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PRICE'S LAW FOR SPIN FIELDS ON A SCHWARZSCHILD BACKGROUND

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Abstract. In this work, we give a proof of the globally sharp asymptotic profiles for the spin-$\mathfrak{s}$ fields on a Schwarzschild background, including the scalar field ($\mathfrak{s} = 0$), the Maxwell field ($\mathfrak{s} = \pm 1$) and the linearized gravity ($\mathfrak{s} = \pm 2$). The conjectured Price's law in the physics literature which predicts the sharp estimates of the spin $s = \pm\mathfrak{s}$ components towards the future null infinity as well as in a compact region is shown. Further, we confirm the heuristic claim by Barack and Ori that the spin $+1, +2$ components have an extra power of decay at the event horizon than the conjectured Price's law. The asymptotics are derived via a unified, detailed analysis of the Teukolsky master equation that is satisfied by all these components.

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1. Introduction

The metric of a Schwarzschild black hole spacetime [67] takes the form of

$$g_{ab} = -2l_{(a}n_{b)} + 2m_{(a}\bar{m}_{b)}, \tag{1.1}$$

where $(l^a, n^a, m^a, \bar{m}^a)$ is a Hawking–Hartle (H–H) null tetrad [39] and reads in the Boyer-Lindquist coordinates (t, r, θ, ϕ) [17]

$$l^a = \tfrac{1}{\sqrt{2}}(1, \mu, 0, 0),\ n^a = \tfrac{1}{\sqrt{2}}(\tfrac{1}{\mu}, -1, 0, 0),\ m^a = \tfrac{1}{\sqrt{2}r}\left(0, 0, 1, \tfrac{i}{\sin\theta}\right),\ \bar{m}^a = \tfrac{1}{\sqrt{2}r}\left(0, 0, 1, \tfrac{-i}{\sin\theta}\right). \tag{1.2}$$

Here, the function $\mu = \mu(r, M) = \Delta r^{-2}$ with $\Delta = \Delta(r, M) = r^2 - 2Mr$ and M is the mass of the black hole. The larger root $r = 2M$ of the function Δ is the location of the event horizon $\mathcal{H}$, and we define the domain of outer communication (DOC), denoted as $\mathcal{D}$, of a Schwarzschild black hole spacetime to be the closure of $\{(t, r, \theta, \phi) \in \mathbb{R} \times (2M, \infty) \times S^2\}$ in the Kruskal maximal extension. We consider in this work only the future Cauchy problem and denote the future event horizon and the future null infinity as $\mathcal{H}^+$ and $\mathcal{I}^+$, respectively. Denote v and u the forward and retarded time, respectively, and define τ a hyperboloidal time function such that the level sets of the time function are spacelike hypersurfaces,[1] cross $\mathcal{H}^+$ regularly, and are aymptotic $\mathcal{I}^+$ for large r. We define the coordinate system $(\tau, \rho = r, \theta, \phi)$ the hyperboloidal coordinates and denote the level sets of τ as Σ_τ. See Section 2.1.

1.1. **Price's law.** This work is for the spin-$\mathfrak{s}$ fields, which correspond to the scalar field, the Maxwell field and the linearized gravity in the case of $\mathfrak{s} = 0, 1$ and 2, respectively. The scalar field, denoted as Υ_0, solves the scalar wave equation $\Box_g \Upsilon_0 = 0$. The Maxwell field, a real two-form $\mathbf{F}_{\alpha\beta}$, satisfies the Maxwell equations

$$\nabla^\alpha \mathbf{F}_{\alpha\beta} = 0, \qquad \nabla_{[\gamma}\mathbf{F}_{\alpha\beta]} = 0. \tag{1.3}$$

In the Newman–Penrose (N–P) formalism [60, 61], one projects the Maxwell field onto the H–H tetrad (1.2) and obtains the N–P components of the Maxwell field

$$\Upsilon_{+1} = \mathbf{F}_{\mu\nu}l^\mu m^\nu, \quad \Upsilon_{0,\mathrm{mid}} = \mathbf{F}_{\mu\nu}(l^\mu n^\nu + \bar{m}^\mu m^\nu), \quad \Upsilon_{-1} = \mathbf{F}_{\mu\nu}\bar{m}^\mu n^\nu. \tag{1.4}$$

The lower index of each N–P component indicates its spin weight, and we call the N–P components Υ_{+1} and Υ_{-1} as the spin $+1$ and -1 components of the Maxwell field, respectively, and $\Upsilon_{0,\mathrm{mid}}$ as the middle component. In the same way, we define the N–P components of the linearized gravity by projecting the linearized Weyl tensor $\mathbf{W}_{\alpha\beta\gamma\delta}$ onto the H–H tetrad: the spin $+2$ and -2 components of the linearized gravity are given by

$$\Upsilon_{+2} = \mathbf{W}_{lmlm}, \qquad \Upsilon_{-2} = \mathbf{W}_{n\bar{m}n\bar{m}}. \tag{1.5}$$

The above defined spin s components Υ_s $(s = 0, \pm1, \pm2)$ are regular and non-degenerate at the future event horizon. Throughout this work, we always denote s the spin weight and $\mathfrak{s} = |s|$.

As is well-known, these massless spin-s fields will develop power tails in the future development in the DOC of a Schwarzschild spacetime. Physically, this is due to the backscattering caused by an effective curvature potential which is in turn related to the non-vanishing middle N–P component of the Weyl curvature tensor of the Schwarzschild background, and, in view of the expectation that these linear models provide high accuracy approximation for the nonlinear dynamics, these tails are intimately related to many important problems in General Relativity, such as the black hole exterior (in-)stability problem and the Strong Cosmic Censorship conjecture for the black hole interior. These

[1] In fact, the estimates in our main theorem 1.1 are still valid if one portion or the whole of the level sets of τ is null.

power tails are predicted by Price in [63, 64], and they are conjectured to be sharp and are now called the **Price's law**. In addition, the tails of a fixed ℓ mode are also discussed therein. We shall also remark that these Price's law are conjectured to be sharp as both an *upper* and a *lower* bound. Furthermore, Barack–Ori claimed in [12] that the positive spin components for both the Maxwell field and the linearized gravity have an extra v^{-1} decay at the event horizon than the conjectured Price's law, which means the sharp decay for $(\Upsilon_{+\mathfrak{s}})^{\ell=\ell_0}$, $\mathfrak{s}=1,2$ on event horizon shall be $v^{-4-2\ell_0}$. The conjectured Price's law and the heuristic argument of Barack–Ori together for both the spin s component Υ_s and its fixed $\ell=\ell_0$ mode $(\Upsilon_s)^{\ell=\ell_0}$ are summarized in Table 1 provided the initial data decay sufficiently fast (or have compact support) on an initial future Cauchy surface terminating at spatial infinity. **When there is no confusions, we shall call the summarized sharp decay rates in Table 1 as the Price's law, albeit it is in fact a combination of the conjectured Price's law in [63, 64] and the predicted asymptotics for the positive spin components by Barack–Ori in [12].** There are also works discussing the power tails in a Kerr spacetime; we guide the readers to [42, 12, 35, 21] and the references therein.

	towards $\mathcal{I}^+$	in a compact region	on $\mathcal{H}^+$
Υ_s	$r^{-1-\mathfrak{s}-s}u^{-2-\mathfrak{s}+s}$	$v^{-3-2\mathfrak{s}}$	$v^{-3-2\mathfrak{s}-\zeta(s)}$
$(\Upsilon_s)^{\ell=\ell_0}$	$r^{-1-\mathfrak{s}-s}u^{-2-\ell_0+s}$	$v^{-3-2\ell_0}$	$v^{-3-2\ell_0-\zeta(s)}$
total decay power	$-3-\mathfrak{s}-\max\{\ell_0,\mathfrak{s}\}$	$-3-2\max\{\ell_0,\mathfrak{s}\}$	$-3-2\max\{\ell_0,\mathfrak{s}\}-\zeta(s)$

TABLE 1. The Conjectured Price's law for the spin $s=\pm\mathfrak{s}$ components on Schwarzschild are as above. The original Price's law, or the sharp decay rates predicted by Price in [63, 64], contains the ones towards $\mathcal{I}^+$ and in a compact region. The last column about the sharp decay rates on $\mathcal{H}^+$ contains the predicted decay rates by Barack–Ori [12] for the positive spin component, with $\zeta(s)$ equals 1 if $s>0$ and 0 if $s\leq 0$.

1.2. Main results.

Our main results give an affirmative answer to the conjecture Price's law in [63, 64] and Barach-Ori's claim in [12] for the spin s components on a Schwarzschild background by providing a rigorous mathematical proof.

Theorem 1.1. *[Rough version of the asymptotic profiles of the spin s components.] Let $j\in\mathbb{N}$. Let $\mathfrak{s}=0,1,2$, and let $\ell_0\geq\mathfrak{s}$. Let function $h_{\mathfrak{s},\ell_0}$ be defined as in Definition 3.34. Let $\tau_0\geq 1$.*

(1) Assume the initial data of the spin $s=\pm\mathfrak{s}$ components on Σ_{τ_0} are smooth, decay sufficiently fast (or of compact support), and are supported on $\ell\geq\ell_0$ modes. Let $\varepsilon_0>0$ be arbitrary. Then in the future domain of dependence of Σ_{τ_0}, there exists a constant $\varepsilon>0$, constants $C_{\mathfrak{s},1}$, $C_{\mathfrak{s},2}$ and $C_{\mathfrak{s},3}$ which can be calculated from the initial data, and $D<\infty$ which depends only on the initial data such that in the region $\{\rho\geq\tau^{1+\varepsilon_0}\}$,

$$\left|\partial_\tau^j\Upsilon_s - C_{\mathfrak{s},1}v^{-1-\mathfrak{s}-s}\tau^{-\ell_0+s-2-j}\right| \leq Dv^{-1-\mathfrak{s}-s}\tau^{-\ell_0+s-2-j-\varepsilon}, \tag{1.6a}$$

in the region $\{\rho\leq\tau^{1-\varepsilon_0}\}$,

$$\left|\partial_\tau^j\Upsilon_s - C_{\mathfrak{s},2}\mu^{\frac{s+\mathfrak{s}}{2}}r^{\ell_0-\mathfrak{s}}h_{\mathfrak{s},\ell_0}\tau^{-2\ell_0-3-j}\right| \leq D\tau^{-2\ell_0-3-j-\varepsilon}, \tag{1.6b}$$

and on the future event horizon, for $\mathfrak{s}=1,2$,

$$\left|\partial_\tau^j\Upsilon_{+\mathfrak{s}}\big|_{\mathcal{H}^+} - C_{\mathfrak{s},3}v^{-2\ell_0-4-j}\right| \leq Dv^{-2\ell_0-4-j-\varepsilon}. \tag{1.6c}$$

(2) Assume the initial data for the spin $s=\pm\mathfrak{s}$ components on $t=\tau_0$ hypersurface is smooth and compactly supported in a compact region of $(2M,\infty)$, and assume they are supported on $\ell\geq\ell_0$ modes. Let function $h_{\mathfrak{s},\ell_0-\mathfrak{s},\ell_0-\mathfrak{s}}$ be determined as in Proposition 3.7 with $h_{\mathfrak{s},0,0}=-2(\mathfrak{s}+1)(2\mathfrak{s}+1)Mr^{-1}$, and let

$$\mathbb{Q}_{\ell=\ell_0} = \frac{(-1)^{\ell_0-\mathfrak{s}+1}(\ell_0-\mathfrak{s})!}{(2\ell_0+1)(2\ell_0+2)}\lim_{r\to\infty}(rh_{\mathfrak{s},\ell_0-\mathfrak{s},\ell_0-\mathfrak{s}})$$

$$\times \int_{2M}^{\infty} \mu^{-1} h_{\mathfrak{s},\ell_0} r^{\ell_0+\mathfrak{s}} \big(r^2 \partial_t (\Upsilon_{+\mathfrak{s}})^{\ell=\ell_0} + 2\mathfrak{s}(r-3M)(\Upsilon_{+\mathfrak{s}})^{\ell=\ell_0}\big) \mathrm{d}r. \tag{1.7}$$

Let $\varepsilon_0 > 0$ be arbitrary. Then in the future domain of dependence of Σ_{τ_0}, there exists $\varepsilon > 0$ and $D < \infty$ such that in the region $\{\rho \geq \tau^{1+\varepsilon_0}\}$,

$$\left| \partial_\tau^j \Upsilon_s - (-1)^{j+1} 2^{\ell_0+\mathfrak{s}+2} \prod_{i=\ell_0+\mathfrak{s}+1}^{2\ell_0+1} i^{-1} \mathbb{Q}_{\ell=\ell_0} v^{-1-\mathfrak{s}-s} \tau^{-\ell_0+s-2-j} \right| \leq D v^{-1-\mathfrak{s}-s} \tau^{-\ell_0+s-2-j-\varepsilon}, \tag{1.8a}$$

in the region $\{\rho \leq \tau^{1-\varepsilon_0}\}$,

$$\left| \partial_\tau^j \Upsilon_s - (-1)^{j+1} 2^{2\ell_0+2} \prod_{i=\ell_0+\mathfrak{s}+1}^{2\ell+1} i^{-1} \prod_{n=2\ell_0+2}^{2\ell_0+j+2} n \mathbb{Q}_{\ell=\ell_0} \mu^{\frac{s+\mathfrak{s}}{2}} r^{\ell_0-\mathfrak{s}} h_{\mathfrak{s},\ell_0} \tau^{-2\ell_0-3-j} \right| \leq D \tau^{-2\ell_0-3-j-\varepsilon}, \tag{1.8b}$$

and on the future event horizon, for $\mathfrak{s} = 1, 2$,

$$\left| \partial_\tau^j \Upsilon_{+\mathfrak{s}} \Big|_{\mathcal{H}^+} - \frac{(-1)^{\mathfrak{s}+j+1} 2^{2\ell_0+3} \mathfrak{s} (\ell_0-\mathfrak{s})! (2M)^{\ell_0-\mathfrak{s}+1} h_{\mathfrak{s},\ell_0}(2M)}{(2\ell_0+1)!} \prod_{n=2\ell_0+2}^{2\ell_0+j+3} n \mathbb{Q}_{\ell=\ell_0} v^{-2\ell_0-4-j} \right|$$
$$\leq D v^{-2\ell_0-4-j-\varepsilon}. \tag{1.8c}$$

Remark 1.2.

- The results in point 1 of this theorem verify the sharp decay rates for the spin $s = \pm\mathfrak{s}$ components on a Schwarzschild background presented in the above Table 1. In particular, it suggests that the correct asymptotic decay rates should be a combination of the conjectured Price's law in [63, 64] and the heuristic claim of Barack–Ori in [12];
- The function $r^{\ell-\mathfrak{s}} h_{\mathfrak{s},\ell_0}$ is a zero energy mode to the equation of an $\ell = \ell_0$ mode of the spin $-\mathfrak{s}$ component $\Upsilon_{-\mathfrak{s}}$, with the limit $\lim_{r\to\infty} h_{\mathfrak{s},\ell_0} = 1$;
- The precise version of point 1 is in Theorem 4.9, where the conditions for the initial data are explicitly imposed, the constant coefficients $C_{\mathfrak{s},1}$, $C_{\mathfrak{s},2}$ and $C_{\mathfrak{s},3}$ of the asymptotic profiles are explicitly expressed in terms of the initial data and from the explicit form of the wave equation satisfied by the spin s components, and the constant D is expressed in terms of some finite regularity norm of the initial data. Further, the global asymptotic profiles are computed in Theorem 4.9, not merely in the regions $\{\rho \geq \tau^{1+\varepsilon_0}\}$ and $\{\rho \leq \tau^{1-\varepsilon_0}\}$ as in point 1;
- In point 1, by reading off the values of $C_{\mathfrak{s},1}$, $C_{\mathfrak{s},2}$ and $C_{\mathfrak{s},3}$ in terms of the Newman–Penrose constants $\mathbb{Q}_{g+}^{(m,\ell_0)}$ and $\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell_0+1)}$ in Theorem 4.9 and from the expressions of $\mathbb{Q}_{g+}^{(m,\ell_0)}$ and $\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell_0+1)}$ in Definition 3.22 and Lemma 5.4, one finds that the asymptotics are generically sharp;
- Point 2 is an immediate corollary of Theorem 5.7. Meanwhile, one can in fact obtain the globally sharp asymptotic profiles for the spin $\pm\mathfrak{s}$ components. Similarly, the asymptotics in point 2 are generically sharp;
- The proof in this work also implies that in point 1, if the coefficients of the asymptotic profiles are both vanishing, the decay rate will be faster by an extra τ^{-1} globally;
- Further, in point 2, one finds the decay rate can be improved if and only if the integral in the second line of (1.7) vanishes. For the scalar field ($\mathfrak{s} = 0$), it is manifest that if, additionally, the time derivative of the initial data vanishes, then the decay rate is faster by an extra τ^{-1} globally. Note that this has been verified in [40, 11] as well. For $\mathfrak{s} = 1, 2$, this is in fact compatible with the expectation in [47, 65] that solutions to the Regge–Wheeler equation with smooth, compactly supported in a compact region in $(2M, +\infty)$, static (in the sense that its time derivative on the initial hypersurface $t = \tau_0$ vanishes) initial data will develop power tails $\tau^{-2\ell_0-4}$ in a finite radius region. This can be seen as follows: it is certain $\mathfrak{s}$-order derivative of the spin s component which satisfies the Regge–Wheeler equation [66], and we can use the wave equation satisfied by the spin s component to rewrite its time derivative as a combination of total radial derivatives, a first order time derivative and the component itself, which will then yield that the integral in the second line of (1.7) vanishes;

- As a byproduct, we have also obtained the global asymptotic profile of the middle component of the Maxwell field. See Theorem 5.7. This thus yields that the middle component asymptotically decays to a static Coulomb solution, which equals r^{-2} times a complex constant that can be calculated from any sphere of the initial hypersurface, at a rate of decay as shown in Theorem 5.7;
- Although we consider only $\mathfrak{s} = 0, 1, 2$ cases in this work, the proof in this work suggests that the Price's law shall also hold true for the spin $s = \pm\mathfrak{s}$ components of the spin-$\mathfrak{s}$ field for arbitrary $\mathfrak{s} \in \mathbb{N}$. Additionally, the method developed in this work can be applied together with the results in [55] to obtain the Price's law, or the global asymptotic profiles, for the massless Dirac field, which is also called as the massless spin-$\frac{1}{2}$ field. Further, as pointed out in [55], Barach–Ori's claim about faster decay for the positive integer-spin component on the future event horizon can not be extended to the Dirac field.

Remark 1.3. It is without much effort to extract from the proof a general sharp decay estimate for a wave equation with a potential: Assume φ solves $\Box_g\varphi + \mathbf{P}\varphi = 0$ in a Schwarzschild spacetime, where $\mathbf{P} = \mathbf{P}(r) = O(r^{-1})$ is smooth and $\{\lim_{r\to\infty}(r^2\partial_r)^i\mathbf{P}\}_{i=0,\ldots,n}$ exist for a sufficiently large n, assume there is a uniform energy boundedness and an integrated local energy decay (Morawetz) estimate for φ, and assume the initial dat on Σ_{τ_0} are smooth, decay sufficiently fast, and are supported on $\ell \geq \ell_0$ modes, $\ell_0 \in \mathbb{N}$, then the statements in point 1 of Theorem 1.1 with $\mathfrak{s} = 0$ hold true for φ and the coefficients in the asymptotic profiles can be calculated explicitly from the initial data and the information of the potential $\mathbf{P}$.

1.3. **Outline of the proof.** In this subsection, we make a comparison to other existed results, outline the methods and ideas, and in the end, propose the future applications.

1.3.1. *Overview of the approach.* In [9, 8], Angelopoulos–Aretakis–Gajic initiated works aiming to prove the Price's law $v^{-1}\tau^{-2}$ for the scalar field on a Reissner–Nördstrom background by the following steps:

A) prove uniform energy boundedness and the Morawetz estimate;

B) obtain the almost Price's law in both the nonvanishing Newman–Penrose constant (NVNPC) case and the vanishing Newman–Penrose constant (VNPC) case via

  B1) a definition of the Newman–Penrose constant for the spherically symmetric $\ell = 0$ mode, which is a conserved quantity at null infinity;

  B2) an extended r^p hierarchy [26] of spatially weighted energy estimates for an optimal range of the r-weight p;

C) achieve the sharp asymptotic $cv^{-1}\tau^{-1}$ in the NVNPC case;

D) in the VNPC case, derive the leading asymptotic $cv^{-1}\tau^{-2}$ for the $\ell = 0$ mode by defining the "time integral" which solves the scalar wave equation and whose time derivative equals the scalar field.

In particular, the step A) and a significant part of step B) have been well-developed for the wave equation, both linear and nonlinear, in black hole spacetimes in the past two decades.

In our work, we consider arbitrary modes of the spin s components, and, without loss of much information, we shall consider here only an arbitrary (m, ℓ) mode of the spin s components and the dependence on m and ℓ may be suppressed. One has to generalize the above approach to some extend and develop some novel methods and ideas. We provide an overview of the approach in the present work which can be basically divided into the following steps:

A') prove uniform energy boundedness and the Morawetz estimate for the spin s components;

B') obtain the almost Price's law via

  B1') a derivation of extended wave systems, and a definition of the corresponding Newman–Penrose constant for the (m, ℓ) mode, which is also a conserved quantity at null infinity;

  B2') an extended r^p hierarchy [26] of spatially weighted energy estimates for the extended wave systems for an optimal[2] range of the r-weight p;

  B3') obtaining the almost Price's law in the exterior region $\{\rho \geq \tau\}$;

[2] What we mean by "optimal" here is in the sense that the obtained p range has a sharp upper bound.

B4') obtaining the almost Price's law in the interior region $\{\rho \leq \tau\}$;

C') derive the sharp asymptotics in the NVNPC case;

D') derive the asymptotic profiles in the VNPC case by defining the time integral for the spin s component.

We shall emphasize that all the estimates are derived through a unified analysis of the *Teukolsky master equation* (TME)[3] [74], a wave equation satisfied by all the spin s components. See Section 2.4. In fact, this equation is satisfied by the spin s components in any subextremal or extremal Kerr spacetime. It has shown its enormous importance in recent works [24, 53, 23, 3, 34, 50] in proving the linear or nonlinear stability of black hole spacetimes and recent works [52, 53] towards the (almost) Price's law for non-zero spin fields.

In the following, we discuss each step, contrast our method with the one outlined above in [9, 8], the one developed in [54, 55], as well as the ones in other related works, and present the new ideas in this work.

Step A'). As is mentioned above, the starting point is to achieve the energy and Morawetz estimates for solutions to the TME. These estimates have been proven in a Schwarzschild spacetime for the scalar field in [16, 25] and for the Maxwell field and the linearized gravity in [24, 62], where a key point is to use certain differential transformations due to Chandrasekhar [18]. Extensions to the Kerr spacetimes include [73, 4, 27] for the scalar field and [52, 53, 23] for non-zero spin fields. For the non-zero spin fields, we start with the energy and Morawetz estimates in [52, 53] where the author uses the standard technique in estimating the scalar field to treat the coupled wave systems

$$\mathbf{WS}_s = \left\{\text{the wave system of } \{\hat{\mathcal{V}}^i(\mu^{\mathfrak{s}}\Upsilon_{-\mathfrak{s}})\}_{i=0,\dots,\mathfrak{s}} \text{ or } \{(r^2Y)^i\Upsilon_{+\mathfrak{s}}\}_{i=0,\dots,\mathfrak{s}}\right\}, \tag{1.9}$$

where $\hat{\mathcal{V}} = r^2\hat{V}$, and $Y = \mu^{-1}\partial_t - \partial_r$ and $\hat{V} = \mu^{-1}V = \mu^{-1}\partial_t + \partial_r$ are the ingoing and outgoing principle null vectors, respectively. In particular, the wave equations of $\hat{\mathcal{V}}^{\mathfrak{s}}(\mu^{\mathfrak{s}}\Upsilon_{-\mathfrak{s}})$ and $(r^2Y)^{\mathfrak{s}}\Upsilon_{+\mathfrak{s}}$ are the Regge–Wheeler equation.

Step B'). This step has much difference from the step B) where $\ell = \mathfrak{s} = 0$. The above Morawetz estimates have already encoded most of the local information, and it remains to extract out the information in a large radius region. The substeps B1'), B2'), B3') have followed closely the approach in [54] which treats the Maxwell field in Schwarzschild. See also [55].

Substep B1'). The spin-weighted spherical Laplacian has a nonpositive eigenvalue when acting on a fixed (m, ℓ) mode, and to make use of this eigenvalue gap away from zero, one can in fact commute $\hat{\mathcal{V}}$ further j times, $j \in \mathbb{N}$, with the equation of $\hat{\mathcal{V}}^{\mathfrak{s}}(\mu^{\mathfrak{s}}\Upsilon_{-\mathfrak{s}})$ and obtain an extended system

$$\mathbf{WS}^{(j)}_{-\mathfrak{s}} = \left\{\text{the system of wave equations of } \{\Phi^{(i)}_{-\mathfrak{s}}\}_{i=0,\dots,j+\mathfrak{s}}\right\}.$$

Here, $\Phi^{(i)}_{-\mathfrak{s}} \triangleq \hat{\mathcal{V}}^i(\mu^{\mathfrak{s}}\Upsilon_{-\mathfrak{s}})$, and each wave equation $\Phi^{(i)}_{-\mathfrak{s}}$ can be written in the following form

$$\begin{aligned}&-\mu Y\hat{\mathcal{V}}\Phi^{(i)}_{-\mathfrak{s}} - (\ell - \mathfrak{s} + i + 1)(\ell + \mathfrak{s} - i)\Phi^{(i)}_{-\mathfrak{s}}\\&\quad - 2(i - \mathfrak{s} + 1)(r - 3M)r^{-2}\hat{\mathcal{V}}\Phi^{(i)}_{-\mathfrak{s}} + 2i(i - \mathfrak{s})(i - 2\mathfrak{s})M\Phi^{(i-1)}_{-\mathfrak{s}} = 0.\end{aligned} \tag{1.10}$$

It is crucial in the later application of the r^p method that the coefficient of the $\Phi^{(i)}_{-\mathfrak{s}}$ term is non-positive, and this requirement, which is satisfied only for $i \leq \ell + \mathfrak{s}$, imposes an upper bound for the value of j, that is, $j \leq \ell$.

To properly define a conserved quantity, the N–P constant, at null infinity, we aim to find a scalar $\widetilde{\Phi}^{(\ell+\mathfrak{s})}_{-\mathfrak{s}}$ whose equation is in the following form

$$-\mu Y\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell+\mathfrak{s})}_{-\mathfrak{s}} = O(r^{-1})\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell+\mathfrak{s})}_{-\mathfrak{s}} + \sum_{i=0}^{\ell+\mathfrak{s}} O(r^{-1})\Phi^{(i)}_{-\mathfrak{s}}. \tag{1.11}$$

The right-hand side will have a zero limit towards null infinity, hence $\lim_{\rho\to\infty}\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell+\mathfrak{s})}_{-\mathfrak{s}}$ is a constant independent of time τ; we call it the N–P constant for this (m, ℓ) mode. Such a scalar $\widetilde{\Phi}^{(\ell+\mathfrak{s})}_{-\mathfrak{s}}$ is

[3] In some works, it is called as *Teukolsky equation* instead.

constructed from a linear combination of $\{\Phi^{(i)}_{-\mathfrak{s}}\}_{i=0,\dots,\ell+\mathfrak{s}}$ in [54] for the Maxwell field ($\mathfrak{s}=1$), and, in particular, the scalar $\widetilde{\Phi}^{(\ell+\mathfrak{s})}_{-\mathfrak{s}}$ is unique in all such linear combinations with constant coefficients up to an overall nonvanishing constant factor. Here, we follow the idea therein and extend to any $\mathfrak{s}\in\mathbb{N}$. More generally, one can construct a set of scalars $\{\widetilde{\Phi}^{(i)}_{-\mathfrak{s}}\}_{i=0,\dots,j+\mathfrak{s}}$, each of which takes the form $\widetilde{\Phi}^{(i)}_{-\mathfrak{s}}=\Phi^{(i)}_{-\mathfrak{s}}+\sum_{j=0}^{i-1}c_{\mathfrak{s},i,j}M^{i-j}\widetilde{\Phi}^{(j)}_{-\mathfrak{s}}$, $c_{\mathfrak{s},i,j}$ being constants, such that they satisfy

$$-\mu Y\hat{\mathcal{V}}\widetilde{\Phi}^{(i)}_{-\mathfrak{s}}-\frac{2(i-\mathfrak{s}+1)(r-3M)}{r^2}\hat{\mathcal{V}}\widetilde{\Phi}^{(i)}_{-\mathfrak{s}}-(\ell-\mathfrak{s}+i+1)(\ell+\mathfrak{s}-i)\widetilde{\Phi}^{(i)}_{-\mathfrak{s}}+\sum_{j=0}^{i}O(r^{-1})\widetilde{\Phi}^{(j)}_{-\mathfrak{s}}=0.$$

Note that $\widetilde{\Phi}^{(i)}_{-\mathfrak{s}}=\Phi^{(i)}_{-\mathfrak{s}}$ for $i\in\{0,\dots,2\mathfrak{s}\}$. As a result, we obtain different extended wave systems for $j=0,\dots,\ell$:

$$\widetilde{\mathbf{WS}}^{(j)}_{-\mathfrak{s}}=\Big\{\text{the system of wave equations of }\{\widetilde{\Phi}^{(i)}_{-\mathfrak{s}}\}_{i=0,\dots,j+\mathfrak{s}}\Big\}.$$

Analogously, one can define scalars $\{\Phi^{(i)}_{+\mathfrak{s}}\}_{i=0,\dots,\ell-\mathfrak{s}}$ and $\{\widetilde{\Phi}^{(i)}_{+\mathfrak{s}}\}_{i=0,\dots,\ell-\mathfrak{s}}$ such that their equations are similar to the above ones for $\{\Phi^{(i)}_{-\mathfrak{s}}\}_{i=2\mathfrak{s},\dots,\ell-\mathfrak{s}}$ and $\{\widetilde{\Phi}^{(i)}_{-\mathfrak{s}}\}_{i=2\mathfrak{s},\dots,\ell-\mathfrak{s}}$, respectively, as well as the wave systems $\mathbf{WS}^{(j)}_{+\mathfrak{s}}$ and $\widetilde{\mathbf{WS}}^{(j)}_{+\mathfrak{s}}$ for $j=0,\dots,\ell-\mathfrak{s}$. Note that the equation for $\widetilde{\Phi}^{(i)}_{+\mathfrak{s}}$ is exactly the same as the one for $\widetilde{\Phi}^{(i+2\mathfrak{s})}_{-\mathfrak{s}}$, that is, for any $i\in\{0,\dots,\ell-\mathfrak{s}\}$,

$$-\mu Y\hat{\mathcal{V}}\widetilde{\Phi}^{(i)}_{+\mathfrak{s}}-\frac{2(i+\mathfrak{s}+1)(r-3M)}{r^2}\hat{\mathcal{V}}\widetilde{\Phi}^{(i)}_{+\mathfrak{s}}-(\ell+\mathfrak{s}+i+1)(\ell+\mathfrak{s}-i)\widetilde{\Phi}^{(i)}_{+\mathfrak{s}}+\sum_{j=0}^{i}O(r^{-1})\widetilde{\Phi}^{(j)}_{+\mathfrak{s}}=0. \tag{1.12}$$

Let us finally remark that the Newman–Penrose constant for the (m,ℓ) mode of the spin $+\mathfrak{s}$ component is simply a nonvanishing constant multiple of the one for the spin $-\mathfrak{s}$ component.

Substep B2'). We are ready to apply the r^p method to achieve the energy decay. For $i\in\{\mathfrak{s},\dots,\ell+\mathfrak{s}\}$, the r^p estimates with $p\in[0,2]$ can be easily proven for each wave equation of $\widetilde{\Phi}^{(i)}_{-\mathfrak{s}}$, and one can obtain τ^{-2} decay for the $p=0$ weighted energy of $\widetilde{\Phi}^{(i)}_{-\mathfrak{s}}$ in terms of the $p=2$ weighted energy of $\widetilde{\Phi}^{(i)}_{-\mathfrak{s}}$ which in turn, by definition, is equivalent to the $p=0$ weighted energy of $\widetilde{\Phi}^{(i+1)}_{-\mathfrak{s}}$. This thus yields $\tau^{-2\ell}$ decay for the $p=0$ weighted energy of $\widetilde{\Phi}^{(\mathfrak{s})}_{-\mathfrak{s}}$ in terms of the $p=2$ weighted energy of $\widetilde{\Phi}^{(\ell+\mathfrak{s})}_{-\mathfrak{s}}$ with a loss in the order of regularity. For the equation of $\widetilde{\Phi}^{(\ell+\mathfrak{s})}_{-\mathfrak{s}}$, because of the vanishing constant coefficient of $\widetilde{\Phi}^{(\ell+\mathfrak{s})}_{-\mathfrak{s}}$ in (1.11), one can in fact extend the r^p estimates from $p\in[0,2]$ to $p\in[0,3)$. This provides an optimal upper bound for the weight p in the NVNPC case since the energy is infinite for the $p=3$ weighted energy. In total, we achieve $\tau^{-2\ell-1+2\varepsilon}$ decay for the weighted energy of $\widetilde{\Phi}^{(\mathfrak{s})}_{-\mathfrak{s}}$ in terms of the $p=3-2\varepsilon$ weighted energy of $\widetilde{\Phi}^{(\ell+\mathfrak{s})}_{-\mathfrak{s}}$. Additionally, we remark that for any $j\in\mathbb{N}$, there is an extra τ^{-2j} decay, that is, $\tau^{-2\ell-1-2j+2\varepsilon}$ decay in total, for the $p=0$ weighted energy of $\partial_\tau^j\widetilde{\Phi}^{(\mathfrak{s})}_{-\mathfrak{s}}$ in terms of the $p=3-2\varepsilon$ weighted energy of $\widetilde{\Phi}^{(\ell+\mathfrak{s})}_{-\mathfrak{s}}$, but with a further loss of regularity. There are analogous energy decay estimates for the spin $+\mathfrak{s}$ component.

In the VNPC case, we can derive the r^p estimates for $p\in[0,5)$ for the wave equation of $\widetilde{\Phi}^{(\ell+\mathfrak{s})}_{-\mathfrak{s}}$, hence the above energy decay will be faster by an extra τ^{-2} decay.

Substep B3'). In the NVNPC case, the above energy decay estimate yields $v^{-1}\tau^{-\ell-1-j+\varepsilon}$ globally pointwise decay for the scalars $\{\partial_\tau^j(r^2V)^i\Upsilon_{-\mathfrak{s}}\}_{i=0,\dots,\mathfrak{s}}$. In the exterior region $\{\rho\geq\tau\}$, one needs to gain extra $\tau^{-\mathfrak{s}}$ decay for $\Upsilon_{-\mathfrak{s}}$, and this is done by appealing to a derivation of an elliptic systems from the wave system of $\{\Phi^{(i)}_{-\mathfrak{s}}\}_{i=0,\dots,\mathfrak{s}-1}$. Consider only the most complicated case $\mathfrak{s}=2$. By equation (1.10), the wave system can be rewritten as

$$\mathbf{A}(\Phi^{(0)}_{-2},\Phi^{(1)}_{-2})^T=O(r^{-1})(rV\Phi^{(1)}_{-2},rV\Phi^{(2)}_{-2})+O(r^{-1})(\Phi^{(1)}_{-2},\Phi^{(2)}_{-2})+O(1)(\partial_\tau\Phi^{(1)}_{-2},\partial_\tau\Phi^{(2)}_{-2}),$$

where $\mathbf{A}$ is a strongly elliptic 2×2 matrix. This clearly implies an extra τ^{-1} decay for $\{\Phi^{(i)}_{-2}\}_{i=0,1}$, and one can derive an elliptic equation for $\Phi^{(0)}_{-2}$ to achieve a further τ^{-1} decay for $\Phi^{(0)}_{-2}$. To summarize, we obtain $v^{-1}\tau^{-\ell-\mathfrak{s}-1-j+\varepsilon}$ pointwise decay for the scalars $\partial_\tau^j\Upsilon_{-\mathfrak{s}}$ in the exterior region. This is exactly the almost Price's law in the exterior region in the NVNPC case. A similar argument works in the VNPC case and one obtains extra τ^{-1} pointwise decay.

The $v^{-1-2\mathfrak{s}}\tau^{-\ell+\mathfrak{s}-1-j+\varepsilon}$ pointwise decay estimates for the scalars $\partial_\tau^j\Upsilon_{+\mathfrak{s}}$ are then derived from the above energy decay estimates for the spin $+\mathfrak{s}$ component in Substep B2') and the pointwise decay for $\partial_\tau^j\Upsilon_{-\mathfrak{s}}$ together with an application of the **Teukolsky–Starobinsky identities** (TSI) [76, 70], which are two $2\mathfrak{s}$-order differential identities between the spin $\pm\mathfrak{s}$ components. See Section 3.5 for the TSI. The TSI allow one to derive certain estimates for one spin component from the estimates of the other spin component, and they are frequently used and play a vital role in each of the following steps as well.

Substep B4') This substep is fundamentally different from the other works. It suffices to consider the spin $-\mathfrak{s}$ component, as the estimates for the spin $+\mathfrak{s}$ component can be achieved via the TSI.

A key ingredient is to define a scalar $\varphi_{-\mathfrak{s},\ell} = (h_{\mathfrak{s},\ell}r^{\ell-\mathfrak{s}})^{-1}\Upsilon_{-\mathfrak{s}}$ such that its equation takes the form of

$$\partial_\rho(r^{2\ell+2}\mu^{1+\mathfrak{s}}h_{\mathfrak{s},\ell}^2\partial_\rho\varphi_{-\mathfrak{s},\ell}) = \partial_\tau H_{\varphi_{-\mathfrak{s},\ell}}, \tag{1.13}$$

where $H_{\varphi_{-\mathfrak{s},\ell}}$ is an expression of $\varphi_{-\mathfrak{s},\ell}, \partial_\rho\varphi_{-\mathfrak{s},\ell}$ and $\partial_\tau\varphi_{-\mathfrak{s},\ell}$. Recall from Remark 1.2 that $h_{\mathfrak{s},\ell}r^{\ell-\mathfrak{s}}$ is a zero energy mode solution to the TME of $\Upsilon_{-\mathfrak{s}}$. This is essentially a degenerate (at horizon) elliptic equation with the RHS viewed as a source. One can derive elliptic estimates for this equation which basically says that 2β-weighted energy of $\partial_\tau^j\varphi_{-\mathfrak{s},\ell}$ in the interior region is bounded by a weighted energy of the source, which is in turn bounded by $2\beta+2$-weighted energy of $\partial_\tau^{j+1}\varphi_{-\mathfrak{s},\ell}$, plus some boundary flux at $\rho=\tau$, for any $\beta\in[0,\ell-\mathfrak{s}-1]$. A simple iteration in β for these elliptic estimates then yields $\tau^{-3-4\ell-2j+2\varepsilon}$ decay for the energy $\int_{2M}^{\rho}(|\partial_\tau^j\varphi_{-\mathfrak{s},\ell}|^2+|\partial_\tau^j(r\partial_\rho\varphi_{-\mathfrak{s},\ell})|^2)\mathrm{d}\rho$ in terms of the $p=3-2\varepsilon$ weighted energy of $\widetilde{\Phi}_{-\mathfrak{s}}^{(\ell+\mathfrak{s})}$; therefore, in the NVNPC case, we prove $\tau^{-2-2\ell-j+\varepsilon}$ pointwise decay for $\partial_\tau^j\varphi_{-\mathfrak{s},\ell}$ in the interior region, and in the VNPC case, the decay is faster by τ^{-1} in terms of the $p=5-2\varepsilon$ weighted energy of $\widetilde{\Phi}_{-\mathfrak{s}}^{(\ell+\mathfrak{s})}$. As an extra benefit, we achieve an extra τ^{-1} decay for $\partial_\tau^j\partial_\rho\varphi_{-\mathfrak{s},\ell}$ compared to $\partial_\tau^j\varphi_{-\mathfrak{s},\ell}$ in both the NVNPC and VNPC cases.

Step C') First, we follow the idea in [8] to derive the asymptotic profiles for $\partial_\tau^{j_1}\partial_u^{j_2}\big((v-u)^{-2\ell}\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}\big)$ in the region $\{v-u\geq v^\alpha\}$ where α is a constant strictly less than 1 but supposed to be sufficiently close to 1. This is done as follows: One can first integrate along constant v for the wave equation (1.12) of $\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}$ to obtain the asymptotic profile of $V\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}$, then integrate along constant u from the hypersurface $\{v-u=v^\alpha\}$ to achieve the asymptotic profile of $\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}$. We can commute ∂_v with the wave equation for $\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}$ and run through the above argument again, which then enables us to derive the asymptotic profiles of $\partial_\tau^{j_1}\partial_u^{j_2}\big((v-u)^{-2\ell}\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}\big)$.

In the next step, we make use of equation (1.12) for $i\in\{0,\dots,\ell-\mathfrak{s}-1\}$ and obtain the following system

$$\begin{aligned}&(\ell-i-\mathfrak{s})(\ell+i+\mathfrak{s}+1)(v-u)^{-2(i+\mathfrak{s})}\widetilde{\Phi}_{+\mathfrak{s}}^{(i)}\\ &=\ -2(v-u)^2\partial_u\big((v-u)^{-2(i+\mathfrak{s}+1)}\widetilde{\Phi}_{+\mathfrak{s}}^{(i+1)}\big)\\ &\quad+(v-u)^{-2(i+\mathfrak{s})}\Big(O(1)\partial_u\widetilde{\Phi}_{+\mathfrak{s}}^{(i)}+\sum_{j=0}^{i}O(r^{-1})\widetilde{\Phi}_{+\mathfrak{s}}^{(j)}+O(r^{-1})r^{-1}\log r\widetilde{\Phi}_{+\mathfrak{s}}^{(i+1)}\Big).\end{aligned}$$

The last line has faster fall-off in the region $v-u\geq v^\alpha$, hence, one can determine the asymptotic profiles for $(v-u)^{-2(i+\mathfrak{s})}\widetilde{\Phi}_{+\mathfrak{s}}^{(i)}$ iteratively, including in particular the one for $(v-u)^{-2\mathfrak{s}}\widetilde{\Phi}_{+\mathfrak{s}}^{(0)}\sim 2^{-2\mathfrak{s}}r\Upsilon_{+\mathfrak{s}}$, which then yields the asymptotic profile of $\partial_\tau^j\Upsilon_{+\mathfrak{s}}$ in the region $\{v-u\geq v^\alpha\}$. In the remaining region $\{v-u\leq v^\alpha\}$, one can integrate $\partial_\rho\partial_\tau^j\varphi_{-\mathfrak{s},\ell}$ from the hypersurface $\{v-u=v^\alpha\}$ and make use of the better decay for $\partial_\rho\partial_\tau^j\varphi_{-\mathfrak{s},\ell}$ proven in Substep B4'); we thus conclude that the asymptotic profile of $\partial_\tau^j\varphi_{-\mathfrak{s},\ell}$ remains the same in the region $\{v-u\leq v^\alpha\}$. We shall comment that arbitrary $j\in\mathbb{N}$ times ∂_τ derivative yields an extra τ^{-j} pointwise decay, a fact which will show its importance in the next step. An application of the TSI then yields the asymptotics of the spin $+\mathfrak{s}$ component. In particular, for the spin $+\mathfrak{s}$ component, both the conjectured Price's law outside the black hole and the claim of Barack–Ori about faster decay on $\mathcal{H}^+$ can be shown via the TSI.

Step D') The basic idea in this step is in the same spirit of the one in [8] by reducing the VNPC problem to a NVNPC problem via defining a time integral of the solution. Again, in view of the TSI, we consider only the spin $+\mathfrak{s}$ component. The time integral is defined such that it solves also the TME of the spin $+\mathfrak{s}$ component and, more importantly, the time derivative of the time integral equals the spin $+\mathfrak{s}$ component. By this property, one gains an extra τ^{-1} decay for the spin $+\mathfrak{s}$ component than the decay of the time integral, hence than the one in NVNPC case. The main task of the remaining discussions is to calculate the associated N–P constant of the time integral.

1.3.2. *Remarks on other most relevant works.* Throughout the above discussions, we have been contrasting our results with the ones in [9, 8, 54] for a couple of reasons: on one hand, our approach follows partly from the ones in these works; on the other hand, we can contrast ours with these works in different steps in the proof easily since they are all based on the vector field method. We shall now briefly discuss the other works aiming at proving the Price's law and make comparisons with our results.

The work [58] by Metcalfe–Tataru–Tohaneanu followed the authors' earlier results [72, 57] and derived, under assumptions of both integrated local energy decay estimates and stationary local energy bounds, a global $v^{-2-s}\tau^{-2+s}$ decay for the spin $s = \pm 1$ components and the middle component, for which $s = 0$, of the Maxwell field in a class of non-stationary asymptotically flat spacetimes. The backgrounds under consideration in this work include in particular the Schwarzschild spacetimes and the family of the Kerr spacetimes, and the assumptions are known to hold true in a Schwarzschild spacetime or a slowly rotating Kerr spacetime after subtracting the static/stationary Coulomb solution but unknown in more general spacetimes. If in a Schwarzschild or a Kerr spacetime, the proven decay rates in [58] are slower than the Price's law by τ^{-1}.

The works [28, 29] by Donninger–Schlag–Soffer treated a Regge–Wheeler equation by constructing the Green's function for its solution and obtained on Schwarzschild $t^{-2\ell-2}$ for a fixed ℓ mode of the solution, t^{-3} decay for the scalar field, t^{-4} and t^{-6} decay for the part of Maxwell field and linearized gravity which solves this equation, respectively. Further, they showed that the decay rate is faster by t^{-1} for initially static initial data. As was discussed in Remark 1.2, by applying certain derivatives, known as the Chandrasekhar transformation [18], on the spin s components, the obtained scalars satisfy the Regge–Wheeler equation, the potential in which is propositional to s^2. These estimates are valid in a compact region but are not uniform in the future Cauchy development of a future Cauchy surface; the decay estimates are sharp for the entire scalar field, but have one less power of decay in time compared to Price's law in a compact region in the remaining cases.

Very recently, much important progress were made in proving the Price's law for the spin-$\mathfrak{s}$ fields on Schwarzschild, Reissner–Nördstrom, and Kerr backgrounds. For the scalar field, Hintz [40] computed the $v^{-1}\tau^{-2}$ leading order term on both Schwarzschild and subextremal Kerr spacetimes and obtained $t^{-2\ell_0-3}$ sharp asymptotics for $\ell \geq \ell_0$ modes in a compact region on Schwarzschild; Angelopoulos–Aretakis–Gajic derived in [10] the asymptotic profiles of the $\ell = 0$, $\ell = 1$, and $\ell \geq 2$ modes in a subextremal Kerr spacetime and computed in [11] the $v^{-1}\tau^{-2\ell_0-2}$ asymptotics for $\ell \geq \ell_0$ modes on a subextremal Reissner–Nordström background. For non-zero spin fields, in an earlier work [54] of the first author of our current work, $v^{-2-s}\tau^{-\frac{3}{2}+s}$ decay in non-static Kerr and $v^{-2-s}\tau^{-3+s+\epsilon}$ decay in Schwarzschild towards a stationary/static Coulomb solution are proven, and it also proves the almost Price's law $v^{-2-s}\tau^{-2-\ell_0+s+\varepsilon}$ for any $\ell \geq \ell_0$ modes for the Maxwell field in the region $\rho \geq \tau$ on a Schwarzschild background; the authors of this current work obtained in [55] the energy and Morawetz estimates and calculated the asymptotic profiles with decay $v^{-\frac{3}{2}-s}\tau^{-\frac{5}{2}+s}$ for the spin $s = \pm\frac{1}{2}$ components of the massless Dirac field on Schwarzschild.

1.3.3. *Future applications.* We propose some future applications of the methods developed here:

(1) a generalization of the results herein to the Reissner–Nordström spacetime, which will be a cornerstone in a proof of the Strong Cosmic Censorship for the linearized gravity of this spacetime. We note from [34] that the spin ± 2 components of the linearized gravity and the spin ± 1 components of the Maxwell field in their TME-like wave equations. Given now a unified treatment for the TME of both the Maxwell field and the linearized gravity in a Schwarzschild spacetime, we expect that the Price's law, or the asymptotic profiles, can also

be proven for both the linearized gravity and the Maxwell field in a Reissner–Nordström spacetime;

(2) a generalization to the non-static Kerr spacetimes where the angular momentum per mass $\mathbf{a} \neq 0$. In the nonvanishing $\mathbf{a}$ case, decoupling between different spherical symmetric modes is no longer valid in the evolution. It is quite interesting to analyze this coupling between different modes and investigate how the parameter $\mathbf{a}$ affects the asymptotic profiles of the spin s components or of a higher mode of them.

1.4. **Other relevant works.** Apart from the above mentioned works which are most relevant to the Price's law topic, we list here some other related works.

For the works on wave equations in Minkowski, we refer to the pioneering ones [59, 49, 19, 20, 51] and the references therein. There is a large amount of works in the literature about scalar field on a black hole background: Morawetz estimates as well as pointwise decay estimates are obtained in Schwarzschild in [14, 15, 16, 25] using a Morawetz type multiplier, in slowly rotating Kerr in [4, 73], and further extended to subextremal Kerr in [27]. Strichartz estimates are shown in [56, 77]. See also [31] for local decay of the scalar field on Kerr.

Morawetz estimates and decay estimates for Maxwell field are obtained in a Schwarzschild spacetime in [13, 62], in some general family of spherically symmetric stationary spacetimes in [71], and in slowly rotating Kerr in [5, 52]. We note also that a conserved, positive definite energy has been constructed for the Maxwell field in Schwarzschild in [2] and on Kerr and Kerr-de Sitter backgrounds but under axial symmetry in [36, 37]. Morawetz estimates and decay estimates for the spin ± 2 components of the linearized gravity in a slowly rotating Kerr spacetime are proven in [53, 23]. See also [32, 69] for some results in a subextremal Kerr spacetime. The mode stability results in a Kerr spacetime have been obtained for the scalar field in [68] and for general spin-$\mathfrak{s}$ fields in [78, 7, 22].

Linear stability of a Schwarzschild or a subextremal Reissner–Nordström spacetime has been shown by [24, 45, 46, 43, 44, 34]. See also [6]. Linear stability of a slowly rotating Kerr spacetime is proven in [3, 38].

Finally, we mention two nonlinear stability results [50, 41]: the first one for Schwarzschild under polarized axisymmetry, and the second one for slowly rotating Kerr-de Sitter spacetimes.

Overview of the paper. We give some preliminaries in Section 2, and then obtain the almost Price's law in Section 3.1. In Section 4, we prove the asymptotic profiles for the spin s components in the nonvanishing Newman–Penrose constant case. In the end, Section 5 is devoted to proving the asymptotic profiles in the vanishing Newman–Penrose constant case, which thus completes the proof of our main Theorem 1.1.

2. Preliminaries

2.1. **Coordinates and foliation of the spacetime.** Define a tortoise coordinate r^* by

$$\mathrm{d}r^* = \mu^{-1}\mathrm{d}r, \qquad r^*(3M) = 0. \tag{2.1}$$

Denote the forward double null coordinate $v = t + r^*$. Define a *hyperboloidal coordinate system* $(\tau, \rho, \theta, \phi)$ as in [3], where $\tau = v - h_{\text{hyp}}$ and $h_{\text{hyp}} = h_{\text{hyp}}(r)$, such that the level sets of the time function τ are strictly spacelike with

$$c(M)r^{-2} \leq -g(\nabla\tau, \nabla\tau) \leq C(M)r^{-2} \tag{2.2}$$

for two positive universal constants $c(M)$ and $C(M)$ and they cross the future event horizon regularly and are asymptotic to future null infinity $\mathcal{I}^+$, and for large r, $1 \lesssim \lim_{\rho\to\infty} r^2(\partial_r h_{\text{hyp}} - 2\mu^{-1})|_{\Sigma_\tau} < \infty$.

Let Σ_τ be the constant τ hypersurface in the domain of outer communication $\mathcal{D}$. Let $\tau_0 \geq 1$, and let Σ_{τ_0} be our initial hypersurface on which the initial data are imposed. For any $\tau_0 \leq \tau_1 < \tau_2$, let $\mathcal{D}_{\tau_1,\tau_2}$, $\mathcal{I}^+_{\tau_1,\tau_2}$ and $\mathcal{H}^+_{\tau_1,\tau_2}$ be the truncated parts of $\mathcal{D}$, $\mathcal{I}^+$ and $\mathcal{H}^+$ on $\tau \in [\tau_1, \tau_2]$, respectively. See Figures 1 and 2.

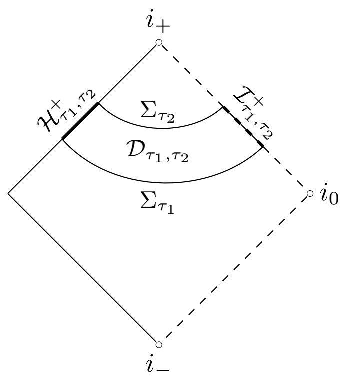

FIGURE 1. Hyperboloidal foliation and some related definitions.

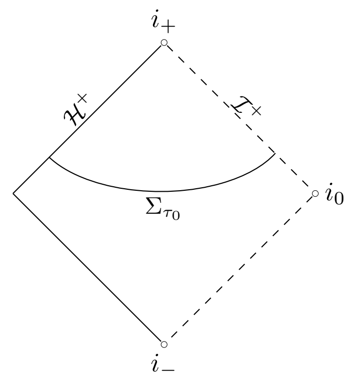

FIGURE 2. Initial hypersurface Σ_{τ_0}.

2.2. **General conventions.** $\mathbb{N}$ is denoted as the natural number set $\{0, 1, \ldots\}$, $\mathbb{N}^+$ the positive natural number set, $\mathbb{Z}^+$ the positive integer set, $\mathbb{R}$ the real number set, and $\mathbb{R}^+$ the positive real number set. Denote $\Re(\cdot)$ as the real part.

LHS and RHS are short for left-hand side and right-hand side, respectively.

Constants in this work may depend on the hyperboloidal foliation via the function h_{hyp}. For simplicity, we shall always suppress this dependence throughout this work as one can fix this function once for all. For the same reason, the dependence on the mass parameter M is always suppressed as well.

We denote a universal constant by C if it depends only on the hyperboloidal foliation and the mass M. If a universal constant depends on a set of parameters $\mathbf{P}$, we denote it by $C(\mathbf{P})$. We use regularity parameters, generally denoted by k, and k' which is a universal constant. Also, $k'(\mathbf{P})$ means a regularity constant depending on the parameters in the set $\mathbf{P}$.

We say $F_1 \lesssim F_2$ if there exists a universal constant C such that $F_1 \leq CF_2$. Similarly for $F_1 \gtrsim F_2$. If both $F_1 \lesssim F_2$ and $F_1 \gtrsim F_2$ hold, we say $F_1 \sim F_2$.

Let $\mathbf{P}$ be a set of parameters. We say $F_1 \lesssim_{\mathbf{P}} F_2$ if there exists a universal constant $C(\mathbf{P})$ such that $F_1 \leq C(\mathbf{P})F_2$. Similarly for $F_1 \gtrsim_{\mathbf{P}} F_2$. If both $F_1 \lesssim_{\mathbf{P}} F_2$ and $F_1 \gtrsim_{\mathbf{P}} F_2$ hold, then we say $F_1 \sim_{\mathbf{P}} F_2$.

For any $\alpha \in \mathbb{R}^+$, we say a function $f(r, \theta, \phi)$ is $O(r^{-\alpha})$ if for any $j \in \mathbb{N}$, $|(\partial_r)^j f_2| \leq C_j r^{-\alpha-j}$. Further, we say a function $f(r, \theta, \phi)$ is $O(1)$ if $|f| \leq C_0$ and $|(\partial_r)^j f| \leq C_j r^{-1-j}$ for any $j \in \mathbb{N}^+$.

For any $x \in \mathbb{R}$, let the Japanese bracket be defined by $\langle x \rangle = \sqrt{x^2 + 1}$.

Let χ_1 be a standard smooth cutoff function which is decreasing, 1 on $(-\infty, 0)$, and 0 on $(1, \infty)$, and let $\chi = \chi_1((R_0 - r)/M)$ with R_0 suitably large and to be fixed in the proof. So $\chi = 1$ for $r \geq R_0$ and vanishes identically for $r \leq R_0 - M$.

2.3. **Basic definitions.**

Definition 2.1.
- Define functions related to the hyperboloidal foliation

$$H_{\text{hyp}} = 2\mu^{-1} - \partial_r h_{\text{hyp}}, \quad \tilde{H}_{\text{hyp}} = 2\mu^{-1} - H_{\text{hyp}} = \partial_r h_{\text{hyp}}. \tag{2.3}$$

- Define $\mathrm{d}^2\mu = \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi$.

Definition 2.2.
- Define the vector fields

$$Y = \sqrt{2} n^a \partial_a, \qquad V = \sqrt{2} l^a \partial_a, \qquad \hat{V} = \mu^{-1} V. \tag{2.4}$$

- Define the *double null coordinates*

$$u = t - r^*, \qquad v = t + r^*. \tag{2.5}$$

Remark 2.3.
- One can express Y and $\hat{V}$ as

$$Y = -\partial_\rho + (2\mu^{-1} - H_{\text{hyp}})\partial_\tau, \quad \hat{V} = \partial_\rho + H_{\text{hyp}}\partial_\tau. \tag{2.6}$$

Moreover, we have

$$\partial_u = \frac{1}{2}\mu Y, \qquad \partial_v = \frac{1}{2}V = \frac{1}{2}\mu\hat{V}. \tag{2.7}$$

- By the choice of the hyperboloidal coordinates,

$$r^2 H_{\text{hyp}} \lesssim 1 \quad \text{for } r \text{ large}, \quad \text{and} \quad |H_{\text{hyp}} - 2\mu^{-1}| \lesssim 1 \quad \text{as } r \to r_+. \tag{2.8}$$

Definition 2.4.
- A scalar which has proper spin weight and zero boost weight in the sense of Geroch, Held and Penrose [33] is called a *spin-weighted scalar*.[4] Unless otherwise stated, we shall always denote s the spin weight, and we may call a spin-weighted scalar with spin weight s as a *spin s scalar*.
- A differential operator is a *spin-weighted operator* if it takes a spin-weighted scalar to a spin-weighted scalar.
- Let $\mathbb{X} = \{X_1, X_2, \ldots, X_n\}$, $n \in \mathbb{N}^+$, be a set of spin-weighted operators, and let a multi-index $\mathbf{a}$ be an ordered set $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ with all $a_i \in \{1, \ldots, n\}$. Define $|\mathbf{a}| = m$, and define $\mathbb{X}^{\mathbf{a}} = X_{a_1} X_{a_2} \cdots X_{a_m}$ if $m \in \mathbb{N}^+$ and $\mathbb{X}^{\mathbf{a}}$ as the identity operator if $m = 0$. Let φ be a spin-weighted scalar, and define its pointwise norm of order m, $m \in \mathbb{N}$, as

$$|\varphi|_{m,\mathbb{X}} = \sqrt{\sum_{|\mathbf{a}| \leq m} |\mathbb{X}^{\mathbf{a}}\varphi|^2}. \tag{2.9}$$

Definition 2.5. Let φ be a spin s scalar. Let the *spherical edth operators* $\mathring{\eth}$ and $\mathring{\eth}'$ be given in B–L coordinates by

$$\mathring{\eth}\varphi = \partial_\theta\varphi + icsc\theta\partial_\phi\varphi - scot\theta\varphi, \qquad \mathring{\eth}'\varphi = \partial_\theta\varphi - i\csc\theta\partial_\phi\varphi + scot\theta\varphi. \tag{2.10}$$

Remark 2.6. If φ is a spin s scalar, then $\mathring{\eth}\varphi$ and $\mathring{\eth}'\varphi$ are spin $s+1$ and $s-1$ scalars, respectively. That is, $\mathring{\eth}$ increases the spin weight by 1, while $\mathring{\eth}'$ decreases the spin weight by 1.

Definition 2.7. Let φ be a spin s scalar. Define a set of operators

$$\mathbb{B} = \{Y, V, r^{-1}\mathring{\eth}, r^{-1}\mathring{\eth}'\} \tag{2.11a}$$

adapted to the Hawking–Hartle tetrad, and its rescaled one

$$\widetilde{\mathbb{B}} = \{rY, rV, \mathring{\eth}, \mathring{\eth}'\}. \tag{2.11b}$$

Define a set of operators

$$\mathbb{D} = \{Y, rV, \mathring{\eth}, \mathring{\eth}'\} \tag{2.11c}$$

which is adapted to the hyperboloidal foliation and will be the set of commutators.

Now we are able to define energy norms and (spacetime) Morawetz norms.

Definition 2.8. Let φ be a spin-weighted scalar. Let $k \in \mathbb{N}$ and $\gamma \in \mathbb{R}$. Let Ω be a 4-dimensional subspace of the DOC and let Σ be a 3-dimensional space that can be parameterized by (ρ, θ, ϕ). Define

$$\|\varphi\|^2_{W^k_\gamma(\Omega)} = \int_\Omega r^\gamma |\varphi|^2_{k,\mathbb{D}} \mathrm{d}\tau \wedge \mathrm{d}\rho \wedge \mathrm{d}^2\mu, \tag{2.12a}$$

$$\|\varphi\|^2_{W^k_\gamma(\Sigma)} = \int_\Sigma r^\gamma |\varphi|^2_{k,\mathbb{D}} \mathrm{d}\rho \wedge \mathrm{d}^2\mu. \tag{2.12b}$$

[4] In particular, the spin-weighted scalars are sections of complex line bundles.

2.4. **Teukolsky master equation.**

Definition 2.9. Define two rescaled spin $\pm\mathfrak{s}$ components

$$\psi_{+\mathfrak{s}} = r^{2\mathfrak{s}}\Upsilon_{+\mathfrak{s}}, \qquad \psi_{-\mathfrak{s}} = \Upsilon_{-\mathfrak{s}}. \tag{2.13}$$

Define their radiation field

$$\Psi_{+\mathfrak{s}} = r\psi_{+\mathfrak{s}}, \qquad \Psi_{-\mathfrak{s}} = r\psi_{-\mathfrak{s}}. \tag{2.14}$$

Teukolsky [75] found that the scalars ψ_s in a Schwarzschild spacetime satisfy the celebrated *Teukolsky Master Equation* (TME), a separable, decoupled wave equation, which takes the following form in Boyer-Lindquist coordinates:

$$\begin{aligned}&\left(-\mu^{-1}r^2\partial_t^2 + \partial_r(\Delta\partial_r) + \frac{1}{\sin^2\theta}\partial_\phi^2 + \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{2is\cos\theta}{\sin^2\theta}\partial_\phi - (s^2\cot^2\theta + s)\right)\psi_s\\ &= -2s((r-M)Y - 2r\partial_t)\psi_s.\end{aligned} \tag{2.15}$$

This is a spin-weighted wave equation in the sense that the operator on the LHS of (2.15), being equal to $-\mu^{-1}r^2\partial_t^2 + \partial_r(\Delta\partial_r) + \mathring{\eth}\mathring{\eth}'$, is a spin-weighted wave operator. Such a TME is actually derived in [75] for general half integer spin fields on a larger family of spacetimes–the Kerr family of spacetimes [48], and it serves as a starting model for quite many results in obtaining quantitative estimates for these fields, including the scalar field, the Maxwell field and the linearized gravity.

2.5. **Spin-weighted spherical harmonic decomposition.** Recall that $\{Y^s_{m,\ell}(\cos\theta)e^{im\phi}\}_{m,\ell}$ are the eigenfunctions, called as the "spin-weighted spherical harmonics," of a self-adjoint operator $\mathring{\eth}'\mathring{\eth}$ and have eigenvalues $-\Lambda_{\ell,s} = -(\ell-s)(\ell+s+1)$ defined by

$$\mathring{\eth}'\mathring{\eth}(Y^s_{m,\ell}(\cos\theta)e^{im\phi}) = -\Lambda_{\ell,s}Y^s_{m,\ell}(\cos\theta)e^{im\phi}. \tag{2.16}$$

They form a complete orthonormal basis on $L^2(\mathrm{d}^2\mu)$. Further,

$$\begin{aligned}\mathring{\eth}(Y^s_{m,\ell}(\cos\theta)e^{im\phi}) &= -\sqrt{(\ell-s)(\ell+s+1)}Y^{s+1}_{m,\ell}(\cos\theta)e^{im\phi},\\ \mathring{\eth}'(Y^s_{m,\ell}(\cos\theta)e^{im\phi}) &= \sqrt{(\ell+s)(\ell-s+1)}Y^{s-1}_{m,\ell}(\cos\theta)e^{im\phi}.\end{aligned} \tag{2.17}$$

Definition 2.10. For any spin s scalar φ, let $(\varphi)^{\ell=\ell_0}$ and $(\varphi)_{m,\ell_0}$, $m \in \{-\ell_0, -\ell_0+1, \ldots, \ell_0\}$, be defined such that the following decompositions hold in $L^2(S^2)$:

$$\varphi = \sum_{\ell_0=|s|}^{\infty}(\varphi)^{\ell=\ell_0}, \tag{2.18a}$$

$$(\varphi)^{\ell=\ell_0} = \sum_{m=-\ell_0}^{\ell_0}(\varphi)_{m,\ell_0}Y^s_{m,\ell_0}(\cos\theta)e^{im\phi}. \tag{2.18b}$$

In particular, by definition,

$$\mathring{\eth}\mathring{\eth}'(\varphi)^{\ell=\ell_0} = -(\ell_0+s)(\ell_0-s+1)(\varphi)^{\ell=\ell_0}, \quad \mathring{\eth}'\mathring{\eth}(\varphi)^{\ell=\ell_0} = -(\ell_0-s)(\ell_0+s+1)(\varphi)^{\ell=\ell_0}. \tag{2.19}$$

Lemma 2.11. *Let φ be a spin s scalar, then*

$$\int_{S^2}\left(|\mathring{\eth}'\varphi|^2 - (s+|s|)|\varphi|^2\right)\mathrm{d}^2\mu = \int_{S^2}\left(|\mathring{\eth}\varphi|^2 - (|s|-s)|\varphi|^2\right)\mathrm{d}^2\mu \geq 0. \tag{2.20}$$

If φ is a spin s scalar and supported on $\ell \geq \ell_0$ modes, then

$$\int_{S^2}\left(|\mathring{\eth}'\varphi|^2 - (\ell_0+s)(\ell_0-s+1)|\varphi|^2\right)\mathrm{d}^2\mu = \int_{S^2}\left(|\mathring{\eth}\varphi|^2 - (\ell_0-s)(\ell_0+s+1)|\varphi|^2\right)\mathrm{d}^2\mu \geq 0. \tag{2.21}$$

2.6. **The full Maxwell system of equations.** The full system of the Maxwell equations (1.3) can be written as a system of first-order differential equations for the spin s components:

$$\mathring{\eth}\Upsilon_{0,\text{mid}} = 2Y\left(r^{-1}\psi_{+1}\right), \tag{2.22a}$$

$$\mathring{\eth}'\Upsilon_{0,\text{mid}} = 2\mu^{-1}V\left(\mu r\psi_{-1}\right), \tag{2.22b}$$

$$V\Upsilon_{0,\text{mid}} = 2\mathring{\eth}'\left(r^{-3}\psi_{+1}\right), \tag{2.22c}$$

$$Y\Upsilon_{0,\text{mid}} = 2\mathring{\eth}\left(r^{-1}\psi_{-1}\right). \tag{2.22d}$$

Definition 2.12. Let ${}^\star\mathbf{F}$ be the Hodge dual of the Maxwell field $\mathbf{F}$. Define the electronic and magnetic charges of a Maxwell field by

$$q_{\mathbf{E}} = \frac{1}{4\pi}\int_{S^2(\tau,\rho)} {}^\star\mathbf{F}, \qquad q_{\mathbf{B}} = \frac{1}{4\pi}\int_{S^2(\tau,\rho)} \mathbf{F}. \tag{2.23}$$

The following lemma is a standard statement and taken from [5, Proposition 2]. See also [54]. This is to decompose a Maxwell field into a static part and a radiative part.

Lemma 2.13. *The Maxwell field in a Schwarzschild spacetime can be decomposed into*

$$\mathbf{F} = \mathbf{F}_{sta} + \mathbf{F}_{rad}, \tag{2.24}$$

where the part $\mathbf{F}_{rad}$ is the non-charged radiative part of the Maxwell field and the other part $\mathbf{F}_{sta}$ is the charged static Coulomb part, such that

(1) $\mathbf{F}_{sta}$ and $\mathbf{F}_{rad}$ are both solutions to the Maxwell equations, and the N-P components of them satisfy

$$\Upsilon_{+1}(\mathbf{F}_{sta}) = \Upsilon_{-1}(\mathbf{F}_{sta}) = 0, \qquad \Upsilon_{0,mid}(\mathbf{F}_{sta}) = r^{-2}\left(q_{\mathbf{E}} + iq_{\mathbf{B}}\right), \tag{2.25a}$$

$$\Upsilon_{+1}(\mathbf{F}_{rad}) = \Upsilon_{+1}(\mathbf{F}), \qquad \Upsilon_{-1}(\mathbf{F}_{rad}) = \Upsilon_{-1}(\mathbf{F}); \tag{2.25b}$$

(2) The charges $q_{\mathbf{E}}$ and $q_{\mathbf{B}}$ are constants at all spheres $\mathbb{S}^2(\tau,\rho)$ for any $\tau \in \mathbb{R}$ and $\rho \geq 2M$ and can be calculated from the initial data;
(3) For any closed 2-surface, say S^2, $\int_{S^2}\mathbf{F}_{rad} = \int_{S^2}{}^\star\mathbf{F}_{rad} = 0$;
(4) $\partial_\tau\mathbf{F}_{sta} = 0$.

Remark 2.14. By such a decomposition, one can easily calculate the charged static Coulomb part of the Maxwell field from the initial data, and the estimates for the non-charged radiative part can be derived from the system (2.22) given estimates for the spin ± 1 components.

2.7. **Basic estimates.** We switch to stating some basic estimates.
The following simple Hardy's inequality will be useful.

Lemma 2.15. *Let φ be a spin s scalar. Then for any $r' > 2M$,*

$$\int_{2M}^{r'}|\varphi|^2\mathrm{d}r \lesssim \int_{2M}^{r'}\mu^2r^2|\partial_r\varphi|^2\mathrm{d}r + (r'-2M)|\varphi(r')|^2. \tag{2.26}$$

If, moreover, $\lim_{r\to\infty} r|\varphi|^2 = 0$, then

$$\int_{2M}^{\infty}|\varphi|^2\mathrm{d}r \lesssim \int_{2M}^{\infty}\mu^2r^2|\partial_r\varphi|^2\mathrm{d}r. \tag{2.27}$$

Proof. It follows easily by integrating the following equation

$$\partial_r((r-2M)|\varphi|^2) = |\varphi|^2 + 2(r-2M)\Re(\bar{\varphi}\partial_r\varphi) \tag{2.28}$$

from $2M$ to r' and applying the Cauchy-Schwarz inequality to the last product term. □

We will also use the following standard Hardy's inequality cited from [3, Lemma 4.30].

Lemma 2.16 (One-dimensional Hardy estimates). *Let $\alpha \in \mathbb{R}\setminus\{0\}$ and $h : [r_0, r_1] \to \mathbb{R}$ be a C^1 function.*

(1) If $r_0^\alpha |h(r_0)|^2 \leq D_0$ and $\alpha<0$, then

$$-2\alpha^{-1} r_1^\alpha |h(r_1)|^2 + \int_{r_0}^{r_1} r^{\alpha-1}|h(r)|^2 \mathrm{d}r \leq \frac{4}{\alpha^2}\int_{r_0}^{r_1} r^{\alpha+1}|\partial_r h(r)|^2\mathrm{d}r - 2\alpha^{-1}D_0; \tag{2.29a}$$

(2) If $r_1^\alpha |h(r_1)|^2 \leq D_0$ and $\alpha>0$, then

$$2\alpha^{-1} r_0^\alpha |h(r_0)|^2 + \int_{r_0}^{r_1} r^{\alpha-1}|h(r)|^2 \mathrm{d}r \leq \frac{4}{\alpha^2}\int_{r_0}^{r_1} r^{\alpha+1}|\partial_r h(r)|^2\mathrm{d}r + 2\alpha^{-1}D_0. \tag{2.29b}$$

Recall the following Sobolev-type estimates from [3, Lemmas 4.32 and 4.33].

Lemma 2.17. *Let φ be a spin s scalar. Then*

$$\sup_{\Sigma_\tau}|\varphi|^2 \lesssim_s \|\varphi\|^2_{W^3_{-1}(\Sigma_\tau)}. \tag{2.30}$$

If $\alpha\in(0,1]$, then

$$\sup_{\Sigma_\tau}|\varphi|^2 \lesssim_{s,\alpha} (\|\varphi\|^2_{W^3_{-2}(\Sigma_\tau)} + \|rV\varphi\|^2_{W^2_{-1-\alpha}(\Sigma_\tau)})^{\frac{1}{2}} (\|\varphi\|^2_{W^3_{-2}(\Sigma_\tau)} + \|rV\varphi\|^2_{W^2_{-1+\alpha}(\Sigma_\tau)})^{\frac{1}{2}}. \tag{2.31}$$

If $\lim_{\tau\to\infty}|r^{-1}\varphi| = 0$ pointwise in (ρ,θ,ϕ), then

$$|r^{-1}\varphi|^2 \lesssim_s \|\varphi\|_{W^3_{-3}(\mathcal{D}_{\tau,\infty})}\|\partial_\tau\varphi\|_{W^3_{-3}(\mathcal{D}_{\tau,\infty})}. \tag{2.32}$$

3. Almost Price's law

3.1. **Energy and Morawetz estimates.** We briefly review the energy and Morawetz estimates for the TME in this subsection.

Definition 3.1. With $\hat{V}$ defined as in Definition 2.2, we define the operator

$$\hat{\mathcal{V}} = r^2\hat{V}. \tag{3.1}$$

Let $i\in\mathbb{N}$ and define the following scalars constructed from the spin $\pm\mathfrak{s}$ components

$$\Xi^{(0)}_{+\mathfrak{s}} = r^{1-2\mathfrak{s}}\psi_{+\mathfrak{s}}, \qquad \Xi^{(i)}_{+\mathfrak{s}} = (-r^2Y)^i\Xi^{(0)}_{+\mathfrak{s}}, \tag{3.2a}$$

$$\Phi^{(0)}_{-\mathfrak{s}} = \mu^{\mathfrak{s}} r\psi_{-\mathfrak{s}}, \qquad \Phi^{(i)}_{-\mathfrak{s}} = \hat{\mathcal{V}}^i\Phi^{(0)}_{-\mathfrak{s}}. \tag{3.2b}$$

Lemma 3.2.

- *Let $\mathfrak{s}=1$. The equations of $\Phi^{(i)}_{-1}$, $i=0,1,2$, are*

$$(-r^2YV + \mathring{\eth}\mathring{\eth}' - 2)\Phi^{(0)}_{-1} = -2(r-3M)r^{-2}\Phi^{(1)}_{-1}, \tag{3.3a}$$

$$(-r^2YV + \mathring{\eth}\mathring{\eth}' - 2)\Phi^{(1)}_{-1} = 0, \tag{3.3b}$$

$$(-r^2YV + \mathring{\eth}\mathring{\eth}' - 12Mr^{-1})\Phi^{(2)}_{-1} - 2(r-3M)r^{-2}\hat{\mathcal{V}}\Phi^{(2)}_{-1} = 0. \tag{3.3c}$$

  The wave equations for $\{\Xi^{(i)}_{+1}\}_{i=0,1}$ are the same as (3.3a)–(3.3c) by replacing $\mathring{\eth}\mathring{\eth}'$ and $\{\Phi^{(i)}_{-1}\}_{i=0,1}$ by $\mathring{\eth}'\mathring{\eth}$ and $\{\Xi^{(i)}_{-1}\}_{i=0,1}$, respectively.

- *Let $\mathfrak{s}=2$. The equations of $\Phi^{(i)}_{-2}$, $i=0,1,2,3,4$, are*

$$(-r^2YV + \mathring{\eth}\mathring{\eth}' - 4 - 6Mr^{-1})\Phi^{(0)}_{-2} = -4(r-3M)r^{-2}\Phi^{(1)}_{-2}, \tag{3.4a}$$

$$(-r^2YV + \mathring{\eth}\mathring{\eth}' - 6 + 6Mr^{-1})\Phi^{(1)}_{-2} = -2(r-3M)r^{-2}\Phi^{(2)}_{-2} - 6M\Phi^{(0)}_{-2}, \tag{3.4b}$$

$$(-r^2YV + \mathring{\eth}\mathring{\eth}' - 6 + 6Mr^{-1})\Phi^{(2)}_{-2} = 0, \tag{3.4c}$$

$$(-r^2YV + \mathring{\eth}\mathring{\eth}' - 4 - 6Mr^{-1})\Phi^{(3)}_{-2} - 2(r-3M)r^{-2}\hat{\mathcal{V}}\Phi^{(3)}_{-2} = 6M\Phi^{(2)}_{-2}, \tag{3.4d}$$

$$(-r^2YV + \mathring{\eth}\mathring{\eth}' - 30Mr^{-1})\Phi^{(4)}_{-2} - 4(r-3M)r^{-2}\hat{\mathcal{V}}\Phi^{(4)}_{-2} = 0. \tag{3.4e}$$

  The wave equations for $\{\Xi^{(i)}_{+2}\}_{i=0,1,2}$ are the same as (3.4a)–(3.4c) by replacing $\mathring{\eth}\mathring{\eth}'$ and $\{\Phi^{(i)}_{-2}\}_{i=0,1,2}$ by $\mathring{\eth}'\mathring{\eth}$ and $\{\Xi^{(i)}_{-2}\}_{i=0,1,2}$, respectively.

Proof. The governing equations for the spin $+\mathfrak{s}$ scalars can be similarly derived as the ones for the spin $-\mathfrak{s}$ scalars, hence we focus only on the spin $-\mathfrak{s}$ scalars. Equations (3.3a) and (3.4a) are easily obtained from the TME (2.15) since the definition (3.2b) of $\Phi^{(0)}_{-\mathfrak{s}}$ involves only a rescaling of $\psi_{-\mathfrak{s}}$. By utilizing the following commutation relation

$$[\hat{\mathcal{V}}, -r^2YV]\varphi = -\hat{\mathcal{V}}\left(\frac{2(r-3M)}{r^2}\hat{\mathcal{V}}\varphi\right) = -\frac{2(r-3M)}{r^2}\hat{\mathcal{V}}^2\varphi + (2-12Mr^{-1})\hat{\mathcal{V}}\varphi, \tag{3.5}$$

the derivations of the other equations are straightforward. □

Equations of $\Phi^{(\mathfrak{s})}_{-\mathfrak{s}}$ and $\Xi^{(\mathfrak{s})}_{+\mathfrak{s}}$ are the scalar wave equation for $\mathfrak{s}=0$, the Fackerell–Ipser equation [30] for $\mathfrak{s}=1$ and the Regge–Wheeler equation [66] for $\mathfrak{s}=2$,[5] respectively. These equations can be treated in a similar way to obtain for the spin-weighted scalar under consideration a uniform bound of a nondegenerate energy and an integrated local decay estimate, also called a Morawetz estimate. We call these two types of estimates together as the energy and Morawetz estimate. One can also treat the wave systems of $\{\Phi^{(i)}_{-\mathfrak{s}}\}_{i=0,\dots,\mathfrak{s}}$ and $\{\Xi^{(i)}_{+\mathfrak{s}}\}_{i=0,\dots,\mathfrak{s}}$ and arrive at the energy and Morawetz estimates for both spin components. The following is a summary of the energy and Morawetz estimates proven for different spin fields in [25, 52, 53].

Theorem 3.3. *[Energy and Morawetz estimates for TME] Let $\mathfrak{s}+1 \le k \in \mathbb{N}^+$. There exists a universal constant $C = C(k)$ such that the following energy and Morawetz estimate holds in the region $\mathcal{D}_{\tau_1,\tau_2}$ for any $\tau_0 \le \tau_1 < \tau_2$ for the spin $s = \pm\mathfrak{s}$ components:*

$$\mathbf{BE}^k_{\Sigma_{\tau_1}}(\Psi_s) + \int_{\tau_1}^{\tau_2} \mathbf{BM}^k_{\Sigma_\tau}(\Psi_s)\mathrm{d}\tau \le C\mathbf{BE}^k_{\Sigma_{\tau_1}}(\Psi_s), \tag{3.6}$$

where for any $\tau \ge \tau_0$, the basic energies are

$$\mathbf{BE}^k_{\Sigma_\tau}(\Psi_{+1}) = \sum_{i=0,\dots,\mathfrak{s}}\sum_{|\mathbf{a}|\le k-\mathfrak{s}-1} \|\mathbb{B}^{\mathbf{a}}\Xi^{(i)}_{+\mathfrak{s}}\|^2_{W^1_{-2}(\Sigma_\tau)}$$

$$\mathbf{BE}^k_{\Sigma_\tau}(\Psi_{-1}) = \sum_{i=0,\dots,\mathfrak{s}}\sum_{|\mathbf{a}|\le k-\mathfrak{s}-1} \|\mathbb{B}^{\mathbf{a}}(\mu^{-\mathfrak{s}+i}\Phi^{(i)}_{-\mathfrak{s}})\|^2_{W^1_{-2}(\Sigma_\tau)},$$

and the basic Morawetz densities (in time) for any $\tau_2 > \tau_1 \ge \tau_0$ are

$$\mathbf{BM}^k_{\Sigma_\tau}(\Psi_{+\mathfrak{s}}) = \sum_{i=0,\dots,\mathfrak{s}}\sum_{|\mathbf{a}|\le k-\mathfrak{s}-1}\Big(\|\mathbb{B}^{\mathbf{a}}\Xi^{(i)}_{+\mathfrak{s}}\|^2_{W^0_{-3}(\Sigma_\tau)} + \|\mathbb{B}^{\mathbf{a}}\widetilde{\mathbb{B}}(\Xi^{(i)}_{+\mathfrak{s}})\|^2_{W^0_{-4}(\Sigma_\tau\cap\{r\ge 4M\})}\Big)$$

$$\mathbf{BM}^k_{\Sigma_\tau}(\Psi_{-\mathfrak{s}}) = \sum_{i=0,\dots,\mathfrak{s}}\sum_{|\mathbf{a}|\le k-\mathfrak{s}-1}\Big(\|\mathbb{B}^{\mathbf{a}}(\mu^{-\mathfrak{s}+i}\Phi^{(i)}_{-\mathfrak{s}})\|^2_{W^0_{-3}(\Sigma_\tau)} + \|\mathbb{B}^{\mathbf{a}}\widetilde{\mathbb{B}}(\mu^{-\mathfrak{s}+i}\Phi^{(i)}_{-\mathfrak{s}})\|^2_{W^0_{-4}(\Sigma_\tau\cap\{r\ge 4M\})}\Big).$$

Remark 3.4.
(1) The presence of a cutoff integral region $\Sigma_\tau \cap \{r \ge 4M\}$, instead of Σ_τ, in the expressions of the basic Morawetz densities is due to the trapping phenomenon at the trapped surface $r = 3M$ where one has to loss derivatives.
(2) The presence of the factor $\mu^{-\mathfrak{s}+i}$ in the expressions of the basic energy and the basic Morawetz density for the spin $s = -\mathfrak{s}$ component is such that the scalars $\mu^{-\mathfrak{s}+i}\Phi^{(i)}_{-\mathfrak{s}}$ are regular at future event horizon.

3.2. **Extended wave systems.** To obtain estimates close to the Price's law, we have to first derive further wave equations and add them to the wave systems in Lemma 3.2, after which we can achieve estimates for the obtained extended, larger wave systems. This follows closely [54] where the Maxwell field ($\mathfrak{s}=1$) is treated.

Definition 3.5. Let $i \in \mathbb{N}$ and define for the spin s components the following spin s scalars

$$\Phi^{(0)}_{+\mathfrak{s}} = \mu^{-\mathfrak{s}}r\psi_{+\mathfrak{s}}, \qquad \Phi^{(i)}_{+\mathfrak{s}} = \hat{\mathcal{V}}^i\Phi^{(0)}_{+\mathfrak{s}}, \tag{3.7a}$$

$$\Phi^{(0)}_{-\mathfrak{s}} = \mu^{\mathfrak{s}}r\psi_{-\mathfrak{s}}, \qquad \Phi^{(i)}_{-\mathfrak{s}} = \hat{\mathcal{V}}^i\Phi^{(0)}_{-\mathfrak{s}}. \tag{3.7b}$$

[5]In some works, these three equations are put in a wave form with an $\mathfrak{s}$-dependent potential and are all called the Regge–Wheeler equation.

Definition 3.6. Let $i \in \mathbb{N}$. Define $f_{\mathfrak{s},i,1} = i(i+2\mathfrak{s}+1)$, $f_{\mathfrak{s},i,2} = -(f_{\mathfrak{s},i+1,1} - f_{\mathfrak{s},i,1}) = -2(\mathfrak{s}+i+1)$, $f_{\mathfrak{s},i,3} = f_{\mathfrak{s},i,1} + \frac{1}{3}(\mathfrak{s}+1)(2\mathfrak{s}+1)$, $g_{\mathfrak{s},i} = 6\sum\limits_{j=0}^{i-1} f_{\mathfrak{s},j,3} = 2i(i+\mathfrak{s})(i+2\mathfrak{s})$. Further, let $x_{\mathfrak{s},i+1,i} = \frac{g_{\mathfrak{s},i+1}}{f_{\mathfrak{s},i+1,1}-f_{\mathfrak{s},i,1}} = (i+1)(i+2\mathfrak{s}+1)$ and $x_{\mathfrak{s},i+1,j} = -\frac{g_{\mathfrak{s},i+1}x_{\mathfrak{s},i,j}}{f_{\mathfrak{s},i+1,1}-f_{\mathfrak{s},j,1}}$ for $0 \le j \le i-1$, and define

$$\widetilde{\Phi}^{(0)}_{+\mathfrak{s}} = \Phi^{(0)}_{+\mathfrak{s}}, \qquad \widetilde{\Phi}^{(i+1)}_{+\mathfrak{s}} = \Phi^{(i+1)}_{+\mathfrak{s}} + \sum_{j=0}^{i} x_{\mathfrak{s},i+1,j} M^{i+1-j} \widetilde{\Phi}^{(j)}_{+\mathfrak{s}}, \tag{3.8a}$$

$$\widetilde{\Phi}^{(2\mathfrak{s})}_{-\mathfrak{s}} = \Phi^{(2\mathfrak{s})}_{-\mathfrak{s}}, \qquad \widetilde{\Phi}^{(i+2\mathfrak{s}+1)}_{-\mathfrak{s}} = \Phi^{(i+2\mathfrak{s}+1)}_{-1} + \sum_{j=0}^{i} x_{\mathfrak{s},i+1,j} M^{i+1-j} \widetilde{\Phi}^{(j+2\mathfrak{s})}_{-1}. \tag{3.8b}$$

Proposition 3.7. *Let $i \in \mathbb{N}$.*

(1) *The equation of $\Phi^{(0)}_{+\mathfrak{s}}$ is*

$$-\mu Y\hat{\mathcal{V}}\Phi^{(0)}_{+\mathfrak{s}} + \mathring{\eth}'\mathring{\eth}\Phi^{(0)}_{+\mathfrak{s}} - 2(\mathfrak{s}+1)(r-3M)r^{-2}\hat{\mathcal{V}}\Phi^{(0)}_{+\mathfrak{s}} - 2(\mathfrak{s}+1)(2\mathfrak{s}+1)Mr^{-1}\Phi^{(0)}_{+\mathfrak{s}} = 0, \tag{3.9}$$

the equation of $\Phi^{(i)}_{+\mathfrak{s}}$ is

$$-\mu Y\hat{\mathcal{V}}\Phi^{(i)}_{+\mathfrak{s}} + (\mathring{\eth}'\mathring{\eth} + f_{\mathfrak{s},i,1})\Phi^{(i)}_{+\mathfrak{s}} + f_{\mathfrak{s},i,2}(r-3M)r^{-2}\hat{\mathcal{V}}\Phi^{(i)}_{+\mathfrak{s}} - 6f_{\mathfrak{s},i,3}Mr^{-1}\Phi^{(i)}_{+\mathfrak{s}} + g_{\mathfrak{s},i}M\Phi^{(i-1)}_{+\mathfrak{s}} = 0, \tag{3.10}$$

and the equation of $\widetilde{\Phi}^{(i)}_{+\mathfrak{s}}$ is

$$-\mu Y\hat{\mathcal{V}}\widetilde{\Phi}^{(i)}_{+\mathfrak{s}} + (\mathring{\eth}'\mathring{\eth} + f_{\mathfrak{s},i,1})\widetilde{\Phi}^{(i)}_{+\mathfrak{s}} + f_{\mathfrak{s},i,2}(r-3M)r^{-2}\hat{\mathcal{V}}\widetilde{\Phi}^{(i)}_{+\mathfrak{s}} + \sum_{j=0}^{i} h_{\mathfrak{s},i,j}\Phi^{(j)}_{+\mathfrak{s}} = 0, \tag{3.11}$$

with $h_{\mathfrak{s},i,j} = O(r^{-1})$ for all $j \in \{0,1,\ldots,i\}$. Moreover, all the functions $h_{\mathfrak{s},i,j}$ can be determined by the following relation

$$\begin{aligned}\sum_{j=0}^{i+1} h_{\mathfrak{s},i+1,j}\Phi^{(j)}_{+\mathfrak{s}} = {} & -(r-3M)r^{-2}\sum_{j=0}^{i}(f_{\mathfrak{s},i+1,2} - f_{\mathfrak{s},j,2})x_{\mathfrak{s},i+1,j}M^{i+1-j}\hat{\mathcal{V}}\widetilde{\Phi}^{(j)}_{+\mathfrak{s}} \\ & + \sum_{j=0}^{i} x_{\mathfrak{s},i+1,j}M^{i+1-j}\sum_{j'=0}^{j} h_{\mathfrak{s},j,j'}\Phi^{(j')}_{+\mathfrak{s}}\end{aligned} \tag{3.12}$$

together with $h_{\mathfrak{s},0,0} = -2(\mathfrak{s}+1)(2\mathfrak{s}+1)Mr^{-1}$.

(2) *The equation of $\Phi^{(2\mathfrak{s})}_{-\mathfrak{s}}$ is*

$$-\mu Y\hat{\mathcal{V}}\Phi^{(2\mathfrak{s})}_{-\mathfrak{s}} + \mathring{\eth}\mathring{\eth}'\Phi^{(2\mathfrak{s})}_{-\mathfrak{s}} - 2(\mathfrak{s}+1)(r-3M)r^{-2}\hat{\mathcal{V}}\Phi^{(2\mathfrak{s})}_{-\mathfrak{s}} - 2(\mathfrak{s}+1)(2\mathfrak{s}+1)Mr^{-1}\Phi^{(2\mathfrak{s})}_{-\mathfrak{s}} = 0, \tag{3.13}$$

and the equation of $\Phi^{(i+2\mathfrak{s})}_{-\mathfrak{s}}$ is

$$-\mu Y\hat{\mathcal{V}}\Phi^{(i+2\mathfrak{s})}_{-\mathfrak{s}} + (\mathring{\eth}\mathring{\eth}' + f_{\mathfrak{s},i,1})\Phi^{(i+2\mathfrak{s})}_{-\mathfrak{s}} + f_{\mathfrak{s},i,2}(r-3M)r^{-2}\hat{\mathcal{V}}\Phi^{(i+2\mathfrak{s})}_{-\mathfrak{s}} - 6f_{\mathfrak{s},i,3}Mr^{-1}\Phi^{(i+2\mathfrak{s})}_{-\mathfrak{s}} + g_{\mathfrak{s},i}M\Phi^{(i+2\mathfrak{s}-1)}_{-\mathfrak{s}} = 0, \tag{3.14}$$

and the equation of $\widetilde{\Phi}^{(i+2\mathfrak{s})}_{-\mathfrak{s}}$ is

$$-\mu Y\hat{\mathcal{V}}\widetilde{\Phi}^{(i+2\mathfrak{s})}_{-\mathfrak{s}} + (\mathring{\eth}\mathring{\eth}' + f_{\mathfrak{s},i,1})\widetilde{\Phi}^{(i+2\mathfrak{s})}_{-\mathfrak{s}} + f_{\mathfrak{s},i,2}(r-3M)r^{-2}\hat{\mathcal{V}}\widetilde{\Phi}^{(i+2\mathfrak{s})}_{-\mathfrak{s}} + \sum_{j=0}^{i} h_{\mathfrak{s},i,j}\Phi^{(j+2\mathfrak{s})}_{-\mathfrak{s}} = 0, \tag{3.15}$$

with $h_{\mathfrak{s},i,j}$ being of the same expression as the ones in (3.11) *and satisfying $h_{\mathfrak{s},i,j} = O(r^{-1})$ for all $j \in \{0,1,\ldots,i\}$.*

Proof. By the TME (2.15), one has

$$\big(-r^2YV + \mathring{\eth}'\mathring{\eth} - 2(s+1)Mr^{-1}\big)(r\psi_s) = -2s((r-M)Y - 2r\partial_t)(r\psi_s). \tag{3.16}$$

In view of the definition (3.7a), one therefore reaches

$$-r^2YV\Phi^{(0)}_{+\mathfrak{s}} + \mathring{\eth}'\mathring{\eth}\Phi^{(0)}_{+\mathfrak{s}} - 2\mathfrak{s}(r-3M)r^{-2}\hat{\mathcal{V}}\Phi^{(0)}_{+\mathfrak{s}} - 2(\mathfrak{s}+1)(2\mathfrak{s}+1)Mr^{-1}\Phi^{(0)}_{+\mathfrak{s}} = 0, \tag{3.17}$$

which is exactly equation (3.9).

Commuting equation (3.17) with $\hat{\mathcal{V}}^i$ and using the commutator (3.5), we can inductively show

$$-r^2YV\Phi_{+\mathfrak{s}}^{(i)}+(\mathring{\eth}'\mathring{\eth}+f_{\mathfrak{s},i,1})\Phi_{+\mathfrak{s}}^{(i)}+(f_{\mathfrak{s},i,2}+2)(r-3M)r^{-2}\hat{\mathcal{V}}\Phi_{+\mathfrak{s}}^{(i)}-6f_{\mathfrak{s},i,3}Mr^{-1}\Phi_{+\mathfrak{s}}^{(i)}+g_{\mathfrak{s},i}M\Phi_{+\mathfrak{s}}^{(i-1)}=0. \tag{3.18}$$

This thus proves (3.10) for general $i\in\mathbb{N}$.

We prove equation (3.11) by induction. Assume it holds for $\widetilde{\Phi}_{+\mathfrak{s}}^{(j)}$ for all $1\leq j\leq i$, it suffices to show that it holds also for $\widetilde{\Phi}_{+\mathfrak{s}}^{(i+1)}$. By adding an $x_{\mathfrak{s},i+1,j}M^{i+1-j}$ multiple of equation (3.11) for $\widetilde{\Phi}_{+\mathfrak{s}}^{(j)}$ for all $j=0,1,\ldots,i$ to equation (3.10) of $\Phi_{+\mathfrak{s}}^{(i+1)}$, one obtains

$$\begin{aligned}
&-\mu Y\hat{\mathcal{V}}\widetilde{\Phi}_{+\mathfrak{s}}^{(i+1)}+(\mathring{\eth}'\mathring{\eth}+f_{i+1,1})\widetilde{\Phi}_{+\mathfrak{s}}^{(i+1)}+f_{\mathfrak{s},i+1,2}(r-3M)r^{-2}\hat{\mathcal{V}}\widetilde{\Phi}_{+\mathfrak{s}}^{(i+1)}-6f_{\mathfrak{s},i+1,3}Mr^{-1}\widetilde{\Phi}_{+\mathfrak{s}}^{(i+1)}\\
&-\sum_{j=0}^{i}x_{\mathfrak{s},i+1,j}M^{i+1-j}(f_{\mathfrak{s},i+1,1}-f_{\mathfrak{s},j,1})\widetilde{\Phi}_{+\mathfrak{s}}^{(j)}+g_{\mathfrak{s},i+1}M\Phi_{+\mathfrak{s}}^{(i)}\\
&-(r-3M)r^{-2}\sum_{j=0}^{i}(f_{\mathfrak{s},i+1,2}-f_{\mathfrak{s},j,2})x_{\mathfrak{s},i+1,j}M^{i+1-j}\hat{\mathcal{V}}\widetilde{\Phi}_{+\mathfrak{s}}^{(j)}+\sum_{j=0}^{i}x_{\mathfrak{s},i+1,j}M^{i+1-j}\sum_{j'=0}^{j}h_{\mathfrak{s},j,j'}\Phi_{+\mathfrak{s}}^{(j')}=0.
\end{aligned} \tag{3.19}$$

We replace $\Phi_{+\mathfrak{s}}^{(i)}=\widetilde{\Phi}_{+\mathfrak{s}}^{(i)}-\sum\limits_{j=0}^{i-1}x_{\mathfrak{s},i,j}M^{i-j}\widetilde{\Phi}_{+\mathfrak{s}}^{(j)}$ in the last term of the second line and find the second line equals $\sum\limits_{j=1}^{i}e_{\mathfrak{s},i+1,j}M^{i+1-j}\widetilde{\Phi}_{+\mathfrak{s}}^{(j)}$ with

$$e_{\mathfrak{s},i+1,i}=-x_{\mathfrak{s},i+1,i}(f_{\mathfrak{s},i+1,1}-f_{\mathfrak{s},i,1})+g_{\mathfrak{s},i+1}, \tag{3.20a}$$

$$e_{\mathfrak{s},i+1,j}=-x_{\mathfrak{s},i+1,j}(f_{\mathfrak{s},i+1,1}-f_{\mathfrak{s},j,1})-g_{\mathfrak{s},i+1}x_{\mathfrak{s},i,j},\quad \text{for}\quad 0\leq j\leq i-1. \tag{3.20b}$$

All these $\{e_{\mathfrak{s},i+1,j}\}_{j=0,1,\ldots,i}$ identically vanish by the choices of $\{x_{\mathfrak{s},i+1,j}\}_{j=0,1,\ldots,i}$ made in Definition 3.6, hence the entire second line of (3.19) vanishes. One can rewrite $\hat{\mathcal{V}}\widetilde{\Phi}_{+\mathfrak{s}}^{(j)}$ using Definition 3.6 as a weighted sum of $\{\widetilde{\Phi}_{+\mathfrak{s}}^{(j')}\}|_{j'=0,1,\ldots,j+1}$ with all coefficients being $O(1)$, and by denoting all the terms in the last line on the LHS of (3.19) as $\sum\limits_{j=0}^{i+1}h_{\mathfrak{s},i+1,j}\Phi_{+\mathfrak{s}}^{(j)}$, one finds $h_{\mathfrak{s},i+1,j}=O(r^{-1})$ for all $j\in\{0,1,\ldots,i+1\}$. All these together then prove equation (3.11) for $\widetilde{\Phi}_{+\mathfrak{s}}^{(i+1)}$.

For the spin $-\mathfrak{s}$ component, equation (3.13) comes directly from (3.3c) and (3.4e). Given that equations (3.13) and (3.9) have the same form, the derivation of equations (3.14) and (3.15) is immediate. □

3.3. **Initial energies.** We define a few initial energies in this subsection, which will be utilized frequently in the rest of this work.

Definition 3.8. Let $n\in\mathbb{N}$ and $n\geq\min\{0,s\}$, and let $k\geq n$. Let $\tau\geq\tau_0$. Let $\widetilde{\Phi}_{+\mathfrak{s}}^{(i)}$ and $\widetilde{\Phi}_{-\mathfrak{s}}^{(i)}$ be defined as in Definition 3.6. Define on Σ_τ an energy of the spin $s=+\mathfrak{s}$ component

$$\mathbb{I}_{\Sigma_\tau}^{n,k,\alpha}[\Psi_{+\mathfrak{s}}]=\begin{cases}\sum_{i=0}^{n-\mathfrak{s}}\|(r^2V)^i\Psi_{+\mathfrak{s}}\|^2_{W_{-2}^{k-i}(\Sigma_\tau)}+\|rV\widetilde{\Phi}_{+\mathfrak{s}}^{(n-\mathfrak{s})}\|^2_{W_{\alpha-2}^{k-n}(\Sigma_\tau\cap\{\rho\geq 4M\})}, & \alpha>2;\\ \sum_{i=0}^{n-\mathfrak{s}}\|(r^2V)^i\Psi_{+\mathfrak{s}}\|^2_{W_{-2}^{k-i}(\Sigma_\tau)}+\|rV\Phi_{+\mathfrak{s}}^{(n-\mathfrak{s})}\|^2_{W_{\alpha-2}^{k-n}(\Sigma_\tau\cap\{\rho\geq 4M\})}, & \alpha\leq 2,\end{cases} \tag{3.21a}$$

and an energy of the spin $s=-\mathfrak{s}$ component

$$\mathbb{I}_{\Sigma_\tau}^{n,k,\alpha}[\Psi_{-\mathfrak{s}}]=\begin{cases}\sum_{i=0}^{n+\mathfrak{s}}\|(r^2V)^i\Psi_{-\mathfrak{s}}\|^2_{W_{-2}^{k+\mathfrak{s}-i}(\Sigma_\tau)}+\|rV\widetilde{\Phi}_{-\mathfrak{s}}^{(n+\mathfrak{s})}\|^2_{W_{\alpha-2}^{k-n}(\Sigma_\tau\cap\{\rho\geq 4M\})}, & \alpha>2;\\ \sum_{i=0}^{n+\mathfrak{s}}\|(r^2V)^i\Psi_{-\mathfrak{s}}\|^2_{W_{-2}^{k+\mathfrak{s}-i}(\Sigma_\tau)}+\|rV\Phi_{-\mathfrak{s}}^{(n+\mathfrak{s})}\|^2_{W_{\alpha-2}^{k-n}(\Sigma_\tau\cap\{\rho\geq 4M\})}, & \alpha\leq 2.\end{cases} \tag{3.21b}$$

Definition 3.9. Define initial energies

$$\mathbb{I}_{\mathrm{NV},\Sigma_{\tau_0}}^{\ell\geq\ell_0,k,\vartheta}[\Psi_{+\mathfrak{s}}]=\mathbb{I}_{\Sigma_{\tau_0}}^{\ell_0,k,3-\vartheta}[(\Psi_{+\mathfrak{s}})^{\ell=\ell_0}]+\sum_{\ell'=\ell_0+1}^{2\ell_0+1}\mathbb{I}_{\Sigma_{\tau_0}}^{\ell',k,1+\vartheta}[(\Psi_{+\mathfrak{s}})^{\ell=\ell'}]$$

$$+ \mathbb{I}^{2\ell_0+2,k,1+\vartheta}_{\Sigma_{\tau_0}}[(\Psi_{+\mathfrak{s}})^{\ell\geq 2\ell_0+2})], \tag{3.22}$$

$$\mathbb{I}^{\ell\geq\ell_0,k,\vartheta}_{\mathrm{V},\Sigma_{\tau_0}}[\Psi_{+\mathfrak{s}}] = \mathbb{I}^{\ell_0,k,5-\vartheta}_{\Sigma_{\tau_0}}[(\Psi_{+\mathfrak{s}})^{\ell=\ell_0})] + \mathbb{I}^{\ell_0+1,k,3-\vartheta}_{\Sigma_{\tau_0}}[(\Psi_{+\mathfrak{s}})^{\ell=\ell_0+1})]$$

$$+ \sum_{\ell'=\ell_0+2}^{2\ell_0+2} \mathbb{I}^{\ell',k,1+\vartheta}_{\Sigma_{\tau_0}}[(\Psi_{+\mathfrak{s}})^{\ell=\ell'})] + \mathbb{I}^{2\ell_0+3,k,1+\vartheta}_{\Sigma_{\tau_0}}[(\Psi_{+\mathfrak{s}})^{\ell\geq 2\ell_0+3})]. \tag{3.23}$$

3.4. **Weak decay estimates.** We now use the wave systems derived in Sections 3.1–3.2 together with the energy and Morawetz estimates in Theorem 3.3 to achieve decay estimates. These pointwise decay estimates are by no means optimal and will be improved in the later sections; hence we call them "*weak decay estimates.*"

Proposition 3.10. ***[Weak decay estimates for the spin $\pm\mathfrak{s}$ components]***

(1) If the spin $s = \pm\mathfrak{s}$ components are supported on a fixed ℓ mode, $\ell \geq \mathfrak{s}$, then for any $j \in \mathbb{N}$ and $(i,p) \in \{\max\{0,s\} < i \leq \ell, 0 \leq p < 5\} \cup \{i = \max\{0,s\}, p > 1\}$, there exists a $k' = k'(j,i)$ such that

$$|\partial_\tau^j (r^{-1}\Psi_{+\mathfrak{s}})|_{k,\mathbb{D}} \lesssim (\mathbb{I}^{i,k+k',p}_{\Sigma_\tau}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}} v^{-1}\tau^{-(p-1)/2-(i-s)-j}, \tag{3.24a}$$

$$\sum_{i=0}^{\mathfrak{s}} |\partial_\tau^j (r^{-1}(r^2V)^i\Psi_{-\mathfrak{s}})|_{k,\mathbb{D}} \lesssim (\mathbb{I}^{i,k+k',p}_{\Sigma_\tau}[\Psi_{-\mathfrak{s}}])^{\frac{1}{2}} v^{-1}\tau^{-(p-1)/2-i-j}. \tag{3.24b}$$

(2) If the spin $s = \pm\mathfrak{s}$ components are supported on $\ell \geq \ell_0$ modes, $\ell_0 \geq \mathfrak{s}+1$, then for any $j \in \mathbb{N}$ and $0 \leq p \leq 2$, there exists a $k' = k'(j,\ell_0)$ such that

$$|\partial_\tau^j (r^{-1}\Psi_{+\mathfrak{s}})|_{k,\mathbb{D}} \lesssim (\mathbb{I}^{\ell_0,k+k',p}_{\Sigma_\tau}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}} v^{-1}\tau^{-(p-1)/2-\ell_0-j}, \tag{3.25a}$$

$$\sum_{n=0}^{\mathfrak{s}} |\partial_\tau^j (r^{-1}(r^2V)^n\Psi_{-\mathfrak{s}})|_{k,\mathbb{D}} \lesssim (\mathbb{I}^{\ell_0,k+k',p}_{\Sigma_\tau}[\Psi_{-\mathfrak{s}}])^{\frac{1}{2}} v^{-1}\tau^{-(p-1)/2-\ell_0-j}. \tag{3.25b}$$

We state here an r^p estimate for a general spin-weighted wave equation, which is an analog of the one first proven for spin-0 scalar field in [26]. See also [3, 54, 55] where r^p estimates are derived for a general spin-weighted wave equation on Schwarzschild or Kerr.

Proposition 3.11. *Let $k \in \mathbb{N}$, $\mathfrak{s} \in \mathbb{N}$, and $p \in \mathbb{R}^+ \cup \{0\}$. Let $0 < \delta, \varsigma, \varepsilon < 1/2$ be arbitrary. Let φ and $\vartheta = \vartheta(\varphi)$ be spin s scalars satisfying*

$$-r^2YV\varphi + (\mathring{\eth}\mathring{\eth}' - b_0)\varphi - b_V V\varphi = \vartheta. \tag{3.26}$$

Let b_V and b_0 be smooth real functions of r such that

(1) $\exists b_{V,-1} \in \mathbb{R}^+ \cup \{0\}$ such that $b_V = b_{V,-1}r + O(1)$, and

(2) $\exists b_{0,0} \in \mathbb{R}$ such that $b_0 = b_{0,0} + O(r^{-1})$ and the eigenvalues of $\mathring{\eth}\mathring{\eth}' - b_{0,0}$ are non-positive, i.e. $\int_{\mathbb{S}^2}(|\mathring{\eth}'\varphi|^2 + b_{0,0}|\varphi|^2)\mathrm{d}^2\mu \geq 0$.

Then there is a constant $\hat{R}_0 = \hat{R}_0(p, b_0, b_V)$ such that for all $R_0 \geq \hat{R}_0$ and $\tau_2 > \tau_1 \geq \tau_0$,

- *for $p = 0$ and $b_{V,-1} > 0$,*

$$\|\varphi\|^2_{W^{k+1}_{-2}(\Sigma^{R_0}_{\tau_2})} + \|\varphi\|^2_{W^{k+1}_{-3}(\mathcal{D}^{R_0}_{\tau_1,\tau_2})} \lesssim_{[R_0-M,R_0]} \|\varphi\|^2_{W^{k+1}_{-2}(\Sigma^{R_0}_{\tau_1})} + \|\vartheta\|^2_{W^k_{-3}(\mathcal{D}^{R_0}_{\tau_1,\tau_2})}; \tag{3.27a}$$

- *for $p = 0$ and $b_{V,-1} = 0$,*

$$\|\varphi\|^2_{W^{k+1}_{-2}(\Sigma^{R_0}_{\tau_2})} + \|\varphi\|^2_{W^{k+1}_{-3}(\mathcal{D}^{R_0}_{\tau_1,\tau_2})} \lesssim_{[R_0-M,R_0]} \|\varphi\|^2_{W^{k+1}_{-2}(\Sigma^{R_0}_{\tau_1})} + \|rV\varphi\|^2_{W^k_{-3}(\mathcal{D}^{R_0}_{\tau_1,\tau_2})} + \|\vartheta\|^2_{W^k_{-3}(\mathcal{D}^{R_0}_{\tau_1,\tau_2})}; \tag{3.27b}$$

- *for $p \in (0,2)$,*

$$\|rV\varphi\|^2_{W^k_{p-2}(\Sigma^{\geq R_0}_{\tau_2})} + \|\varphi\|^2_{W^{k+1}_{-2}(\Sigma^{\geq R_0}_{\tau_2})} + \|\varphi\|^2_{W^{k+1}_{p-3}(\mathcal{D}^{\geq R_0}_{\tau_1,\tau_2})} + \|Y\varphi\|^2_{W^k_{-1-\delta}(\mathcal{D}^{\geq R_0}_{\tau_1,\tau_2})}$$

$$\lesssim_{[R_0-M,R_0]} \|rV\varphi\|^2_{W^k_{p-2}(\Sigma^{\geq R_0}_{\tau_1})} + \|\varphi\|^2_{W^{k+1}_{-2}(\Sigma^{\geq R_0}_{\tau_1})} + \|\vartheta\|^2_{W^k_{p-3}(\mathcal{D}^{\geq R_0}_{\tau_1,\tau_2})}; \tag{3.27c}$$

- *for $p=2$, both of the following two estimates hold:*

$$\begin{aligned}&\|rV\varphi\|^2_{W_0^k(\Sigma_{\tau_2}^{\geq R_0})}+\|\varphi\|^2_{W_{-2}^{k+1}(\Sigma_{\tau_2}^{\geq R_0})}+\|\varphi\|^2_{W_{-1-\delta}^{k+1}(\mathcal{D}_{\tau_1,\tau_2}^{\geq R_0})}+\|rV\varphi\|^2_{W_{-1}^k(\mathcal{D}_{\tau_1,\tau_2}^{\geq R_0})}\\&\quad\lesssim_{[R_0-M,R_0]}\|rV\varphi\|^2_{W_0^k(\Sigma_{\tau_1}^{\geq R_0})}+\|\varphi\|^2_{W_{-2}^{k+1}(\Sigma_{\tau_1}^{\geq R_0})}+\|\vartheta\|^2_{W_{-1}^k(\mathcal{D}_{\tau_1,\tau_2}^{\geq R_0})};\end{aligned}\tag{3.27d}$$

$$\begin{aligned}&\|rV\varphi\|^2_{W_0^k(\Sigma_{\tau_2}^{\geq R_0})}+\|\varphi\|^2_{W_{-2}^{k+1}(\Sigma_{\tau_2}^{\geq R_0})}+\|\varphi\|^2_{W_{-1-\delta}^{k+1}(\mathcal{D}_{\tau_1,\tau_2}^{\geq R_0})}+\|rV\varphi\|^2_{W_{-1}^k(\mathcal{D}_{\tau_1,\tau_2}^{\geq R_0})}\\&\quad\lesssim_{[R_0-M,R_0]}\|rV\varphi\|^2_{W_0^k(\Sigma_{\tau_1}^{\geq R_0})}+\|\varphi\|^2_{W_{-2}^{k+1}(\Sigma_{\tau_1}^{\geq R_0})}\\&\quad+\varepsilon\int_{\tau_1}^{\tau_2}\langle\tau-\tau_1\rangle^{-1-\varsigma}\|rV\varphi\|^2_{W_0^k(\Sigma_\tau^{\geq R_0})}\mathrm{d}\tau+\frac{1}{\varepsilon}\int_{\tau_1}^{\tau_2}\langle\tau-\tau_1\rangle^{1+\varsigma}\|\vartheta\|^2_{W_{-2}^k(\Sigma_\tau^{\geq R_0})}\mathrm{d}\tau.\end{aligned}\tag{3.27e}$$

If, in addition, the eigenvalue of of $\mathring{\eth}\mathring{\eth}'-b_{0,0}$ acting on φ vanishes, then for $p\geq 2$, both of the following estimates hold:

$$\begin{aligned}&\|rV\varphi\|^2_{W_{p-2}^k(\Sigma_{\tau_2}^{\geq R_0})}+\|\varphi\|^2_{W_{-2}^{k+1}(\Sigma_{\tau_2}^{\geq R_0})}+\|\varphi\|^2_{W_{-1-\delta}^{k+1}(\mathcal{D}_{\tau_1,\tau_2}^{\geq R_0})}+\|rV\varphi\|^2_{W_{p-3}^k(\mathcal{D}_{\tau_1,\tau_2}^{\geq R_0})}\\&\quad\lesssim_{[R_0-M,R_0]}\|rV\varphi\|^2_{W_{p-2}^k(\Sigma_{\tau_1}^{\geq R_0})}+\|\varphi\|^2_{W_{-2}^{k+1}(\Sigma_{\tau_1}^{\geq R_0})}\\&\quad+\Big(\|\vartheta\|^2_{W_{p-3}^k(\mathcal{D}_{\tau_1,\tau_2}^{\geq R_0})}+\|\varphi\|^2_{W_{p-5}^k(\mathcal{D}_{\tau_1,\tau_2}^{\geq R_0})}\Big);\end{aligned}\tag{3.28}$$

$$\begin{aligned}&\|rV\varphi\|^2_{W_{p-2}^k(\Sigma_{\tau_2}^{\geq R_0})}+\|\varphi\|^2_{W_{-2}^{k+1}(\Sigma_{\tau_2}^{\geq R_0})}+\|\varphi\|^2_{W_{-1-\delta}^{k+1}(\mathcal{D}_{\tau_1,\tau_2}^{\geq R_0})}+\|rV\varphi\|^2_{W_{p-3}^k(\mathcal{D}_{\tau_1,\tau_2}^{\geq R_0})}\\&\quad\lesssim_{[R_0-M,R_0]}\|rV\varphi\|^2_{W_{p-2}^k(\Sigma_{\tau_1}^{\geq R_0})}+\|\varphi\|^2_{W_{-2}^{k+1}(\Sigma_{\tau_1}^{\geq R_0})}\\&\quad+\varepsilon\int_{\tau_1}^{\tau_2}\langle\tau-\tau_1\rangle^{-1-\varsigma}\|rV\varphi\|^2_{W_{p-2}^k(\Sigma_\tau^{\geq R_0})}\mathrm{d}\tau+\frac{1}{\varepsilon}\int_{\tau_1}^{\tau_2}\langle\tau-\tau_1\rangle^{1+\varsigma}\Big(\|\vartheta\|^2_{W_{p-4}^k(\Sigma_\tau^{\geq R_0})}+\|\varphi\|^2_{W_{p-6}^k(\Sigma_\tau^{\geq R_0})}\Big)\mathrm{d}\tau.\end{aligned}\tag{3.29}$$

In all the above estimates, the integral terms $\|\varphi\|^2_{W_0^{k+1}(\Sigma_{\tau_2}^{R_0-M,R_0})}+\|\varphi\|^2_{W_0^{k+1}(\Sigma_{\tau_1}^{R_0-M,R_0})}+\|\varphi\|^2_{W_0^{k+1}(\mathcal{D}_{\tau_1,\tau_2}^{R_0-M,R_0})}+\|\vartheta\|^2_{W_0^k(\mathcal{D}_{\tau_1,\tau_2}^{R_0-M,R_0})}$ supported on $[R_0-M,R_0]$ are implicit in the symbol $\lesssim_{[R_0-M,R_0]}$.

Proof. The estimates (3.27) are the same as the ones proven in [55, Proposition 2.15] and hence the proof is omitted. If the eigenvalue of of $\mathring{\eth}\mathring{\eth}'-b_{0,0}$ acting on φ vanishes, the wave equation (3.26) reduces to

$$-r^2YV\varphi-b_VV\varphi=\vartheta+(b_0-b_{0,0})\varphi=\vartheta+O(r^{-1})\varphi\doteq\tilde{\vartheta}.\tag{3.30}$$

By the same approach as proving the estimates (3.27)[6], we arrive at

$$\begin{aligned}&\|rV\varphi\|^2_{W_{p-2}^k(\Sigma_{\tau_2}^{\geq R_0})}+\|\varphi\|^2_{W_{-2}^{k+1}(\Sigma_{\tau_2}^{\geq R_0})}+\|\varphi\|^2_{W_{-1-\delta}^{k+1}(\mathcal{D}_{\tau_1,\tau_2}^{\geq R_0})}+\|rV\varphi\|^2_{W_{p-3}^k(\mathcal{D}_{\tau_1,\tau_2}^{\geq R_0})}\\&\quad\lesssim_{[R_0-M,R_0]}\|rV\varphi\|^2_{W_{p-2}^k(\Sigma_{\tau_1}^{\geq R_0})}+\|\varphi\|^2_{W_{-2}^{k+1}(\Sigma_{\tau_1}^{\geq R_0})}+\sum_{|\mathbf{a}|\leq k}\bigg|\int_{\mathcal{D}_{\tau_1,\tau_2}^{R_0}}r^{p-2}\Re\Big(V\overline{\mathbb{D}^{\mathbf{a}}\varphi}\mathbb{D}^{\mathbf{a}}\tilde{\vartheta}\Big)\mathrm{d}^4\mu\bigg|.\end{aligned}\tag{3.31}$$

The estimates (3.28) and (3.29) follow easily by applying the Cauchy–Schwarz inequality to the last term of (3.31). □

3.4.1. *Energy decay estimates for the spin $+\mathfrak{s}$ component.* Throughout this subsubsection, we focus on a fixed ℓ mode, and, unless otherwise stated, we drop the subscript indicating the ℓ mode in the scalars constructed from the spin $+\mathfrak{s}$ component.

Definition 3.12. Define $F(k,i,p,\tau,\Psi_{+\mathfrak{s}})$ as follows:

- if $i\leq\ell-\mathfrak{s}$,

$$F(k,i,p,\tau,\Psi_{+\mathfrak{s}})=\sum_{j=0}^{i}\|\widetilde{\Phi}^{(j)}_{+\mathfrak{s}}\|^2_{W_{-3}^{k-\mathfrak{s}-j}(\Sigma_\tau^{\geq R_0})}+\mathbf{BM}^k_{\Sigma_\tau}(\Psi_{+\mathfrak{s}}),\qquad\text{for } p=-1,\tag{3.32a}$$

$$F(k,i,p,\tau,\Psi_{+\mathfrak{s}})=0,\qquad\text{for } p\in(-1,0),\tag{3.32b}$$

[6]This is basically to multiply this equation by $-2\chi_{R_0}r^{p-2}V\bar{\varphi}$, take the real part, and integrate over $\mathcal{D}_{\tau_1,\tau_2}$ with the volume element $\mathrm{d}^4\mu$

$$F(k,i,p,\tau,\Psi_{+\mathfrak{s}}) = \sum_{j=0}^{i}\Big(\|rV\widetilde{\Phi}^{(j)}_{+\mathfrak{s}}\|^2_{W^{k-\mathfrak{s}-j-1}_{p-2}(\Sigma^{\geq R_0}_\tau)} + \|\widetilde{\Phi}^{(j)}_{+\mathfrak{s}}\|^2_{W^{k-\mathfrak{s}-j}_{-2}(\Sigma^{\geq R_0}_\tau)}\Big) + \mathbf{BE}^k_{\Sigma_\tau}(\Psi_{+\mathfrak{s}}), \quad \text{for } p\in[0,2]; \tag{3.32c}$$

- additionally, for $i=\ell-\mathfrak{s}$ and $p\in[2,5)$,

$$\begin{aligned} F(k,\ell-\mathfrak{s},p,\tau,\Psi_{+\mathfrak{s}}) = {} & \sum_{j=0}^{\ell-\mathfrak{s}-1}\Big(\|rV\widetilde{\Phi}^{(j)}_{+\mathfrak{s}}\|^2_{W^{k-\mathfrak{s}-j-1}_{0}(\Sigma^{\geq R_0}_\tau)} + \|\widetilde{\Phi}^{(j)}_{+\mathfrak{s}}\|^2_{W^{k-\mathfrak{s}-j}_{-2}(\Sigma^{\geq R_0}_\tau)}\Big) \\ & + \|rV\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}\|^2_{W^{k-\ell-1}_{p-2}(\Sigma^{\geq R_0}_\tau)} + \|\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}\|^2_{W^{k-\ell}_{-2}(\Sigma^{\geq R_0}_\tau)} + \mathbf{BE}^k_{\Sigma_\tau}(\Psi_{+\mathfrak{s}}). \end{aligned} \tag{3.33}$$

We similarly define $F(k,i,p,\tau,\partial^j_\tau\Psi_{+\mathfrak{s}})$, $j\in\mathbb{N}$, with all the scalars in the Sobolev norms acted by ∂^j_τ.

Remark 3.13. By Definition 3.6, one has $\widetilde{\Phi}^{(j)}_{+\mathfrak{s}} = \hat{\mathcal{V}}\Phi^{(j-1)}_{+\mathfrak{s}} + \sum_{j'=0}^{j-1}O(1)\widetilde{\Phi}^{(j')}_{+\mathfrak{s}}$, hence, in the case that $i=\ell-\mathfrak{s}$ and $p=2$, the two definitions (3.32c) and (3.33) are equivalent.

We shall now prove global r^p estimates for the spin $+\mathfrak{s}$ component.

Proposition 3.14. *For all $0\leq i\leq \ell-\mathfrak{s}$, $p\in[0,2]$ and $\tau_2>\tau_1\geq\tau_0$,*

$$F(k,i,p,\tau_2,\Psi_{+\mathfrak{s}}) + \int_{\tau_1}^{\tau_2} F(k-1,i,p-1,\tau,\Psi_{+\mathfrak{s}})\mathrm{d}\tau \lesssim F(k,i,p,\tau_1,\Psi_{+\mathfrak{s}}). \tag{3.34}$$

Moreover, this estimate is also valid in the case that $i=\ell-\mathfrak{s}$ and $p\in[2,4)$.

Proof. We put equation (3.11) satisfied by the scalars constructed from the spin $+\mathfrak{s}$ component into the form of (3.26). Since the eigenvalue of $\mathring{\eth}'\mathring{\eth}$ acting on an ℓ mode is $-(\ell-\mathfrak{s})(\ell+\mathfrak{s}+1)$, one finds that the first assumption is trivially satisfied and the second assumption holds true as long as $i\leq\ell-\mathfrak{s}$. In particular, $b_{V,-1}=0$ happens only when $\mathfrak{s}=i=0$, and $b_{0,0}=0$ only when $i=\ell-\mathfrak{s}$. Based on these observations, we apply the corresponding estimates in Proposition 3.11 to equation (3.11) in different cases.

Consider first the case that $i\leq\ell-\mathfrak{s}$ and $p\in[0,2]$. As discussed above, $b_{V,-1}=0$ holds only when $\mathfrak{s}=i=0$. If $\mathfrak{s}=i=0$, then the estimate (3.34) with $p=0$ holds in view of the energy and Morawetz estimate (3.6). In the remaining cases, it holds that $b_{V,-1}>0$. We apply to equation (3.11) with $\varphi=\widetilde{\Phi}^{(i')}_{+\mathfrak{s}}$ the estimate (3.27a) for $p=0$, the estimate (3.27c) for $p\in(0,2)$, and the estimate (3.27d) for $p=2$ to achieve

$$\begin{aligned} & \|\widetilde{\Phi}^{(i')}_{+\mathfrak{s}}\|^2_{W^{k-\mathfrak{s}-i'}_{-2}(\Sigma^{\geq R_0}_{\tau_2})} + \|\widetilde{\Phi}^{(i')}_{+\mathfrak{s}}\|^2_{W^{k-\mathfrak{s}-i'}_{-3}(\mathcal{D}^{\geq R_0}_{\tau_1,\tau_2})} \\ & \quad \lesssim_{[R_0-M,R_0]} \|\widetilde{\Phi}^{(i')}_{+\mathfrak{s}}\|^2_{W^{k-\mathfrak{s}-i'}_{-2}(\Sigma^{\geq R_0}_{\tau_1})} + \|\vartheta(\widetilde{\Phi}^{(i')}_{+\mathfrak{s}})\|^2_{W^{k-\mathfrak{s}-i'-1}_{-3}(\mathcal{D}^{\geq R_0}_{\tau_1,\tau_2})}, \end{aligned} \tag{3.35a}$$

$$\begin{aligned} & \|rV\widetilde{\Phi}^{(i')}_{+\mathfrak{s}}\|^2_{W^{k-\mathfrak{s}-i'-1}_{p-2}(\Sigma^{\geq R_0}_{\tau_2})} + \|\widetilde{\Phi}^{(i')}_{+\mathfrak{s}}\|^2_{W^{k-\mathfrak{s}-i'}_{-2}(\Sigma^{\geq R_0}_{\tau_2})} + \|\widetilde{\Phi}^{(i')}_{+\mathfrak{s}}\|^2_{W^{k-\mathfrak{s}-i'}_{p-3}(\mathcal{D}^{\geq R_0}_{\tau_1,\tau_2})} + \|Y\widetilde{\Phi}^{(i')}_{+\mathfrak{s}}\|^2_{W^{k-\mathfrak{s}-i'-1}_{-1-\delta}(\mathcal{D}^{\geq R_0}_{\tau_1,\tau_2})} \\ & \quad \lesssim_{[R_0-M,R_0]} \|rV\widetilde{\Phi}^{(i')}_{+\mathfrak{s}}\|^2_{W^{k-\mathfrak{s}-i'-1}_{p-2}(\Sigma^{\geq R_0}_{\tau_1})} + \|\widetilde{\Phi}^{(i')}_{+\mathfrak{s}}\|^2_{W^{k-\mathfrak{s}-i'}_{-2}(\Sigma^{\geq R_0}_{\tau_1})} + \|\vartheta(\widetilde{\Phi}^{(i')}_{+\mathfrak{s}})\|^2_{W^{k-\mathfrak{s}-i'-1}_{p-3}(\mathcal{D}^{\geq R_0}_{\tau_1,\tau_2})}, \end{aligned} \tag{3.35b}$$

$$\begin{aligned} & \|rV\widetilde{\Phi}^{(i')}_{+\mathfrak{s}}\|^2_{W^{k-\mathfrak{s}-i'-1}_{0}(\Sigma^{\geq R_0}_{\tau_2})} + \|\widetilde{\Phi}^{(i')}_{+\mathfrak{s}}\|^2_{W^{k-\mathfrak{s}-i'}_{-2}(\Sigma^{\geq R_0}_{\tau_2})} + \|\widetilde{\Phi}^{(i')}_{+\mathfrak{s}}\|^2_{W^{k-\mathfrak{s}-i'}_{-1-\delta}(\mathcal{D}^{\geq R_0}_{\tau_1,\tau_2})} + \|rV\widetilde{\Phi}^{(i')}_{+\mathfrak{s}}\|^2_{W^{k-\mathfrak{s}-i'-1}_{-1}(\mathcal{D}^{\geq R_0}_{\tau_1,\tau_2})} \\ & \quad \lesssim_{[R_0-M,R_0]} \|rV\widetilde{\Phi}^{(i')}_{+\mathfrak{s}}\|^2_{W^{k-\mathfrak{s}-i'-1}_{0}(\Sigma^{\geq R_0}_{\tau_1})} + \|\widetilde{\Phi}^{(i')}_{+\mathfrak{s}}\|^2_{W^{k-\mathfrak{s}-i'}_{-2}(\Sigma^{\geq R_0}_{\tau_1})} + \|\vartheta(\widetilde{\Phi}^{(i')}_{+\mathfrak{s}})\|^2_{W^{k-\mathfrak{s}-i'-1}_{-1}(\mathcal{D}^{\geq R_0}_{\tau_1,\tau_2})}, \end{aligned} \tag{3.35c}$$

respectively. In view of the expression $\vartheta(\widetilde{\Phi}^{(i')}_{+\mathfrak{s}}) = \sum_{j=0}^{i'}O(r^{-1})\widetilde{\Phi}^{(j)}_{+\mathfrak{s}}$, one finds on the RHS,

$$\begin{aligned} \|\vartheta(\widetilde{\Phi}^{(i')}_{+\mathfrak{s}})\|^2_{W^{k-\mathfrak{s}-i'-1}_{p-3}(\mathcal{D}^{\geq R_0}_{\tau_1,\tau_2})} & \lesssim \sum_{j=0}^{i'}\|\widetilde{\Phi}^{(j)}_{+\mathfrak{s}}\|^2_{W^{k-\mathfrak{s}-j-1}_{p-5}(\mathcal{D}^{\geq R_0}_{\tau_1,\tau_2})} \\ & \lesssim R_0^{-2}\|\widetilde{\Phi}^{(i')}_{+\mathfrak{s}}\|^2_{W^{k-\mathfrak{s}-i'-1}_{p-3}(\mathcal{D}^{\geq R_0}_{\tau_1,\tau_2})} + \sum_{j=0}^{i'-1}\|\widetilde{\Phi}^{(j)}_{+\mathfrak{s}}\|^2_{W^{k-\mathfrak{s}-j-1}_{p-5}(\mathcal{D}^{\geq R_0}_{\tau_1,\tau_2})}, \end{aligned} \tag{3.36}$$

where the term with R_0^{-2} coefficient can be absorbed by choosing $R_0 \geq \hat{R}_0$ sufficiently large. We can then take a weighted sum of the estimates (3.35) for $i' \in \{0, 1, \dots, i\}$ such that the last term in (3.36) is aslo absorbed, and the error terms which are supported on $[R_0 - M, R_0]$ and implicit in the symbol $\lesssim_{[R_0-M,R_0]}$ can be controlled by adding in a sufficient multiple of the energy and Morawetz estimate (3.6). This thus completes the proof of (3.34) for $p \in [0, 2]$.

Consider next the case that $i = \ell - \mathfrak{s}$ and $2 \leq p < 4$. The steps are the same as the above for $p \in [0, 2]$ except that we are now applying the estimate (3.28) for $p \in [2, 4)$ to equation (3.11) with $i = \ell - \mathfrak{s}$; we arrive at

$$\begin{aligned} &\|rV\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}\|^2_{W^{k-\ell-1}_{p-2}(\Sigma^{\geq R_0}_{\tau_2})} + \|\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}\|^2_{W^{k-\ell}_{-2}(\Sigma^{\geq R_0}_{\tau_2})} + \|\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}\|^2_{W^{k-\ell}_{-1-\delta}(\mathcal{D}^{\geq R_0}_{\tau_1,\tau_2})} + \|rV\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}\|^2_{W^{k-\ell-1}_{p-3}(\mathcal{D}^{\geq R_0}_{\tau_1,\tau_2})} \\ &\quad\lesssim_{[R_0-M,R_0]} \|rV\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}\|^2_{W^{k-\ell-1}_{p-2}(\Sigma^{\geq R_0}_{\tau_1})} + \|\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}\|^2_{W^{k-\ell}_{-2}(\Sigma^{\geq R_0}_{\tau_1})} + \|\vartheta(\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})})\|^2_{W^{k-\ell-1}_{p-3}(\mathcal{D}^{\geq R_0}_{\tau_1,\tau_2})}. \end{aligned} \tag{3.37}$$

It remains to estimate $\|\vartheta(\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})})\|^2_{W^{k-\ell-1}_{p-3}(\mathcal{D}^{\geq R_0}_{\tau_1,\tau_2})}$ by

$$\begin{aligned} \|\vartheta(\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})})\|^2_{W^{k-\ell-1}_{p-3}(\mathcal{D}^{\geq R_0}_{\tau_1,\tau_2})} &\lesssim \sum_{j=0}^{\ell-\mathfrak{s}} \|\widetilde{\Phi}_{+\mathfrak{s}}^{(j)}\|^2_{W^{k-\mathfrak{s}-j-1}_{p-5}(\mathcal{D}^{\geq R_0}_{\tau_1,\tau_2})} \\ &\lesssim R_0^{p+\delta-4}\Big(\|\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}\|^2_{W^{k-\ell-1}_{-1-\delta}(\mathcal{D}^{\geq R_0}_{\tau_1,\tau_2})} + \sum_{j=0}^{\ell-\mathfrak{s}-1}\|\widetilde{\Phi}_{+\mathfrak{s}}^{(j)}\|^2_{W^{k-\mathfrak{s}-j-1}_{-1-\delta}(\mathcal{D}^{\geq R_0}_{\tau_1,\tau_2})}\Big). \end{aligned} \tag{3.38}$$

Thus, by adding a suitable weighted sum of the estimate (3.35c) for $i \in \{0, 1, \dots, \ell - \mathfrak{s}\}$ (e.g. by taking the coefficients of this weighted sum to satisfy $C_0 \gg C_1 \gg \dots \gg C_{\ell-\mathfrak{s}} \gg 1$) to the estimate (3.38) and taking δ sufficiently small compared to $4-p$, the error terms $\sum_{i'=0}^{\ell-\mathfrak{s}} C_{i'} \|\vartheta(\widetilde{\Phi}_{+\mathfrak{s}}^{(i')})\|^2_{W^{k-\mathfrak{s}-i'-1}_{-1}(\mathcal{D}^{\geq R_0}_{\tau_1,\tau_2})} + \|\vartheta(\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})})\|^2_{W^{k-\ell-1}_{p-3}(\mathcal{D}^{\geq R_0}_{\tau_1,\tau_2})}$ are absorbed. In the end, we add the energy and Morawetz estimate (3.6) to bound the error terms implicit in the symbol $\lesssim_{[R_0-M,R_0]}$, thus proving the estimate (3.34) in the case that $i = \ell - \mathfrak{s}$ and $p \in [2, 4)$. ☐

Corollary 3.15. *Let $i_1, i_2 \in \mathbb{N}$ and $p_1, p_2 \in \mathbb{R}^+ \cup \{0\}$ be such that either of the following holds:*

- $i_1 = i_2 < \ell - \mathfrak{s}$ *and* $p_1 \leq p_2 \leq 2$*;*
- $i_1 < i_2 < \ell - \mathfrak{s}$, $p_1 \leq 2$, $p_2 \leq 2$*;*
- $i_1 = i_2 = \ell - \mathfrak{s}$, $p_1 \leq p_2 < 4$*;*
- $i_1 < i_2 = \ell - \mathfrak{s}$, $p_1 \leq 2$, $p_2 < 4$.

Then there exists a constant $k' = k'(j, i_2 - i_1)$, which grows linearly in its arguments, such that for any $\tau_2 > \tau_1 \geq \tau_0$,

$$\begin{aligned} &F(k, i_1, p_1, \tau_2, \partial_\tau^j \Psi_{+\mathfrak{s}}) + \int_{\tau_2}^{\infty} F(k-1, i_1, p_1 - 1, \tau', \partial_\tau^j \Psi_{+\mathfrak{s}}) \mathrm{d}\tau' \\ &\quad\lesssim \langle \tau_2 - \tau_1 \rangle^{-2(i_2 - i_1) - (p_2 - p_1) - 2j} F(k + k', i_2, p_2, \tau_1, \Psi_{+\mathfrak{s}}). \end{aligned} \tag{3.39}$$

Proof. An application of [3, Lemma 5.2] to the estimates in Proposition 3.14 yields that for any $0 \leq i \leq \ell - \mathfrak{s}$, $0 \leq p_1 \leq p_2 \leq 2$, and $\tau_2 > \tau_1 \geq \tau_0$,

$$F(k-2, i, p_1, \tau_2, \Psi_{+\mathfrak{s}}) \lesssim \langle \tau_2 - \tau_1 \rangle^{-(p_2 - p_1)} F(k, i, p_2, \tau_1, \Psi_{+\mathfrak{s}}), \tag{3.40}$$

and for $i = \ell - \mathfrak{s}$, $0 \leq p_1 \leq p_2 < 4$, and $\tau_2 > \tau_1 \geq \tau_0$,

$$F(k-4, \ell - \mathfrak{s}, p_1, \tau_2, \Psi_{+\mathfrak{s}}) \lesssim \langle \tau_2 - \tau_1 \rangle^{-(p_2 - p_1)} F(k, \ell - \mathfrak{s}, p_2, \tau_1, \Psi_{+\mathfrak{s}}), \tag{3.41}$$

Note that by Definition 3.6, we have for any $0 \leq j \leq \ell - \mathfrak{s}$ that $\widetilde{\Phi}_{+\mathfrak{s}}^{(i)} = \hat{\mathcal{V}}\Phi_{+\mathfrak{s}}^{(i-1)} + \sum_{i'=0}^{i-1} O(1)\widetilde{\Phi}_{+\mathfrak{s}}^{(i')}$, therefore,

$$F(k, i-1, 2, \tau, \Psi_{+\mathfrak{s}}) \sim F(k, i, 0, \tau, \Psi_{+\mathfrak{s}}). \tag{3.42}$$

Combining the above estimates then prove the estimate (3.39) for $j = 0$.

We prove the general j case of the estimate (3.39) by induction. Assume it holds for j, and it suffices to prove the $j + 1$ case. Since ∂_τ is a Killing vector field and commutes with the wave equations of $\widetilde{\Phi}^{(i)}_{+\mathfrak{s}}$, the above estimates in this proof are still valid by replacing $\Psi_{+\mathfrak{s}}$ with $\partial_\tau^j\Psi_{+\mathfrak{s}}$, which thus yields

$$F(k,i_1,p_1,\tau_2,\partial_\tau^{j+1}\Psi_{+\mathfrak{s}}) \lesssim \langle\tau_2-\tau_1\rangle^{-(2-p_1)}F(k+k',i_1,2,\tau_2-(\tau_2-\tau_1)/3,\partial_\tau^{j+1}\Psi_{+\mathfrak{s}}). \tag{3.43}$$

We use equation (3.11) and the expression $Y=\mu^{-1}\big(2\partial_\tau - V\big)$ away from horizon to rewrite $r^2V\partial_\tau\widetilde{\Phi}^{(i')}_{+\mathfrak{s}}$ as a weighted sum of $(rV)^2\widetilde{\Phi}^{(i')}_{+\mathfrak{s}}$, $((\ell-\mathfrak{s})(\ell+\mathfrak{s}+1)-i'(i'+2\mathfrak{s}+1))\widetilde{\Phi}^{(i')}_{+\mathfrak{s}}$, $rV\widetilde{\Phi}^{(i')}_{+\mathfrak{s}}$, and $\sum_{j=0}^{i'}O(r^{-1})\widetilde{\Phi}^{(j)}_{+\mathfrak{s}}$ all with $O(1)$ coefficients. Therefore, in the case that $0\le i\le \ell-\mathfrak{s}$,

$$\begin{aligned}F(k,i,2,\tau,\partial_\tau^{j+1}\Psi_{+\mathfrak{s}}) &\lesssim \sum_{i'=0}^{i}\|r^2V\partial_\tau^{j+1}\widetilde{\Phi}^{(i')}_{+\mathfrak{s}}\|^2_{W^{k-\mathfrak{s}-i'-1}_{-2}(\Sigma_\tau^{\geq R_0})}+F(k+1,i,0,\tau,\partial_\tau^j\Psi_{+\mathfrak{s}})\\ &\lesssim F(k+1,i,0,\tau,\partial_\tau^j\Psi_{+\mathfrak{s}});\end{aligned} \tag{3.44}$$

and in the other case that $i=\ell-\mathfrak{s}$, $2\le p<4$,

$$\begin{aligned}F(k,\ell-\mathfrak{s},p,\tau,\partial_\tau^{j+1}\Psi_{+1}) &\lesssim \|r^2V\partial_\tau^{j+1}\widetilde{\Phi}^{(i')}_{+\mathfrak{s}}\|^2_{W^{k-\mathfrak{s}-i'-1}_{p-4}(\Sigma_\tau^{\geq R_0})}+F(k+1,\ell-\mathfrak{s},0,\tau,\partial_\tau^j\Psi_{+1})\\ &\lesssim F(k+1,i,p-2,\tau,\partial_\tau^j\Psi_{+1})\end{aligned} \tag{3.45}$$

where we have used in the last step that the coefficient $(\ell-\mathfrak{s})(\ell+\mathfrak{s}+1)-i(i+2\mathfrak{s}+1)=0$. Consequently, the RHS of (3.43) is bounded by

$$\begin{aligned}&\langle\tau_2-\tau_1\rangle^{-(2-p_1)}F(k+k',i_1,0,\tau_2-\tfrac{2}{3}(\tau_2-\tau_1),\partial_\tau^j\Psi_{+\mathfrak{s}})\\ &\quad\lesssim \langle\tau_2-\tau_1\rangle^{-2(i_2-i_1)-(p_2-p_1)-2(j+1)}F(k+k',i_2,p_2,\tau_1,\Psi_{+\mathfrak{s}}),\end{aligned} \tag{3.46}$$

where the last step follows by induction. This closes the proof. □

For the convenience of the latter discussions, we shall extend the results in the above corollary to the ones in the following lemma.

Lemma 3.16. *Let $i_1,i_2\in\mathbb{N}$ and $p_1,p_2\in\mathbb{R}^+\cup\{0\}$ be such that either of the following holds:*

- *$i_1=i_2<\ell-\mathfrak{s}$ and $p_1\le p_2\le 2$;*
- *$i_1<i_2<\ell-\mathfrak{s}$, $p_1\le 2$, $p_2\le 2$;*
- *$i_1=i_2=\ell-\mathfrak{s}$, $p_1\le p_2<5$;*
- *$i_1<i_2=\ell-\mathfrak{s}$, $p_1\le 2$, $p_2<5$.*

Then there exists a constant $k'=k'(j,i_2-i_1)$, which grows linearly in its arguments, such that for any $\tau_2>\tau_1\ge\tau_0$,

$$\begin{aligned}&F(k,i_1,p_1,\tau_2,\partial_\tau^j\Psi_{+\mathfrak{s}})+\int_{\tau_2}^{\infty}F(k-1,i_1,p_1-1,\tau',\partial_\tau^j\Psi_{+\mathfrak{s}})\mathrm{d}\tau'\\ &\quad\lesssim \langle\tau_2-\tau_1\rangle^{-2(i_2-i_1)-(p_2-p_1)-2j}F(k+k',i_2,p_2,\tau_1,\Psi_{+\mathfrak{s}}).\end{aligned} \tag{3.47}$$

Proof. As can be seen from the proof of Corollary 3.15, in order to prove this lemma, it suffices to extend the estimate (3.34) to the case that $i=\ell-\mathfrak{s}$ and $p\in[2,5)$. Specifically, we aim to prove that for any $p\in[4,5)$ and $\tau_2>\tau_1\ge\tau_0$,

$$F(k-k',\ell-\mathfrak{s},p,\tau_2,\Psi_{+\mathfrak{s}})+\int_{\tau_1}^{\tau_2}F(k-k'-1,\ell-\mathfrak{s},p-1,\tau,\Psi_{+\mathfrak{s}})\mathrm{d}\tau\lesssim F(k,\ell-\mathfrak{s},p,\tau_1,\Psi_{+\mathfrak{s}}), \tag{3.48}$$

where k' is a finite universal constant.

By applying (3.29) to equation (3.11) with $i=\ell-\mathfrak{s}$, and together with the expression $\vartheta(\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}})=\sum_{i=0}^{\ell-\mathfrak{s}}O(r^{-1})\widetilde{\Phi}^{(i)}_{+\mathfrak{s}}$ which follows from equation (3.11), we get for $p\in[4,5)$ and any $\varsigma>0$ that

$$\begin{aligned}&\|rV\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}\|^2_{W^{k-\ell-1}_{p-2}(\Sigma_{\tau_2}^{\geq R_0})}+\|\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}\|^2_{W^{k-\ell}_{-2}(\Sigma_{\tau_2}^{\geq R_0})}+\|\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}\|^2_{W^{k-\ell}_{-1-\delta}(\mathcal{D}_{\tau_1,\tau_2}^{\geq R_0})}+\|rV\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}\|^2_{W^{k-\ell-1}_{p-3}(\mathcal{D}_{\tau_1,\tau_2}^{\geq R_0})}\\ &\lesssim_{[R_0-M,R_0]}\|rV\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}\|^2_{W^{k-\ell-1}_{p-2}(\Sigma_{\tau_1}^{\geq R_0})}+\|\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}\|^2_{W^{k-\ell}_{-2}(\Sigma_{\tau_1}^{\geq R_0})}\\ &+\varepsilon\int_{\tau_1}^{\tau_2}\langle\tau-\tau_1\rangle^{-1-\varsigma}\|rV\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}\|^2_{W^{k-\ell-1}_{p-2}(\Sigma_\tau^{\geq R_0})}\mathrm{d}\tau\end{aligned}$$

$$
\begin{aligned}
&+\varepsilon^{-1}\int_{\tau_1}^{\tau_2}\langle\tau-\tau_1\rangle^{1+\varsigma}\Big(\|\vartheta(\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})})\|^2_{W^{k-\ell-s}_{p-4}(\Sigma_\tau^{\geq R_0})}+\|\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}\|^2_{W^{k-\ell-1}_{p-6}(\Sigma_\tau^{\geq R_0})}\Big)\mathrm{d}\tau\\
&\lesssim_{[R_0-M,R_0]}\|rV\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}\|^2_{W^{k-\ell-1}_{p-2}(\Sigma_{\tau_1}^{\geq R_0})}+\|\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}\|^2_{W^{k-\ell}_{-2}(\Sigma_{\tau_1}^{\geq R_0})}\\
&+\varepsilon^{-1}\sum_{i=0}^{\ell-\mathfrak{s}}\int_{\tau_1}^{\tau_2}\langle\tau-\tau_1\rangle^{1+\varsigma}\|\widetilde{\Phi}_{+\mathfrak{s}}^{(i)}\|^2_{W^{k-\ell-1}_{p-6}(\Sigma_\tau^{\geq R_0})}\mathrm{d}\tau+\frac{\varepsilon}{\varsigma}\sup_{\tau\in[\tau_1,\tau_2]}\|rV\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}\|^2_{W^{k-\ell-1}_{p-2}(\Sigma_\tau^{\geq R_0})}.
\end{aligned}\tag{3.49}
$$

The last term can be absorbed by taking a supreme norm over the LHS and choosing ε sufficiently small compared to ς, thus

$$
\begin{aligned}
&\|rV\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}\|^2_{W^{k-\ell-1}_{p-2}(\Sigma_{\tau_2}^{\geq R_0})}+\|\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}\|^2_{W^{k-\ell}_{-2}(\Sigma_{\tau_2}^{\geq R_0})}+\|\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}\|^2_{W^{k-\ell}_{-1-\delta}(\mathcal{D}_{\tau_1,\tau_2}^{\geq R_0})}+\|rV\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}\|^2_{W^{k-\ell-1}_{p-3}(\mathcal{D}_{\tau_1,\tau_2}^{\geq R_0})}\\
&\lesssim_{[R_0-M,R_0]}\|rV\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}\|^2_{W^{k-\ell-1}_{p-2}(\Sigma_{\tau_1}^{\geq R_0})}+\|\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}\|^2_{W^{k-\ell}_{-2}(\Sigma_{\tau_1}^{\geq R_0})}+\frac{1}{\varepsilon}\sum_{i=0}^{\ell-\mathfrak{s}}\int_{\tau_1}^{\tau_2}\langle\tau-\tau_1\rangle^{1+\varsigma}\|\widetilde{\Phi}_{+\mathfrak{s}}^{(i)}\|^2_{W^{k-\ell-1}_{p-6}(\Sigma_\tau^{\geq R_0})}\mathrm{d}\tau.
\end{aligned}\tag{3.50}
$$

From the Hardy's inequality (2.29a) and the proven (3.39) for $i_1=i_2=\ell-\mathfrak{s}$, $p_1=p-4$ and $p_2=3.5$, we have for $p\in[4,5)$ that

$$
\begin{aligned}
&\sum_{i=0}^{\ell-\mathfrak{s}}\int_{\tau_1}^{\tau_2}\langle\tau-\tau_1\rangle^{1+\varsigma}\|\widetilde{\Phi}_{+\mathfrak{s}}^{(i)}\|^2_{W^{k-\ell-1}_{p-6}(\Sigma_\tau^{\geq R_0})}\mathrm{d}\tau\\
&\lesssim\int_{\tau_1}^{\tau_2}\langle\tau-\tau_1\rangle^{1+\varsigma}F(k,\ell-\mathfrak{s},p-4,\tau,\Psi_{+\mathfrak{s}})\mathrm{d}\tau\\
&\lesssim\int_{\tau_1}^{\tau_2}\langle\tau-\tau_1\rangle^{1+\varsigma}\langle\tau-\tau_1\rangle^{p-15/2}\mathrm{d}\tau F(k+4,\ell-\mathfrak{s},3.5,\tau_1,\Psi_{+\mathfrak{s}})\\
&\lesssim F(k+4,\ell-\mathfrak{s},3.5,\tau_1,\Psi_{+\mathfrak{s}}).
\end{aligned}\tag{3.51}
$$

Plugging this estimate into (3.50) and adding the estimate (3.34) together, we prove the estimate (3.48) for any $p\in[4,5)$ and $\tau_2>\tau_1\geq\tau_0$, with $k'=4$. □

3.4.2. *Energy decay estimates for the spin $-\mathfrak{s}$ component.* Throughout this subsubsection, we focus on a fixed ℓ mode, and, unless otherwise stated, we drop the subscript indicating the ℓ mode in the scalars constructed from the spin $-\mathfrak{s}$ component.

Definition 3.17. Define $F(k,i,p,\tau,\Psi_{-\mathfrak{s}})$ as follows:

- if $i\leq\ell$,

$$
F(k,i,p,\tau,\Psi_{-\mathfrak{s}})=0,\qquad\text{for } p\in[-1,0),
$$

$$
F(k,i,p,\tau,\Psi_{-\mathfrak{s}})=\sum_{j=0}^{i+\mathfrak{s}}\Big(\|rV\widetilde{\Phi}_{-\mathfrak{s}}^{(j)}\|^2_{W^{k-s-j-1}_{p-2}(\Sigma_\tau^{\geq R_0})}+\|\widetilde{\Phi}_{-\mathfrak{s}}^{(j)}\|^2_{W^{k-s-j}_{-2}(\Sigma_\tau^{\geq R_0})}\Big)+\mathbf{BE}^k_{\Sigma_\tau}(\Psi_{-\mathfrak{s}}),\quad\text{for } p\in[0,2];
$$

- additionally, for $i=\ell$ and $p\in[2,5)$,

$$
\begin{aligned}
F(k,\ell,p,\tau,\Psi_{-\mathfrak{s}})=&\sum_{j=0}^{\ell+\mathfrak{s}-1}\Big(\|rV\widetilde{\Phi}_{-\mathfrak{s}}^{(j)}\|^2_{W^{k-s-j-1}_{0}(\Sigma_\tau^{\geq R_0})}+\|\widetilde{\Phi}_{-\mathfrak{s}}^{(j)}\|^2_{W^{k-s-j}_{-2}(\Sigma_\tau^{\geq R_0})}\Big)\\
&+\|rV\widetilde{\Phi}_{-\mathfrak{s}}^{(\ell+\mathfrak{s})}\|^2_{W^{k-\ell-2s-1}_{p-2}(\Sigma_\tau^{\geq R_0})}+\|\widetilde{\Phi}_{-\mathfrak{s}}^{(\ell+\mathfrak{s})}\|^2_{W^{k-\ell-2s}_{-2}(\Sigma_\tau^{\geq R_0})}+\mathbf{BE}^k_{\Sigma_\tau}(\Psi_{-\mathfrak{s}}).
\end{aligned}
$$

For $j\in\mathbb{N}$, we similarly define $F(k,i,p,\tau,\partial_\tau^j\Psi_{-1})$ with all the scalars in the Sobolev norms acted by ∂_τ^j.

Proposition 3.18. *For all $i<\mathfrak{s}$, $p\in[0,2]$ and $\tau_2>\tau_1\geq\tau_0$, there exists a nonnegative universal constant $k'=k'(\mathfrak{s})$ such that*

$$
F(k,i,p,\tau_2,\Psi_{-\mathfrak{s}})+\int_{\tau_1}^{\tau_2}F(k-1,i,p-1,\tau,\Psi_{-\mathfrak{s}})\mathrm{d}\tau\lesssim F(k+k',i,p,\tau_1,\Psi_{-\mathfrak{s}}).\tag{3.52}
$$

Moreover, in both the case that $0 \le i_1 < i_2 < \mathfrak{s}$, $0 \le p_1, p_2 \le 2$ *and the case that* $0 \le i_1 = i_2 < \mathfrak{s}$, $0 \le p_1 \le p_2 \le 2$, *there exists a nonnegative universal constant* $k' = k'(j, i_2 - i_1)$, *which grows linearly in its arguments, such that for any* $j \in \mathbb{N}$ *and* $\tau_2 > \tau_1 \ge \tau_0$,

$$F(k, i_1, p_1, \tau_2, \partial_\tau^j \Psi_{-\mathfrak{s}}) \lesssim \langle \tau_2 - \tau_1 \rangle^{-2(i_2-i_1)-(p_2-p_1)-2j} F(k+k', i_2, p_2, \tau_1, \Psi_{-\mathfrak{s}}). \tag{3.53}$$

Proof. In the case of $\mathfrak{s} = 0$, this has been proven in Proposition 3.14. It remains to prove the estimates for $\mathfrak{s} = 1, 2$, and we consider the $\mathfrak{s} = 1$ and $\mathfrak{s} = 2$ cases separately.

For $\mathfrak{s} = 1$ case, we first consider the system of equations (3.3a) and (3.3b). These two subequations can be both put into the form of equation (3.26) and the assumptions in Proposition 3.11 are satisfied. In particular, $\vartheta(\Phi_{-1}^{(0)}) = 2(r-3M)r^{-2}\Phi_{-1}^{(1)}$ and $\vartheta(\Phi_{-1}^{(1)}) = 0$. We apply to these two subequations the estimates of both (3.27c) and (3.27d) and sum them up, and the error term $\|\vartheta(\Phi_{-1}^{(0)})\|^2_{W^k_{p-3}(\mathcal{D}^{\ge R_0}_{\tau_1,\tau_2})} \lesssim \|\Phi_{-1}^{(1)}\|^2_{W^k_{-3}(\mathcal{D}^{\ge R_0}_{\tau_1,\tau_2})}$ can clearly be absorbed by taking $R_0 \ge \hat{R}_0$ sufficiently large. Thus, this proves the estimate (3.52) with $k' = 0$ for all $i < 1$, $p \in [0, 2]$ and $\tau_2 > \tau_1 \ge \tau_0$, where the $p = 0$ case follows from the energy and Morawetz estimate (3.6).

Consider next the $\mathfrak{s} = 2$ case. By exactly the same argument as for the $\mathfrak{s} = 1$ case, one can show the estimate (3.52) with $k' = 0$ for all $i = 0$, $p \in [0, 2]$ and $\tau_2 > \tau_1 \ge \tau_0$. Moreover, the estimate (3.52) with $k' = 0$ for $i = 1$, $p = 0$ and $\tau_2 > \tau_1 \ge \tau_0$ trivially holds due to the relation

$$F(k, 1, 0, \tau, \Psi_{-2}) \sim F(k, 0, 2, \tau, \Psi_{-2}) \tag{3.54}$$

which is valid by Definition 3.1. By putting quation (3.4d) of $\Phi_{-2}^{(3)}$ in the form of (3.26), one finds $\vartheta(\Phi_{-2}^{(3)}) = -6M\Phi_{-2}^{(2)}$. We first apply the estimate (3.27c) and achieve for any $p \in (0, 2)$ and $\tau_2 > \tau_1 \ge \tau_0$ that

$$\begin{aligned} &\|rV\Phi_{-2}^{(3)}\|^2_{W^{k-6}_{p-2}(\Sigma^{\ge R_0}_{\tau_2})} + \|\Phi_{-2}^{(3)}\|^2_{W^{k-5}_{-2}(\Sigma^{\ge R_0}_{\tau_2})} + \|\Phi_{-2}^{(3)}\|^2_{W^{k-5}_{p-3}(\mathcal{D}^{\ge R_0}_{\tau_1,\tau_2})} \\ &\lesssim_{[R_0-M,R_0]} \|rV\Phi_{-2}^{(3)}\|^2_{W^{k-6}_{p-2}(\Sigma^{\ge R_0}_{\tau_1})} + \|\Phi_{-2}^{(3)}\|^2_{W^{k-5}_{-2}(\Sigma^{\ge R_0}_{\tau_1})} + \|\Phi_{-2}^{(2)}\|^2_{W^{k-6}_{p-3}(\mathcal{D}^{\ge R_0}_{\tau_1,\tau_2})}. \end{aligned} \tag{3.55}$$

By the Hardy's inequality (2.29a), we have for the last term

$$\|\Phi_{-2}^{(2)}\|^2_{W^{k-6}_{p-3}(\mathcal{D}^{\ge R_0}_{\tau_1,\tau_2})} \lesssim_{[R_0-M,R_0]} \|rV\Phi_{-2}^{(2)}\|^2_{W^{k-6}_{p-3}(\mathcal{D}^{\ge R_0}_{\tau_1,\tau_2})} \lesssim_{[R_0-M,R_0]} \|\Phi_{-2}^{(3)}\|^2_{W^{k-6}_{p-5}(\mathcal{D}^{\ge R_0}_{\tau_1,\tau_2})} \tag{3.56}$$

which can thus be absorbed. Hence, we prove the estimate (3.52) with $k' = 0$ for all $i = 1$, $p \in [0, 2)$ and $\tau_2 > \tau_1 \ge \tau_0$. Similarly to the proof of Corollary 3.15, an application of [3, Lemma 5.2] yields that there exists a nonnegative universal constant k' such that for all $p = 2$ and $\tau_2 > \tau_1 \ge \tau_0$,

$$F(k, 0, 0, \tau_2, \Psi_{-2}) \lesssim \langle \tau_2 - \tau_1 \rangle^{-2-p} F(k+k', 1, p, \tau_1, \Psi_{-2}). \tag{3.57}$$

We then apply the estimate (3.27e) to equation (3.4d) of $\Phi_{-2}^{(3)}$ and achieve for any $p \in (0, 2)$ and $\tau_2 > \tau_1 \ge \tau_0$ that

$$\begin{aligned} &\|rV\Phi_{-2}^{(3)}\|^2_{W^{k-6}_{0}(\Sigma^{\ge R_0}_{\tau_2})} + \|\Phi_{-2}^{(3)}\|^2_{W^{k-5}_{-2}(\Sigma^{\ge R_0}_{\tau_2})} + \|\Phi_{-2}^{(3)}\|^2_{W^{k-5}_{-1}(\mathcal{D}^{\ge R_0}_{\tau_1,\tau_2})} \\ &\lesssim_{[R_0-M,R_0]} \|rV\Phi_{-2}^{(3)}\|^2_{W^{k-6}_{0}(\Sigma^{\ge R_0}_{\tau_1})} + \|\Phi_{-2}^{(3)}\|^2_{W^{k-5}_{-2}(\Sigma^{\ge R_0}_{\tau_1})} \\ &+ \varepsilon \int_{\tau_1}^{\tau_2} \langle \tau - \tau_1 \rangle^{-1-\varsigma} \|rV\Phi_{-2}^{(3)}\|^2_{W^{k-6}_{0}(\Sigma^{\ge R_0}_{\tau})} \mathrm{d}\tau + \frac{1}{\varepsilon} \int_{\tau_1}^{\tau_2} \langle \tau - \tau_1 \rangle^{1+\varsigma} \|\Phi_{-2}^{(2)}\|^2_{W^{k-6}_{-2}(\Sigma^{\ge R_0}_{\tau})} \mathrm{d}\tau. \end{aligned} \tag{3.58}$$

The second last term on the RHS of (3.58) is absorbed by taking a supreme norm over the LHS and taking ε small enough, and, using the estimate (3.57) with $p = 1.5$, the last term is bounded by $\frac{1}{\varepsilon}F(k+k', 1, 1.5, \tau_1, \Psi_{-2})$. Together with the proven estimate (3.52) with $k' = 0$ for all $i = 1$, $p \in [0, 2)$ and $\tau_2 > \tau_1 \ge \tau_0$, we conclude the $i = 1$, $p = 2$ case of (3.52), and, hence, complete the proof of (3.52).

The estimate (3.53) for the $j = 0$ case follows simply by applying [3, Lemma 5.2] to (3.52) and the equivalence relation (3.54). The general j case holds by the same way of arguing as in Corollary 3.15. □

Lemma 3.19. *Let* $i_1, i_2 \in \mathbb{N}$ *and* $p_1, p_2 \in \mathbb{R}^+ \cup \{0\}$ *be such that either of the following holds:*

- $i_1 = i_2 < \ell$ *and* $p_1 \le p_2 \le 2$;

- $i_1 < i_2 < \ell$, $p_1 \le 2$, $p_2 \le 2$;
- $i_1 = i_2 = \ell$, $p_1 \le p_2 < 5$;
- $i_1 < i_2 = \ell$, $p_1 \le 2$, $p_2 < 5$.

Then there exists a constant $k' = k'(j, i_2 - i_1)$, which grows linearly in its arguments, such that for any $\tau_2 > \tau_1 \ge \tau_0$,

$$\begin{aligned} &F(k, i_1, p_1, \tau_2, \partial_\tau^j \Psi_{-\mathfrak{s}}) + \int_{\tau_2}^{\infty} F(k-1, i_1, p_1 - 1, \tau', \partial_\tau^j \Psi_{-\mathfrak{s}}) \mathrm{d}\tau' \\ &\lesssim \langle \tau_2 - \tau_1 \rangle^{-2(i_2 - i_1) - (p_2 - p_1) - 2j} F(k + k', i_2, p_2, \tau_1, \Psi_{-\mathfrak{s}}). \end{aligned} \tag{3.59}$$

Proof. For each $j \in \mathbb{N}$, $\widetilde{\Phi}_{+\mathfrak{s}}^{(j)}$ and $\widetilde{\Phi}_{-\mathfrak{s}}^{(j+2\mathfrak{s})}$ satisfy the same equations, therefore, analogous to the discussions for the spin $-\mathfrak{s}$ component, in particular, analogous to the statement in Lemma 3.16, the statements in Lemma 3.19 are valid but under an additional, common assumption that $i_1 \ge \mathfrak{s}$. Meanwhile, the case that $i_1 < \mathfrak{s}$ and $i_2 < \mathfrak{s}$ has been treated in Proposition 3.18. In the remainder case that $i_1 < \mathfrak{s}$ and $i_2 \ge \mathfrak{s}$, we have from Proposition 3.18 that there exists a $k' = k'(j, \mathfrak{s} - i_1)$ such that

$$\begin{aligned} F(k, i_1, p_1, \tau_2, \partial_\tau^j \Psi_{-\mathfrak{s}}) &\lesssim \langle \tau_2 - \tau_1 \rangle^{-2(\mathfrak{s} - 1 - i_1) - (2 - p_1) - 2j} F(k + k', \mathfrak{s} - 1, 2, \tau_1 + (\tau_2 - \tau_1)/2, \Psi_{-\mathfrak{s}}) \\ &\lesssim \langle \tau_2 - \tau_1 \rangle^{-2(\mathfrak{s} - i_1) + p_1 - 2j} F(k + k', \mathfrak{s}, 0, \tau_1 + (\tau_2 - \tau_1)/2, \Psi_{-\mathfrak{s}}), \end{aligned}$$

where the last step is due to the relation $F(k, \mathfrak{s} - 1, 2, \tau, \Psi_{-\mathfrak{s}}) \sim F(k, \mathfrak{s}, 0, \tau, \Psi_{-\mathfrak{s}})$ by Definition 3.1. On the other hand, the above discussions yield that there exists a $k' = k'(i_2 - \mathfrak{s})$ such that

$$F(k, \mathfrak{s}, 0, \tau_1 + (\tau_2 - \tau_1)/2, \Psi_{-\mathfrak{s}}) \lesssim \langle \tau_2 - \tau_1 \rangle^{-2(i_2 - \mathfrak{s}) - p_2} F(k + k', i_2, p_2, \tau_1, \Psi_{-\mathfrak{s}})$$

Combining these two estimates then implies the desired estimate. □

3.4.3. *Proof of the weak decay estimates in Proposition 3.10.* Given the energy decay estimates above, we can now prove Proposition 3.10. We notice by definition that there exists a universal constant $k' \ge 0$ such that for any $p \ge 0$, the following holds

$$F(k - k', i - s, p, \tau, \Psi_{+\mathfrak{s}}) \lesssim \mathbb{I}_{\Sigma_\tau}^{i,k,p}[\Psi_{+\mathfrak{s}}] \lesssim F(k + k', i, p, \tau, \Psi_{+\mathfrak{s}}), \quad \text{for } \mathfrak{s} \le i \le \ell, \tag{3.60a}$$

$$F(k - k', i, p, \tau, \Psi_{-\mathfrak{s}}) \lesssim \mathbb{I}_{\Sigma_\tau}^{i,k,p}[\Psi_{-\mathfrak{s}}] \lesssim F(k + k', i, p, \tau, \Psi_{-\mathfrak{s}}), \quad \text{for } 0 \le i \le \ell. \tag{3.60b}$$

Suppose the spin $+\mathfrak{s}$ component is supported on a single ℓ mode. For any $j \in \mathbb{N}$ and $\tau_2 > \tau_1 \ge \tau_0$, we utilize the estimates in Lemma 3.16 and conclude

- the estimate (2.32) implies that for any $0 \le i \le \ell - \mathfrak{s}$, $0 \le p < 5$,

$$\begin{aligned} |\partial_\tau^j (r^{-1} \Psi_{+\mathfrak{s}})|_{k,\mathbb{D}} &\lesssim \bigg(\int_\tau^\infty F(k + k', 0, -1, \tau', \partial_\tau^j \Psi_{+\mathfrak{s}}) \mathrm{d}\tau' \int_\tau^\infty F(k + k', 0, -1, \tau', \partial_\tau^{j+1} \Psi_{+\mathfrak{s}}) \mathrm{d}\tau' \bigg)^{\frac{1}{4}} \\ &\lesssim (F(k + k', i, p, \tau_0, \Psi_{+\mathfrak{s}}))^{\frac{1}{2}} \tau^{-(p+1)/2 - i - j}. \end{aligned} \tag{3.61a}$$

- the estimate (2.30) implies that for any $0 \le i \le \ell - \mathfrak{s}$, $0 \le p < 5$,

$$|\partial_\tau^j (r^{-1/2} \Psi_{+\mathfrak{s}})|_{k,\mathbb{D}} \lesssim \big(F(k + k', 0, 0, \tau, \partial_\tau^j \Psi_{+\mathfrak{s}}) \big)^{\frac{1}{2}} \lesssim (F(k + k', i, p, \tau_0, \Psi_{+\mathfrak{s}}))^{\frac{1}{2}} \tau^{-p/2 - i - j}, \tag{3.61b}$$

- the estimate (2.31) with $\alpha = \vartheta$, ϑ sufficiently small, implies that for any $(i, p) \in \{0 < i \le \ell - \mathfrak{s}, 0 \le p < 5\} \cup \{i = 0, p > 1\}$

$$\begin{aligned} |\partial_\tau^j \Psi_{+\mathfrak{s}}|_{k,\mathbb{D}} &\lesssim (F(k + k', 0, 1 - \vartheta, \tau, \Psi_{+\mathfrak{s}}) F(k + k', 0, 1 + \vartheta, \tau, \Psi_{+\mathfrak{s}}))^{\frac{1}{4}} \\ &\lesssim \big(F(k + k', 0, 0, \tau, \partial_\tau^j \Psi_{+\mathfrak{s}}) \big)^{\frac{1}{2}} \tau^{-(p-1)/2 - i - j}, \end{aligned} \tag{3.61c}$$

Combining the above three estimates together yields that there exists a $k' = k'(j, i)$ such that for any $j \in \mathbb{N}$ and $(i, p) \in \{0 < i \le \ell - \mathfrak{s}, 0 \le p < 5\} \cup \{i = 0, p > 1\}$,

$$|\partial_\tau^j (r^{-1} \Psi_{+\mathfrak{s}})|_{k,\mathbb{D}} \lesssim (F(k + k', i, p, \tau_0, \Psi_{+\mathfrak{s}}))^{\frac{1}{2}} v^{-1} \tau^{-(p-1)/2 - i - j}. \tag{3.62}$$

Suppose the spin $-\mathfrak{s}$ component is supported on a single ℓ mode. Similarly, we employ the estimates in Lemma 3.19 together with the estimates (2.30)–(2.32) to conclude that that there exists a $k' = k'(j,i)$ such that for any $j \in \mathbb{N}$ and $(i,p) \in \{0 < i \leq \ell, 0 \leq p < 5\} \cup \{i = 0, p > 1\}$,

$$\sum_{n=0}^{\mathfrak{s}} |\partial_\tau^j (r^{-1}(r^2V)^n \Psi_{-\mathfrak{s}})|_{k,\mathbb{D}} \lesssim (F(k+k', i, p, \tau_0, \Psi_{-\mathfrak{s}}))^{\frac{1}{2}} v^{-1} \tau^{-(p-1)/2-i-j}. \tag{3.63}$$

Taking into account the relation (3.60), the above thus proves the estimate (3.24).

We next consider the case that the spin $+\mathfrak{s}$ component is supported on $\ell \geq \ell_0$ modes with $\ell_0 \geq \mathfrak{s}+1$, and the case for the spin $-\mathfrak{s}$ component can be analogously treated. In fact, the estimates in Proposition 3.11 can be applied to equation (3.11), hence the estimate (3.34) holds for all $0 \leq i \leq \ell - \mathfrak{s}$, $p \in [0,2]$ and $\tau_2 > \tau_1 \geq \tau_0$. By going through the remaining proof in Section 3.4.1 then yields point 2 of Proposition 3.10.

Remark 3.20. The above proof also yields that if the spin $s = \pm\mathfrak{s}$ components are supported on a fixed ℓ mode, $\ell \geq \mathfrak{s}$, then for any j, $0 \leq i \leq \ell - \mathfrak{s}$ and $0 \leq p < 5$, there exists a $k' = k'(j,i)$ such that in the region $r \geq 4M$,

$$|\partial_\tau^j (r^{-1}\Phi_{+\mathfrak{s}}^{(i)})|_{k,\mathbb{D}} \lesssim (\mathbb{I}_{\Sigma_\tau}^{\ell,k+k',p}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}} v^{-1} \tau^{-(p-1)/2-(\ell-\mathfrak{s}-i)-j}. \tag{3.64}$$

3.5. **Teukolsky–Starobinsky identities.** It is surprising that the spin $\pm\mathfrak{s}$ components are related to each other by purely differential identities–the *Teukolsky-Starobinsky identities* (TSI), originating from [76, 70]. See also the covariant form of these identities in [1]. We state explicitly these TSI in terms of our terminologies, and these identities will be crucial in the later sections.

Lemma 3.21. *(1) There are the following TSI for the spin ± 1 components of the Maxwell field*

$$(\mathring{\eth}')^2(\Delta^{-1}\psi_{+1}) = \hat{V}^2(\Delta\psi_{-1}), \tag{3.65a}$$

$$(\mathring{\eth})^2\psi_{-1} = Y^2\psi_{+1}. \tag{3.65b}$$

The first one (3.65a) *can also be written as*

$$\Phi_{-1}^{(2)} = (\mathring{\eth}')^2\Phi_{+1}^{(0)}. \tag{3.66}$$

If restricted to a fixed ℓ mode, then equations (3.65a) *become*

$$\ell(\ell+1)\Phi_{+1}^{(0)} = \Phi_{-1}^{(2)}, \tag{3.67a}$$

$$\ell(\ell+1)\psi_{-1} = Y^2\psi_{+1}. \tag{3.67b}$$

(2) There are the following TSI for the spin ± 2 components of the linearized gravity:

$$\mu^2(\hat{V})^4(\mu^2 r^4\psi_{-2}) = (\mathring{\eth}')^4(r^{-4}\psi_{+2}) - 12M\overline{\partial_\tau(r^{-4}\psi_{+2})}, \tag{3.68a}$$

$$Y^4(\psi_{+2}) = (\mathring{\eth})^4\psi_{-2} + 12M\overline{\partial_\tau\psi_{-2}}. \tag{3.68b}$$

Moreover, equation (3.68a) *can also be written as*

$$\Phi_{-2}^{(4)} = (\mathring{\eth}')^4\Phi_{+2}^{(0)} - 12M\overline{\partial_\tau\Phi_{+2}^{(0)}}. \tag{3.69}$$

If restricted to a fixed ℓ mode, then equations (3.68) *become*

$$(\ell-1)\ell(\ell+1)(\ell+2)\Phi_{+2}^{(0)} = \Phi_{-2}^{(4)} + 12M\overline{\partial_\tau\Phi_{+2}^{(0)}}, \tag{3.70a}$$

$$(\ell-1)\ell(\ell+1)(\ell+2)\psi_{-2} = Y^4\psi_{+2} - 12M\overline{\partial_\tau\psi_{-2}}. \tag{3.70b}$$

Proof. For the spin ± 1 components, these TSI can be easily derived from the Maxwell system of equations (2.22): the identity (3.65b) can be obtained by applying $\frac{1}{2}r^{-1}Y$ to (2.22a) and using (2.22d); the other identity (3.65a) can be obtained from (3.65b) by interchanging l and n and $\mathring{\eth}$ and $\mathring{\eth}'$. Equation (3.66) is straightforward from identity (3.65a). When considering a fixed ℓ mode, equations (3.67a) and (3.67b) are immediate from (3.66) and (3.65b), respectively, together with the eigenvalues of $(\mathring{\eth}')^2$ and $(\mathring{\eth})^2$ in (2.17).

For the spin ± 2 components, the equations in [3, Equations (3.26)] can be rewritten in the form

$$Y^4(\kappa_1^4\Upsilon_{+2}) = r^{-4}(\mathring{\eth})^4(\kappa_1^4\Upsilon_{-2}) + \frac{M}{27}\partial_\tau\overline{\Upsilon_{-2}}, \tag{3.71a}$$

$$(\hat{V})^4(\mu^2\kappa_1^4\Upsilon_{-2}) = \mu^{-2}\left(r^{-4}(\mathring{\eth}')^4(\kappa_1^4\Upsilon_{+2}) - \frac{M}{27}\partial_\tau\overline{\Upsilon_{+2}}\right), \tag{3.71b}$$

where $\kappa_1 = -\frac{1}{3}r$ in Schwarzschild spacetime. By substituting in $\Upsilon_{+2} = r^{-4}\psi_{+2}$ and $\Upsilon_{-2} = \psi_{-2}$, one arrives at equations (3.68).

Using Definition 3.1, equation (3.68a) becomes

$$r^5(\hat{V})^4(r^3\Phi_{-2}^{(0)}) = (\mathring{\eth}')^4\Phi_{+2}^{(0)} - 12M\overline{\partial_\tau\Phi_{+2}^{(0)}}. \tag{3.72}$$

Expanding out the LHS and using Definition 3.1 again, one finds the LHS equals $\Phi_{-2}^{(4)}$, hence completing the proof of (3.69). If restricted to to a fixed ℓ mode, since we have from (2.17) that the eigenvalues of $(\mathring{\eth}')^4$ acting on the spin $+2$ component and $(\mathring{\eth})^4$ acting on the spin -2 component are both $(\ell-1)\ell(\ell+1)(\ell+2)$, equations (3.70a) and (3.70b) are thus easily justified from (3.68b)and (3.68a). $\square$

3.6. **Newman–Penrose constants.** We construct the N–P constants for the spin $\pm\mathfrak{s}$ components in this subsection. As will be seen below, these N–P constants are in terms of fixed modes of the components. The vanishing or nonvanishing property of these constants determines the energy decay rates, and these constants are of paramount importance in characterising the precise asymptotics of any fixed mode or the field itself in proving the Price's law.

Definition 3.22. Let $\ell \in \mathbb{N}$ and $\ell \geq \mathfrak{s}$. Define the (m,ℓ)-th N–P constants of the spin $s = +\mathfrak{s}$ and $s = -\mathfrak{s}$ components to be $\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)} = \lim_{\rho\to\infty} \hat{\mathcal{V}}(\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})})_{m,\ell}$ and $\mathbb{Q}_{-\mathfrak{s}}^{(m,\ell)} = \lim_{\rho\to\infty} \hat{\mathcal{V}}(\widetilde{\Phi}_{-\mathfrak{s}}^{(\ell+\mathfrak{s})})_{m,\ell}$, respectively, where $(\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})})_{m,\ell}$ and $(\widetilde{\Phi}_{-\mathfrak{s}}^{(\ell+\mathfrak{s})})_{m,\ell}$ are defined as in Definition 3.6 from the (m,ℓ) mode of the spin $+\mathfrak{s}$ and $-\mathfrak{s}$ components, respectively.

Remark 3.23. These N–P constants are defined in such a way that only the (m,ℓ) mode is relevant to the (m,ℓ)-th N–P constant. Hence, in the rest of this subsection, we will always assume that the spin $\pm\mathfrak{s}$ components are supported only on the (m,ℓ) mode.

Remark 3.24. As will be shown in Proposition 3.26, these N–P constants are independent of τ under some general conditions, a fact which justifies they are indeed real constants.

Proposition 3.25. *Let $\ell \geq \mathfrak{s}$ and $k \in \mathbb{N}$. Let $k' = k'(\ell) > 0$ be suitably large.*

(1) Assume $\mathbb{I}_{\Sigma_{\tau_0}}^{\ell,k+k',0}[\Psi_{+\mathfrak{s}}] < \infty$ which is defined as in Definition 3.8.

(i) If $\lim_{r\to\infty}\sum_{j=0}^{\ell-\mathfrak{s}}|\Phi_{+\mathfrak{s}}^{(j)}|_{k,\mathbb{D}}|_{\Sigma_{\tau_0}} < \infty$, then for any $\tau \geq \tau_0$, $\lim_{r\to\infty}\sum_{j=0}^{\ell-\mathfrak{s}}|\Phi_{+\mathfrak{s}}^{(j)}|_{k,\mathbb{D}}|_{\Sigma_\tau} < \infty$. The same statement holds if one replaces all $\Phi_{+\mathfrak{s}}^{(j)}$ by $\widetilde{\Phi}_{+\mathfrak{s}}^{(j)}$;

(ii) If $\lim_{r\to\infty}\left(\sum_{j=0}^{\ell-\mathfrak{s}}|\Phi_{+\mathfrak{s}}^{(j)}|_{k,\mathbb{D}}|_{\Sigma_{\tau_0}} + r^{-\alpha}|\Phi_{+\mathfrak{s}}^{(\ell-\mathfrak{s}+1)}|_{k,\mathbb{D}}|_{\Sigma_{\tau_0}}\right) < \infty$ for some $\alpha \in [0,2]$, then for any $\tau \geq \tau_0$, $\lim_{r\to\infty}\left(\sum_{j=0}^{\ell-\mathfrak{s}}|\Phi_{+\mathfrak{s}}^{(j)}|_{k,\mathbb{D}}|_{\Sigma_\tau} + r^{-\alpha}|\Phi_{+\mathfrak{s}}^{(\ell-\mathfrak{s}+1)}|_{k,\mathbb{D}}|_{\Sigma_\tau}\right) < \infty$. The same statement holds if one replaces all $\Phi_{+\mathfrak{s}}^{(j)}$ and $\Phi_{+\mathfrak{s}}^{(\ell-\mathfrak{s}+1)}$ by $\widetilde{\Phi}_{+\mathfrak{s}}^{(j)}$ and $\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s}+1)}$, respectively;

(2) Assume $\mathbb{I}_{\Sigma_{\tau_0}}^{\ell,k+k',0}[\Psi_{-\mathfrak{s}}] < \infty$ which is defined as in Definition 3.8.

(iii) If $\lim_{r\to\infty}\sum_{j=0}^{\ell+\mathfrak{s}}|\Phi_{-\mathfrak{s}}^{(j)}|_{k,\mathbb{D}}|_{\Sigma_{\tau_0}} < \infty$, then for any $\tau \geq \tau_0$, $\lim_{r\to\infty}\sum_{j=0}^{\ell+\mathfrak{s}}|\Phi_{-\mathfrak{s}}^{(j)}|_{k,\mathbb{D}}|_{\Sigma_\tau} < \infty$. The same statement holds if one replaces all $\Phi_{-\mathfrak{s}}^{(j)}$ by $\widetilde{\Phi}_{-\mathfrak{s}}^{(j)}$;

(iv) *If* $\lim_{r\to\infty}\Big(\sum_{j=0}^{\ell+\mathfrak{s}}|\Phi_{-\mathfrak{s}}^{(j)}|_{k,\mathbb{D}}|_{\Sigma_{\tau_0}}+r^{-\alpha}|\Phi_{-\mathfrak{s}}^{(\ell+\mathfrak{s}+1)}|_{k,\mathbb{D}}|_{\Sigma_{\tau_0}}\Big)<\infty$ *for some* $\alpha\in[0,2]$*, then for any* $\tau\geq\tau_0$*,* $\lim_{r\to\infty}\Big(\sum_{j=0}^{\ell+\mathfrak{s}}|\Phi_{-\mathfrak{s}}^{(j)}|_{k,\mathbb{D}}|_{\Sigma_{\tau}}+r^{-\alpha}|\Phi_{-\mathfrak{s}}^{(\ell+\mathfrak{s}+1)}|_{k,\mathbb{D}}|_{\Sigma_{\tau}}\Big)<\infty$*. The same statement holds if one replaces all* $\Phi_{-\mathfrak{s}}^{(j)}$ *and* $\Phi_{-\mathfrak{s}}^{(\ell+\mathfrak{s}+1)}$ *by* $\widetilde{\Phi}_{-\mathfrak{s}}^{(j)}$ *and* $\widetilde{\Phi}_{-\mathfrak{s}}^{(\ell+\mathfrak{s}+1)}$*, respectively.*

Proof. The assumption $\mathbb{I}_{\Sigma_{\tau_0}}^{\ell,k+k',0}[\Psi_{+\mathfrak{s}}]<\infty$ in particular yields that for any $\tau\geq\tau_0$ and any $1\leq j\leq\ell$,

$$\|\Psi_{+\mathfrak{s}}\|^2_{W^k_{-2}(\Sigma_\tau)}+\sum_{j=0}^{\ell-\mathfrak{s}}\|\Phi_{+\mathfrak{s}}^{(j)}\|^2_{W^k_{-2}(\Sigma_\tau^{\geq 4M})}<\infty \tag{3.73}$$

and

$$\sup_{\Sigma_\tau}\int_{S^2}r^{-1}|\Psi_{+\mathfrak{s}}|^2_{k,\mathbb{D}}\mathrm{d}^2\mu+\sup_{\Sigma_\tau\cap\{\rho\geq 4M\}}\sum_{j=0}^{\ell-\mathfrak{s}}\int_{S^2}r^{-1}|\Phi_{+\mathfrak{s}}^{(j)}|^2_{k,\mathbb{D}}\mathrm{d}^2\mu<\infty. \tag{3.74}$$

Note that the first estimate (3.73) follows from Lemma 3.16 and the relation (3.60), and the second estimate (3.74) follows from the Sobolev-type estimate (2.30) together with the estimate (3.73). The rest of the proof is similar to the one of [9, Propositions 3.4 and 3.5] and we omit it. □

Proposition 3.26. *Let* $\ell\geq\mathfrak{s}$*, and let* $k'=k'(\ell)>0$ *be suitably large.*

(i) *Assume* $\mathbb{I}_{\Sigma_{\tau_0}}^{\ell,k',0}[\Psi_{+\mathfrak{s}}]<\infty$*, and assume* $\lim_{r\to\infty}\sum_{j=0}^{\ell-\mathfrak{s}}(|\widetilde{\Phi}_{+\mathfrak{s}}^{(j)}|+|\hat{\mathcal{V}}\widetilde{\Phi}_{+\mathfrak{s}}^{(j)}|)|_{\Sigma_{\tau_0}}<\infty$*, then the* (m,ℓ)*-th N–P constant* $\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)}$ *is finite and independent of* τ*;*

(ii) *Assume* $\mathbb{I}_{\Sigma_{\tau_0}}^{\ell,k',0}[\Psi_{-\mathfrak{s}}]<\infty$*, and assume* $\lim_{r\to\infty}\sum_{j=0}^{\ell+\mathfrak{s}}(|\widetilde{\Phi}_{-1}^{(j)}|+|\hat{\mathcal{V}}\widetilde{\Phi}_{-\mathfrak{s}}^{(j)}|)|_{\Sigma_{\tau_0}}<\infty$*, then the* (m,ℓ)*-th N–P constant* $\mathbb{Q}_{-\mathfrak{s}}^{(m,\ell)}$ *is finite and independent of* τ*.*

Proof. Recall that we consider only the (m,ℓ) mode of the spin $\pm\mathfrak{s}$ component. We have from Proposition 3.7 the following equations for $\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}$ and $\widetilde{\Phi}_{-\mathfrak{s}}^{(\ell+\mathfrak{s})}$:

$$-2\partial_u\hat{\mathcal{V}}\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}-2(\ell+1)(r-3M)r^{-2}\hat{\mathcal{V}}\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}+\sum_{j=0}^{\ell-\mathfrak{s}}O(r^{-1})\widetilde{\Phi}_{+\mathfrak{s}}^{(j)}=0, \tag{3.75a}$$

$$-2\partial_u\hat{\mathcal{V}}\widetilde{\Phi}_{-\mathfrak{s}}^{(\ell+\mathfrak{s})}-2(\ell+1)(r-3M)r^{-2}\hat{\mathcal{V}}\widetilde{\Phi}_{-\mathfrak{s}}^{(\ell+\mathfrak{s})}+\sum_{j=2\mathfrak{s}}^{\ell+\mathfrak{s}}O(r^{-1})\widetilde{\Phi}_{-\mathfrak{s}}^{(j)}=0. \tag{3.75b}$$

Consider the spin $+\mathfrak{s}$ component case, the spin $-\mathfrak{s}$ component case being treated in the same fashion. Since by Proposition 3.25, $\lim_{r\to\infty}\big(\sum_{j=0}^{\ell-\mathfrak{s}}|\widetilde{\Phi}_{+\mathfrak{s}}^{(j)}|+|\hat{\mathcal{V}}\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}|_{1,\mathbb{D}}\big)|_{\Sigma_\tau}<\infty$ for any $\tau\geq\tau_0$, we conclude from (3.75a) that

$$\lim_{\rho\to\infty}\partial_\tau\hat{\mathcal{V}}\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}|_{\Sigma_\tau}=0. \tag{3.76}$$

By the bounded convergence theorem, the statement follows. □

Given the above results, we can now show that the (m,ℓ)-th N–P constants for the spin $+\mathfrak{s}$ and $-\mathfrak{s}$ components are related to each other by a nonzero constant factor depending only on ℓ and $\mathfrak{s}$.

Corollary 3.27. *Let* $\ell\geq\mathfrak{s}$*. Assume* $\mathbb{I}_{\Sigma_{\tau_0}}^{\ell,k',0}[\Psi_{+\mathfrak{s}}]<\infty$ *for a suitably large* $k'=k'(\ell)>0$*, and assume* $\lim_{r\to\infty}\sum_{j=0}^{\ell-\mathfrak{s}}(|\widetilde{\Phi}_{+\mathfrak{s}}^{(j)}|+|\hat{\mathcal{V}}\widetilde{\Phi}_{+\mathfrak{s}}^{(j)}|)|_{\Sigma_{\tau_0}}<\infty$*. Then*

(i) $\mathbb{Q}_{-\mathfrak{s}}^{(m,\ell)}=\prod_{i=\ell-\mathfrak{s}+1}^{\ell+\mathfrak{s}}i\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)}$*;*

(ii) *if* $\mathbb{Q}_{-\mathfrak{s}}^{(m,\ell)}$ *vanishes, then* $\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)}$ *vanishes, and vice versa.*

Proof. Equation (3.67a) is $\ell(\ell+1)\Phi^{(0)}_{+1} = \Phi^{(2)}_{-1}$ and equation (3.70a) is $(\ell-1)\ell(\ell+1)(\ell+2)\Phi^{(0)}_{+2} = \Phi^{(4)}_{-2} + 12M\overline{\partial_\tau \Phi^{(0)}_{+2}}$, which thus yield

$$(\ell+1)\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell-1)}_{+1} = \hat{\mathcal{V}}\widetilde{\Phi}^{(\ell+1)}_{-1}, \tag{3.77}$$

$$(\ell-1)\ell(\ell+1)(\ell+2)\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell-2)}_{+2} = \hat{\mathcal{V}}\widetilde{\Phi}^{(\ell+2)}_{-2} + 12M\overline{\partial_\tau \hat{\mathcal{V}}\widetilde{\Phi}^{(\ell-2)}_{+2}}. \tag{3.78}$$

By Proposition 3.26 and (3.76), the statement (i) follows. The statement (ii) is manifest true. □

Proposition 3.28. *Let $\ell \geq \mathfrak{s}$, $\alpha \in [0,1]$ be arbitrary, and $k \in \mathbb{N}$. Assume the (m,ℓ)-th N–P constant $\mathbb{Q}^{(m,\ell)}_{-\mathfrak{s}}$ vanishes.*

- (i) *If $\mathbb{I}^{\ell,k+k',0}_{\Sigma_{\tau_0}}[\Psi_{-\mathfrak{s}}] + \lim\limits_{r\to\infty}\Big(\sum\limits_{j=0}^{\ell+\mathfrak{s}} |\widetilde{\Phi}^{(j)}_{-\mathfrak{s}}|_{k,\mathbb{D}} + |r^{1-\alpha}\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell+\mathfrak{s})}_{-\mathfrak{s}}|_{k,\mathbb{D}}\Big)|_{\Sigma_{\tau_0}} < \infty$ for a suitably large $k' = k'(\ell)$, then there is a constant $C_\ell(\tau) < \infty$ such that for any $\tau \geq \tau_0$, $\lim\limits_{r\to\infty}|r^{1-\alpha}\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell+\mathfrak{s})}_{-\mathfrak{s}}|_{k,\mathbb{D}}|_{\Sigma_\tau} < C_\ell(\tau)$. In particular, if $\alpha > 0$, then $\lim\limits_{r\to\infty}|r^{1-\alpha}\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell+\mathfrak{s})}_{-\mathfrak{s}}|_{k,\mathbb{D}}|_{\Sigma_\tau}$ is independent of τ;*
- (ii) *If $\mathbb{I}^{\ell,k+k',0}_{\Sigma_{\tau_0}}[\Psi_{+\mathfrak{s}}] + \lim\limits_{r\to\infty}\Big(\sum\limits_{j=0}^{\ell-\mathfrak{s}} |\widetilde{\Phi}^{(j)}_{+\mathfrak{s}}|_{k,\mathbb{D}} + |r^{1-\alpha}\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}|_{k,\mathbb{D}}\Big)|_{\Sigma_{\tau_0}} < \infty$ for a suitably large $k' = k'(\ell)$, then there is a constant $C_\ell(\tau) < \infty$ such that for any $\tau \geq \tau_0$, $\lim\limits_{r\to\infty}|r^{1-\alpha}\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}|_{k,\mathbb{D}}|_{\Sigma_\tau} < C_\ell(\tau)$. In particular, if $\alpha > 0$, then $\lim\limits_{r\to\infty}|r^{1-\alpha}\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}|_{k,\mathbb{D}}|_{\Sigma_\tau}$ is independent of τ.*

Proof. By Corollary 3.27, the (m,ℓ)-th N–P constant $\mathbb{Q}^{(m,\ell)}_{+\mathfrak{s}}$ of the spin $+\mathfrak{s}$ component vanishes as well. We show the statements only for the spin $-\mathfrak{s}$ component, the proof of spin $+\mathfrak{s}$ component being the same. The scalar $\widetilde{\Phi}^{(\ell+\mathfrak{s})}_{-\mathfrak{s}}$ satisfies equation (3.75b), and hence performing an $r^{1-\alpha}$ rescaling gives

$$-2\partial_\tau(r^{1-\alpha}\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell+\mathfrak{s})}_{-\mathfrak{s}}) = O(r^{-\alpha})rV\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell+\mathfrak{s})}_{-\mathfrak{s}} + O(r^{-\alpha})\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell+\mathfrak{s})}_{-\mathfrak{s}} + \sum_{j=2\mathfrak{s}}^{\ell+\mathfrak{s}} O(r^{-\alpha})\widetilde{\Phi}^{(j)}_{-\mathfrak{s}}. \tag{3.79}$$

In the case that $\alpha > 0$, we have from the assumption of vanishing (m,ℓ)-th N–P constant and Propositions 3.25 and 3.26 that the limit of the RHS on Σ_τ as $\rho \to \infty$ is zero for any $\tau \geq \tau_0$, hence one obtains $\lim\limits_{r\to\infty}\partial_\tau(r^{1-\alpha}\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell+\mathfrak{s})}_{-\mathfrak{s}})|_{\Sigma_\tau} = 0$. The statements for $\alpha > 0$ then follow from the bounded convergence theorem. For $\alpha = 0$, the limit of the RHS on Σ_τ as $\rho \to \infty$ is now bounded by a τ-dependent constant, hence $\lim\limits_{r\to\infty}|r\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell+\mathfrak{s})}_{-\mathfrak{s}}|_{k,\mathbb{D}}|_{\Sigma_\tau} < C_\ell(\tau)$. □

3.7. Almost Price's law for a fixed mode in the exterior region $\{\rho \geq \tau\}$.

Lemma 3.29. *Assume the the spin $\pm\mathfrak{s}$ components are supported on $\ell \geq \ell_0$ mode, $\ell_0 \geq \mathfrak{s}$. There exists universal constants $C = C(\vartheta, j, k, \ell_0) > 0$ and $k' = k'(j, \ell_0) > 0$ such that for any $\tau \geq \tau_0$ and $p \in [0,5)$,*

$$C^{-1}\mathbb{I}^{\ell_0,k-k',p}_{\Sigma_\tau}[\Psi_{+\mathfrak{s}}] \leq \mathbb{I}^{\ell_0,k,p}_{\Sigma_\tau}[\Psi_{-\mathfrak{s}}] \leq C\mathbb{I}^{\ell_0,k+k',p}_{\Sigma_\tau}[\Psi_{+\mathfrak{s}}]. \tag{3.80}$$

Proof. This follows easily from Definition 3.8 and the TSI between the spin $+\mathfrak{s}$ and $-\mathfrak{s}$ components in Lemma 3.21. □

Proposition 3.30. *Assume the the spin $\pm\mathfrak{s}$ components are supported on a single (m,ℓ) mode, $\ell \geq \mathfrak{s}$ and $m \in \{-\ell, -\ell+1, \dots, \ell\}$. Let the (m,ℓ)-th N–P constants $\mathbb{Q}^{(m,\ell)}_{-\mathfrak{s}}$ be defined as in Definition 3.22. Let $\vartheta \in (0, 1/2)$ be arbitrary, and let $j \in \mathbb{N}$.*

- (i) *If the ℓ-th N–P constant $\mathbb{Q}^{(m,\ell)}_{-\mathfrak{s}}$ is nonzero, there exist universal constants $C = C(\vartheta, j, k, \ell)$ and $k' = k'(j,\ell) > 0$ such that in the exterior region $\{\rho \geq \tau\}$,*

$$|\partial^j_\tau(r^{-2\mathfrak{s}-1}\Psi_{+\mathfrak{s}})|_{k-k',\mathbb{D}} \leq Cv^{-1-2\mathfrak{s}}\tau^{-\frac{2\ell-2\mathfrak{s}-\vartheta}{2}-j-1}(\mathbb{I}^{\ell,k,3-\vartheta}_{\Sigma_{\tau_0}}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}}, \tag{3.81a}$$

$$|\partial^j_\tau(r^{-1}\Psi_{-\mathfrak{s}})|_{k-k',\mathbb{D}} \leq Cv^{-1}\tau^{-\frac{2\ell+2\mathfrak{s}-\vartheta}{2}-j-1}(\mathbb{I}^{\ell,k,3-\vartheta}_{\Sigma_{\tau_0}}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}}, \tag{3.81b}$$

and for any $0 \le i \le \ell - \mathfrak{s}$,

$$|\partial_\tau^j (r^{-1}\Phi_{+\mathfrak{s}}^{(i)})|_{k-k',\mathbb{D}} \le C v^{-1} \tau^{-\frac{2\ell-2\mathfrak{s}-2i-\vartheta}{2}-j-1} (\mathbb{I}^{\ell,k,3-\vartheta}_{\Sigma_{\tau_0}}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}}. \tag{3.82}$$

(*ii*) *If the* ℓ*-th N–P constant* $\mathbb{Q}^{(m,i)}_{-\mathfrak{s}}$ *equals zero, there exist universal constants* $C = C(\vartheta, j, k, \ell)$ *and* $k' = k'(j,\ell) > 0$ *such that in the exterior region* $\{\rho \ge \tau\}$,

$$|\partial_\tau^j (r^{-2\mathfrak{s}-1}\Psi_{+\mathfrak{s}})|_{k-k',\mathbb{D}} \le C v^{-1-2\mathfrak{s}} \tau^{-\frac{2\ell-2\mathfrak{s}-\vartheta}{2}-j-2} (\mathbb{I}^{\ell,k,5-\vartheta}_{\Sigma_{\tau_0}}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}}, \tag{3.83a}$$

$$|\partial_\tau^j (r^{-1}\Psi_{-\mathfrak{s}})|_{k-k',\mathbb{D}} \le C v^{-1} \tau^{-\frac{2\ell+2\mathfrak{s}-\vartheta}{2}-j-2} (\mathbb{I}^{\ell,k,5-\vartheta}_{\Sigma_{\tau_0}}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}}. \tag{3.83b}$$

Remark 3.31.
- These decay estimates, compared to the predicted Price's law, have only a $\tau^{\vartheta/2}$ loss, with $\vartheta > 0$ arbitrarily small. This explains why they are called *almost Price's law*.
- In the case (i) that $\mathbb{Q}^{(m,\ell)}_{-\mathfrak{s}}$ is nonzero, the energies $\mathbb{I}^{\ell,k,p}_{\Sigma_{\tau_0}}[\Psi_{+\mathfrak{s}}] = \infty$ and $\mathbb{I}^{\ell,k,p}_{\Sigma_{\tau_0}}[\Psi_{-\mathfrak{s}}] = \infty$ for $p \ge 3$ by Definition 3.8. This is why these decay estimates are almost sharp.

Proof. Consider first the case that the ℓ-th N–P constant $\mathbb{Q}^{(m,\ell)}_{-\mathfrak{s}}$ is nonvanishing. We have from Proposition 3.10 with $p = 3 - \vartheta$ that

$$|\partial_\tau^j (r^{-1-2\mathfrak{s}}\Psi_{+\mathfrak{s}})|_{k,\mathbb{D}} \le C(\vartheta, j, k, \ell) (\mathbb{I}^{\ell,k+k',3-\vartheta}_{\Sigma_\tau}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}} r^{-2\mathfrak{s}} v^{-1} \tau^{-(2-\vartheta)/2-(\ell-\mathfrak{s})-j}, \tag{3.84a}$$

$$\sum_{n=0}^{\mathfrak{s}} |\partial_\tau^j (r^{-1}(r^2V)^n\Psi_{-\mathfrak{s}})|_{k,\mathbb{D}} \le C(\vartheta, j, k, \ell) (\mathbb{I}^{\ell,k+k',3-\vartheta}_{\Sigma_\tau}[\Psi_{-\mathfrak{s}}])^{\frac{1}{2}} v^{-1} \tau^{-(2-\vartheta)/2-\ell-j}. \tag{3.84b}$$

This already implies the $\mathfrak{s} = 0$ case of (3.81). The estimate (3.84b) for the spin $-\mathfrak{s}$ component can be improved in the following way. For $\mathfrak{s} = 1$, we have from equation (3.3a) that in the exterior region,

$$\ell(\ell+1)\Phi_{-1}^{(0)} = 2(r-3M)r^{-2}\Phi_{-1}^{(1)} - r^2YV\Phi_{-1}^{(0)} = O(r^{-1})\Phi_{-1}^{(1)} + O(r^{-1})rV\Phi_{-1}^{(1)} + O(1)\partial_\tau\Phi_{-1}^{(1)}.$$

Since the RHS has further τ^{-1} decay in the exterior region compared to $\Phi_{-1}^{(1)}$, the estimate (3.81b) thus follows. For $\mathfrak{s} = 2$, we have from (3.4a) and (3.4b) that in the exterior region,

$$((\ell-1)(\ell+2) + 6Mr^{-1})\Phi_{-2}^{(0)} - 4(r-3M)r^{-2}\Phi_{-2}^{(1)} = -r^2YV\Phi_{-2}^{(0)} = O(r^{-1})\Phi_{-2}^{(1)} + O(r^{-1})rV\Phi_{-2}^{(1)} + O(1)\partial_\tau\Phi_{-2}^{(1)}, \tag{3.85a}$$

$$(\ell(\ell+1) - 6Mr^{-1})\Phi_{-2}^{(1)} - 6M\Phi_{-2}^{(0)} = 2(r-3M)r^{-2}\Phi_{-2}^{(2)} - r^2YV\Phi_{-2}^{(1)} = O(r^{-1})\Phi_{-2}^{(2)} + O(r^{-1})rV\Phi_{-2}^{(2)} + O(1)\partial_\tau\Phi_{-2}^{(2)}. \tag{3.85b}$$

When viewing equations (3.85) as a coupled system of equations of $(\Phi_{-2}^{(0)}, \Phi_{-2}^{(1)})$, the matrix operator of the LHS, $\begin{pmatrix} (\ell-1)(\ell+2)+6Mr^{-1} & -4(r-3M)r^{-2} \\ -6M & \ell(\ell+1)-6Mr^{-1} \end{pmatrix}$, has determinant not less than $24 - 12Mr^{-1} + 36M^2r^{-2}$ and is clearly positive definite for $r \ge 2M$, and the RHS has further τ^{-1} decay compared to $\Phi_{-2}^{(2)}$. An elliptic estimate then yields

$$\sum_{n=0}^{1} |\partial_\tau^j (r^{-1}\Phi_{-2}^{(n)})|_{k,\mathbb{D}} \lesssim (\mathbb{I}^{\ell,k+k',3-\vartheta}_{\Sigma_\tau}[\Psi_{-\mathfrak{s}}])^{\frac{1}{2}} v^{-1} \tau^{-(2-\vartheta)/2-1-\ell-j}. \tag{3.86}$$

Plugging this estimate back into (3.85a) and moving the term $-4(r-3M)r^{-2}\Phi_{-2}^{(1)}$ in (3.85a) from the LHS to the RHS, we then conclude (3.81b).

The estimate (3.81a), in the exterior region $\{\rho \ge \tau\}$ is trivial in view of the estimate (3.84a) since $r \sim v$. Meanwhile, the estimate (3.82) holds true in the exterior region $\{\rho \ge \tau\}$ in view of Remark 3.20.

Consider next the case that the ℓ-th N–P constant $\mathbb{Q}^{(m,\ell)}_{-\mathfrak{s}}$ vanishes. By Proposition 3.10 with $p = 5 - \vartheta$ instead, and together with Lemma 3.29, the proof proceeds in the same way as in the above case. ∎

Proposition 3.32. *Assume the the spin* $\pm\mathfrak{s}$ *components are supported on* $\ell \ge \ell_0$ *modes,* $\ell_0 \ge \mathfrak{s}$. *Let the* (m, ℓ_0)*-th N–P constants* $\mathbb{Q}^{(m,\ell_0)}_{-\mathfrak{s}}$ *for the* (m, ℓ_0) *mode of the spin* $+\mathfrak{s}$ *component be defined as in Definition 3.22. Let* $\vartheta \in (0, 1/2)$ *be arbitrary, and let* $j \in \mathbb{N}$.

(1) If not all of the (m,ℓ_0)-th N–P constants $\mathbb{Q}^{(m,\ell_0)}_{-\mathfrak{s}}$, $m=-\ell_0,\ldots,\ell_0$, are nonzero, there exist universal constants $C=C(\vartheta,j,k,\ell_0)$ and $k'=k'(j,\ell_0)>0$ such that in the exterior region $\{\rho\geq\tau\}$,

$$|\partial_\tau^j(r^{-2\mathfrak{s}-1}\Psi_{+\mathfrak{s}})|_{k-k',\mathbb{D}}\leq Cv^{-1-2\mathfrak{s}}\tau^{-\frac{2\ell-2\mathfrak{s}-\vartheta}{2}-j-1}(\mathbb{I}^{\ell\geq\ell_0,k,\vartheta}_{NV,\Sigma_{\tau_0}}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}}, \tag{3.87a}$$

$$|\partial_\tau^j(r^{-1}\Psi_{-\mathfrak{s}})|_{k-k',\mathbb{D}}\leq Cv^{-1}\tau^{-\frac{2\ell_0+2\mathfrak{s}-\vartheta}{2}-j-1}(\mathbb{I}^{\ell\geq\ell_0,k,\vartheta}_{NV,\Sigma_{\tau_0}}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}}, \tag{3.87b}$$

and

$$|\partial_\tau^j(r^{-2\mathfrak{s}-1}(\Psi_{+\mathfrak{s}})^{\ell\geq\ell_0+1})|_{k-k',\mathbb{D}}\leq Cv^{-1-2\mathfrak{s}}\tau^{-\frac{2\ell-2\mathfrak{s}+\vartheta}{2}-j-1}(\mathbb{I}^{\ell\geq\ell_0,k,\vartheta}_{NV,\Sigma_{\tau_0}}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}}, \tag{3.88a}$$

$$|\partial_\tau^j(r^{-1}(\Psi_{-\mathfrak{s}})^{\ell\geq\ell_0+1})|_{k-k',\mathbb{D}}\leq Cv^{-1}\tau^{-\frac{2\ell_0+2\mathfrak{s}+\vartheta}{2}-j-1}(\mathbb{I}^{\ell\geq\ell_0,k,\vartheta}_{NV,\Sigma_{\tau_0}}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}}; \tag{3.88b}$$

(2) If all the (m,ℓ_0)-th N–P constant $\mathbb{Q}^{(m,\ell_0)}_{-\mathfrak{s}}$, $m=-\ell_0,\ldots,\ell_0$, are zero, there exist universal constants $C=C(\vartheta,j,k,\ell_0)$ and $k'=k'(j,\ell_0)>0$ such that in the exterior region $\{\rho\geq\tau\}$,

$$|\partial_\tau^j(r^{-2\mathfrak{s}-1}\Psi_{+\mathfrak{s}})|_{k-k',\mathbb{D}}\leq Cv^{-1-2\mathfrak{s}}\tau^{-\frac{2\ell_0-2\mathfrak{s}-\vartheta}{2}-j-2}(\mathbb{I}^{\ell\geq\ell_0,k,\vartheta}_{V,\Sigma_{\tau_0}}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}}, \tag{3.89a}$$

$$|\partial_\tau^j(r^{-1}\Psi_{-\mathfrak{s}})|_{k-k',\mathbb{D}}\leq Cv^{-1}\tau^{-\frac{2\ell_0+2\mathfrak{s}-\vartheta}{2}-j-2}(\mathbb{I}^{\ell\geq\ell_0,k,\vartheta}_{V,\Sigma_{\tau_0}}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}}, \tag{3.89b}$$

and

$$|\partial_\tau^j(r^{-2\mathfrak{s}-1}(\Psi_{+\mathfrak{s}})^{\ell\geq\ell_0+2})|_{k-k',\mathbb{D}}\leq Cv^{-1-2\mathfrak{s}}\tau^{-\frac{2\ell-2\mathfrak{s}+\vartheta}{2}-j-2}(\mathbb{I}^{\ell\geq\ell_0,k,\vartheta}_{NV,\Sigma_{\tau_0}}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}}, \tag{3.90a}$$

$$|\partial_\tau^j(r^{-1}(\Psi_{-\mathfrak{s}})^{\ell\geq\ell_0+2})|_{k-k',\mathbb{D}}\leq Cv^{-1}\tau^{-\frac{2\ell_0+2\mathfrak{s}+\vartheta}{2}-j-2}(\mathbb{I}^{\ell\geq\ell_0,k,\vartheta}_{NV,\Sigma_{\tau_0}}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}}. \tag{3.90b}$$

Proof. If the spin $+\mathfrak{s}$ component is supported on $\ell\geq\ell_0$ modes, we can decompose it as in Definition 2.10 into

$$(\Psi_{+\mathfrak{s}})_{\ell\geq\ell_0}=\sum_{i=\ell_0}^{\infty}(\Psi_{+\mathfrak{s}})^{\ell=i}=\sum_{i=\ell_0}^{\infty}\sum_{m=-i}^{i}(\Psi_{+\mathfrak{s}})_{(m,i)}Y^s_{m,i}(\cos\theta)e^{im\phi}, \tag{3.91}$$

which holds true in $L^2(S^2)$. Consider first the case 1. For each $(\Psi_{+\mathfrak{s}})_{m,\ell'}$, the estimates in Proposition 3.30 can be applied; while for the remainder $(\Psi_{+\mathfrak{s}})^{\ell\geq\ell_0+1}$, we can use the estimate (3.25) with $p=1+\vartheta$ in Proposition 3.10 to achieve the decay estimate (3.88a). These two together then yield the estimate (3.87a). Consider next the case 2. We apply Proposition 3.30 to each $(\Psi_{+\mathfrak{s}})_{m,\ell_0}$, the estimate (3.87a) to each $(\Psi_{+\mathfrak{s}})_{m,\ell_0+1}$ and the estimate (3.88a) to $(\Psi_{+\mathfrak{s}})^{\ell\geq\ell_0+2}$, which proves (3.89a) and (3.90a). The estimates for the spin $-\mathfrak{s}$ component are similarly derived. □

3.8. **Equations for a fixed mode.**

Lemma 3.33. *Let the spin $s=\pm\mathfrak{s}$ components be supported on an (m,ℓ) mode, $\ell\geq\mathfrak{s}$. Let*

$$\hat{\varphi}_{s,\ell}=r^{-(\ell+s)}\psi_{s,\ell}. \tag{3.92}$$

Then its governing equation is

$$\partial_\rho\left(r^{2\ell+2}\mu^{1-s}\partial_\rho\hat{\varphi}_{s,\ell}\right)-2\ell(\ell+s)M\mu^{-s}r^{2\ell-1}\hat{\varphi}_{s,\ell}=\partial_\tau H_{\hat{\varphi}_{s,\ell}}, \tag{3.93}$$

where

$$\begin{aligned}H_{\hat{\varphi}_{s,\ell}}=&-\mu^{-s+1}r^{2\ell+2}(H_{hyp}-\tilde{H}_{hyp})\partial_\rho\hat{\varphi}_{s,\ell}+\mu^{-s+1}r^{2\ell+2}H_{hyp}\tilde{H}_{hyp}\partial_\tau\hat{\varphi}_{s,\ell}\\&+\mu^{-s}r^{2\ell}[2(\ell+1)\mu r\tilde{H}_{hyp}-2(s-1)M\tilde{H}_{hyp}+\mu r^2\partial_r\tilde{H}_{hyp}-2(\ell-s+1)r]\hat{\varphi}_{s,\ell}\\=&\ r^{\ell-s+1}\big\{\mu\tilde{H}_{hyp}\hat{V}(\mu^{-s}r\psi_{s,\ell})-\mu H_{hyp}\partial_\rho(\mu^{-s}r\psi_{s,\ell})\\&+([2sr-2M(3s+1)]H_{hyp}r^{-2}-\mu\partial_rH_{hyp})\mu^{-s}r\psi_{s,\ell}\big\}.\end{aligned} \tag{3.94}$$

Proof. For a fixed (m,ℓ) mode, the TME (2.15) can be simplified to

$$-\mu^{-1}V(\mu r^2Y\psi_{s,\ell})-(\ell(\ell+1)-(s^2-s))\psi_{s,\ell}+2s(r-M)Y\psi_{s,\ell}-(4s-2)r\partial_\tau\psi_{s,\ell}=0. \tag{3.95}$$

This is equivalent to

$$-r^2VY\psi_{s,\ell}-(\ell+s)(\ell-s+1)\psi_{s,\ell}+2(s-1)(r-M)Y\psi_{s,\ell}-(4s-2)r\partial_\tau\psi_{s,\ell}=0. \tag{3.96}$$

Let $\psi_{s,\ell} = r^\lambda \hat{\varphi}_{s,\ell}$, then

$$-r^2 VY\psi_{s,\ell} = -r^{2+\lambda} VY\hat{\varphi}_{s,\ell} + \lambda r^{\lambda+1}(V-\mu Y)\hat{\varphi}_{s,\ell} + \lambda(\lambda-1)\mu r^\lambda \hat{\varphi}_{s,\ell},$$
$$2(s-1)(r-M)Y\psi_{s,\ell} = 2(s-1)(r-M)r^\lambda Y\hat{\varphi}_{s,\ell} - 2\lambda(s-1)(r-M)r^{\lambda-1}\hat{\varphi}_{s,\ell}.$$

Hence by letting $\lambda = \ell + s$, one obtains the following equation for $\hat{\varphi}_{s,\ell}$

$$\begin{aligned}&-r^3VY\hat{\varphi}_{s,\ell} + (\ell+s)r^2(V-\mu Y)\hat{\varphi}_{s,\ell} + 2(s-1)(r-M)rY\hat{\varphi}_{s,\ell} - (4s-2)r^2\partial_\tau\hat{\varphi}_{s,\ell}\\ &+[-(\ell+s)(\ell-s+1)r + (\ell+s)(\ell+s-1)\mu r - 2(\ell+s)(s-1)(r-M)]\hat{\varphi}_{s,\ell} = 0.\end{aligned} \tag{3.97}$$

Calculating the coefficients and plugging in the expressions (2.6), we arrive at

$$\begin{aligned}&\mu r^3\partial_\rho^2\hat{\varphi}_{s,\ell} + [2(\ell+s)\mu r^2 - 2(s-1)(r-M)r]\partial_\rho\hat{\varphi}_{s,\ell} - 2M\ell(\ell+s)\hat{\varphi}_{s,\ell}\\ &+\mu r^3(H_{\text{hyp}} - \tilde{H}_{\text{hyp}})\partial_\tau\partial_\rho\hat{\varphi}_{s,\ell} - \mu r^3 H_{\text{hyp}}\tilde{H}_{\text{hyp}}\partial_\tau^2\hat{\varphi}_{s,\ell}\\ &+[-2(\ell+s)\mu r^2\tilde{H}_{\text{hyp}} + 2(s-1)(r-M)r\tilde{H}_{\text{hyp}} - \mu r^3\partial_r\tilde{H}_{\text{hyp}} + 2(\ell-s+1)r^2]\partial_\tau\hat{\varphi}_{s,\ell} = 0.\end{aligned} \tag{3.98}$$

A further rescaling yields equation (3.93). □

It is convenient and important in later discussions to introduce a further rescaled quantity of $\hat{\varphi}_{s,\ell}$ such that the ODE of this new rescaled quantity has no zeroth order term. Let us first define a function $h_{\mathfrak{s},\ell}$.

Definition 3.34. Define

$$h_{\mathfrak{s},\ell} = \sum_{i=0}^{\ell-\mathfrak{s}} h_{\mathfrak{s},\ell}^{(i)} M^i r^{-i} \tag{3.99}$$

with

$$h_{\mathfrak{s},\ell}^{(0)} = 1, \tag{3.100a}$$
$$h_{\mathfrak{s},\ell}^{(i)} = -2h_{\mathfrak{s},\ell}^{(i-1)}\frac{(\ell-i-\mathfrak{s}+1)(\ell-i+1)}{i(2\ell+1-i)}, \qquad i = 1,2,\ldots,\ell-\mathfrak{s}. \tag{3.100b}$$

Proposition 3.35. *Let the spin $\pm\mathfrak{s}$ components be supported on an (m,ℓ) mode, $\ell \geq \mathfrak{s}$. Define*

$$\varphi_{s,\ell} = (\mu^{\frac{s+\mathfrak{s}}{2}} h_{\mathfrak{s},\ell})^{-1}\hat{\varphi}_{s,\ell} = (\mu^{\frac{s+\mathfrak{s}}{2}} h_{\mathfrak{s},\ell} r^{\ell+s})^{-1}\psi_{s,\ell}. \tag{3.101}$$

Then the scalar $\varphi_{s,\ell}$ satisfies

$$\partial_\rho\left(r^{2\ell+2}\mu^{1+\mathfrak{s}} h_{\mathfrak{s},\ell}^2 \partial_\rho\varphi_{s,\ell}\right) = \partial_\tau H_{\varphi_{s,\ell}}, \tag{3.102}$$

with $H_{\varphi_{s,\ell}} = \mu^{\frac{s+\mathfrak{s}}{2}} h_{\mathfrak{s},\ell} H_{\hat{\varphi}_{s,\ell}}$ and $H_{\hat{\varphi}_{s,\ell}}$ as defined in (3.94).

Proof. Given a general second order ODE

$$\partial_\rho(f_1\partial_\rho\hat{\varphi}) + f_2\hat{\varphi} = F_1, \tag{3.103}$$

our aim is to define $\varphi = h_1^{-1}\hat{\varphi}$ such that it satisfies an ODE with no zeroth order term. By this definition of φ, the above equation becomes

$$\partial_\rho(f_1h_1\partial_\rho\varphi) + f_1\partial_\rho h_1\partial_\rho\varphi + [\partial_\rho(f_1\partial_\rho h_1) + f_2h_1]\varphi = F_1, \tag{3.104}$$

or, equivalently

$$\partial_\rho(f_1h_1^2\partial_\rho\varphi) + h_1[\partial_\rho(f_1\partial_\rho h_1) + f_2h_1]\varphi = h_1F_1. \tag{3.105}$$

Recall our goal is to make the coefficient of φ term vanishes, hence, it suffices that the function h_1 satisfies a second order ODE

$$\partial_\rho(f_1\partial_\rho h_1) + f_2h_1 = 0. \tag{3.106}$$

Consider first $s \leq 0$ case. In view of the above argument and equation (3.93), the function $h_{\mathfrak{s},\ell}$ in the definition (3.101) is required to satisfy

$$\partial_\rho\left(r^{2\ell+2}\mu^{1+\mathfrak{s}}\partial_\rho h_{\mathfrak{s},\ell}\right) - 2\ell(\ell-\mathfrak{s})M\mu^{\mathfrak{s}}r^{2\ell-1}h_{\mathfrak{s},\ell} = 0, \tag{3.107}$$

and by doing so, equation (3.93) reduces to

$$\partial_\rho \left(r^{2\ell+2}\mu^{1+\mathfrak{s}}h_{\mathfrak{s},\ell}^2\partial_\rho\varphi_{s,\ell}\right) = \partial_\tau \left(h_{\mathfrak{s},\ell}H_{\hat{\varphi}_{s,\ell}}\right) \tag{3.108}$$

which is exactly (3.102).

Turn to $s > 0$ case. Similarly, we need to choose a function h_1 in the expression $\varphi_{+\mathfrak{s},\ell} = h_1^{-1}\hat{\varphi}_{+\mathfrak{s},\ell}$ such that

$$\partial_\rho \left(r^{2\ell+2}\mu^{1-\mathfrak{s}}\partial_\rho h_1\right) - 2\ell(\ell+\mathfrak{s})M\mu^{-\mathfrak{s}}r^{2\ell-1}h_1 = 0, \tag{3.109}$$

Some direct calculations show that if $h_{\mathfrak{s},\ell}$ solves equation (3.107), then $h_1 = \mu^{\mathfrak{s}}h_{\mathfrak{s},\ell}$ solves equation (3.109). Therefore, upon making such a choice of $h_1 = \mu^{\mathfrak{s}}h_{\mathfrak{s},\ell}$, equation (3.105) becomes (3.93), which in turn takes the form of (3.102).

It remains to check that the function $h_{\mathfrak{s},\ell}$ in Definition 3.34 solves equation (3.107). By plugging the expression (3.99) of $h_{\mathfrak{s},\ell}$ into equation (3.107), it becomes

$$\sum_{i=0}^{\ell-\mathfrak{s}} \left(i(2\ell+1-i)h_{\mathfrak{s},\ell}^{(i)}M^i r^{2\ell-i} + 2(\ell-i)(\ell-i-\mathfrak{s})h_{\mathfrak{s},\ell}^{(i)}M^{i+1}r^{2\ell-i-1}\right) = 0. \tag{3.110}$$

This equation is clearly satisfied upon the choices of $h_{\mathfrak{s},\ell}$ made in Definition 3.34. □

We list a few properties of the function $h_{\mathfrak{s},\ell}$.

Proposition 3.36. *Let function $h_{\mathfrak{s},\ell}$ be defined as in Definition 3.34. Then,*

(1) the function $h_{\mathfrak{s},\ell}$ satisfies

$$\partial_\rho \left(r^{2\ell+2}\mu^{1+\mathfrak{s}}\partial_\rho h_{\mathfrak{s},\ell}\right) - 2\ell(\ell-\mathfrak{s})M\mu^{\mathfrak{s}}r^{2\ell-1}h_{\mathfrak{s},\ell} = 0; \tag{3.111}$$

(2) $r^{\ell-\mathfrak{s}}h_{\mathfrak{s},\ell}$ is a zero energy mode of the TME (2.15) *for the spin $-\mathfrak{s}$ component $\psi_{-\mathfrak{s}}$;*

(3) the function $h_{\mathfrak{s},\ell}$ satisfies

$$(r^2\partial_\rho)^{2\mathfrak{s}}(\mu^{\mathfrak{s}}r^{\ell-\mathfrak{s}+1}h_{\mathfrak{s},\ell}) = (\ell-\mathfrak{s}+1)\cdots(\ell+\mathfrak{s})r^{\ell+\mathfrak{s}+1}h_{\mathfrak{s},\ell}; \tag{3.112}$$

(4) the function $h_{\mathfrak{s},\ell}$ in Proposition 3.35 has no zero root in $r \in [2M, \infty)$.

Remark 3.37. Equation (3.112) can in fact be derived from the TSI. We will not discuss in more details but give a simple remark that equation (3.112) holds true for any $\mathfrak{s} \in \mathbb{N}$.

Proof. Equation (3.111) in the first point is already shown in the proof of Proposition 3.35.

Turn to the second point. Since $h_{\mathfrak{s},\ell}$ solves (3.111), it is a zero energy mode of equation (3.93), which thus yields $r^{\ell-\mathfrak{s}}h_{\mathfrak{s},\ell}$ is a zero energy mode of the TME (2.15) of $\psi_{-\mathfrak{s}}$.

For the third point, equation (3.112) can be verified by direct calculations using equation (3.111) for $\mathfrak{s} = 1, 2$.

It remains to show the fourth point. Recall that $\lim_{r\to\infty} h_{\mathfrak{s},\ell} = 1$, hence if $\ell = \mathfrak{s}$, then clearly $h_{\mathfrak{s},\ell} = 1$ and the claim trivially holds. As a result, it suffices to show the nonvanishing property of function $h_{\mathfrak{s},\ell}$ in $[2M, \infty)$ in the case that $\ell \geq \mathfrak{s}+1$.

By equation (3.111) satisfied by function $h_{\mathfrak{s},\ell}$, we have

$$\mu r^{2\ell+2}\partial_\rho^2 h_{\mathfrak{s},\ell} + [2(\ell+1)r^{2\ell+1} - 2M(2\ell+1-\mathfrak{s})r^{2\ell}]\partial_\rho h_{\mathfrak{s},\ell} - 2M\ell(\ell-\mathfrak{s})r^{2\ell-1}h_{\mathfrak{s},\ell} = 0. \tag{3.113}$$

As a result, function $h_{\mathfrak{s},\ell}$ cannot reach its nonnegative maximum or nonpositive minimum in $\rho \in (2M, +\infty)$.

In addition, we claim $h_{\mathfrak{s},\ell}(2M) > 0$. Suppose $h_{\mathfrak{s},\ell}(2M) < 0$, then the above equation (3.113) implies $\partial_\rho h_{\mathfrak{s},\ell}(2M) < 0$, which means that $h_{\mathfrak{s},\ell}$ must reach its negative minimum in $(2M, +\infty)$. This is in contradiction with the above argument. This thus yields the claim $h_{\mathfrak{s},\ell}(2M) \geq 0$. Instead, assume $h_{\mathfrak{s},\ell}(2M) = 0$, then we have $\partial_\rho h_{\mathfrak{s},\ell}(2M) = 0$ by equation (3.113). We can take further derivatives on (3.113) to obtain for any $i \geq 0$,

$$\mu r^3\partial_\rho^{i+2}h_{\mathfrak{s},\ell} + [(2\ell+2+3i)r^2 - 2M(2\ell+1-\mathfrak{s}+2i)r]\partial_\rho^{i+1}h_{\mathfrak{s},\ell} + \text{lower order derivatives} = 0.$$

We thus have $\partial_\rho^i h_{\mathfrak{s},\ell}(2M) = 0$ for all $i \geq 0$, which is clearly in contradiction with the expression of $h_{\mathfrak{s},\ell}$ in Proposition 3.35. Hence, $h_{\mathfrak{s},\ell}(2M) > 0$, and $\partial_\rho h_{\mathfrak{s},\ell}(2M) > 0$ by (3.113).

In the end, since $h^{(1)}_{\mathfrak{s},\ell} = -(\ell-1) < 0$, we have $\partial_\rho h_{\mathfrak{s},\ell} > 0$ for sufficiently large ρ. By the above conclusions that $\partial_\rho h_{\mathfrak{s},\ell}(2M) > 0$ and $h_{\mathfrak{s},\ell}$ cannot reach its nonnegative maximum in $(2M,\infty)$, function $h_{\mathfrak{s},\ell}$ must be monotonically increasing in $[2M,\infty)$. The nonvanishing property of function $h_{\mathfrak{s},\ell}$ in $[2M,\infty)$ then follows easily from $h_{\mathfrak{s},\ell}(2M) > 0$. □

3.9. **Almost Price's law for a fixed ℓ mode in the interior region $\{\rho \le \tau\}$.** Throughout this subsection, we shall always assume that the spin $s = \pm\mathfrak{s}$ components are supported on a fixed (m,ℓ) mode, with $\ell \ge \mathfrak{s}$. The main content of this subsection is devoted to obtaining in the interior region almost Price's law for the spin $-\mathfrak{s}$ component, and almost Price's law for the spin $+\mathfrak{s}$ component is then achieved by the TSI in Lemma 3.21.

Recall that the scalar $\hat{\varphi}_{-\mathfrak{s},\ell}$ satisfies equation (3.93), which is

$$\partial_\rho\left(r^{2\ell+2}\mu^{1+\mathfrak{s}}\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell}\right) - 2M\mu^{\mathfrak{s}}\lambda_{\mathfrak{s}}r^{2\ell-1}\hat{\varphi}_{-\mathfrak{s},\ell} = \partial_\tau H_{\hat{\varphi}_{-\mathfrak{s},\ell}}, \tag{3.114}$$

with $\hat{\varphi}_{-\mathfrak{s},\ell} = r^{-\ell+\mathfrak{s}}\psi_{-\mathfrak{s}}$, $\lambda_{\mathfrak{s}} = \ell(\ell-\mathfrak{s})$, and $H_{\hat{\varphi}_{-\mathfrak{s},\ell}}$ being defined as in (3.94) for $s = -\mathfrak{s}$. A key estimate is the following

Lemma 3.38. *Let $k \in \mathbb{N}$ and $\beta \in [0,(2\ell+1)/2]$ be arbitraty. Let $\hat{\varphi}_{-\mathfrak{s},\ell} = r^{-\ell+\mathfrak{s}}\psi_{-\mathfrak{s}}$. Define $\not{\mathbb{D}} = \{\partial_\tau, \rho\partial_\rho\}$. Then there exists a constant $C = C(k,j)$ such that*

$$\begin{aligned}
&\int_{2M}^{\tau}\left(r^{2\beta}|\hat{\varphi}_{-\mathfrak{s},\ell}|^2_{k,\not{\mathbb{D}}} + r^{2\beta+2}|\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell}|^2_{k,\not{\mathbb{D}}} + \mu^2r^{2\beta+4}|\partial_\rho^2\hat{\varphi}_{-\mathfrak{s},\ell}|^2_{k,\not{\mathbb{D}}}\right)\mathrm{d}\rho\\
&\le C\Bigg(\int_{2M}^{\tau}\left(r^{2\beta+4}|\partial_\tau\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell}|^2_{k,\not{\mathbb{D}}} + r^{2\beta}|\partial_\tau^2\hat{\varphi}_{-\mathfrak{s},\ell}|^2_{k,\not{\mathbb{D}}} + r^{2\beta+2}|\partial_\tau\hat{\varphi}_{-\mathfrak{s},\ell}|^2_{k,\not{\mathbb{D}}}\right)\mathrm{d}\rho\\
&\qquad + \left(r^{2\beta+1}|\hat{\varphi}_{-\mathfrak{s},\ell}|^2_{k,\not{\mathbb{D}}} + r^{2\beta+2}|\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell}|^2_{k,\not{\mathbb{D}}}\right)|_{\rho=\tau}\Bigg).
\end{aligned}\tag{3.115}$$

Proof. Taking a modulus square of both sides and multiplying by $r^{-4\ell}f^2$, we arrive at

$$\begin{aligned}
r^{-4\ell}f^2|\partial_\tau H_{\hat{\varphi}_{-\mathfrak{s},\ell}}|^2 &= f^2r^{-4\ell}|\partial_\rho(\mu^{1+\mathfrak{s}}r^{2\ell+2}\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell})|^2 + 4M^2\lambda_{\mathfrak{s}}^2\mu^{2\mathfrak{s}}f^2r^{-2}|\hat{\varphi}_{-\mathfrak{s},\ell}|^2\\
&\quad - 4M\lambda_{\mathfrak{s}}\mu^{\mathfrak{s}}f^2r^{-2\ell-1}\Re\big(\partial_\rho(\mu^{1+\mathfrak{s}}r^{2\ell+2}\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell})\overline{\hat{\varphi}_{-\mathfrak{s},\ell}}\big)\\
&\triangleq I_1 + I_2 + I_3.
\end{aligned}\tag{3.116}$$

Consider the term I_3. We use the Leibniz rule to obtain

$$\begin{aligned}
I_3 &= \partial_\rho\Big(-4M\lambda_{\mathfrak{s}}f^2\mu^{1+2\mathfrak{s}}r\Re(\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell}\overline{\hat{\varphi}_{-\mathfrak{s},\ell}})\Big) + 4M\lambda_{\mathfrak{s}}f^2\mu^{1+2\mathfrak{s}}r|\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell}|^2\\
&\quad + 4M\lambda_{\mathfrak{s}}\partial_r(\mu^{\mathfrak{s}}f^2r^{-2\ell-1})\mu^{1+\mathfrak{s}}r^{2\ell+2}\Re(\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell}\overline{\hat{\varphi}_{-\mathfrak{s},\ell}})\\
&= \partial_\rho\Big(-4M\lambda_{\mathfrak{s}}f^2\mu^{1+2\mathfrak{s}}r\Re(\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell}\overline{\hat{\varphi}_{-\mathfrak{s},\ell}})\Big) + \partial_\rho\Big(2M\lambda_{\mathfrak{s}}\partial_r(\mu^{\mathfrak{s}}f^2r^{-2\ell-1})\mu^{1+\mathfrak{s}}r^{2\ell+2}|\hat{\varphi}_{-\mathfrak{s},\ell}|^2\Big)\\
&\quad + 4M\lambda_{\mathfrak{s}}f^2\mu^{1+2\mathfrak{s}}r|\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell}|^2 - 2M\lambda_{\mathfrak{s}}\partial_r\Big(\partial_r(\mu^{\mathfrak{s}}f^2r^{-2\ell-1})\mu^{1+\mathfrak{s}}r^{2\ell+2}\Big)|\hat{\varphi}_{-\mathfrak{s},\ell}|^2.
\end{aligned}\tag{3.117}$$

Choose $f^2 = \mu^{-2\mathfrak{s}}r^{2\beta}$, with β to be fixed. For the last term, it equals

$$\begin{aligned}
&2M\lambda_{\mathfrak{s}}\partial_r\Big((2\ell+1-2\beta)\mu r^{\beta} + 2M\mathfrak{s}r^{2\beta-1}\Big)|\hat{\varphi}_{-\mathfrak{s},\ell}|^2\\
&= 2M\lambda_{\mathfrak{s}}\Big(\beta(2\ell+1-2\beta)\mu r^{2\beta-1} + 2M((2\ell+1-2\beta)+\mathfrak{s}(2\beta-1))r^{2\beta-2}\Big)|\hat{\varphi}_{-\mathfrak{s},\ell}|^2,
\end{aligned}\tag{3.118}$$

and the coefficient is nonnegative for any $\beta \in [0,(2\ell+1)/2]$. As a result, we obtain

$$\begin{aligned}
&\mu^{-2\mathfrak{s}}r^{-4\ell}r^{2\beta}|\partial_\tau H_{\hat{\varphi}_{-\mathfrak{s},\ell}}|^2\\
&= \partial_\rho\Big(-4M\lambda_{\mathfrak{s}}\mu r^{2\beta+1}\Re(\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell}\overline{\hat{\varphi}_{-\mathfrak{s},\ell}})\Big) + \partial_\rho\Big(2M\lambda_{\mathfrak{s}}\partial_r(\mu^{-\mathfrak{s}}r^{-2\ell-2\beta-1})\mu^{1+\mathfrak{s}}r^{2\ell+2}|\hat{\varphi}_{-\mathfrak{s},\ell}|^2\Big)\\
&\quad + \mu^{-2\mathfrak{s}}r^{-4\ell+2\beta}|\partial_\rho(\mu^{1+\mathfrak{s}}r^{2\ell+2}\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell})|^2 + 4M\lambda_{\mathfrak{s}}\mu r^{2\beta+1}|\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell}|^2\\
&\quad + 2M\lambda_{\mathfrak{s}}\Big(2\beta(2\ell+1-2\beta)\mu r^{2\beta-1} + 2M((2\ell+1-2\beta)+\mathfrak{s}(2\beta-1)+\lambda_{\mathfrak{s}})r^{2\beta-2}\Big)|\hat{\varphi}_{-\mathfrak{s},\ell}|^2.
\end{aligned}\tag{3.119}$$

Note in particular that the coefficients of the last three terms are all nonnegative.

On any Σ_τ, we integrate this equation (3.119) in ρ from $2M$ to τ. By the Cauchy–Schwarz inequality, the integral of the LHS is bounded by

$$C\int_{2M}^{\tau}\left(r^{2\beta+4}|\partial_\tau\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell}|^2+r^{2\beta}|\partial_\tau^2\hat{\varphi}_{-\mathfrak{s},\ell}|^2+r^{2\beta+2}|\partial_\tau\hat{\varphi}_{-\mathfrak{s},\ell}|^2\right)\mathrm{d}\rho; \tag{3.120}$$

for the integral of the RHS, the boundary term $(-4M\lambda_{\mathfrak{s}}\mu r^{2\beta+1}\Re(\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell}\overline{\hat{\varphi}_{-\mathfrak{s},\ell}}))|_{\rho=2M}$ vanishes due to the presence of the factor μ, hence it is bounded below by

$$\begin{aligned}
&c\int_{2M}^{\tau}\left(\mu^{-2\mathfrak{s}}r^{-4\ell+2\beta}|\partial_\rho(\mu^{1+\mathfrak{s}}r^{2\ell+2}\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell})|^2+\mu M\lambda_{\mathfrak{s}}r^{2\beta+1}|\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell}|^2+M^2\lambda_{\mathfrak{s}}r^{2\beta-2}|\hat{\varphi}_{-\mathfrak{s},\ell}|^2\right)\mathrm{d}\rho\\
&+c\lambda_{\mathfrak{s}}(|\hat{\varphi}_{-\mathfrak{s},\ell}|^2)|_{\rho=2M}-CM\lambda_{\mathfrak{s}}(r^{2\beta}|\hat{\varphi}_{-\mathfrak{s},\ell}|^2+r^{2\beta+2}|\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell}|^2)|_{\rho=\tau}.
\end{aligned}\tag{3.121}$$

Furthermore, in view of the following inequality

$$\int_{2M}^{\rho'}r^{2\beta+2}|\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell}|^2\mathrm{d}\rho+(\mu r^{2\beta+3}|\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell}|^2)|_{\rho=\rho'}\leq C\int_{2M}^{\rho'}\mu^{-2\mathfrak{s}}r^{-4\ell+2\beta}|\partial_\rho(\mu^{1+\mathfrak{s}}r^{2\ell+2}\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell})|^2\mathrm{d}\rho$$

which holds for any $\rho'>2M$ and follows simply by integrating

$$\begin{aligned}
&\partial_\rho(\mu^{-2\mathfrak{s}-1}r^{-4\ell+2\beta-1}|\mu^{1+\mathfrak{s}}r^{2\ell+2}\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell}|^2)\\
&=-[(2\mathfrak{s}+1)\mu^{-2\mathfrak{s}-2}2Mr^{-1}+(4\ell-2\beta+1)\mu^{-2\mathfrak{s}-1}]r^{-4\ell+2\beta-2}|\mu^{1+\mathfrak{s}}r^{2\ell+2}\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell}|^2\\
&\quad+2\mu^{-2\mathfrak{s}-1}r^{-4\ell+2\beta-1}\Re\big(\partial_\rho(\mu^{1+\mathfrak{s}}r^{2\ell+2}\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell})\mu^{1+\mathfrak{s}}r^{2\ell+2}\overline{\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell}}\big)
\end{aligned}$$

in ρ from $2M$ to ρ', and the following Hardy-type inequality

$$\int_{2M}^{\rho'}r^{2\beta}|\hat{\varphi}_{-\mathfrak{s},\ell}|^2\mathrm{d}\rho\leq C\int_{2M}^{\rho'}\mu^2r^{2\beta+2}|\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell}|^2\mathrm{d}\rho+C(\mu r^{2\beta+1}|\hat{\varphi}_{-\mathfrak{s},\ell}|^2)|_{\rho=\rho'}$$

which follows from integrating in ρ from $2M$ to ρ' for the following equality

$$\partial_\rho(\mu r^{2\beta+1}|\hat{\varphi}_{-\mathfrak{s},\ell}|^2)=(2Mr^{-1}+(2\beta+1)\mu)r^{2\beta}|\hat{\varphi}_{-\mathfrak{s},\ell}|^2+2\mu r^{2\beta+1}\Re(\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell}\overline{\hat{\varphi}_{-\mathfrak{s},\ell}}),$$

the first line of (3.121) is bounded below by

$$c\int_{2M}^{\tau}\left(r^{2\beta}|\hat{\varphi}_{-\mathfrak{s},\ell}|^2+r^{2\beta+2}|\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell}|^2+\mu^2r^{2\beta+4}|\partial_\rho^2\hat{\varphi}_{-\mathfrak{s},\ell}|^2\right)\mathrm{d}\rho-C(r^{2\beta+1}|\hat{\varphi}_{-\mathfrak{s},\ell}|^2)|_{\rho=\tau}. \tag{3.122}$$

Consequently, we arrive at

$$\begin{aligned}
&\int_{2M}^{\tau}\left(r^{2\beta}|\hat{\varphi}_{-\mathfrak{s},\ell}|^2+r^{2\beta+2}|\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell}|^2+\mu^{-2\mathfrak{s}}r^{-4\ell+2\beta}|\partial_\rho(\mu^{1+\mathfrak{s}}r^{2\ell+2}\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell})|^2\right)\mathrm{d}\rho\\
&\lesssim\int_{2M}^{\tau}\left(r^{2\beta+4}|\partial_\tau\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell}|^2+r^{2\beta}|\partial_\tau^2\hat{\varphi}_{-\mathfrak{s},\ell}|^2+r^{2\beta+2}|\partial_\tau\hat{\varphi}_{-\mathfrak{s},\ell}|^2\right)\mathrm{d}\rho+(r^{2\beta+1}|\hat{\varphi}_{-\mathfrak{s},\ell}|^2+r^{2\beta+2}|\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell}|^2)|_{\rho=\tau}.
\end{aligned}$$

This thus proves (3.115) for $k=0$.

Since ∂_τ commutes with equation (3.114), it suffices to prove the estimate (3.115) with $\mathbb{D}$ replaced by $\{\rho\partial_\rho\}$. We prove it by induction in k, that is, assuming it holds for $k=n-1$, $n\in\mathbb{N}^+$, we prove for $k=n$. We multiply both sides of equation (3.114) by $\mu^{-\mathfrak{s}}$ and then commute $\rho\partial_\rho$, and since

$$\begin{aligned}
r\partial_\rho\Big(\mu^{-\mathfrak{s}}\partial_\rho(r^{2\ell+2}\mu^{1+\mathfrak{s}}\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell})\Big)&=\mu^{-(\mathfrak{s}+1)}\partial_\rho\Big(r^{2\ell+2}\mu^{1+(\mathfrak{s}+1)}\partial_\rho(r\partial_\rho\hat{\varphi}_{-\mathfrak{s},\ell})\Big)\\
&\quad+\big(O_\infty(1)\mu r^{2\ell+2}\partial_\rho^2+O_\infty(1)r^{2\ell+1}\partial_\rho+O_\infty(1)r^{2\ell}\big)\hat{\varphi}_{-\mathfrak{s},\ell},
\end{aligned}\tag{3.123}$$

where $O_\infty(1)$ are $O(1)$ functions and smooth everywhere in $\rho\in[2M,\infty)$, we obtain for any $n\in\mathbb{N}^+$ that

$$\begin{aligned}
&\mu^{-(\mathfrak{s}+n)}\partial_\rho\Big(r^{2\ell+2}\mu^{1+(\mathfrak{s}+n)}\partial_\rho((r\partial_\rho)^n\hat{\varphi}_{-\mathfrak{s},\ell})\Big)-2M\lambda_{\mathfrak{s}}r^{2\ell-1}(r\partial_\rho)^n\hat{\varphi}_{-\mathfrak{s},\ell}\\
&=\partial_\tau(r\partial_\rho)^nH_{\hat{\varphi}_{-\mathfrak{s},\ell}}+r^{2\ell}\Bigg(\sum_{i=0}^{n}O_\infty(1)(r\partial_\rho)^i\hat{\varphi}_{-\mathfrak{s},\ell}+O_\infty(1)\mu(r\partial_\rho)^{n+1}\hat{\varphi}_{-\mathfrak{s},\ell}\Bigg).
\end{aligned}\tag{3.124}$$

Similarly to the $k=0$ case, we take a modulus square of both sides, multiply by $r^{-4\ell+2\beta}$, and integrate in ρ from $2M$ to τ. The integral arising from the last term of (3.124) is controlled by the assumption in the induction, and the analysis for the integral coming from the LHS of (3.124) is exactly the same as the above one in treating the $k = 0$ case, which thus completes the proof. $\square$

Proposition 3.39. *Assume the the spin $s = \pm\mathfrak{s}$ components are supported on a single (m, ℓ) mode, $\ell \geq \mathfrak{s}$ and $m \in \{-\ell, -\ell + 1, \ldots, \ell\}$. Let the N–P constants $\mathbb{Q}^{(m,\ell)}_{-\mathfrak{s}}$ be defined as in Definition 3.22, and let $\varphi_{s,\ell}$ be defined as in Proposition 3.35. Let $\vartheta \in (0, 1/2)$ be arbitrary, and let $j \in \mathbb{N}$ and $k \in \mathbb{N}$.*

(i) If the ℓ-th N–P constant $\mathbb{Q}^{(m,\ell)}_{-\mathfrak{s}}$ is nonzero, there exist universal constants $C = C(k, j, \ell, \vartheta)$ and $k' = k'(k, j, \ell, \vartheta)$ such that in the interior region $\{\rho \leq \tau\}$,

$$|\partial_\tau^j (r^{-\ell-\mathfrak{s}-1}\Psi_{+\mathfrak{s}})|_{k,\mathbb{D}} \leq C\tau^{-2-2\ell-j+\frac{\vartheta}{2}} (\mathbb{I}^{\ell_0,k+k',3-\vartheta}_{\Sigma_{\tau_0}}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}}, \tag{3.125a}$$

$$|\partial_\tau^j \varphi_{-\mathfrak{s},\ell}|_{k,\mathbb{D}} \leq C\tau^{-2-2\ell-j+\frac{\vartheta}{2}} (\mathbb{I}^{\ell,k+k',3-\vartheta}_{\Sigma_{\tau_0}}[\Psi_{-\mathfrak{s}}])^{\frac{1}{2}}, \tag{3.125b}$$

$$|\partial_\tau^j \partial_\rho \varphi_{-\mathfrak{s},\ell}|_{k,\mathbb{D}} \leq C\tau^{-3-2\ell-j+\frac{\vartheta}{2}} (\mathbb{I}^{\ell,k+k',3-\vartheta}_{\Sigma_{\tau_0}}[\Psi_{-\mathfrak{s}}])^{\frac{1}{2}}. \tag{3.125c}$$

(ii) If the ℓ-th N–P constant $\mathbb{Q}^{(m,\ell)}_{-\mathfrak{s}}$ is zero, there exist universal constants $C = C(k, j, \ell, \vartheta)$ and $k' = k'(k, j, \ell, \vartheta)$ such that in the interior region $\{\rho \leq \tau\}$,

$$|\partial_\tau^j (r^{-\ell-\mathfrak{s}-1}\Psi_{+\mathfrak{s}})|_{k,\mathbb{D}} \leq C\tau^{-3-2\ell-j+\frac{\vartheta}{2}} (\mathbb{I}^{\ell_0,k+k',5-\vartheta}_{\Sigma_{\tau_0}}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}}, \tag{3.126a}$$

$$|\partial_\tau^j \varphi_{-\mathfrak{s},\ell}|_{k,\mathbb{D}} \leq C\tau^{-3-2\ell-j+\frac{\vartheta}{2}} (\mathbb{I}^{\ell,k+k',5-\vartheta}_{\Sigma_{\tau_0}}[\Psi_{-\mathfrak{s}}])^{\frac{1}{2}}, \tag{3.126b}$$

$$|\partial_\tau^j \partial_\rho \varphi_{-\mathfrak{s},\ell}|_{k,\mathbb{D}} \leq C\tau^{-4-2\ell-j+\frac{\vartheta}{2}} (\mathbb{I}^{\ell,k+k',5-\vartheta}_{\Sigma_{\tau_0}}[\Psi_{-\mathfrak{s}}])^{\frac{1}{2}}. \tag{3.126c}$$

Proof. We consider only the case that the ℓ-th N–P constant $\mathbb{Q}^{(m,\ell)}_{-\mathfrak{s}} \neq 0$, and the other case that this constant vanishes can be analogously treated.

By the pointwise estimates in Proposition 3.30, the estimate (3.115) implies that for any $j \in \mathbb{N}$, $k \in \mathbb{N}$, $\beta \in [0, \ell+\frac{1}{2}]$ and $\vartheta \in (0, \frac{1}{2})$, there exist universal constants $C = C(k, j, \vartheta)$ and $k' = k'(k, j, \ell, \vartheta)$ such that

$$\|\partial_\tau^j \psi_{-\mathfrak{s}}\|^2_{W^k_{2(-\ell+\mathfrak{s}+\beta)}(\Sigma^{\leq\tau}_{\tau})} \leq C\|\partial_\tau^{j+1}\psi_{-\mathfrak{s}}\|^2_{W^k_{2(-\ell+\mathfrak{s}+\beta+1)}(\Sigma^{\leq\tau}_{\tau})} + C\tau^{-3+2\beta-4\ell-2j+\vartheta}\mathbb{I}^{\ell,k+k',3-\vartheta}_{\Sigma_{\tau_0}}[\Psi_{-\mathfrak{s}}]. \tag{3.127}$$

We use this estimate for all $\beta \in \{0, 1, \ldots, \ell - \mathfrak{s} - 1\}$ iteratively: we first prove the estimate for the term on the LHS with $\beta = \ell - \mathfrak{s} - 1$; then iteratively, the estimate for the LHS with $\beta = \beta_0$, which gives an estimate for the first term on the RHS with $\beta = \beta_0 - 1$, yields the estimate for the LHS with $\beta = \beta_0 - 1$. Specifically, the pointwise estimates in Proposition 3.30 imply that the first term on the RHS of (3.127) is bounded by $C(k, j, \ell, \vartheta)\tau^{-3+2\beta-4\ell-2j+\vartheta}\mathbb{I}^{\ell,k+k',3-\vartheta}_{\Sigma_{\tau_0}}[\Psi_{-\mathfrak{s}}]$, hence the term on the LHS is bounded by $C(k, j, \ell, \vartheta)\tau^{-3+2\beta-4\ell-2j+\vartheta}\mathbb{I}^{\ell,k+k',3-\vartheta}_{\Sigma_{\tau_0}}[\Psi_{-\mathfrak{s}}]$ as well. Iteratively, we eventually conclude

$$\|\partial_\tau^j \Psi_{-\mathfrak{s}}\|^2_{W^k_{2(-\ell+\mathfrak{s}-1)}(\Sigma^{\leq\tau}_{\tau})} \leq C\tau^{-3-4\ell-2j+\vartheta}\mathbb{I}^{\ell,k+k',3-\vartheta}_{\Sigma_{\tau_0}}[\Psi_{-\mathfrak{s}}]. \tag{3.128}$$

Applying the Sobolve-type inequality (2.30) to this energy decay estimate, one obtains in the interior region $\{\rho \leq \tau\}$ that

$$|\partial_\tau^j \hat{\varphi}_{-\mathfrak{s},\ell}|_{k,\mathbb{D}} \leq C(k, j, \ell, \vartheta) r^{-\frac{1}{2}} \tau^{-\frac{3-\vartheta}{2}-2\ell-j} (\mathbb{I}^{\ell,k+k'(k,j,\ell,\vartheta),3-\vartheta}_{\Sigma_{\tau_0}}[\Psi_{-\mathfrak{s}}])^{\frac{1}{2}}. \tag{3.129}$$

From Proposition 3.35, with $h_{\mathfrak{s},\ell}$ defined as in Definition 3.34, the scalar $\varphi_{-\mathfrak{s},\ell} = h^{-1}_{\mathfrak{s},\ell}\hat{\varphi}_{-\mathfrak{s},\ell}$ satisfies

$$\partial_\rho (r^{2\ell+2}\mu^{1+\mathfrak{s}} h^2_{\mathfrak{s},\ell}\partial_\rho \varphi_{-\mathfrak{s},\ell}) = h_{\mathfrak{s},\ell}\partial_\tau H_{\hat{\varphi}_{s,\ell}}. \tag{3.130}$$

Recall from Proposition 3.36 that there exist two positive universal constant c and C such that $c \leq h_{\mathfrak{s},\ell} \leq C$. We integrate this equation from $\rho = 2M$ to any ρ with $\rho \leq \tau$, and the boundary term at horizon vanishes because of the degenerate factor $\mu^{1+\mathfrak{s}}$, thus arriving at

$$\mu^{1+\mathfrak{s}} r^{2\ell+2} |\partial_\tau^j \partial_\rho \varphi_{-\mathfrak{s},\ell}|_{k,\mathbb{D}} \lesssim_k \mu^{1+\mathfrak{s}} r^{2\ell+2} |\partial_\tau^{j+1} \hat{\varphi}_{-\mathfrak{s},\ell}|_{k,\mathbb{D}}. \tag{3.131}$$

By utilizing the pointwise decay estimate (3.129) to estimate the RHS, we achieve a better decay estimate for $\partial_\tau^j\partial_\rho\varphi_{-\mathfrak{s},\ell}$:

$$|\partial_\tau^j\partial_\rho\varphi_{-\mathfrak{s},\ell}|_{k,\mathbb{D}} \leq C(k,j,\ell,\vartheta)r^{-\frac{1}{2}}\tau^{-\frac{5-\vartheta}{2}-2\ell-j}(\mathbb{I}^{\ell,k+k'(k,j,\ell,\vartheta),3-\vartheta}_{\Sigma_{\tau_0}}[\Psi_{-\mathfrak{s}}])^{\frac{1}{2}}. \tag{3.132}$$

We can now integrate $\partial_\rho\varphi_{-\mathfrak{s},\ell}$ from $\rho=\tau$ to ρ and use the estimate (3.129) to estimate the boundary term at $\rho=\tau$ and the estimate (3.132) to estimate the integral of $\partial_\rho\varphi_{-\mathfrak{s},\ell}$. By doing so, we prove (3.125b). Finally, we repeat this step of proving better decay estimate (3.132) for $\partial_\tau^j\partial_\rho\varphi_{-\mathfrak{s},\ell}$ by using the estimate (3.125b), which then proves (3.125c).

In the end, the approach of showing the estimate (3.125a) for the spin $+\mathfrak{s}$ component is the same as the one in the proof of Proposition 3.30 where the decay of the spin $+\mathfrak{s}$ is proven by the estimate of the spin $-\mathfrak{s}$ together with an application of the TSI (3.70a), hence we omit it. □

Proposition 3.40. *Assume the the spin $s=\pm\mathfrak{s}$ components are supported on $\ell\geq\ell_0$ modes, $\ell_0\geq\mathfrak{s}$. For each $m\in\{-\ell_0,-\ell_0+1,\ldots,\ell_0\}$, let the N–P constants $\mathbb{Q}^{(m,\ell_0)}_{-\mathfrak{s}}$ for the (m,ℓ_0) mode of the spin $+\mathfrak{s}$ component be defined as in Definition 3.22. Let $\vartheta\in(0,1/2)$ be arbitrary, and let $j\in\mathbb{N}$ and $k\in\mathbb{N}$.*

(i) *If not all of the (m,ℓ_0)-th N–P constant $\mathbb{Q}^{(m,\ell_0)}_{-\mathfrak{s}}$ are zero, there exist universal constants $C=C(k,j,\ell_0,\vartheta)$ and $k'=k'(k,j,\ell_0,\vartheta)$ such that in the interior region $\{\rho\leq\tau\}$,*

$$|\partial_\tau^j(r^{-\ell_0-s-1}\Psi_s)|_{k,\mathbb{D}} \leq C\tau^{-2-2\ell_0-j+\frac{\vartheta}{2}}(\mathbb{I}^{\ell\geq\ell_0,k+k',\vartheta}_{NV,\Sigma_{\tau_0}}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}}. \tag{3.133}$$

(ii) *If all of the (m,ℓ_0)-th N–P constant $\mathbb{Q}^{(m,\ell_0)}_{-\mathfrak{s}}$ are zero, there exist universal constants $C=C(k,j,\ell_0,\vartheta)$ and $k'=k'(k,j,\ell_0,\vartheta)$ such that in the interior region $\{\rho\leq\tau\}$,*

$$|\partial_\tau^j(r^{-\ell_0-s-1}\Psi_s)|_{k,\mathbb{D}} \leq C\tau^{-3-2\ell_0-j+\frac{\vartheta}{2}}(\mathbb{I}^{\ell\geq\ell_0,k+k',\vartheta}_{V,\Sigma_{\tau_0}}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}}. \tag{3.134}$$

Proof. It suffices to prove the estimate for the spin $-\mathfrak{s}$ component in the case that not all of the (m,ℓ_0)-th N–P constant $\mathbb{Q}^{(m,\ell_0)}_{-\mathfrak{s}}$ are zero, since the estimate for the spin $+\mathfrak{s}$ component can be obtained via the TSI and the proof for the other case that all of the (m,ℓ_0)-th N–P constant $\mathbb{Q}^{(m,\ell_0)}_{-\mathfrak{s}}$ are zero is analogous. For each (m,ℓ_0) mode, its estimate has been obtained in Proposition 3.39. For each (m,ℓ') mode, $\ell_0+1\leq\ell\leq2\ell_0+1$, the proof in the previous sections also implies a slightly different decay estimate from (3.81b)

$$|\partial_\tau^j(r^{-1}(\Psi_{-\mathfrak{s}})_{m,\ell})|_{k-k',\mathbb{D}} \leq Cv^{-1}\tau^{-\frac{2\ell+2\mathfrak{s}-2+\vartheta}{2}-j-1}(\mathbb{I}^{\ell,k,1+\vartheta}_{\Sigma_{\tau_0}}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}}. \tag{3.135}$$

By applying Lemma 3.38 and going through the proof of Proposition 3.39, we obtain an analogous estimate as (3.128)

$$\|\partial_\tau^j(\Psi_{-\mathfrak{s}})_{m,\ell}\|^2_{W^k_{2(-\ell+\mathfrak{s}-1)}(\Sigma^{\leq\tau}_\tau)} \leq C\tau^{-1-4\ell-2j-\vartheta}\mathbb{I}^{\ell,k+k',1+\vartheta}_{\Sigma_{\tau_0}}[\Psi_{-\mathfrak{s}}], \tag{3.136}$$

and thus an analog of the estimate (3.125b):

$$|\partial_\tau^j(r^{-\ell+\mathfrak{s}-1}(\Psi_{-\mathfrak{s}})_{m,\ell})|_{k,\mathbb{D}} \leq C\tau^{-1-2\ell-j-\frac{\vartheta}{2}}(\mathbb{I}^{\ell,k+k',1+\vartheta}_{\Sigma_{\tau_0}}[\Psi_{-\mathfrak{s}}])^{\frac{1}{2}}. \tag{3.137}$$

Together with the weak decay estimate (3.25) applied to the remainder $(\Psi_{-\mathfrak{s}})^{\ell\geq2\ell_0+2}$, and using the Plancherel's lemma, this yields in the interior region $\{\rho\leq\tau\}$,

$$|\partial_\tau^j(r^{-\ell_0+\mathfrak{s}-1}(\Psi_{-\mathfrak{s}})^{\ell\geq\ell_0+1})|_{k,\mathbb{D}} \leq C\tau^{-3-2\ell_0-j-\frac{\vartheta}{2}}(\mathbb{I}^{\ell\geq\ell_0,k+k',\vartheta}_{\mathrm{NV},\Sigma_{\tau_0}}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}}, \tag{3.138a}$$

the decay of which is faster than the one of $r^{-\ell_0+\mathfrak{s}-1}(\Psi_{-\mathfrak{s}})_{m,\ell_0}$; Similarly, one has for the spin $+\mathfrak{s}$ component that in the interior region $\{\rho\leq\tau\}$,

$$|\partial_\tau^j(r^{-\ell_0-\mathfrak{s}-1}(\Psi_{+\mathfrak{s}})^{\ell\geq\ell_0+1})|_{k,\mathbb{D}} \leq C\tau^{-3-2\ell_0-j-\frac{\vartheta}{2}}(\mathbb{I}^{\ell\geq\ell_0,k+k',\vartheta}_{\mathrm{NV},\Sigma_{\tau_0}}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}}, \tag{3.138b}$$

Adding this estimate for $\ell\geq\ell_0+1$ to the one of the (m,ℓ_0) mode $r^{-\ell_0+\mathfrak{s}-1}(\Psi_{-\mathfrak{s}})_{m,\ell_0}$ then yields the desired estimate. □

4. Price's law under nonvanishing Newman–Penrose constant condition

We will frequently use the double null coordinates (u, v, θ, ϕ), and the DOC is divided into different regions as in Figure 3, where $\gamma_\alpha = \{v - u = v^\alpha\}$ for an $\alpha \in (\frac{1}{2}, 1)$. The following lemma lists all relations and estimates among u, v, r, and τ that are utilized in these different regions. The proof is simple and omitted.

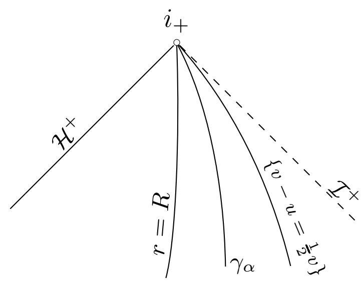

Figure 3. Useful hypersurfaces

Lemma 4.1. *For any $\alpha \in (\frac{1}{2}, 1)$, let $\gamma_\alpha = \{v - u = v^\alpha\}$. For any u and v, let $u_{\gamma_\alpha}(v)$ and $v_{\gamma_\alpha}(u)$ be such that $(u_{\gamma_\alpha}(v), v), (u, v_{\gamma_\alpha}(u)) \in \gamma_\alpha$. In the region $v - u \geq v^\alpha$,*

$$r \gtrsim v^\alpha + u^\alpha, \tag{4.1a}$$

$$|u - v_{\gamma_\alpha}(u)| \lesssim u^\alpha, \tag{4.1b}$$

$$|2r - (v - u)| \lesssim \log(r - 2M); \tag{4.1c}$$

in the region $\{v - u \geq v^\alpha\} \cap \{v - u \geq \frac{v}{2}\}$,

$$v + u \lesssim r \lesssim v; \tag{4.1d}$$

in the region $\{v - u \geq v^\alpha\} \cap \{v - u \leq \frac{v}{2}\}$,

$$u \sim v, \quad r \gtrsim v^\alpha; \tag{4.1e}$$

in the region $\{r \geq R\} \cap \{v - u \leq v^\alpha\}$,

$$r \lesssim \min\{v^\alpha, u^\alpha\}; \tag{4.1f}$$

in the region $\{2M \leq r \leq R\}$, there is a constant C_R depending on R such that

$$|v|_{\Sigma_\tau}(R) - v|_{\Sigma_\tau}(r)| + |v - \tau| \leq C_R. \tag{4.1g}$$

On Σ_{τ_0}, for r large,

$$|r^{-1}v - 2 - 4Mr^{-1}\log(r - 2M)| \lesssim r^{-1}. \tag{4.2}$$

Last, we remark that throughout the subsections 4.1 and 4.2, we consider only a fixed (m, ℓ) mode of the spin $\pm\mathfrak{s}$ components, and the dependence on m, ℓ may be suppressed.

4.1. **Price's law in a region $\{v - u \geq v^\alpha\}$.**

For any $\delta \in (0, \frac{1}{2})$, we denote

$$\mathbf{E}_{\mathrm{NV},\ell} = (\mathbb{I}^{\ell,k,3-\delta}_{\Sigma_\tau}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}} + |\mathbb{Q}^{(m,\ell)}_{+\mathfrak{s}}| + D_0, \tag{4.3}$$

where $\mathbb{I}^{\ell,k,3-\delta}_{\Sigma_\tau}[\Psi_{+\mathfrak{s}}]$ is defined as in Definition 3.8 with δ is to be fixed in the proof, D_0 is a constant appearing in the assumption below, and we have suppressed the dependence on the mode parameter m, the regularity parameter k which only depends on ℓ, and the constant δ.

Assumption 4.2 (Initial data assumption to order i)**.** Let $i \in \mathbb{N}$. Assume on Σ_{τ_0} that there are constants $\mathbb{Q}^{(m,\ell)}_{+\mathfrak{s}} \in \mathbb{R} \setminus \{0\}$, $\beta \in (0, \frac{1}{2})$ and $0 \leq D_0 < \infty$ such that for all $0 \leq i' \leq i$ and $r \geq 10M$,

$$\left|\partial_\rho^{i'}\left(\hat{V}\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}(\tau_0, v) - \rho^{-2}\mathbb{Q}^{(m,\ell)}_{+\mathfrak{s}}\right)\right| \lesssim D_0\rho^{-2-\beta-i'}. \tag{4.4}$$

Definition 4.3. For any spin-weighted scalar φ, we call $(\varphi)_{\mathrm{AP}}$ the "asymptotic profile" of φ if there exists a $\varepsilon > 0$ such that

$$|\varphi - (\varphi)_{\mathrm{AP}}| \lesssim \tau^{-\varepsilon}(\mathbb{Q}^{(m,\ell)}_{+\mathfrak{s}})^{-1}\mathbf{E}_{\mathrm{NV},\ell} \cdot (\varphi)_{\mathrm{AP}}. \tag{4.5}$$

Remark 4.4. In the latter discussions, the function $(\varphi)_{\mathrm{AP}}$ always contains a factor $\mathbb{Q}^{(m,\ell)}_{+\mathfrak{s}}$, and this is the reason for the presence of the factor $(\mathbb{Q}^{(m,\ell)}_{+\mathfrak{s}})^{-1}\mathbf{E}_{\mathrm{NV},\ell}$ on the RHS here.

The proof is divided into three steps to obtain the Price's law in this region.

Step 1. Decay of $V\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}$ by integrating its wave equation over constant v

Proposition 4.5. *Assume the initial data assumption 4.2 holds to order i_0, then for δ sufficiently small and $\alpha = \alpha(\delta)$ sufficiently close to 1, there exists an $\varepsilon > 0$ such that in the region $v - u \geq v^{\alpha}$,*

$$\left|V\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}} - 4(v-u)^{2\ell}v^{-2\ell-2}\mathbb{Q}^{(m,\ell)}_{+\mathfrak{s}}\right| \lesssim (v-u)^{2\ell}v^{-2\ell-2-\varepsilon}\mathbf{E}_{\mathrm{NV},\ell}. \tag{4.6}$$

Proof. Recall the wave equations of $\widetilde{\Phi}^{(i)}_{+\mathfrak{s}}$ in Proposition 3.7: for $i = \ell - \mathfrak{s}$,

$$-\mu Y\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}} - 2(\ell+1)(r-3M)r^{-2}\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}} + \sum_{j=0}^{\ell-\mathfrak{s}} O(r^{-1})\Phi^{(j)}_{+\mathfrak{s}} = 0. \tag{4.7}$$

One can rewrite it as

$$-\mu^{-\ell}r^{2\ell+2}Y(\mu^{\ell}r^{-2\ell}V\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}) + \sum_{j=0}^{\ell-\mathfrak{s}} O(r^{-1})\Phi^{(j)}_{+\mathfrak{s}} = 0, \tag{4.8}$$

or equivalently,

$$\partial_u\big(\mu^{\ell}r^{-2\ell}V\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}\big) = \sum_{j=0}^{\ell-\mathfrak{s}} O(r^{-2\ell-3})\Phi^{(j)}_{+\mathfrak{s}}. \tag{4.9}$$

Integrating this equation along constant v from the intersection point with Σ_{τ_0} and using the estimates in Propositions 3.30 and 3.39, we achieve

$$\begin{aligned}
&\left|(\mu^{\ell}r^{-2\ell}v^{2\ell+2}V\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}})(u,v) - (\mu^{\ell}r^{-2\ell}v^{2\ell+2}V\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}})(u_{\Sigma_{\tau_0}}(v),v)\right|\\
&\lesssim v^{2\ell+2}\int_{u_{\Sigma_{\tau_0}}(v)}^{u}\big(r^{-2\ell-2}v^{-1}\tau^{-1+\delta/2}\big)(u',v)\mathrm{d}u'(\mathbb{I}^{\ell,k,3-\delta}_{\Sigma_\tau}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}}\\
&\lesssim v^{-\eta}(\mathbb{I}^{\ell,k,3-\delta}_{\Sigma_\tau}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}},
\end{aligned} \tag{4.10}$$

with $\eta = (2\ell+2)\alpha - (2\ell+1) - \delta/2$. Meanwhile, we utilize (4.2) to obtain

$$|\mu^{\ell}r^{-2\ell}v^{2\ell+2}V\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}(u_{\Sigma_{\tau_0}}(v),v) - 2^{2\ell+2}\mathbb{Q}^{(m,\ell)}_{+\mathfrak{s}}| \lesssim v^{-\beta}D_0 + v^{-\alpha}\log v|\mathbb{Q}^{(m,\ell)}_{+\mathfrak{s}}|. \tag{4.11}$$

The above two estimates together then yield

$$\left|V\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}} - 4(v-u)^{2\ell}v^{-2\ell-2}\mathbb{Q}^{(m,\ell)}_{+\mathfrak{s}}\right| \lesssim \big(r^{2\ell-1}\log r v^{-2\ell-2} + r^{2\ell}v^{-2\ell-2}(v^{-\beta}+v^{-\eta})\big)\mathbf{E}_{\mathrm{NV},\ell}. \tag{4.12}$$

By taking δ sufficiently small and α sufficiently close to 1, this proves the estimate (4.6). □

Step 2. Decay of $\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}$ and its derivatives in $\{v-u \geq v^{\alpha}\}$.

We then derive the asymptotics of $\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}$ and its derivatives. To obtain the asymptotics for $\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}$, one integrates along $u = \mathit{const}$ as in Figure 4 to obtain

$$\begin{aligned}
\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}(u,v) &= \widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}(u,v_{\gamma_\alpha}(u)) + \frac{1}{2}\int_{v_{\gamma_\alpha}(u)}^{v} V\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}(u,v')\mathrm{d}v'\\
&= \widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}(u,v_{\gamma_\alpha}(u)) + \frac{1}{2}\int_{v_{\gamma_\alpha}(u)}^{v}\Big(V\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}} - 4(v-u)^{2\ell}v^{-2\ell-2}\mathbb{Q}^{(m,\ell)}_{+\mathfrak{s}}\Big)(u,v')\mathrm{d}v'\\
&\quad + 2\mathbb{Q}^{(m,\ell)}_{+\mathfrak{s}}\int_{v_{\gamma_\alpha}(u)}^{v}(v'-u)^{2\ell}(v')^{-2\ell-2}\mathrm{d}v'.
\end{aligned} \tag{4.13}$$

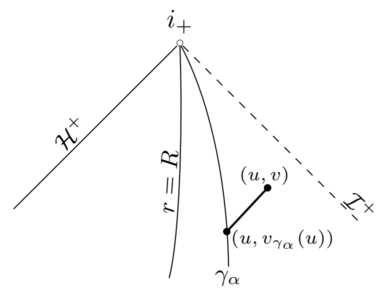

FIGURE 4. For any point (u,v) in $\{r \geq R\} \cap \{v-u \geq v^\alpha\}$, i.e. $v \geq v_{\gamma_\alpha}(u)$, one integrates along $u = const$ from $(u, v_{\gamma_\alpha}(u)) \in \gamma_\alpha$.

By an integration by parts, one has

$$\int_{v_{\gamma_\alpha}(u)}^{v} (v'-u)^{2\ell}(v')^{-2\ell-1}\mathrm{d}v' = \frac{1}{2\ell+1}(v'-u)^{2\ell+1}(v')^{-2\ell-1}|_{v_{\gamma_\alpha}(u)}^{v} + \int_{v_{\gamma_\alpha}(u)}^{v}(v'-u)^{2\ell+1}(v')^{-2\ell-2}\mathrm{d}v',$$

and moving the last term to the LHS yields

$$\begin{aligned}\int_{v_{\gamma_\alpha}(u)}^{v} 2(v'-u)^{2\ell}(v')^{-2\ell-2}\mathrm{d}v' &= \frac{2}{2\ell+1}(v'-u)^{2\ell+1}u^{-1}(v')^{-2\ell-1}|_{v_{\gamma_\alpha}(u)}^{v}\\ &= \frac{2}{2\ell+1}\frac{(v-u)^{2\ell+1}}{uv^{2\ell+1}} + O(u^{-2\ell-2+(2\ell+1)\alpha}).\end{aligned} \tag{4.14}$$

Hence, we conclude

$$\left|2\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)}\int_{v_{\gamma_\alpha}(u)}^{v}((v-u)^{2\ell}v^{-2\ell-2})(u,v')\mathrm{d}v' - \frac{2}{2\ell+1}\frac{(v-u)^{2\ell+1}}{uv^{2\ell+1}}\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)}\right| \lesssim u^{-2\ell-2+(2\ell+1)\alpha}|\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)}|.$$

The coefficient of the RHS is of lower order than the $\frac{(v-u)^{2\ell+1}}{uv^{2\ell+1}}$ behaviour by requiring α sufficiently close to 1. Meanwhile, the second last integral on the RHS of (4.13) is of lower order than $\frac{(v-u)^{2\ell+1}}{uv^{2\ell+1}}$ as well from (4.6), and $|\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}(u, v_{\gamma_\alpha}(u))| \lesssim rv^{-1}\tau^{-1+\delta/2}$ which is also of lower order than $\frac{(v-u)^{2\ell+1}}{uv^{2\ell+1}}$ by choosing δ sufficiently small and α sufficiently close to 1. In total, there exists an $\varepsilon > 0$ such that

$$\left|\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})} - \frac{2}{2\ell+1}\frac{(v-u)^{2\ell+1}}{uv^{2\ell+1}}\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)}\right| \lesssim \frac{(v-u)^{2\ell+1}}{uv^{2\ell+1}}\tau^{-\varepsilon}\mathbf{E}_{\mathrm{NV},\ell}. \tag{4.15}$$

By further commuting (4.9) with ∂_v^i $(i \leq i_0)$ implies

$$\partial_u\Big(\partial_v^i\big(\mu^\ell r^{-2\ell}V\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}\big)\Big) = \sum_{x=0}^{i}\sum_{j=0}^{\ell-\mathfrak{s}} O(r^{-2\ell-3-i})(r\partial_v)^x\Phi_{+\mathfrak{s}}^{(j)}. \tag{4.16}$$

We integrate along constant v from Σ_{τ_0}, and from (4.1c) and the almost Price's law estimates in Propositions 3.30 and 3.39, we can choose δ sufficiently small and $\alpha = \alpha(i,\delta)$ sufficiently close to 1 such that there exists an $\varepsilon = \varepsilon(\delta,\alpha,i) > 0$

$$\left|\Big(\partial_v^i((v-u)^{-2\ell}V\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})})\Big)(u,v) - \Big(\partial_v^i((v-u)^{-2\ell}V\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})})\Big)(u_{\Sigma_{\tau_0}}(v),v)\right| \lesssim v^{-i-2\ell-2-\varepsilon}\mathbf{E}_{\mathrm{NV},\ell}. \tag{4.17}$$

By the initial data assumption 4.2 to order i_0, for any $i \leq i_0$, we can take $\alpha = \alpha(i,\beta)$ sufficiently close to 1 and achieve

$$\left|\Big(\partial_v^i((v-u)^{-2\ell}V\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})})\Big)(u_{\Sigma_{\tau_0}}(v),v) - 4\partial_v^i(v^{-2\ell-2})\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)}\right| \lesssim v^{-i-2\ell-2-\varepsilon}\mathbf{E}_{\mathrm{NV},\ell}. \tag{4.18}$$

As a result,

$$\left|\Big(\partial_v^i((v-u)^{-2\ell}V\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}) - 4\partial_v^i(v^{-2\ell-2})\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)}\Big)(u,v)\right| \lesssim v^{-i-2\ell-2-\varepsilon}\mathbf{E}_{\mathrm{NV},\ell}. \tag{4.19}$$

This together with (4.15) implies for any $i \leq i_0$,

$$\left| \partial_v^i \left((v-u)^{-2\ell} \widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})} - \frac{2}{2\ell+1} \frac{v-u}{uv^{2\ell+1}} \mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)} \right) \right| \lesssim v^{-i-2\ell-1-\varepsilon} \mathbf{E}_{\mathrm{NV},\ell}. \tag{4.20}$$

In the above estimate (4.19), we can replace one ∂_v using $\partial_v = \partial_\tau - \partial_u$ and utilize equation (4.16) to estimate $\partial_u \partial_v^{i-1}((v-u)^{-2\ell} V \widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})})$, and this enables us to obtain for any $i \leq i_0$ that

$$\left| \left(\partial_v^{i-1}((v-u)^{-2\ell} V \partial_\tau \widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}) - 4\partial_v^{i-1} \partial_\tau (v^{-2\ell-2}) \mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)} \right)(u,v) \right| \lesssim v^{-i-2\ell-2-\varepsilon} \mathbf{E}_{\mathrm{NV},\ell}. \tag{4.21}$$

We can repeat this and conclude that for any $0 \leq j \leq i$,

$$\left| \left(\partial_v^{i-j}((v-u)^{-2\ell} V \partial_\tau^j \widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}) - 4\partial_v^{i-j} \partial_\tau^j (v^{-2\ell-2}) \mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)} \right)(u,v) \right| \lesssim v^{-i-2\ell-2-\varepsilon} \mathbf{E}_{\mathrm{NV},\ell}, \tag{4.22}$$

that is,

$$\left| \left(\partial_v^{i-j} \left((v-u)^{-2\ell} \partial_\tau^j \left[\partial_v \widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})} - 2(v-u)^{2\ell} v^{-2\ell-2} \mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)} \right] \right) \right)(u,v) \right| \lesssim v^{-i-2\ell-2-\varepsilon} \mathbf{E}_{\mathrm{NV},\ell}. \tag{4.23}$$

In particular, for $j = i$, we have

$$\left| \left(\partial_v \partial_\tau^i \widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})} - 2\partial_\tau^i ((v-u)^{2\ell} v^{-2\ell-2}) \mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)} \right)(u,v) \right| \lesssim (v-u)^{2\ell} v^{-i-2\ell-2-\varepsilon} \mathbf{E}_{\mathrm{NV},\ell}. \tag{4.24}$$

We now integrate $\partial_v \partial_\tau^i \widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}$ on constant-u hypersurface from the intersection point $(u, v_{\gamma_\alpha}(u))$ with γ_α:

$$\partial_\tau^i \widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}(u,v) - \partial_\tau^i \widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}(u, v_{\gamma_\alpha}(u)) = \int_{v_{\gamma_\alpha}(u)}^{v} \partial_v \partial_\tau^i \widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}(u,v') \mathrm{d}v'. \tag{4.25}$$

To estimate the RHS, we substitute (4.24) in and use the same way of arguing as in proving (4.14) to conclude

$$\left| \int_{v_{\gamma_\alpha}(u)}^{v} \partial_v \partial_\tau^i \widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}(u,v') \mathrm{d}v' - \frac{2}{2\ell+1} \partial_\tau^i \left(\frac{(v-u)^{2\ell+1}}{uv^{2\ell+1}} \right) \mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)} \right| \lesssim (v-u)^{2\ell} v^{-i-2\ell-1-\varepsilon} \mathbf{E}_{\mathrm{NV},\ell}. \tag{4.26}$$

For the second term on the LHS of (4.25), which is evaluated at $(u, v_{\gamma_\alpha}(u))$, one can use $\partial_u = \partial_\tau - \frac{1}{2r} rV$ and the estimates in Proposition 3.39 to achieve

$$\begin{aligned} \left| \partial_\tau^i \widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}(u, v_{\gamma_\alpha}(u)) \right| &\lesssim (r(u, v_{\gamma_\alpha}(u)))^{2\ell+1} u^{-i-2\ell-2+\delta/2} \mathbf{E}_{\mathrm{NV},\ell} \\ &\lesssim u^{(2\ell+1)\alpha + (-i-2\ell-2+\delta/2)} \mathbf{E}_{\mathrm{NV},\ell}. \end{aligned} \tag{4.27}$$

These together imply that for any $i \leq i_0$,

$$\begin{aligned} &\left| \partial_\tau^i \left((v-u)^{-2\ell} \widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})} - \frac{2}{2\ell+1} \frac{v-u}{uv^{2\ell+1}} \mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)} \right) \right| \\ &\lesssim v^{-i-2\ell-1-\varepsilon} \mathbf{E}_{\mathrm{NV},\ell} + (v-u)^{-2\ell} u^{(2\ell+1)\alpha + (-i-2\ell-2+\delta/2)} \mathbf{E}_{\mathrm{NV},\ell}. \end{aligned} \tag{4.28}$$

The last term can be easily verified to be bounded by $Cv^{-2\ell} \tau^{-i-1-\varepsilon} \mathbf{E}_{\mathrm{NV},\ell}$ by considering two cases $\{v - u \geq \frac{1}{2}v\}$ and $\{v - u \leq \frac{1}{2}v\}$ separately if taking δ sufficiently small and α sufficiently close to 1.

Using $\partial_u = \partial_\tau - \partial_v$ and combining the estimates (4.20) and (4.28), we finally conclude

Proposition 4.6. *Let the initial data condition 4.2 to order i_0 hold true. Let $j_1, j_2, j_3 \in \mathbb{N}$ and $j_1 + j_2 + j_3 \leq i_0$. There exists a sufficiently small δ and an α sufficiently close to 1 such that in the region $\{v - u \geq v^\alpha\}$,*

$$(\partial_\tau^{j_1} \partial_u^{j_2} \partial_v^{j_3} ((v-u)^{-2\ell} \widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}))_{AP} = \frac{2\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)}}{2\ell+1} \partial_\tau^{j_1} \partial_u^{j_2} \partial_v^{j_3} \left(\frac{v-u}{uv^{2\ell+1}} \right). \tag{4.29}$$

Step 3. Decay of all $\widetilde{\Phi}_{+\mathfrak{s}}^{(i)}$ and spin $\pm\mathfrak{s}$ components in $\{v - u \geq v^{\alpha'}\}$.

Utilizing the above asymptotics of $\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}$, we can derive the precise asymptotics of all $\widetilde{\Phi}_{+\mathfrak{s}}^{(i)}$, $i = 0, 1, \ldots, \ell - \mathfrak{s}$, hence also of the spin $+\mathfrak{s}$ component itself. The asymptotics of the spin $-\mathfrak{s}$ component is then calculated by the TSI in Section 3.5.

Theorem 4.7. *Let $j \in \mathbb{N}$ and $|\mathbf{a}| \leq j$. Let $\mathbb{P} = \{\partial_\tau, \partial_u, \partial_v\}$. Let the initial data condition 4.2 to order $j + \ell + \mathfrak{s}$ hold true. Then, there exists an α sufficiently close to 1 such that in the region $\{v - u \geq v^\alpha\}$, the asymptotics $(\mathbb{P}^{\mathbf{a}}(r^{-2\mathfrak{s}}\psi_{+\mathfrak{s}}))_{AP}$ are*

$$(\mathbb{P}^{\mathbf{a}}(r^{-2\mathfrak{s}}\psi_{+\mathfrak{s}}))_{AP} = 2^{\ell+\mathfrak{s}+2} \prod_{i=\ell+\mathfrak{s}+1}^{2\ell+1} i^{-1}\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)}\mathbb{P}^{\mathbf{a}}\big((v-u)^{\ell-\mathfrak{s}}u^{-\ell+\mathfrak{s}-1}v^{-\ell-\mathfrak{s}-1}\big) \tag{4.30}$$

and the asymptotics $(\mathbb{P}^{\mathbf{a}}\psi_{-\mathfrak{s}})_{AP}$ are

$$(\mathbb{P}^{\mathbf{a}}\psi_{-\mathfrak{s}})_{AP} = 2^{\ell+\mathfrak{s}+2} \prod_{i=\ell-\mathfrak{s}+1}^{2\ell+1} j^{-1}\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)}\mathbb{P}^{\mathbf{a}}\partial_u^{2\mathfrak{s}}\big((v-u)^{\ell+\mathfrak{s}}u^{-\ell+\mathfrak{s}-1}v^{-\ell-\mathfrak{s}-1}\big). \tag{4.31}$$

Proof. Recall from equation (3.11) that for all $i \in \{0, 1, \dots, \ell - \mathfrak{s} - 1\}$,

$$-\mu Y\hat{\mathcal{V}}\widetilde{\Phi}_{+\mathfrak{s}}^{(i)} + (\mathring{\eth}'\mathring{\eth} + f_{\mathfrak{s},i,1})\widetilde{\Phi}_{+\mathfrak{s}}^{(i)} + f_{\mathfrak{s},i,2}(r-3M)r^{-2}\hat{\mathcal{V}}\widetilde{\Phi}_{+\mathfrak{s}}^{(i)} + \sum_{j=0}^{i} O(r^{-1})\Phi_{+\mathfrak{s}}^{(j)} = 0. \tag{4.32}$$

We substitute in the values of $f_{\mathfrak{s},i,1}$ and $f_{\mathfrak{s},i,2}$ and put the above equation into the following form

$$\begin{aligned} &-\mu Y\widetilde{\Phi}_{+\mathfrak{s}}^{(i+1)} - (\ell - i - \mathfrak{s})(\ell + i + \mathfrak{s} + 1)\widetilde{\Phi}_{+\mathfrak{s}}^{(i)} - 2(i+\mathfrak{s}+1)(r-3M)r^{-2}\widetilde{\Phi}_{+\mathfrak{s}}^{(i+1)} \\ &= \sum_{j=0}^{i}\Big(O(r^{-1})\widetilde{\Phi}_{+\mathfrak{s}}^{(j)} + O(1)Y\widetilde{\Phi}_{+\mathfrak{s}}^{(j)}\Big). \end{aligned} \tag{4.33}$$

This gives

$$\begin{aligned} &(\ell - i - \mathfrak{s})(\ell + i + \mathfrak{s} + 1)(v-u)^{-2(i+\mathfrak{s})}\widetilde{\Phi}_{+\mathfrak{s}}^{(i)} \\ &= (v-u)^{-2(i+\mathfrak{s})}\Big(-2\partial_u\widetilde{\Phi}_{+\mathfrak{s}}^{(i+1)} - 2(i+\mathfrak{s}+1)r^{-1}\widetilde{\Phi}_{+\mathfrak{s}}^{(i+1)} \\ &\quad + \sum_{j=0}^{i}\big(O(1)\partial_u\widetilde{\Phi}_{+\mathfrak{s}}^{(i)} + O(r^{-1})\widetilde{\Phi}_{+\mathfrak{s}}^{(j)}\big) + O(r^{-2})\widetilde{\Phi}_{+\mathfrak{s}}^{(i+1)}\Big) \\ &= -2(v-u)^2\partial_u\Big((v-u)^{-2(i+\mathfrak{s}+1)}\widetilde{\Phi}_{+\mathfrak{s}}^{(i+1)}\Big) \\ &\quad + (v-u)^{-2(i+\mathfrak{s})}\Big(O(1)\partial_u\widetilde{\Phi}_{+\mathfrak{s}}^{(i)} + \sum_{j=0}^{i} O(r^{-1})\widetilde{\Phi}_{+\mathfrak{s}}^{(j)} + O(r^{-1})r^{-1}\log r\widetilde{\Phi}_{+\mathfrak{s}}^{(i+1)}\Big). \end{aligned} \tag{4.34}$$

All the terms in the last line have (at least) extra $\tau^{-\varepsilon}$ decay compared to the second last line, thus the asymptotics $(\mathbb{P}^{\mathbf{a}}\widetilde{\Phi}_{+\mathfrak{s}}^{(i)})_{\text{asymp}}$ for all $i \in \{0, 1, \dots, \ell - \mathfrak{s}\}$ are determined by iteratively solving the following equations

$$\begin{aligned} (\mathbb{P}^{\mathbf{a}}((v-u)^{-2\ell}\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})}))_{\text{AP}} &= \frac{2\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)}}{2\ell+1}\mathbb{P}^{\mathbf{a}}\Big(\frac{v-u}{uv^{2\ell+1}}\Big), \\ (\mathbb{P}^{\mathbf{a}}((v-u)^{-2(i+\mathfrak{s})}\widetilde{\Phi}_{+\mathfrak{s}}^{(i)}))_{\text{AP}} &= \frac{-2(v-u)^2(\partial_u\mathbb{P}^{\mathbf{a}}(v-u)^{-2(i+\mathfrak{s}+1)}\widetilde{\Phi}_{+\mathfrak{s}}^{(i+1)}))_{\text{AP}}}{(\ell-i-\mathfrak{s})(\ell+i+\mathfrak{s}+1)}, \quad 0 \leq i \leq \ell - \mathfrak{s} - 1. \end{aligned}$$

Solving these equations yields

$$(\mathbb{P}^{\mathbf{a}}((v-u)^{-2\mathfrak{s}}\widetilde{\Phi}_{+\mathfrak{s}}^{(0)}))_{\text{AP}} = 2^{\ell-\mathfrak{s}+1} \prod_{i=\ell+\mathfrak{s}+1}^{2\ell+1} i^{-1}\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)}\mathbb{P}^{\mathbf{a}}\bigg(\Big(\frac{v-u}{u}\Big)^{\ell-\mathfrak{s}+1}v^{-\ell-\mathfrak{s}-1}\bigg). \tag{4.35}$$

Since $r^{-2\mathfrak{s}}\psi_{+\mathfrak{s}} = \mu^{\mathfrak{s}}r^{-2\mathfrak{s}-1}\widetilde{\Phi}_{+\mathfrak{s}}^{(0)}$, equation (4.30) holds.

Consider next the asymptotics of the spin $-\mathfrak{s}$ component. As mentioned already, the asymptotics of the spin $-\mathfrak{s}$ component can be calculated explicitly from the TSI (3.67b) and the already proven asymptotics of the spin $+\mathfrak{s}$ component. In view of the TSI (3.67b) and (3.70b), and since the last term $-12M\overline{\partial_\tau\psi_{-2}}$ in (3.70b) has (at least) faster $\tau^{-1+\varepsilon}$ decay than ψ_{-2}, one has

$$(\mathbb{P}^{\mathbf{a}}\psi_{-\mathfrak{s}})_{\text{AP}} = \frac{1}{(\ell-\mathfrak{s}+1)\cdots(\ell+\mathfrak{s})}\mathbb{P}^{\mathbf{a}}((Y^{2\mathfrak{s}}\psi_{+\mathfrak{s}})_{\text{AP}}). \tag{4.36}$$

One can expand $Y^{2\mathfrak{s}}\psi_{+\mathfrak{s}} = 2^{2\mathfrak{s}}\partial_u^{2\mathfrak{s}}\psi_{+\mathfrak{s}} + \sum_{i=1}^{2\mathfrak{s}} O(r^{-1-i})\partial_u^{2\mathfrak{s}-i}\psi_{+\mathfrak{s}}$ and the last term clearly has faster decay than the term $2^{2\mathfrak{s}}\partial_u^{2\mathfrak{s}}\psi_{+\mathfrak{s}}$ in the region $\{v-u\geq v^\alpha\}$, hence this yields

$$\begin{aligned}(\mathbb{P}^{\mathbf{a}}\psi_{-\mathfrak{s}})_{\mathrm{AP}} &= \frac{2^{2\mathfrak{s}}}{(\ell-\mathfrak{s}+1)\cdots(\ell+\mathfrak{s})}\mathbb{P}^{\mathbf{a}}((\partial_u^{2\mathfrak{s}}\psi_{+\mathfrak{s}})_{\mathrm{AP}})\\ &= \frac{1}{(\ell-\mathfrak{s}+1)\cdots(\ell+\mathfrak{s})}(\mathbb{P}^{\mathbf{a}}(\partial_u^{2\mathfrak{s}}((v-u)^{2\mathfrak{s}}(v-u)^{-2\mathfrak{s}}\psi_{+\mathfrak{s}}))_{\mathrm{AP}}).\end{aligned} \tag{4.37}$$

In view of equation (4.30), this proves (4.31). □

4.2. **Price's law in the region $\{v-u\leq v^\alpha\}$.** The estimates are first proven for the spin $-\mathfrak{s}$ component, and these yield the estimates for the spin $+\mathfrak{s}$ component via the TSI of Section 3.5.

Theorem 4.8. *Let $j\in\mathbb{N}$. Let the initial data condition 4.2 to order $j+\ell+\mathfrak{s}$ hold true. Then, there exists an $\alpha\in(\frac{1}{2},1)$ sufficiently close to 1 such that in the region $\{v-u\leq v^\alpha\}$, the asymptotics $(\partial_\tau^j(r^{-2\mathfrak{s}}\psi_{+\mathfrak{s}}))_{AF}$ and $(\partial_\tau^j\psi_{-\mathfrak{s}})_{AF}$ are*

$$(\partial_\tau^j\psi_{-\mathfrak{s}}(\tau,\rho))_{AP} = (-1)^j 2^{2\ell+2}\prod_{i=\ell+\mathfrak{s}+1}^{2\ell+1} i^{-1}\frac{(2\ell+j+1)!}{(2\ell+1)!}\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)}\rho^{\ell-\mathfrak{s}}h_{\mathfrak{s},\ell}\tau^{-2\ell-j-2}, \tag{4.38a}$$

$$(\partial_\tau^j(r^{-2\mathfrak{s}}\psi_{+\mathfrak{s}}(\tau,\rho)))_{AP} = (-1)^j 2^{2\ell+2}\prod_{i=\ell+\mathfrak{s}+1}^{2\ell+1} i^{-1}\frac{(2\ell+j+1)!}{(2\ell+1)!}\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)}\mu^{\mathfrak{s}}\rho^{\ell-\mathfrak{s}}h_{\mathfrak{s},\ell}\tau^{-2\ell-j-2}, \tag{4.38b}$$

and for $\mathfrak{s}\neq 0$,

$$(\partial_\tau^j(r^{-2\mathfrak{s}}\psi_{+\mathfrak{s}}(\tau,\rho)))_{AP}\Big|_{\mathcal{H}^+} = \frac{(-1)^{\mathfrak{s}+j}2^{2\ell+3}\mathfrak{s}(\ell-\mathfrak{s})!(2\ell+j+2)!}{((2\ell+1)!)^2}\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)}(2M)^{\ell-\mathfrak{s}+1}h_{\mathfrak{s},\ell}(2M)v^{-2\ell-j-3}. \tag{4.38c}$$

Proof. Consider first the spin $-\mathfrak{s}$ component $\psi_{-\mathfrak{s}}$. We have achieved in Proposition 3.39 that

$$|\partial_\tau^j\partial_\rho\varphi_{-\mathfrak{s},\ell}| \lesssim \tau^{-2\ell-j-3+\delta/2}\mathbf{E}_{\mathrm{NV},\ell}. \tag{4.39}$$

Therefore, by integrating $\partial_\rho\varphi_{-\mathfrak{s},\ell}$ from any point $(\tau,\rho')\in\{v-u\leq v^\alpha\}$ along constant τ up to the intersection point with the curve γ_α, one finds for small enough δ that

$$\left|\int_{\rho'}^{\rho_{\gamma_\alpha}(\tau)}\partial_\tau^j\partial_\rho\varphi_{-1,\ell}\mathrm{d}\rho\right| \lesssim \tau^{-2\ell-j-3+\delta/2+\alpha}\mathbf{E}_{\mathrm{NV},\ell} \lesssim \tau^{-2\ell-j-2-\varepsilon}\mathbf{E}_{\mathrm{NV},\ell}. \tag{4.40}$$

From the definition (3.101) of $\varphi_{-\mathfrak{s},\ell}$ and equation (4.31), we obtain

$$\begin{aligned}(\partial_\tau^j\varphi_{-\mathfrak{s},\ell})_{\mathrm{AP}}\big|_{\gamma_\alpha} &= 2^{2\ell+2}\prod_{i=\ell-\mathfrak{s}+1}^{2\ell+1} i^{-1}\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)}\partial_\tau^j\big((v-u)^{-\ell+\mathfrak{s}}\partial_u^{2\mathfrak{s}}\big((v-u)^{\ell+\mathfrak{s}}u^{-\ell+\mathfrak{s}-1}v^{-\ell-\mathfrak{s}-1}\big)\big)\big|_{\gamma_\alpha}\\ &= 2^{2\ell+2}\prod_{i=\ell+\mathfrak{s}+1}^{2\ell+1} i^{-1}\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)}\partial_\tau^j\big(u^{-\ell+\mathfrak{s}-1}v^{-\ell-\mathfrak{s}-1}\big)\big|_{\gamma_\alpha}\\ &= (-1)^j 2^{2\ell+2}\prod_{i=\ell+\mathfrak{s}+1}^{2\ell+1} i^{-1}\prod_{n=2\ell+2}^{2\ell+j+1} n\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)}\tau^{-2\ell-j-2}.\end{aligned} \tag{4.41}$$

From this equation and the estimate (4.40), it holds at any point $(\tau,\rho)\in\{v-u\leq v^\alpha\}$ that

$$(\partial_\tau^j\varphi_{-\mathfrak{s},\ell}(\tau,\rho))_{\mathrm{AP}} = (-1)^j 2^{2\ell+2}\prod_{i=\ell+\mathfrak{s}+1}^{2\ell+1} i^{-1}\prod_{n=2\ell+2}^{2\ell+j+1} n\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)}\tau^{-2\ell-j-2}. \tag{4.42}$$

Equation (4.38a) thus follows from this equation together with the definition (3.101) .

Consider next the spin $+\mathfrak{s}$ component. We can obtain its asymptotics by utilizing the TSI (3.67a) and (3.70a) together with the proven estimates for the spin $-\mathfrak{s}$ component. Since the last term

$12M\overline{\partial_\tau\Phi_{+2}^{(0)}}$ on the RHS of (3.70a) has faster decay in τ than $\Phi_{+2}^{(0)}$, the TSI (3.67a) and (3.70a) can be written as

$$(\ell-\mathfrak{s}+1)\cdots(\ell+\mathfrak{s})(\partial_\tau^j(r^{-2\mathfrak{s}}\psi_{+\mathfrak{s}}))_{\mathrm{AP}} = (\partial_\tau^j(\mu^{\mathfrak{s}}r^{-2\mathfrak{s}-1}\Phi_{-\mathfrak{s}}^{(2\mathfrak{s})}))_{\mathrm{AP}}. \tag{4.43}$$

Note that $\Phi_{-\mathfrak{s}}^{(2\mathfrak{s})} = (r^2\hat{V})^{2\mathfrak{s}}\Phi_{-\mathfrak{s}}^{(0)} = (r^2\hat{V})^{2\mathfrak{s}}(\mu^{\mathfrak{s}}r\psi_{-\mathfrak{s}}) = (r^2(\partial_\rho + H_{\mathrm{hyp}}\partial_\tau))^{2\mathfrak{s}}(\mu^{\mathfrak{s}}r^{\ell+1-\mathfrak{s}}h_{\mathfrak{s},\ell}\varphi_{-\mathfrak{s},\ell})$, hence we can expand out $\partial_\tau^j(\mu^{\mathfrak{s}}r^{-2\mathfrak{s}-1}\Phi_{-\mathfrak{s}}^{(2\mathfrak{s})})$ as follows:

$$\begin{aligned}\partial_\tau^j(\mu^{\mathfrak{s}}r^{-2\mathfrak{s}-1}\Phi_{-\mathfrak{s}}^{(2\mathfrak{s})}) &= \mu^{\mathfrak{s}}r^{-2\mathfrak{s}-1}(r^2(\partial_\rho + H_{\mathrm{hyp}}\partial_\tau))^{2\mathfrak{s}}(\mu^{\mathfrak{s}}r^{\ell+1-\mathfrak{s}}h_{\mathfrak{s},\ell}\partial_\tau^j\varphi_{-\mathfrak{s},\ell})\\ &= \sum_{j_1\geq 0, j_2\geq 0, 1\leq j_1+j_2\leq 2\mathfrak{s}} O(1)r^{\ell-\mathfrak{s}+j_1+j_2}\partial_\rho^{j_1}\partial_\tau^{j_2+j}\varphi_{-\mathfrak{s},\ell}\\ &\quad + \mu^{\mathfrak{s}}r^{-2\mathfrak{s}-1}(r^2\partial_\rho)^{2\mathfrak{s}}(\mu^{\mathfrak{s}}r^{\ell+1-\mathfrak{s}}h_{\mathfrak{s},\ell})\partial_\tau^j\varphi_{-\mathfrak{s},\ell}.\end{aligned} \tag{4.44}$$

By the estimates (4.39), the second last line has decay $\tau^{-2\ell-j-2-\varepsilon}$ in the region $\{v-u\leq v^\alpha\}$. Therefore, to show the estimate (4.38b), it suffices to prove

$$\left([(\ell-\mathfrak{s}+1)\cdots(\ell+\mathfrak{s})]^{-1}\mu^{\mathfrak{s}}r^{-2\mathfrak{s}-1}(r^2\partial_\rho)^{2\mathfrak{s}}(\mu^{\mathfrak{s}}r^{\ell+1-\mathfrak{s}}h_{\mathfrak{s},\ell})\partial_\tau^j\varphi_{-\mathfrak{s},\ell}\right)_{\mathrm{AP}} = \text{RHS of (4.38b)}. \tag{4.45}$$

In view of (4.42), it in turn suffices to prove

$$[(\ell-\mathfrak{s}+1)\cdots(\ell+\mathfrak{s})]^{-1}\mu^{\mathfrak{s}}r^{-2\mathfrak{s}-1}(r^2\partial_\rho)^{2\mathfrak{s}}(\mu^{\mathfrak{s}}r^{\ell+1-\mathfrak{s}}h_{\mathfrak{s},\ell}) = \mu^{\mathfrak{s}}r^{\ell-\mathfrak{s}}h_{\mathfrak{s},\ell}. \tag{4.46}$$

This is exactly (3.112), hence this proves (4.38b).

To compute the asymptotics for the spin $+\mathfrak{s}$ component on $\mathcal{H}^+$, we shall use again equation (4.43). Recall that $\partial_\tau^j(\mu^{\mathfrak{s}}r^{-2\mathfrak{s}-1}\Phi_{-\mathfrak{s}}^{(2\mathfrak{s})}) = \partial_\tau^j(\mu^{\mathfrak{s}}r^{-2\mathfrak{s}-1}(\mu^{-1}r^2V)^{2\mathfrak{s}}(\mu^{\mathfrak{s}}r\psi_{-\mathfrak{s}}))$. Hence, for $\rho\geq 2M$, for $\mathfrak{s}=1$, this becomes

$$\begin{aligned}\partial_\tau^j(\mu r^{-3}\Phi_{-1}^{(2)}) &= \partial_\tau^j(\mu r^{-3}(\mu^{-1}r^2V)^2(\mu r\psi_{-1}))\\ &= \partial_\tau^j(r^{-1}V(r^2\psi_{-1} + r^3V\psi_{-1}))\\ &= \partial_\tau^j(2\mu r\psi_{-1} + rV\psi_{-1} + 3\mu rV\psi_{-1} + r^2V^2\psi_{-1})\\ &= r\partial_\tau^j V\psi_{-1} + \mu\sum_{i=0,1} O(1)r^i\partial_\tau^j V^i\psi_{-1} + O(1)\partial_\tau^j r^2V^2\psi_{-1},\end{aligned} \tag{4.47a}$$

and for $\mathfrak{s}=2$, it becomes

$$\begin{aligned}&\partial_\tau^j(\mu^2r^{-5}\Phi_{-2}^{(4)})\\ &= \partial_\tau^j(\mu^2r^{-5}(\mu^{-1}r^2V)^4(\mu^2r\psi_{-2}))\\ &= \partial_\tau^j\left[\mu r^{-3}V(\mu^{-1}r^2V)^2(r^2\partial_r(\mu^2r)\psi_{-2} + \mu r^3V\psi_{-2})\right]\\ &= \partial_\tau^j\left[\mu r^{-3}V(\mu^{-1}r^2V)(r^2\partial_r(r^2\partial_r(\mu^2r))\psi_{-2} + \mu^{-1}r^4\partial_r(\mu^2r)V\psi_{-2} + r^2\partial_r(\mu r^3)V\psi_{-2} + r^5V^2\psi_{-2})\right]\\ &= \mu^2r^{-3}\partial_r(\mu^{-1}r^4\partial_r(r^2\partial_r(\mu^2r)))V\partial_\tau^j\psi_{-2} + \mu\sum_{i=0,1}O(1)r^i\partial_\tau^jV^i\psi_{-2} + \sum_{i=2}^4 O(1)\partial_\tau^j r^iV^i\psi_{-2}.\end{aligned} \tag{4.47b}$$

On the event horizon, we have $\mu=0$ and $V=2\partial_\tau$, hence the second last term of both (4.47a) and (4.47b) vanishes and the last term of both (4.47a) and (4.47b) has $v^{-2\ell-j-4}$ decay by the asymptotics (4.38a) of $\psi_{-\mathfrak{s}}$. Since the coefficient of the first term in the last line of (4.47b), $\mu^2r^{-3}\partial_r(\mu^{-1}r^4\partial_r(r^2\partial_r(\mu^2r)))$, equals $-4M$ when on $\mathcal{H}^+$, we obtain, by the above discussions and from equation (4.43), that

$$(\ell-\mathfrak{s}+1)\cdots(\ell+\mathfrak{s})(\partial_\tau^j(r^{-2\mathfrak{s}}\psi_{+\mathfrak{s}}))_{\mathrm{AP}}\big|_{\mathcal{H}^+} = (-1)^{\mathfrak{s}-1}4\mathfrak{s}M(\partial_\tau^{j+1}\psi_{-\mathfrak{s}})_{\mathrm{AP}}\big|_{\mathcal{H}^+}. \tag{4.48}$$

By the asymptotics (4.38a) of $\psi_{-\mathfrak{s}}$ on $\mathcal{H}^+$, this thus yields the asymptotic (4.38c) for the spin $+\mathfrak{s}$ component on $\mathcal{H}^+$. □

4.3. **Global Price's law estimates.** We collect the statements in Theorems 4.7 and 4.8 and summarize as follows:

Theorem 4.9. *Let $j \in \mathbb{N}$ and let $\mathfrak{s} = 0, 1, 2$. Assume the spin $s = \pm\mathfrak{s}$ components are supported on a single (m, ℓ) mode, $\ell \geq \mathfrak{s}$. Let the initial data condition 4.2 to order $j + \ell + \mathfrak{s}$ hold true. Let function $h_{\mathfrak{s},\ell}$ and scalars $\widetilde{\Phi}^{(i)}_{+\mathfrak{s}}$ and $\widetilde{\Phi}^{(i)}_{-\mathfrak{s}}$ be defined as in Definitions 3.34 and 3.6, respectively.*

Then there exists an $\delta > 0$, $k = k(\ell, j)$, $\varepsilon = \varepsilon(j, \ell, \delta) > 0$ and $C = C(k, \ell, j, \delta)$ such that for any $\tau \geq \tau_0$,

$$\begin{aligned} &\left|\partial_\tau^j \Upsilon_{-\mathfrak{s}} - 2^{2\ell+2} \prod_{i=\ell-\mathfrak{s}+1}^{2\ell+1} i^{-1} \mathbb{Q}^{(m,\ell)}_{+\mathfrak{s}} h_{\mathfrak{s},\ell} Y^{-\mathfrak{s}}_{m,\ell}(\cos\theta) e^{im\phi} \partial_\tau^j (Y^{2\mathfrak{s}}(r^{\ell+\mathfrak{s}} \tau^{-\ell+\mathfrak{s}-1} v^{-\ell-\mathfrak{s}-1}))\right| \\ &\quad \leq C\tau^{-\varepsilon} \partial_\tau^j (Y^{2\mathfrak{s}}(r^{\ell+\mathfrak{s}} \tau^{-\ell+\mathfrak{s}-1} v^{-\ell-\mathfrak{s}-1})) \mathbf{E}_{\mathrm{NV},\ell}, \end{aligned} \tag{4.49a}$$

$$\begin{aligned} &\left|\partial_\tau^j \Upsilon_{+\mathfrak{s}} - 2^{2\ell+2} \prod_{i=\ell+\mathfrak{s}+1}^{2\ell+1} i^{-1} \mathbb{Q}^{(m,\ell)}_{+\mathfrak{s}} \mu^{\mathfrak{s}} h_{\mathfrak{s},\ell} Y^{\mathfrak{s}}_{m,\ell}(\cos\theta) e^{im\phi} \partial_\tau^j (r^{\ell-\mathfrak{s}} \tau^{-\ell+\mathfrak{s}-1} v^{-\ell-\mathfrak{s}-1})\right| \\ &\quad \leq C\tau^{-\varepsilon} \partial_\tau^j (r^{\ell-\mathfrak{s}} \tau^{-\ell+\mathfrak{s}-1} v^{-\ell-\mathfrak{s}-1}) \mathbf{E}_{\mathrm{NV},\ell}, \end{aligned} \tag{4.49b}$$

and on the future event horizon $\mathcal{H}^+$, for $\mathfrak{s} \neq 0$,

$$\begin{aligned} &\left|\partial_\tau^j \Upsilon_{+\mathfrak{s}}\big|_{\mathcal{H}^+} - \frac{(-1)^{\mathfrak{s}+j} 2^{2\ell+3} \mathfrak{s}(\ell-\mathfrak{s})!}{(2\ell+1)!} \prod_{n=2\ell+2}^{2\ell+j+2} n \mathbb{Q}^{(m,\ell)}_{+\mathfrak{s}} Y^{\mathfrak{s}}_{m,\ell}(\cos\theta) e^{im\phi} (2M)^{\ell-\mathfrak{s}+1} h_{\mathfrak{s},\ell}(2M) v^{-2\ell-j-3}\right| \\ &\leq C v^{-2\ell-j-3-\varepsilon} \mathbf{E}_{\mathrm{NV},\ell}, \end{aligned} \tag{4.49c}$$

with $\mathbf{E}_{\mathrm{NV},\ell} = (\mathbb{I}^{\ell,k,3-\delta}_{\Sigma_{\tau_0}}[(\Psi_{+\mathfrak{s}})_{(m,\ell)}])^{\frac{1}{2}} + |\mathbb{Q}^{(m,\ell)}_{+\mathfrak{s}}| + D_0$ and $\mathbb{I}^{\ell,k,3-\delta}_{\Sigma_{\tau_0}}[(\Psi_{+\mathfrak{s}})_{(m,\ell)}]$ is defined as in Definition 3.8. In particular,

(1) *in the region $\{\rho \geq \tau^{1+\varepsilon_0}\}$ for an arbitrary $\varepsilon_0 > 0$, then*

$$\begin{aligned} &\left|\partial_\tau^j \Upsilon_s - (-1)^j 2^{\ell+\mathfrak{s}+2} \prod_{i=\ell+\mathfrak{s}+1}^{2\ell+1} i^{-1} \mathbb{Q}^{(m,\ell)}_{+\mathfrak{s}} Y^s_{m,\ell}(\cos\theta) e^{im\phi} v^{-1-\mathfrak{s}-s} \tau^{-\ell+s-j-1}\right| \\ &\quad \leq C(\tau^{-\varepsilon} + \tau^{-\varepsilon_0}) v^{-1-\mathfrak{s}-s} \tau^{-\ell+s-j-1} \mathbf{E}_{\mathrm{NV},\ell}; \end{aligned} \tag{4.50a}$$

(2) *in the region $\{\rho \leq \tau^{1-\varepsilon_0}\}$ for an arbitrary $\varepsilon_0 > 0$, then for $s = \pm\mathfrak{s}$,*

$$\begin{aligned} &\left|\partial_\tau^j \Upsilon_s - (-1)^j 2^{2\ell+2} \prod_{i=\ell+\mathfrak{s}+1}^{2\ell+1} i^{-1} \frac{(2\ell+1+j)!}{(2\ell+1)!} \mathbb{Q}^{(m,\ell)}_{+\mathfrak{s}} \mu^{\frac{s+\mathfrak{s}}{2}} r^{\ell-\mathfrak{s}} h_{\mathfrak{s},\ell} Y^s_{m,\ell}(\cos\theta) e^{im\phi} \tau^{-2\ell-j-2}\right| \\ &\quad \leq C(\tau^{-\varepsilon} + \tau^{-\varepsilon_0}) \tau^{-2\ell-j-2} \mathbf{E}_{\mathrm{NV},\ell}. \end{aligned} \tag{4.50b}$$

Additionally, for the middle component $\Upsilon_{0,mid}$ of the Maxwell field, let $q_{\mathbf{E}}$ and $q_{\mathbf{B}}$ be defined as in Lemma 2.13, then

(*i*) *for any $\tau \geq \tau_0$ and $\rho > 2M$,*

$$\begin{aligned} &\left|\partial_\tau^j \left(\Upsilon_{0,mid} - r^{-2}(q_{\mathbf{E}} + iq_{\mathbf{B}})\right) + \frac{2^{2\ell+3}}{\sqrt{\ell(\ell+1)}} \prod_{i=\ell+2}^{2\ell+1} i^{-1} \mathbb{Q}^{(m,\ell)}_{+1} Y^0_{m,\ell}(\cos\theta) e^{im\phi} \partial_\tau^j (Y(\mu h_{1,\ell} r^\ell \tau^{-\ell} v^{-\ell-2}))\right| \\ &\leq C\tau^{-\varepsilon} \partial_\tau^j (Y(\mu r^\ell \tau^{-\ell} v^{-\ell-2})) \mathbf{E}_{\mathrm{NV},\ell}; \end{aligned} \tag{4.51a}$$

(*ii*) *in the region $\{\rho \geq \tau^{1+\varepsilon_0}\}$ for an arbitrary $\varepsilon_0 > 0$,*

$$\begin{aligned} &\left|\partial_\tau^j \left(\Upsilon_{0,mid} - r^{-2}(q_{\mathbf{E}} + iq_{\mathbf{B}})\right) - \frac{(-1)^j 2^{\ell+3}}{\sqrt{\ell(\ell+1)}} \prod_{i=\ell+2}^{2\ell+1} i^{-1} \prod_{n=\ell}^{\ell+j} n \sum_{m=-\ell}^{\ell} \mathbb{Q}^{(m,\ell)}_{+1} Y^0_{m,\ell}(\cos\theta) e^{im\phi} v^{-2} \tau^{-\ell-j-1}\right| \\ &\leq C(\tau^{-\varepsilon} + \tau^{-\varepsilon_0}) v^{-2} \tau^{-\ell-j-1} \mathbf{E}_{\mathrm{NV},\ell}; \end{aligned} \tag{4.51b}$$

(iii) *in the region* $\{2M \leq \rho \leq \tau^{1-\varepsilon_0}\}$ *for an arbitrary* $\varepsilon_0 > 0$,

$$\left|\partial_\tau^j \left(\Upsilon_{0,mid} - r^{-2}\left(q_{\mathbf{E}} + iq_{\mathbf{B}}\right)\right) - (-1)^j \frac{2^{2\ell+3}}{\sqrt{\ell(\ell+1)}} \prod_{i=\ell+2}^{2\ell+1} i^{-1} \frac{(2\ell+1+j)!}{(2\ell+1)!} \right.$$
$$\left. \times \sum_{m=-\ell}^{\ell} \mathbb{Q}_{+1}^{(m,\ell)} \partial_\rho(\mu h_{1,\ell} r^\ell) Y_{m,\ell}^0(\cos\theta) e^{im\phi} \tau^{-2\ell-j-2}\right|$$
$$\leq C(\tau^{-\varepsilon} + \tau^{-\varepsilon_0})\tau^{-2\ell-j-2}\mathbf{E}_{\mathrm{NV},\ell}. \tag{4.51c}$$

Proof. The estimates (4.49) and (4.50) are immediate from the estimates in Theorems 4.7 and 4.8. For the middle component of the Maxwell field, we can decompose it as in Lemma 2.13 into $\Upsilon_{0,\mathrm{mid}} = \Upsilon_{0,\mathrm{mid}}(\mathbf{F}_{\mathrm{sta}}) + \Upsilon_{0,\mathrm{mid}}(\mathbf{F}_{\mathrm{rad}})$, where $\Upsilon_{0,\mathrm{mid}}(\mathbf{F}_{\mathrm{sta}}) = r^{-2}(q_{\mathbf{E}} + iq_{\mathbf{B}})$. The Maxwell equation (2.22a) and (2.17) together thus give $-\sqrt{\ell(\ell+1)}\Upsilon_{0,\mathrm{mid}}(\mathbf{F}_{\mathrm{rad}}) = 2Y(r^{-1}\psi_{+1})$. We then use the estimates for $r^{-2}\psi_{+1}$ in Theorems 4.7 and 4.8 and conclude the estimate (4.51a) for $\rho > 2M$. The other two estimates (4.51b) and (4.51c) for $\rho > 2M$ are direct consequence of the estimate (4.51a). In the end, we use that on $\mathcal{H}^+$, $\sqrt{\ell(\ell+1)}(\Upsilon_{0,\mathrm{mid}}(\mathbf{F}_{\mathrm{rad}})_{m,\ell} = 2\mu^{-1}V(\mu r(\psi_{-1})_{m,\ell}) = 2(\psi_{-1})_{m,\ell} + 2rV(\psi_{-1})_{m,\ell} = 2(\psi_{-1})_{m,\ell} + 8M(\partial_\tau\psi_{-1})_{m,\ell}$ which comes from the Maxwell equation (2.22b) and (2.17), and this together with (4.38a) clearly yields (4.51c) on $\mathcal{H}^+$. □

We can also obtain estimates if the spin $\pm\mathfrak{s}$ components are supported on $\ell \geq \ell_0$ modes, $\ell_0 \geq \mathfrak{s}$.

Theorem 4.10. *Let* $j \in \mathbb{N}$ *and let* $\mathfrak{s} = 0, 1, 2$. *Assume the spin* $s = \pm\mathfrak{s}$ *components are supported on* $\ell \geq \ell_0$ *modes,* $\ell_0 \geq \mathfrak{s}$. *Let function* $h_{\mathfrak{s},\ell}$ *and scalars* $\widetilde{\Phi}_{+\mathfrak{s}}^{(i)}$ *and* $\widetilde{\Phi}_{-\mathfrak{s}}^{(i)}$ *be defined as in Definitions 3.34 and 3.6, respectively. Assume on* Σ_{τ_0} *that there are constants* $\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)} \in \mathbb{R}$, $\beta \in (0, \frac{1}{2})$ *and* $0 \leq D_0 < \infty$ *such that for all* $0 \leq i \leq j + \ell + \mathfrak{s}$, $m \in \{-\ell_0, -\ell_0+1, \ldots, \ell_0\}$ *and* $r \geq 10M$,

$$\left|\partial_\rho^i\left(\hat{V}(\widetilde{\Phi}_{+\mathfrak{s}}^{(\ell-\mathfrak{s})})_{m,\ell_0} - \rho^{-2}\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell_0)}\right)\right| \lesssim D_0\rho^{-2-\beta-i}. \tag{4.52}$$

Then there exists an $\delta > 0$, $k = k(\ell_0, j)$, $\varepsilon = \varepsilon(j, \ell_0, \delta) > 0$ *and* $C = C(k, \ell_0, j, \delta)$ *such that for any* $\tau \geq \tau_0$,

$$\left|\partial_\tau^j \Upsilon_{-\mathfrak{s}} - 2^{2\ell_0+2} \prod_{i=\ell_0-\mathfrak{s}+1}^{2\ell_0+1} i^{-1} \sum_{m=-\ell_0}^{\ell_0} \mathbb{Q}_{+\mathfrak{s}}^{(m,\ell_0)} h_{\mathfrak{s},\ell_0} Y_{m,\ell_0}^{-\mathfrak{s}}(\cos\theta) e^{im\phi} \partial_\tau^j (Y^{2\mathfrak{s}}(r^{\ell_0+\mathfrak{s}}\tau^{-\ell_0+\mathfrak{s}-1}v^{-\ell_0-\mathfrak{s}-1}))\right|$$
$$\leq C\tau^{-\varepsilon}\partial_\tau^j(Y^{2\mathfrak{s}}(r^{\ell_0+\mathfrak{s}}\tau^{-\ell_0+\mathfrak{s}-1}v^{-\ell_0-\mathfrak{s}-1}))\mathbf{E}_{\mathrm{NV}}, \tag{4.53a}$$

$$\left|\partial_\tau^j \Upsilon_{+\mathfrak{s}} - 2^{2\ell_0+2} \prod_{i=\ell_0+\mathfrak{s}+1}^{2\ell_0+1} i^{-1} \sum_{m=-\ell_0}^{\ell_0} \mathbb{Q}_{+\mathfrak{s}}^{(m,\ell_0)} \mu^{\mathfrak{s}} h_{\mathfrak{s},\ell_0} Y_{m,\ell_0}^{\mathfrak{s}}(\cos\theta) e^{im\phi} \partial_\tau^j (r^{\ell_0-\mathfrak{s}}\tau^{-\ell_0+\mathfrak{s}-1}v^{-\ell_0-\mathfrak{s}-1})\right|$$
$$\leq C\tau^{-\varepsilon}\partial_\tau^j(r^{\ell_0-\mathfrak{s}}\tau^{-\ell_0+\mathfrak{s}-1}v^{-\ell_0-\mathfrak{s}-1})\mathbf{E}_{\mathrm{NV}}, \tag{4.53b}$$

with $\mathbf{E}_{\mathrm{NV}} = (\mathbb{I}_{NV,\Sigma_{\tau_0}}^{\ell\geq\ell_0,k,\delta}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}} + \sum_{m=-\ell_0}^{\ell_0}|\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell_0)}| + D_0$ *and* $\mathbb{I}_{NV,\Sigma_{\tau_0}}^{\ell\geq\ell_0,k,\delta}[\Psi_{+\mathfrak{s}}]$ *is defined as in Definition 3.9. In particular,*

(1) *in the region* $\{\rho \geq \tau^{1+\varepsilon_0}\}$ *for an arbitrary* $\varepsilon_0 > 0$, *then*

$$\left|\partial_\tau^j \Upsilon_s - (-1)^j 2^{\ell_0+\mathfrak{s}+2} \prod_{i=\ell_0+\mathfrak{s}+1}^{2\ell_0+1} i^{-1} \sum_{m=-\ell_0}^{\ell_0} \mathbb{Q}_{+\mathfrak{s}}^{(m,\ell_0)} Y_{m,\ell_0}^s(\cos\theta) e^{im\phi} v^{-1-\mathfrak{s}-s}\tau^{-\ell_0+s-j-1}\right|$$
$$\leq C(\tau^{-\varepsilon} + \tau^{-\varepsilon_0})v^{-1-\mathfrak{s}-s}\tau^{-\ell_0+s-j-1}\mathbf{E}_{\mathrm{NV}}; \tag{4.54a}$$

(2) *in the region* $\{\rho \leq \tau^{1-\varepsilon_0}\}$ *for an arbitrary* $\varepsilon_0 > 0$,

$$\left|\partial_\tau^j \Upsilon_s - (-1)^j 2^{2\ell_0+2} \prod_{i=\ell_0+\mathfrak{s}+1}^{2\ell+1} i^{-1} \frac{(2\ell_0+1+j)!}{(2\ell_0+1)!} \sum_{m=-\ell_0}^{\ell_0} \mathbb{Q}_{+\mathfrak{s}}^{(m,\ell_0)} \mu^{\frac{s+\mathfrak{s}}{2}} r^{\ell_0-\mathfrak{s}} h_{\mathfrak{s},\ell_0} Y_{m,\ell_0}^s(\cos\theta) e^{im\phi} \tau^{-2\ell_0-j-2}\right|$$
$$\leq C(\tau^{-\varepsilon} + \tau^{-\varepsilon_0})\tau^{-2\ell_0-j-2}\mathbf{E}_{\mathrm{NV}}, \tag{4.54b}$$

(3) *on the future event horizon, for* $\mathfrak{s}\neq 0$,

$$\left|\partial_\tau^j\Upsilon_{+\mathfrak{s}}\big|_{\mathcal{H}^+}-\frac{(-1)^{\mathfrak{s}+j}2^{2\ell_0+3}\mathfrak{s}(\ell_0-\mathfrak{s})!}{(2\ell_0+1)!}\prod_{n=2\ell_0+2}^{2\ell_0+j+2}n\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell_0)}Y_{m,\ell_0}^{\mathfrak{s}}(\cos\theta)e^{im\phi}(2M)^{\ell_0-\mathfrak{s}+1}h_{\mathfrak{s},\ell_0}(2M)v^{-2\ell_0-j-3}\right|$$
$$\leq Cv^{-2\ell_0-j-3-\varepsilon}\mathbf{E}_{\mathrm{NV},\ell}, \tag{4.54c}$$

Additionally, for the middle component $\Upsilon_{0,mid}$ *of the Maxwell field, let* $q_{\mathbf{E}}$ *and* $q_{\mathbf{B}}$ *be defined as in Lemma 2.13, then*

(i) *for any* $\tau\geq\tau_0$ *and* $\rho>2M$,

$$\left|\partial_\tau^j\left(\Upsilon_{0,mid}-r^{-2}\left(q_{\mathbf{E}}+iq_{\mathbf{B}}\right)\right)+\frac{2^{2\ell_0+3}}{\sqrt{\ell_0(\ell_0+1)}}\prod_{i=\ell_0+2}^{2\ell_0+1}i^{-1}\sum_{m=-\ell_0}^{\ell_0}\mathbb{Q}_{+1}^{(m,\ell_0)}Y_{m,\ell_0}^{0}(\cos\theta)e^{im\phi}\partial_\tau^j(Y(\mu h_{1,\ell_0}r^{\ell_0}\tau^{-\ell_0}v^{-\ell_0-2}))\right|$$
$$\leq C\tau^{-\varepsilon}\partial_\tau^j(Y(r^{\ell_0}\tau^{-\ell_0}v^{-\ell_0-2}))\mathbf{E}_{\mathrm{NV}}; \tag{4.55a}$$

(ii) *in the region* $\{\rho\geq\tau^{1+\varepsilon_0}\}$ *for an arbitrary* $\varepsilon_0>0$,

$$\left|\partial_\tau^j\left(\Upsilon_{0,mid}-r^{-2}\left(q_{\mathbf{E}}+iq_{\mathbf{B}}\right)\right)-\frac{(-1)^j2^{2\ell_0+3}}{\sqrt{\ell_0(\ell_0+1)}}\prod_{i=\ell_0+2}^{2\ell_0+1}i^{-1}\prod_{n=\ell_0}^{\ell_0+j}n\sum_{m=-\ell_0}^{\ell_0}\mathbb{Q}_{+1}^{(m,\ell_0)}Y_{m,\ell_0}^{0}(\cos\theta)e^{im\phi}v^{-2}\tau^{-\ell_0-j-1}\right|$$
$$\leq C(\tau^{-\varepsilon}+\tau^{-\varepsilon_0})v^{-2}\tau^{-\ell_0-j-1}\mathbf{E}_{\mathrm{NV}}; \tag{4.55b}$$

(iii) *in the region* $\{2M\leq\rho\leq\tau^{1-\varepsilon_0}\}$ *for an arbitrary* $\varepsilon_0>0$,

$$\begin{aligned}&\Bigg|\partial_\tau^j\left(\Upsilon_{0,mid}-r^{-2}\left(q_{\mathbf{E}}+iq_{\mathbf{B}}\right)\right)-(-1)^j\frac{2^{2\ell_0+3}}{\sqrt{\ell_0(\ell_0+1)}}\prod_{i=\ell_0+2}^{2\ell_0+1}i^{-1}\frac{(2\ell_0+1+j)!}{(2\ell_0+1)!}\\&\quad\times\sum_{m=-\ell_0}^{\ell_0}\mathbb{Q}_{+1}^{(m,\ell_0)}\partial_\rho(\mu h_{1,\ell_0}r^{\ell_0})Y_{m,\ell_0}^{0}(\cos\theta)e^{im\phi}\tau^{-2\ell_0-j-2}\Bigg|\\&\leq C(\tau^{-\varepsilon}+\tau^{-\varepsilon_0})\tau^{-2\ell_0-j-2}\mathbf{E}_{\mathrm{NV}}.\end{aligned} \tag{4.55c}$$

Proof. By decomposing the spin $\pm\mathfrak{s}$ components into $\ell=\ell_0$ mode and $\ell\geq\ell_0+1$ modes and using the previous theorem 4.9 to achieve the asymptotics for its (m,ℓ_0) mode, it suffices to show that $|\partial_\tau^j(\Upsilon_s)^{\ell\geq\ell_0+1}|$ are bounded by the RHS of each estimate in (4.53), and the rest estimates follow easily. This fact is in turn implied by the estimates (3.88) in the exterior region and (3.138) in the interior region. □

5. Price's law under vanishing Newman–Penrose constant condition

This section differs from the previous section in the sense that we assume the Newman–Penrose constant therein, i.e., $\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell)}$, vanishes. Under this vanishing Newman–Penrose constant condition, and assuming further initial data condition, we prove the asymptotic profiles for the spin $\pm\mathfrak{s}$ components. In Section 5.1, we define the time integral of each fixed (m,ℓ) mode of the spin $+\mathfrak{s}$ component and compute the expressions for its derivatives. Then, in Section 5.2, we calculate the Newman–Penrose constant for the time integral and bound the initial norms of the time integral in terms of initial norm of the (m,ℓ) mode of the spin $+\mathfrak{s}$ component. Finally, we apply the results in Section 4.3 to the time integral and derive the asymptotic profiles of the spin $\pm\mathfrak{s}$ components in Section 5.3.

5.1. **Time integral.** In this subsection, we will always assume that the spin $+\mathfrak{s}$ component is supported on an (m,ℓ) mode, and without confusion, we might suppress the dependence on m and ℓ unless specified. In addition, we denote $(\psi_{+\mathfrak{s}})_{m,\ell}$ by $\psi_{+\mathfrak{s},\ell}$. Recall equation (3.102) satisfied by $\psi_{+\mathfrak{s},\ell}$, for $\mathfrak{s}=0,1,2$,

$$\partial_\rho\left(r^{2\ell+2}\mu^{1+\mathfrak{s}}h_{\mathfrak{s},\ell}^2\partial_\rho(\mu^{-\mathfrak{s}}h_{\mathfrak{s},\ell}^{-1}r^{-\ell-\mathfrak{s}}\psi_{+\mathfrak{s},\ell})\right)=r^{\ell-\mathfrak{s}-1}\partial_\tau(\mathcal{H}[\psi_{+\mathfrak{s},\ell}]),$$

where

$$\begin{aligned}\mathcal{H}[\psi_{+\mathfrak{s},\ell}] =&\mu^{\mathfrak{s}} h_{\mathfrak{s},\ell}\Big\{\mu\tilde{H}_{\mathrm{hyp}}\hat{\mathcal{V}}(\mu^{-\mathfrak{s}}r\psi_{+\mathfrak{s},\ell}) - \mu H_{\mathrm{hyp}} r^2\partial_\rho(\mu^{-\mathfrak{s}}r\psi_{+\mathfrak{s},\ell})\\ &+ \big[\big(2\mathfrak{s}r - 2M(3\mathfrak{s}+1)\big)H_{\mathrm{hyp}} - \mu r^2\partial_r H_{\mathrm{hyp}}\big]\mu^{-\mathfrak{s}}r\psi_{+\mathfrak{s},\ell}\Big\}.\end{aligned} \tag{5.1}$$

Lemma 5.1. *For $\mathfrak{s}=0,1,2$, there exists a unique smooth $g_{+\mathfrak{s},\ell}$ to the following equation*

$$\partial_\rho\left(r^{2\ell+2}\mu^{1+\mathfrak{s}}h_{\mathfrak{s},\ell}^2\partial_\rho(\mu^{-\mathfrak{s}}h_{\mathfrak{s},\ell}^{-1}r^{-(\ell+\mathfrak{s})}\psi_{+\mathfrak{s},\ell})\right) = r^{\ell-\mathfrak{s}-1}\mathcal{H}[\psi_{+\mathfrak{s},\ell}] \tag{5.2}$$

which satisfies both

$$\lim_{\rho\to\infty} r^{-\ell-\mathfrak{s}}g_{+\mathfrak{s},\ell}\big|_{\Sigma_{\tau_0}} = 0 \tag{5.3}$$

and

$$\partial_\tau g_{+\mathfrak{s},\ell} = \psi_{+\mathfrak{s},\ell}. \tag{5.4}$$

Furthermore, we have, on Σ_{τ_0},

$$g_{+\mathfrak{s},\ell}(\rho) = -\mu^{\mathfrak{s}}h_{\mathfrak{s},\ell}\rho^{\ell+\mathfrak{s}}\int_\rho^{+\infty}\left\{r^{-2\ell-2}h_{\mathfrak{s},\ell}^{-2}\mu^{-1-\mathfrak{s}}\left[\mathfrak{c}_{+\mathfrak{s},\ell} + \int_{2M}^{\rho_1} r^{\ell-\mathfrak{s}-1}\mathcal{H}[\psi_{+\mathfrak{s},\ell}](\tau_0,\rho_2)\mathrm{d}\rho_2\right]\right\}\mathrm{d}\rho_1, \tag{5.5}$$

where the constants $\mathfrak{c}_{+\mathfrak{s},\ell}$ are

$$\mathfrak{c}_{+0,\ell} = 0, \tag{5.6a}$$

$$\mathfrak{c}_{+1,\ell} = \frac{(2M)^{\ell-1}\mathcal{H}[\psi_{+\mathfrak{s},\ell}]\big|_{\rho=2M}}{\ell(\ell+1)}, \tag{5.6b}$$

$$\mathfrak{c}_{+2,\ell} = \frac{2(2M)^{\ell-2}}{3(\ell+2)(\ell+1)\ell(\ell-1)}\big((4\ell^2+\ell+3)\mathcal{H}[\psi_{+\mathfrak{s},\ell}]\big|_{\rho=2M} - 6M\partial_\rho\mathcal{H}[\psi_{+\mathfrak{s},\ell}]\big|_{\rho=2M}\big). \tag{5.6c}$$

Remark 5.2. By equations (5.2) and (5.4), $g_{+\mathfrak{s},\ell}$ satisfies the same TME as the one of $\psi_{+\mathfrak{s},\ell}$, that is, $g_{+\mathfrak{s},\ell}Y^{\mathfrak{s}}_{m,\ell}(\cos\theta)e^{im\phi}$ is also a solution to the TME (2.15) for $s=+\mathfrak{s}$.

Proof. It suffices to uniquely determine $g_{+\mathfrak{s},\ell}$ on Σ_{τ_0}, which means, we need to solve $g_{+\mathfrak{s},\ell}$ by the following equation on Σ_{τ_0}

$$\partial_\rho\left(r^{2\ell+2}\mu^{1+\mathfrak{s}}h_{\mathfrak{s},\ell}^2\partial_\rho(\tilde{h}_{\mathfrak{s},\ell}^{-1}g_{+\mathfrak{s},\ell})\right) = r^{\ell-\mathfrak{s}-1}\mathcal{H}[\psi_{+\mathfrak{s},\ell}] \tag{5.7}$$

with the asymptotic condition (5.3) and the smoothness up to $\rho=2M$, where $\tilde{h}_{\mathfrak{s},\ell} = \mu^{\mathfrak{s}}h_{\mathfrak{s},\ell}r^{\ell+\mathfrak{s}}$. By integrating (5.7) from ρ_1 to ρ_2 where $2M<\rho_1<\rho_2$, we get

$$r^{2\ell+2}\mu^{1+\mathfrak{s}}h_{\mathfrak{s},\ell}^2\partial_\rho(\tilde{h}_{\mathfrak{s},\ell}^{-1}g_{+\mathfrak{s},\ell})(\rho_2) - r^{2\ell+2}\mu^{1+\mathfrak{s}}h_{\mathfrak{s},\ell}^2\partial_\rho(\tilde{h}_{\mathfrak{s},\ell}^{-1}g_{+\mathfrak{s},\ell})(\rho_1) = \int_{\rho_1}^{\rho_2} r^{\ell-\mathfrak{s}-1}\mathcal{H}[\psi_{+\mathfrak{s},\ell}](\rho)\mathrm{d}\rho. \tag{5.8}$$

Taking $\rho_1\to 2M$, and by the fact that $\mathcal{H}[\psi_{+\mathfrak{s},\ell}]$ is smooth away from $\rho=2M$ (one can easy check this by the definition of $\mathcal{H}[\psi_{+\mathfrak{s},\ell}]$ in (5.1) and the smoothness of $\psi_{+\mathfrak{s},\ell}$ on Σ_{τ_0}), we have for any $\rho>2M$,

$$r^{2\ell+2}\mu^{1+\mathfrak{s}}h_{\mathfrak{s},\ell}^2\partial_\rho(\tilde{h}_{\mathfrak{s},\ell}^{-1}g_{+\mathfrak{s},\ell}) - \int_{2M}^{\rho}\big(r^{\ell-\mathfrak{s}-1}\mathcal{H}[\psi_{+\mathfrak{s},\ell}]\big)(\rho_1)\mathrm{d}\rho_1 = \mathfrak{c}_{+\mathfrak{s},\ell}, \tag{5.9}$$

for some constant $\mathfrak{c}_{+\mathfrak{s},\ell}$ to be determined. We integrate (5.9) again and obtain for $2M<\rho_1<\rho_2$,

$$\tilde{h}_{\mathfrak{s},\ell}^{-1}g_{+\mathfrak{s},\ell}(\rho_2) - \tilde{h}_{\mathfrak{s},\ell}^{-1}g_{+\mathfrak{s},\ell}(\rho_1) = \int_{\rho_1}^{\rho_2} r^{-2\ell-2}\mu^{-1-\mathfrak{s}}h_{\mathfrak{s},\ell}^{-2}\Big\{\mathfrak{c}_{+\mathfrak{s},\ell} + \int_{2M}^{\rho}\big(r^{\ell-\mathfrak{s}-1}\mathcal{H}[\psi_{+\mathfrak{s},\ell}]\big)(\rho_3)\mathrm{d}\rho_3\Big\}\mathrm{d}\rho. \tag{5.10}$$

Thus, one easily finds that $g_{+\mathfrak{s},\ell}$ is smooth in $(2M,+\infty)$.

We shall now determine the values of the constants $\mathfrak{c}_{+\mathfrak{s},\ell}$ such that $g_{+\mathfrak{s},\ell}$ is smooth up to and including $\rho=2M$.

First, consider $\mathfrak{s}=0$ case. By requiring $g_{+\mathfrak{s},\ell}$ to be C^1 at $\rho=2M$, one has from (5.9) that

$$\mathfrak{c}_{+\mathfrak{s},\ell} = \lim_{\rho\to 2M} r^{2\ell+2}\mu^{1+\mathfrak{s}}h_{\mathfrak{s},\ell}^2\partial_\rho(\tilde{h}_{\mathfrak{s},\ell}^{-1}g_{+\mathfrak{s},\ell}) = 0. \tag{5.11}$$

Then, by (5.10) and since $\mu^{-1}\int_{2M}^{\rho}\big(r^{\ell-\mathfrak{s}-1}\mathcal{H}[\psi_{+\mathfrak{s},\ell}]\big)(\rho_1)\mathrm{d}\rho_1$ is smooth at $\rho=2M$, $g_{+\mathfrak{s},\ell}$ can be smoothly extended to $\rho=2M$. In (5.10), taking $\rho_1\to 2M$, $\rho_2\to+\infty$, and by (5.3), we have

$$\lim_{\rho_1\to 2M}\big(\tilde{h}_{\mathfrak{s},\ell}^{-1}g_{+\mathfrak{s},\ell}\big)(\rho_1)=-\int_{2M}^{+\infty}\Big\{r^{-2\ell-2}h_{\mathfrak{s},\ell}^{-2}\mu^{-1}\int_{2M}^{\rho}\big(r^{\ell-1}\mathcal{H}[\psi_{+\mathfrak{s},\ell}]\big)(\rho_3)\mathrm{d}\rho_3\Big\}\mathrm{d}\rho, \tag{5.12}$$

where the RHS of (5.12) is integrable. Together with (5.10) and (5.12), we get

$$\tilde{h}_{\mathfrak{s},\ell}^{-1}g_{+\mathfrak{s},\ell}(\rho)=-\int_{\rho}^{+\infty}\Big\{r^{-2\ell-2}h_{\mathfrak{s},\ell}^{-2}\mu^{-1}\int_{2M}^{\rho_1}\big(r^{\ell-1}\mathcal{H}[\psi_{+\mathfrak{s},\ell}]\big)(\rho_2)\mathrm{d}\rho_2\Big\}\mathrm{d}\rho_1. \tag{5.13}$$

This is (5.5) in the case $\mathfrak{s}=0$.

Second, we show that $g_{+\mathfrak{s},\ell}$ can be continuously extended to $2M$ with any $\mathfrak{c}_{+\mathfrak{s},\ell}$ for $\mathfrak{s}=1,2$. In fact, for $\mathfrak{s}=1,2$,

$$\begin{aligned}\lim_{\rho\to 2M}g_{+\mathfrak{s},\ell}&=\lim_{\rho\to 2M}\frac{\tilde{h}_{\mathfrak{s},\ell}^{-1}g_{+\mathfrak{s},\ell}}{\tilde{h}_{\mathfrak{s},\ell}^{-1}}\\&=\lim_{\rho\to 2M}\frac{\partial_\rho(\tilde{h}_{\mathfrak{s},\ell}^{-1}g_{+\mathfrak{s},\ell})}{-\tilde{h}_{\mathfrak{s},\ell}^{-2}\partial_\rho\tilde{h}_{\mathfrak{s},\ell}}\\&=\lim_{\rho\to 2M}-\frac{r^{2(\mathfrak{s}-1)}\mu^{\mathfrak{s}-1}(\mathfrak{c}_{+\mathfrak{s},\ell}+\int_{2M}^{\rho}\big(r^{\ell-\mathfrak{s}-1}\mathcal{H}[\psi_{+\mathfrak{s},\ell}]\big)(\rho_1)\mathrm{d}\rho_1)}{\partial_\rho\tilde{h}_{+\mathfrak{s},\ell}}\\&=-\,\mathfrak{c}_{+\mathfrak{s},\ell}\mathfrak{s}^{-1}(h_{\mathfrak{s},\ell}(2M))^{-1}(2M)^{\mathfrak{s}-\ell-1}:=\mathfrak{g}_{+\mathfrak{s},\ell}^{(0)},\end{aligned} \tag{5.14}$$

where we have used the L'Hôpital's rule in the second step and (5.9) in the third step. Further, from (5.9), we have, for $\rho_2>\rho_1>2M$,

$$\partial_\rho\big(\tilde{h}_{\mathfrak{s},\ell}^{-1}(g_{+\mathfrak{s},\ell}-\mathfrak{g}_{+\mathfrak{s},\ell}^{(0)})\big)=r^{-2\ell-2}h_{\mathfrak{s},\ell}^{-2}\mu^{-1-\mathfrak{s}}W_1(\rho), \tag{5.15}$$

where

$$W_1(\rho)\triangleq\mathfrak{c}_{+\mathfrak{s},\ell}+\mathfrak{g}_{+\mathfrak{s},\ell}^{(0)}r^{2-2\mathfrak{s}}\mu^{1-\mathfrak{s}}\partial_\rho(\tilde{h}_{\mathfrak{s},\ell})+\int_{2M}^{\rho}r^{\ell-\mathfrak{s}-1}\mathcal{H}[\psi_{+\mathfrak{s},\ell}](\rho_1)\mathrm{d}\rho_1. \tag{5.16}$$

We now determine the values of $\mathfrak{c}_{+\mathfrak{s},\ell}$ such that $g_{+\mathfrak{s},\ell}$ can be smoothly extended to $2M$ for $\mathfrak{s}=1$. For $\rho_2>\rho_1>2M$, we have by integrating (5.15) that

$$\big(\tilde{h}_{\mathfrak{s},\ell}^{-1}(g_{+\mathfrak{s},\ell}-\mathfrak{g}_{+\mathfrak{s},\ell}^{(0)})\big)(\rho_1)=\big(\tilde{h}_{\mathfrak{s},\ell}^{-1}(g_{+\mathfrak{s},\ell}-\mathfrak{g}_{+\mathfrak{s},\ell}^{(0)})\big)(\rho_2)-\int_{\rho_1}^{\rho_2}r^{-2\ell-2}h_{\mathfrak{s},\ell}^{-2}\mu^{-2}W_1(\rho)\mathrm{d}\rho. \tag{5.17}$$

From (5.16), $W_1(\rho)=\big(\mathfrak{g}_{+\mathfrak{s},\ell}^{(0)}\partial_\rho^2(\tilde{h}_{\mathfrak{s},\ell})(2M)+(2M)^{\ell-2}\mathcal{H}[\psi_{+\mathfrak{s},\ell}](2M)\big)\cdot(\rho-2M)+O((\rho-2M)^2)$ as $\rho\to 2M$. Further, notice from equation (3.111) solved by $h_{\mathfrak{s},\ell}$ that $\partial_\rho^2\tilde{h}_{\mathfrak{s},\ell}=\ell(\ell+1)r^{\ell-1}h_{\mathfrak{s},\ell}$. Hence, by taking $\mathfrak{g}_{+\mathfrak{s},\ell}^{(0)}=-\frac{(2M)^{\ell-2}\mathcal{H}[\psi_{+\mathfrak{s},\ell}](2M)}{\partial_\rho^2(\tilde{h}_{\mathfrak{s},\ell})(2M)}=-\frac{\mathcal{H}[\psi_{+\mathfrak{s},\ell}](2M)}{2\ell(\ell+1)Mh_{\mathfrak{s},\ell}(2M)}$, or equivalently, from (5.14), taking

$$\mathfrak{c}_{+\mathfrak{s},\ell}=\frac{(2M)^{\ell-1}\mathcal{H}[\psi_{+\mathfrak{s},\ell}](2M)}{\ell(\ell+1)}, \tag{5.18}$$

then $\mu^{-2}W_1$ is smooth in $[2M,+\infty)$, which thus yields $g_{+\mathfrak{s},\ell}$ is smooth at $2M$ from (5.15). Furthermore, By (5.15) and the assumption (5.3), we have

$$\lim_{\rho\to 2M}\tilde{h}_{\mathfrak{s},\ell}^{-1}(g_{+\mathfrak{s},\ell}-\mathfrak{g}_{+\mathfrak{s},\ell}^{(0)})=-\int_{2M}^{+\infty}r^{-2\ell-2}h_{\mathfrak{s},\ell}^{-2}\mu^{-2}W_1\mathrm{d}\rho. \tag{5.19}$$

Using (5.17) again, we have

$$\tilde{h}_{\mathfrak{s},\ell}^{-1}(g_{+\mathfrak{s},\ell}-\mathfrak{g}_{+\mathfrak{s},\ell}^{(0)})=-\int_{\rho}^{+\infty}\big(r^{-2\ell-2}h_{\mathfrak{s},\ell}^{-2}\mu^{-2}W_1\big)(\rho_1)\mathrm{d}\rho_1. \tag{5.20}$$

This corresponds to (5.5) in the case $\mathfrak{s}=1$.

Finally, we discuss the case $\mathfrak{s}=2$. Similarly as proving the continuity of $g_{+\mathfrak{s},\ell}$ at $2M$ in the case $\mathfrak{s}=1$ via equation (5.14) , we can show that $g_{+\mathfrak{s},\ell}$ is actually C^1 at $2M$ for any constant $\mathfrak{c}_{+\mathfrak{s},\ell}$. In fact, for $\mathfrak{s}=2$,

$$\begin{aligned}\lim_{\rho\to 2M}\frac{g_{+\mathfrak{s},\ell}(\rho)-\mathfrak{g}^{(0)}_{+\mathfrak{s},\ell}}{\rho-2M}&=\lim_{\rho\to 2M}\frac{\tilde{h}^{-1}_{\mathfrak{s},\ell}(g_{+\mathfrak{s},\ell}-\mathfrak{g}^{(0)}_{+\mathfrak{s},\ell})}{\mu r\tilde{h}^{-1}_{\mathfrak{s},\ell}}\\&=\lim_{\rho\to 2M}\frac{\partial_\rho(\tilde{h}^{-1}_{\mathfrak{s},\ell}(g_{+\mathfrak{s},\ell}-\mathfrak{g}^{(0)}_{+\mathfrak{s},\ell}))}{\partial_\rho(\mu r\tilde{h}^{-1}_{\mathfrak{s},\ell})}\\&=\lim_{\rho\to 2M}\frac{r^{-2\ell-2}\mu^{-1-\mathfrak{s}}h^{-2}_{\mathfrak{s},\ell}W_1(\rho)}{\partial_\rho(\mu r\tilde{h}^{-1}_{\mathfrak{s},\ell})}\\&=\frac{r^{\ell-\mathfrak{s}-1}\mathcal{H}[\psi_{+\mathfrak{s},\ell}]+\mathfrak{g}^{(0)}_{+\mathfrak{s},\ell}\partial_\rho(r^{-2}\mu^{-1}\partial_\rho(\tilde{h}_{+\mathfrak{s},\ell}))}{-2Mr^{\ell-2}h_{\mathfrak{s},\ell}}\Big|_{\rho=2M}:=\mathfrak{g}^{(1)}_{+\mathfrak{s},\ell},\end{aligned}\tag{5.21}$$

where we have used the L'Hôpital's rule in the second step and (5.15) in the third step. Furthermore, by (5.9), we have, for $\rho_2>\rho_1>2M$,

$$\begin{aligned}\mu r\tilde{h}^{-1}_{\mathfrak{s},\ell}\Big(\frac{g_{+\mathfrak{s},\ell}-\mathfrak{g}^{(0)}_{+\mathfrak{s},\ell}}{\mu r}-\mathfrak{g}^{(1)}_{+\mathfrak{s},\ell}\Big)(\rho_1)=&\mu r\tilde{h}^{-1}_{\mathfrak{s},\ell}\Big(\frac{g_{+\mathfrak{s},\ell}-\mathfrak{g}^{(0)}_{+\mathfrak{s},\ell}}{\mu r}-\mathfrak{g}^{(1)}_{+\mathfrak{s},\ell}\Big)(\rho_2)\\&-\int_{\rho_1}^{\rho_2}r^{-2\ell-2}\mu^{-3}h^{-2}_{\mathfrak{s},\ell}W_2(\rho)\mathrm{d}\rho,\end{aligned}\tag{5.22}$$

where

$$\begin{aligned}W_2(\rho)\triangleq{}&\mathfrak{c}_{+\mathfrak{s},\ell}+\mathfrak{g}^{(0)}_{+\mathfrak{s},\ell}r^{2-2\mathfrak{s}}\mu^{1-\mathfrak{s}}\partial_\rho(\tilde{h}_{\mathfrak{s},\ell})\\&-\mathfrak{g}^{(1)}_{+\mathfrak{s},\ell}r^{2\ell+2}\mu^{1+\mathfrak{s}}h^2_{\mathfrak{s},\ell}\partial_\rho(\mu r\tilde{h}^{-1}_{\mathfrak{s},\ell})+\int_{2M}^{\rho}\big(r^{\ell-\mathfrak{s}-1}\mathcal{H}[\psi_{+\mathfrak{s},\ell}]\big)(\rho_1)\mathrm{d}\rho_1.\end{aligned}\tag{5.23}$$

Using $\tilde{h}_{\mathfrak{s},\ell}=\mu^{\mathfrak{s}}h_{\mathfrak{s},\ell}r^{\ell+\mathfrak{s}}$, we obtain for $\rho-2M$ small,

$$\begin{aligned}W_2(\rho)={}&\mathfrak{c}_{+\mathfrak{s},\ell}+\mathfrak{g}^{(0)}_{+\mathfrak{s},\ell}\big(r^{-2}\mu^{-1}\partial_\rho(\tilde{h}_{\mathfrak{s},\ell})\big)(2M)+(\rho-2M)\cdot\Big[\mathfrak{g}^{(0)}_{+\mathfrak{s},\ell}\big(\partial_\rho(r^{-2}\mu^{-1}\partial_\rho(\tilde{h}_{\mathfrak{s},\ell}))\big)(2M)\\&+\mathfrak{g}^{(1)}_{+\mathfrak{s},\ell}(2M)^{\ell-1}h_{\mathfrak{s},\ell}(2M)+\big(r^{\ell-3}\mathcal{H}[\psi_{+\mathfrak{s},\ell}]\big)(2M)\Big]\\&+\frac{1}{2}(\rho-2M)^2\cdot\Big[\mathfrak{g}^{(0)}_{+\mathfrak{s},\ell}\big(\partial^2_\rho(r^{-2}\mu^{-1}\partial_\rho(\tilde{h}_{\mathfrak{s},\ell}))\big)(2M)+\big(\partial_\rho(r^{\ell-3}\mathcal{H}[\psi_{+\mathfrak{s},\ell}])\big)(2M)\\&-2\mathfrak{g}^{(1)}_{+\mathfrak{s},\ell}\big(\partial_\rho(r^{2\ell+1}\mu^2h^2_{\mathfrak{s},\ell}\partial_\rho(\mu r\tilde{h}^{-1}_{\mathfrak{s},\ell}))\big)(2M)\Big]+O((\rho-2M)^3).\end{aligned}\tag{5.24}$$

We utilize then equations (5.21) and (5.14) to find the first two lines on the RHS of the above equation are vanishing identically. By requiring the $(\rho-2M)^2$ terms vanish, one then finds that $\mu^{-3}W_2$ is smooth in $[2M,+\infty)$, and, from (5.22), $g_{+\mathfrak{s},\ell}$ is thus smooth in $[2M,+\infty)$. This requirement is equivalent to setting the coefficient of $\frac{1}{2}(\rho-2M)^2$ term on the RHS of (5.24) to be zero, and, plugging in the expression (5.21) of $\mathfrak{g}^{(1)}_{+\mathfrak{s},\ell}$, it reduces to imposing the following condition

$$\begin{aligned}&\mathfrak{g}^{(0)}_{+\mathfrak{s},\ell}\big(\partial^2_\rho(r^{-2}\mu^{-1}\partial_\rho(\tilde{h}_{\mathfrak{s},\ell}))\big)(2M)+\big(\partial_\rho(r^{\ell-3}\mathcal{H}[\psi_{+\mathfrak{s},\ell}])\big)(2M)\\&=2\mathfrak{g}^{(1)}_{+\mathfrak{s},\ell}\big(\partial_\rho(r^{\ell-1}h_{\mathfrak{s},\ell}-\mu^{-1}r^{-2}\partial_\rho\tilde{h}_{\mathfrak{s},\ell})\big)(2M)\\&=2\big(\partial_\rho(r^{\ell-1}h_{\mathfrak{s},\ell}-\mu^{-1}r^{-2}\partial_\rho\tilde{h}_{\mathfrak{s},\ell})\big)(2M)\times\frac{r^{\ell-\mathfrak{s}-1}\mathcal{H}[\psi_{+\mathfrak{s},\ell}]+\mathfrak{g}^{(0)}_{+\mathfrak{s},\ell}\partial_\rho(r^{-2}\mu^{-1}\partial_\rho(\tilde{h}_{+\mathfrak{s},\ell}))}{-2Mr^{\ell-2}h_{\mathfrak{s},\ell}}\Big|_{\rho=2M}.\end{aligned}$$

We use equation (3.111) solved by $h_{\mathfrak{s},\ell}$, which in particular implies $\partial_\rho(\mu^{-1}r^{-2}\partial_\rho\tilde{h}_{\mathfrak{s},\ell})=(\ell+2)(\ell-1)r^{\ell-2}h_{\mathfrak{s},\ell}$ and $6M(\partial_\rho h_{\mathfrak{s},\ell})(2M)=\ell(\ell-2)h_{\mathfrak{s},\ell}(2M)$, to solve this equation, and then use (5.14) to obtain the values of $\mathfrak{g}^{(0)}_{+\mathfrak{s},\ell}$ and $\mathfrak{c}_{+\mathfrak{s},\ell}$:

$$\mathfrak{g}^{(0)}_{+\mathfrak{s},\ell}=-\frac{1}{(\ell+2)(\ell-1)}\Big(\frac{r^2h_{\mathfrak{s},\ell}\partial_\rho(r^{-1}\mathcal{H}[\psi_{+\mathfrak{s},\ell}])+((\ell-2\ell^2)h_{\mathfrak{s},\ell}+2r\partial_\rho h_{\mathfrak{s},\ell})\mathcal{H}[\psi_{+\mathfrak{s},\ell}]}{h_{\mathfrak{s},\ell}\big[3r^2\partial_\rho h_{\mathfrak{s},\ell}+(\ell-2\ell^2)h_{\mathfrak{s},\ell}r\big]}\Big)\Big|_{\rho=2M},$$

$$\mathfrak{c}_{+\mathfrak{s},\ell} = \frac{2(2M)^{\ell-1}}{(\ell+2)(\ell-1)}\bigg(\frac{r^2 h_{\mathfrak{s},\ell}\partial_\rho(r^{-1}\mathcal{H}[\psi_{+\mathfrak{s},\ell}]) + ((\ell-2\ell^2)h_{\mathfrak{s},\ell} + 2r\partial_\rho h_{\mathfrak{s},\ell})\mathcal{H}[\psi_{+\mathfrak{s},\ell}]}{3r^2\partial_\rho h_{\mathfrak{s},\ell} + (\ell-2\ell^2)h_{\mathfrak{s},\ell}r}\bigg)\bigg|_{\rho=2M}$$
$$= \frac{2(2M)^{\ell-2}}{3(\ell+2)(\ell+1)\ell(\ell-1)}\big((4\ell^2+\ell+3)\mathcal{H}[\psi_{+\mathfrak{s},\ell}](2M) - 6M(\partial_\rho\mathcal{H}[\psi_{+\mathfrak{s},\ell}])(2M)\big). \tag{5.25}$$

By (5.3), we have

$$\lim_{\rho\to 2M}\mu r\tilde{h}_{\mathfrak{s},\ell}^{-1}\Big(\frac{g_{+\mathfrak{s},\ell} - \mathfrak{g}_{+\mathfrak{s},\ell}^{(0)}}{\mu r} - \mathfrak{g}_{+\mathfrak{s},\ell}^{(1)}\Big)(\rho) = -\int_{2M}^{+\infty} r^{-2\ell-2}\mu^{-3}h_{\mathfrak{s},\ell}^{-2}W_2(\rho)\mathrm{d}\rho, \tag{5.26}$$

and then,

$$\tilde{h}_{\mathfrak{s},\ell}^{-1}(g_{+\mathfrak{s},\ell} - \mathfrak{g}_{+\mathfrak{s},\ell}^{(0)} - \mu r\mathfrak{g}_{+\mathfrak{s},\ell}^{(1)})(\rho) = -\int_{\rho}^{+\infty}\big(r^{-2\ell-2}\mu^{-3}h_{\mathfrak{s},\ell}^{-2}W_2\big)(\rho_1)\mathrm{d}\rho_1, \tag{5.27}$$

which is equality (5.5) for $\mathfrak{s}=2$. □

As an immediate consequence, we compute the derivatives of $g_{+\mathfrak{s},\ell}$ here in terms of $\psi_{+\mathfrak{s},\ell}$.

Lemma 5.3. *Assume $(r^2\partial_\rho)^{i'}(\mu^{-\mathfrak{s}}r\psi_{+\mathfrak{s},\ell})$, $0\le i'\le \ell-\mathfrak{s}+1$, are bounded near infinity. Then for any $0\le i\le \ell-\mathfrak{s}$,*

$$\begin{aligned}
&(r^2\partial_\rho)^i(\mu^{-\mathfrak{s}}rg_{+\mathfrak{s},\ell})\\
&= -\sum_{j=0}^{i}\sum_{n=0}^{j} C_i^j C_j^n[(r^2\partial_\rho)^{i-j}h_{\mathfrak{s},\ell}]\rho^{\ell+\mathfrak{s}+1+j}\int_\rho^\infty\bigg(\Big[(r^2\partial_\rho)^{j-n}(h_{\mathfrak{s},\ell}^{-2}\mu^{-1-\mathfrak{s}})\Big]r^{-2\ell-j+n-2}\\
&\qquad\times\bigg\{\mathfrak{a}_{\ell-\mathfrak{s},n} + \int_{2M}^{r}\big(r^{\ell-\mathfrak{s}-n-1}(r^2\partial_\rho)^n\mathcal{H}[\psi_{+\mathfrak{s},\ell}]\big)(\rho_1)\mathrm{d}\rho_1\bigg\}\bigg)(\rho_2)\mathrm{d}\rho_2\\
&\quad - \mathfrak{c}_{+\mathfrak{s},\ell}(r^2\partial_\rho)^i\Big\{h_{\mathfrak{s},\ell}\rho^{\ell+\mathfrak{s}+1}\int_\rho^\infty r^{-2\ell-2}h_{\mathfrak{s},\ell}^{-2}\mu^{-1-\mathfrak{s}}\mathrm{d}r\Big\},
\end{aligned} \tag{5.28}$$

where $\mathfrak{a}_{\ell-\mathfrak{s},0}=0$, $\mathfrak{a}_{\ell-\mathfrak{s},1}=(2M)^{\ell-\mathfrak{s}}\mathcal{H}[\psi_{+\mathfrak{s},\ell}](2M)$, and for $2\le n\le \ell-\mathfrak{s}$,

$$\begin{aligned}
\mathfrak{a}_{\ell-\mathfrak{s},n} =& (2M)^{\ell-\mathfrak{s}-n+1}(r^2\partial_\rho)^{n-1}\mathcal{H}[\psi_{+\mathfrak{s},\ell}](2M)\\
&+\sum_{j=0}^{n-2}(-1)^{n-j-1}\frac{(\ell-\mathfrak{s}-(j+1))!}{(\ell-\mathfrak{s}-n)!}(2M)^{\ell-\mathfrak{s}-j}(r^2\partial_\rho)^j\mathcal{H}[\psi_{+\mathfrak{s},\ell}](2M).
\end{aligned} \tag{5.29}$$

Proof. Let $E(\rho) = h_{\mathfrak{s},\ell}^{-2}\mu^{-1-\mathfrak{s}}\rho^{-\ell+\mathfrak{s}}\int_{2M}^{\rho}\big(r^{\ell-\mathfrak{s}-1}\mathcal{H}[\psi_{+\mathfrak{s},\ell}]\big)(\rho_1)\mathrm{d}\rho_1$, then $E(\rho)$ is bounded near infinity by assumption. By this definition and equation (5.5), we have

$$\begin{aligned}
\mu^{-\mathfrak{s}}rg_{+\mathfrak{s},\ell} =& -h_{\mathfrak{s},\ell}\rho^{\ell+\mathfrak{s}+1}\int_\rho^\infty\big(r^{-\ell-\mathfrak{s}-2}E\big)(\rho_1)\mathrm{d}\rho_1\\
&-\mathfrak{c}_{+\mathfrak{s},\ell}h_{\mathfrak{s},\ell}\rho^{\ell+\mathfrak{s}+1}\int_\rho^\infty\big(r^{-2\ell-2}h_{\mathfrak{s},\ell}^{-2}\mu^{-1-\mathfrak{s}}\big)(\rho_1)\mathrm{d}\rho_1.
\end{aligned} \tag{5.30}$$

This proves the $i=0$ case of (5.28). One can perform integration by parts and obtain

$$\begin{aligned}
&r^2\partial_\rho\bigg(\rho^{-\ell+\mathfrak{s}}\int_{2M}^{\rho}\big(r^{\ell-\mathfrak{s}-1}\mathcal{H}[\psi_{+\mathfrak{s},\ell}]\big)(\rho_1)\mathrm{d}\rho_1\bigg)\\
&= (2M)^{\ell-s}\mathcal{H}[\psi_{+\mathfrak{s},\ell}](2M)\rho^{-\ell+\mathfrak{s}+1} + \rho^{-\ell+\mathfrak{s}+1}\int_{2M}^{\rho}\big(r^{\ell-\mathfrak{s}-2}r^2\partial_\rho\mathcal{H}[\psi_{+\mathfrak{s},\ell}]\big)(\rho_1)\mathrm{d}\rho_1.
\end{aligned} \tag{5.31}$$

A simple induction then yields that for any $j\in\mathbb{N}^+$,

$$\begin{aligned}
&(r^2\partial_\rho)^j\bigg\{\rho^{-\ell+\mathfrak{s}}\int_{2M}^{\rho}\big(r^{\ell-\mathfrak{s}-1}\mathcal{H}[\psi_{+\mathfrak{s},\ell}]\big)(\rho_1)\mathrm{d}\rho_1\bigg\}\\
&= \rho^{-\ell+\mathfrak{s}+j}\bigg\{\mathfrak{a}_{\ell-\mathfrak{s},j} + \int_{2M}^{\rho}\big(r^{\ell-\mathfrak{s}-j-1}(r^2\partial_\rho)^j\mathcal{H}[\psi_{+\mathfrak{s},\ell}]\big)(\rho_1)\mathrm{d}\rho_1\bigg\}.
\end{aligned} \tag{5.32}$$

Further, we have for any $j \leq \ell - \mathfrak{s}$,

$$(r^2\partial_\rho)^j \bigg\{ \rho^{\ell+\mathfrak{s}+1} \int_\rho^\infty \big(r^{-\ell-\mathfrak{s}-2}E\big)(\rho_1)\mathrm{d}\rho_1 \bigg\} = \rho^{\ell+\mathfrak{s}+1+j} \int_\rho^\infty \big(r^{-(\ell+\mathfrak{s}+2+j)}(r^2\partial_\rho)^j E\big)(\rho_1)\mathrm{d}\rho_1, \quad (5.33)$$

where we have used integration by parts and $\lim_{\rho\to\infty} r^{-\ell-\mathfrak{s}-1+j}(r^2\partial_\rho)^j E(\rho) = 0$ which yields the boundary term at ∞ vanishes. By taking $(r^2\partial_\rho)^i$ derivative over equation (5.30) and utilizing equations (5.32) and (5.33), (5.28) then follows. □

5.2. **Newman–Penrose constant of the time integral $(g_{+\mathfrak{s}})_{m,\ell}$.** Consider a fixed (m,ℓ) mode of the spin $+\mathfrak{s}$ component. Recall from Remark 5.2 that $(g_{+\mathfrak{s}})_{m,\ell} Y^{\mathfrak{s}}_{m,\ell}(\cos\theta)e^{im\phi}$ is also a solution to the TME, hence $\Phi^{(0)}_{(g_{+\mathfrak{s}})_{m,\ell}} = \mu^{-\mathfrak{s}} r (g_{+\mathfrak{s}})_{m,\ell}$ is a solution to (3.9). One can then similarly define $\Phi^{(i)}_{(g_{+\mathfrak{s}})_{m,\ell}}$ and $\widetilde{\Phi}^{(i)}_{(g_{+\mathfrak{s}})_{m,\ell}}$ as in Definition 3.6. As a result, the scalars $\Phi^{(i)}_{(g_{+\mathfrak{s}})_{m,\ell}}$ and $\widetilde{\Phi}^{(i)}_{(g_{+\mathfrak{s}})_{m,\ell}}$ solve equations (3.10) and (3.11), respectively, that is,

$$\begin{aligned} &-\mu Y \hat{\mathcal{V}} \Phi^{(i)}_{(g_{+\mathfrak{s}})_{m,\ell}} + (\mathring{\eth}'\mathring{\eth} + f_{\mathfrak{s},i,1})\Phi^{(i)}_{(g_{+\mathfrak{s}})_{m,\ell}} \\ &\qquad + f_{\mathfrak{s},i,2}(r-3M)r^{-2}\hat{\mathcal{V}}\Phi^{(i)}_{(g_{+\mathfrak{s}})_{m,\ell}} - 6 f_{\mathfrak{s},i,3} M r^{-1} \Phi^{(i)}_{(g_{+\mathfrak{s}})_{m,\ell}} + g_{\mathfrak{s},i} M \Phi^{(i-1)}_{(g_{+\mathfrak{s}})_{m,\ell}} = 0, \end{aligned} \quad (5.34a)$$

$$-\mu Y \hat{\mathcal{V}} \widetilde{\Phi}^{(i)}_{(g_{+\mathfrak{s}})_{m,\ell}} + (\mathring{\eth}'\mathring{\eth} + f_{\mathfrak{s},i,1})\widetilde{\Phi}^{(i)}_{(g_{+\mathfrak{s}})_{m,\ell}} + f_{\mathfrak{s},i,2}(r-3M)r^{-2}\hat{\mathcal{V}}\widetilde{\Phi}^{(i)}_{(g_{+\mathfrak{s}})_{m,\ell}} + \sum_{j=0}^{i} h_{\mathfrak{s},i,j} \Phi^{(j)}_{(g_{+\mathfrak{s}})_{m,\ell}} = 0. \quad (5.34b)$$

Lemma 5.4. *Let $\mathfrak{a}_{\ell-\mathfrak{s},\ell-\mathfrak{s}}$, $\mathfrak{c}_{+\mathfrak{s},\ell}$, and $\mathcal{H}[\psi_{+\mathfrak{s},\ell}]$ be defined as in Lemma 5.3, equation (5.6), and equation (5.1), respectively. Assume on Σ_{τ_0} that the limits $\lim_{\rho\to\infty} r\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}$ and $\lim_{\rho\to\infty}(r^2\partial_\rho)^{i'}(\mu^{-\mathfrak{s}} r \psi_{+\mathfrak{s},\ell})$, $0 \leq i' \leq \ell - \mathfrak{s} + 1$, exist and are finite. Then, the limits $\{\lim_{\rho\to\infty}(r^2\partial_\rho)^i \Phi^{(0)}_{(g_{+\mathfrak{s}})_{m,\ell}}\}_{i=0,1,\dots,\ell-\mathfrak{s}}$ and $\{\lim_{\rho\to\infty}\Phi^{(i)}_{(g_{+\mathfrak{s}})_{m,\ell}}\}_{i=0,1,\dots,\ell-\mathfrak{s}}$ exist and are finite, and*

$$\begin{aligned} &\lim_{\rho\to\infty}(r^2\partial_\rho)^i \Phi^{(0)}_{(g_{+\mathfrak{s}})_{m,\ell}} \\ &= \sum_{j=0}^{i}\sum_{n=0}^{j} C_i^j C_j^m (-1)^{i-j+1}(i-j)! M^{i-j} h^{(i-j)}_{\mathfrak{s},\ell} (\ell+\mathfrak{s}+j+1)^{-1}(2\ell+j-n+1)^{-1} \\ &\quad \times \lim_{\rho\to\infty}\big[(r^2\partial_\rho)^{j-n}(h^{-2}_{\mathfrak{s},\ell}\mu^{-1-s})\big] \lim_{\rho\to\infty}(r^2\partial_\rho)^n \mathcal{H}[\psi_{+\mathfrak{s},\ell}], \quad \textit{for } 0 \leq i \leq \ell-\mathfrak{s}-1, \end{aligned} \quad (5.35a)$$

$$\begin{aligned} &\lim_{\rho\to\infty}(r^2\partial_\rho)^{\ell-\mathfrak{s}} \Phi^{(0)}_{(g_{+\mathfrak{s}})_{m,\ell}} \\ &= \sum_{j=0}^{i}\sum_{n=0}^{\min(j,\ell-\mathfrak{s}-1)} C_i^j C_j^n (-1)^{i-j+1}(i-j)! M^{i-j} h^{(i-j)}_{\mathfrak{s},\ell} (\ell+\mathfrak{s}+j+1)^{-1}(2\ell+j-n+1)^{-1} \\ &\quad \times \lim_{\rho\to\infty}\big[(r^2\partial_\rho)^{j-n}(h^{-2}_{\mathfrak{s},\ell}\mu^{-1-s})\big] \lim_{\rho\to\infty}(r^2\partial_\rho)^n \mathcal{H}[\psi_{+\mathfrak{s},\ell}] - (-1)^{\ell-\mathfrak{s}}\frac{\mathfrak{c}_{+\mathfrak{s},\ell}}{2\ell+1}(\ell-\mathfrak{s})! \\ &\quad - \frac{1}{2\ell+1}\mathfrak{a}_{\ell-\mathfrak{s},\ell-\mathfrak{s}} - \frac{1}{2\ell+1}\int_{2M}^\infty \rho^{-1}(r^2\partial_\rho)^{\ell-\mathfrak{s}}\mathcal{H}[\psi_{+\mathfrak{s},\ell}]\mathrm{d}\rho, \end{aligned} \quad (5.35b)$$

and the limits $\lim_{\rho\to\infty}\Phi^{(j)}_{(g_{+\mathfrak{s}})_{m,\ell}}$ with $j \in \{0,1,\dots,\ell-\mathfrak{s}\}$ are computed via (5.35) and

$$\Phi^{(i)}_{(g_{+\mathfrak{s}})_{m,\ell}} = \hat{\mathcal{V}}^i \Phi^{(0)}_{(g_{+\mathfrak{s}})_{m,\ell}} = (r^2\partial_\rho)^i \Phi^{(0)}_{(g_{+\mathfrak{s}})_{m,\ell}} + \mathcal{E}^i[\Phi^{(0)}_{+\mathfrak{s},\ell}], \quad (5.36)$$

where $\mathcal{E}^0[\Phi^{(0)}_{+\mathfrak{s},\ell}] = 0$ and $\mathcal{E}^i[\Phi^{(0)}_{+\mathfrak{s},\ell}] = r^2 H_{hyp}(r^2\partial_\rho)^{i-1}\Phi^{(0)}_{+\mathfrak{s},\ell} + \hat{\mathcal{V}}\mathcal{E}^{i-1}[\Phi^{(0)}_{+\mathfrak{s},\ell}]$.

The N–P constant of the time integral $(g_{+\mathfrak{s}})_{m,\ell}$ is

$$\mathbb{Q}^{(m,\ell)}_{(g_{+\mathfrak{s}})_{m,\ell}} = \frac{1}{2\ell+2}\sum_{j=0}^{\ell-\mathfrak{s}}\Big(\lim_{\rho\to\infty}(r h_{\mathfrak{s},\ell-\mathfrak{s},j}) \times \lim_{\rho\to\infty}\Phi^{(j)}_{(g_{+\mathfrak{s}})_{m,\ell}}\Big) - \frac{2}{2\ell+2}\lim_{\rho\to\infty} r\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}, \quad (5.37)$$

where the limits $\lim_{\rho\to\infty}(r h_{\mathfrak{s},\ell-\mathfrak{s},j})$ are calculated from Proposition 3.7.

Proof. One can easily see from (5.28) that for $i \le \ell - \mathfrak{s} - 1$, the limit $\lim_{\rho\to\infty}(r^2\partial_\rho)^i \Phi^{(0)}_{(g_{+\mathfrak{s}})_{m,\ell}}$ exists and is finite. For $i = \ell - \mathfrak{s}$, the limit of the RHS exists by direct inspection except for one sub-term with $n = k = i = \ell - \mathfrak{s}$:

$$\lim_{\rho\to\infty} -h_{\mathfrak{s},\ell}\rho^{2\ell+1}\int_\rho^\infty \left(h^{-2}_{\mathfrak{s},\ell}\mu^{-1-s}r^{-2l-2}\int_{2M}^r \left(r^{-1}(r^2\partial_\rho)^{\ell-\mathfrak{s}}\mathcal{H}[\psi_{+\mathfrak{s},\ell}]\right)(r')\mathrm{d}r'\right)(\rho_1)\mathrm{d}\rho_1. \tag{5.38}$$

By expanding $(r^2\partial_\rho)^{\ell-\mathfrak{s}}\mathcal{H}[\psi_{+\mathfrak{s},\ell}] = e_{\ell-\mathfrak{s}+1}(r)\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}} + \sum_{j=0}^{\ell-\mathfrak{s}} e_j(r)\widetilde{\Phi}^{(j)}_{+\mathfrak{s}}$, where we have utilized the expression (2.6) and equation (3.10) to rewrite $\partial_\tau\widetilde{\Phi}^{(j)}_{+\mathfrak{s}}$, it is manifest that all $\{e_j\}_{j=0,\dots,\ell-\mathfrak{s}+1}$ are $O(1)$ and comparing the coefficient of the highest order derivative term gives $e_{\ell-\mathfrak{s}+1} = 2 + O(r^{-1})$, thus the integral from $2M$ to r may lead to a $\log r$ growth and the limit (5.38) goes to infinity. However, a direct, but tedious, computation shows that $e_{\ell-\mathfrak{s}+1} = O(1)$ and $\{e_j\}_{j=0,\dots,\ell-\mathfrak{s}+1}$ are $O(r^{-1})$. Hence, $(r^2\partial_\rho)^{\ell-\mathfrak{s}}\mathcal{H} = (2r^{-1} + O(r^{-2}))r\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}} + \sum_{j=0}^{\ell-\mathfrak{s}} O(r^{-1})\widetilde{\Phi}^{(j)}_{+\mathfrak{s}} = (2r^{-1} + O(r^{-2}))r\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}} + \sum_{j=0}^{\ell-\mathfrak{s}} O(r^{-1})(r^2\partial_\rho)^j\Phi^{(0)}_{+\mathfrak{s}}$, where we have used in the second step the relation $\widetilde{\Phi}^{(j)}_{+\mathfrak{s}} = \sum_{n=0}^{j} O(1)(r^2\partial_\rho)^n\Phi^{(0)}_{+\mathfrak{s}}$ which in turn follows by the definition of $\widetilde{\Phi}^{(j)}_{+\mathfrak{s}}$ and an application of equation (3.10) to rewrite $\partial_\tau\Phi^{(n)}_{+\mathfrak{s}}$, and this then guarantees the existence and finiteness of the limit (5.38). Meanwhile, this also yields that the limits $\{\lim_{\rho\to\infty}\widetilde{\Phi}^{(j)}_{+\mathfrak{s}}\}_{j=0,1,\dots,\ell-\mathfrak{s}}$ exist.

A different way of proving the finiteness of the limit $\lim_{\rho\to\infty}\Phi^{(\ell-\mathfrak{s})}_{(g_{+\mathfrak{s}})_{m,\ell}}$ is to start with equation (5.34a) with $i = \ell - \mathfrak{s} - 1$, which is

$$-\mu Y\Phi^{(\ell-\mathfrak{s})}_{(g_{+\mathfrak{s}})_{m,\ell}} - 2\ell\Phi^{(\ell-\mathfrak{s}-1)}_{(g_{+\mathfrak{s}})_{m,\ell}} - 2\ell(r-3M)r^{-2}\Phi^{(\ell-\mathfrak{s})}_{(g_{+\mathfrak{s}})_{m,\ell}} - 6f_{\mathfrak{s},\ell-\mathfrak{s}-1,3}Mr^{-1}\Phi^{(\ell-\mathfrak{s}-1)}_{(g_{+\mathfrak{s}})_{m,\ell}} + g_{\mathfrak{s},\ell-\mathfrak{s}-1}M\Phi^{(\ell-\mathfrak{s}-2)}_{(g_{+\mathfrak{s}})_{m,\ell}} = 0.$$

Since $-\mu Y = -(2-\mu H)\partial_\tau + \mu\partial_\rho$ and $\partial_\tau\Phi^{(i)}_{(g_{+\mathfrak{s}})_{m,\ell}} = \Phi^{(i)}_{+\mathfrak{s}}$, this equation can be written as

$$\begin{aligned}\partial_\rho((\mu r^{-2})^\ell\Phi^{(\ell-\mathfrak{s})}_{(g_{+\mathfrak{s}})_{m,\ell}}) = \mu^{\ell-1}r^{-2\ell}\Big((2-\mu H)\Phi^{(\ell-\mathfrak{s})}_{+\mathfrak{s}} + 2\ell\Phi^{(\ell-\mathfrak{s}-1)}_{(g_{+\mathfrak{s}})_{m,\ell}} - 2(\ell-\mathfrak{s}-1)(\ell-1)(\ell+\mathfrak{s}-1)M\Phi^{(\ell-\mathfrak{s}-2)}_{(g_{+\mathfrak{s}})_{m,\ell}}\Big)\\ + 6f_{\mathfrak{s},\ell-\mathfrak{s}-1,3}M\mu^{\ell-1}r^{-2\ell-1}\Phi^{(\ell-\mathfrak{s}-1)}_{(g_{+\mathfrak{s}})_{m,\ell}}.\end{aligned} \tag{5.39}$$

By (2.6) and (2.8), we have, for any $j \ge 1$,

$$\hat{\mathcal{V}}^j = (r^2\partial_\rho + r^2H_{\text{hyp}}\partial_\tau)^j = \sum_{p+q\le j}(a^j_{p,q} + O(r^{-1}))(r^2\partial_\rho)^p\partial^q_\tau \qquad \text{as } r\to\infty, \tag{5.40}$$

and, for any $i \le \ell - \mathfrak{s} - 1$ and $n \ge 0$, $(r^2\partial_\rho)^i\partial^n_\tau\Phi^{(0)}_{(g_{+\mathfrak{s}})_{m,\ell}}$ are equal to $b_{i,n} + O(r^{-1})$ as $r\to\infty$, where $b_{i,n}$ are finite constants. Hence, the RHS of the first line of (5.39) equals $r^{-2\ell}(c + O(r^{-1}))$ for some constant c for r large. We now prove $c = 0$. In fact, the LHS of (5.39) equals

$$\begin{aligned}&2\ell\rho^{-2\ell-1}\cdot\rho^{2\ell+1}\int_\rho^\infty\left(h^{-2}_{\mathfrak{s},\ell}\mu^{-1-s}r^{-2l-2}\int_{2M}^r\left(r^{-1}(r^2\partial_\rho)^{\ell-\mathfrak{s}}\mathcal{H}[\psi_{+\mathfrak{s},\ell}]\right)(r')\mathrm{d}r'\right)(\rho_1)\mathrm{d}\rho_1\\ &-\rho^{-2\ell}\partial_\rho\left(\rho^{2\ell+1}\int_\rho^\infty\left(h^{-2}_{\mathfrak{s},\ell}\mu^{-1-s}r^{-2l-2}\int_{2M}^r\left(r^{-1}(r^2\partial_\rho)^{\ell-\mathfrak{s}}\mathcal{H}[\psi_{+\mathfrak{s},\ell}]\right)(r')\mathrm{d}r'\right)(\rho_1)\mathrm{d}\rho_1\right) + O(\rho^{-2\ell-1}).\end{aligned}$$

This yields $\lim\limits_{\rho\to\infty}\rho^{2\ell}\times$ [left-hand side of (5.39)] $= 0$, hence, by multiplying (5.39) by $r^{2\ell}$ and taking $r\to\infty$, we get $c = 0$. That is, right-hand side of (5.39) $= O(\rho^{-2\ell-1})$. We integrate equation (5.39) from ∞ and thus justify that the limit $\lim_{\rho\to\infty}\Phi^{(\ell-\mathfrak{s})}_{(g_{+\mathfrak{s}})_{m,\ell}}$ exists and is finite.

We turn to computing the limits $\lim_{\rho\to\infty}(r^2\partial_\rho)^i\Phi^{(0)}_{(g_{+\mathfrak{s}})_{m,\ell}}$ for $i = 0,\dots,\ell-\mathfrak{s}$. By Definition 3.34 of the function $h_{\mathfrak{s},\ell}$, we have $(r^2\partial_\rho)^{i-j}h_{\mathfrak{s},\ell} = (-1)^{i-j}(i-j)!M^{i-j}h^{(i-j)}_{\mathfrak{s},\ell}$. Meanwhile,

$$\begin{aligned}&\lim_{\rho\to\infty}\rho^{\ell+\mathfrak{s}+1+j}\int_\rho^\infty\left[(r^2\partial_\rho)^{j-n}(h^{-2}_{\mathfrak{s},\ell}\mu^{-1-\mathfrak{s}})\right]r^{-2\ell-j+n-2}\mathfrak{a}_{\ell-\mathfrak{s},n}\mathrm{d}r\\ &\quad= \begin{cases}0, & \text{if } n\le\ell-\mathfrak{s}-1;\\ \frac{1}{2\ell+1}\mathfrak{a}_{\ell-\mathfrak{s},\ell-\mathfrak{s}}, & \text{if } n=j=i=\ell-\mathfrak{s},\end{cases}\end{aligned}$$

$$\lim_{\rho\to\infty}-\mathfrak{c}_{+\mathfrak{s},\ell}(r^2\partial_\rho)^i\left(h_{\mathfrak{s},\ell}\rho^{\ell+\mathfrak{s}+1}\int_\rho^\infty r^{-2\ell-2}h^{-2}_{\mathfrak{s},\ell}\mu^{-1-\mathfrak{s}}\mathrm{d}r\right) = \begin{cases}0, & \text{if } i\le\ell-\mathfrak{s}-1;\\ (-1)^{\ell-\mathfrak{s}+1}\frac{\mathfrak{c}_{+\mathfrak{s},\ell}}{2\ell+1}(\ell-\mathfrak{s})!, & \text{if } i=\ell-\mathfrak{s}.\end{cases}$$

Further, for $n \le \ell - \mathfrak{s} - 1$,

$$\begin{aligned}
&\lim_{\rho_1 \to \infty} (\rho_1)^{\ell+\mathfrak{s}+1+j} \int_{\rho_1}^{\infty} \big[(r^2\partial_\rho)^{j-n}(h_{\mathfrak{s},\ell}^{-2}\mu^{-1-\mathfrak{s}})\big] r^{-2\ell-j+n-2} \int_{2M}^{\rho} (r')^{\ell-\mathfrak{s}-n-1}((r^2\partial_\rho)^n \mathcal{H}[\psi_{+\mathfrak{s},\ell}])(r')\mathrm{d}r'\mathrm{d}\rho \\
&= \frac{1}{(\ell+\mathfrak{s}+j+1)(2\ell+j-n+1)} \lim_{\rho\to\infty} \big[(r^2\partial_\rho)^{j-n}(h_{\mathfrak{s},\ell}^{-2}\mu^{-1-\mathfrak{s}})\big] \lim_{\rho\to\infty} (r^2\partial_\rho)^n \mathcal{H}[\psi_{+\mathfrak{s},\ell}],
\end{aligned}$$

and for $n = j = \ell - \mathfrak{s}$,

$$\begin{aligned}
&\lim_{\rho_1 \to \infty} (\rho_1)^{\ell+\mathfrak{s}+1+j} \int_{\rho_1}^{\infty} \big[(r^2\partial_\rho)^{j-n}(h_{\mathfrak{s},\ell}^{-2}\mu^{-1-\mathfrak{s}})\big] r^{-2\ell-j+n-2} \int_{2M}^{\rho} (r')^{\ell-\mathfrak{s}-n-1}((r^2\partial_\rho)^n \mathcal{H}[\psi_{+\mathfrak{s},\ell}])(r')\mathrm{d}r'\mathrm{d}\rho \\
&= \lim_{\rho_1\to\infty} (\rho_1)^{2\ell+1} \int_{\rho_1}^{\infty} h_{\mathfrak{s},\ell}^{-2}\mu^{-1-\mathfrak{s}} r^{-2\ell-2} \int_{2M}^{\rho} (r')^{-1}((r^2\partial_\rho)^{\ell-\mathfrak{s}} \mathcal{H}[\psi_{+\mathfrak{s},\ell}])(r')\mathrm{d}r'\mathrm{d}\rho \\
&= \frac{1}{2\ell+1} \int_{2M}^{\infty} \rho^{-1}(r^2\partial_\rho)^{\ell-\mathfrak{s}} \mathcal{H}[\psi_{+\mathfrak{s},\ell}]\mathrm{d}\rho.
\end{aligned}$$

In view of the above limits, equations (5.35) thus follow.

In the end, we estimate $\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{(g_{+\mathfrak{s}})_{m,\ell}}$, the limit of which as $\rho \to \infty$ equals the Newman–Penrose constant of the time integral $(g_{+\mathfrak{s}})_{m,\ell}$. We use equation (5.34b) with $i = \ell - \mathfrak{s}$ and utilize equation (2.6) to write $-\mu Y = -(2 - \mu H_{\text{hyp}})\partial_\tau + \mu\partial_\rho$, and in view of $\partial_\tau \widetilde{\Phi}^{(i)}_{(g_{+\mathfrak{s}})_{m,\ell}} = \widetilde{\Phi}^{(i)}_{+\mathfrak{s}}$ and the expressions of $f_{\mathfrak{s},i,1}$ and $f_{\mathfrak{s},i,2}$, we achieve

$$\mu\partial_\rho \hat{\mathcal{V}}\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{(g_{+\mathfrak{s}})_{m,\ell}} - (2-\mu H_{\text{hyp}})\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}} - 2(\ell+1)(r-3M)r^{-2}\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{(g_{+\mathfrak{s}})_{m,\ell}} + \sum_{j=0}^{\ell-\mathfrak{s}} h_{\mathfrak{s},\ell-\mathfrak{s},j}\Phi^{(j)}_{(g_{+\mathfrak{s}})_{m,\ell}} = 0.$$

This can be rewritten as

$$\partial_\rho\big((\mu r^{-2})^{\ell+1}\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{(g_{+\mathfrak{s}})_{m,\ell}}\big) = \mu^\ell r^{-2\ell-2}\Big((2-\mu H_{\text{hyp}})\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}} - \sum_{j=0}^{\ell-\mathfrak{s}} h_{\mathfrak{s},\ell-\mathfrak{s},j}\Phi^{(j)}_{(g_{+\mathfrak{s}})_{m,\ell}}\Big). \tag{5.41}$$

By integrating from ∞, we obtain

$$\begin{aligned}
\lim_{\rho\to\infty} \hat{\mathcal{V}}\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{(g_{+\mathfrak{s}})_{m,\ell}} &= -\lim_{\rho'\to\infty}(\rho')^{2\ell+2}\int_{\rho'}^{\infty} \mu^\ell r^{-2\ell-2}\Big((2-\mu H_{\text{hyp}})\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}} - \sum_{j=0}^{\ell-\mathfrak{s}} h_{\mathfrak{s},\ell-\mathfrak{s},j}\Phi^{(j)}_{(g_{+\mathfrak{s}})_{m,\ell}}\Big)\mathrm{d}\rho \\
&= -\frac{2}{2\ell+2}\lim_{\rho\to\infty} r\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}} + \lim_{\rho'\to\infty}(\rho')^{2\ell+2}\int_{\rho'}^{\infty}\mu^\ell r^{-2\ell-2}\Big(\sum_{j=0}^{\ell-\mathfrak{s}} h_{\mathfrak{s},\ell-\mathfrak{s},j}\Phi^{(j)}_{(g_{+\mathfrak{s}})_{m,\ell}}\Big)\mathrm{d}\rho.
\end{aligned} \tag{5.42}$$

For the the last term, since the limits $\lim_{\rho\to\infty}(r^2\partial_\rho)^j\Phi^{(0)}_{(g_{+\mathfrak{s}})_{m,\ell}}$ exist and are finite for all $j = 0, 1, \ldots, \ell - \mathfrak{s}$ and equation (5.36) follows easily from equation (2.6) and Definition 3.1, we conclude the limits $\lim_{\rho\to\infty}\Phi^{(j)}_{(g_{+\mathfrak{s}})_{m,\ell}}$ exist and are finite for all $j = 0, 1, \ldots, \ell - \mathfrak{s}$. Hence we obtain

$$\lim_{\rho'\to\infty}(\rho')^{2\ell+2}\int_{\rho'}^{\infty}\mu^\ell r^{-2\ell-2}\Big(\sum_{j=0}^{i} h_{\mathfrak{s},\ell-\mathfrak{s},j}\Phi^{(j)}_{(g_{+\mathfrak{s}})_{m,\ell}}\Big)\mathrm{d}\rho = \frac{1}{2\ell+2}\sum_{j=0}^{\ell-\mathfrak{s}}\Big(\lim_{\rho\to\infty}(rh_{\mathfrak{s},\ell-\mathfrak{s},j}) \times \lim_{\rho\to\infty}\Phi^{(j)}_{(g_{+\mathfrak{s}})_{m,\ell}}\Big).$$

These yield equation (5.37).

□

Lemma 5.5.

- *For any $\delta \in (0, \frac{1}{2})$,*

$$\mathbb{I}^{\ell,k,3-\delta}_{\Sigma_{\tau_0}}[\Psi_{(g_{+\mathfrak{s}})_{m,\ell}}] \lesssim_k \mathbb{I}^{\ell,k+1,5-\delta}_{\Sigma_{\tau_0}}[(\Psi_{+\mathfrak{s}})_{(m,\ell)}] \tag{5.43}$$

- *Assume on Σ_{τ_0} that the limit $\lim_{\rho\to\infty} r\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}$ exists and is finite and, for any $i \in \mathbb{N}$, there are constants $\beta \in (0, \frac{1}{2})$ and $D_0 \ge 0$ such that for all $0 \le i' \le i+1$ and $r \ge R$,*

$$\Big|\partial_\rho^{i'}\Big(\hat{V}\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}} - \rho^{-3}\lim_{\rho\to\infty} r\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}\Big)\Big| \lesssim D_0\rho^{-3-\beta-i'}. \tag{5.44}$$

Then, for all $0 \le i' \le i$ and $r \ge R$,

$$\left|\partial_\rho^{i'}\left(\hat{V}\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{(g_{+\mathfrak{s}})_{m,\ell}} - \rho^{-2}\mathbb{Q}^{(m,\ell)}_{(g_{+\mathfrak{s}})_{m,\ell}}\right)\right| \lesssim D_0'\rho^{-2-\beta-i'}, \tag{5.45}$$

with $D_0' = D_0 + (\mathbb{I}^{\ell,i+k',5-\delta}_{\Sigma_{\tau_0}}[(\Psi_{+\mathfrak{s}})_{(m,\ell)}])^{\frac{1}{2}}$ for some large enough k'.

Proof. It is manifest from Lemmas 5.28 and 5.4 that

$$\sum_{i=0}^{\ell-\mathfrak{s}}\|(r^2V)^i\Psi_{(g_{+\mathfrak{s}})_{m,\ell}}\|^2_{W^{k-i}_{-2}(\Sigma_{\tau_0})} \lesssim \mathbb{I}^{\ell,k+1,5-\delta}_{\Sigma_{\tau_0}}[(\Psi_{+\mathfrak{s}})_{(m,\ell)}]. \tag{5.46}$$

Meanwhile, by examining the proof of Lemma 5.4, it also implies

$$\begin{aligned}\|rV\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{(g_{+\mathfrak{s}})_{m,\ell}}\|^2_{W^{k-\ell}_{3-\delta-2}(\Sigma_{\tau_0}\cap\{\rho\ge 4M\})} &\lesssim \sum_{i=0}^{\ell-\mathfrak{s}}\|(r^2V)^i\Psi_{+\mathfrak{s}}\|^2_{W^{k-i+1}_{-2}(\Sigma_{\tau_0})} + \|rV\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}\|^2_{W^{k-\ell+1}_{5-\delta-2}(\Sigma_{\tau_0}\cap\{\rho\ge 4M\})}\\ &= \mathbb{I}^{\ell,k+1,5-\delta}_{\Sigma_{\tau_0}}[(\Psi_{+\mathfrak{s}})_{(m,\ell)}]. \end{aligned}\tag{5.47}$$

These two estimates together yield (5.43). The second claim also follows from the proof of Lemma 5.4. ☐

5.3. **Global asymptotic profiles of the spin $\pm\mathfrak{s}$ components.** We first state the Price's law in the case that the spin $\pm\mathfrak{s}$ components are supported on a single (m,ℓ) mode.

Theorem 5.6. *Let $j \in \mathbb{N}^+$ and let $\mathfrak{s} = 0,1,2$. Suppose the spin $s = \pm\mathfrak{s}$ components are supported on an (m,ℓ) mode, $\ell \ge \mathfrak{s}$. Let function $h_{\mathfrak{s},\ell}$ and scalars $\widetilde{\Phi}^{(i)}_{+\mathfrak{s}}$ and $\widetilde{\Phi}^{(i)}_{-\mathfrak{s}}$ be defined as in Definitions 3.34 and 3.6, respectively. Assume on Σ_{τ_0} that there are constants $D_1 \in \mathbb{R}$, $\beta \in (0,\frac{1}{2})$ and $0 \le D_0 < \infty$ such that for all $0 \le i \le j+\ell+\mathfrak{s}$ and $r \ge 10M$,*

$$\left|\partial_\rho^{i}\left(\hat{V}\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}} - \rho^{-3}\lim_{\rho\to\infty} r\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}\right)\right| \lesssim D_0\rho^{-3-\beta-i}. \tag{5.48}$$

Then all the statements in Theorem 4.9 hold true by making the following replacements:

$$\begin{aligned}&\partial_\tau^j\Upsilon_{-\mathfrak{s}} \to \partial_\tau^{j-1}\Upsilon_{-\mathfrak{s}}, \quad \partial_\tau^j\Upsilon_{+\mathfrak{s}} \to \partial_\tau^{j-1}\Upsilon_{+\mathfrak{s}}, \quad \partial_\tau^j\Upsilon_s \to \partial_\tau^{j-1}\Upsilon_s, \quad \mathbb{Q}^{(m,\ell)}_{+\mathfrak{s}} \to \mathbb{Q}^{(m,\ell)}_{g_+},\\ &\partial_\tau^j\left(\Upsilon_{0,mid} - r^{-2}\left(q_{\mathbf{E}} + iq_{\mathbf{B}}\right)\right) \to \partial_\tau^{j-1}\left(\Upsilon_{0,mid} - r^{-2}\left(q_{\mathbf{E}} + iq_{\mathbf{B}}\right)\right), \quad \mathbf{E}_{\mathrm{NV},\ell} \to \mathbf{E}_{\mathrm{V},\ell},\end{aligned}\tag{5.49}$$

where $\mathbf{E}_{\mathrm{V},\ell} = (\mathbb{I}^{\ell,k,5-\delta}_{\Sigma_{\tau_0}}[(\Psi_{+\mathfrak{s}})_{(m,\ell)}])^{\frac{1}{2}} + |\mathbb{Q}^{(m,\ell)}_{g_+}| + D_0$, $\mathbb{I}^{\ell,k,5-\delta}_{\Sigma_{\tau_0}}[(\Psi_{+\mathfrak{s}})_{(m,\ell)}]$ is defined as in Definition 3.8, and the N–P constant $\mathbb{Q}^{(m,\ell)}_{g_+}$ of the time integral $(g_{+\mathfrak{s}})_{m,\ell}$ can be calculated as in Lemma 5.4.

Proof. By Lemma 5.5, we can apply Theorem 4.9 to the time integral $(g_{+\mathfrak{s}})_{m,\ell}$. The statements for the spin $+\mathfrak{s}$ component then follows simply from $(\psi_{+\mathfrak{s}})_{(m,\ell)} = \partial_\tau(g_{+\mathfrak{s}})_{m,\ell}$. The estimates for the spin $-\mathfrak{s}$ component can be proven by utilizing the TSI in Lemma 3.21. The estimates for the middle component of the Maxwell field are obtained in the same way as in the proof of Theorem 4.9 . ☐

We also obtain the asymptotic profiles for the spin $\pm\mathfrak{s}$ components if they are supported on $\ell \ge \ell_0$ modes.

Theorem 5.7. *Let $j \in \mathbb{N}^+$ and let $\mathfrak{s} = 0,1,2$. Suppose the spin $s = \pm\mathfrak{s}$ components are supported on $\ell \ge \ell_0$ mode, $\ell_0 \ge \mathfrak{s}$. Let function $h_{\mathfrak{s},\ell}$ and scalars $\widetilde{\Phi}^{(i)}_{+\mathfrak{s}}$ and $\widetilde{\Phi}^{(i)}_{-\mathfrak{s}}$ be defined as in Definitions 3.34 and 3.6, respectively. Assume on Σ_{τ_0} that there are constants $\mathbb{Q}^{(m,\ell_0+1)}_{+\mathfrak{s}} \in \mathbb{R}$, $\beta \in (0,\frac{1}{2})$ and $0 \le D_0 < \infty$ such that for any $r \ge 10M$,*

$$\left|\partial_\rho^{i}\left(\hat{V}(\widetilde{\Phi}^{(\ell_0-\mathfrak{s})}_{+\mathfrak{s}})_{m,\ell_0} - \rho^{-3}\lim_{\rho\to\infty} r\hat{\mathcal{V}}\widetilde{\Phi}^{(\ell-\mathfrak{s})}_{+\mathfrak{s}}\right)\right| \lesssim D_0\rho^{-3-\beta-i}, \quad \forall 0 \le i \le j+\ell_0+\mathfrak{s}, |m| \le \ell_0, \tag{5.50a}$$

$$\left|\partial_\rho^{i}\left(\hat{V}(\widetilde{\Phi}^{(\ell_0+1-\mathfrak{s})}_{+\mathfrak{s}})_{m,\ell_0+1} - \rho^{-2}\mathbb{Q}^{(m,\ell_0+1)}_{+\mathfrak{s}}\right)\right| \lesssim D_0\rho^{-2-\beta-i}, \quad \forall 0 \le i \le j+\ell_0+\mathfrak{s}-1, |m| \le \ell_0+1. \tag{5.50b}$$

Then there exists an $\delta > 0$, $k = k(\ell_0, j)$, $\varepsilon = \varepsilon(j, \ell_0, \delta) > 0$ and $C = C(k, \ell_0, j, \delta)$ such that for any $\tau \geq \tau_0$,

$$\Big|\partial_\tau^{j-1}\Upsilon_{-\mathfrak{s}} - 2^{2\ell_0+2}\prod_{i=\ell_0-\mathfrak{s}+1}^{2\ell_0+1} i^{-1}\sum_{m=-\ell_0}^{\ell_0}\mathbb{Q}_{g+}^{(m,\ell_0)}h_{\mathfrak{s},\ell_0}Y_{m,\ell_0}^{-\mathfrak{s}}(\cos\theta)e^{im\phi}\partial_\tau^j(Y^{2\mathfrak{s}}(r^{\ell_0+\mathfrak{s}}\tau^{-\ell_0+\mathfrak{s}-1}v^{-\ell_0-\mathfrak{s}-1}))$$
$$- 2^{2\ell_0+4}\prod_{i=\ell_0-\mathfrak{s}+2}^{2\ell_0+3} i^{-1}\sum_{m=-\ell_0-1}^{\ell_0+1}\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell_0+1)}h_{\mathfrak{s},\ell_0+1}Y_{m,\ell_0+1}^{-\mathfrak{s}}(\cos\theta)e^{im\phi}\partial_\tau^{j-1}(Y^{2\mathfrak{s}}(r^{\ell_0+1+\mathfrak{s}}\tau^{-\ell_0+\mathfrak{s}-2}v^{-\ell_0-\mathfrak{s}-2}))\Big|$$
$$\leq C\tau^{-\varepsilon}\big[\partial_\tau^j(Y^{2\mathfrak{s}}(r^{\ell_0+\mathfrak{s}}\tau^{-\ell_0+\mathfrak{s}-1}v^{-\ell_0-\mathfrak{s}-1})) + \partial_\tau^{j-1}(Y^{2\mathfrak{s}}(r^{\ell_0+1+\mathfrak{s}}\tau^{-\ell_0+\mathfrak{s}-2}v^{-\ell_0-\mathfrak{s}-2}))\big]\mathbf{E}_{\mathrm{V}}, \quad (5.51a)$$

$$\Big|\partial_\tau^{j-1}\Upsilon_{+\mathfrak{s}} - 2^{2\ell_0+2}\prod_{i=\ell_0+\mathfrak{s}+1}^{2\ell_0+1} i^{-1}\sum_{m=-\ell_0}^{\ell_0}\mathbb{Q}_{g+}^{(m,\ell_0)}\mu^{\mathfrak{s}}h_{\mathfrak{s},\ell_0}Y_{m,\ell_0}^{\mathfrak{s}}(\cos\theta)e^{im\phi}\partial_\tau^j(r^{\ell_0-\mathfrak{s}}\tau^{-\ell_0+\mathfrak{s}-1}v^{-\ell_0-\mathfrak{s}-1})$$
$$- 2^{2\ell_0+4}\prod_{i=\ell_0+\mathfrak{s}+2}^{2\ell_0+3} i^{-1}\sum_{m=-\ell_0-1}^{\ell_0+1}\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell_0+1)}\mu^{\mathfrak{s}}h_{\mathfrak{s},\ell_0+1}Y_{m,\ell_0+1}^{\mathfrak{s}}(\cos\theta)e^{im\phi}\partial_\tau^{j-1}(r^{\ell_0+1-\mathfrak{s}}\tau^{-\ell_0+\mathfrak{s}-2}v^{-\ell_0-\mathfrak{s}-2})\Big|$$
$$\leq C\tau^{-\varepsilon}\big[\partial_\tau^j(r^{\ell_0-\mathfrak{s}}\tau^{-\ell_0+\mathfrak{s}-1}v^{-\ell_0-\mathfrak{s}-1}) + \partial_\tau^{j-1}(r^{\ell_0+1-\mathfrak{s}}\tau^{-\ell_0+\mathfrak{s}-2}v^{-\ell_0-\mathfrak{s}-2})\big]\mathbf{E}_{\mathrm{V}}, \quad (5.51b)$$

with $\mathbf{E}_{\mathrm{V}} = (\mathbb{I}_{V,\Sigma_{\tau_0}}^{\ell\geq\ell_0,k,\delta}[\Psi_{+\mathfrak{s}}])^{\frac{1}{2}} + \sum_{m=-\ell_0}^{\ell_0}|\mathbb{Q}_{g+}^{(m,\ell_0)}| + \sum_{m=-\ell_0-1}^{\ell_0+1}|\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell_0+1)}| + D_0$ and $\mathbb{I}_{V,\Sigma_{\tau_0}}^{\ell\geq\ell_0,k,\delta}[\Psi_{+\mathfrak{s}}]$ is defined as in Definition 3.9. In particular,

(1) *in the region $\{\rho \geq \tau^{1+\varepsilon_0}\}$ for an arbitrary $\varepsilon_0 > 0$, then*

$$\Big|\partial_\tau^{j-1}\Upsilon_s - \Big[(-1)^j 2^{\ell_0+\mathfrak{s}+2}\prod_{i=\ell_0+\mathfrak{s}+1}^{2\ell_0+1} i^{-1}\sum_{m=-\ell_0}^{\ell_0}\mathbb{Q}_{g+}^{(m,\ell_0)}Y_{m,\ell_0}^{s}(\cos\theta)e^{im\phi}v^{-1-\mathfrak{s}-s}\tau^{-\ell_0+s-2-(j-1)}$$
$$+ (-1)^{j-1}2^{\ell_0+\mathfrak{s}+3}\prod_{i=\ell_0+\mathfrak{s}+2}^{2\ell_0+3} i^{-1}\sum_{m=-\ell_0-1}^{\ell_0+1}\mathbb{Q}_{+\mathfrak{s}}^{(m,\ell_0+1)}Y_{m,\ell_0+1}^{s}(\cos\theta)e^{im\phi}v^{-1-\mathfrak{s}-s}\tau^{-\ell_0+s-2-(j-1)}\Big]\Big|$$
$$\leq C(\tau^{-\varepsilon} + \tau^{-\varepsilon_0})v^{-1-\mathfrak{s}-s}\tau^{-\ell_0+s-2-(j-1)}\mathbf{E}_{\mathrm{V}}; \quad (5.52a)$$

(2) *in the region $\{\rho \leq \tau^{1-\varepsilon_0}\}$ for an arbitrary $\varepsilon_0 > 0$, then for $s = \pm\mathfrak{s}$,*

$$\Big|\partial_\tau^{j-1}\Upsilon_s - (-1)^j 2^{2\ell_0+2}\prod_{i=\ell_0+\mathfrak{s}+1}^{2\ell+1} i^{-1}\prod_{n=2\ell_0+2}^{2\ell_0+j+1} n\sum_{m=-\ell_0}^{\ell_0}\mathbb{Q}_{g+}^{(m,\ell_0)}\mu^{\frac{s+\mathfrak{s}}{2}}r^{\ell_0-\mathfrak{s}}h_{\mathfrak{s},\ell_0}Y_{m,\ell_0}^{s}(\cos\theta)e^{im\phi}\tau^{-2\ell_0-3-(j-1)}\Big|$$
$$\leq C(\tau^{-\varepsilon} + \tau^{-\varepsilon_0})\tau^{-2\ell_0-3-(j-1)}\mathbf{E}_{\mathrm{V}}, \quad (5.52b)$$

(3) *on the future event horizon, for $\mathfrak{s} \neq 0$,*

$$\Big|\partial_\tau^{j-1}\Upsilon_{+\mathfrak{s}}\big|_{\mathcal{H}^+} - \frac{(-1)^{\mathfrak{s}+j}2^{2\ell_0+3}\mathfrak{s}(\ell_0-\mathfrak{s})!(2M)^{\ell_0-\mathfrak{s}+1}h_{\mathfrak{s},\ell_0}(2M)}{(2\ell_0+1)!}\prod_{n=2\ell_0+2}^{2\ell_0+j+2} n\sum_{m=-\ell_0}^{\ell_0}\mathbb{Q}_{g+}^{(m,\ell_0)}Y_{m,\ell_0}^{s}(\cos\theta)e^{im\phi}v^{-2\ell_0-4-(j-1)}\Big|$$
$$\leq Cv^{-2\ell_0-4-(j-1)-\varepsilon}\mathbf{E}_{\mathrm{V}}. \quad (5.52c)$$

Additionally, for the middle component $\Upsilon_{0,mid}$ of the Maxwell field, let $q_{\mathbf{E}}$ and $q_{\mathbf{B}}$ be defined as in Lemma 2.13, then

(i) *for any $\tau \geq \tau_0$ and $\rho > 2M$,*

$$\Big|\partial_\tau^{j-1}\left(\Upsilon_{0,mid} - r^{-2}\left(q_{\mathbf{E}} + iq_{\mathbf{B}}\right)\right) + \frac{2^{2\ell_0+3}}{\sqrt{\ell_0(\ell_0+1)}}\prod_{i=\ell_0+2}^{2\ell_0+1} i^{-1}\sum_{m=-\ell_0}^{\ell_0}\mathbb{Q}_{+1}^{(m,\ell_0)}Y_{m,\ell_0}^{0}(\cos\theta)e^{im\phi}\partial_\tau^j(Y(\mu h_{1,\ell_0}r^{\ell_0}\tau^{-\ell_0}v^{-\ell_0-2}))$$
$$+ \frac{2^{2\ell_0+5}}{\sqrt{(\ell_0+1)(\ell_0+2)}}\prod_{i=\ell_0+3}^{2\ell_0+3} i^{-1}\sum_{m=-\ell_0-1}^{\ell_0+1}\mathbb{Q}_{+1}^{(m,\ell_0+1)}Y_{m,\ell_0+1}^{0}(\cos\theta)e^{im\phi}\partial_\tau^{j-1}(Y(\mu h_{1,\ell_0+1}r^{\ell_0+1}\tau^{-\ell_0-1}v^{-\ell_0-3}))\Big|$$
$$\leq C\tau^{-\varepsilon}\big[\partial_\tau^j(Y(r^{\ell_0}\tau^{-\ell_0}v^{-\ell_0-2})) + \partial_\tau^{j-1}(Y(r^{\ell_0+1}\tau^{-\ell_0-1}v^{-\ell_0-3}))\big]\mathbf{E}_{\mathrm{V}}; \quad (5.53a)$$

(ii) *in the region* $\{\rho \geq \tau^{1+\varepsilon_0}\}$ *for an arbitrary* $\varepsilon_0 > 0$,

$$\bigg|\partial_\tau^{j-1}\left(\Upsilon_{0,mid} - r^{-2}\left(q_{\mathbf{E}} + iq_{\mathbf{B}}\right)\right) - \bigg[\frac{(-1)^j 2^{\ell_0+3}}{\sqrt{\ell_0(\ell_0+1)}}\prod_{i=\ell_0+2}^{2\ell_0+1} i^{-1}\prod_{n=\ell_0}^{\ell_0+j} n \sum_{m=-\ell_0}^{\ell_0}\mathbb{Q}_{g+}^{(m,\ell_0)} Y_{m,\ell_0}^0(\cos\theta)e^{im\phi}$$
$$+ \frac{(-1)^{j-1}2^{\ell_0+4}}{\sqrt{(\ell_0+1)(\ell_0+2)}}\prod_{i=\ell_0+3}^{2\ell_0+3} i^{-1}\prod_{n=\ell_0+1}^{\ell_0+j} n \sum_{m=-\ell_0-1}^{\ell_0+1}\mathbb{Q}_{+1}^{(m,\ell_0+1)} Y_{m,\ell_0+1}^0(\cos\theta)e^{im\phi}\bigg] v^{-2}\tau^{-\ell_0-j-1}\bigg|$$
$$\leq C(\tau^{-\varepsilon} + \tau^{-\varepsilon_0})v^{-2}\tau^{-\ell_0-2-(j-1)}\mathbf{E}_{\mathrm{V}}; \tag{5.53b}$$

(iii) *in the region* $\{2M \leq \rho \leq \tau^{1-\varepsilon_0}\}$ *for an arbitrary* $\varepsilon_0 > 0$,

$$\bigg|\partial_\tau^{j-1}\left(\Upsilon_{0,mid} - r^{-2}\left(q_{\mathbf{E}} + iq_{\mathbf{B}}\right)\right) - (-1)^j\frac{2^{2\ell_0+3}}{\sqrt{\ell_0(\ell_0+1)}}\prod_{i=\ell_0+2}^{2\ell_0+1} i^{-1}\prod_{n=2\ell_0+2}^{2\ell_0+1+j} n$$
$$\times\sum_{m=-\ell_0}^{\ell_0}\mathbb{Q}_{g+}^{(m,\ell_0)}\partial_\rho(\mu h_{1,\ell_0} r^{\ell_0}) Y_{m,\ell_0}^0(\cos\theta)e^{im\phi}\tau^{-2\ell_0-3-(j-1)}\bigg|$$
$$\leq C(\tau^{-\varepsilon} + \tau^{-\varepsilon_0})\tau^{-2\ell_0-3-(j-1)}\mathbf{E}_{\mathrm{V}}. \tag{5.53c}$$

Proof. To prove this theorem, it suffices to prove the estimates (5.51). One can apply Theorem 5.6 to estimate each (m, ℓ_0) mode and Theorem 4.10 to estimate the remainder $\ell \geq \ell_0 + 1$ modes, which thus completes the proof. □

In the end, we give a proof of point 2 in Theorem 1.1. If the initial data of the spin s components on $t = \tau_0$ hypersurface are compactly supported in a compact region of $(2M, \infty)$, we can choose Σ_{τ_0} such that it coincides with $t = \tau_0$ hypersurface in the compact support of the initial data, and on Σ_{τ_0}, if away from the intersection of these two hypersurfaces, the initial data vanish identically. Then, one finds by (5.6) that all $\mathfrak{c}_{+\mathfrak{s},\ell} = 0$. Since $H_{\mathrm{hyp}} = \tilde{H}_{\mathrm{hyp}} = \partial_r h_{\mathrm{hyp}} = \mu^{-1}$, we have from (5.1) that $\mathcal{H}[\psi_{+\mathfrak{s},\ell}] = \mu^{-1}h_{\mathfrak{s},\ell}r^{2\mathfrak{s}+1}(r^2\partial_t(\Upsilon_{+1})_{m,\ell} + 2\mathfrak{s}(r-3M)(\Upsilon_{+1})_{m,\ell})$. By Lemma 5.4, the N–P constant of the time integral $(g_{+\mathfrak{s}})_{m,\ell}$ is

$$\begin{aligned}
\mathbb{Q}^{(m,\ell)}_{(g_{+\mathfrak{s}})_{m,\ell}} &= \frac{1}{2\ell+2}\lim_{r\to\infty}(rh_{\mathfrak{s},\ell-\mathfrak{s},\ell-\mathfrak{s}}) \times \lim_{r\to\infty}(r^2\partial_r)^{\ell-\mathfrak{s}}\Phi^{(0)}_{(g_{+\mathfrak{s}})_{m,\ell}}\\
&= -\frac{1}{(2\ell+1)(2\ell+2)}\lim_{r\to\infty}(rh_{\mathfrak{s},\ell-\mathfrak{s},\ell-\mathfrak{s}}) \times \int_{2M}^{\infty} r^{-1}(r^2\partial_r)^{\ell-\mathfrak{s}}\mathcal{H}[\psi_{+\mathfrak{s},\ell}]\mathrm{d}r\\
&= -\frac{1}{(2\ell+1)(2\ell+2)}\lim_{r\to\infty}(rh_{\mathfrak{s},\ell_0-\mathfrak{s},\ell_0-\mathfrak{s}})\\
&\quad\times\int_{2M}^{\infty} r^{-1}(r^2\partial_r)^{\ell-\mathfrak{s}}\Big(\mu^{-1}h_{\mathfrak{s},\ell}r^{2\mathfrak{s}+1}(r^2\partial_t(\Upsilon_{+1})_{m,\ell} + 2\mathfrak{s}(r-3M)(\Upsilon_{+1})_{m,\ell})\Big)\mathrm{d}r\\
&= \frac{(-1)^{\ell-\mathfrak{s}+1}(\ell-\mathfrak{s})!}{(2\ell+1)(2\ell+2)}\lim_{r\to\infty}(rh_{\mathfrak{s},\ell_0-\mathfrak{s},\ell_0-\mathfrak{s}})\\
&\quad\times\int_{2M}^{\infty}\mu^{-1}h_{\mathfrak{s},\ell}r^{\ell+\mathfrak{s}}(r^2\partial_t(\Upsilon_{+1})_{m,\ell} + 2\mathfrak{s}(r-3M)(\Upsilon_{+1})_{m,\ell})\mathrm{d}r,
\end{aligned} \tag{5.54}$$

where in the last step we have used integration by parts in r and the assumption that the initial data is supported on a compact region in $(2M, \infty)$. We then have from Theorem 5.7 that in the region $\rho \geq \tau^{1+\varepsilon_0}$,

$$\bigg|\partial_\tau^j\Upsilon_s - (-1)^{j+1}2^{\ell_0+\mathfrak{s}+2}\prod_{i=\ell_0+\mathfrak{s}+1}^{2\ell_0+1} i^{-1}\sum_{m=-\ell_0}^{\ell_0}\mathbb{Q}^{(m,\ell)}_{(g_{+\mathfrak{s}})_{m,\ell}} Y^s_{m,\ell_0}(\cos\theta)e^{im\phi}v^{-1-\mathfrak{s}-s}\tau^{-\ell_0+s-2-j}\bigg|$$
$$\leq Dv^{-1-\mathfrak{s}-s}\tau^{-\ell_0+s-2-j-\varepsilon}, \tag{5.55}$$

in the region $\rho \leq \tau^{1-\varepsilon_0}$,

$$\bigg|\partial_\tau^j\Upsilon_s - (-1)^{j+1}2^{2\ell_0+2}\prod_{i=\ell_0+\mathfrak{s}+1}^{2\ell+1} i^{-1}\prod_{n=2\ell_0+2}^{2\ell_0+j+2} n\sum_{m=-\ell_0}^{\ell_0}\mathbb{Q}^{(m,\ell)}_{(g_{+\mathfrak{s}})_{m,\ell}}\mu^{\frac{s+\mathfrak{s}}{2}}r^{\ell_0-\mathfrak{s}}h_{\mathfrak{s},\ell_0}Y^s_{m,\ell_0}(\cos\theta)e^{im\phi}\tau^{-2\ell_0-3-j}\bigg|$$

$$\leq D\tau^{-2\ell_0-3-j-\varepsilon}, \tag{5.56}$$

and on the future event horizon, for $\mathfrak{s} \neq 0$,

$$\left|\partial_\tau^j \Upsilon_{+\mathfrak{s}}\Big|_{\mathcal{H}^+} - \frac{(-1)^{\mathfrak{s}+j+1}2^{2\ell_0+3}\mathfrak{s}(\ell_0-\mathfrak{s})!(2M)^{\ell_0-\mathfrak{s}+1}h_{\mathfrak{s},\ell_0}(2M)}{(2\ell_0+1)!}\prod_{n=2\ell_0+2}^{2\ell_0+j+3} n \sum_{m=-\ell_0}^{\ell_0} \mathbb{Q}^{(m,\ell_0)}_{(g_{+\mathfrak{s}})_{m,\ell}} Y^s_{m,\ell_0}(\cos\theta)e^{im\phi}v^{-2\ell_0-4-j}\right|$$
$$\leq Dv^{-2\ell_0-4-j-\varepsilon}. \tag{5.57}$$

Combining the above discussions then completes the proof of point 2 in Theorem 1.1.

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