Quantum advantage and stability to errors in analogue quantum simulators

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Several quantum hardware platforms, while being unable to perform fully fault-tolerant quantum computation, can still be operated as analogue quantum simulators for addressing many-body problems. However, due to the presence of errors, it is not clear to what extent those devices can provide us with an advantage with respect to classical computers. In this work we consider the use of noisy analogue quantum simulators for computing physically relevant properties of many-body systems both in equilibrium and undergoing dynamics. We first formulate a system-size independent notion of stability against extensive errors, which we prove for Gaussian fermion models, as well as for a restricted class of spin systems. Remarkably, for the Gaussian fermion models, our analysis shows the stability of critical (gapless) models at zero temperature which have long-range correlations. Furthermore, we analyze how this stability may lead to a quantum advantage, for the problem of computing the thermodynamic limits of many-body models, in the presence of a constant error rate and without any explicit error correction.

I. INTRODUCTION

Quantum information processing systems hold the promise of solving a number of problems in physics and computer science faster than their classical counterparts [1, 2]. However, most quantum algorithms with theoretical performance guarantees require a fault-tolerant quantum computer [3–5]. While in principle possible, implementing a fault tolerant quantum computer is a technological challenge that could still take a long time to solve. This has motivated several investigations trying to identify both quantum algorithms, as well as physically relevant computational problems, that can be addressed by quantum hardware in the near term and without any explicit error correction.

Analog quantum simulators, wherein a target Hamiltonian is mimicked by an experimentally controllable system, have shown some promise in solving problems arising in many-body physics in the near term [6–8]. From a theoretical standpoint, several previous works have developed the notion of universal quantum simulators [9–11]. More practically, they offer several distinct advantages in solving many-body problems. First, in many of those problems, there is no need for a universal gate set to implement the target Hamiltonian, and thus the requirements are much milder than those to build a quantum computer. Furthermore, the simulation can be performed just by letting the system evolve, without the need to Trotterize the evolution into a quantum circuit, which typically incurs in a rapid proliferation of errors during the algorithm [7, 8, 12–14]. Moreover, since one is typically interested in the thermodynamic limit, this only requires obtaining the expectation values of certain intensive quantities, like the energy or magnetization per lattice site. Even if the quantum state of the simulator is orthogonal to the ideal one, those quantities may still be correct, or have a small error, raising the expectation that the degree of robustness required to be much lower than for other problems in quantum computing.

These expectations make quantum simulators very promising in providing some advantage with respect to classical computers when addressing typical quantum many-body problems that appear in different branches of physics. However, the standard notion of quantum advantage [15–20], that deals with the computational effort required to solve a problem as a function of the system size, must be revisited since one may be interested in the thermodynamic limit as opposed to a finite system [21, 22]. Additionally, the fact that quantum simulators do not implement error correction casts doubts on their applicability and has to be carefully taken into account. More specifically, analog quantum simulators in most applications try to implement a target Hamiltonian H which is typically geometrically local i.e. $H = \sum_\alpha h_\alpha$, with $\|h_\alpha\| \leq 1$. Each term h_α is implemented by engineering local interactions between different subsystems in the quantum simulator. In practice, however, each of these local interactions are slightly different from their target. Consequently, the quantum simulator implements a perturbed Hamiltonian $H' = H + \delta \sum_\alpha v_\alpha$, where $\|v_\alpha\| \leq 1$ and δ measures the "hardware error" incurred by the simulator locally. While a well designed experimental setup can, in principle, achieve $\delta \ll 1$, the total error between H' and H grows with n, the number of qubits in the quantum simulator (i.e. the problem size) — $\|H' - H\| \sim \delta\text{poly}(n)$. Due to this proliferation of errors, the error in the state of the quantum simulator, relative to the target state, both for dynamics or thermal equilibrium, will only be small if $\delta < O(1/\text{poly}(n))$ — this could potentially limit the applicability to quantum simulators to only small-scale problems.

In this paper we show that many physically relevant

problems are stable and avoid this worst-case proliferation of errors, and thus can be addressed with the help of quantum simulators without implementing any error correction. We also propose a notion of quantum advantage, in the presence of errors, for such problems, where the figure of merit is the computational time to obtain an intensive quantity in the thermodynamic limit with a hardware-limited precision.

In order to address the stability with respect to noise, we consider the quantum simulation of both equilibrium and non-equilibrium problems. We first study geometrically local Gaussian fermion models, and show that $k-$local observables (or their weighted averages) are stable for the problem of constant-time dynamics. Furthermore, when considering the ground state and Gibbs state of Gaussian fermion models, we show that translationally invariant $k-$local observables are stable without any restrictive assumptions on the spectrum of the model — our results hold not only for gapped models, but also for gapless models and without any assumption on the frustration freeness of the model. Finally, we also consider spin systems with geometrically local interactions, and show that well-known locality properties [23–27] of these models imply stability of local observables in constant-time dynamics, ground states and Gibbs states but with more restrictive assumptions on the model (e.g. the Hamiltonian being stably gapped for ground states, or exhibiting exponential clustering of correlations for Gibbs states). Additionally, for Gaussian fermion models, we numerically verify that the the ground state problem is, in fact, stable to errors in both the gapped and gapless cases, as is theoretically predicted by our stability analysis. Based on these results, we hypothesize that many physically relevant, gapped or gapless, models could potentially be within the reach of near-term analogue quantum simulators.

Finally, we turn to the question of whether the class of problems that are stable under errors on quantum simulators also provide some advantage over the best known classical algorithms for solving the same problems. Here, instead of framing the algorithm run-time as a function of system-size (i.e. the number of spins/fermions) and the desired precision, we treat the system-size as a truncation parameter introduced in approximating the more physically relevant thermodynamic limit [21, 22]. This allows us to express the run-time of the classical and quantum algorithms in terms of a single parameter — the precision in the approximated thermodynamic limit. We argue that for many problems for which we have evidence of stability to errors, we also expect an advantage, with respect to precision, in using the quantum simulator — in the absence of errors, for the problem of constant-time dynamics this advantage is superpolynomial with respect to precision and exponential with respect to evolution time in two or higher dimensional local Hamiltonians, and for ground state computation it is exponential in desired precision. In the presence of errors, there is a fundamental limit to the precision that can be achieved by the quantum simulator which is determined by the hardware error — the advantage of the quantum simulator can then be assessed by scalings of the classical run-time required to achieve this hardware-limited precision with the hardware error.

II. STABILITY OF ANALOGUE QUANTUM SIMULATORS

Abstractly, an analogue quantum simulation can be considered to configure a target Hamiltonian H on $n-$spins and measure an observable O on a state ρ_H associated with this Hamiltonian (e.g. a state obtained from an initial product state after evolution for finite time under H, or the Gibbs state/ground state of H). The target Hamiltonian can be expressed as a sum of terms each of which correspond to interactions between groups of spins i.e.

$$H = \sum_\Lambda h_\Lambda. \tag{1}$$

In an experimentally realistic setting, in the process of configuring this target Hamiltonian, an error can be incurred in each term i.e. the Hamiltonian that is implemented on the hardware is instead given by

$$H' = \sum_\Lambda h'_\Lambda, \tag{2}$$

where we are only guaranteed that $\|h_\Lambda - h'_\Lambda\| \leq \delta$ for all Λ and for some δ — δ can be considered to be a measure of the 'hardware' error incurred in the quantum simulator. We note that since the number of terms in the Hamiltonian in Eqs. 1 and 2 will typically be $\text{poly}(n)$, we can only guarantee that $\|H - H'\| \leq \delta\text{poly}(n)$ — consequently, for the analogue quantum simulator to be accurate without error correction, we would require the hardware error to scale as $1/\text{poly}(n)$. This would typically make it intractable to obtain thermodynamic limits accurately with analogue quantum simulators since the hardware error would have to be made very small.

An alternative viewpoint would be to ask if there are certain interesting many-body problems for which a good estimate for the thermodynamic limit can be produced with a hardware with constant errors. This motivates us to look for 'quantum simulation tasks' which are stable to an error in each term of the Hamiltonian, as made precise in the following definition.

Definition 1 (Stable quantum simulation task). *The quantum simulation task on n spins of measuring an observable O_n in a state ρ_{H_n} associated with a target Hamiltonian H_n is said to be stable if the corresponding state $\rho_{H'_n}$ for the perturbed Hamiltonian H'_n satisfies*

$$|\text{Tr}(O_n \rho_{H_n}) - \text{Tr}(O_n \rho_{H'_n})| \leq f(\delta),$$

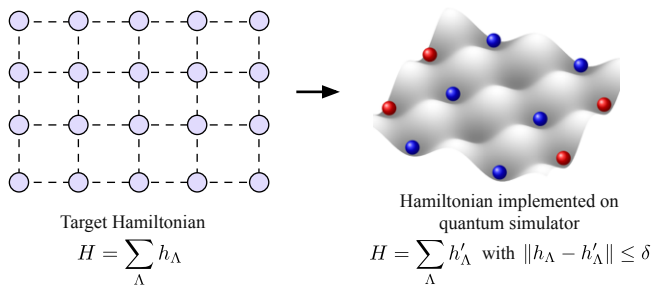*for some f, independent of n, such that $f(\delta) \to 0$ as $\delta \to 0$.*

FIG. 1. Schematic depiction of our error model for analogue quantum simulator. A target Hamiltonian H, expressed as sum of Hamiltonian terms modelling interactions between groups of spins, when implemented on an analogue quantum simulator would have an error δ per term. The parameter δ thus controls the accuracy of the hardware implementation.

Thus, if a quantum simulation task is stable as per this definition, we can hope to be able to estimate the thermodynamic limit of the observable on a quantum simulator with the hardware error determined entirely by the precision desired in the thermodynamic limit. These problems would not require the hardware error to be scaled down with system size even in the absence of error correction, and we can consider them to be problems that analogue quantum simulators can conceivable solve in the near term.

The stability of quantum simulation tasks for many-body physics problems, while hard to prove rigorously, can be intuitively understood. For simplicity, consider the specific setting in which both the target Hamiltonian H and $V = H' - H$ are translationally invariant — in this case, if the thermodynamic limit of an observable (e.g. a local observable) O in a non-equilibrium or equilibrium state associated with the translationally invariant Hamiltonian $H(s) = H + sV$ exists and is a continuous function of s, then it can be seen that it is indeed stable in the sense of definition 1. However, this argument falls short of a full proof since in an actual experiment, even for target Hamiltonians that are translationally invariant, the presence of errors can make it translationally varying thus making it hard to define its thermodynamic limit. In particular, disorder can induce Anderson or many-body localization which can even result in local observables being unstable. In the following sections, however, we show that for several many-body problems, physically interesting observables and order parameters are stable even to translationally varying errors in the Hamiltonian.

III. STABILITY OF GAUSSIAN FERMION MODELS

To make further progress on whether many-body simulation tasks can be considered to be stable in the sense defined above, we consider physically relevant setting of free-fermion models. We will consider fermions arranged

on a $d-$dimensional lattice with L sites in each direction $\mathbb{Z}_L^d$, and at each site we have D fermionic modes — we denote by c_x^α for $x \in \mathbb{Z}_L^d, \alpha \in \{1, 2 \ldots 2D\}$ the Majorana operators associated with each site x. We consider a target Hamiltonian with local interactions given by

$$H = \sum_{\substack{x,y \in \mathbb{Z}_L^d \\ d(x,y) \leq R}} \sum_{\alpha,\beta=1}^{2D} h_{x,y}^{\alpha,\beta} c_x^\alpha c_y^\beta. \tag{3}$$

where R is the range of interaction, and we assume that all interaction strengths are bounded by a constant, $|h_{x,y}^{\alpha,\beta}| \leq J$. Due to local hardware errors, the quantum simulator instead implements a perturbed free-fermion Hamiltonian H',

$$H' = \sum_{\substack{x,y \in \mathbb{Z}_L^d \\ d(x,y) \leq R}} \sum_{\alpha,\beta=1}^{2D} h_{x,y}^{\prime\alpha,\beta} c_x^\alpha c_y^\beta, \tag{4}$$

such that

$$|h_{x,y}^{\alpha,\beta} - h_{x,y}^{\prime\alpha,\beta}| \leq \delta.$$

Generically, we expect $\|H - H'\| = \Theta(n)$, where $n = 2DL^d$ is the total number of fermions. However, in the following subsections, we show the stability of the quantum simulation task of measuring several important classes of observables, which include typical intensive order parameters, in both dynamical and equilibrium states of these models.

A. Finite-time dynamics

We are interested in the expectation value of Gaussian observables O which are either $k-$local, i.e. they act on a set $\mathcal{S} \subseteq \mathbb{Z}_L^d$ of k sites

$$O_0 = \sum_{x,y \in \mathcal{S}} \sum_{\alpha,\beta=1}^{2D} o_{x,y}^{\alpha,\beta} c_x^\alpha c_y^\beta, \tag{5}$$

or weighted averages of $k-$local Gaussian observables i.e. are of the form $\sum_{i=1}^{M} w_i O_i$, where O_i is of the form Eq. 5, $\sum_{i=1}^{M} |w_i| = 1$ and M can possibly grow with $n \sim L^d$.

We consider an arbitrary Gaussian initial state, and let the target state ρ_H be the state obtained on evolving it with H for time t. We show in Appendix B that

Proposition 1. *The quantum simulation task of measuring $k-$local Gaussian observables, or their weighted sums, for constant-time dynamics under a spatially local Gaussian Hamiltonian is stable with $f(\delta) = O(t\delta)$.*

We point out that the dependence of the error between the observable in perturbed and unperturbed models on t is independent of the dimensionality

of the lattice d — this result is thus stronger than what would be expected simply from locality, wherein the error would be expected to grow as $t \times$ (Number of sites in the light cone at time t) $\propto t^{d+1}$ — we revisit this in section IV.

B. Equilibrium

We next study the stability properties of the ground state as well as the Gibbs state. We again consider the target Hamiltonian described while studying the stability of constant-time dynamics, but make an additional physically reasonable assumption on the density of modes of the target Hamiltonian H.

Assumption 1. *We assume that the number of eigenfrequencies n_η of H, which are eigenvalues of the matrix $h_{x,y}^{\alpha,\beta}$ defining the target Hamiltonian, lying in the interval $[-\eta, \eta]$ satisfy the upper bound*

$$n_\eta \leq n f_h(\eta) + \kappa(\eta, n), \qquad (6)$$

where $n = DL^d$ is the number of fermionic modes, $f_h(\eta)$ is some function which $\to 0$ as $\eta \to 0$ and $\kappa(\eta, n)$ is $o(n)$ for any fixed η.

Physically, this assumption formalizes the expectation that any frequency interval would have an extensive number of eigenmodes lying within it — we expect this to be true for most physically relevant models. In particular, for translationally invariant models, such an estimate would be true with $f_h(\eta)$ being governed by the derivatives of the dispersion relation of the free-fermion bounds near $\omega = 0$.

The observables we consider while analyzing ground states are translationally invariant Gaussian observables generated $k-$locally, i.e., if O_0 is a $k-$local observable of the form of Eq. 5, then we consider observables of the form

$$O = \frac{1}{n} \sum_{x \in \mathbb{Z}_L^d} \tau_x(O_0),$$

where τ_x is the observable O_0 translated by x. Considering the target state ρ_H to be the ground state of the free-fermion Hamiltonian, we then obtain the following proposition (proved in appendix C)

Proposition 2. *The quantum simulation task of measuring translationally invariant Gaussian observables generated $k-$locally, in the ground state of a spatially local free-fermion model whose density of modes satisfies Eq. 6 is stable with $f(\delta) = O(\sqrt{\delta}) + f_h(O(\delta^{1/4}))$.*

We note that the stability result above holds very generally, with only a mild assumption on the density of modes of the free-fermion model. In particular, it holds for free-fermion models which are not gapped, i.e. the energy separation between the ground state and the first excited state decreases with n, as long as the observable under consideration is translationally invariant.

The translational invariance of the observables considered here is key to obtaining the stability result for ground states — translationally varying observables need not be stable, even if they are intensive and spatially local. A simple counter-example here is that of Anderson localization — consider H to be a 1D translationally invariant tight-binding model i.e. $H = \sum_{i=1}^{n-1}(a_{i+1}^\dagger a_i + a_i^\dagger a_{i+1})$, with errors $\sum_{i=1}^n \delta_i a_i^\dagger a_i$ where v_i is chosen uniformly at random between $[-\delta, \delta]$. In the absence of error, the ground state of H is completely delocalized across the spin-chain. In the presence of errors, no matter how small, this model is known to be localized. Now, for every $\delta_1, \delta_2 \ldots \delta_n$ and , consider the translationally varying intensive observable $O_{\delta_1, \delta_2 \ldots \delta_n}$ given by the average particle numbers on $\Theta(1/\delta)$ sites around the site where the ground state is localized. This observable, when measured in the delocalized ground state of the unperturbed Hamiltonian H, yields an expected value of 0 as $n \to \infty$. On the other hand, in the ground state of the perturbed localized model it will yield an expected value of $\Theta(1)$. Thus, not all translationally varying observables can be stable, even if we restrict ourselves to free-fermion models, with the observables being intensive and spatially local.

Finally, we consider the Gibbs state, $\rho_H = e^{-\beta H}/\text{Tr}(e^{-\beta H})$ where the inverse-temperature β is a constant independent of n, corresponding to the spatially local free-fermion Hamiltonian above, and again study the stability of translationally invariant Gaussian observables that are generated $k-$locally. We show in appendix D that

Proposition 3. *The quantum simulation task of measuring translationally invariant Gaussian observables generated $k-$locally, in the Gibbs state at inverse-temperature β of a spatially local free-fermion is stable with $f(\delta) = O(\beta\sqrt{\delta})$.*

We point out that, in contrast to the corresponding result for ground state, this stability result corresponding to the Gibbs state does not rely on an assumption on the density of modes of the target Hamiltonian. However, $f(\delta)$ grows with β, so this result does not directly imply the stability of the ground state since β would in general have to be increased with n for the Gibbs state to approximate the ground state.

As an illustrative example to corroborate proposition 2, we numerically study the ground state of the free fermionic Su–Schrieffer–Heeger (SSH) model on n fermions with periodic boundary condtion:

$$H_{\text{SSH}}[J] \equiv \sum_{i=1}^n t_i a_i^\dagger a_{i+1} + \text{H.c.}, \qquad t_i = \begin{cases} 1 & i \text{ odd}, \\ J & i \text{ even}. \end{cases} \quad (7)$$

where $a_{n+1} \equiv a_n$. This model displays a (topological) phase transition at $J = 1$, where the gap closes as $1/n$, and is gapped otherwise. We consider measuring the en-
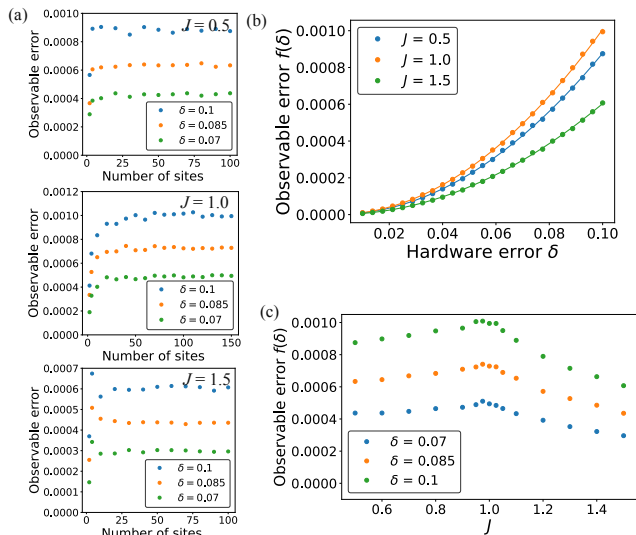
FIG. 2. Numerical study of the error in intensive translationally varying observable in the SSH model. The observable that we study here is $O = H_{\mathrm{SSH}}[J]/n$, where $H_{\mathrm{SSH}}[J]$ is the Hamiltonian of the ideal SSH model (Eq. 7) (a) The error in the expected value of the observable O in the ground state between the perturbed and unperturbed Hamiltonians, as a function of δ, the hardware-error, and the number of sites n. For both gapped ($J = 0.5, 1.5$) and gapless ($J = 1.0$) cases, we see that the error in O becomes independent of n as $N \to \infty$. (b) Numerically extracted error between the perturbed and unperturbed models for $n \to \infty$ as a function of δ, and its fit with δ^2. (c) The error between the perturbed and unperturbed model as a function of J — for the same hardware error δ, this error peaks at $J = 1$ which is also the point at which the gap in the unperturbed model closes. All the errors are computed by averaging over 500 random instances of perturbed models.

ergy density $H_{\mathrm{SSH}}[J]/n$ of the unperturbed Hamiltonian in the presence and absence of errors. Figure 2(a) shows impact of changing system size on this energy density — we see that for both gapped ($J = 0.5, 1.5$) and gapless ($J = 1.0$) cases, the errors in the energy density becomes independent of n as $n \to \infty$, verifying the expectation in proposition 2. Furthermore, we show the error in the energy density for large n as a function of δ in Fig. 2(b) and see that, consistent with proposition 2, this error $\to 0$ as $\delta \to 0$. Finally, Fig. 2(c) shows this error as a function of J — we see the error peak near $J = 1$ (i.e. the point where the gap in the Hamiltonian closes), and that it is smaller for values of J where the model is gapped.

IV. STABILITY IN QUANTUM SPIN SYSTEMS

While for free fermion models, we could prove tight stability results with minimal assumptions on the model, looser stability results hold for quantum spin systems under more restrictive assumptions on their many-body spectrum. In this section, we outline several quantum simulation tasks whose stability is a consequence of several locality results that have already been established in the many-body literature [23–30]. Our emphasis here is to cross-examine these results from the standpoint of quantum simulation, and discuss their implications on noisy quantum simulators.

We restrict ourselves to spin systems that are spatially local — n spins are arranged on a lattice $\mathfrak{L} \subset \mathbb{Z}^d$ in $d-$dimensions, and the Hamiltonian is of the form

$$H = \sum_{x \in \mathfrak{L}} h_x,$$

where h_x only acts on spins within a distance R from x, and $\|h_x\| \leq J$ for all x. We consider different physically relevant observables, and states (either ground states, Gibbs states or states generated by constant-time dynamics) that are associated with H, and study the stability of the quantum simulation problem of computing a local observable in these states.

A. Finite time dynamics

Consider first the setting where an initial state $(|0\rangle \langle 0|)^{\otimes n}$ is evolved under the Hamiltonian H for a time t that is independent of n — we are thus interested in $\rho_H = e^{-iHt}(|0\rangle \langle 0|)^{\otimes n}e^{iHt}$. We consider observables O that are either local (i.e. they only act non-trivially on an $n-$independent subset of spins), or of the form

$$O = O_1 O_2 \ldots O_k, \tag{8}$$

where $O_1, O_2 \ldots O_k$ are local and k is independent of n, or

$$O = \sum_{i=1}^{M} w_i O_i,$$

where $\sum_{i=1}^{M} |w_i| = 1$, O_i are of the form of Eq. 8 and M can possibly grow with n. For these observables, the stability of this quantum simulation task can be stated:

Proposition 4. *The quantum simulation task of measuring $k-$local observables, or their weighted averages, for constant-time dynamics under a spatially local Hamiltonian is stable with $f(\delta) = O(t^{d+1}\delta) + O(t\delta \log^d(\delta^{-1}))$.*

The proof of this result, provided in appendix E is straightforward and uses the Lieb-Robinson's bounds [23, 24] to approximate the Heisenberg picture evolution of local observables with that corresponding to the Hamiltonian truncated within their light cones, and then uses a perturbation theory bound on this truncated Hamiltonian.

Note also that for large t, the error between the target observable and the observable measured on the quantum simulator grows as t^{d+1} — this is looser than the corresponding result in Gaussian fermion models (proposition

1), where the error grows only as t. Furthermore, since this error bound becomes loose with t, it prevents us from using this result to understand the stability of quantum simulation tasks which are aimed to studying the ground state properties of many-body Hamiltonians, and use the adiabatic algorithms [7] that evolve a Hamiltonian for $t \sim \text{poly}(n)$. To address these problems, we consider the stability of the ground state and Gibbs states of local Hamiltonians using a different approach.

B. Equilibrium

We next study the stability of the task of simulating the ground state and Gibbs states of H. We first consider the problem of measuring k−local observables, and their weighted averages, in the ground state. We assume that H is gapped i.e. the energy difference between the ground state and the first excited state is $\geq \Delta$ independent of n. Furthermore, we also assume that the Hamiltonian remains gapped in the presence of errors i.e. any Hamiltonian H' such that

$$H' = \sum_{x \in \mathfrak{L}} h'_x \text{ with } \|h_x - h'_x\| \leq \delta \text{ for all } x, \qquad (9)$$

is gapped with gap Δ — we refer to such a target Hamiltonian H to be *stably gapped* with gap Δ. We point out that the stability of the gap in the presence of errors or perturbations has only been shown for certain frustration free models with local topological order [31–34], although we posit it as a reasonable physical assumption. The stability of this quantum simulation task is a direct consequence of the spectral flow method, or quasi-adiabatic continuation, proposed by Hastings and co-workers [25, 26] which shows that map taking ground state of H to the ground state of H' can be well-approximated by a local unitary. We thus obtain the following proposition and we include a proof of this in appendix F.

Proposition 5. *The quantum simulation task of measuring k-local observables, or their weighted averages, in the ground state of stably gapped spatially local Hamiltonians is stable with $f(\delta) = O(\delta)$.*

We point out, again, that the choice of observables here is crucial to having a stable quantum simulation problem — even for stably gapped Hamiltonians, non-local observables would in general not yield a stable quantum simulation task. A counter-example here would be the unperturbed Hamiltonian $H = \sum_{i=1}^{n} Z_i$, and a perturbed Hamiltonian $H' = \sum_{i=1}^{n} Z_i + \delta X_i$. It can be seen that both these models are gapped for small δ — the gap of H is 2, and gap of H' is $2\sqrt{1+\delta^2}$. However, the non-local observable $O = (|0\rangle \langle 0|)^{\otimes n}$, when measured in the ground state of H is 1 and when measured in the ground state of H' evaluates to $(1+\delta^2)^{-n/2}$ — thus, at any $\delta > 0$, the error in this observable $\rightarrow 1$ as $n \rightarrow \infty$ showing that it is unstable in the sense of definition 1.

We next consider the Gibbs state of H at some temperature β independent of n, and assume that the Gibbs state has an exponential clustering of correlation [27] i.e. for any two observables A, B separated by distance l,

$$|\langle A \otimes B \rangle - \langle A \rangle \langle B \rangle| \leq \|A\| \|B\| O(e^{-c_2 l}),$$

for some constant c_2. Furthermore, as in the case of ground states, we assume that this exponential clustering of correlations is stable i.e. for any perturbation H' of H of the form of Eq. 9, this exponential clustering property holds. In this case, we again obtain that the problem of measuring 1−local observables and their weighted averages is stable.

Proposition 6. *The quantum simulation task of measuring 1-local observables, or their weighted averages, in the Gibbs state of spatially local Hamiltonians with stable exponential clustering of correlations is stable with $f(\delta) = O(\log^{1-1/d}(1/\delta)e^{-\Omega(\log^{1/d}(1/\delta))})$.*

V. DISCUSSION: QUANTUM ADVANTAGE IN THE THERMODYNAMIC LIMIT

In many problems in physics, the quantities of interest are the value of certain intensive observables in the thermodynamic limit i.e. when the system size $n \rightarrow \infty$. Thus, typical complexity arguments behind quantum advantage that rely on the scaling of the computational effort with the system size have to be revised and adapted to this limit. One way of doing that is to concentrate on the computational complexity of obtaining the thermodynamic limit as a function of a prescribed error, ε [21, 22]. A quantum advantage can then be obtained if the computational time on a quantum computer required to reach a precision of ε scales more favorably with it than with a classical computer. For instance, in a quantum computer this scaling can be $T_{\mathrm{Q}} = \text{poly}(1/\varepsilon)$ while in a classical one $T_{\mathrm{cl}} = \exp(\log^2(1/\varepsilon))$ or $T_{\mathrm{cl}} = \exp(O(1/\varepsilon))$, in which case we will respectively have a superpolynomial or exponential quantum advantage. Furthermore, even this notion of advantage relies on error-free computation — for quantum simulation in the presence of errors, a new notion of quantum advantage needs to be introduced.

In this section we discuss potential advantage in quantum simulations for dynamics and ground state problems. First, we will consider the ideal (and unrealistic) situation where there are no errors. For dynamical problems we will show that a quantum advantage can be reached for the problem of computing the thermodynamic limit in the sense described above. For ground state problems, we will adopt a different perspective than the one analyzed in Refs. [21, 22], where they consider the most general scenario and even a quantum computer may require an exponential time in terms of the required error. Instead, we will consider promise problems where

the many-body Hamiltonian fulfills physically reasonable assumptions and discuss potential quantum advantage. Most of our discussion is based on technical results related to locality estimates and analysis of adiabatic quantum algorithms that can already be found in literature — we focus on describing the implications of these results for quantum simulation.

Next, we will consider the more realistic scenario where the quantum simulator is subject to errors as analyzed in the previous sections. There, we will argue that quantum advantage has to be based on the classical computational time required to obtain the value of the intensive observable in the thermodynamic limit with the same error as with the quantum simulator. Within this framework, we will rely on the robustness results of the previous sections, to argue that quantum advantage can persist for stable quantum simulation tasks even in the presence of errors.

A. Quantum advantage for ideal quantum simulators

Let us consider a many-body model defined as a family of Hamiltonians $\{H_n\}_{n\in\mathbb{N}}$ and observables $\{O_n\}_{n\in\mathbb{N}}$, where H_n, O_n act on $n-$spins. We are interested in the expected value of O_n in a many body quantum state generated by the Hamiltonian H_n, ρ_{H_n}. We furthermore assume that the models and observables under consideration have a well-defined thermodynamic limit i.e.

$$O^* := \lim_{n\to\infty} \mathrm{Tr}(\rho_{H_n} O_n) \qquad (10)$$

exists.

Examples of physically relevant ρ_{H_n} would include the states obtained on evolving an initial state of n spins after some time t, the Gibbs state corresponding to H_n or the ground state of H_n. We will mostly focus on families of local Hamiltonians

$$H_n = \sum_{x\in\mathbb{Z}_{L_n}^d} h_x, \qquad (11)$$

where h_x acts on spins in a cube of unit length with x being its lower left corner and satisfies $\|h_x\| \leq J$ for some $J > 0$. Typically, one considers translationally invariant models, where h_x is just some h_0 translated to the position x. We will also only consider here observables O_n are then acting on a single or few lattice sites.

Now, given a precision ε, we can then choose $n(\varepsilon)$ such that

$$|O^* - \mathrm{Tr}(O_n \rho_{H_n})| \leq \varepsilon, \qquad (12)$$

i.e. approximate the thermodynamic limit by a finite-size problem. The run-time of a quantum simulation or a classical simulation for the finite-size problem can thus be expressed in terms of the precision ε demanded in the thermodynamic limit. This allows us to then compare
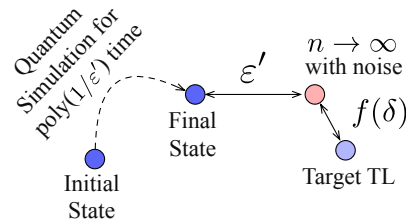

FIG. 3. An erroneous quantum simulator can obtain the thermodynamic limit of the perturbed model to a precision ε' in time $\mathrm{poly}(1/\varepsilon')$ — this thermodynamic limit, however, can have an error $f(\delta)$ from the target thermodynamic limit in the presence of hardware error δ.

the scalings of the run-time of these algorithms with the precision ε, and declare an algorithm *to have an advantage in precision* compared to others depending on their respective scaling.

We note that there are classical algorithms which directly operate in the thermodynamic limit [35, 36]. We will not consider them here, since for ground state problem in dimensions higher than two, it is not possible to give rigorous scalings, and for dynamical ones they give the same scaling as the ones considered here.

1. Finite time quantum dynamics

We consider first an initial product state $|0\rangle^{\otimes n}$, and for $t > 0$, we take $\rho_{H_n} = e^{-iH_n t}\big(|0\rangle\langle 0|\big)^{\otimes n}e^{iH_n t}$. The observable of interest is a *local observable O*.

The existence of the thermodynamic limit is obtained directly using the Lieb-Robinson bounds [23, 24], and it also characterizes the error between the thermodynamic limit and its finite-size approximation.

Lemma 1. *For the problem of computing a local observable O^* after evolving $|0\rangle^{\otimes n}$ for finite-time t with respect to a nearest-neighbour Hamiltonian exists and fulfills Eq. 12 for $n = \Theta(\log^d(1/\varepsilon) + t^d)$.*

On a quantum simulator, one would evolve $n = \Theta(\log^d(1/\varepsilon) + t^d)$ qubits for a time t and measure the observable. The procedure would be repeated M times, so that the error in the observable would decrease as $1/M^2$. In order to obtain an error ε, the computation time would then be $O(t/\varepsilon^2)$

On a classical computer, in general, we would have to compute the quantum state on $O(\log^d(1/\varepsilon) + t^d)$ spins. Using exact methods, this would require a run-time that scales as $2^{O(\log^d(\varepsilon^{-1})) + t^d}$. For a fixed time t, in two or higher dimension, this run-time would be superpolynomial in $1/\varepsilon$ and exponential in t. One could also use other classical methods that obtain a better scaling with the required precision. For instance, in Ref. [37] a method based on cluster expansion was analyzed and which the

computational time scales as poly(n), although super-exponentially with time. Compared to such a classical algorithm, the quantum advantage is super-exponential with respect to evolution time t.

2. Ground state

Consider next the problem of estimating local observables in the ground state of many-body Hamiltonians in the thermodynamic limit. The convergence rate of a finite-size approximation of a local observable to its thermodynamic limit for the ground state problem is expected to depend on whether the model is gapped (and hence the ground state has exponentially decaying correlations [28, 29]) or gapless. While it is generally hard to rigorously characterize the rate of convergence of a finite-size approximation to the thermodynamic limit for ground states, it is physically reasonable to consider the following cases:

- **Logarithmic Convergence:** $n = \log^d(1/\varepsilon)$, where d is the lattice dimension.

- **Power-law Convergence:** $n = \text{poly}(1/\varepsilon)$.

The first case is expected to hold for gapped models. In fact, under the local topological quantum order condition [31, 33, 34], one can easily show that this is the case. The second case is expected to hold for critical (gapless) models. For instance, for the Gaussian fermionic Hamiltonians analyzed in the previous section this is the case, under very general conditions for the Fermi surface.

Furthermore, to ensure that there is a quantum algorithm that reaches the ground state we will assume that H_n is adiabatically connected to a family of Hamiltonians H_n^0 with efficiently preparable ground states with a minimal gap, Δ_n along the adiabatic path, fulfilling

- **Power-law Gap:** $\Delta_n \geq 1/\text{poly}(n)$.

This ensures that using the adiabatic algorithm one can reach the ground state within an error ε in a time $T_Q = \text{poly}(n, 1/\varepsilon)$, or $T_Q = \text{poly}(1/\varepsilon)$ if framed entirely in terms of the precision of the thermodynamic limit[1]. This scaling of the gap is expected to hold for critical systems as well.

We expect that, generically, classical algorithms to compute a ground state observable would be no better than exactly diagonalizing a finite-size Hamiltonian. Thus, under the Power-Law Gap and Logarithmic convergence conditions, a classical computer would

———

[1] Note that we could have also considered a constant gap, in which case, at least under certain further assumptions on the Hamiltonian, it is provably possible to reach the ground state in $T_Q = \text{polylog}(n, 1/\varepsilon)$ [38].

require time $\exp(O(\log^d(1/\varepsilon)))$ and thus one would obtain a superpolynomial quantum advantage using a quantum simulator. Instead, under the Power-Law Convergence conditions, a classical computer would require time $\exp(O(\text{poly}(1/\varepsilon)))$ thus yielding an exponential quantum advantage using a quantum simulator.

B. Quantum advantage for noisy quantum simulators

As argued above, in the absence of errors, we expect a quantum simulator to provide an advantage over classical algorithms for computing thermodynamic limits. The presence of errors, however, sets a limit on the precision that can be obtained by quantum simulators. In the error models analyzed in previous sections, such a limit is determined by the hardware error δ. In time $\text{poly}(1/\varepsilon')$, the quantum simulator is expected to compute the thermodynamic limit of the perturbed model to a precision ε' — the precision of the target thermodynamic limit obtained is thus upper bounded by

$$\varepsilon \leq O(\max(\varepsilon', f(\delta))),$$

where $f(\delta)$ is given in Definition 1. From this, we immediately see that in the presence of hardware errors, we do not gain any benefit in precision by running the quantum simulator beyond a time needed to obtain $\varepsilon' = f(\delta)$ with which we can expect to compute the target thermodynamic limit to a precision of $O(f(\delta))$. Based on the stability analysis of the previous subsections, we generically expect $f(\delta) = \text{poly}(\delta)$ for most stable many-body simulation tasks, and thus to be able to obtain the thermodynamic limit to a precision of $O(\text{poly}(\delta))$, determined entirely by the hardware error δ, in quantum-simulation time $O(\text{poly}(1/\delta))$.

A numerical illustration of this analysis is shown in Fig. 4 — here, we use the adiabatic quantum algorithm to find the energy density observable in the ground state of the critical SSH model (i.e. Eq. 7 with $J = 1$). Figure 4(a) shows the convergence of the energy density observable, in the absence of errors, to its thermodynamic limit — we see that a power-law convergence is obtained, as physically expected for gapless models. In Fig. 4(b), we use a system-size that yields a precision of $O(f(\delta))$, as determined by the stability bounds on the ground-state of this model, and simulate an adiabatic algorithm to find the ground state in the presence of hardware error. We see that, in the presence of errors, the accuracy in the achieved precision is fundamentally limited by the hardware precision δ — Fig. 4 shows the run-time of the adiabatic algorithm as a function of this hardware-limited precision. We see that this run-time scales polynomially with $1/\varepsilon$, where ε is the hardware-limited precision that is achieved by the adiabatic algorithm.

To define a notion of advantage in the presence of noise, we can now compare the classical and quantum run-times
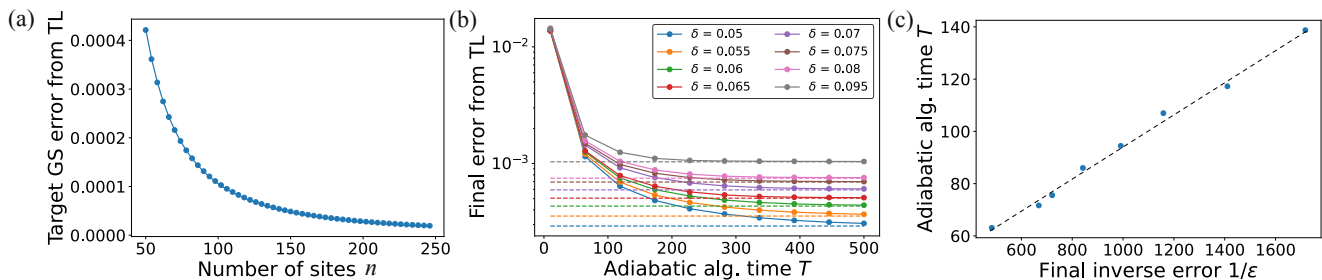
FIG. 4. Numerical study of quantum adiabatic algorithm in the presence of error. We consider using the adiabatic algorithm to find the energy density observable for the critical SSH model in the thermodynamic limit (TL). (a) Convergence of the energy density to the thermodynamic limit as $n \to \infty$ — the scaling of ε with n reveals a power-law scaling that is expected for gapless models. (b) The adiabatic algorithm in the presence of hardware errors — the quantity being plotted is the error of the noisy adiabatic algorithm from the thermodynamic limit of the noiseless model. The precision achieved by the adiabatic algorithm is fundamentally limited by hardware errors. (c) The final precision (ε) achieved by an adiabatic algorithm in the presence of errors as a function of the adiabatic algorithm run-time, confirming that $T \sim \text{poly}(1/\varepsilon)$ as expected from our analysis. Thus, on decreasing the hardware error, the error achievable by the noisy quantum simulator decreases and the run-time of the quantum algorithm increases at-most polynomially.

needed to achieve this hardware-limited precision. Assuming $f(\delta) = \text{poly}(\delta)$, it follows from the discussion in the previous subsection that we would need classical run-times that are either superpolynomial or exponential in $\text{poly}(1/\delta)$ to achieve the precision that can be achieved by quantum simulators in time $\text{poly}(1/\delta)$. If δ is decreased by a constant factor, then the run-time of the quantum simulator will only increase at-most polynomially with this factor, while the run-time of the classical simulator will increase by a super-polynomial or exponential factor.

More specifically, in the case of dynamics, we can evolve the noisy quantum-simulator for time $t = O(\delta^{-1/(d+2)})$ and, from proposition 4, we obtain an error in local observable that $\to 0$ as $\text{poly}(\delta)$. All known classical algorithms, based either on exact methods or cluster expansion, will have a run-time which is $\exp(O(\text{poly}(1/\delta)))$ thus one would obtain an exponential quantum advantage in the sense of the previous paragraph. Similary, for the ground state, depending on whether we have logarithmic or power-law convergence, one would obtain a superpolynomial or an exponential quantum advantage. We summarize our expectation of noisy quantum advantage for the quantum simulation task of dynamics and ground states in the table 1. This table is based on both the stability results provided in the previous two sections, as well as the scaling conjectured above for the run-times of classical algorithms.

VI. CONCLUSION

We have considered both the stability and quantum advantage of using near-term analogue quantum simulators for thermodynamic limits of many-body problems in physics. Based on both existing theoretical results in many-body literature, and new technical results for free-fermion models, we argue that many physically relevant many-body problems are stable to a constant rate of er-

ror on the quantum hardware being used to solve them and thus are accessible in near-term experiments. We also hypothesize that these algorithms have an advantage, with respect to the obtained precision, in computing thermodynamic limits of many-body problems. Our formulation and results provides some evidence for near-term analogue quantum simulators being useful for solving many-body problems.

Extending the stability results for gapless models to the case of quantum spins, or non-Gaussian fermionic systems is an immediate open problem suggested by our work. While previous work by Hastings [25] indicates that, under some assumption on the density of states of the many-body model, such a stability result could hold for gapless spin systems, it would be interesting to see if restricting observables to being translationally invariant could help improve these results. Furthermore, we have assumed a coherent error model for analyzing stability of quantum simulation task, future directions could include expanding these results to incoherent error models. Finally, any progress on a rigorous analysis of impact of errors on adiabatic quantum algorithms would help lay current experimental efforts on a strong theoretical foundation.

Problem	Stability	Quantum Run-Time w.r.t hardware-limited precision	Classical Run-Time w.r.t. hardware-limited precision	Conjectured Advantage with respect to hardware errors
Constant time dynamics for $t = \Theta(\delta^{-1/(d+1)})$	Provable with $f(\delta) = \tilde{O}(\text{poly}(\delta))$	$\tilde{O}(\text{poly}(1/\delta))$	$\exp(O(\text{poly}(1/\delta)))$	Exponential
Ground states of stably gapped Hamiltonians with logarithmic convergence and power-law gap	Provable with $f(\delta) = O(\delta)$	$O(\text{poly}(1/\delta))$	$\exp(O(\log^d(\delta^{-1}))$	Superpolynomial
Ground states of gapless Hamiltonians with power-law convergence and power-law gap	Conjectured from free-fermions with $f(\delta) = O(\text{poly}(\delta))$	$O(\text{poly}(1/\delta))$	$\exp(O(1/\delta))$	Exponential

TABLE I. Summary of stability, hardware-limited precision, classical and quantum run-times for thermodynamic limits in many-body problems with respect to the hardware error δ, together with their conjectured quantum advantage in the presence of noise.

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Appendix A: Notational preliminaries

The following is a list of notations that we will use for the mathematical proofs in the following appendices. Here v denotes a vector, and A a matrix.

- $\|v\|_p \equiv \left(\sum_i v_i^p\right)^{\frac{1}{p}}$ denotes the p-norm of the vector v ($p = 2$ is the Euclidean norm, for which we drop the subscript, $\|\cdot\| \equiv \|\cdot\|_2$). In the $p \to \infty$ limit, it becomes the max norm, $\|v\|_\infty \equiv \max_i |v_i|$.

- $\|A\|_{\mathrm{op}}$ is the operator norm of A, i.e. the norm induced by the Euclidean vector norm, or the ∞-Schatten norm. $\|A\|_{\mathrm{op},1}$ denotes the trace norm, i.e. the 1-Schatten norm of A.

- $\mathrm{vec}(A)$ denotes the vectorization of A, i.e. the vector whose components are the matrix elements of A.

- Unless otherwise mentioned, $\|v\|$, where v is a vector, will denote their ℓ^2 norm and $\|O\|$, where O is an operator, will be its operator norm.

In the proofs relating to gaussian fermion models, for a hermitian operator O expressed as a quadratic form over the Majorana operators c_x^α,

$$O = \sum_{x,y \in \mathbb{Z}_L^d} \sum_{\alpha,\beta=1}^{2D} o_{x,y}^{\alpha,\beta} c_x^\alpha c_y^\beta,$$

we will denote by $\tilde{O}$ the matrix of coefficients $o_{x,y}^{\alpha,\beta}$, with the indices (x,α) corresponding to the rows and (y,β) corresponding to the columns. We will assume, without loss of generality and unless otherwise mentioned, that $\tilde{O}$ is a hermitian matrix with purely imaginary matrix elements.

Appendix B: Proof of proposition 1 (Dynamics of free-fermion models)

We will need the following lemma on matrix norms:

Lemma 2. *Let M be an $n \times n$ matrix such that (i) $|M_{ij}| \le \delta, \forall i, j$, and (ii) any row or column of M has at most r nonzero elements. Then $\|M\|_{\mathrm{op}} \le r\delta$.*

Proof. Let $v = (v_i)$ be a vector, and denote $\mathcal{R}_i \equiv \{j | M_{ij} \ne 0\}, \mathcal{C}_j \equiv \{i | M_{ij} \ne 0\}$. Then,

$$
\begin{aligned}
\|Mv\|^2 &= \sum_{i=1}^{n} \left| \sum_{j \in \mathcal{R}_i} A_{ij} v_j \right|^2 \\
&\le \sum_{i=1}^{n} \left(\sum_{j \in \mathcal{R}_i} |A_{ij}|^2 \sum_{k \in \mathcal{R}_i} |v_k|^2 \right) \\
&\le r\delta^2 \sum_{i=1}^{n} \sum_{k \in \mathcal{R}_i} |v_k|^2 = r\delta^2 \sum_{k=1}^{n} \sum_{i \in \mathcal{C}_k} |v_k|^2 \\
&\le r^2 \delta^2 \|v\|^2 \implies \|M\|_{\mathrm{op}} \le r\delta.
\end{aligned}
$$

$\square$

Lemma 3. *Given bounded Hermitian operators H and H', for any bounded operator O*

$$
\left\| e^{iH't} O e^{-iH't} - e^{iHt} O e^{-iHt} \right\|_{\mathrm{op}} \le 2\|O\|_{\mathrm{op}} \|H - H'\|_{\mathrm{op}} t.
$$

Proof. Consider the operator $\tilde{O}(t) \equiv e^{-iHt} e^{iH't} O e^{-iH't} e^{iHt}$. Note that

$$
\frac{d}{dt} \tilde{O}(t) = i \left[e^{-iHt}(H'-H) e^{iH't} O e^{-iH't} e^{iHt} - e^{-iHt} e^{iH't} O e^{-iH't} (H'-H) e^{iHt} \right],
$$

and consequently,

$$
\left\| \frac{d}{dt} \tilde{O}(t) \right\| \le 2\|H - H'\| \|O\|.
$$

We then immediately obtain that

$$
\left\| e^{iH't} O e^{-iH't} - e^{iHt} O e^{-iHt} \right\| \le \int_0^t \left\| \frac{d}{ds} \tilde{O}(s) \right\| ds \le 2\|O\| \|H - H'\| t.
$$

$\square$

Proof (of proposition 1). Let $\tilde{H}, \tilde{O}$ be the $2DL^d \times 2DL^d$ matrices with elements $h_{x,y}^{\alpha,\beta}, o_{x,y}^{\alpha,\beta}$ (see Eqs. (3)-(5)), and let Γ_0 be the correlation matrix in the Majorana basis corresponding to the initial state ρ_0,

$$
(\Gamma_0)_{x,y}^{\alpha,\beta} \equiv \frac{1}{2} \mathrm{Tr} \left(\rho_0 [c_x^\alpha, c_y^\beta] \right). \tag{B1}
$$

After evolving for time t, the expectation values of the unperturbed and perturbed observable are given by

$$
\langle O(t) \rangle_H = \mathrm{Tr} \left(\Gamma_0 e^{i\tilde{H}t} \tilde{O} e^{-i\tilde{H}t} \right) \text{ and } \langle O(t) \rangle_{H'} = \mathrm{Tr} \left(\Gamma_0 e^{i\tilde{H}'t} \tilde{O} e^{-i\tilde{H}'t} \right).
$$

Using $|\mathrm{Tr}\, A| \le \mathrm{Tr}\, |A| = \|A\|_{\mathrm{op},1}$, and Hölder's inequality, $\|AB\|_{\mathrm{op},1} \le \|A\|_{\mathrm{op},1} \|B\|_{\mathrm{op}}$, we have

$$
|\langle O(t) \rangle_H - \langle O(t) \rangle_{H'}| \le \|\tilde{O}\|_{\mathrm{op},1} \|e^{-i\tilde{H}t} \Gamma_0 e^{i\tilde{H}t} - e^{-i\tilde{H}'t} \Gamma_0 e^{i\tilde{H}'t}\| \le 2t \|\tilde{O}\|_{\mathrm{op},1} \|\Gamma_0\| \|\tilde{H} - \tilde{H}'\|,
$$

where the second inequality follows from lemma 3. It only remains to prove that the last expression is independent of lattice-size L. Observe that since O only acts on k sites, $\tilde{O}$ has at most $2kD$ nonzero eigenvalues, thus

$$
\|\tilde{O}\|_{\mathrm{op},1} \le 2kD \|\tilde{O}\|.
$$

Furthermore, for any correlation matrix, $\|\Gamma_0\| \le 1$. Finally, by assumption, $\tilde{H} - \tilde{H}'$ is a matrix that satisfies the conditions of lemma 2 (its elements are bounded by δ, and due to H, H' being local, it has at most $2DR^d$ nonzero elements per row and column), hence

$$
\|\tilde{H} - \tilde{H}'\| \le 2DR^d \delta \tag{B2}
$$

and the proposition follows. $\square$

Appendix C: Proof of proposition 2 (Ground states of local gaussian fermionic models)

In this appendix we will prove the stability of the expectation value of a translationally invariant, k-locally generated Gaussian observable on the ground state of a quadratic Hamiltonian. We first provide a lemma that uses the translation invariance of a local observable to provide an error bound.

Lemma 4. *Consider a quadratic operator O which is translationally invariant and expressible as*

$$O = \frac{1}{n} \sum_{x \in \mathbb{Z}_L^d} \tau_x(O_0),$$

where $n = L^d$ is the number of sites in $\mathbb{Z}_L^d$, O_0 is a quadratic operator with a support on at most k sites and τ_x is a super-operator that translates an operator by x, then for any quadratic operator A_0,

$$\left| \mathrm{Tr}(O^\dagger A_0) \right| \leq \frac{4D^2 k}{n} \|\tilde{O}_0\| \|\tilde{A}_0\|_{\mathrm{op},1}.$$

Proof. We note that

$$\mathrm{Tr}(\tilde{O}^\dagger \tilde{A}_0) = \frac{1}{n} \sum_{x \in \mathbb{Z}_L^d} \mathrm{Tr}\big(\tilde{O}_0 \tau_x^\dagger(\tilde{A}_0)\big) = \frac{1}{n} \sum_{x \in \mathbb{Z}^d} \mathrm{Tr}\big(\tilde{O}_0 \tau_{-x}(\tilde{A}_0)\big).$$

Define $A \equiv n^{-1} \sum_{x \in \mathbb{Z}_L^d} \tau_{-x}(A_0)$. We note that A is translationally invariant on the underlying lattice — consequently, if F is the $n \times n$ Fourier transform matrix, then $F_D = F \otimes I_{2D}$ block diagonalizes $\tilde{A}$ i.e. $F_D \tilde{A} F_D^\dagger$ will be a block diagonal matrix with n blocks of size $2D \times 2D$. Then we use

$$\left| \mathrm{Tr}(\tilde{O}^\dagger \tilde{A}_0) \right| = \left| \mathrm{Tr}(F_D^\dagger \tilde{O}^\dagger F_D F_D^\dagger \tilde{A} F_D) \right| \leq \left\| \mathrm{vec}\big(F_D^\dagger \tilde{O}_0 F_D\big) \right\|_\infty \left\| \mathrm{vec}\big(F_D^\dagger \tilde{A} F_D\big) \right\|_1$$

where we have applied Hölder's inequality to the norms of the vectorized matrices. Now we bound each of the factors in the right hand side. Since the operator O_0 has support only k sites, $\tilde{O}_0$ only has $2Dk \times 2Dk$ non-zero elements. Suppose that Π_{O_0} is a diagonal matrix with 1s on the entries that correspond to non-zero elements of $\tilde{O}_0$ — it then follows that $\tilde{O}_0 = \Pi_{O_0} \tilde{O}_0 \Pi_{O_0}$. We further note that if f_i is the ith column of F_D then

$$\left\| \mathrm{vec}\big(F_D^\dagger \tilde{O}_0 F_D\big) \right\|_\infty = \sup_{i,j} \left| f_i^\dagger \Pi_{O_0} \tilde{O}_0 \Pi_{O_0} f_j \right| \leq \|\tilde{O}_0\| \sup_{i,j} \|\Pi_{O_0} f_i\| \|\Pi_{O_0} f_j\| = \frac{2Dk}{n} \|\tilde{O}_0\|,$$

where we have used that each entry of F_D has magnitude $1/\sqrt{n}$ since it is the Fourier transform matrix. Next, since $F_D^\dagger \tilde{A} F_D$ is block diagonal with N $2D \times 2D$ blocks, and labelling by $A_1, A_2 \ldots A_N$ these blocks, we obtain that

$$\left\| \mathrm{vec}(F_D^\dagger \tilde{A} F_D) \right\|_1 = \sum_{i=1}^N \|\mathrm{vec}(A_i)\|_1 \leq 2D \sum_{i=1}^N \|A_i\|_{\mathrm{op},1} = 2D \left\| F_D^\dagger \tilde{A} F_D \right\|_{\mathrm{op},1} = 2D \left\| \tilde{A} \right\|_{\mathrm{op},1}.$$

where we have used $\|\mathrm{vec}(M)\|_1 \leq n \|M\|_{\mathrm{op},1}$ for an $n \times n$ matrix[2]. Finally, since $A = \sum_{x \in \mathbb{Z}_L^d} \tau_{-x}(A_0)/N$, it follows that $\|\tilde{A}\|_{\mathrm{op},1} \leq \|\tilde{A}_0\|_{\mathrm{op},1}$. Combining the above estimates, the lemma statement follows. $\square$

The correlation matrix Γ of the ground state of a quadratic Hamiltonian H with matrix of coefficients $\tilde{H}$ (see Eq. (3)) is given by

$$\Gamma = \mathrm{sign}(\tilde{H}),$$

where $\mathrm{sign}(x) = x/|x|$ for $x \neq 0$ and 0 for $x = 0$[3]. The sign function applied on a matrix is to be understood as an operator function i.e. as a function acting on the eigenvalues of the argument while keeping the eigenvectors

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[2] To see this, let $\sigma_{ij} \equiv \mathrm{sign}(M_{ji})$. Then $\|\sigma\|_{\mathrm{op}} \leq n \|\mathrm{vec}(\sigma)\|_\infty = n$, and $\|\mathrm{vec}(M)\|_1 = \mathrm{Tr}(\sigma M) \leq \|\sigma\|_{\mathrm{op}} \|M\|_{\mathrm{op},1} \leq n \|M\|_{\mathrm{op},1}$.

[3] The reader may be familiar with the equivalent formulation in terms of complex fermions, where the function to be applied to the Hamiltonian matrix to obtain the correlation matrix of the ground state is of the Heaviside type, such that it populates negative energy states and depopulates positive energy states. The function of the sign function in the language of Majorana fermions is exactly analogous.
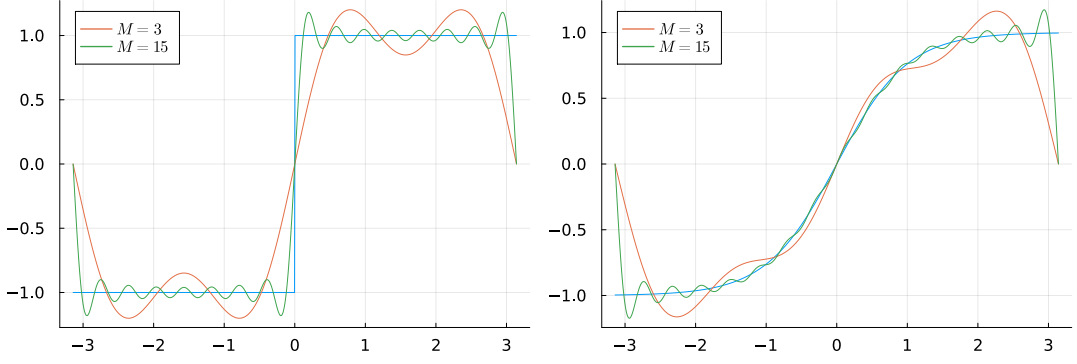
FIG. 5. (left) Truncated Fourier series approximation $\text{sign}_M(x)$ to the $\text{sign}(x)$ function, used in the proof of proposition 2. (right) Truncated Fourier series approximation $t_M(x)$ to the $\tanh(\beta x)$ function (for $\beta = 1$), used in the proof of proposition 3.

unchanged. Our proof will rely on a Fourier series approximation to the sign function. Within the interval $(-\pi, \pi)$, we will investigate the approximation of $\text{sign}(x)$ with $\text{sign}_M(x)$, where

$$\text{sign}_M(x) = \sum_{n=-M}^{M} c_n e^{inx} \text{ where } c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{sign}(x)e^{-inx}dx.$$

To analyze the error between $\text{sign}_M(x)$ and $\text{sign}(x)$, it is convenient to express $\text{sign}_M(x)$ in terms of the Dirichlet kernel,

$$\text{sign}_M(x) \equiv \int_{-\pi}^{\pi} D_M(x - y)\text{sign}(y)dy,$$

where

$$D_M(x) \equiv \frac{1}{2\pi} \sum_{n=-M}^{M} e^{-inx} = \frac{1}{2\pi} \frac{\sin[(M + 1/2)x]}{\sin(x/2)}.$$

Below, we provide two technical lemmas about the sign_M function — one that quantifies the approximation error between it and the exact sign function, and the next that quantifies the maximum value of the sign_M function. Both of these lemmas will be used for the perturbation theory analysis of the free-fermion ground state problem.

Lemma 5. *For all $\eta \leq |x| \leq \pi - \eta$ and $M > 0$,*

$$\left|\text{sign}(x) - \text{sign}_M(x)\right| \leq \frac{1}{M} + \frac{1}{M\eta}.$$

Proof. We first consider $x \in [\eta, \pi]$. We note that

$$\text{sign}_M(x) = \int_0^{\pi} D_M(x - y)dy - \int_0^{\pi} D_M(x + y)dy.$$

Now, since $\int_{-\pi}^{\pi} D_M(y)dy = 1$, we obtain that

$$\int_0^{\pi} D_M(x - y)dy = 1 - \int_0^{\pi} D_M(x + y)dy,$$

and thus

$$|\text{sign}_M(x) - \text{sign}(x)| = 2\left|\int_0^{\pi} D_M(x + y)dy\right|.$$

Next, we apply integration by parts to obtain

$$\int_0^{\pi} D_M(x + y)dy = \frac{1}{\pi(2M + 1)}\left(\frac{\cos((M + 1/2)(\pi + x))}{\cos(x/2)} + \frac{\cos((M + 1/2)(\pi + x))}{\sin(x/2)} - \frac{1}{2}\int_0^{\pi} \frac{\cos((M + 1/2)(x + y))\cos((x + y)/2)}{\sin^2((x + y)/2)}dy\right),$$

and therefore

$$\left| \int_0^\pi D_M(x+y)dy \right| \leq \frac{1}{\pi(2M+1)} \left(\frac{1}{|\cos(x/2)|} + \frac{1}{|\sin(x/2)|} + \frac{1}{2} \int_0^\pi \frac{|\cos((x+y)/2)|dy}{\sin^2((x+y)/2)} \right)$$

$$\leq \frac{2}{\pi(2M+1)} \left(\frac{1}{|\cos(x/2)|} + \frac{1}{|\sin(x/2)|} - 1 \right),$$

where in the last step we have used the integral

$$\frac{1}{2} \int_x^{\pi+x} \frac{|\cos(y/2)|}{\sin^2(y/2)} dy = \frac{1}{\sin(x/2)} + \frac{1}{\cos(x/2)} - 2.$$

Now, for $x \in (\eta, \pi/2)$, $|\cos(x/2)| \geq 1/\sqrt{2}$ and $|\sin(x/2)| \geq x/\pi \geq \eta/\pi$. Therefore, we obtain that

$$\left| \int_0^\pi D_M(x+y)dy \right| \leq \frac{2}{\pi(2M+1)} \left(\sqrt{2} + \frac{\pi}{\eta} - 1 \right).$$

While this bound is true for $x \in [\eta, \pi/2]$, we note that both $\text{sign}, \text{sign}_M$ satisfy $f(x) = f(\pi - x)$ for $x \in [0, \pi]$ and consequently this bound also holds for $x \in [\pi/2, \pi - \eta]$. Finally, since for both $\text{sign}, \text{sign}_M$, $f(x) = -f(-x)$, it follows that this bound holds for $[-\pi + \eta, -\eta] \cup [\eta, \pi - \eta]$. $\square$

Lemma 6. *For all $x \in [-\pi, \pi]$,*

$$|\text{sign}_M(x)| \leq 5.$$

Proof. This proof is an adaptation of the standard technique based on Riemann integration that is used to treat Gibbs phenomena in Fourier analysis. We repurpose that technique to provide error bounds as a function of M instead of just concentrating on the asymptotic limit $M \to \infty$. Again, we only consider $x \in [0, \pi/2]$, and extend the bound on $|\text{sign}_M(x)|$ to the remaining interval by symmetry. We divide the interval $[0, \pi/2]$ into $[0, \alpha_0/M] \cup [\alpha_0/M, \pi/2]$, where α_0 is a constant that we pick later.

Consider first $x \in [\alpha_0/M, \pi/2]$. An application of lemma 5 yields

$$|\text{sign}_M(x)| \leq 1 + \frac{2}{\pi(2M+1)} \left(\sqrt{2} - 1 + \frac{\pi M}{\alpha_0} \right).$$

For large M, this bound scales as $\sim 1/\alpha_0$ and thus does not allow us to provide an upper bound on $\text{sign}_M(x)$ for x close to 0. For this, we use the representation of $\text{sign}_M(x)$ as a Fourier series which approximates a Riemann integral of $\sin(t)/t$. Consider $x \in [0, \alpha_0/M)$ and let $\alpha = xM$ $(\alpha \leq \alpha_0)$. Note that

$$\text{sign}_M(x) = \frac{2}{\pi} \sum_{k \in [1,M] | k \text{ is odd}} \frac{2}{k} \sin\left(\frac{k\alpha}{M} \right)$$

To bound the term in the summation, we observe that it is an approximation of the Riemann integral of $\sin(\alpha x)/x$ in the interval $[0, 1]$. In particular, since $\sup_{x \in \mathbb{R}} |(\sin x/x)'| \leq 2$, Taylor's theorem yields that

$$\left| \sum_{k \in [1,M] | k \text{ is odd}} \frac{2}{k} \sin\left(\frac{k\alpha}{M} \right) - \int_0^1 \frac{\sin \alpha x}{x} dx \right| \leq \frac{4\alpha^2}{M} \leq \frac{4\alpha_0^2}{M}.$$

Finally, we note that

$$\int_0^1 \frac{\sin \alpha x}{x} dx \leq \alpha \leq \alpha_0.$$

Thus, we obtain that for $x \in [0, \alpha_0/M)$,

$$|\text{sign}_M(x)| \leq \alpha_0 + \frac{4\alpha_0^2}{M}.$$

Thus, for the entire interval $[0, \pi/2]$, we obtain that

$$|\text{sign}_M(x)| \leq \max\left(\alpha_0 + \frac{4\alpha_0^2}{M}, 1 + \frac{2}{\pi(2M+1)}\left(\sqrt{2} - 1 + \frac{\pi M}{\alpha_0}\right)\right).$$

Since this holds for any α_0, we choose $\alpha_0 = 1$. We then obtain that

$$|\text{sign}_M(x)| \leq \max\left(1 + \frac{4}{M}, 1 + \frac{2}{\pi(2M+1)}\left(\sqrt{2} - 1 + \pi M\right)\right) \leq 5 \text{ for } M \geq 1.$$

$\square$

Proof (of proposition 2). The expectation value of the observable O in the ground state of the Hamiltonian H is given by

$$\langle O \rangle_H = \text{Tr}\big(\tilde{O}\, \text{sign}(\tilde{H})\big), \ \langle O \rangle_{H'} = \text{Tr}\big(\tilde{O}\, \text{sign}(\tilde{H}')\big).$$

Without loss of generality, we will assume that $\tilde{H}, \tilde{H}'$ are normalized so that $\|\tilde{H}\|, \|\tilde{H}'\| \leq \frac{\pi}{2}$. This way all the eigenfrequencies lie in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Note that lemma 2 guarantees that this can be done with a constant normalization factor, i.e. one that does not depend on the system size, and does not change the ground state (note however that δ and f_h would have to be rescaled accordingly). Now, from lemma 4, it follows that

$$|\langle O \rangle_H - \langle O \rangle_{H'}| \leq \frac{4D^2 k}{n}\|\tilde{O}_0\|\|\text{sign}(\tilde{H}) - \text{sign}(\tilde{H}')\|_{\text{op},1}.$$

Furthermore,

$$\|\text{sign}(\tilde{H}) - \text{sign}(\tilde{H}')\|_{\text{op},1} \leq$$
$$\|\text{sign}(\tilde{H}) - \text{sign}_M(\tilde{H})\|_{\text{op},1} + \|\text{sign}(\tilde{H}') - \text{sign}_M(\tilde{H}')\|_{\text{op},1} + \|\text{sign}_M(\tilde{H}) - \text{sign}_M(\tilde{H}')\|_{\text{op},1}.$$

We bound each term on the right hand side separately. Consider $\|\text{sign}(\tilde{H}) - \text{sign}_M(\tilde{H})\|_{\text{op},1}$ — denoting by λ_i the eigenvalues of $\tilde{H}$ and for any $\eta > 0$, we can express it as

$$\|\text{sign}(\tilde{H}) - \text{sign}_M(\tilde{H})\|_{\text{op},1} = \sum_{i|\lambda_i \in [-\eta, \eta]} |\text{sign}(\lambda_i) - \text{sign}_M(\lambda_i)| + \sum_{i|\lambda_i \notin [-\eta, \eta]} |\text{sign}(\lambda_i) - \text{sign}_M(\lambda_i)|.$$

The motivation behind splitting the error into these two terms is that, within the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, the approximation of $\text{sign}(\lambda)$ by $\text{sign}_M(\lambda)$ is only good outside the neighbourhood of 0 (see Fig. 5) — consequently, we treat the eigenvalues of $\tilde{H}$ which lie within η radius of 0 separately from the rest. It now follows that from assumption 1 and lemma 6 that

$$\sum_{i|\lambda_i \in [-\eta, \eta]} |\text{sign}(\lambda_i) - \text{sign}_M(\lambda_i)| \leq 6n f_h(\eta) + 6\kappa(\eta, n).$$

Furthermore, from lemma 5,

$$\sum_{i|\lambda_i \notin [-\eta, \eta]} |\text{sign}(\lambda_i) - \text{sign}_M(\lambda_i)| \leq \frac{n}{M}\left(1 + \frac{1}{\eta}\right).$$

Therefore, we obtain that

$$\frac{1}{n}\|\text{sign}(\tilde{H}) - \text{sign}_M(\tilde{H})\|_{\text{op},1} \leq 6f_h(\eta) + 6\frac{\kappa(\eta, n)}{n} + \frac{1}{M}\left(1 + \frac{1}{\eta}\right).$$

We can similarly analyze $\|\text{sign}(\tilde{H}') - \text{sign}(\tilde{H}')\|_{\text{op},1}$. Denote by λ_i' the eigenvalues of $\tilde{H}'$ — it follows from Weyl's theorem that $|\lambda_i - \lambda_i'| \leq \|\tilde{H} - \tilde{H}'\|_{\text{op}} \leq 2DR^d\delta$ (see Eq. (B2)). Consequently, for sufficiently small, but $\Theta(1)$, δ, we obtain that

$$\sum_{i|\lambda_i' \in [-\eta, \eta]} |\text{sign}(\lambda_i') - \text{sign}_M(\lambda_i')| \leq 6n f_h(\eta + 2DR^d\delta) + 6\kappa(\eta + 2DR^d\delta, n),$$

and

$$\sum_{i|\lambda_i'\notin[-\eta,\eta]} |\mathrm{sign}(\lambda_i') - \mathrm{sign}_M(\lambda_i')| \le \frac{n}{M}\left(1 + \frac{1}{\eta}\right).$$

Finally, we consider $\|\mathrm{sign}_M(H) - \mathrm{sign}_M(H')\|_{\mathrm{op},1} \le n\|\mathrm{sign}_M(H) - \mathrm{sign}_M(H')\|$. Now, denoting by $\{c_m\}_{m\in\mathbb{Z}}$ the Fourier series components of sign function, then

$$\|\mathrm{sign}_M(H) - \mathrm{sign}_M(H')\| \le \sum_{m=-M}^{M} |c_m|\|e^{im\tilde{H}} - e^{im\tilde{H}'}\| \le \sum_{m=-M}^{M} |mc_m|\|\tilde{H} - \tilde{H}'\|.$$

Using the explicit expression for c_m, we can immediately conclude that $|mc_m| = 2/\pi$ when m is odd, and 0 when m is even. Therefore, we obtain that

$$\|\mathrm{sign}_M(H) - \mathrm{sign}_M(H')\| \le \frac{2(M+1)}{\pi}\|\tilde{H} - \tilde{H}'\| \le \frac{2(M+1)}{\pi}2DR^d\delta.$$

Combining all of these estimates, we obtain that

$$\frac{1}{n}\|\mathrm{sign}(\tilde{H}) - \mathrm{sign}(\tilde{H}')\|_{\mathrm{op},1} \le cM\delta + \frac{2}{M}\left(1 + \frac{1}{\eta}\right) + 6\big(f_h(\eta) + f_h(\eta + c'\delta)\big) + 6\left(\frac{\kappa(\eta,n)}{n} + \frac{\kappa(\eta + c'\delta,n)}{n}\right).$$

with c, c' constants. Since this is valid for any η and M, choosing $M = \delta^{-1/2}$ and $\eta = \delta^{1/4}$, we obtain the proposition. $\square$

Appendix D: Proof of proposition 3 (Gibbs state of free-fermion models)

The correlation matrix of a thermal state of a quadratic Hamiltonian can be written in terms of the coefficient matrix H of the latter as $\Gamma = \tanh(\beta H)$. Note that the $\beta \to \infty$ limit yields the sign function, which was used in the previous appendix to compute the ground state correlation matrix. Indeed, the reasoning here will be similar to that of appendix C, replacing the sign function with the hyperbolic tangent. The next couple of lemmas discuss the Fourier series approximation of $\tanh\beta x$, defined as

$$t_M(x) \equiv \sum_{n=-M}^{M} c_n e^{inx}, \quad \text{where } c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tanh\beta x e^{-inx}dx.$$

Lemma 7. *For $M \ge 1$, and $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,*

$$|t_M(x) - \tanh\beta x| \le \frac{q(\beta)}{M},$$

where $q(\beta) \equiv 12\pi^2\beta^3 + 2\pi^2\beta^2 + \left(2 + \frac{\pi^2}{2}\right)\beta + \left(\frac{4\sqrt{2}}{\pi} + \frac{\pi^2}{2}\right) = O(\beta^3)$.

Proof. We fix the value of β and let $t(x)$ be the 2π-periodic extension of $\tanh\beta x$

$$t(x) \equiv \tanh\beta(x - 2n\pi), \qquad x - 2n\pi \in [-\pi, \pi], \quad n \in \mathbb{Z} \tag{D1}$$

Once again, it will be convenient to represent $t_M(x)$ in terms of the Dirichlet kernel D_M. We note,

$$t_M(x) = \int_{-\pi}^{\pi} D_M(x-y)t(y)dy = \int_{-\pi}^{\pi} D_M(y)t(x-y)dy = \int_{-\pi}^{\pi} D_M(y)t(x+y)dy,$$

and therefore, using that the Dirichlet kernel is normalized, we write

$$t(x) - t_M(x) = \frac{1}{2}\int_{-\pi}^{\pi} D_M(y)\left(2t(x) - t(x-y) - t(x+y)\right)dy = \int_{0}^{\pi} D_M(y)f_x(y)dy,$$

where in the last step we have defined $f_x(y) \equiv 2t(x) - t(x - y) - t(x + y)$. In the integration interval $[0, \pi]$, $f_x(y)$ is piecewise smooth with a single jump discontinuity at $y = \pi - x$. We thus split the integral into the two intervals $[0, \pi - x]$ and $[\pi - x, \pi]$ and apply integration by parts in each of them. For the first one,

$$\int_0^{\pi-x} D_M(y) f_x(y) dy = -\frac{1}{\pi} \frac{\cos\left(\left(M + \frac{1}{2}\right) y\right)}{2M + 1} \frac{f_x(y)}{\sin \frac{y}{2}} \Big|_{y=0}^{\pi-x} + \frac{1}{(2M+1)\pi} \int_0^{\pi-x} g_x(y) \frac{\cos\left(\left(M + \frac{1}{2}\right) y\right)}{\sin^2 \frac{y}{2}} dy$$

where $g_x(y) \equiv 2 \sin \frac{y}{2} f_x'(y) - \cos \frac{y}{2} f_x(y)$. To bound this expression, we will use the following properties of the functions $f_x(y), g_x(y)$ on the interval $[0, \pi - x]$, where they are smooth:

$$f_x(0) = f_x'(0) = 0, |f_x(y)| \leq 4, |f_x'(y)| \leq 2\beta, |f_x''(y)| \leq 4\beta^2, |f_x'''(y)| \leq 12\beta^3,$$
$$g_x(0) = g_x'(0) = 0, |g_x''(y)| \leq 24\beta^3 + 4\beta^2 + \beta + 1.$$

These bounds follow from direct computation, and in the case of $g_x(y)$ they are easiest to see when expressed in terms of $f_x(y)$. They imply (via Taylor's theorem with second order remainder) that

$$|g_x(y)| \leq (24\beta^3 + 4\beta^2 + \beta + 1)\frac{y^2}{2}$$

which together with $\sin^2(y) \geq \frac{y^2}{\pi^2}$ will allow us to bound the integral. Putting it all together, we have

$$\left|\int_0^{\pi-x} D_M(y) f_x(y) dy\right| \leq \frac{4\sqrt{2}}{(2M+1)\pi} + \frac{\pi^2}{(2M+1)}(24\beta^3 + 4\beta^2 + \beta + 1)$$

Now we proceed on to the second interval $y \in [\pi - x, \pi]$ and similarly integrate by parts,

$$\int_{\pi-x}^{\pi} D_M(y) f_x(y) dy = -\frac{1}{\pi} \frac{\cos\left(\left(M + \frac{1}{2}\right) y\right)}{2M + 1} \frac{f_x(y)}{\sin \frac{y}{2}} \Big|_{y=\pi-x}^{\pi} + \frac{1}{(2M+1)\pi} \int_{\pi-x}^{\pi} g_x(y) \frac{\cos\left(\left(M + \frac{1}{2}\right) y\right)}{\sin^2 \frac{y}{2}} dy.$$

Now the bound on $g_x(y)$ from Taylor's theorem no longer holds, due to the discontinuity, but since $y = 0$ is not in the integration interval, we can just use the constant bound $|g(x)| \leq 4\beta + 4$ to obtain

$$\left|\int_{\pi-x}^{\pi} D_M(y) f_x(y) dy\right| \leq \frac{4\sqrt{2}}{(2M+1)\pi} + \frac{4}{(2M+1)}(\beta + 1),$$

and putting everything together the lemma follows. $\qquad\square$

Lemma 8. *If $\{c_n\}_{n \in \mathbb{Z}}$ are the Fourier series coefficients of $\tanh \beta x$ in the interval $[-\pi, \pi]$, then for $M \geq 1$*

$$\sum_{n=-M}^{M} |n c_n| \leq 2M(\beta + 1).$$

Proof. This follows by a straightforward manipulation of c_n — note that $c_0 = 0$, and for $n \neq 0$, we obtain from integration by parts that

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tanh \beta x \, e^{-inx} dx = \frac{1}{2\pi} \left(\frac{2i}{n} \tanh \beta\pi \, e^{-in\pi} + \frac{\beta}{in} \int_{-\pi}^{\pi} \frac{e^{-inx}}{\cosh^2 \beta x} dx\right).$$

Consequently,

$$|c_n| \leq \frac{1}{2\pi}\left(\frac{2}{n} + \frac{2\pi\beta}{n}\right) \leq \frac{\beta + 1}{n}.$$

From this bound, the lemma follows. $\qquad\square$

Proof (of proposition 3). We bound the error between $\langle O \rangle_{H,\beta}$ and $\langle O \rangle_{H',\beta}$ using the same procedure as for the ground state (see appendix C) — the proof simplifies significantly because $\tanh \beta x$ does not have a discontinuity near $x = 0$ (unlike the sign function). From lemma 4 it follows that

$$|\langle O \rangle_{H,\beta} - \langle O \rangle_{H',\beta}| \leq \frac{4D^2 k}{n} \|\tilde{O}_0\| \|\tanh \beta \tilde{H} - \tanh \beta \tilde{H}'\|_{\text{op},1}.$$

We again split

$$\|\tanh \beta \tilde{H} - \tanh \beta \tilde{H}'\|_{\text{op},1} \leq$$
$$\|\tanh \beta \tilde{H} - t_M(\tilde{H})\|_{\text{op},1} + \|\tanh \beta \tilde{H}' - t_M(\tilde{H}')\|_{\text{op},1} + \|t_M(\tilde{H}') - t_M(\tilde{H})\|_{\text{op},1}$$

We will assume once again that $\|H\|, \|H'\| \leq \frac{\pi}{2}$, so that from lemma 7, it follows that

$$\|\tanh \beta \tilde{H} - t_M(\tilde{H})\|_{\text{op},1}, \|\tanh \beta \tilde{H}' - t_M(\tilde{H}')\|_{\text{op},1} \leq \frac{n q(\beta)}{M},$$

and

$$\|t_M(\tilde{H}) - t_M(\tilde{H}')\|_{\text{op},1} \leq n \|t_M(\tilde{H}) - t_M(\tilde{H}')\|_{\text{op}} \leq n \sum_{m=-M}^{M} |c_m| \|e^{im\tilde{H}} - e^{im\tilde{H}'}\|_{\text{op}}.$$

Furthermore, from lemmas 3 and 2 we have $\|e^{im\tilde{H}} - e^{im\tilde{H}'}\|_{\text{op}} \leq mc\delta$, where $c = 2DR^d$. Thus, from lemma 8, it follows that

$$\|t_M(\tilde{H}) - t_M(\tilde{H}')\|_{\text{op},1} \leq 2nM(\beta+1)c\delta.$$

Thus, we obtain that for any $M > 1$,

$$|\langle O \rangle_{H,\beta} - \langle O \rangle_{H',\beta}| \leq 4D^2 k \|O_0\|_{\text{op}} \left(\frac{2q(\beta)}{M} + 2(\beta+1)cM\delta \right).$$

choosing $M = \sqrt{q(\beta)/c(\beta+1)\delta}$, we obtain the result. $\qquad\square$

Appendix E: Proof of proposition 4 (Dynamics of locally interacting spin systems)

We will need the following two lemmas:

Lemma 9 (Lieb-Robinson bounds, Ref. [24, 39]). *Given a Hamiltonian defined on a lattice $\mathbb{Z}_L^d$, $H = \sum_{\alpha \in \mathbb{Z}_L^d s} h_\alpha$ such that $\|h_\alpha\| \leq J$ and h_α acts on sites α' such that $d(\alpha, \alpha') \leq R$, then there exist positive constants μ, v that depend only on the lattice such that*

(a) *For any two operators A, B with support S_A, S_B and $l = d(S_A, S_B)$,*

$$\|[A(t), B]\| \leq |S_A| \|A\| \|B\| e^{-\mu l} \left(e^{vJ|t|} - 1 \right),$$

where $A(t) = e^{iHt} A e^{-iHt}$.

(b) *For any local operator O with support S_O, and for $l > 0$,*

$$\|O(t) - O_l(t)\| \leq \|O\| |S_O| e^{-\mu l} \left(e^{vJt} - 1 \right),$$

where $O_l(t) = e^{iH_l t} O e^{-iH_l t}$ with $H_l = H - \sum_{\alpha | d(S_{h_\alpha}, S_O) \geq l} h_\alpha$ being the restriction of the Hamiltonian to a region within distance l of S_O.

Proof (of proposition 4). We consider the Hamiltonians H and H' represented as

$$H = \sum_{\alpha \in \mathbb{Z}_L^d} h_\alpha, \qquad H' = \sum_{\alpha \in \mathbb{Z}_L^d} h'_\alpha.$$

We have, by assumption, $\|h_\alpha - h'_\alpha\|_{\text{op}} \leq \varepsilon$. Thus if $\varepsilon < J$, then for all $\alpha \in \mathbb{Z}_L^d$, $\|h_\alpha\|, \|h'_\alpha\| \leq 2J$. Now,

$$|\langle O \rangle_{H,t} - \langle O \rangle_{H',t}| \leq \sum_{K \in \{H, H'\}} \left\| e^{iKt} O e^{-iKt} - e^{iK_l t} O e^{-iK_l t} \right\| + \left\| e^{iH_l t} O e^{-iH_l t} - e^{iH'_l t} O e^{-iH'_l t} \right\|.$$

We note that by simply counting the number of sites in the support of the truncated Hamiltonian,

$$\|H_l - H'_l\| \leq (R_O + 2R + 2l)^d \varepsilon,$$

where $R_O = \mathrm{diam}(S_O)$. From lemma 3, it then follows that

$$\left\| e^{iH_l t} O e^{-iH_l t} - e^{iH'_l t} O e^{-iH'_l t} \right\| \leq 2\varepsilon t \|O\| (R_O + 2R + 2l)^d.$$

Using this together with the Lieb-Robinson bounds (lemma 9), we obtain that

$$|\langle O \rangle_{H,t} - \langle O \rangle_{H',t}| \leq 2\|O\| |S_O| e^{-\mu l} \left(e^{2vJt} - 1 \right) + 2\varepsilon t \|O\| (R_O + 2R + 2l)^d.$$

Choosing $l = 2vJt/\mu + \log(2J/\varepsilon)/\mu$, we obtain that

$$|\langle O \rangle_{H,t} - \langle O \rangle_{H',t}| \leq \|O\| \left[\frac{\varepsilon}{J} |S_O| \left(1 - e^{-2vJt} \right) + 2\varepsilon t \left(R_O + 2R + \frac{4vJt}{\mu} + \frac{2}{\mu} \log \left(\frac{2J}{\varepsilon} \right) \right)^d \right],$$

which proves the proposition statement. $\qquad\square$

Appendix F: Proof of proposition 5 (Ground states of gapped local Hamiltonians)

We will apply the formalism developed in Ref. for spectral flows for families of gapped Hamiltonians. We are interested in a target spatially local Hamiltonian H, expressed as

$$H = \sum_{x \in \mathfrak{L}} h_x,$$

where h_x acts only on spins with a distance R of $x \in \mathfrak{L}$, and $\|h_x\| \leq 1$. The implemented Hamiltonian H' is assumed to have a similar form,

$$H' = \sum_{x \in \mathfrak{L}} \left(h_x + v_x \right),$$

where $\|v_x\| \leq \varepsilon$ for all $x \in \mathfrak{L}$. We assume that H is stably gapped with gap Δ i.e. any H' of the above form has an energy gap between the ground state and the first excited state that is larger than Δ. We consider the family of Hamiltonians, H_s, for $s \in [0,1]$, defined by

$$H_s = H + s(H' - H) = \sum_{x \in \mathfrak{L}} h_x + sv_x,$$

and note that the assumption of being stably gapped is equivalent to H_s being gapped, with the gap being larger than Δ, for all $s \in [0,1]$. Now, the spectral flow method allows us to construct a unitary $U(s)$ that relates the ground state $|G_{s=0}\rangle$ of $H_{s=0} = H$ to the ground state $|G_s\rangle$ of H_s as provided in the following lemma.

Lemma 10 (From Ref. [26]). *Consider the unitary $U(s)$ obtained from*

$$\frac{d}{ds} U(s) = iD(s)U(s) \text{ where } D(s) = \int_{-\infty}^{\infty} W_\Delta(t) e^{-itH_s} (H' - H) e^{itH_s} dt,$$

where $W_\Delta \in L^1(\mathbb{R})$ is a real valued odd function which satisfies

(a) $|W_\Delta(t)|$ is bounded and satisfies

$$\|W_\Delta\|_\infty = \sup_{t \in \mathbb{R}} |W_\Delta(t)| = \frac{1}{2}. \tag{F1}$$

(b) For $t > 0$, the function $I_\Delta(t) = \int_t^\infty |W_\Delta(s)| ds$ satisfies

$$I_\Delta(t) \leq G(\Delta t), \tag{F2}$$

where $G(x)$ falls off faster than any polynomial as $x \to \infty$.

Then, $|G_s\rangle = U(s)|G_{s=0}\rangle$, where $|G_s\rangle$ is the ground state of $H(s)$.

Proof (of proposition 5). Using this result, we can straightforwardly show the stability of the quantum simulation task of computing a local observable in the ground state of H. To see this, we note that

$$|\langle G_0| O |G_0\rangle - \langle G_s| O |G_s\rangle| = \left|\langle G_0| \left(O - U^\dagger(s)OU(s)\right) |G_0\rangle\right| \leq \|O - U^\dagger(s)OU(s)\| \leq \int_0^s \|[O, D(s')]\| ds'.$$

It then remains to bound $\|[O, D(s')]\|$ — we can do this by following lemma 4.7 in Ref. [26], and we reproduce this below — we start by noting that

$$\|[A, D(s')]\| \leq \sum_{x \in \mathfrak{L}} \left\|\int_{-\infty}^\infty W_\gamma(t)[A, e^{itH_s}v_x, e^{-itH_s}]dt\right\|.$$

For each term in this summation, we further split the integral and bound it as

$$\left\|\int_{-\infty}^\infty W_\Delta(t)[A, e^{itH_s}v_x, e^{-itH_s}]dt\right\| \leq \left\|\int_{|t|\leq T_x} W_\Delta(t)[A, e^{itH_s}v_x e^{-itH_s}]dt\right\| + \left\|\int_{|t|>T_x} W_\Delta(t)[A, e^{itH_s}v_x e^{-itH_s}]dt\right\|.$$

For the first term, which only concerns with $|t| \leq T_x$, we use the Lieb Robinson's bound (lemma 9) and Eq. F1 to obtain

$$\left\|\int_{|t|\leq T_x} W_\Delta(t)[O, e^{itH_s}v_x e^{-itH_s}]dt\right\| \leq \|O\|\|v_x\||S_O|e^{-\mu d(S_O, S_{v_x})} \int_0^{T_x}(e^{vt}-1)dt \leq \|O\|\|v_x\||S_O|\frac{e^{-\mu d(S_O, S_{v_x})}e^{vT_x}}{v}.$$

For the second term for $|t| \geq T_x$, we use Eq. F2 together with the fact that W_Δ is an odd function and the simple bound $\|[O, e^{itH_s}v_x e^{-itH_s}]\| \leq 2\|O\|\|v_x\|$ to obtain that

$$\left\|\int_{|t|\geq T_x} W_\Delta(t)[O, e^{itH_s}v_x e^{-itH_s}]dt\right\| \leq 2\|O\|\|v_x\| \int_{|t|\geq T_x} |W_\Delta(t)| \leq 2\|O\|\|v_x\|G(\Delta T_x),$$

Note that T_x can be arbitrary in the above two estimates — choosing $T_x = \mu d(S_O, S_{v_x})/2v$, we obtain that

$$\left\|\int_{-\infty}^\infty W_\Delta(t)[O, e^{itH_s}v_x, e^{-itH_s}]dt\right\| \leq \|O\|\|v_x\|\left[\frac{|S_O|}{v}e^{-\mu d(S_O, S_{v_x})/2} + 2G\left(\frac{\Delta\mu}{2v}d(S_O, S_{v_x})\right)\right],$$

and therefore, for all $s' \in [0, s]$, we obtain that bound

$$\|[O, D(s')]\| \leq \|O\|\varepsilon \sum_{x \in \mathfrak{L}} \left[\frac{|S_O|}{v}e^{-\mu d(S_O, S_{v_x})/2} + 2G\left(\frac{\Delta\mu}{2v}d(S_O, S_{v_x})\right)\right].$$

Noting that the summand in the above expression decreases faster than any polynomial in $d(S_O, S_{v_x})$, we see that it will be upper bounded by a constant independent of the size of the lattice $\mathfrak{L}$, thus independent of n. This proves the proposition. $\qquad\square$

Appendix G: Proof of proposition 6 (Gibbs state with exponential clustering of correlations)

We begin by presenting a proof of a standard bound on the perturbation of Gibb's states of a Hamiltonian. This can be found in , and we reproduce it here for the convenience of the reader.

Lemma 11. *Suppose H and V are two Hermitian bounded operators, then for any $\beta \geq 0$:*

(a) *The partition functions satisfy*

$$\mathrm{Tr}(e^{-\beta(H+V)}) \leq \mathrm{Tr}(e^{-\beta H})e^{\beta\|V\|}$$

(b) *For any $O \succeq 0$, it follows that*

$$\mathrm{Tr}(Oe^{-\beta(H+V)}) \leq \mathrm{Tr}(Oe^{-\beta H})\exp\left(e^{\beta(\|H\|+\|V\|)}\|V\|\right).$$

Proof. (a) We will use the Duhamel's formula, which states that for any differentiable bounded operator $F(t)$,

$$\frac{d}{dt}e^{F(t)} = \int_0^1 e^{(1-u)F(t)}\frac{dF(t)}{dt}e^{uF(t)}dt.$$

Defining $H(s) = H + sV$, we note from the Duhamel's formula that

$$\left|\frac{d}{ds}\text{Tr}\left(e^{-\beta H(s)}\right)\right| = \beta\left|\int_0^1 \text{Tr}\left(e^{-(1-u)\beta H(s)}Ve^{-u\beta H(s)}ds\right)du\right| = \beta\left|\int_0^1 \text{Tr}\left(e^{-\beta H(s)}V\right)du\right| \le \beta\|V\|_{\text{op}}\|e^{-\beta H(s)}\|_{\text{op},1},$$

where we have used the Holder's inequality in the last step. Noting that $\|e^{-\beta H(s)}\|_{\text{op},1} = \text{Tr}(e^{-\beta H(s)})$, we obtain that

$$\left|\frac{d}{ds}\text{Tr}\left(e^{-\beta H(s)}\right)\right| \le \beta\|V\|_{\text{op}}\text{Tr}\left(e^{-\beta H(s)}\right)$$

Therefore,

$$|\log\text{Tr}(e^{-\beta(H+V)}) - \log\text{Tr}(e^{-\beta H})| \le \int_0^1 \left|\frac{d}{ds}\log\text{Tr}(e^{-\beta H(s)})\right|ds = \int_0^1 \left|\frac{1}{\text{Tr}(e^{-\beta H(s)})}\frac{d}{ds}\text{Tr}(e^{-\beta H(s)})\right|ds \le \beta\|V\|_{\text{op}}.$$

Thus, we obtain the lemma statement.

(b) We again use the Duhamel's formula to obtain

$$\left|\frac{d}{ds}\text{Tr}(Oe^{-\beta H(s)})\right| = \beta\left|\int_0^1 \text{Tr}\left(Oe^{-(1-u)\beta H(s)}Ve^{-u\beta H(s)}\right)du\right| \le \beta\int_0^1 \left|\text{Tr}\left(Oe^{-(1-u)\beta H(s)}Ve^{-u\beta H(s)}\right)\right|du$$

We note that

$$\left|\text{Tr}\left(Oe^{-(1-u)\beta H(s)}Ve^{-u\beta H(s)}\right)\right| = \left|\text{Tr}\left(e^{-\beta H(s)/2}Oe^{-\beta H(s)/2}e^{-(1/2-u)\beta H(s)}Ve^{-(u-1/2)\beta H(s)}\right)\right|,$$
$$\le \text{Tr}\left(e^{-\beta H(s)/2}Oe^{-\beta H(s)/2}\right)\|e^{-(1/2-u)\beta H(s)}Ve^{-(u-1/2)\beta H(s)}\|,$$
$$\le \text{Tr}\left(Oe^{-\beta H(s)}\right)\|V\|e^{\beta(\|H\|+s\|V\|)}.$$

where we have used the fact that since $u \in [0, 1]$, $|u - 1/2| \le 1/2$. Therefore, we obtain that

$$\left|\frac{d}{ds}\log\text{Tr}\left(Oe^{-\beta H(s)}\right)\right| \le \|V\|e^{\beta\|H\|}e^{\beta s\|V\|} \implies \left|\log\text{Tr}(Oe^{-\beta H}) - \log\text{Tr}(Oe^{-\beta(H+V)})\right| \le \|V\|e^{\beta(\|H\|+\|V\|)}.$$

This estimate yields the lemma statement. $\qquad\square$

Lemma 12. *Given bounded hermitian operators H and V, and any bounded hermitian operator (observable) O,*

$$\left|\text{Tr}\left(\frac{Oe^{-\beta H}}{Z_H(\beta)}\right) - \text{Tr}\left(\frac{Oe^{-\beta(H+V)}}{Z_{H+V}(\beta)}\right)\right| \le 2\|O\|_{\text{op}}\left(\left|\exp(e^{\beta(\|H\|+\|V\|)}\|V\|) - 1\right| + \left|\exp(\beta\|V\|) - 1\right|\right)$$

Proof. This is a straightforward application of lemma 11. We denote by $H' = H + V$. For simplicity, we will analyze the operator $O' = O + \|O\|I - O' \succeq 0$ and

$$\text{Tr}\left(\frac{O'e^{-\beta H}}{Z_H(\beta)}\right) - \text{Tr}\left(\frac{O'e^{-\beta H}}{Z_H(\beta)}\right) = \text{Tr}\left(\frac{Oe^{-\beta H}}{Z_H(\beta)}\right) - \text{Tr}\left(\frac{Oe^{-\beta H}}{Z_H(\beta)}\right).$$

We begin by noting that

$$\left|\text{Tr}\left(\frac{O'e^{-\beta H}}{Z_H(\beta)}\right) - \text{Tr}\left(\frac{O'e^{-\beta H'}}{Z_{H'}(\beta)}\right)\right| \le \frac{\text{Tr}(O'e^{-\beta H})}{Z_H(\beta)}\left|1 - \frac{\text{Tr}(O'e^{-\beta H'})}{\text{Tr}(O'e^{-\beta H})}\right| + \frac{\text{Tr}(O'e^{-\beta H'})}{Z_{H'}(\beta)}\left|1 - \frac{Z_{H'}(\beta)}{Z_H(\beta)}\right|$$

Noting that

$$\frac{\text{Tr}(O'e^{-\beta H})}{Z_H(\beta)}, \frac{\text{Tr}(O'e^{-\beta H'})}{Z_{H'}(\beta)} \le \|O'\|_{\text{op}} \le 2\|O\|_{\text{op}},$$

we obtain that

$$\left|\mathrm{Tr}\left(\frac{O'e^{-\beta H}}{Z_H(\beta)}\right) - \mathrm{Tr}\left(\frac{O'e^{-\beta H'}}{Z_{H'}(\beta)}\right)\right| \le 2\|O\|_{\mathrm{op}}\left(\left|\exp(e^{\beta(\|H\|+\|V\|)}\|V\|) - 1\right| + \left|\exp(\beta\|V\|) - 1\right|\right)$$

$\square$

We next need the notion of exponentially-clustered correlations in a Gibb's state — which we reproduce below from Ref. [27]. We will consider Hamiltonians on $\mathfrak{L} \subset \mathbb{Z}^d$ expressed as

$$H = \sum_{x \in \mathfrak{L}} h_x,$$

where h_x acts only on spins within a distance R of $x \in \mathfrak{L}$. We will denote by $\mathrm{supp}(h_x) \subseteq \mathfrak{L}$ the support of h_x. Given $X \subseteq \mathfrak{L}$, we denote by H_X the operator

$$H_X = \sum_{x|\mathrm{supp}(h_x) \subseteq X} h_x,$$

i.e. H_X is the Hamiltonian H obtained on restricting H to the set X.

Definition 2. *A local Hamiltonian H is said to have exponential clustering of correlations at inverse-temperature β if $\exists c_1, c_2 > 0$ such that for all operators $X \subset \mathfrak{L}$ and A, B with $\mathrm{supp}(A), \mathrm{supp}(B) \subset X$ with $d(\mathrm{supp}(A), \mathrm{supp}(B)) \ge l$,*

$$\left|\mathrm{Tr}\big(A \otimes B\sigma_X(\beta)\big) - \mathrm{Tr}\big(A\sigma_X(\beta)\big)\mathrm{Tr}\big(B\sigma_X(\beta)\big)\right| \le c_2\|A\|\|B\|e^{-c_1 l},$$

where $\sigma_X(\beta) = e^{-\beta H_X}/\mathrm{Tr}[e^{-\beta H_X}]$ is the Gibb's state corresponding to H_X at inverse-temperature β.

An important property of Hamiltonians with exponential clustering of correlations, which relies on quantum belief propagation [40] and is proved in Ref. [27], is that local observables can be estimated locally.

Lemma 13 (From Ref. [27]). *Suppose H is a local Hamiltonian on a finite lattice $\mathfrak{L} \subset \mathbb{Z}^d$ with exponential clustering of correlations at inverse temperature β. If $\mathfrak{L} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ such that $\mathrm{dist}(\mathcal{A}, \mathcal{C}) \ge l$, then $\exists c_1', c_2'$ such that*

$$\|\mathrm{Tr}_{\mathcal{B},\mathcal{C}}(\sigma_{\mathfrak{L}}(\beta)) - \mathrm{Tr}_{\mathcal{B}}(\sigma_{\mathcal{A} \cup \mathcal{B}}(\beta))\|_{\mathrm{tr}} \le |\partial\mathcal{C}|c_2'e^{-c_1' l},$$

where $\sigma_X(\beta)$ is the Gibb's state corresponding to H_X and $\partial\mathcal{C}$ is the boundary between B, C.

Proof (of proposition 6). We assume that both H and H' have exponential clustering of correlations and satisfy lemma 13. Suppose O is a local observable with support S_O and consider $\mathcal{B}$ to be a region around S_O and $\mathcal{C}$ be the remainder of the lattice. We also assume that $d(\mathcal{C}, S_O) \ge l$, for some l to be chosen later. We denote by $\sigma_l(\beta)$ and $\sigma_l'(\beta)$ the Gibb's state, at inverse temperature β, corresponding to $H_{S_O \cup \mathcal{B}}$ and $H'_{S_O \cup \mathcal{B}}$ respectively, and by $\sigma(\beta), \sigma'(\beta)$ the Gibb's state corresponding to H and H'. Now, from lemma 13 it follows that

$$|\mathrm{Tr}(O\sigma(\beta)) - \mathrm{Tr}(O\sigma_l(\beta))|, |\mathrm{Tr}(O\sigma'(\beta)) - \mathrm{Tr}(O\sigma_l'(\beta))| \le \|O\|d(2l + R_O)^{d-1}c_2'e^{-c_1' l},$$

where $R_O = \mathrm{diam}(S_O)$ and we have used that $|\partial\mathcal{C}| \le d \times \mathrm{diam}(S_O \cup \mathcal{B})^{d-1} \le d(2l + R_O)^{d-1}$. Furthermore, lemma 12 can be used to bound $|\mathrm{Tr}(\sigma_l(\beta)) - \mathrm{Tr}(O\sigma_l'(\beta))|$. We note that

$$\|H_{S_O \cup \mathcal{B}} - H'_{S_O \cup \mathcal{B}}\| \le \delta(2l + R_O)^d \text{ and } \|H_{S_O \cup \mathcal{B}}\| \le (2l + R_O)^d.$$

Therefore,

$$|\mathrm{Tr}(\sigma_l(\beta)) - \mathrm{Tr}(O\sigma_l'(\beta))| \le 2\|O\|O(e^{\beta(2l + R_O)^d}(2l + R_O)^d\delta).$$

Thus, from the triangle inequality we obtain the bound that

$$|\mathrm{Tr}(\sigma(\beta)) - \mathrm{Tr}(O\sigma'(\beta))| \le \|O\|\left[O\left((2l + R_O)^{d-1}e^{-c_1' l}\right) + O\left(e^{\beta(2l + R_O)^d}(2l + R_O)^d\delta\right)\right].$$

Choosing $2l + R_O = \Theta(\log^{1/d}(1/\sqrt{\delta}))$, we obtain that

$$|\mathrm{Tr}(O\sigma(\beta)) - \mathrm{Tr}(O\sigma'(\beta))| \le \|O\|O(\log^{1-1/d}(1/\delta)e^{-\Omega(\log^{1/d}(1/\delta))}),$$

which proves the lemma statement. $\square$