial_{\dot\alpha}. \tag{3.5}$$

An appropriate module of a spin-s representation has to contain states with helicities from $-s$ to $+s$. This can be achieved by considering a set of functions

$$\Phi^s(Y,\tau) = \{\Phi_{\alpha(m),\dot\alpha(n)}(\tau)(y^\alpha)^m(\bar{y}^{\dot\alpha})^n, \quad (m+n) \geq 2s, \quad |m-n| \leq 2s\}, \tag{3.6}$$

where we make use of condensed notations for symmetrized indices

$$v_{\alpha(m)} = v_{(\alpha_1\alpha_2\ldots\alpha_m)}, \quad (y^\alpha)^m = y^{\alpha_1} y^{\alpha_2} \ldots y^{\alpha_m}. \tag{3.7}$$

The module (3.6) can be also represented as

$$\Phi^s(Y,\tau) = \Phi_{A(2s)}(y\bar{y},\tau)(Y^A)^{2s}, \tag{3.8}$$

where A is a Majorana index taking four values $\{1, 2, \dot{1}, \dot{2}\}$. This form is visually more similar to the standard Minkowski approach, where an integer spin-s module is a rank-s tensor field $\phi_{a(s)}(x)$. It should be stressed however, that in (3.8) 'external' Y-s and 'internal' y-s and $\bar{y}$-s are on a completely equal footing, as seen from (3.6). And $2s$ explicit spinors and indices in (3.8) are highlighted only in order to show restrictions on the number of y and $\bar{y}$ and play no special role otherwise.

Now one has to find an expression for the momentum operator $P_{\alpha\dot\beta}$. The most general Ansatz is

$$P_{\alpha\dot\beta} = a_{N,\bar{N}} \partial_\alpha \bar\partial_{\dot\beta} + b_{N,\bar{N}} y_\alpha \bar{y}_{\dot\beta} + c_{N,\bar{N}} y_\alpha \bar\partial_{\dot\beta} + \bar{c}_{N,\bar{N}} \partial_\alpha \bar{y}_{\dot\beta}, \tag{3.9}$$

where Lorentz-invariant coefficients $a, b, c, \bar{c}$ are built out of Euler operators (3.5), as well as of τ and τ-derivatives. (3.9) automatically satisfies (2.4) and (2.5), so the only equation to be solved is (2.6). It can be equivalently reformulated in terms of two conjugate equations

$$P_{\alpha\dot\beta} P_{\alpha\dot\gamma} \epsilon^{\dot\beta\dot\gamma} = 0, \tag{3.10}$$

$$P_{\beta\dot\alpha}P_{\gamma\dot\alpha}\epsilon^{\beta\gamma}=0. \tag{3.11}$$

Substituting (3.9), they lead to the following constraints

$$(\bar N+2)a_{N,\bar N}\bar c_{N+1,\bar N+1}-\bar N a_{N+1,\bar N-1}\bar c_{N,\bar N}=0, \tag{3.12}$$
$$(\bar N+2)b_{N-1,\bar N+1}c_{N,\bar N}-\bar N b_{N,\bar N}\bar c_{N-1,\bar N-1}=0, \tag{3.13}$$
$$(\bar N+2)a_{N,\bar N}b_{N+1,\bar N+1}-\bar N a_{N-1,\bar N-1}b_{N,\bar N}+(\bar N+2)c_{N,\bar N}\bar c_{N-1,\bar N+1}-\bar N\bar c_{N,\bar N}c_{N+1,\bar N-1}=0, \tag{3.14}$$

plus three conjugate equations with $N\leftrightarrow\bar N$, $c\leftrightarrow\bar c$ interchanged. In addition, one has to ensure that the action of (3.9) does not lead outside the module (3.6). This means that only those solutions are suitable that satisfy

$$a_{N,\bar N}|_{\varsigma=s-1}=0,\quad c_{N,\bar N}|_{\chi=s+1}=0,\quad \bar c_{N,\bar N}|_{\chi=-s-1}=0, \tag{3.15}$$

where ς and χ are important linear combinations of Euler operators (3.5), which we actively use below,

$$\varsigma=\frac{N+\bar N}{2},\quad \chi=\frac{N-\bar N}{2}. \tag{3.16}$$

Any solution of (3.12)-(3.14) respecting boundary conditions (3.15) defines some representation of the Poincaré algebra. But many of these representations are equivalent, and this allows one to put some further constraints.

First, we restrict τ-dependence and provide a 'separation of variables' Y and τ. Specifically, we require the operator P^2 to be Y-independent, so that the mass-shell equation (2.9) becomes an ODE in τ. In addition, we demand τ to enter (3.9) only through this P^2-combination.

Second, we require $P_{\alpha\dot\beta}$ to allow for a usual integration by parts rule

$$\int d\tau\int d^4Y f(Y,\tau)P_{\alpha\dot\beta}g(Y,\tau)=-\int d\tau\int d^4Y g(Y,\tau)P_{\alpha\dot\beta}f(Y,\tau). \tag{3.17}$$

To this end one notes that (assuming that one can neglect boundary terms)

$$\int d^4Y(y^\alpha\partial_\alpha f(Y))g(Y)=\int d^4Y((\partial_\alpha y^\alpha-2)f(Y))g(Y)=-\int d^4Y f(Y)(y^\alpha\partial_\alpha+2)g(Y), \tag{3.18}$$

which allows one to formulate general rules

$$\int Nf\cdot g=-\int f\cdot(N+2)g,\quad \int\bar N f\cdot g=-\int f\cdot(\bar N+2)g,\quad \int\varsigma f\cdot g=-\int f\cdot(\varsigma+2)g,\quad \int\chi f\cdot g=-\int f\cdot\chi g. \tag{3.19}$$

These constraints significantly restrict the space of solutions to (3.10)-(3.11), though still do not fix it unambiguously. We pick up the following particular solution

$$\begin{aligned}-iP_{\alpha\dot\beta} &= \frac{(\varsigma-s+1)(\varsigma+s+2)(\varsigma+3/2)}{(N+1)(N+2)(\bar N+1)(\bar N+2)}\partial_\alpha\bar\partial_{\dot\beta}-\frac{P^2}{(\varsigma+1/2)}y_\alpha\bar y_{\dot\beta}+\\ &+\frac{1}{(\bar N+1)(\bar N+2)}[(\chi+s)(\chi-s-1)\Pi^+-P^2\Pi^{-0}]y_\alpha\bar\partial_{\dot\beta}+\\ &+\frac{1}{(N+1)(N+2)}[(\chi-s)(\chi+s+1)\Pi^--P^2\Pi^{+0}]\partial_\alpha\bar y_{\dot\beta},\end{aligned} \tag{3.20}$$

where projectors Π on different χ-components are introduced as

$$\Pi^+ F_\chi(Y) = \begin{cases} F_\chi(Y), & \chi > 0 \\ 0, & \chi \leq 0 \end{cases}; \qquad \Pi^- F_\chi(Y) = \begin{cases} F_\chi(Y), & \chi < 0 \\ 0, & \chi \geq 0 \end{cases}; \tag{3.21}$$

$$\Pi^{+0} F_\chi(Y) = \begin{cases} F_\chi(Y), & \chi \geq 0 \\ 0, & \chi < 0 \end{cases}; \qquad \Pi^{-0} F_\chi(Y) = \begin{cases} F_\chi(Y), & \chi \leq 0 \\ 0, & \chi > 0 \end{cases}. \tag{3.22}$$

Expression (3.20) for P contains manifestly and self-consistently its own square P^2, which is Y-independent by construction. P^2 is also required to be even under integration by parts in order to provide (3.17).

Now for the Pauli–Lubanski pseudovector (2.10) one has

$$\begin{aligned} -iW_{\alpha\dot\beta} &= -\chi\frac{(\varsigma - s + 1)(\varsigma + s + 2)(\varsigma + 3/2)}{(N+1)(N+2)(\bar N + 1)(\bar N + 2)}\partial_\alpha\bar\partial_{\dot\beta} - \chi\frac{P^2}{(\varsigma + 1/2)}y_\alpha\bar y_{\dot\beta} + \\ &+\frac{(\varsigma+1)}{(\bar N + 1)(\bar N + 2)}[(\chi + s)(\chi - s - 1)\Pi^+ - P^2\Pi^{-0}]y_\alpha\bar\partial_{\dot\beta} - \\ &-\frac{(\varsigma+1)}{(N + 1)(N + 2)}[(\chi - s)(\chi + s + 1)\Pi^- - P^2\Pi^{+0}]\partial_\alpha\bar y_{\dot\beta}, \end{aligned} \tag{3.23}$$

with its square being

$$W^2 = -P^2 s(s+1). \tag{3.24}$$

In the case $P^2 = 0$ one finds that $P_{\alpha\dot\beta}$ and $W_{\alpha\dot\beta}$ are proportional to each other whenever the module contains components with $|\chi| = s$ only, in which case

$$-iP^{m=0}_{\alpha\dot\beta} = \frac{(\varsigma - s + 1)(\varsigma + s + 2)(\varsigma + 3/2)}{(N+1)(N+2)(\bar N + 1)(\bar N + 2)}\partial_\alpha\bar\partial_{\dot\beta}, \quad W^{m=0}_{\alpha\dot\beta} = -\chi P^{m=0}_{\alpha\dot\beta}, \tag{3.25}$$

that corresponds to two $\pm s$ helicities (2.12) of the massless field.

Thus, operators (3.1), (3.2) and (3.20) indeed correctly determine a spin-s representation on the module (3.6) after fixing the value of P^2. In the massless case $P^2 = 0$ one also has to reduce the module, leaving only $\pm s$ helicities, which corresponds to setting $|m - n| = 2s$ instead of $|m - n| \leq 2s$ in (3.6), or having, instead of (3.8),

$$\Phi^s_{m=0}(Y,\tau) = \Phi_{\alpha(2s)}(y\bar y,\tau)(y^\alpha)^{2s} \oplus \bar\Phi_{\dot\alpha(2s)}(y\bar y,\tau)(\bar y^{\dot\alpha})^{2s}. \tag{3.26}$$

Now, in order to formulate an action principle, one has to realize P^2 as a differential operator. As mentioned previously, it must be τ-dependent only and even under integration by parts, but completely unrestricted otherwise. This means that in our construction Klein–Gordon equation (2.9) can be implemented in many different ways. In the next Section we consider one of the simplest possibilities.

4 Free action, e.o.m. and inner product

First we consider the massive case. We take

$$P^2 = -\frac{\partial^2}{\partial\tau^2}. \tag{4.1}$$

Then a Poincaré-invariant action for a spin-s mass-m field is simply

$$S=\frac{1}{2}\sum_{\chi=-s}^{s}\int d^4Y\int d\tau(\dot{\Phi}_\chi^2-m^2\Phi_\chi^2), \tag{4.2}$$

where the dot means a τ-derivative and Φ_χ means a subspace of the spin-s module (3.6) of the definite helicity-χ

$$\Phi_\chi(Y,\tau)=\Phi_{\alpha(s+\chi),\dot{\beta}(s-\chi)}(y\bar{y},\tau)(y^\alpha)^{s+\chi}(y^{\dot{\beta}})^{s-\chi}. \tag{4.3}$$

Poincaré-invariance of the action (4.2) is guaranteed by the integration-by-parts property (3.17), which is obvious for M and $\bar{M}$ (3.1)-(3.2) as well.

The action (4.2) leads to an e.o.m.

$$\ddot{\Phi}_\chi+m^2\Phi_\chi=0. \tag{4.4}$$

Its general solution is

$$\Phi_\chi(Y,\tau)=e^{-im\tau}f_\chi(Y)+e^{im\tau}g_\chi(Y), \tag{4.5}$$

where the only requirement to Y-functions f and g is to belong to helicity-χ subspace. Thus, from the point of view of (4.4), Y are coordinates on the subspace of Cauchy data, while e.o.m. determines the evolution in τ-direction.

A Poincaré-invariant inner product for the on-shell states is

$$(\Phi_\chi,\Psi_{\chi'})=i\int d^4Y(\bar{\Phi}\dot{\Psi}-\Psi\dot{\bar{\Phi}})\delta_{\chi,\chi'}. \tag{4.6}$$

It is τ-independent due to (4.4) and positive-definite for a 'positive-mass' subspace of (4.5) with $g=0$. The states with the same Y-dependence but with different mass signs are orthogonal.

The split of the on-shell space into two subspaces, corresponding to 'positive-mass' f and 'negative-mass' g contributions in (4.6), is reminiscent to the split into positive-energy and negative-energy branches in the standard QFT. However, establishing the rigorous relation between two these phenomenae requires a separate thorough analysis which we leave for the future study. Let us note, however, that in our case the split, being determined by τ-dependence, is manifestly Lorentz-invariant.

Now we move to the massless case. Here using (4.1) potentially leads to problems: the general solution to (4.4) with $m=0$ is an arbitrary linear function of τ, so all on-shell states either have zero norm with respect to (4.6) or are unbounded in τ, which may be unpleasant.

This can be easily fixed by introducing a mass-dimension parameter μ and deforming (4.1) to

$$P^2=-\frac{\partial^2}{\partial\tau^2}-\mu^2. \tag{4.7}$$

Then the zero-mass action becomes

$$S=\frac{1}{2}\sum_{\chi=-s}^{s}\int d^4Y\int d\tau(\dot{\Phi}_\chi^2-\mu^2\Phi_\chi^2), \tag{4.8}$$

and e.o.m. now are

$$\ddot{\Phi}_\chi+\mu^2\Phi_\chi=0, \tag{4.9}$$

so the general solution is

$$\Phi_\chi(Y,\tau) = e^{-i\mu\tau} f_\chi(Y) + e^{i\mu\tau} g_\chi(Y), \tag{4.10}$$

and one has τ-bounded functions and the split into two branches again.

As said before, in the massless case one also has to reduce the module, leaving only $|\chi| = s$ components, (3.26). Intermediate components $|\chi| < s$ are necessary to provide off-shell Poincaré invariance of the action (4.8), but on shell $|\chi| = s$ components decouple into closed subspaces.

It should be stressed that the equation (4.9) describes a massless field, $m = 0$. The parameter μ does not shift the value of the mass, it only deforms the functional dependence of P^2 on τ. In particular, μ enters directly the expression for the off-shell momentum generator (3.20) through (4.7). In principle, it can be introduced for the massive fields as well. Practically, the parameter μ plays the role of a manifestly Poincaré-invariant IR-regulator. The possibility of such deformation relies on the large freedom in choosing the differential realization of the P^2 and is specific to the presented construction. In particular, it is unclear how to locally deform the momentum operator (2.14) of a covariant QFT to have $P^2 = -\Box + \mu^2$.

Let us also give a brief comment on the issue of locality of the constructed representations. As seen from (3.20), the translations, as opposite to the rotations (3.1)-(3.2), are realized non-locally: Y-differential operators N and $\bar{N}$ enter (3.20) in a non-polynomial way. But a crucial feature is that the translations are local in τ, so one cannot e.g. shift the pole of the propagator by means of Poincaré-transformations. So the evolution in τ is completely local, while transformations on the Cauchy hypersurface with coordinates Y are non-local.

5 Momentum eigenstates

Having formulated the classical action and e.o.m., the next natural step is to look for various partial solutions to them. Of special importance are solutions that correspond to momentum eigenstates. We restrict ourselves here to the simplest cases of a scalar field and massless arbitrary spin fields, for which the momentum operator takes a particularly simple form.

5.1 Scalar field

Let us construct momentum eigenstates for the scalar field $s = 0$. In this case the module (3.6) is

$$\Phi^{s=0}(Y,\tau) = \Phi(y\bar{y},\tau), \tag{5.1}$$

and the momentum operator (3.20) reduces to

$$P^{s=0}_{\alpha\dot{\alpha}} = \frac{i(\varsigma+3/2)}{(\varsigma+1)(\varsigma+2)}\partial_\alpha\bar{\partial}_{\dot{\alpha}} + \frac{i}{(\varsigma+1/2)} y_\alpha\bar{y}_{\dot{\alpha}}\frac{\partial^2}{\partial\tau^2}. \tag{5.2}$$

We have to solve an equation

$$P_{\alpha\dot{\beta}}\Phi_p(Y,\tau) = p_{\alpha\dot{\beta}}\Phi_p(Y,\tau) \tag{5.3}$$

with some momentum $p_{\alpha\dot{\beta}}$, $p^2 = m^2$.

A natural Ansatz is

$$\Phi_p(Y,\tau) = \Phi_p(-ip_{\alpha\dot{\alpha}}y^\alpha\bar{y}^{\dot{\alpha}})e^{\pm im\tau}, \tag{5.4}$$

where τ-dependence gets fixed by the general solution (4.5) and $p_{\alpha\dot{\alpha}}y^\alpha\bar{y}^{\dot{\alpha}}$ is the only available Lorentz-invariant combination involving Y.

Using that

$$\partial_\alpha \bar{\partial}_{\dot\alpha} f(z_{\beta\dot\beta} y^\beta \bar{y}^{\dot\beta}) = z_{\alpha\dot\alpha}(\varsigma + 1) f' - z^2 y_\alpha \bar{y}_{\dot\alpha} f'', \tag{5.5}$$

where the prime means the derivative with respect to the entire argument of f, one can rewrite (5.3) as an ODE with respect to the variable $u = -ip_{\alpha\dot\alpha} y^\alpha \bar{y}^{\dot\alpha}$

$$u\Phi''(u) + (\frac{3}{2} - u)\Phi'(u) - 2\Phi(u) = 0. \tag{5.6}$$

This arises from the terms in (5.3), proportional to $p_{\alpha\dot\beta}$. Strictly speaking, there is one more ODE coming from (5.3), which is generated by terms proportional to $y_\alpha \bar{y}_{\dot\alpha}$, but it represents a differential consequence of (5.6).

(5.6) is the Kummer's equation. Its solution regular at $u = 0$ is the confluent hypergeometric function

$$\Phi(u) = {}_1F_1(2; \frac{3}{2}; u). \tag{5.7}$$

Thus, momentum-$p_{\alpha\dot\beta}$ eigenstate of the scalar field is

$$\Phi_p(Y, \tau) = {}_1F_1(2; \frac{3}{2}; -ipy\bar{y}) e^{\pm im\tau}. \tag{5.8}$$

5.2 Massless fields

For a massless spin-s field the module is (3.26). It contains two $\pm s$ helicities and for both of them the momentum operator reduces to

$$P^{m=0}_{\alpha\dot\beta} = \frac{i(\varsigma + 3/2)}{(\varsigma + s + 1)(\varsigma - s + 2)} \partial_\alpha \bar{\partial}_{\dot\beta}. \tag{5.9}$$

Introducing a polarization vector $\varepsilon_{\alpha\dot\beta}$, orthogonal to the null momentum $p_{\alpha\dot\beta}$, $p^2 = 0$,

$$\varepsilon_{\alpha\dot\beta} p^{\alpha\dot\beta} = 0, \tag{5.10}$$

we choose following Ansätze for negative and positive helicites

$$\Phi^-_{p,\varepsilon}(Y, \tau) = (i\varepsilon_{\alpha\dot\beta} p_\alpha{}^{\dot\beta} y^\alpha y^\alpha)^s \Psi(-ipy\bar{y}) e^{\pm i\mu\tau}, \tag{5.11}$$

$$\Phi^+_{p,\varepsilon}(Y, \tau) = (i\varepsilon_{\beta\dot\alpha} p^\beta{}_{\dot\alpha} \bar{y}^{\dot\alpha} \bar{y}^{\dot\alpha})^s \Psi(-ipy\bar{y}) e^{\pm i\mu\tau}. \tag{5.12}$$

Here we made use of a μ-deformed realization of P^2 (4.7). Then for Ψ one gets, analogously to the scalar field case, the following Kummer's equation

$$u\Psi''(u) + (\frac{3}{2} + s - u)\Psi'(u) - 2\Psi(u) = 0 \tag{5.13}$$

whose regular at $u = 0$ solution is

$$\Psi(u) = {}_1F_1(2; \frac{3}{2} + s; u). \tag{5.14}$$

6 Conclusion

In the paper we proposed a new way of implementing bosonic UIR of $4d$ Poincaré group. We presented them as bunches of scalar fields on $5d$ space with coordinates $\{Y^A, \tau\}$ and found appropriate realizations for Poincaré generators. These realizations possess some distinguishing features: the mass operator P^2 is independent of spinor coordinates Y, so that equations of motion become ODE in a Lorentz-invariant proper time τ and follow from a simple manifestly Poincaré-invariant action. Thus, our construction demonstrates a separation of variables: e.o.m. governs the evolution in τ, while Y parameterize the space of Cauchy data. The translation generators are local differential operators in τ, but non-local in Y, hence τ-evolution is local, while translations act non-locally on the Cauchy hypersurface spanned by Y.

The simple form of e.o.m. allowed us to write down their general solutions. Those contain two branches, corresponding to different sign-dependence on τ, similarly to positive and negative energy branches in the standard QFT approach. We found a Poincaré-invariant inner product, which is positive-definite for one of the branches.

For massless fields we modified the mass operator by introducing an IR-regulator. This allowed us to have bounded in τ solutions and the split into two branches. This modification is manifestly Poincaré-invariant and is possible due to the large ambiguity in the form of the mass operator, caused by the separation of variables. Our construction is non-gauge, as we work directly with helicity-expanded fields: the bunch of scalar fields mentioned before represents a bunch of helicities of a spin-s representation, connected by Poincaré transformations. On the zero-mass shell $\pm s$-helicity components form closed subrepresentations, so 'gauge-fixing' reduces to direct putting all intermediate-helicity components to zero.

We also found the momentum-eigenstate solutions for the simplest cases of a scalar field and massless fields. They have the form of the confluent hypergeometric functions.

The construction, proposed in the paper, poses many problems for further research. One of the most urgent is to develop appropriate canonical structures and to define an analogue of the canonical quantization procedure, regarding that some necessary elements are already presented (a classical action, distinguished in a Lorentz-invariant way coordinate τ that governs the evolution, two branches of classical solutions etc). Other interesting directions include considering fermionic and infinite-spin representations as well as supersymmetric extensions, generalizations to (A)dS backgrounds and, the most important, introducing interactions. The problem of interactions, in its turn, immediately rise many questions: can one formulate a systematic procedure of looking for Poincaré-invariant vertices? what happens to the separation of τ and Y variables at the nonlinear level? how does the Y-nonlocality of Poincaré transformations affect the perturbative analysis? One may hope that answering these questions will provide us with new powerful formalism for studying higher-spin theories.

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