Knizhnik–Zamolodchikov functor for degenerate double affine Hecke algebras : algebraic theory

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Abstract

In this article, we define an algebraic version of the Knizhnik–Zamolodchikov functor for the degenerate double affine Hecke algebras (a.k.a. trigonometric Cherednik algebras). We compare it with the KZ monodromy functor constructed by Varagnolo–Vasserot. We prove the double centraliser property for our functor and give a characterisation of its kernel. We establish these results for a family of algebras, called quiver double Hecke algebras, which includes the degenerate double affine Hecke algebras as special cases.

Introduction

Degenerate double affine Hecke algebras

The degenerate double affine Hecke algebras (dDAHA), also known as trigonometric Cherednik algebras, were introduced by I. Cherednik in his study of integration of the trigonometric form of the Knizhnik–Zamolodchikov equations (KZ) [8].

The degenerate double affine Hecke algebras, unlike their non-degenerate version and its rational degeneration, are not "symmetric": it contains a polynomial subalgebra and a Laurent polynomial subalgebra. Due to this asymmetry, one can adopt two different points of view to study the dDAHA: either viewing it

(i) as the algebra generated by regular functions on a torus $T^\vee$ attached to a root system R, the Weyl group of R acting the torus $T^\vee$ and the trigonometric Dunkl operators on it, or

(ii) as the algebra generated by Demazure-like difference operators on E, where E is an affine space which carries an affine root system; this is the affine version of the graded affine Hecke algebras of G. Lusztig [25].

The former approach allows one to apply various techniques of $\mathcal{D}$-modules, symplectic geometry and is closer to the theory of rational Cherednik algebras [14, 1]; the latter approach allows one to apply cohomological, K-theoretic or sheaf-theoretic methods [10, 35], and is closer to the (non-degenerate) double affine Hecke algebras.

In the present work, we will adopt the second approach most of the time. We show that with this point of view, the dDAHAs can be easily generalised and are quite flexible in the choice of parameters. We show also that some of the features from first approach can be recovered with the second approach, namely the integration of the KZ equations.

Quiver Hecke algebras

The quiver Hecke algebras, also known as Khovanov–Lauda–Rouquier algebras, were introduced in [21] and [30]. They were introduced in the purpose of categorifying the Drinfel'd–Jimbo quantum groups for Kac–Moody algebras as well as their integrable representations.

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It was proven by Brundan–Kleshchev–McNamara [6] and Kato [20] that quiver Hecke algebras for Dynkin quivers of finite ADE types have pretty nice homological properties. Retrospectively speaking, they proved that the categories of graded modules over these algebras carry an *affine highest weight structure* in the sense of [22]. As a consequence, these algebras have finite global dimension. However, once one goes beyond the family of finite type, the quiver Hecke algebras often have infinite global dimension. The simplest example would be the cyclic quivers of length ≥ 2. According to the result of Brundan–Kleshchev [5] and Rouquier [30], the quiver Hecke algebras of cyclic quivers are equivalent to affine Hecke algebras for GL_n with parameter at roots of unity. The representation theory of affine Hecke algebras at roots of unity is known to share several features of the modular representation theory finite groups. Notably, there are fewer simple modules in the modular case than there are in the ordinary case.

One approach to the modular representation theory is to resolve this lack of simple objects by finding a larger, but better behaved category, of which the modular category is a quotient. In the case of modular representation theory of symmetric groups, one uses the Schur algebras as resolution via the Schur–Weyl duality. In the same spirit, for Hecke algebras of complex reflection groups, the rational Cherednik algebras provide resolution, as it was first established in [16]. For affine Hecke algebras, the resolution would be the degenerate double affine Hecke algebras. This perspective appeared in [34], where degenerate DAHAs are viewed as replacement for affine q-Schur algebras in relation with affine Hecke algebras *cf.* §2.2 and §4. We will introduce a new family of algebras, called *quiver double Hecke algebras*, which we believe to play the rôle of "resolution" for quiver Hecke algebras.

Results of the present article

Let (V, R) be an irreducible finite root system and let (E, S) be its affinisation (the definition is recalled in §1.1). In particular, $E \cong V$ is a euclidean affine space. We fix a basis $\Delta_0 \subset R$, which extends in a standard way to an affine basis $\Delta \subset S$. The affine Weyl group W_S is generated by affine simple reflections s_a for $a \in \Delta$ and the finite Weyl group $W_R \subset W_S$ is the subgroup generated by s_a for $a \in \Delta_0$. The extended affine Weyl group $\widetilde{W}_S$ acts on S.

The degenerate double affine Hecke algebra attached to (E, S) is given by $\mathbb{H} = \mathbf{C}W_S \otimes \mathbf{C}[E]$ as vector space. The multiplication of $\mathbb{H}$ depends on a function $h : S \longrightarrow \mathbf{C}$, called parameters, see §2 for the precise definition. For $\lambda \in E$, let $\mathcal{O}_\lambda(\mathbb{H})$ denote the category of finitely generated $\mathbb{H}$-modules on which the subalgebra $\mathbf{C}[E]$ acts locally finitely with eigenvalues lying in the orbit $W_S \cdot \lambda \subset E$.

The affine Hecke algebra attached to (V, R) given by $\mathbb{K} = H_R \otimes \mathbf{C}[T]$, where H_R is the Iwahori-Hecke algebra of type (W_R, Δ_0) and $\mathbf{C}[T]$ is the group algebra of the weight lattice of the root system (V, R). See §3.1 for the precise definition. For $\ell \in V$, let $\mathcal{O}_\ell(\mathbb{K})$ denote the category of finite-dimensional $\mathbb{K}$-modules on which the subalgebra $\mathbf{C}[T]$ acts with eigenvalues lying in the orbit $W_R \cdot \ell \subset T$.

There is an exponential map $\exp : E \longrightarrow T$. Fix $\lambda_0 \in E$ and let $\ell_0 = \exp(\lambda_0) \in T$. Denote by $\mathbb{V} : \mathcal{O}_{\lambda_0}(\mathbb{H}) \longrightarrow \mathcal{O}_{\ell_0}(\mathbb{K})$ the monodromy functor for the Knizhnik–Zamolodchikov equations introduced by Varagnolo–Vasserot in [34]. We show in Proposition 27 that M is a quotient functor. The first main result is the following:

Theorem A (=Definition 96+Proposition 29)**.** *There is a quotient functor* $\mathbf{V} : \mathcal{O}_{\lambda_0}(\mathbb{H}) \longrightarrow \mathcal{O}_{\ell_0}(\mathbb{K})$ *defined in algebraic terms such that*

$$\ker \mathbf{V} = \ker \mathbb{V}.$$

We expect that there exists an isomorphism $\mathbf{V} \cong \mathbb{V}$. In order to construct $\mathbf{V}$, we introduce in §2.5 and §3.4 two auxiliary algebras $\mathbf{H}_{\lambda_0}$ and $\mathbf{K}_{\ell_0}$ and show in Proposition 12 and Proposition 17 that $\mathbf{H}_{\lambda_0}$ and $\mathbf{K}_{\ell_0}$ are Morita-equivalent respectively to $\mathcal{O}_{\lambda_0}(\mathbb{H})$ and $\mathcal{O}_{\ell_0}(\mathbb{K})$. By analysing the structure of the quiver-Hecke-like algebras $\mathbf{H}_{\lambda_0}$ and $\mathbf{K}_{\ell_0}$, we show in Theorem 82 that there exists an idempotent $\mathbf{e}_\gamma \in \mathbf{H}_{\lambda_0}$ such that the idempotent subalgebra $\mathbf{e}_\gamma \mathbf{H}_{\lambda_0} \mathbf{e}_\gamma$ is isomorphic to $\mathbf{K}_{\ell_0}$. This allows us to define the functor $\mathbf{V}$ as the idempotent truncation by $\mathbf{e}_\gamma$.

The second main result concerns $\mathbf{V}$:

Theorem B (=Theorem 105+Theorem 108)**.** *The following statements hold:*

(i) The functor $\mathbf{V}$ *satisfies the double centraliser property (i.e. fully faithful on projective objects) after passing to a suitable completion of* $\mathcal{O}_{\lambda_0}(\mathbb{H})$ *and* $\mathcal{O}_{\ell_0}(\mathbb{K})$.

(ii) *The kernel* $\ker \mathbf{V}$ *is the Serre subcategory generated by simple objects* $L \in \mathcal{O}_{\lambda_0}(\mathbb{H})$ *such that the projective envelope of* L *in the completion of* $\mathcal{O}_{\lambda_0}(\mathbb{H})$ *is not relatively injective with respect to the categorical centre* $\mathrm{Z}(\mathcal{O}_{\lambda_0}(\mathbb{H}))$*.*

Notice that by the comparison result Theorem A, the statements of Theorem B also hold for $\mathbb{V}$. The second statement of Theorem B implies in particular that the subcategory $\ker \mathbf{V}$ is an invariant of the category $\mathcal{O}_{\lambda_0}(\mathbb{H})$. In fact, we construct $\mathbf{V}$ and establish Theorem B for a greater family of algebras, *quiver double Hecke algebras*, which are introduced in §6.3. This family of algebras seems to be related to a localised Iwahori version of Coulomb branch algebras of Braverman–Finkelberg–Nakajima [3] for semisimple groups.

Related works

As mentioned above, the algebra $\mathbf{A}^\omega$ that we introduce in Part II is expected to be related to Iwahori version of the quantised Coulomb branch algebras. There exist in the literature some works on the representation theory of such algebras with an approach similar to ours.

In [37], B. Webster studied a module category of the rational Cherednik algebra for the complex reflection group $G(\ell, 1, n)$ whose objects admit a weight decomposition for the action of a polynomial subalgebra defined by Dunkl–Opdam [13]. He introduced an algebraic version of the KZ functor and he classified the simple objects of that category. The results were later generalised in [23], to the rational Cherednik algebra for $G(\ell, d, n)$.

Our construction of KZ functor $\mathbf{V}$ can be regarded as a variant of theirs. One can expect that their functor also satisfies the properties listed in Theorem B.

Organisation

This paper is composed of two parts. The first part serves mainly as preliminary materials and motivation for the second part. The proof of most of the statements in the first part can be found in the literature [25, 9, 28, 34, 33].

We review briefly the affine root systems in §1.1, the dDAHAs in §2 and the affine Hecke algebras (AHA) in §3.1. We introduce the idempotent form of these algebras, each controlling a block of the category $\mathcal{O}$ of both algebras. The definition of idempotent forms is a straightforward generalisation of the result of Brundan–Kleshchev [5] and Rouquier [30] on the equivalence between affine Hecke algebras for GL_n and quiver Hecke algebras for linear and cyclic quivers.

We recall in §4 the monodromy functor $\mathbb{V}$ introduced in [34] as the trigonometric counterpart of the KZ functor of [16]. We prove that it is a quotient functor in the sense of Gabriel.

We discuss in §5 the relations between the monodromy functor $\mathbb{V}$ and the functor $\mathbf{V}$, which will be defined in algebraic terms in §10.6.

In the second part we introduce quiver double Hecke algebras (QDHA). They can be viewed as a generalisation of degenerate double affine Hecke algebras (dDAHA) or as an affinisation of quiver Hecke algebras (QHA).

In §6, we introduce the quiver double Hecke algebras $\mathbf{A}^\omega$ attached to an affine root system (E, S) with spectrum being a W_S-orbit in E and with parameter ω. We define the filtration by length on $\mathbf{A}^\omega$ in §6.4 and prove the basis theorem in §6.5 with this filtration. We study the associated graded $\mathrm{gr}^F \mathbf{A}^\omega$ of the filtration by length in §6.6.

In §7, we study the categories of graded and ungraded $\mathbf{A}^\omega$-modules. We introduce in §7.6 a functor of induction from the quiver Hecke algebras attached to the finite root system (V, R) underlying (E, S).

In §8, we study good filtrations on $\mathbf{A}^\omega$-modules and use it to define the Gelfand–Kirillov dimension of an $\mathbf{A}^\omega$-module. We prove that "induced $\mathbf{A}^\omega$-modules" are of maximal Gelfand–Kirillov dimension.

In §9, we introduce the quiver Hecke algebra $\mathbf{B}^\omega$ attached to a finite root system (V, R) and with parameter ω. We prove a basis theorem for $\mathbf{B}^\omega$ and we introduce a Frobenius form on $\mathbf{B}^\omega$.

In §10, we prove that the algebra $\mathbf{B}^\omega$ is isomorphic to an idempotent subalgebra of $\mathbf{A}^\omega$. We use this isomorphism to define the Knizhnik–Zamolodchikov functor $\mathbf{V}$, which is a quotient functor. We give characterisations for the kernel of $\mathbf{V}$ in §10.7 and §10.9. The double centraliser property for $\mathbf{V}$ is proven in §10.8.

In Appendix A, we collect some basic facts about the category of pro-objects of abelian categories, which are used to construct completions of the categories $\mathcal{O}_{\lambda_0}(\mathbb{H})$ and $\mathcal{O}_{\ell_0}(\mathbb{K})$.

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Part I
Degenerate double affine Hecke algebras

1 Reminder on affine root systems

We review the notion of affine root systems. The reference is [27].

1.1 Affine reflections on euclidean spaces

Let E be an affine euclidean space of dimension $n > 0$ and let V be its vector space of translations. In particular, V is equipped with a positive definite scalar product $\langle -, - \rangle : V \times V \longrightarrow \mathbf{R}$. The dual space V^* is identified with V via the scalar product $\langle -, - \rangle$. Let $\mathbf{R}[E]^{\leq 1}$ be the space of affine functions on E. We have a map of differential $\partial : \mathbf{R}[E]^{\leq 1} \longrightarrow V^*$ whose kernel is the set of constant functions. The space $\mathbf{R}[E]^{\leq 1}$ is equipped with a symmetric bilinear form $\langle f, g \rangle = \langle \partial f, \partial g \rangle$. For any non-constant function $f \in \mathbf{R}[E]^{\leq 1}$, let $f^\vee = 2f/|f|^2$ and define the reflection with respect to the zero hyperplane of f:

$$s_f : E \longrightarrow E, \quad s_f(x) = x - f^\vee(x)\partial f$$

and

$$s_f : \mathbf{R}[E]^{\leq 1} \longrightarrow \mathbf{R}[E]^{\leq 1}, \quad s_f(g) = g - \langle f^\vee, g \rangle f.$$

It extends to an automorphism of the ring of $\mathbf{C}$-valued polynomial functions $s_f : \mathbf{C}[E] \longrightarrow \mathbf{C}[E]$.

1.2 Affine root systems

An affine root system on E is the pair (E, S), where $S \subset \mathbf{R}[E]^{\leq 1}$ is a subset satisfying the following conditions:

(i) S spans $\mathbf{R}[E]^{\leq 1}$ and the elements of S are non-constant functions on E;

(ii) $s_a(b) \in S$ for all $a, b \in S$;

(iii) $\langle a^\vee, b \rangle \in \mathbf{Z}$ for all $a, b \in S$;

(iv) the group W_S of auto-isometries on E generated by $\{s_a \,;\, a \in S\}$ acts properly on E.

The group W_S is called the affine Weyl group (or simply the Weyl group of S). An affine root system (E, S) is called irreducible if there is no partition $S = S_1 \sqcup S_2$ with $\langle -, - \rangle |_{S_1 \times S_2} = 0$ and $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$; it is called reduced if $a \in S$ implies $2a \notin S$.

Let (E, S) be an affine root system. The set $R = \partial(S) \subset V^*$ is a finite root system on V. Let $P = P_R \subset V$ denote the weight lattice, $Q_R = \mathbf{Z}R$ the root lattice, $P^\vee = P_R^\vee$ the coweight lattice and $Q^\vee = Q_R^\vee = \mathbf{Z}R^\vee$ the coroot lattice.

Conversely, let (V, R) be an irreducible finite root system, reduced or not. Define $R_{\mathrm{red}} = R \setminus 2R$ to be the set of indivisible roots. Let $P = P_R$ be the weight lattice and $Q = Q_R$ the root lattice; we define the affinisation of (V, R) to be the affine root system (E, S) with $E = V$ and

$$S = \langle \alpha + n \,;\, n \in \mathbf{Z}, \alpha \in R_{\mathrm{red}} \rangle \sqcup \langle \alpha + 2m + 1 \,;\, m \in \mathbf{Z}, \alpha \in R \cap 2R \rangle.$$

Given a basis $\Delta_0 \subset R$, we form $\Delta = \Delta_0 \cup \{a_0\}$, where $a_0 = 1 - \theta$ with $\theta \in R$ being the highest root with respect to the basis Δ_0.

1.3 Affine Weyl group

Let (E, S) be the affinisation of (V, R), which is an irreducible reduced affine root system. A basis of S is an $\mathbf{R}$-linearly independent subset $\Delta \subset S$ such that the following conditions are satisfied:

(i) $S \subset \mathbf{N}\Delta \cup -\mathbf{N}\Delta$;

(ii) the set $\bigcap_{a\in\Delta} \{x \in E \; ; \; a(x) > 0\}$ is non-empty.

The W_S-action on S induces a simple transitive W_S-action on the set of bases of S. Upon fixing a basis Δ of S, let $S^+ = S \cap \mathbf{N}\Delta$ and $S^- = S \cap -\mathbf{N}\Delta$ denote the sets of positive and negative roots.

The parabolic Coxeter subgroup $W_R = \langle s_a \; ; \; a \in \Delta_0\rangle$ of W_S can be identified with the Weyl group of the finite root system (V, R) and there is an isomorphism

$$\begin{aligned} Q_R^\vee \rtimes W_R &\cong W_S \\ (\mu, w) &\mapsto X^\mu w, \end{aligned}$$

where the element X^μ acts on S by $a \mapsto a - \langle \partial a, \mu\rangle$. The extended affine Weyl group is defined to be $\tilde{W}_S = P^\vee \rtimes W_R$. It acts on S by extending the W_S-action by the same formula $X^\mu a = a - \langle \partial a, \mu\rangle$ for $\mu \in P^\vee$.

The length function is defined to be

$$\ell : \tilde{W}_S \longrightarrow \mathbf{N}, \quad \ell(w) = \#\left(S^+ \cap w^{-1}S^-\right).$$

It extends the usual length function on the Coxeter group W_S with respect to the set of generators $\{s_a\}_{a\in\Delta}$. We will need the following formula for the length function.

Proposition 1. *For $\mu \in P^\vee$ and $w \in W_R$, we have*

$$\ell(wX^\mu) = \sum_{\alpha \in R^+_{\mathrm{red}} \cap w^{-1}R^-_{\mathrm{red}}} |\langle\alpha, \mu\rangle + 1| + \sum_{\alpha \in R^+_{\mathrm{red}} \cap w^{-1}R^+_{\mathrm{red}}} |\langle\alpha, \mu\rangle| + \sum_{\alpha\in R^+\cap 2R} \frac{|\langle\alpha, \mu\rangle|}{2}.$$

$$\ell(X^\mu w) = \sum_{\alpha \in R^+_{\mathrm{red}} \cap wR^-_{\mathrm{red}}} |\langle\alpha, \mu\rangle - 1| + \sum_{\alpha \in R^+_{\mathrm{red}} \cap wR^+_{\mathrm{red}}} |\langle\alpha, \mu\rangle| + \sum_{\alpha\in R^+\cap 2R} \frac{|\langle\alpha, \mu\rangle|}{2}.$$

These formulae can be obtained by counting the set $S^+ \cap w^{-1}S^-$ along the fibres of the differential map $\partial : S \to R$. □

1.4 Alcoves

For each affine root $a \in S$, let $H_a = \{\lambda \in E \; ; \; a(\lambda) = 0\}$ be the vanishing locus of a. The affine hyperplanes $\{H_a\}_{a\in S}$ yield a simplicial cellular decomposition of E. The open cells are called **alcoves**. Thus the set of connected components

$$\pi_0\left(E \setminus \bigcup_{a\in S} H_a\right)$$

is the set of alcoves. The affine Weyl group W_S acts simply transitively on it. When a basis $\Delta \subset S$ is given, the fundamental alcove is defined to be $\nu_0 = \bigcap_{a\in\Delta} \{x \in E \; ; \; a(x) > 0\}$.

2 Reminder on degenerate double affine Hecke algebra

Let (E, S, Δ) be an irreducible reduced affine root system with a basis. We define in this section the degenerate double affine Hecke algebra $\mathbb{H}$ attached to (E, S, Δ) and its idempotent form $\mathbf{H}_{\lambda_0}$, which is a block algebra for the category $\mathcal{O}$ of $\mathbb{H}$.

2.1 Degenerate double affine Hecke algebra $\mathbb{H}$

Let $h = \{h_a\}_{a\in S}$ be a $\tilde{W}_S$-invariant family of complex numbers. The degenerate double affine Hecke algebra with parameters h attached to the affine root system S is the associative unital $\mathbf{C}$-algebra on the vector space $\mathbb{H} = \mathbf{C}W_S \otimes \mathbf{C}[E]$ whose multiplication satisfies following properties:

- Each of the subspaces $\mathbf{C}W_S$ and $\mathbf{C}[E]$ is given the usual ring structure, so that they are subalgebras of $\mathbb{H}$.
- $w \in \mathbf{C}W_S$ and $f \in \mathbf{C}[E]$ multiply by juxtaposition: $(w \otimes 1)(1 \otimes f) = w \otimes f$.
- $a \in \Delta$ and $f \in \mathbf{C}[E]$ satisfy the equation:

$$(s_a \otimes 1)(1 \otimes f) - (1 \otimes s_a(f))(s_a \otimes 1) = 1 \otimes h_a \frac{f - s_a(f)}{a}.$$

2.2 Global dimension of $\mathbb{H}$

Put a filtration F on $\mathbb{H}$ as follows:

$$F_{\leq -1}\mathbb{H} = 0, \quad F_{\leq 0}\mathbb{H} = \mathbf{C}W_S, \quad F_{\leq 1}\mathbb{H} = (F_{\leq 0}\mathbb{H})\,\mathbf{C}[E]^{\leq 1}, \quad F_{\leq n}\mathbb{H} = (F_{\leq 1}\mathbb{H})^n, \quad n \geq 2.$$

Namely, $\mathbb{H}$ is filtered by its polynomial part $\mathbf{C}[E]$. The filtration F is compatible with the multiplication and its associated graded ring is given by the skew tensor product $\operatorname{gr}^F \mathbb{H} \cong \mathbf{C}W_R \ltimes (\mathbf{C}Q^\vee \otimes \mathbf{C}[V])$. Since $\dim.\mathrm{gl}\,\mathbb{H} \leq \dim.\mathrm{gl}\ \operatorname{gr}^F \mathbb{H}$ ([18, D.2.6]) and since $\dim.\mathrm{gl}\,\mathbf{C}W_R \ltimes (\mathbf{C}Q^\vee \otimes \mathbf{C}[V]) = 2r$, where $r = \operatorname{rk} S = \dim E$, we have the following:

Proposition 2. *The global dimension of $\mathbb{H}$ is at most $2r$.* □

2.3 Category $\mathcal{O}$

For each $\lambda \in E_{\mathbf{C}}$, let $\mathfrak{m}_\lambda \subset \mathbf{C}[E]$ be the defining ideal of the closed point $\lambda \in E$. Given any module $M \in \mathbb{H}$-Mod, for each $\lambda \in E_{\mathbf{C}}$ consider the generalised λ-weight space in M:

$$M_\lambda = \bigcup_{N\geq 0} \left\{a \in M \ ; \ \mathfrak{m}_\lambda^N a = 0\right\}.$$

For any $\lambda_0 \in E_{\mathbf{C}}$, we define $\mathcal{O}_{\lambda_0}(\mathbb{H})$ to be the full subcategory of finitely generated left $\mathbb{H}$-modules $\mathbb{H}$-mod consisting of those M such that

$$M = \bigoplus_{\lambda \in W_S\,\lambda_0} M_\lambda.$$

In other words, the polynomial subalgebra $\mathbf{C}[E]$ acts locally finitely on M with eigenvalues in the W_S-orbit of $\lambda_0 \in E_{\mathbf{C}}$.

From the triangular decomposition $\mathbb{H} = \mathbf{C}Q^\vee \otimes \mathbf{C}W_R \otimes \mathbf{C}[E]$, we deduce the following:

Proposition 3. *For every $\lambda_0 \in E$, every object of $\mathcal{O}_{\lambda_0}(\mathbb{H})$ is a coherent $\mathbf{C}Q^\vee$-module.* □

2.4 Block algebra $\mathbb{H}^\wedge_{\lambda_0}$

In order to study the category $\mathcal{O}_{\lambda_0}(\mathbb{H})$, it is often useful to consider a certain completion of the polynomial part $\mathbf{C}[E]$ at the orbit $W_S\lambda_0 \subset E$. The completion of $\mathbb{H}$ that we will consider is similar to the one from [12] in the context of Gelfand–Zetlin algebras. A similar construction has been employed in [33] for double affine Hecke algebras.

Fix once and for all $\lambda_0 \in E_{\mathbf{C}}$. Define for each $\lambda \in W_S\lambda_0$ a polynomial ring $\mathrm{Pol}_\lambda = \mathbf{C}[V]$ and let $\mathrm{Pol} = \bigoplus_{\lambda\in W_S\lambda_0} \mathrm{Pol}_\lambda$. Define the completion

$$\widehat{\mathrm{Pol}}_\lambda = \varprojlim_N \mathrm{Pol}_\lambda / \mathfrak{m}_0^N \,\mathrm{Pol}_\lambda = \mathbf{C}[\![V]\!], \quad \widehat{\mathrm{Pol}} = \bigoplus_{\lambda\in W_S\lambda_0} \widehat{\mathrm{Pol}}_\lambda,$$

where $\mathfrak{m}_0 \subset \mathrm{Pol}_\lambda$ is the defining ideal of $0 \in V$. The completion $\widehat{\mathrm{Pol}}_\lambda$ is equipped with the $\mathfrak{m}_0$-adic topology and $\widehat{\mathrm{Pol}}$ is equipped with the colimit topology.

For $\lambda \in W_S\lambda_0$, the translation $\lambda_* : V_{\mathbf{C}} \xrightarrow{\lambda +} E_{\mathbf{C}}$ yields an isomorphism

$$\lambda^* : \mathbf{C}[E] \cong \mathbf{C}[V] = \mathrm{Pol}_\lambda\,.$$

We define an action of $\mathbb{H}$ on Pol:

$$\psi = (\psi_\lambda)_\lambda : \mathbb{H} \longrightarrow \mathrm{End}^{\mathrm{cont}}(\widehat{\mathrm{Pol}}), \quad \psi_\lambda : \mathbb{H} \longrightarrow \mathrm{Hom}^{\mathrm{cont}}(\widehat{\mathrm{Pol}}_\lambda, \widehat{\mathrm{Pol}}).$$

by setting, for $f \in \mathbf{C}[E]$ and $a \in \Delta$,

$$\psi_\lambda(f) = \lambda^* f \tag{4}$$

$$\psi_\lambda(s_a - 1) = \begin{cases} -\frac{\lambda^*(a-h_a)}{\lambda^* a}(s_{\partial a} - 1) \in \mathrm{Hom}^{\mathrm{cont}}(\widehat{\mathrm{Pol}}_\lambda, \widehat{\mathrm{Pol}}_\lambda) & a(\lambda) = 0 \\ \frac{\lambda^*(h_a - a)}{\lambda^* a} - \frac{(s_a\lambda)^*(a-h_a)}{(s_a\lambda)^* a} s_{\partial a} \in \mathrm{Hom}^{\mathrm{cont}}(\widehat{\mathrm{Pol}}_\lambda, \widehat{\mathrm{Pol}}_\lambda + \widehat{\mathrm{Pol}}_{s_a\lambda}) & a(\lambda) \neq 0 \end{cases}. \tag{5}$$

Lemma 6. *The map ψ defines a faithful continuous action of $\mathbb{H}$ on $\widehat{\mathrm{Pol}}$.* ☐

Let $\mathbb{H}^\wedge_{\lambda_0} \subset \mathrm{End}^{\mathrm{cont}}(\widehat{\mathrm{Pol}})$ be the closure of the image of ψ. It has a set of topological generators which reflects better than $\mathbb{H}$ the weight-space decomposition of objects of $\mathcal{O}_{\lambda_0}(\mathbb{H})$. For $\lambda \in W_S\lambda_0$, we define a function $\mathrm{ord}_\lambda : S^+ \longrightarrow \mathbf{Z}_{\geq -1}$ by

$$\mathrm{ord}_\lambda(a) = \mathrm{ord}_{z=a(\lambda)}(z - h_a)z^{-1}. \tag{7}$$

Lemma 8. *The topological algebra $\mathbb{H}^\wedge_{\lambda_0}$ is topologically generated by the following elements:*

(i) for each $\lambda \in W_S\lambda_0$, the projector $\mathbf{e}(\lambda) : \widehat{\mathrm{Pol}} \longrightarrow \widehat{\mathrm{Pol}}_\lambda \subset \widehat{\mathrm{Pol}}$,

(ii) the polynomial ring $\mathbf{C}[V]$, which acts diagonally on $\widehat{\mathrm{Pol}}$ by multiplication on each factor $\widehat{\mathrm{Pol}}_\lambda = \mathbf{C}[\![V]\!]$,

(iii) for each $a \in \Delta$ an operator $\tau_a = \sum_{\lambda \in W_S\lambda_0} \tau_a \mathbf{e}(\lambda) : \widehat{\mathrm{Pol}} \longrightarrow \widehat{\mathrm{Pol}}$, where

$$\tau_a \mathbf{e}(\lambda) : \widehat{\mathrm{Pol}}_\lambda \longrightarrow \widehat{\mathrm{Pol}}_{s_a\lambda}, \quad \tau_a f = \begin{cases} (\partial a)^{-1}(s_{\partial a}(f) - f) & \mathrm{ord}_\lambda(a) = -1 \\ (\partial a)^{\mathrm{ord}_\lambda(a)} s_{\partial a}(f) & \mathrm{ord}_\lambda(a) \geq 0 \end{cases} \tag{9}$$

for $f \in \widehat{\mathrm{Pol}}_\lambda = \mathbf{C}[\![V]\!]$, where $\partial a \in R$ is the differential of $a \in S$ and $s_{\partial a} : \mathbf{C}[\![V]\!] \longrightarrow \mathbf{C}[\![V]\!]$ is the reflection with respect to the finite root $\partial a \in R$, see §1.1.

Proof. Let $A \subset \mathrm{End}^{\mathrm{cont}}(\widehat{\mathrm{Pol}})$ denote the closure of the subalgebra generated by the three set of operators $\mathbf{e}(\lambda)$, $\mathbf{C}[V]$ and τ_a. We need to show that $A = \mathbb{H}^\wedge_{\lambda_0}$.

Consider the restriction $\psi\,|_{\mathbf{C}[E]}$. It factorises as

$$\mathbf{C}[E] \longrightarrow \prod_{\lambda \in W_S\lambda_0} \varprojlim_k \mathbf{C}[E]/\mathfrak{m}_\lambda^k \xrightarrow{\cong} \prod_\lambda \widehat{\mathrm{Pol}}_\lambda,$$

where $\mathfrak{m}_\lambda \subset \mathbf{C}[E]$ is the defining ideal of the closed point $\lambda \in E$. The Chinese remainder theorem implies that the map has dense image. In particular, $\mathbf{e}(\lambda) \in \widehat{\mathrm{Pol}}_\lambda \subset \widehat{\mathrm{Pol}}$ lies in the closure of image for each $\lambda \in W_S\lambda_0$. Therefore, $\mathbb{H}^\wedge_{\lambda_0}$ contains the closure of $\psi(\mathbf{C}[E])\mathbf{e}(\lambda)$ in $\mathrm{Hom}^{\mathrm{cont}}(\widehat{\mathrm{Pol}}, \widehat{\mathrm{Pol}})$; the latter is equal to the algebra $\mathbf{C}[\![V]\!]\mathbf{e}(\lambda)$ which acts on $\widehat{\mathrm{Pol}}_\lambda$ by multiplication. Thus we have $\widehat{\mathrm{Pol}} \subset A$ and $\widehat{\mathrm{Pol}} \subset \mathbb{H}^\wedge_{\lambda_0}$. It remains to show that $\{\psi(s_a)\}_{a\in\Delta}$ lies in A and $\{\tau_a\}_{a\in\Delta}$ lies in $\mathbb{H}^\wedge_{\lambda_0}$.

For each $a \in \Delta$ and $\lambda \in W_S\lambda_0$, by comparison of the formulae (4) and (9), we see that the elements $\mathbf{e}(s_a\lambda)\psi(s_a - 1)\mathbf{e}(\lambda)$ and $\tau_a\mathbf{e}(\lambda)$ generate the same cyclic left $\mathbf{C}[\![V]\!]$-submodule of $\mathrm{Hom}^{\mathrm{cont}}(\widehat{\mathrm{Pol}}_\lambda, \widehat{\mathrm{Pol}}_{s_a\lambda})$. In particular, $\tau_a\mathbf{e}(\lambda)$ lies in $\mathbb{H}^\wedge_{\lambda_0}$ and conversely, $\mathbf{e}(s_a\lambda)\psi(s_a - 1)\mathbf{e}(\lambda)$ lies in A. For $\lambda \in W_S\lambda_0$ such that $s_a\lambda = \lambda$, we have $\psi(s_a - 1)\mathbf{e}(\lambda) = \mathbf{e}(\lambda)\psi(s_a - 1)\mathbf{e}(\lambda) \in A$. For $\lambda \in W_S\lambda_0$ such that $s_a\lambda \neq \lambda$, we have

$$\psi(s_a - 1)\mathbf{e}(\lambda) = \mathbf{e}(\lambda)\psi(s_a - 1)\mathbf{e}(\lambda) + \mathbf{e}(s_a\lambda)\psi(s_a - 1)\mathbf{e}(\lambda),$$
$$\mathbf{e}(\lambda)\psi(s_a - 1)\mathbf{e}(\lambda) = \frac{\lambda^*(h_a - a)}{\lambda^* a}, \quad \mathbf{e}(s_a\lambda)\psi(s_a - 1)\mathbf{e}(\lambda) = -\frac{(s_a\lambda)^*(a - h_a)}{(s_a\lambda)^* a} s_{\partial a};$$

since $\lambda^*(h_a - a)/\lambda^* a \in \mathbf{C}[\![V]\!]\mathbf{e}(\lambda) \subset A$, it follows that $\psi(s_a - 1)\mathbf{e}(\lambda) \in A$. Summing over the idempotents, we obtain

$$\psi(s_a - 1) = \sum_{\lambda \in W_S\lambda_0} \psi(s_a - 1)\mathbf{e}(\lambda) \in A, \quad \tau_a = \sum_{\lambda \in W_S\lambda_0} \tau_a\mathbf{e}(\lambda) \in \mathbb{H}^\wedge_{\lambda_0}.$$

The result follows. ☐

Let $\mathbb{H}^\wedge_{\lambda_0}\text{-mod}^{\mathrm{sm}}$ be the category of finitely generated $\mathbb{H}^\wedge_{\lambda_0}$-modules M such that for each element $m \in M$, the annihilator $\mathrm{ann}_{\mathbb{H}^\wedge_{\lambda_0}}(m)$ is an open left ideal of $\mathbb{H}^\wedge_{\lambda_0}$. Notice that these conditions imply

$$M = \bigoplus_{\lambda \in W_S\lambda_0} \mathbf{e}(\lambda)M \quad \text{and} \quad \dim \mathbf{e}(\lambda)M < \infty, \quad \text{for } M \in \mathbb{H}^\wedge_{\lambda_0}\text{-mod}^{\mathrm{sm}}.$$

Lemma 10. *The restriction ψ^* yields an equivalence of categories*

$$\mathbb{H}^\wedge_{\lambda_0}\text{-mod}^{\mathrm{sm}} \cong \mathcal{O}_{\lambda_0}(\mathbb{H}). \qquad \square$$

2.5 Idempotent form $\mathbf{H}_{\lambda_0}$

In view of Lemma 10 and Lemma 8, in order to study the block $\mathcal{O}_{\lambda_0}(\mathbb{H})$, it is convenient to consider the subalgebra generated by the generators given in Lemma 8.

Observe that the operators $\{\mathbf{e}(\lambda)\}_{\lambda \in W_S\lambda_0}$, $\mathbf{C}[V]$ and $\{\tau_a\}_{a\in\Delta}$ preserve the dense submodule $\mathrm{Pol} \subset \widehat{\mathrm{Pol}}$. Let $\mathbf{H}_{\lambda_0}$ be the associative (non-unital) subalgebra of $\mathrm{End}_{\mathbf{C}}(\mathrm{Pol})$ generated these operators. Let $\mathbf{H}_{\lambda_0}\text{-mod}_0$ be the category of finitely generated $\mathbf{H}_{\lambda_0}\text{-mod}_0$-modules M such that $M = \bigoplus_{\lambda\in W_S\lambda_0} \mathbf{e}(\lambda)M$ and such that the subspace $V^* \subset \mathbf{C}[V]$ acts locally nilpotently on M.

Lemma 11. *There is a natural inclusion $\mathbf{H}_{\lambda_0} \hookrightarrow \mathbb{H}^\wedge_{\lambda_0}$ with dense image, which induces an equivalence of categories by pulling back the module-structure:*

$$\mathbb{H}^\wedge_{\lambda_0}\text{-mod}^{\mathrm{sm}} \xrightarrow{\cong} \mathbf{H}_{\lambda_0}\text{-mod}_0 .$$

Proof. By the density of the submodule $\mathrm{Pol} \subset \widehat{\mathrm{Pol}}$ and Lemma 8, there is a unique inclusion $\mathbf{H}_{\lambda_0} \hookrightarrow \mathbb{H}^\wedge_{\lambda_0}$ with dense image which fixes the generators $\{\mathbf{e}(\lambda)\}_{\lambda\in W_S\lambda_0}$, $\mathbf{C}[V]$ and $\{\tau_a\}_{a\in\Delta}$. The assertion on the equivalence of category follows straightforward from the density. $\square$

Combining the equivalences of Lemma 10 and Lemma 11, we obtain the following result:

Proposition 12. *There is an equivalence of categories*

$$\mathcal{O}_{\lambda_0}(\mathbb{H}) \cong \mathbf{H}_{\lambda_0}\text{-mod}_0 . \qquad \square$$

Remark 13. *In §6, we will attach to each family of functions $\{\omega_\lambda\}_{\lambda\in W_S\lambda_0}$ an algebra $\mathbf{A}^\omega$. We will study them in a larger generality. The algebra $\mathbf{H}_{\lambda_0}$ is the special case where $\omega_\lambda = \mathrm{ord}_\lambda$ for $\lambda \in W_S\lambda_0$.*

2.6 Central subalgebra $\mathcal{Z}^\wedge$

For $\lambda \in W_S\lambda_0$, let W_λ denote the stabiliser of λ in W_S. The stabiliser W_λ is a finite parabolic subgroup of the Coxeter group W_S. The affine Weyl group W_S acts on the vector space $V_{\mathbf{C}}$ via the finite quotient* $\partial^W : W_S \longrightarrow W_S/Q^\vee \cong W_R$. Let $\mathcal{Z}^\wedge = \mathbf{C}[\![V]\!]^{W_{\lambda_0}}$ be the ring of W_{λ_0}-invariant formal power series. Since W_{λ_0} acts by reflections on V, the ring $\mathcal{Z}^\wedge$ is a complete regular local ring. Let $\mathfrak{m}_{\mathcal{Z}} \subset \mathcal{Z}^\wedge$ be the maximal ideal.

For each $\lambda \in W_S\lambda_0$, we define a homomorphism $\mathcal{Z}^\wedge \longrightarrow \widehat{\mathrm{Pol}}_\lambda$: choosing a $w \in W_S$ such that $w\lambda_0 = \lambda$, we let $f \mapsto w(f) \in \mathbf{C}[\![V]\!]^{W_{w\lambda}} \subset \widehat{\mathrm{Pol}}_\lambda$. This map is clearly independent of the choice of w and it identify $\mathcal{Z}^\wedge$ with the invariant subspace $\mathbf{C}[\![V]\!]^{W_{w\lambda}}$. The space $\widehat{\mathrm{Pol}}$ is regarded as a $\mathcal{Z}^\wedge$-module via the diagonal action. It is easy to observe that $\mathcal{Z}^\wedge$ lies in the centre of $\mathbb{H}^\wedge_{\lambda_0}$.

Remark 14. *One can show that $\mathcal{Z}^\wedge$ coincides with the centre of $\mathbb{H}^\wedge_{\lambda_0}$; however, we do not need this fact.*

3 Reminder on affine Hecke algebra

We keep the notation (V, R, Δ_0), (E, S, Δ) and $h = \{h_a\}_{a\in S}$ as above.

*The notation is chosen so that $(\partial^W w)(\partial a) = \partial(wa)$ for $a \in S$ and $w \in W_S$ as well as $\partial^W s_a = s_{\partial a}$ for $a \in S$.

3.1 Extended affine Hecke algebras

Put

$$v = \{v_\alpha\}_{\alpha \in R}, \quad v_\alpha = \begin{cases} \exp(\pi i h_\alpha) & \alpha \in R_{\mathrm{red}} \\ \exp(\pi i h_{\alpha+1}) & \alpha \in R \cap 2R \end{cases}.$$

Recall that $\tilde{W}_S^\vee = P \rtimes W_R$ is the dual extended affine Weyl group (we identify W_R with $W_{R^\vee}$ via the correspondence $s_\alpha \leftrightarrow s_{\alpha^\vee}$). Define the extended affine braid group $\mathfrak{B}_S$ for the dual root system $(V^*, R^\vee)$ to be the group generated by T_w for $w \in \tilde{W}_S^\vee$ with the following relation for each $y, w \in \tilde{W}_S^\vee$:

$$T_y T_w = T_{yw}, \quad \text{if } \ell(yw) = \ell(y) + \ell(w).$$

The extended affine Hecke algebra in parameters v, denoted by $\mathbb{K}$, is the quotient of the group algebra $\mathbf{C}\mathfrak{B}_S$ by the following relations for $\alpha \in \Delta_0$, in the case where R is reduced:

$$(T_{s_\alpha} - v_\alpha^2)(T_{s_{\theta^\vee}} + 1) = 0, \quad (T_{s_0} - v_\theta^2)(T_{s_0} + 1) = 0$$

where $s_0 \in \tilde{W}_S^\vee$ is the reflection with respect to the affine simple root and $\theta \in R^+$ is the highest root. In the case where R is non-reduced, let $\beta \in \Delta_0$ be the simple root such that $2\beta \in R$. Let $\mathbb{K}$ be the quotient of $\mathbf{C}\mathfrak{B}_S$ by the following relations for $\alpha \in \Delta_0 \setminus \{\beta\}$:

$$\begin{aligned} &(T_{s_\alpha} - v_\alpha^2)(T_{s_\alpha} + 1) = 0 \\ &(T_{s_\beta} - v_\beta^2 v_\theta)(T_{s_\beta} + 1) = 0 \\ &(T_{s_0} - v_\beta^2 v_\theta^{-1})(T_{s_0} + 1) = 0. \end{aligned}$$

3.2 Bernstein–Lusztig presentation

Choose a square root $v_\theta^{1/2}$ of v_θ. Define a group homomorphism $v : \mathfrak{B}_S \longrightarrow \mathbf{C}^\times$ by setting $v(s_\alpha) = v_\alpha$ for $\alpha \in \Delta_0$ and $v(s_0) = v_\theta$ in the case where R is reduced; $v(s_\alpha) = v_\alpha$, $v(s_\beta) = v_\beta v_\theta^{1/2}$ and $v(s_0) = v_\beta v_\theta^{-1/2}$ in the case where R is non-reduced and $\beta \in \Delta_0$ with $2\beta \in R$ and $\alpha \in \Delta_0 \setminus \{\beta\}$.

There is a subalgebra $\mathbf{C}P \subset \mathbb{K}$ given by $\mu \mapsto v(\mu) T_\mu$ for $\mu \in P \subset \tilde{W}_S^\vee$ dominant with respect to the basis Δ_0. For $\beta \in P$ in general, we decompose it into $\beta = \beta' - \beta''$ with β' and β'' dominant and set $Y^\beta = T_{\beta'} T_{\beta''}^{-1}$. Then there is a decomposition

$$\mathbb{K} = H_R \otimes \mathbf{C}P,$$

where H_R is the subalgebra generated by $\{T_{s_\alpha}\}_{a \in \Delta_0}$ and $\mathbf{C}P$ is the subalgebra generated by $\{Y^\beta\}_{\beta \in P}$, with the following commutation relations: for each $f \in \mathbf{C}P$,

$$T_{s_\alpha} f - s_\alpha(f) T_{s_\alpha} = (v_\alpha^2 - 1) \frac{f - s_\alpha(f)}{1 - Y^{-\alpha}}, \qquad \alpha \in \Delta_0,\ 2\alpha \notin R \tag{15}$$

$$T_{s_\beta} f - s_\beta(f) T_{s_\beta} = \left((v_\beta^2 v_\theta - 1) + \left(v_\beta^2 - v_\theta\right) Y^{-\beta} \right) \frac{f - s_\beta(f)}{1 - Y^{-2\beta}}, \qquad \beta \in \Delta_0,\ 2\beta \in R. \tag{16}$$

3.3 Finite dimensional modules

Let T be the torus defined by $T = Q^\vee \otimes \mathbf{C}^\times$ so that $Q^\vee = \mathbf{X}_*(T)$ is its group of cocharacters and $P = \mathbf{X}^*(T)$ is its group of characters. We view $\mathbf{C}[T] = \mathbf{C}P$ as a subalgebra of $\mathbb{K}$.

For each $\ell \in T$, let $\mathfrak{m}_\ell \subset \mathbf{C}P$ denote the defining ideal of the closed point ℓ, which is generated by $Y^\beta - Y^\beta(\ell) \in \mathbf{C}P$ for all $\beta \in P$. Given any module $M \in \mathbb{K}$-Mod, consider for each $\ell \in T$ the generalised ℓ-weight space in M of the action of the subalgebra $\mathbf{C}P \subset \mathbb{K}$:

$$M_\ell = \bigcup_{N \geq 0} \left\{ a \in M \ ; \ \mathfrak{m}_\ell^N a = 0 \right\}.$$

For any $\ell_0 \in T$, we define $\mathcal{O}_{\ell_0}(\mathbb{K})$ to be the full subcategory of $\mathbb{K}$-mod consisting of those $M \in \mathbb{K}$-mod which admit a decomposition by weight:

$$M = \bigoplus_{\ell \in W \ell_0} M_\ell.$$

3.4 Idempotent form $\mathbf{K}_{\ell_0}$

Fix $\ell_0 \in T$. As in the case of $\mathbb{H}$, we define an algebra which is more adapted to the study of the block $\mathcal{O}_{\ell_0}(\mathbb{K})$. Define for each $\ell \in W_R\ell_0$ a polynomial ring $\mathrm{Pol}_\ell = \mathbf{C}[V]$ and let $\mathrm{Pol} = \bigoplus_{\ell \in W_R\ell_0} \mathrm{Pol}_\ell$. For each ℓ, define $\mathbf{e}(\ell) : \mathrm{Pol} \longrightarrow \mathrm{Pol}$ to be the idempotent linear endomorphism of projection onto the factor Pol_ℓ. Let $R^+_{\mathrm{red}} = R^+ \setminus 2R^+$ denote the set of indivisible positive roots. In view of (15), for $\ell \in W_R\ell_0$, we define a function $\mathrm{ord}_\ell : R^+_{\mathrm{red}} \longrightarrow \mathbf{Z}$:

$$\mathrm{ord}_\ell(\alpha) = \begin{cases} \mathrm{ord}_{z=Y^\alpha(\ell)}(z - v_\alpha^2)(z-1)^{-1} & 2\alpha \notin R \\ \mathrm{ord}_{z=Y^\alpha(\ell)}(z - v_\alpha^2)(z+v_\theta)(z^2-1)^{-1} & 2\alpha \in R. \end{cases}$$

For each $\alpha \in \Delta_0$ and $\ell \in W_R\ell_0$, we define an operator $\tau_\alpha \mathbf{e}(\ell) : \mathrm{Pol}_\ell \longrightarrow \mathrm{Pol}_{s_\alpha \ell}$ by

$$\tau_\alpha \mathbf{e}(\ell) = \begin{cases} \alpha^{-1}(s_\alpha - 1) & \mathrm{ord}_\ell(\alpha) = -1 \\ \alpha^{\mathrm{ord}_\ell(\alpha)} s_\alpha & \mathrm{ord}_\ell(\alpha) \geq 0 \end{cases}.$$

Here $s_\alpha : \mathbf{C}[V] \longrightarrow \mathbf{C}[V]$ is the reflection with respect to α.

Let $\mathbf{K}_{\ell_0}$ be the associative subalgebra of $\mathrm{End}_{\mathbf{C}}(\mathrm{Pol})$ generated by $f\mathbf{e}(\ell)$ and $\tau_\alpha \mathbf{e}(\ell)$ for $f \in \mathbf{C}[V]$, $\alpha \in \Delta_0$ and $\ell \in W_R\ell_0$. Let $\mathbf{K}_{\ell_0}$-mod_0 be the category of finitely generated $\mathbf{K}_{\lambda_0}$-mod_0-modules M such that the subspace $V^* \subset \mathbf{C}[V]$ acts locally nilpotently on M. Same arguments as Lemma 10 and Lemma 11 show that:

Proposition 17. *There is an equivalence of categories*

$$\mathcal{O}_{\ell_0}(\mathbb{K}) \cong \mathbf{K}_{\ell_0}\text{-}\mathrm{mod}_0 .$$ □

Remark 18. *In §9, we will attach to each family of functions $\{\omega_\ell\}_{\ell \in W_R\ell_0}$ an algebra $\mathbf{B}^\omega$. The algebra $\mathbf{K}_{\lambda_0}$ is the special case of $\mathbf{B}^\omega$ with $\omega_\ell = \mathrm{ord}_\ell$ for $\ell \in W_R\ell_0$.*

4 The monodromy functor $\mathbb{V}$

In this section, we review the construction of the monodromy functor of [34], which is a trigonometric analogue of the Knizhnik–Zamolodchikov functor introduced in [16] for rational Cherednik algebras. We prove in Proposition 27 that this functor is a quotient functor.

Keep the notation (E, S, Δ) and $a_0 \in \Delta$ as above. In addition, we fix $\lambda_0 \in E_{\mathbf{C}}$. Consider the following exponential map

$$\begin{gathered} E_{\mathbf{C}} \cong V_{\mathbf{C}} = Q^\vee \otimes \mathbf{C} \xrightarrow{\exp} Q^\vee \otimes \mathbf{C}^\times = T \\ \mu \otimes r \mapsto \mu \otimes e^{2\pi i r}. \end{gathered} \tag{19}$$

Put $\ell_0 = \exp(\lambda_0)$. For simplifying the notation, denote $\mathcal{C}_0 = \mathcal{O}_{\lambda_0}(\mathbb{H})$ and $\mathcal{B}_0 = \mathcal{O}_{\ell_0}(\mathbb{K})$.

4.1 Dunkl operators

Consider the dual torus $T^\vee = P \otimes \mathbf{C}^\times$. The ring of regular functions $\mathbf{C}[T^\vee]$ is isomorphic to the group algebra of the coroot lattice $\mathbf{C}Q^\vee$:

$$\begin{gathered} \mathbf{C}Q^\vee \xrightarrow{\cong} \mathbf{C}[T^\vee] \\ Q^\vee \ni \mu \mapsto X^\mu \end{gathered}$$

For each $\xi \in V^*$, let $\partial_\xi \in \Gamma(T^\vee, \mathcal{T}_{T^\vee})^{T^\vee}$ be the translation-invariant vector field on $T^\vee$ such that $\partial_\xi |_e = \xi$ under the isomorphism $\mathcal{T}_{T^\vee} |_e \cong V$. We view ∂_ξ as a linear differential operator on $T^\vee$, so that $\partial_\xi(X^\mu) = \langle \xi, \mu \rangle X^\mu$ for each $\mu \in Q^\vee$.

The regular part of $T^\vee$ is defined as $T^\vee_\circ = \bigcap_{\alpha \in R^+} \left\{ X^{\alpha^\vee} \neq 1 \right\} \subset T^\vee$. Let $\mathcal{D}(T^\vee_\circ)$ denote the ring of algebraic differential operators on $T^\vee_\circ$.

For $\xi \in V^*$, the trigonometric Dunkl operator $D_\xi : \mathbf{C}[T^\vee] \longrightarrow \mathbf{C}[T^\vee]$ is the $\mathbf{C}$-linear operator defined as follows:

$$D_\xi(f) = \partial_\xi(f) - \sum_{\alpha \in R^+} h_\alpha \langle \xi, \alpha^\vee \rangle \frac{f - s_\alpha(f)}{1 - X^{-\alpha^\vee}} + \langle \xi, \rho_h^\vee \rangle f, \quad \rho_h^\vee = \frac{1}{2} \sum_{\alpha \in \Delta^+} h_\alpha \alpha^\vee \in V_{\mathbf{C}}.$$

We consider D_ξ as an element of $\mathcal{D}(T_\circ^\vee) \rtimes W_R$.

According to [34, 4.1], the following homomorphism of $\mathbf{C}$-algebras

$$\begin{aligned} \mathbf{C}[T^\vee] \otimes \mathbf{C}W_R \otimes \mathbf{C}[V] = \mathbb{H} &\longrightarrow \mathcal{D}(T_\circ^\vee) \rtimes W_R \\ X^\mu \otimes w \otimes 1 &\mapsto X^\mu \otimes w \\ 1 \otimes 1 \otimes \xi &\mapsto D_\xi \end{aligned}$$

extends to an isomorphism $\mathbf{C}[T_\circ^\vee] \otimes_{\mathbf{C}[T^\vee]} \mathbb{H} \cong \mathcal{D}(T_\circ^\vee) \rtimes W_R$.

4.2 Monodromy functor $\mathbb{V}$

Let $[T_\circ^\vee/W_R]$ be the quotient stack. According to [17, 2.5], there is an isomorphism between the orbifold fundamental group $\pi_1([T_\circ^\vee/W_R])$ and the extended affine braid group $\mathfrak{B}_S$ from §3.1.

If $M \in \mathcal{O}_{\lambda_0}(\mathbb{H})$, then

$$M_\circ = \mathbf{C}[T_\circ^\vee] \otimes_{\mathbf{C}[T^\vee]} M$$

is a W-equivariant $\mathcal{D}(T_\circ^\vee)$-module, which is in fact an integrable connection with regular singularities. Therefore the monodromy representation on the vector space of flat sections of M on (the universal covering of) the orbifold $[T_\circ^\vee/W]$ defines a $\mathfrak{B}_S$-module, which is denoted by $\mathbb{V}(M)$. It is shown in [34, 5.1] that the $\mathfrak{B}_S$-action on $\mathbb{V}(M)$ factorises through the surjective algebra homomorphism $\mathbf{C}\mathfrak{B}_S \longrightarrow \mathbb{K}$ and yields an exact functor

$$\mathbb{V} : \mathcal{O}_{\lambda_0}(\mathbb{H}) \longrightarrow \mathcal{O}_{\ell_0}(\mathbb{K}).$$

4.3 Central actions of $\mathcal{Z}^\wedge$ intertwined by $\mathbb{V}$

For convenience, we denote $\mathcal{C}_0 = \mathcal{O}_{\lambda_0}(\mathbb{H})$ and $\mathcal{B}_0 = \mathcal{O}_{\ell_0}(\mathbb{K})$. Recall the central subalgebra $\mathcal{Z}^\wedge = \mathbf{C}[\![V]\!]^{W_{\lambda_0}}$ defined in §2.6. Let $\mathrm{Z}(\mathcal{C}_0) = \mathrm{End}(\mathrm{id}_{\mathcal{C}_0})$ and $\mathrm{Z}(\mathcal{B}_0) = \mathrm{End}(\mathrm{id}_{\mathcal{B}_0})$ denote the categorical centres.

Let W_{λ_0} be the stabiliser of $\lambda_0 \in E_\mathbf{C}$ in W_S and let W_{ℓ_0} be the stabiliser of $\ell_0 \in T$ in W_R. Let $\bar\lambda_0$ be the image of λ_0 in $E_\mathbf{C}/W_{\lambda_0}$ and let $\bar\ell_0$ be the image of ℓ_0 in T/W_{ℓ_0}. The exponential map (19) induces an analytic map

$$\exp^{\lambda_0} : E_\mathbf{C}/W_{\lambda_0} \longrightarrow T/W_{\ell_0},$$

which is locally biholomorphic near $\bar\lambda_0$. The push-forward along $\exp^{\lambda_0}$ at $\bar\lambda_0$ yields an isomorphism of complete local rings

$$\exp_*^{\lambda_0} : \mathcal{O}^\wedge_{E_\mathbf{C}/W_{\lambda_0},\bar\lambda_0} \xrightarrow{\sim} \mathcal{O}^\wedge_{T/W_{\ell_0},\bar\ell_0}.$$

Note that $\mathcal{Z}^\wedge \cong \mathcal{O}^\wedge_{E_\mathbf{C}/W_{\lambda_0},\bar\lambda_0}$. For each $w \in W_S$, the action of w on $E_\mathbf{C}$ and on T (the latter via the quotient map ∂^W from §2.6) induces

$$w_* : \mathcal{O}^\wedge_{E_\mathbf{C}/W_{\lambda_0},\bar\lambda_0} \xrightarrow{\sim} \mathcal{O}_{E_\mathbf{C}/W_{w\lambda_0},w\bar\lambda_0}, \quad w_* : \mathcal{O}^\wedge_{T/W_{\ell_0},\bar\ell_0} \cong \mathcal{O}^\wedge_{T/W_{w\ell_0},w\bar\ell_0}.$$

We define homomorphisms $\mathcal{Z}^\wedge \longrightarrow \mathrm{Z}(\mathcal{C}_0)$ and $\mathcal{Z}^\wedge \longrightarrow \mathrm{Z}(\mathcal{B}_0)$ as follows: for any $M \in \mathcal{C}_0$, we decompose $M = \bigoplus_{\lambda \in W_S\lambda_0} M_\lambda$ and for each $\lambda = w\lambda_0$, an element $f \in \mathcal{Z}^\wedge$ acts by $w_* f$ on M_λ. This depends only on the weight λ but not on the choice of w. Similarly, for any $N \in \mathcal{B}_0$, we decompose $N = \bigoplus_{\ell \in W_R\ell_0} N_\ell$. For each $\ell = w\ell_0$, an element $f \in \mathcal{Z}^\wedge$ acts by multiplication by $w_* \exp_*^{\lambda_0} f$ on N_ℓ.

Lemma 20. *The functor $\mathbb{V} : \mathcal{C}_0 \longrightarrow \mathcal{B}_0$ intertwines the $\mathcal{Z}^\wedge$-actions on $\mathcal{C}_0$ and $\mathcal{B}_0$.*

Proof. Recall that the graded affine Hecke algebra is the subalgebra

$$\underline{\mathbb{H}} = \mathbf{C}W_R \otimes \mathrm{Sym}\, V_\mathbf{C}^* \subset \mathbb{H}.$$

For each weight $\lambda \in V_{\mathbf{C}}$, let $\mathcal{O}_\lambda(\underline{\mathbb{H}})$ be the category of finite-dimensional $\underline{\mathbb{H}}$-modules on which the action of the polynomial part $\operatorname{Sym} V_{\mathbf{C}}$ has weights lying in the orbit $W_S\lambda \subset V_{\mathbf{C}}$.

There is a functor of induction

$$\operatorname{Ind}_{\underline{\mathbb{H}}}^{\mathbb{H}} : \underline{\mathbb{H}}\text{-mod} \longrightarrow \mathbb{H}\text{-mod}, \quad \operatorname{Ind}_{\underline{\mathbb{H}}}^{\mathbb{H}} M = \mathbb{H} \otimes_{\underline{\mathbb{H}}} M$$

and for each weight $\lambda \in E_{\mathbf{C}}$, it restricts to

$$\operatorname{Ind}_{\underline{\mathbb{H}}}^{\mathbb{H}} : \mathcal{O}_\lambda(\underline{\mathbb{H}}) \longrightarrow \mathcal{O}_\lambda(\mathbb{H})$$

Let $\mathcal{I} \subset \mathcal{C}_0$ denote the essential image of $\operatorname{Ind}_{\underline{\mathbb{H}}}^{\mathbb{H}}$. It is known that $\mathcal{I}$ generates $\mathcal{C}_0$ — indeed, the module $P(\lambda)_n = \mathbb{H}/\mathbb{H}\cdot\mathfrak{m}_\lambda^n$ lies in $\mathcal{I}$ and the family $\{P(\lambda)_n\}_{n\in\mathbf{N},\ \lambda\in W_S\lambda_0}$ generate $\mathcal{C}_0$. Therefore, it suffices to show that the restriction $\mathbb{V}\mid_{\mathcal{I}}$ intertwines the actions of $\mathcal{Z}^\wedge$. We shall apply the deformation argument from [34, 5.1] to check this statement.

Let $\mathcal{O} = \mathbf{C}[\![\varpi]\!]$ and let $\mathcal{K} = \mathbf{C}(\!(\varpi)\!)$. Let $\varepsilon \in V_{\mathbf{C}}^*$ be any regular coweight and put $\lambda_{0,\mathcal{O}} = \lambda_0 + \varpi\varepsilon \in V_{\mathcal{O}}^*$. Put $\underline{\mathbb{H}}_{\mathcal{O}} = \underline{\mathbb{H}} \otimes \mathcal{O}$ and $\mathbb{K}_{\mathcal{O}} = \mathbb{K} \otimes \mathcal{O}$. For each $\lambda_{\mathcal{O}} \in W_S\lambda_{0,\mathcal{O}}$ and for $n \in \mathbf{Z}_{\geq 1}$, let

$$\begin{aligned}
\mathfrak{m}_{\lambda_{\mathcal{O}}} &= \langle \beta_{\mathcal{O}} - \langle \beta_{\mathcal{O}}, \lambda_{\mathcal{O}}\rangle \ ;\ \beta \in V_{\mathcal{O}}\rangle \subset \operatorname{Sym}_{\mathcal{O}} V_{\mathcal{O}}^*, \quad \mathfrak{m}_{\lambda_{\mathcal{K}}} = \mathfrak{m}_{\lambda_{\mathcal{O}}}[\varpi^{-1}],\\
\mathbb{S}_{\lambda_{\mathcal{O}}^n} &= \operatorname{Sym}_{\mathcal{O}} V_{\mathcal{O}}^*/\mathfrak{m}_{\lambda_{\mathcal{O}}}^n, \quad \mathbb{S}_{\lambda_{\mathcal{K}}^n} = \mathbb{S}_{\lambda_{\mathcal{O}}^n}[\varpi^{-1}],\\
\underline{P}(\lambda_{\mathcal{O}})_n &= \underline{\mathbb{H}}_{\mathcal{O}} \otimes_{\operatorname{Sym}_{\mathcal{O}} V_{\mathcal{O}}^*} \mathbb{S}_{\lambda_{\mathcal{O}}^n}, \quad \underline{P}(\lambda_{\mathcal{K}})_n = \underline{\mathbb{H}}_{\mathcal{K}}[\varpi^{-1}].
\end{aligned}$$

Note that all these objects are flat over $\mathcal{O}$. Let $\underline{P}(\lambda_{\mathcal{O}})_n^\nabla$ be the space of flat sections of the affine Knizhnik–Zamolodchikov equation (AKZ) on the constant vector bundle on $T_\circ^\vee$ of fibre $\underline{P}(\lambda_{\mathcal{O}})_n$. The monodromy representation yields a $\mathbb{K}_{\mathcal{O}} = \mathbb{K} \otimes \mathcal{O}$ action on $\underline{P}(\lambda_{\mathcal{O}})_n^\nabla$.

Since the stabiliser of $\lambda_{\mathcal{O}}$ in W_S is trivial, there is an eigenspace decomposition

$$\underline{P}(\lambda_{\mathcal{K}})_n = \bigoplus_{w\in W_R} (\underline{P}(\lambda_{\mathcal{K}})_n)_{w\lambda_{\mathcal{K}}}, \quad (\underline{P}(\lambda_{\mathcal{K}})_n)_{w\lambda_{\mathcal{K}}} = b_w \mathbb{S}_{\lambda_{\mathcal{K}}^n},$$

where each $b_w\mathbb{S}_{\lambda_{\mathcal{K}}^n}$ is a free $\mathbb{S}_{\lambda_{\mathcal{K}}^n}$-module of rank 1. Consider the boundary point of $T^\vee$:

$$\lim_{n\to+\infty} \exp(ni\rho) = \left(X^{\alpha^\vee} = 0\right)_{\alpha\in R^+}, \quad \text{where } \rho = (1/2)\sum_{\alpha\in R^+} \alpha.$$

Applying the Frobenius method around this point, we obtain a fundamental solution $\{b_w^\nabla\}_{w\in W_R}$ of the AKZ equation on $T_\circ^\vee$ which satisfies

$$b_w^\nabla(\exp(\mu)) = e^{-2\pi i(\langle\mu,\rho_h^\vee\rangle+\mu)} \cdot (b_w + G(\mu))$$
$$\text{for } \mu \in V_{\mathbf{C}}^* \text{ such that } \mathfrak{Im}\,\langle \mu, \alpha^\vee\rangle \gg 0,\ \forall\alpha \in \Delta^+,$$

where $G(\mu)$ is a $\underline{P}(\lambda_{\mathcal{K}})_n$-valued analytic function in μ with such that

$$G(\mu) \longrightarrow 0 \quad \text{when } \mathfrak{Im}\,\langle \alpha, \mu\rangle \longrightarrow +\infty, \forall\alpha \in \Delta^+.$$

The fundamental solution induces an $\mathbb{S}_{\lambda_{\mathcal{K}}^n}$-linear isomorphism

$$\underline{P}(\lambda_{\mathcal{K}})_n \xrightarrow{\sim} \underline{P}(\lambda_{\mathcal{K}})_n^\nabla, \quad b_w \mapsto b_w^\nabla. \tag{21}$$

Under this isomorphism, the monodromy operator on the right-hand side corresponding to $\beta \in X$ is identified with $e^{2\pi i\beta}$ on the left-hand side. Put

$$\mathcal{Z}_{\mathcal{O}}^\wedge = \left((\operatorname{Sym} V_{\mathcal{O}}^*)^{W_{\lambda_0}}\right)_{\bar{\lambda}_{0,\mathcal{O}}}^\wedge, \quad \mathcal{Z}_{\mathcal{K}}^\wedge = \mathcal{Z}_{\mathcal{O}}^\wedge[\varpi^{-1}] \cong (\operatorname{Sym} V_{\mathcal{K}}^*)_{\bar{\lambda}_{0,\mathcal{O}}}^\wedge.$$

We define the action of $\mathcal{Z}_{\mathcal{O}}^\wedge$ and $\mathcal{Z}_{\mathcal{K}}^\wedge$ on $\underline{\mathbb{H}}_{\mathcal{O}}$-modules and $\underline{\mathbb{H}}_{\mathcal{K}}$-modules in a similar way.

Since the action of $\mathcal{Z}_{\mathcal{K}}^\wedge$ on $\underline{P}(\lambda_{\mathcal{K}})_n$ coincides with the action of the polynomial part $\operatorname{Sym} V_{\mathcal{K}}^* \subset \underline{\mathbb{H}}$ up to twists by elements of W_R, the induced action of $\mathcal{Z}_{\mathcal{K}}^\wedge$ on the $\mathbb{K}_{\mathcal{K}}$-module $\underline{P}(\lambda_{\mathcal{K}})_n^\nabla$ is identified with the exponentiation of the action of $\mathcal{Z}_{\mathcal{K}}^\wedge$ on the $P(\lambda_{\mathcal{K}})_n$ under (21).

Since the $\mathcal{O}$-lattices $\underline{P}(\lambda_{\mathcal{O}})_n \subset \underline{P}(\lambda_{\mathcal{K}})_n$ and $\underline{P}(\lambda_{\mathcal{O}})_n^\nabla \subset \underline{P}(\lambda_{\mathcal{K}})_n^\nabla$ are stable under the action of the subring $\mathcal{Z}_{\mathcal{O}}^\wedge \subset \mathcal{Z}_{\mathcal{K}}^\wedge$, the functor $M \mapsto M^\nabla$ also intertwines the two $\mathcal{Z}_{\mathcal{O}}^\wedge$-actions. Put $\underline{P}(\lambda)_n = \underline{P}(\lambda_{\mathcal{O}})_n \otimes_{\mathcal{O}} \mathbf{C}$. Then $\underline{P}(\lambda)_n \mapsto \underline{P}(\lambda)_n^\nabla = \mathbb{V}(P(\lambda)_n)$ also intertwines the two $\mathcal{Z}^\wedge$-actions. Finally, since the family of modules $\underline{P}(\lambda)_n$ for $\lambda \in W_S\lambda_0$ and $n \geq 1$ generates the category $\mathcal{O}_{\lambda_0}(\underline{\mathbb{H}})$, the functor $\mathbb{V}$ restricted to $\mathcal{I}$ intertwine the $\mathcal{Z}^\wedge$-actions as asserted. □

4.4 Completion of categories

Since the affine Hecke algebra $\mathbb{K}$ is of finite rank over its centre, namely $(\mathbf{C}P)^W$, $\mathcal{B}_0 = \mathcal{O}_{\ell_0}(\mathbb{K})$ is equivalent to the category of modules of finite length over some semi-perfect algebra. It is also the case for $\mathcal{C}_0 = \mathcal{O}_{\lambda_0}(\mathbb{H})$. In particular, they are both noetherian-artinian. Consider the category of pro-objects[†] $\mathrm{Pro}(\mathcal{C}_0)$ and $\mathrm{Pro}(\mathcal{B}_0)$. We have two central actions introduced in §4.3

$$\mathcal{Z}^\wedge \longrightarrow \mathrm{End}(\mathrm{id}_{\mathcal{C}_0}) \cong \mathrm{End}(\mathrm{id}_{\mathrm{Pro}(\mathcal{C}_0)})$$
$$\mathcal{Z}^\wedge \longrightarrow \mathrm{End}\,(\mathrm{id}_{\mathcal{B}_0}) \cong \mathrm{End}(\mathrm{id}_{\mathrm{Pro}(\mathcal{B}_0)}).$$

By Lemma 20, the functor $\mathbb{V} : \mathcal{C}_0 \longrightarrow \mathcal{B}_0$ intertwines these $\mathcal{Z}^\wedge$-actions. The extension $\mathbb{V} : \mathrm{Pro}(\mathcal{C}_0) \longrightarrow \mathrm{Pro}(\mathcal{B}_0)$ still intertwines the $\mathcal{Z}^\wedge$-actions.

Define $\mathcal{C} \subset \mathrm{Pro}(\mathcal{C}_0)$ to be the subcategory consisting of objects $M \in \mathrm{Pro}(\mathcal{C}_0)$ such that $M/\mathfrak{m}_{\mathcal{Z}}^k M \in \mathcal{C}_0$ for all $k \geq 0$. Similarly we define $\mathcal{B} \subset \mathrm{Pro}(\mathcal{B}_0)$ to be the subcategory consisting of objects $N \in \mathrm{Pro}(\mathcal{B}_0)$ such that $M/\mathfrak{m}_{\mathcal{Z}}^k M \in \mathcal{C}_0$ for all $k \geq 0$.

Lemma 22. *For each simple object $L \in \mathcal{C}_0$ (resp. $L \in \mathcal{B}_0$), its projective cover $\mathcal{P}(L) \in \mathrm{Pro}(\mathcal{C}_0)$ (resp. $\mathcal{P}(L) \in \mathrm{Pro}(\mathcal{B}_0)$) lies in $\mathcal{C}$ (resp. $\mathcal{B}$).*

Proof. Notice that by the general result Proposition 114, the objects of $\mathcal{C}_0$ (resp. $\mathcal{B}_0$) admit projective covers in $\mathrm{Pro}(\mathcal{C}_0)$ (resp. $\mathrm{Pro}(\mathcal{B}_0)$). The statement holds obviously for $\mathcal{B}_0$ because $\mathbb{K}$ is of finite rank over its centre. For $\mathcal{C}_0$, by Proposition 12, there is an equivalence $\mathcal{C}_0 \cong \mathbf{H}_{\lambda_0}$ and the algebra $\mathbf{H}_{\lambda_0}$ is Morita-equivalent to an algebra of finite rank over its centre, *cf.* §7.4. □

Lemma 23. *The functor $\mathbb{V} : \mathrm{Pro}(\mathcal{C}_0) \longrightarrow \mathrm{Pro}(\mathcal{B}_0)$ restricts to $\mathbb{V} : \mathcal{C} \longrightarrow \mathcal{B}$.*

Proof. If $M \in \mathcal{C}$, then $M/\mathfrak{m}_{\mathcal{Z}}^k M \in \mathcal{C}_0$ and by Lemma 20, $\mathbb{V}(M)/\mathfrak{m}_{\mathcal{Z}}^k \mathbb{V}(M) \cong \mathbb{V}(M/\mathfrak{m}_{\mathcal{Z}}^k M) \in \mathcal{B}_0$. It follows that $\mathbb{V}(M) \in \mathcal{B}$. □

4.5 Right adjoint of $\mathbb{V}$

Recall that $\mathcal{B}_0 = \mathcal{O}_{\ell_0}(\mathbb{K})$ and $\mathcal{C}_0 = \mathcal{O}_{\lambda_0}(\mathbb{H})$.

Lemma 24. *The functor $\mathbb{V} : \mathcal{C}_0 \longrightarrow \mathcal{B}_0$ admits a right adjoint functor $\mathbb{V}^\top : \mathcal{B}_0 \longrightarrow \mathcal{C}_0$.*

Proof. We first define a functor $\mathbb{V}^\top : \mathcal{B}_0 \longrightarrow \mathrm{Ind}(\mathcal{C}_0)$ with natural isomorphisms

$$\mathrm{Hom}_{\mathcal{B}_0}(\mathbb{V}(M), N) \cong \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C}_0)}\left(M, \mathbb{V}^\top(N)\right) \tag{25}$$

for $M \in \mathcal{C}_0$ and $N \in \mathcal{B}_0$. For any $N \in \mathcal{B}_0$, let

$$F_N : \mathcal{C}_0^{\mathrm{op}} \longrightarrow \mathbf{C}\text{-Mod}, \quad F_N : M \mapsto \mathrm{Hom}_{\mathcal{B}_0}(\mathbb{V}(M), N)$$

and let

$$F_N(M)^{\min} = F_N(M) \setminus \bigcup_{0 \neq M' \subset M} F_N(M/M').$$

Here, we regard $F_N(M/M')$ as a subspace of $F_N(M)$ by the right exactness of F_N. Let $\mathcal{I}_N$ be the category whose objects are pairs (M, a), where $M \in \mathcal{C}_0$ and $a \in F_N(M)^{\min}$, and whose morphisms are defined by

$$\mathrm{Hom}_{\mathcal{I}_N}((M,a),(M',a')) = \{f \in \mathrm{Hom}_{\mathcal{C}_0}(M, M')\,;\, F_N(f)(a') = a\}.$$

We set

$$\mathbb{V}^\top(N) = \text{“}\varinjlim_{(M,a)\in\mathcal{I}_N}\text{”}\, M \in \mathrm{Ind}(\mathcal{C}_0).$$

According to [32, 3.5, Lemma 6], $\mathbb{V}^\top(N)$ represents the functor F_N, so $\mathbb{V}^\top$ satisfies the desired adjoint property (25).

Now we show that in fact the object $\mathbb{V}^\top(N)$ in $\mathrm{Ind}(\mathcal{C}_0)$ lies in the subcategory $\mathcal{C}_0$. Let $\mathcal{P}_{\mathcal{C}} \in \mathcal{C}$ be the sum of all projective indecomposable objects (up to isomorphism) of $\mathcal{C}$ so that

[†]The basic properties of categories of pro-objects are reviewed in Appendix A.

for any $M \in \mathcal{C}_0$, the dimension of $\mathrm{Hom}_{\mathcal{C}}(\mathcal{P}_{\mathcal{C}}, M)$ is equal to the length of M. Since $\mathbb{V}(\mathcal{P}_{\mathcal{C}}) \in \mathcal{B}$ is a finitely generated $\mathbb{K}$-module, the vector space $\mathrm{Hom}_{\mathcal{B}}(\mathbb{V}(\mathcal{P}_{\mathcal{C}}), N)$ is finite-dimensional. On the other hand, there are isomorphisms

$$\begin{aligned}
\varinjlim_{\substack{M \subset \mathbb{V}^{\top}(N) \\ M \in \mathcal{C}_0}} \mathrm{Hom}_{\mathcal{C}}(\mathcal{P}_{\mathcal{C}}, M) &\cong \varinjlim_{\substack{M \subset \mathbb{V}^{\top}(N) \\ M \in \mathcal{C}_0}} \varinjlim_{\substack{\mathcal{Q} \subset \mathcal{P}_{\mathcal{C}} \\ \mathcal{P}_{\mathcal{C}}/\mathcal{Q} \in \mathcal{C}_0}} \mathrm{Hom}_{\mathcal{C}_0}(\mathcal{P}_{\mathcal{C}}/\mathcal{Q}, M) \\
&\cong \varinjlim_{\substack{\mathcal{Q} \subset \mathcal{P}_{\mathcal{C}} \\ \mathcal{P}_{\mathcal{C}}/\mathcal{Q} \in \mathcal{C}_0}} \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C}_0)}\left(\mathcal{P}_{\mathcal{C}}/\mathcal{Q}, \mathbb{V}^{\top}(N)\right) \\
&\cong \varinjlim_{\substack{\mathcal{Q} \subset \mathcal{P}_{\mathcal{C}} \\ \mathcal{P}_{\mathcal{C}}/\mathcal{Q} \in \mathcal{C}_0}} \mathrm{Hom}_{\mathcal{B}_0}(\mathbb{V}(\mathcal{P}_{\mathcal{C}}/\mathcal{Q}), N) \\
&\cong \mathrm{Hom}_{\mathrm{Pro}(\mathcal{B}_0)}\left(\text{“}\varprojlim_{\substack{\mathcal{Q} \subset \mathcal{P}_{\mathcal{C}} \\ \mathcal{P}_{\mathcal{C}}/\mathcal{Q} \in \mathcal{C}_0}}\text{”} \mathbb{V}(\mathcal{P}_{\mathcal{C}}/\mathcal{Q}), N \right) \cong \mathrm{Hom}_{\mathcal{B}}(\mathbb{V}(\mathcal{P}_{\mathcal{C}}), N).
\end{aligned} \tag{26}$$

The first and the fourth isomorphisms are due to (110) of Appendix A; the second one is exchanging the order of the two colimits and it holds due to the definition of morphisms between ind-objects; the third one is due to (25); the last one is due to Lemma 23.

Since $N \in \mathcal{B}_0$, there is some integer n such that $\mathfrak{m}_{\mathcal{Z}}^n N = 0$. Since $\mathbb{V}(\mathcal{P}_{\mathcal{C}}) \in \mathcal{B}$, the quotient $\mathbb{V}(P_{\mathcal{C}})/\mathfrak{m}_{\mathcal{Z}}^n \mathbb{V}(P_{\mathcal{C}})$ lies in $\mathcal{B}_0$. Thus the Hom-space

$$\mathrm{Hom}_{\mathcal{B}}(\mathbb{V}(P_{\mathcal{C}}), N) \cong \mathrm{Hom}_{\mathcal{B}_0}(\mathbb{V}(P_{\mathcal{C}})/\mathfrak{m}_{\mathcal{Z}}^n \mathbb{V}(P_{\mathcal{C}}), N)$$

is finite-dimensional. The above isomorphisms (26) imply that the length of the subobjects $M \subset \mathbb{V}^{\top}(N)$ such that $M \in \mathcal{C}_0$ is bounded. It follows that $\mathbb{V}^{\top}(N)$ lies in $\mathcal{C}_0$ by Proposition 112 (iii). Thus $\mathbb{V}^{\top} : \mathcal{B}_0 \longrightarrow \mathcal{C}_0$ is a right adjoint to $\mathbb{V}$.

□

4.6 $\mathbb{V}$ is a quotient functor

Proposition 27. *The monodromy functor $\mathbb{V} : \mathcal{C}_0 \longrightarrow \mathcal{B}_0$ is a quotient functor.*

Proof. Recall that $\mathcal{D}(T_\circ^\vee)$ is the ring of algebraic linear differential operators on the regular part $T_\circ^\vee$ of the dual torus $T^\vee = P \otimes \mathbf{C}^\times$. By construction, the functor $\mathbb{V}$ factorises into the following

$$\begin{array}{ccccc}
\mathbb{H}\text{-Mod} & \xrightarrow{\mathrm{loc}} & \mathcal{D}(T_\circ^\vee) \rtimes W_R\text{-Mod} & & \\
\uparrow & & \uparrow & & \\
\mathcal{C}_0 & \longrightarrow & \mathrm{conn}^{\mathrm{rs}}_{W_R}(T_\circ^\vee) & \xrightarrow{\mathrm{RH}} & \mathbf{C}\mathfrak{B}_S\text{-mod}^{\mathrm{fini}} \\
 & & & & \uparrow \\
 & & \xrightarrow{\qquad \mathbb{V} \qquad} & & \mathcal{B}_0
\end{array}$$

where $\mathrm{conn}^{\mathrm{rs}}_{W_R}(T_\circ^\vee)$ is the subcategory of $\mathcal{D}(T_\circ^\vee) \rtimes W_R$-mod consisting of W_R-equivariant integrable connections on $T_\circ^\vee$ which have regular singularities along the boundary. The arrow in the first line is the localisation functor $\mathrm{loc} = \mathbf{C}[T_\circ^\vee] \otimes_{\mathbf{C}[T^\vee]} -$, whose right adjoint $\mathrm{loc}^{\top}$ is the restriction of the action of $\mathbb{H}_\circ = \mathcal{D}(T_\circ^\vee) \rtimes W_R$ to $\mathbb{H}$. The restriction of loc to $\mathcal{C}_0$ factorises through the inclusion of subcategory

$$\mathrm{conn}^{\mathrm{rs}}_{W_R}(T_\circ^\vee) \hookrightarrow \mathcal{D}(T_\circ^\vee) \rtimes W_R\text{-Mod}$$

and gives the first arrow of the second line. The functor RH is the Riemann–Hilbert correspondence (the Knizhnik–Zamolodchikov equations have regular singularities [28]), due to Deligne [11, 2.17+5.9], between algebraic connections with regular singularities and finite-dimensional representations of the fundamental group $\pi_1([T_\circ^\vee/W_R]) \cong \mathfrak{B}_S$.

We show that $\mathbb{V}$ admits a section functor in the sense of Gabriel [15]. We have shown in Lemma 24 that $\mathbb{V}$ admits a right adjoint functor $\mathbb{V}^{\top}$. The functor $\mathbb{V}^{\top}$ can be described as follows:

$$\mathcal{B}_0 \hookrightarrow \mathbf{C}\mathfrak{B}_S\text{-mod}^{\mathrm{fini}} \cong \mathrm{conn}^{\mathrm{rs}}_{W_R}(T_\circ^\vee) \longrightarrow \mathcal{C}_0,$$

where the last arrow is the functor which sends an object $M \in \mathrm{conn}^{\mathrm{rs}}_{W_R}(T^\vee_\circ)$ to the biggest $\mathbb{H}$-submodule of M which lies in $\mathcal{C}_0$, denoted by $M|_{\mathcal{C}_0} \subset M$. We show that the adjunction counit $\mathbb{V} \circ \mathbb{V}^\top \longrightarrow \mathrm{id}_{\mathcal{B}_0}$ is an isomorphism. We first show that it is a monomorphism: for any $M \in \mathrm{conn}^{\mathrm{rs}}_{W_R}(T^\vee_\circ)$, we have $\mathbf{C}[T^\vee_\circ] \otimes_{\mathbf{C}[T^\vee]} M \cong M$; by the flatness of $\mathbf{C}[T^\vee_\circ]$ over $\mathbf{C}[T^\vee]$, the inclusion $M|_{\mathcal{C}_0} \hookrightarrow M$ gives rise to a monomorphism

$$\mathbf{C}[T^\vee_\circ] \otimes_{\mathbf{C}[T^\vee]} (M|_{\mathcal{C}_0}) \longrightarrow \mathbf{C}[T^\vee_\circ] \otimes_{\mathbf{C}[T^\vee]} M \cong M;$$

composing it with the Riemann–Hilbert correspondence, we see that $\mathbb{V} \circ \mathbb{V}^\top \longrightarrow \mathrm{id}_{\mathcal{B}_0}$ is a monomorphism.

Let $N \in \mathcal{B}_0$. By the exactness of $\mathbb{V}$, to show that the adjunction counit $\mathbb{V}\mathbb{V}^\top N \hookrightarrow N$ is an isomorphism, it suffices to find an $\mathbb{H}$-submodule of $\mathrm{RH}^{-1}(N)$ whose localisation to $T^\vee_\circ$ is equal to $\mathrm{RH}^{-1}(N)$. There exists a surjection

$$\bigoplus_{i \in \mathcal{I}} P(\ell_i)_{n_i} \longrightarrow N$$

where $\mathcal{I}$ is an index set and $P(\ell_i)_{n_i} = \mathbb{K}/\mathbb{K} \cdot \mathfrak{m}^{n_i}_{\ell_i}$. By [34, 5.1 (i)], for each $i \in \mathcal{I}$ there is an induced module $P(\lambda_i)_{n_i} = \mathbb{H}/\mathbb{H} \cdot \mathfrak{m}^{n_i}_{\lambda_i} \in \mathcal{C}_0$ such that $\exp(\lambda_i) = \ell_i$ and $\mathbb{V}\left(P(\lambda_i)_{n_i}\right) \cong P(\ell_i)_{n_i}$. Hence the image of $P(\lambda_i)_{n_i}$ in $\mathrm{RH}^{-1}(N)$ is an $\mathbb{H}$-submodule which satisfies the requirement. We conclude that $\mathbb{V} \circ \mathbb{V}^\top \cong \mathrm{id}_{\mathcal{B}_0}$; therefore $\mathbb{V}^\top$ is a section functor for $\mathbb{V}$.

By the criterion of Gabriel [15, 3.2, Prop 5], $\mathbb{V}$ is a quotient functor. □

5 Comparison of $\mathbb{V}$ and $\mathbf{V}$

5.1 The functors $\mathbb{V}$ and $\mathbf{V}$

In Part II, we will study the idempotent forms $\mathbf{H}_{\lambda_0}$ and $\mathbf{K}_{\ell_0}$ in a broader context, *cf.* Remark 13 and Remark 18. Specifically, in §10.6, we will introduce a quotient functor for graded modules $\mathbf{V} : \mathbf{H}_{\lambda_0}\text{-gmod} \longrightarrow \mathbf{K}_{\ell_0}\text{-gmod}$. It has an ungraded version $\mathbf{V} : \mathbf{H}_{\lambda_0}\text{-mod}_0 \longrightarrow \mathbf{K}_{\lambda_0}\text{-mod}_0$. On the other hand, by Proposition 12 and Proposition 17, we have equivalences of categories $\mathcal{O}_{\lambda_0}(\mathbb{H}) \cong \mathbf{H}_{\lambda_0}\text{-mod}_0$ and $\mathcal{O}_{\ell_0}(\mathbb{K}) \cong \mathbf{K}_{\lambda_0}\text{-mod}_0$. The situation can be depicted in a diagram:

$$\begin{array}{ccc} \mathcal{O}_{\lambda_0}(\mathbb{H}) & \xrightarrow{\quad\mathbb{V}\quad} & \mathcal{O}_{\ell_0}(\mathbb{K}) \\ \downarrow\cong & & \downarrow\cong \\ \mathbf{H}_{\lambda_0}\text{-mod}_0 & \xrightarrow{\mathbf{V}} & \mathbf{K}_{\ell_0}\text{-mod}_0 \end{array}$$

Conjecture 28. *There is an isomorphism of functors* $\mathbb{V} \cong \mathbf{V}$.

In the rest of this section, we use results from Part II to prove a weaker version of this statement.

5.2 Comparison of the kernels

By Proposition 27 and §10.6, the functors $\mathbb{V}$ and $\mathbf{V}$ are already known to be quotient functors. The following proposition generalises a result from [24], where the geometric construction of the dDAHA was used.

Proposition 29. *The kernels* $\ker \mathbb{V}$ *and* $\ker \mathbf{V}$ *are identified via the equivalence* $\mathcal{O}_{\lambda_0}(\mathbb{H}) \cong \mathbf{H}_{\lambda_0}\text{-mod}_0$.

Proof. Let $F : \mathcal{O}_{\lambda_0}(\mathbb{H}) \xrightarrow{\sim} \mathbf{H}_{\lambda_0}\text{-mod}_0$ denote the equivalence from Proposition 12. We show that for every object $M \in \mathcal{O}_{\lambda_0}(\mathbb{H})$, the condition Theorem 97 (iii) for FM implies $\mathbb{V}M = 0$. Let $M = \bigoplus_{\lambda \in W_S\lambda_0} M_\lambda$ be the decomposition into generalised weight spaces of $\mathbf{C}[E]$ and let

$$M_{\leq t} = \bigoplus_{\substack{\lambda \in W_S\lambda_0 \\ \|\lambda\| \leq t}} M_\lambda, \quad \text{for } t \in \mathbf{R}_{\geq 0}.$$

Note that under the equivalence F, the generalised weight space M_λ is identified with $\mathbf{e}(\lambda)F(M_\lambda)$. Following the same arguments as in the proof (iii)⇒(iv) of Theorem 97, we have $s_a M_t \leq M_{t+\delta}$

for every $t \in \mathbf{R}_{\geq 0}$ and $a \in \Delta$. Let $U = \mathbf{C}[E]^{\leq 1} + \sum_{a \in \Delta} \mathbf{C} \cdot s_a \subset \mathbb{H}$ so that U generates $\mathbb{H}$ as $\mathbf{C}$-algebra. Then, by the assumption (iii), we see that for each finite-dimensional subspace $L \subset M$ and each $\varepsilon > 0$,

$$\lim_{n \longrightarrow \infty} \dim \left(U^n L\right) / n^{r-1+\varepsilon} = 0, \quad r = \operatorname{rk} R.$$

Hence we obtain $\dim_{\mathrm{GK},\mathbb{H}} M \leq r-1$, and in particular $\dim_{\mathrm{GK},\mathbf{C}[T^\vee]} M \leq r-1$ for the subalgebra $\mathbf{C}[T^\vee] = \mathbf{C}Q^\vee \subset \mathbb{H}$. As the algebra $\mathbf{C}[T^\vee]$ is commutative and by Proposition 3, M is coherent over $\mathbf{C}[T^\vee]$, the Gelfand–Kirillov dimension of M coincides with the Krull dimension of the subvariety $\operatorname{Supp}_{T^\vee} M \subset T^\vee$. As the localisation of M on the regular part $T^\vee_\circ$ must be locally free, we see that it must be zero since $\dim T^\vee_\circ = r > \dim \operatorname{Supp} M$. Hence $\mathbb{V}M = 0$ by the definition of $\mathbb{V}$. We see that $\ker \mathbf{V} \subset F(\ker \mathbb{V})$.

Since $\mathbf{V}$ and $\mathbb{V}$ are both quotient functors on noetherian-artinian categories, by comparison of the rank of the Grothendieck groups

$$\begin{aligned} \operatorname{rk} K_0 (\ker \mathbb{V}) &= \operatorname{rk} K_0 \left(\mathcal{O}_{\lambda_0}(\mathbb{H})\right) - \operatorname{rk} K_0 \left(\mathcal{O}_{\ell_0}(\mathbb{K})\right) \\ &= \operatorname{rk} K_0 \left(\mathbf{H}_{\lambda_0}\text{-mod}_0\right) - \operatorname{rk} K_0 \left(\mathbf{K}_{\ell_0}\text{-mod}_0\right) = \operatorname{rk} K_0 (\ker \mathbf{V}), \end{aligned}$$

we see that $\ker \mathbf{V} = F(\ker \mathbb{V})$. □

Part II
Quiver Hecke algebras

6 Quiver double Hecke algebra

Fix an irreducible based finite root system (V, R, Δ_0) and let (E, S, Δ) be its affinisation. In this section we will also abbreviate $P = P_R$, $Q = Q_R$, $P^\vee = P^\vee_R$ and $Q^\vee = Q^\vee_R$.

6.1 The polynomial matrix algebra $\mathbf{A}^o$

Fix once and for all $\lambda_0 \in E$. Define for each $\lambda \in W_S\lambda_0$ a polynomial ring $\mathrm{Pol}_\lambda = \mathbf{C}[V]$ and let $\mathrm{Pol}_{W_S\lambda_0} = \bigoplus_{\lambda \in W_S\lambda_0} \mathrm{Pol}_\lambda$. For each λ, define $\mathbf{e}(\lambda) : \mathrm{Pol}_{W_S\lambda_0} \twoheadrightarrow \mathrm{Pol}_\lambda \subset \mathrm{Pol}_{W_S\lambda_0}$ to be the projection onto the factor Pol_λ.

For each $a \in \Delta$, define an operator $\tau^o_a : \mathrm{Pol}_{W_S\lambda_0} \longrightarrow \mathrm{Pol}_{W_S\lambda_0}$ by

$$\tau^o_a = \sum_{\lambda \in W_S\lambda_0} \tau^o_a \mathbf{e}(\lambda), \quad \tau^o_a \mathbf{e}(\lambda) : \mathrm{Pol}_{\lambda_0} \longrightarrow \mathrm{Pol}_{s_a\lambda_0},$$

$$\tau^o_a \mathbf{e}(\lambda) = \begin{cases} (\partial a)^{-1}(s_{\partial a} - 1) & a(\lambda) = 0 \\ s_{\partial a} & a(\lambda) \neq 0 \end{cases}.$$

Here $\partial a \in R$ is the differential of $a \in S$, *cf.* §1.1.

Let $\mathbf{A}^o = \mathbf{A}^o(E, S, \Delta, \lambda_0)$ be the associative (non-unital) subalgebra of $\operatorname{End}_{\mathbf{C}}(\mathrm{Pol}_{W_S\lambda_0})$ generated by $f\mathbf{e}(\lambda)$ and $\tau^o_a\mathbf{e}(\lambda)$ for $f \in \mathbf{C}[V]$, $a \in \Delta$ and $\lambda \in W_S\lambda_0$.

6.2 Centre $\mathcal{Z}$

For $\lambda \in W_S\lambda_0$, let W_λ be the stabiliser of λ in W_S. The stabiliser W_λ is a finite parabolic subgroup of the Coxeter group W_S. The affine Weyl group W_S acts on the vector space V via the finite quotient $\partial^W : W_S \longrightarrow W_S/Q^\vee \cong W_R$. Let $\mathcal{Z} = \mathbf{C}[V]^{W_{\lambda_0}}$ be the ring of W_{λ_0}-invariant polynomials, graded by the degree of monomials. Since W_{λ_0} acts by reflections on V, the ring $\mathcal{Z}$ is a graded polynomial ring. Let $\mathfrak{m}_{\mathcal{Z}} \subset \mathcal{Z}$ be the unique homogeneous maximal ideal.

For each $\lambda \in W_S\lambda_0$, we define a homomorphism $\mathcal{Z} \longrightarrow \mathrm{Pol}_\lambda$: choosing a $w \in W_S$ such that $w\lambda_0 = \lambda$, we let $f \mapsto w(f) \in \mathbf{C}[V]^{W_\lambda} \subset \mathrm{Pol}_\lambda$. This map is clearly independent of the choice of w and it identifies $\mathcal{Z}$ with the invariant subspace $\mathbf{C}[V]^{W_\lambda}$. The infinite sum $\mathrm{Pol}_{W_S\lambda_0}$ is regarded as a $\mathcal{Z}$-module via the diagonal action.

The following are standard results from the invariant theory for reflection groups:

Proposition 30. *The following statements hold:*

(i) For each $\lambda \in W_S\lambda_0$, the $\mathcal{Z}$-module Pol_λ is free of rank $\#W_\lambda = \#W_{\lambda_0}$.

(ii) For any $w \in W_S$, choose a reduced expression $w = s_{a_l} \cdots s_{a_1}$ and put $\tau^o_w \mathbf{e}(\lambda) = \tau^o_{a_l} \cdots \tau^o_{a_1} \mathbf{e}(\lambda)$ for each $\lambda \in W_S\lambda_0$. Then the element $\tau^o_w \mathbf{e}(\lambda)$ is independent of the choice of the reduced expression for w and moreover, there is a decomposition

$$\mathrm{Hom}_{\mathcal{Z}}\left(\mathrm{Pol}_\lambda, \mathrm{Pol}_{W_S\lambda_0}\right) = \bigoplus_{w \in W_S} \tau^o_w \mathbf{C}[V]\mathbf{e}(\lambda).$$

(iii) The $\mathbf{A}^o$-action on $\mathrm{Pol}_{W_S\lambda_0}$ commutes with $\mathcal{Z}$ and yields an isomorphism

$$\mathbf{A}^o \xrightarrow{\sim} \bigoplus_{\lambda \in W_S\lambda_0} \mathrm{Hom}_{\mathcal{Z}}(\mathrm{Pol}_\lambda, \mathrm{Pol}_{W_S\lambda_0})$$

□

6.3 Subalgebras $\mathbf{A}^\omega$ of $\mathbf{A}^o$

Let $\omega = \{\omega_\lambda\}_{\lambda \in W_S\lambda_0}$ be a family of functions $\omega_\lambda : S^+ \longrightarrow \mathbf{Z}_{\geq -1}$ satisfying the following properties:

(i) $\omega_\lambda(a) = -1$ implies $a(\lambda) = 0$;

(ii) for $w \in W_S$ and $b \in S^+ \cap w^{-1}S^+$ we have $\omega_\lambda(b) = \omega_{w\lambda}(wb)$.

One may extend ω_λ to a function $\tilde{\omega}_\lambda : S \longrightarrow \mathbf{Z}_{\geq -1}$ by choosing $w \in W_S$ such that $wa \in S^+$ and setting $\tilde{\omega}_\lambda(a) = \omega_{w\lambda}(wa)$. We require ω to satisfy the following property:

(iii) For some (thus every) $\lambda \in W_S\lambda_0$, the extended function $\tilde{\omega}_\lambda : S \longrightarrow \mathbf{Z}_{\geq -1}$ has finite support.

We call the family $\{\omega_\lambda\}_{\lambda \in W_S\lambda_0}$ a family of order functions. The order functions can be characterised as follows:

Lemma 31. *Every family of order functions $\{\omega_\lambda\}_{\lambda \in W_S\lambda_0}$ is determined by the W_{λ_0}-invariant finitely supported function $\tilde{\omega}_{\lambda_0} : S \longrightarrow \mathbf{Z}_{\geq -1}$ satisfying*

$$\tilde{\omega}_{\lambda_0}(a) = -1 \Rightarrow a(\lambda_0) = 0 \quad \forall a \in S.$$

□

Define an operator $\tau^\omega_a = \sum_{\lambda \in W_S\lambda_0} \tau^\omega_a \mathbf{e}(\lambda) \in \mathrm{End}_{\mathcal{Z}}(\mathrm{Pol}_{W_S\lambda_0})$ with $\tau^\omega_a \mathbf{e}(\lambda) : \mathrm{Pol}_\lambda \longrightarrow \mathrm{Pol}_{s_a\lambda}$ by setting

$$\tau^\omega_a \mathbf{e}(\lambda) = \begin{cases} (\partial\alpha)^{-1}(s_{\partial a} - 1) & \omega_\lambda(a) = -1 \\ (\partial\alpha)^{\omega_\lambda(a)} s_{\partial a} & \omega_\lambda(a) \geq 0 \end{cases}$$

so that $\tau^\omega_a \mathbf{e}(\lambda) \in \mathbf{A}^o$.

Definition 32. *The quiver double Hecke algebra*[‡] *$\mathbf{A}^\omega = \mathbf{A}(E, S, \Delta, \lambda_0, \omega)$ is defined to be the subalgebra of $\mathbf{A}^o$ generated by $\mathbf{C}[V]\mathbf{e}(\lambda)$ and $\tau^\omega_a \mathbf{e}(\lambda)$ for $\lambda \in W_S$ and $a \in \Delta$.*

We also introduce the rational function field and its matrix algebra:

$$\mathrm{Rat}_\lambda = \mathrm{Frac}\,\mathrm{Pol}_\lambda = \mathrm{Pol}_\lambda \otimes_{\mathcal{Z}} \mathrm{Frac}\,\mathcal{Z}, \quad \mathrm{Rat} = \bigoplus_{\lambda \in W_S\lambda_0} \mathrm{Rat}_\lambda$$

$$\mathbf{A}^{-\infty} = \bigoplus_{\lambda \in W_S\lambda_0} \mathrm{Hom}_{\mathrm{Frac}\,\mathcal{Z}}(\mathrm{Rat}_\lambda, \mathrm{Rat}) = \mathbf{A}^o \otimes_{\mathcal{Z}} \mathrm{Frac}\,\mathcal{Z}, \quad \tau^{-\infty}_a = s_a,$$

where Frac means the field of fractions.

Example 33.

(i) Let $o = \left\{a \mapsto -\delta_{a(\lambda)=0}\right\}_{\lambda \in W_S\lambda_0}$ denote the smallest family of order functions. We recover the matrix algebra $\mathbf{A}^o$.

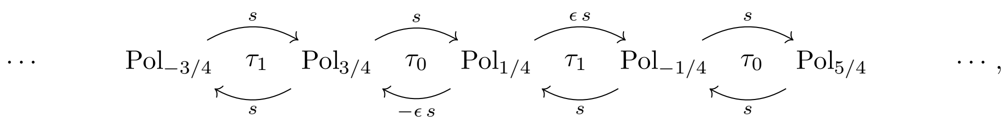
[‡] In this definition, the assumption that $\lambda_0 \in E$ plays no essential role. We could have asked λ_0 to belong to some set on which W_S acts transitively with finite parabolic stabiliser subgroups. However, the euclidean geometry of E will facilitate some arguments.

(ii) *Let $\omega = \{0\}_{\lambda \in W_S \lambda_0}$ be the zero constant function. Then $\mathbf{A}^\omega = \mathrm{Pol}_{W_S \lambda_0} \rtimes W_S$ is the skew tensor product. If $W_{\lambda_0} = 1$, then $\mathbf{A}^\omega = \mathbf{C}[V] \wr W_S$ is the wreath product.*

(iii) *Let $E = \mathbf{R}$, let ϵ be the coordinate function on $\mathbf{R}$ and let $S = \{\pm 2\epsilon\} + \mathbf{Z}$, so that (E, S) is the affine root system of type $A_1^{(1)}$. Choose the basis $\Delta = \{a_1 = 2\epsilon, a_0 = 1 - 2\epsilon\}$. The affine Weyl group W_S is generated by s_0 and s_1, where s_1 (resp. s_0) is the orthogonal reflection with respect to $0 \in E$ (resp. $1/2 \in E$). Set $\lambda_0 = 1/4 \in E$, so that $W_S \lambda_0 = 1/4 + (1/2)\mathbf{Z}$ and $W_{\lambda_0} = 1$. It follows that $\mathrm{Pol}_\lambda = \mathbf{C}[\epsilon]$ for all $\lambda \in W_S \lambda_0$ and $\mathbf{A}^o$ is the matrix algebra over $\mathbf{C}[\epsilon]$ of rank $W_S \lambda_0$.*
Set

$$\tilde{\omega}_{\lambda_0}(a) = \begin{cases} 1 & a \in \Delta \\ 0 & a \in S \setminus \Delta \end{cases}$$

and define the family of order functions $\omega = \{\omega_\lambda\}_{\lambda \in W_S \lambda_0}$ by $\omega_{w\lambda_0}(a) = \tilde{\omega}_{\lambda_0}(w^{-1} a)$. It follows that $\mathbf{A}^\omega$ is equal to the idempotent form of the dDAHA $\mathbf{H}_{\lambda_0}$ introduced in §2.5 with parameter $h_a = 1/2$ for all $a \in S$. We can depict the algebra $\mathbf{A}^\omega$ with the following diagram:



where $s : \mathbf{C}[\epsilon] \longrightarrow \mathbf{C}[\epsilon]$ is given by the substitution $\epsilon \mapsto -\epsilon$.

Remark 34. *We may view $\mathbf{A}^\omega$ as an affinisation of the quiver Hecke algebra $R_\beta(\Gamma)$ attached to a certain quiver $\Gamma = (I, H)$ and a dimension vector $\beta \in \mathbf{N}I$, cf. Remark 73. The parameter ω is an analogue of the polynomials $Q_{i,j}(u, v)$ in Rouquier's definition of quiver Hecke algebras.*

Remark 35. *Following [36, §2.3], one can write down a complete list of relations between the generators $\tau_a^\omega \mathbf{e}(\lambda)$, $\mathbf{C}[V]\mathbf{e}(\lambda)$ for the algebra $\mathbf{A}^\omega$ in the manner of Khovanov–Lauda–Rouquier. The most sophisticated is the braid relation between pairs of generators from $\{\tau_a^\omega \mathbf{e}(\lambda)\}_{a \in \Delta, \lambda \in W_S \lambda_0}$. We will only prove a weaker version of it in Lemma 39, which is enough for our needs.*

6.4 Filtration by length

Definition 36. *We define the filtration by length $\{F_{\le n} \mathbf{A}^\omega\}_{n \in \mathbf{N}}$ on $\mathbf{A}^\omega$ by*

$$F_{\le n} \mathbf{A}^\omega = \sum_{\lambda \in W_S \lambda_0} \sum_{k=0}^{n} \sum_{(a_1, \dots, a_k) \in \Delta^k} \mathbf{C}[V] \tau_{a_1}^\omega \cdots \tau_{a_k}^\omega \mathbf{e}(\lambda).$$

In general, it is hard to express the operators $\tau_{a_1}^\omega \cdots \tau_{a_k}^\omega$; however, the leading term is easy to describe.

Lemma 37. *Let $w = s_{a_l} \cdots s_{a_1}$ be a reduced expression and let $\lambda \in W_S \lambda_0$. Then*

(i) *For any $f \in \mathbf{C}[V]$, and any family of order functions ω there is a commutation relation:*

$$f \tau_{a_l}^\omega \cdots \tau_{a_1}^\omega \mathbf{e}(\lambda) \equiv \tau_{a_l}^\omega \cdots \tau_{a_1}^\omega \, w^{-1}(f) \mathbf{e}(\lambda) \mod F_{\le l-1} \mathbf{A}^\omega.$$

(ii) *For any pair of families of order functions ω and ω' such that $\omega \le \omega'$ (pointwise), there is a congruence relation:*

$$\tau_{a_l}^{\omega'} \cdots \tau_{a_1}^{\omega'} \mathbf{e}(\lambda) \equiv \tau_{a_l}^\omega \cdots \tau_{a_1}^\omega \left(\prod_{b \in S^+ \cap w^{-1} S^-} (-\partial b)^{\omega'_\lambda(b) - \omega_\lambda(b)} \right) \mathbf{e}(\lambda) \mod F_{\le l-1} \mathbf{A}^\omega.$$

Proof. We prove the statement (i) by induction on the length $l = \ell(w)$. It is trivial for $l = 0$. For $l = 1$:

$$(f \tau_a^\omega - \tau_a^\omega \, s_{\partial a}(f)) \mathbf{e}(\lambda) = \begin{cases} (\partial a)^{-1} (s_{\partial a}(f) - f) \mathbf{e}(\lambda) & \omega_\lambda(a) = -1 \\ 0 & \omega_\lambda(a) \ge 0 \end{cases} \tag{38}$$

It belongs to $F_{\leq 0}\mathbf{A}^\omega \mathbf{e}(\lambda) = \mathbf{C}[V]\mathbf{e}(\lambda)$ in both cases.

For $l > 1$, by the induction hypothesis, we get

$$\begin{aligned}
&(f\tau^\omega_{a_l}\cdots\tau^\omega_{a_1} - \tau^\omega_{a_l}\cdots\tau^\omega_{a_1} w^{-1}(f))\mathbf{e}(\lambda)\\
&= (f\tau^\omega_{a_l} - \tau^\omega_{a_l} s_{a_l}(f))\tau^\omega_{a_{l-1}}\cdots\tau^\omega_{a_1}\mathbf{e}(\lambda)\\
&+ \tau^\omega_{a_l}(s_{a_l}(f)\tau^\omega_{a_{l-1}}\cdots\tau^\omega_{a_1} - \tau^\omega_{a_{l-1}}\cdots\tau^\omega_{a_1} w^{-1}(f))\mathbf{e}(\lambda) \in F_{\leq l-1}\mathbf{A}^\omega,
\end{aligned}$$

whence (i).

We prove (ii) by induction on $l = \ell(w)$. Put $w' = s_{a_{l-1}}\cdots s_{a_1}$ and $\lambda' = w'\lambda$. Then

$$\begin{aligned}
&\tau^{\omega'}_{a_l}\cdots\tau^{\omega'}_{a_1}\mathbf{e}(\lambda) = (\partial a_l)^{\omega'_{\lambda'}(a_l)-\omega_{\lambda'}(a_l)}\tau^\omega_{a_l}\tau^{\omega'}_{a_{l-1}}\cdots\tau^{\omega'}_{a_1}\mathbf{e}(\lambda)\\
&= \left((\partial a_l)^{\omega'_{\lambda'}(a_l)-\omega_{\lambda'}(a_l)}\tau^\omega_{a_l} - \tau^\omega_{a_l}(-\partial a_l)^{\omega'_{\lambda'}(a_l)-\omega_{\lambda'}(a_l)}\right)\tau^{\omega'}_{a_{l-1}}\cdots\tau^{\omega'}_{a_1}\mathbf{e}(\lambda)\\
&+ \tau^\omega_{a_l}(-\partial a_l)^{\omega'_{\lambda'}(a_l)-\omega_{\lambda'}(a_l)}\tau^{\omega'}_{a_{l-1}}\cdots\tau^{\omega'}_{a_1}\mathbf{e}(\lambda).
\end{aligned}$$

By (38), the first term belongs to $F_{\leq l-1}\mathbf{A}^\omega$; the second term, by the statement (i) for $w' = a_{i_{l-1}}\cdots a_{i_1}$, satisfies

$$\begin{aligned}
\tau^\omega_{a_l}(-\partial a_l)^{\omega'_{\lambda'}(a_l)-\omega_{\lambda'}(a_l)}\tau^{\omega'}_{a_{l-1}}\cdots\tau^{\omega'}_{a_1}\mathbf{e}(\lambda) &\equiv \tau^\omega_{a_l}\tau^{\omega'}_{a_{l-1}}\cdots\tau^{\omega'}_{a_1} w'^{-1}\left((-\partial a_l)^{\omega'_{\lambda'}(a_l)-\omega_{\lambda'}(a_l)}\right)\mathbf{e}(\lambda)\\
&= \tau^\omega_{a_l}\tau^{\omega'}_{a_{l-1}}\cdots\tau^{\omega'}_{a_1}\left(-\partial(w'^{-1}a_l)\right)^{\omega'_\lambda(w'^{-1}a_l)-\omega_\lambda(w'^{-1}a_l)}\mathbf{e}(\lambda).
\end{aligned}$$

Here we have used the hypothesis that $\omega_\lambda(w'^{-1}a_l) = \omega_{\lambda'}(a_l)$. Using the induction hypothesis, we obtain

$$\begin{aligned}
&\tau^{\omega'}_{a_l}\cdots\tau^{\omega'}_{a_1}\mathbf{e}(\lambda) \equiv \tau^\omega_{a_l}\tau^{\omega'}_{a_{l-1}}\cdots\tau^{\omega'}_{a_1}\left((-\partial(w'^{-1}a_l))^{\omega'_\lambda(w'^{-1}a_l)-\omega_\lambda(w'^{-1}a_l)}\right)\mathbf{e}(\lambda)\\
&\equiv \tau^\omega_{a_l}\cdots\tau^\omega_{a_1}\left(\prod_{b\in S^+\cap w^{-1}S^-}(-\partial b)^{\omega'_\lambda(b)-\omega_\lambda(b)}\right)\mathbf{e}(\lambda).
\end{aligned}$$

The last equation is due to the relation $S^+\cap w^{-1}S^- = S^+\cap w'^{-1}S^-\cup\{w'^{-1}a_l\}$. This proves (ii). □

6.5 Basis theorem

We aim to prove an analogue of Proposition 30 for the subalgebra $\mathbf{A}^\omega\subset\mathbf{A}^o$.

Lemma 39 (braid relation). *For any family of ordered functions $\{\omega_\lambda\}_{\lambda\in W_S\lambda_0}$, the images of the operators $\tau^\omega_a\mathbf{e}(\lambda)$ in $\mathrm{gr}^F\mathbf{A}^\omega$ satisfies the braid relations: for $a,b\in\Delta$ with $a\neq b$, let $m_{a,b}$ be the order of s_as_b in W_S. If $m_{a,b}\neq\infty$, then*

$$\underbrace{\tau^\omega_a\tau^\omega_b\tau^\omega_a\cdots}_{m_{a,b}}\mathbf{e}(\lambda) \equiv \underbrace{\tau^\omega_b\tau^\omega_a\tau^\omega_b\cdots}_{m_{a,b}}\mathbf{e}(\lambda) \qquad \mathrm{mod}\ F_{\leq m_{a,b}-1}\mathbf{A}^\omega.$$

Proof. The statement is empty for $m_{a,b} = \infty$, so we assume $m_{a,b}\neq\infty$. Let $W_{a,b}\subset W_S$ be the parabolic subgroup generated by s_a and s_b, let $w_0\in W_{a,b}$ be the longest element and let $S_{a,b}\subset S$ be the sub-root system spanned by a and b. Let $\mathbf{A}^\omega_{a,b}$ be the subalgebra of $\mathbf{A}^\omega$ generated by $\mathbf{C}[V]\mathbf{e}(\lambda)$, $\tau^\omega_a\mathbf{e}(\lambda)$ and $\tau^\omega_b\mathbf{e}(\lambda)$ for $\lambda\in W_S\lambda_0$ and let $F_{\leq n}\mathbf{A}^\omega_{a,b}$ be the filtration by length defined as in Definition 36. It suffices to show the following

$$\underbrace{\tau^\omega_a\tau^\omega_b\tau^\omega_a\cdots}_{m_{a,b}}\mathbf{e}(\lambda) \equiv \underbrace{\tau^\omega_b\tau^\omega_a\tau^\omega_b\cdots}_{m_{a,b}}\mathbf{e}(\lambda) \qquad \mathrm{mod}\ F_{\leq m_{a,b}-1}\mathbf{A}^\omega_{a,b}.$$

because there is an inclusion $F_{\leq m_{a,b}-1}\mathbf{A}^\omega_{a,b}\subset F_{\leq m_{a,b}-1}\mathbf{A}^\omega$. An analogue of Lemma 37 is valid for this subalgebra with the filtration $F_{\leq n}\mathbf{A}^\omega_{a,b}$.

We first prove the braid relation for the family $\omega' = \{\omega'_\lambda\}_{\lambda\in W_S\lambda_0}$, where $\omega'_\lambda(c) = \max\{\omega_\lambda(c), 0\}$. Since $\omega'_\lambda(c)\geq 0$ for all $c\in S^+_{a,b}$, the braid relation for $\tau^{\omega'}_a$ and $\tau^{\omega'}_b$ follows from the following formula (with similar proof as Lemma 37 (ii)):

$$\underbrace{\tau^{\omega'}_a\tau^{\omega'}_b\tau^{\omega'}_a\cdots}_{m_{a,b}}\mathbf{e}(\lambda) = \underbrace{s_{\partial a}s_{\partial b}s_{\partial b}\cdots}_{m_{a,b}}\prod_{c\in S^+_{a,b}}(-\partial c)^{\omega'_\lambda(c)}\mathbf{e}(\lambda).$$

Let $\mathfrak{d} = \prod_{\substack{c \in S^+_{a,b} \\ \omega_c(\lambda)=-1}} (\partial c)$. By Lemma 37 (ii), we have

$$\underbrace{\tau_a^{\omega'}\tau_b^{\omega'}\tau_a^{\omega'}\cdots}_{m_{a,b}}\mathbf{e}(\lambda) \equiv \underbrace{\tau_a^{\omega}\tau_b^{\omega}\tau_a^{\omega}\cdots}_{m_{a,b}}\mathfrak{d}\,\mathbf{e}(\lambda) \mod F_{\le m_{a,b}-1}\mathbf{A}^{\omega}_{a,b}.$$

Write $X = (\underbrace{\tau_a^{\omega}\tau_b^{\omega}\tau_a^{\omega}\cdots}_{m_{a,b}} - \underbrace{\tau_b^{\omega}\tau_a^{\omega}\tau_b^{\omega}\cdots}_{m_{a,b}})\mathbf{e}(\lambda)$, so that $X\cdot\mathfrak{d} \in \mathbf{e}(w_0\lambda)\left(F_{\le m_{a,b}-1}\mathbf{A}^{\omega}_{a,b}\right)\mathbf{e}(\lambda)$. Moreover, by Lemma 37 (ii), we have

$$X \equiv (\underbrace{\tau_a^{o}\tau_b^{o}\tau_a^{o}\cdots}_{m_{a,b}} - \underbrace{\tau_b^{o}\tau_a^{o}\tau_b^{o}\cdots}_{m_{a,b}}) \prod_{c\in S^+_{a,b}} (-\partial c)^{\omega_\lambda(c)-o_\lambda(c)}\mathbf{e}(\lambda) \mod F_{\le m_{a,b}-1}\mathbf{A}^{o}_{a,b}$$

However, the elements $\tau_a^o\mathbf{e}(\lambda)$ satisfy the braid relations in $\mathbf{A}^o_{a,b}$ by Proposition 30 (ii). It follows that $X \in F_{\le m_{a,b}-1}\mathbf{A}^o_{a,b}$ (notice that $\mathbf{A}^{\omega}_{a,b} \subseteq \mathbf{A}^o_{a,b}$). We claim that for $0 \le j \le m_{a,b}-1$, the quotient $F_{\le j}\mathbf{A}^o_{a,b}\mathbf{e}(\lambda)/F_{\le j}\mathbf{A}^{\omega}_{a,b}\mathbf{e}(\lambda)$ is right $\mathfrak{d}$-torsion-free. This will imply that $X \in F_{\le m_{a,b}-1}\mathbf{A}^{\omega}_{a,b}$ and complete the proof.

We prove the claim by induction on j. For $j=0$, this is obvious since $F_{\le 0}\mathbf{A}^o_{a,b} = F_{\le 0}\mathbf{A}^{\omega}_{a,b}$. Assume $j \in [1, m_{a,b}-1]$. The quotient $\mathrm{gr}^F_j\mathbf{A}^{\omega}_{a,b}\mathbf{e}(\lambda)$ is spanned over $\mathbf{C}[V]$ by $\underbrace{\tau_a^{\omega}\tau_b^{\omega}\tau_a^{\omega}\cdots}_{j}\mathbf{e}(\lambda)$ and $\underbrace{\tau_b^{\omega}\tau_a^{\omega}\tau_b^{\omega}\cdots}_{j}\mathbf{e}(\lambda)$ since any non-reduced word in a, b of length $\le j$ contains consecutive letters aa or bb and since $(\tau_a^{\omega})^2, (\tau_b^{\omega})^2 \in F_{\le 1}\mathbf{A}^{\omega}_{a,b}$. Similarly, $\mathrm{gr}^F_j\mathbf{A}^o_{a,b}\mathbf{e}(\lambda)$ is spanned over $\mathbf{C}[V]$ by $\underbrace{\tau_a^{o}\tau_b^{o}\tau_a^{o}\cdots}_{j}\mathbf{e}(\lambda)$ and $\underbrace{\tau_b^{o}\tau_a^{o}\tau_b^{o}\cdots}_{j}\mathbf{e}(\lambda)$. Moreover, by Proposition 30, $\mathrm{gr}^F_j\mathbf{A}^o_{a,b}\mathbf{e}(\lambda)$ is free of rank 2 over $\mathbf{C}[V]$. Denote $w = \underbrace{s_as_bs_a\cdots}_{j}$. Since $\omega \ge o$, by Lemma 37 (ii), we have

$$\underbrace{\tau_a^{\omega}\tau_b^{\omega}\tau_a^{\omega}\cdots}_{j} \equiv \underbrace{\tau_a^{o}\tau_b^{o}\tau_a^{o}\cdots}_{j}\left(\prod_{c\in S^+_{a,b}\cap w^{-1}S^-_{a,b}} (-\partial c)^{\omega_\lambda(c)-o_\lambda(c)}\right) \mod F_{\le j-1}\mathbf{A}^o_{a,b}.$$

The prime factors of $\mathfrak{d}$ are ∂c for $c \in S^+_{a,b}$ such that $\omega_\lambda(c) = -1$. Therefore $\mathfrak{d}$ and the product

$$\prod_{c\in S^+_{a,b}\cap w^{-1}S^-_{a,b}} (-\partial c)^{\omega_\lambda(c)-o_\lambda(c)}$$

are relatively prime. The same argument applies to the other product $\tau_b^{\omega}\tau_a^{\omega}\tau_b^{\omega}\cdots$.

It follows that $\mathrm{gr}^F_j\mathbf{A}^{\omega}_{a,b}\mathbf{e}(\lambda)$ and $\mathrm{gr}^F_j\mathbf{A}^o_{a,b}\mathbf{e}(\lambda)$ are both free over $\mathbf{C}[V]$ of rank 2, and the matrix representing the $\mathbf{C}[V]$-linear map $\varphi : \mathrm{gr}^F_j\mathbf{A}^{\omega}_{a,b}\mathbf{e}(\lambda) \longrightarrow \mathrm{gr}^F_j\mathbf{A}^o_{a,b}\mathbf{e}(\lambda)$ (which is induced from the inclusion $\mathbf{A}^{\omega}_{a,b}\mathbf{e}(\lambda) \subset \mathbf{A}^o_{a,b}\mathbf{e}(\lambda)$) is diagonal with entries prime to $\mathfrak{d}$. Hence $\operatorname{coker}\varphi$ is $\mathfrak{d}$-torsion free. The snake lemma yields a short exact sequence

$$0 \longrightarrow \frac{F_{\le j-1}\mathbf{A}^o_{a,b}\mathbf{e}(\lambda)}{F_{\le j-1}\mathbf{A}^{\omega}_{a,b}\mathbf{e}(\lambda)} \longrightarrow \frac{F_{\le j}\mathbf{A}^o_{a,b}\mathbf{e}(\lambda)}{F_{\le j}\mathbf{A}^{\omega}_{a,b}\mathbf{e}(\lambda)} \longrightarrow \operatorname{coker}\varphi \longrightarrow 0,$$

in which the first term is also $\mathfrak{d}$-torsion-free by induction hypothesis, and so is the middle term $\mathfrak{d}$-torsion-free, whence the claim is proven. ◻

Theorem 40. *For each $w \in W_S$, choose a reduced expression $w = s_{a_l}\cdots s_{a_1}$ and put $\tau_w^{\omega} = \tau_{a_l}^{\omega}\cdots\tau_{a_1}^{\omega}$. Then there is a decomposition*

$$\mathbf{A}^{\omega} = \bigoplus_{\lambda\in W_S\lambda_0}\bigoplus_{w\in W_S}\mathbf{C}[V]\tau_w^{\omega}\mathbf{e}(\lambda).$$

Proof. By dévissage, it suffices to show that for each $n \in \mathbf{N}$,

$$\mathrm{gr}^F_n\mathbf{A}^{\omega} = \bigoplus_{\lambda\in W_S\lambda_0}\bigoplus_{\substack{w\in W_S \\ \ell(w)=n}}\mathbf{C}[V]\tau_w^{\omega}\mathbf{e}(\lambda).$$

It follows from the braid relations for τ_a^ω in $\mathrm{gr}^F \mathbf{A}^\omega$ proven in Lemma 39 and the fact that $(\tau_a^\omega)^2 \mathbf{e}(\lambda) \in F_{\leq 1}\mathbf{A}^\omega$, that these elements τ_w^ω span $\mathrm{gr}_n^F \mathbf{A}^\omega$. By the invariant theory of reflection groups, the family $\{\tau_w^o \mathbf{e}(\lambda)\}_w$ is free over $\mathbf{C}[V]$ and forms a basis for $\mathrm{End}_{\mathcal{Z}}(\mathrm{Pol}_\lambda)$. In view of Lemma 37 (ii), the matrix of transition between the families $\{\tau_w^o \mathbf{e}(\lambda)\}_w$ and $\{\tau_w^\omega \mathbf{e}(\lambda)\}_w$ is diagonal with non-zero entries; therefore the latter is also free over $\mathbf{C}[V]$. □

Define the filtration $F_{\leq n}\mathbf{A}^{-\infty} = (F_{\leq n}\mathbf{A}^{-o}) \otimes_{\mathcal{Z}} \mathrm{Frac}\,\mathcal{Z} \subset \mathbf{A}^{-\infty}$.

Corollary 41. *For each n, we have*

$$F_{\leq n}\mathbf{A}^\omega = F_{\leq n}\mathbf{A}^{-\infty} \cap \mathbf{A}^\omega.$$

Proof. Let $F'_{\leq n}\mathbf{A}^\omega = F_{\leq n}\mathbf{A}^{-\infty} \cap \mathbf{A}^\omega$. We have $F_{\leq n}\mathbf{A}^\omega \subset F'_{\leq n}\mathbf{A}^\omega$. Fix $\lambda, \lambda' \in W_S\lambda_0$ and denote $A = \mathbf{e}(\lambda')\mathbf{A}^\omega \mathbf{e}(\lambda)$. Put $N = \#\{w \in W_S\,;\, w\lambda = \lambda'\}$, then we have $F_{\leq n}A = A = F_{\leq n}A'$ for $n \geq N$ by Theorem 40. We prove by induction on $k \in [0, N]$ that $F_{\leq N-k}A = F'_{\leq N-k}A$. It is already clear for $k = 0$. Suppose $k \geq 1$. Then we have the obvious diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_{\leq N-k}A & \longrightarrow & F_{\leq N-k+1}A & \longrightarrow & \mathrm{gr}^F_{N-k+1}A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\eta} & & \\ 0 & \longrightarrow & F'_{\leq N-k}A & \longrightarrow & F'_{\leq N-k+1}A & \longrightarrow & \mathrm{gr}^{F'}_{N-k+1}A & \longrightarrow & 0. \end{array}$$

The morphism ψ is an isomorphism by the induction hypothesis and φ is injective. By the snake lemma, we have $\ker\eta \cong \mathrm{coker}\,\varphi$. Theorem 40 implies that $\mathrm{gr}^F_{N-k+1}A$ is $\mathbf{C}[V]$-torsion-free whereas $\mathrm{coker}\,\varphi$ is a $\mathbf{C}[V]$-torsion module. Therefore $\mathrm{coker}\,\varphi = 0$ and φ is an isomorphism. Summing over $\lambda, \lambda' \in W_S\lambda_0$, we obtain $F_{\leq n}\mathbf{A}^\omega = F'_{\leq n}\mathbf{A}^\omega$ for all $n \in \mathbf{N}$. □

Remark 42. *In view of (the proof of) Lemma 39, one can define a "Bruhat filtration" $\{F_{\mathcal{I}}\}_{\mathcal{I}}$ indexed by the order ideals $\mathcal{I}$ of the affine Weyl group W_S with respect to the Bruhat order, so that $F_{\mathcal{I}}\mathbf{A}^\omega$ is spanned by $\mathbf{C}[V]\tau_w^\omega \mathbf{e}(\lambda)$ for $\lambda \in W_S\lambda_0$ and $w \in \mathcal{I}$. Our filtration by length $\{F_{\leq n}\mathbf{A}^\omega\}_{n\in\mathbf{N}}$ can be viewed as part of the Bruhat filtration because we have $F_n\mathbf{A}^\omega = F_{\mathcal{I}_n}\mathbf{A}^\omega$ for $\mathcal{I}_n = \{w \in W_S\,;\, \ell(w) \leq n\}$.*

6.6 The associated graded $\mathrm{gr}^F \mathbf{A}^\omega$

We describe in greater detail the structure of the associated graded $\mathrm{gr}^F \mathbf{A}^\omega$. We establish in Proposition 48 a triangular decomposition for $\mathrm{gr}^F \mathbf{A}^\omega$, which will be used in the proof of Proposition 69. The proof of Lemma 44 is technical. The reader is advised to skip this subsection in the first reading.

Recall the extended affine Weyl group $\tilde{W}_S = P^\vee \rtimes W_R$ defined in §1.3. For $\mu \in P^\vee$, let $w_\mu \in W_R$ be such that $X^\mu w_\mu$ is the minimal element of the coset $X^\mu W_R$. Define the following map of minimal representatives:

$$\theta : P^\vee \longrightarrow \tilde{W}_S, \quad \theta(\mu) = X^\mu w_\mu.$$

In particular, $\theta(Q^\vee) \subset W_S$ coincide with the set of minimal representatives for the quotient W_S/W_R.

Lemma 43. *For each $\mu \in P^\vee$, the element w_μ is characterised by the following property: every positive root $\alpha \in R^+$ satisfies $w_\mu\alpha \in R^-$ if and only if $\langle\alpha, \mu\rangle > 0$.*

Proof. See [7, Proof of 1.4] □

We consider the nil-Hecke algebra $\mathbf{C}[\tilde{W}_S]^{\mathrm{nil}}$ for $\tilde{W}_S$: it is the $\mathbf{C}$-vector space span by the basis $\{[w]^{\mathrm{nil}}\}_{w\in\tilde{W}_S}$ equipped with the following multiplication law

$$[w]^{\mathrm{nil}} \cdot [y]^{\mathrm{nil}} = \begin{cases} [wy]^{\mathrm{nil}} & \text{if } \ell(wy) = \ell(w) + \ell(y) \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{C}[W_S]^{\mathrm{nil}}$ and $\mathbf{C}[W_R]^{\mathrm{nil}}$ be the subspace of $\mathbf{C}[\tilde{W}_S]^{\mathrm{nil}}$ spanned by $\{[w]^{\mathrm{nil}}\}_{w\in W_S}$ and $\{[w]^{\mathrm{nil}}\}_{w\in W_R}$ respectively. These are the nil-Hecke algebras for W_S and W_R.

Let $\mathrm{C}_0 \subset V^*$ denote the fundamental Weyl chamber and $\overline{\mathrm{C}_0}$ its closure in V^*. Let $P^\vee_+ = P^\vee \cap \overline{\mathrm{C}_0}$ (resp. $Q^\vee_+ = Q^\vee \cap \overline{\mathrm{C}_0}$) be the submonoid of $P^\vee$ consisting of dominant coweights (resp. dominant coroots). Let $\mathbf{C}P^\vee_+$ (resp. $\mathbf{C}Q^\vee_+$) denote the monoid algebra of $P^\vee_+$ (resp. $Q^\vee_+$). For $\mu \in P^\vee_+$, let $X^\mu \in \mathbf{C}P^\vee_+$ denote the corresponding element.

We define a map

$$\zeta : \mathbf{C}P^\vee_+ \longrightarrow \mathbf{C}[\tilde{W}_S]^{\mathrm{nil}}, \quad \zeta(X^\mu) = \sum_{\mu' \in W_R \mu} [X^{\mu'}]^{\mathrm{nil}}.$$

Lemma 44. *The following statements hold:*

(i) The map ζ is a ring homomorphism and yield a left $\mathbf{C}P^\vee_+$-module structure on $\mathbf{C}[\tilde{W}_S]^{\mathrm{nil}}$ by left multiplication; moreover, $\mathbf{C}[\tilde{W}_S]^{\mathrm{nil}}$ is a free $(\mathbf{C}P^\vee_+, \mathbf{C}[W_R]^{\mathrm{nil}})$-bimodule of rank $\#W$ and a basis of which is given by $\left\{[\theta(b_w)]^{\mathrm{nil}}\right\}_{w \in W_R}$ with

$$b_w = \sum_{\substack{\alpha \in \Delta_0 \\ s_\alpha w < w}} w^{-1} w_0 \omega^\vee_\alpha \in P^\vee.$$

(ii) The ring $\mathbf{C}Q^\vee_+$ is Cohen–Macaulay and the $\mathbf{C}P^\vee_+$-module structure on $\mathbf{C}[\tilde{W}_S]^{\mathrm{nil}}$ restricts to a $\mathbf{C}Q^\vee_+$-module structure on $\mathbf{C}[W_S]^{\mathrm{nil}}$; moreover, there is a decomposition

$$\mathbf{C}[W_S]^{\mathrm{nil}} = \mathcal{E} \otimes \mathbf{C}[W_R]^{\mathrm{nil}},$$

where $\mathcal{E} \subset \mathbf{C}[W_S]^{\mathrm{nil}}$ is a $\mathbf{C}Q^\vee_+$-direct factor and is a Cohen-Macaulay $\mathbf{C}Q^\vee_+$-module of maximal dimension.

Proof. In view of the length formula Proposition 1 for $\tilde{W}_S$, the condition $\ell(X^{\mu+\nu}) = \ell(X^\mu) + \ell(X^\nu)$ is equivalent to that μ' and ν lie in the closure of the same Weyl chamber. Therefore the map ζ is a ring homomorphism. Define a decreasing filtration $G^\bullet \tilde{W}_S$ by

$$G^k \tilde{W}_S = \bigcup_{\substack{y \in W_R \\ \ell(y) \geq k}} \left\{ v \in \tilde{W}_S \; ; \; \ell(vy^{-1}) = \ell(v) - \ell(y) \right\}.$$

Since $\mathbf{C}[\tilde{W}_S]^{\mathrm{nil}}$ has a canonical basis $\left\{[w]^{\mathrm{nil}}\right\}_{w \in \tilde{W}_S}$, the filtration $G^\bullet \tilde{W}_S$ induces a filtration on $\mathbf{C}[\tilde{W}_S]^{\mathrm{nil}}$, denoted by $G^\bullet \mathbf{C}[\tilde{W}_S]^{\mathrm{nil}}$.

Step 1. We prove that the map

$$P^\vee_+ \times \left\{(w, y) \in (W_R)^2 \; ; \; \ell(y) = k\right\} \xrightarrow{\sim} G^k \tilde{W}_S \setminus G^{k-1} \tilde{W}_S, \quad (\mu, w, y) \mapsto X^{b_w + w^{-1} w_0 \mu} wy$$

is a bijection.

For $\mu \in P^\vee$, let $w_\mu \in W_R$ be the element from Lemma 43. We may partition $P^\vee$ into sub-semigroups :

$$P^\vee = \bigsqcup_{w \in W_R} P^\vee_w, \quad P^\vee_w = \{\mu \in P^\vee \; ; \; w_\mu = w\}.$$

For $w \in W_R$, we have $b_w \in P^\vee_w$ and there is a bijection

$$P^\vee_+ \xrightarrow{\sim} P^\vee_w, \quad \mu \mapsto b_w + w^{-1} w_0 \mu;$$

we can thus express the set $G^k \tilde{W}_S \setminus G^{k-1} \tilde{W}_S$ as

$$G^k \tilde{W}_S \setminus G^{k-1} \tilde{W}_S = \bigsqcup_{\substack{w, y \in W_R \\ \ell(y) = k}} \{X^\mu wy \; ; \; \mu \in P^\vee_w\} = \bigsqcup_{\substack{w, y \in W_R \\ \ell(y) = k}} \left\{X^{b_w + w^{-1} w_0 \mu} wy \; ; \; \mu \in P^\vee_+\right\}.$$

Step 2. We prove that for each $\mu \in P^\vee$ and $w, y \in W_R$, we have $\zeta(X^\mu)[X^{b_w} wy]^{\mathrm{nil}} \in G^{\ell(y)} \mathbf{C}[\tilde{W}_S]^{\mathrm{nil}}$ and

$$\zeta(X^\mu)[X^{b_w} wy]^{\mathrm{nil}} \equiv [X^{b_w + w^{-1} w_0 \mu} wy]^{\mathrm{nil}} \mod G^{\ell(y)+1} \mathbf{C}[\tilde{W}_S]^{\mathrm{nil}}. \tag{45}$$

Indeed, the defining relations of the nil-Hecke algebra $\mathbf{C}[\tilde{W}_S]^{\mathrm{nil}}$ yield

$$\zeta(X^{\mu})[X^{b_w}wy]^{\mathrm{nil}} = \sum_{\substack{\mu'\in W_R\mu \\ \ell(X^{b_w+\mu'}wy)=\ell(X^{b_w}wy)+\ell(X^{\mu'})}} [X^{b_w+\mu'}wy]^{\mathrm{nil}}$$

in $\mathbf{C}[\tilde{W}_S]^{\mathrm{nil}}$. Since

$$\ell(X^{b_w+\mu'}wy) \le \ell(X^{b_w+\mu'}w) + \ell(y) \le \ell(X^{b_w}w) + \ell(X^{\mu'}) + \ell(y) = \ell(X^{b_w}wy) + \ell(X^{\mu'})$$

(the last equality due to Lemma 43), the condition

$$\ell(X^{b_w+\mu'}wy) = \ell(X^{b_w}wy) + \ell(X^{\mu'}) \tag{46}$$

implies that $\ell(X^{b_w+\mu'}wy) = \ell(X^{b_w+\mu'}w) + \ell(y)$ and hence $X^{b_w+\mu'}wy \in G^{\ell(y)}\tilde{W}_S$. It follows that $[X^{b_w+\mu'}wy]^{\mathrm{nil}} \in G^{\ell(y)}\mathbf{C}[\tilde{W}_S]^{\mathrm{nil}}$ for $\mu' \in W_R\mu$ satisfying (46) and $[X^{b_w+\mu'}wy]^{\mathrm{nil}} \in G^{\ell(w)+1}\mathbf{C}[\tilde{W}_S]^{\mathrm{nil}}$ unless $w_{b_w+\mu'} = w$; the latter case happens for the unique element $\mu' = w^{-1}w_0\mu$ in the orbit $W_R\mu$; therefore (45) holds.

Step 3. By Step 1 and Step 2, we see that $G^k\mathbf{C}[\tilde{W}_S]$ is a $\mathbf{C}P_+^\vee$-submodules and the successive quotient $G^k\mathbf{C}[\tilde{W}_S]^{\mathrm{nil}}/G^{k+1}\mathbf{C}[\tilde{W}_S]^{\mathrm{nil}}$ is a free $\mathbf{C}P_+^\vee$-module with a basis formed by the congruence classes of $\left\{[X^{b_w}wy]^{\mathrm{nil}}\right\}_{y,w\in W_R,\,\ell(y)=k}$. It follows that $\left\{[X^{b_w}wy]^{\mathrm{nil}}\right\}_{y,w\in W_R}$ forms a $\mathbf{C}P_+^\vee$-basis for $\mathbf{C}[\tilde{W}_S]^{\mathrm{nil}}$. Since $X^{b_w}w$ is minimal in the coset $X^{b_w}W_R$, we have

$$[X^{b_w}wy]^{\mathrm{nil}} = [X^{b_w}w]^{\mathrm{nil}}\cdot[y]^{\mathrm{nil}}, \quad \text{for } y \in W_R;$$

thus $\left\{[X^{b_w}w]^{\mathrm{nil}}\right\}_{w\in W_R}$ forms a $(\mathbf{C}P_+^\vee, \mathbf{C}[W_R]^{\mathrm{nil}})$-bimodule basis for $\mathbf{C}[\tilde{W}_S]^{\mathrm{nil}}$, whence (i).

Step 4. Let $\mathcal{E}' \subset \mathbf{C}[\tilde{W}_S]^{\mathrm{nil}}$ be the (free) $\mathbf{C}P_+^\vee$-submodule generated by $\left\{[\theta(b_w)]^{\mathrm{nil}}\right\}_{w\in W_R}$ so that, by (i), there is a decomposition $\mathbf{C}[\tilde{W}_S]^{\mathrm{nil}} \cong \mathcal{E}'\otimes\mathbf{C}[W_R]^{\mathrm{nil}}$. Let $\Omega = P/Q$. Define a $\mathbf{C}$-linear action of Ω on $\mathbf{C}[\tilde{W}_S]^{\mathrm{nil}}$ by

$$\begin{aligned}\Omega\times\mathbf{C}[\tilde{W}_S]^{\mathrm{nil}} &\longrightarrow \mathbf{C}[\tilde{W}_S]^{\mathrm{nil}}\\ (\beta, [X^{\mu}w]^{\mathrm{nil}}) &\mapsto e^{2\pi i\langle\beta,\mu\rangle}[X^{\mu}w]^{\mathrm{nil}}, \quad \beta\in\Omega,\ \mu\in P^\vee,\ w\in W_R.\end{aligned}$$

This action preserves the subspace $\mathcal{E}' \subset \mathbf{C}[\tilde{W}_S]^{\mathrm{nil}}$ and fixes $\mathbf{C}[W_R]^{\mathrm{nil}}$ pointwise; hence there is a decomposition of the Ω-fixed subspace

$$\mathbf{C}[W_S]^{\mathrm{nil}} = (\mathbf{C}[\tilde{W}_S]^{\mathrm{nil}})^{\Omega} \cong \mathcal{E}\otimes\mathbf{C}[W_R]^{\mathrm{nil}}, \quad \text{where } \mathcal{E} = (\mathcal{E}')^{\Omega}.$$

It remains to show that $\mathcal{E}$ is a Cohen–Macaulay $\mathbf{C}Q_+^\vee$-module of maximal dimension. Since $\mathbf{C}Q_+^\vee$ is integrally closed and $\mathbf{C}P_+^\vee$ is regular and an integral ring extension of it, by [2, X.2.6,coro 2], $\mathbf{C}P_+^\vee$ is a Cohen–Macaulay $\mathbf{C}Q_+^\vee$-module. Thus $\mathcal{E}'$, being a free $\mathbf{C}P_+^\vee$-module, is Cohen–Macaulay of maximal dimension over $\mathbf{C}Q_+^\vee$. Since $\mathcal{E}$ is a direct factor of $\mathcal{E}'$, so is it Cohen–Macaulay of maximal dimension over $\mathbf{C}Q_+^\vee$, whence (ii).

□

Below, we will work with $\mathrm{gr}^F\mathbf{A}^\omega$ and view the elements $\tau_a^\omega\mathbf{e}(\lambda)$ as in $\mathrm{gr}^F\mathbf{A}^\omega$ for the sake of notational simplicity. View $\mathbf{A}^\omega$ as $(\mathbf{A}^\omega)^{\mathrm{op}}$-module via the right regular representation. The ring $\mathrm{End}_{(\mathrm{gr}^F\mathbf{A}^\omega)^{\mathrm{op}}}(\mathrm{gr}^F\mathbf{A}^\omega)$ can be viewed as a unital completion of $\mathbf{A}^\omega$. Define a $\mathbf{C}$-linear map

$$\Theta : \mathbf{C}[W_S]^{\mathrm{nil}} \longrightarrow \mathrm{End}_{(\mathrm{gr}^F\mathbf{A}^\omega)^{\mathrm{op}}}(\mathrm{gr}^F\mathbf{A}^\omega) = \prod_{\lambda\in W_S\lambda_0}\mathrm{gr}^F\mathbf{A}^\omega\mathbf{e}(\lambda) \tag{47}$$

$$\Theta([w]^{\mathrm{nil}}) = \tau_w^\omega = \sum_{\lambda\in W_S\lambda_0}\tau_w^\omega\mathbf{e}(\lambda), \quad w\in W_S$$

Proposition 48. *There is a triangular decomposition*

$$\mathrm{gr}^F\mathbf{A}^\omega \cong \mathcal{E}\otimes_{\mathbf{C}}\left(\bigoplus_{w\in W_R}\mathbf{C}\tau_w^\omega\right)\otimes_{\mathbf{C}}\left(\bigoplus_{\lambda\in W_S\lambda_0}\mathbf{C}[V]\mathbf{e}(\lambda)\right),$$

where $\mathcal{E}\subset\mathbf{C}[W_S]^{\mathrm{nil}}$ is the $\mathbf{C}Q_+^\vee$-submodule from Lemma 44 (ii).

Proof. From Theorem 40, we see that the $\mathbf{C}[W_S]^{\mathrm{nil}}$-action on $\mathrm{gr}^F \mathbf{A}^\omega$ via Θ yields a decomposition

$$\mathbf{C}[W_S]^{\mathrm{nil}} \otimes \left(\bigoplus_{\lambda \in W_S \lambda_0} \mathbf{C}[V]\mathbf{e}(\lambda) \right) \xrightarrow{\sim} \mathrm{gr}^F \mathbf{A}^\omega, \quad f \otimes b \mapsto \Theta(f)(b).$$

By Lemma 44 (ii), we can further decompose $\mathbf{C}[W_S]^{\mathrm{nil}} = \mathcal{E} \otimes \mathbf{C}[W_R]^{\mathrm{nil}}$. Finally, we have $\Theta(\mathbf{C}[W_R]^{\mathrm{nil}}) = \bigoplus_{w \in W_R} \mathbf{C}\tau_w^\omega$. ☐

7 Module categories of $\mathbf{A}^\omega$

We keep the notation of §6. We put a $\mathbf{Z}$-grading on $\mathbf{A}^\omega$ as follows: the generators are homogeneous: $\deg \alpha\, \mathbf{e}(\lambda) = 2$ for $\alpha \in V^*$ and $\deg \tau_a^\omega \mathbf{e}(\lambda) = \omega_\lambda(a) + \omega_{s_a\lambda}(a)$. If $M = \bigoplus_n M_n$ is a graded vector space, denote by $M\langle m \rangle$ the grading shift given by $M\langle m \rangle_n = M_{m+n}$. For two graded vector spaces M and N, we denote by $\mathrm{Hom}(M, N)$ the space of $\mathbf{C}$-linear maps of degree 0 and $\mathrm{gHom}(M, N) = \bigoplus_{k \in \mathbf{Z}} \mathrm{Hom}(M, N\langle k \rangle)$.

Below, by "modules" we mean left modules. All statements can be turned into those for right modules by means of the anti-involution $\mathbf{A}^\omega \cong (\mathbf{A}^\omega)^{\mathrm{op}}$ defined by $\tau_a^\omega \mathbf{e}(\lambda) \mapsto \tau_a^\omega \mathbf{e}(s_a\lambda)$.

7.1 Graded $\mathbf{A}^\omega$-modules

An $\mathbf{A}^\omega$-module M is called a weight module if there is a decomposition

$$M = \bigoplus_{\lambda \in W_S \lambda_0} \mathbf{e}(\lambda) M.$$

Let $\mathbf{A}^\omega\text{-gMod}$ denote the category of graded weight modules of $\mathbf{A}^\omega$. Let $\mathbf{A}^\omega\text{-gmod} \subset \mathbf{A}^\omega\text{-gMod}$ be the subcategory of compact objects (i.e. $M \in \mathbf{A}^\omega\text{-gmod}$ if $\mathrm{Hom}_{\mathbf{A}^\omega\text{-gMod}}(M, -)$ commutes with filtered colimits) and let $\mathbf{A}^\omega\text{-gmod}_0 \subset \mathbf{A}^\omega\text{-gmod}$ be the subcategory of $\mathfrak{m}_{\mathcal{Z}}$-nilpotent objects. The following lemma is obvious.

Lemma 49. *For every object $M \in \mathbf{A}^\omega\text{-gMod}$ there exists an index set J and two families of integers $\{a_j\}_{j \in J}$ and $\{\lambda_j\}_{j \in J}$ such that there exists an epimorphism in $\mathbf{A}^\omega\text{-gMod}$*

$$\bigoplus_{j=1}^{r} \mathbf{A}^\omega \mathbf{e}(\lambda_j)\langle a_j \rangle \twoheadrightarrow M.$$ ☐

We define a homomorphism of graded rings

$$\mathcal{Z} \longrightarrow \mathrm{gEnd}\,(\mathrm{id}_{\mathbf{A}^\omega\text{-gMod}}) \tag{50}$$

as follows: For every $f \in \mathbf{C}[V]^{W_{\lambda_0}}$ and $w \in W_S$, let f acts on $\mathbf{e}(w\lambda_0)M$ by multiplication with $(\partial w)(f) \in \mathbf{C}[V]^{W_{w\lambda_0}}$.

7.2 Intertwiners

For each $\lambda \in W_S\lambda_0$ and $a \in \Delta$, introduce the following element in $\mathbf{A}^\omega$:

$$\varphi_a \mathbf{e}(\lambda) = \begin{cases} ((\partial a)\tau_a^\omega + 1)\mathbf{e}(\lambda) & \omega_\lambda(a) = -1 \\ \tau_a^\omega \mathbf{e}(\lambda) & \omega_\lambda(a) \geq 0 \end{cases}.$$

It satisfies the following relations:

$$\varphi_a^2 \mathbf{e}(\lambda) = \begin{cases} \mathbf{e}(\lambda) & \omega_\lambda(a) = -1 \\ \pm(\partial a)^{n_{\lambda,a}} \mathbf{e}(\lambda) & \omega_\lambda(a) \geq 0 \end{cases}$$
$$\varphi_a f \mathbf{e}(\lambda) = s_a(f)\varphi_a \mathbf{e}(\lambda) \quad f \in \mathbf{C}[V],$$

where $n_{\lambda,a} = \max(\omega_\lambda(a) + \omega_{s_a\lambda}(-a), 0)$. These elements satisfy the usual braid relations. Thus, we may write $\varphi_w \mathbf{e}(\lambda) = \varphi_{a_l} \cdots \varphi_{a_1} \mathbf{e}(\lambda)$ by choosing any reduced expression $w = s_{a_l} \cdots s_{a_1}$.

Lemma 51. *Let $w \in W_S$ and $a \in \Delta$. Then the right multiplication by the intertwiner φ_a induces an isomorphism of $\mathbf{A}^\omega$-modules*

$$\mathbf{A}^\omega \mathbf{e}(\lambda) \cong \mathbf{A}^\omega \mathbf{e}(s_a\lambda)$$

if $\omega_\lambda(a) + \omega_{s_a\lambda}(-a) \leq 0$.

Proof. The right multiplication by the element $\varphi_a \mathbf{e}(s_a\lambda) = \mathbf{e}(\lambda)\varphi_a\mathbf{e}(s_a\lambda)$ yields $\mathbf{A}^\omega\mathbf{e}(\lambda) \xrightarrow{\sim} \mathbf{A}^\omega\varphi_a\mathbf{e}(s_a\lambda) \xrightarrow{\sim} \mathbf{A}^\omega\varphi_a^2\mathbf{e}(s_a\lambda)$. Hence if $\varphi_a^2\mathbf{e}(\lambda) = f\mathbf{e}(\lambda) \in \mathbf{C}[V]\mathbf{e}(\lambda)$ for $f \in \mathbf{C}[V]$ invertible, then $\varphi_a^2\mathbf{e}(\lambda)$ is an isomorphism. The condition that f be invertible is exactly as stated. Clearly, if $\varphi_a^2\mathbf{e}(\lambda)$ and $\varphi_a^2\mathbf{e}(s_a\lambda)$ are isomorphisms, then so are $\varphi_a\mathbf{e}(\lambda)$ and $\varphi_a\mathbf{e}(s_a\lambda)$. The statement follows. ☐

7.3 Clan decomposition

As in §6.5, we extend ω_{λ_0} to a W_S-invariant function $\tilde{\omega}_{\lambda_0} : S \longrightarrow \mathbf{Z}_{\geq -1}$ and we suppose that the extension $\tilde{\omega}_{\lambda_0}$ has finite support. Consider the following sub-family of hyperplanes

$$\mathfrak{D}^\omega = \{H_a \subset E \,;\, a \in S,\ \tilde{\omega}_{\lambda_0}(a) \geq 1\}.$$

The connected components of the following space

$$E_\circ^\omega = E \setminus \bigcup_{H \in \mathfrak{D}^\omega} H$$

are called clans. Since $\tilde{\omega}_{\lambda_0}$ is supposed to be finitely supported, the family $\mathfrak{D}^\omega$ is finite, the set of connected components $\pi_0(E_\circ^\omega)$ is finite and there are only a finite number of clans.

Let $\mathfrak{C} \subset E_\circ^\omega$ be a clan. Since $E_\circ^\omega$ is the complement of a finite hyperplane arrangement, $\mathfrak{C}$ is a convex polytope. The **salient cone** of $\mathfrak{C}$ is defined to be the convex polyhedral cone $\kappa \subset V$ whose dual cone $\kappa^\vee$ is the cone of linear functions which are bounded from below on $\mathfrak{C}$:

$$\kappa^\vee = \left\{ v \in V^* \,;\, \inf_{x \in \mathfrak{C}} \langle v, x\rangle > -\infty \right\}, \quad \kappa = \kappa^{\vee\vee} = \{x \in V \,;\, \langle v, x\rangle \geq 0,\ \forall v \in \kappa^\vee\}.$$

Then κ is a convex polyhedral generated by a finite subset of $P^\vee$. We say that clan $\mathfrak{C} \subset E_\circ^\omega$ is **generic** if its salient cone is of maximal dimension.

Denote by $\nu_0 \in E$ the fundamental alcove associated with the basis Δ.

Lemma 52. *Let $w \in W_S$ and $a \in \Delta$. Then $w^{-1}\nu_0$ and $w^{-1}s_a\nu_0$ are in the same clan if and only if the intertwiner φ_a induces an isomorphism of $\mathbf{A}^\omega$-modules*

$$\mathbf{A}^\omega\mathbf{e}(w\lambda_0) \cong \mathbf{A}^\omega\mathbf{e}(s_a w\lambda_0).$$

Proof. Using Lemma 51, we have

$$\begin{aligned}\varphi_a^2\mathbf{e}(w\lambda_0) = \mathbf{e}(w\lambda_0) &\Leftrightarrow \omega_{w\lambda_0}(a) + \omega_{s_a w\lambda_0}(-a) \leq 0 \\ &\Leftrightarrow \tilde{\omega}_{\lambda_0}(w^{-1}a) + \tilde{\omega}_{\lambda_0}(-w^{-1}a) \leq 0 \Leftrightarrow H_{wa} \notin \mathfrak{D}^\omega\end{aligned}$$

The last condition is equivalent to that $w^{-1}\nu_0$ and $w^{-1}s_a\nu_0$ belong to the same clan. ☐

The following proposition follows immediately from the above lemma.

Proposition 53. *If $w, w' \in W_S$ are such that $w^{-1}\nu_0$ and $w'^{-1}\nu_0$ lie in the same clan, then right multiplication by the intertwiner $\varphi_{w'w^{-1}}\mathbf{e}(w\lambda)$ yields an isomorphism $\mathbf{A}^\omega\mathbf{e}(w'\lambda) \longrightarrow \mathbf{A}^\omega\mathbf{e}(w\lambda)$.*

☐

Corollary 54. *Let $M \in \mathbf{A}^\omega$-gmod. If $w, w' \in W_S$ are such that $w\nu_0^{-1}$ and $w'\nu_0^{-1}$ lie in the same clan, then multiplication by the intertwiner $\varphi_{w'w^{-1}}\mathbf{e}(w\lambda)$ yields an isomorphism of graded $\mathcal{Z}$-modules $\mathbf{e}(w\lambda_0)M \cong \mathbf{e}(w'\lambda_0)M$. In particular, in this case there is an equality of graded dimensions*

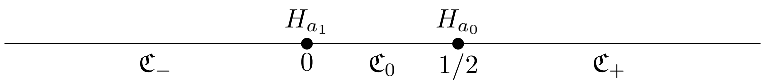
$$\operatorname{gdim} \mathbf{e}(w\lambda_0)M = \operatorname{gdim} \mathbf{e}(w'\lambda_0)M.$$

Proof. Indeed, we have

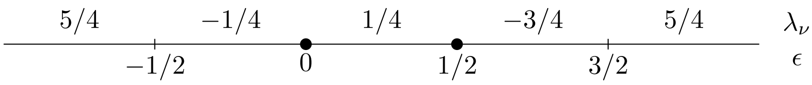
$$\mathbf{e}(w\lambda_0)M \cong \mathrm{Hom}_{\mathbf{A}^\omega}(\mathbf{A}^\omega\mathbf{e}(w\lambda_0), M) \xrightarrow{-\circ\varphi_{w'w^{-1}}} \mathrm{Hom}_{\mathbf{A}^\omega}(\mathbf{A}^\omega\mathbf{e}(w'\lambda_0), M) \cong \mathbf{e}(w\lambda_0)M.$$

□

Example 55. *In the setting of Example 33 (iii), the alcoves in E are of the form $]n, n+1/2[$ for $n \in (1/2)\mathbf{Z}$ and the fundamental alcove is $\nu_0 =]0, 1/2[$. We have $\mathfrak{D}^\omega = \{H_{a_0}, H_{a_1}\}$, with $\{a_0 = 1-2\epsilon, a_1 = 2\epsilon\} = \Delta$. The clan decomposition is depicted as follows:*



The clans $\mathfrak{C}_- =]-\infty, 0[$ and $\mathfrak{C}_+ =]1/2, +\infty[$ are generic whereas the clan $\mathfrak{C}_0 =]0, 1/2[= \nu_0$ is not generic. To each alcove $\nu = w^{-1}\nu_0$ with $w \in W_S$, we attach the element $\lambda_\nu = w\lambda_0 \in E$



In particular, the alcoves $\nu =]1/2, 3/2[$ and $\nu' =]3/2, 5/2[$ lie in the same clan $\mathfrak{C}_+$ with $\lambda_\nu = -3/4$ and $\lambda_{\nu'} = s_0\lambda_\nu = 5/4$. In this case Proposition 53 amount to the fact that the intertwiners $\varphi_{a_0}\mathbf{e}(\lambda_{\nu'}) : \mathbf{A}^\omega\mathbf{e}(\lambda_\nu) \longrightarrow \mathbf{A}^\omega\mathbf{e}(\lambda_{\nu'})$ and $\varphi_{a_0}\mathbf{e}(\lambda_\nu) : \mathbf{A}^\omega\mathbf{e}(\lambda_{\nu'}) \longrightarrow \mathbf{A}^\omega\mathbf{e}(\lambda_\nu)$ are isomorphisms and inverse to each other.

The projective $\mathbf{A}^\omega$-modules $\mathbf{A}^\omega\mathbf{e}(\lambda_\nu)$ are indecomposable and they are non-isomorphic for alcoves ν in the three different clans $\mathfrak{C}_-, \mathfrak{C}_0$ and $\mathfrak{C}_+$. Choose any alcoves $\nu_+ \subset \mathfrak{C}_+$, $\nu_- \subset \mathfrak{C}_-$ and denote $\lambda_+ = \lambda_{\nu_+}, \lambda_- = \lambda_{\nu_-}$, $P_+ = \mathbf{A}^\omega\mathbf{e}(\lambda_+)$,$P_0 = \mathbf{A}^\omega\mathbf{e}(\lambda_0)$ and $P_- = \mathbf{A}^\omega\mathbf{e}(\lambda_-)$. Their simple quotients, denoted by L_+, L_0 and L_-, form a complete collection of simple objects of $\mathbf{A}^\omega$-gmod up to grading shifts. The graded dimension is given by

$$\mathrm{gdim}\,\mathbf{e}(\lambda_\nu)L_* = \begin{cases} 1 & \nu \subseteq \mathfrak{C}_* \\ 0 & \nu \not\subseteq \mathfrak{C}_* \end{cases}, \quad * \in \{+, 0, -\}.$$

In particular, L_+ and L_- are infinite-dimensional and L_0 is finite-dimensional. The cosocle filtrations of P_+, P_0 and P_- are described as follows:

$$P_+ = \begin{bmatrix} L_+ & & \\ & L_0\langle -1\rangle & \\ L_+\langle -2\rangle & & L_-\langle -2\rangle \\ & L_0\langle -3\rangle & \\ L_+\langle -4\rangle & & L_-\langle -4\rangle \\ & \vdots & \end{bmatrix}, \quad P_0 = \begin{bmatrix} & L_0 & \\ L_+\langle -1\rangle & & L_-\langle -1\rangle \\ & L_0\langle -2\rangle & \\ L_+\langle -3\rangle & & L_-\langle -3\rangle \\ & L_0\langle -2\rangle & \\ & \vdots & \end{bmatrix}, \quad P_- = \begin{bmatrix} & & L_- \\ & L_0\langle -1\rangle & \\ L_+\langle -2\rangle & & L_-\langle -2\rangle \\ & L_0\langle -3\rangle & \\ L_+\langle -4\rangle & & L_-\langle -4\rangle \\ & \vdots & \end{bmatrix}.$$

7.4 Basic properties of graded modules of $\mathbf{A}^\omega$

We choose a finite subset $\Sigma \subset W_S$ such that for every clan $\mathfrak{C} \subset E^\omega_\circ$, there exists $w \in \Sigma$ with $w^{-1}\nu_0 \subset \mathfrak{C}$. Set $\mathbf{e}_\Sigma = \sum_{w\in\Sigma}\mathbf{e}(w\lambda_0)$ and $P_\Sigma = \mathbf{A}^\omega\mathbf{e}_\Sigma$.

Lemma 56. *The module P_Σ is a graded compact projective generator of $\mathbf{A}^\omega$-gMod.*

Proof. For any $y \in W_S$, we can find $w \in \Sigma$ such that $y^{-1}\nu_0$ and $w^{-1}\nu_0$ are in the same clan. By Proposition 53, there exists an isomorphism

$$\mathbf{A}^\omega\mathbf{e}(w\lambda_0) \cong \mathbf{A}^\omega\mathbf{e}(y\lambda_0)$$

Since the former is a direct factor of P_Σ, the above isomorphism yields a surjection $P_\Sigma \twoheadrightarrow \mathbf{A}^\omega\mathbf{e}(y\lambda_0)$. Combining this with Lemma 49, we see that P_Σ is a graded generator, which is clearly compact projective. □

Put $A_\Sigma = (\mathrm{gEnd}_{\mathbf{A}^\omega\text{-gMod}}P_\Sigma)^{\mathrm{op}} = \mathbf{e}_\Sigma\mathbf{A}^\omega\mathbf{e}_\Sigma$. It follows from Lemma 56 and the Morita theory that there is a graded equivalence

$$\mathrm{gHom}_{\mathbf{A}^\omega\text{-gMod}}(P_\Sigma, -) : \mathbf{A}^\omega\text{-gMod} \xrightarrow{\sim} A_\Sigma\text{-gMod}, \tag{57}$$

which restricts to an equivalence on the subcategories of compact objects $\mathbf{A}^\omega$-gmod $\xrightarrow{\sim}$ A_Σ-gmod.

Proposition 58. *The following statements hold:*

(i) The category $\mathbf{A}^\omega$-gmod *is noetherian and the subcategory* $\mathbf{A}^\omega$-gmod$_0$ *consists of objects of finite length.*

(ii) For each $M \in \mathbf{A}^\omega$-gmod *and each* $\lambda \in W_S\lambda_0$*, the graded dimension* $\mathrm{gdim}\,\mathbf{e}(\lambda)M$ *is in* $\mathbf{N}(\!(v)\!)$*. Moreover,* $M \in \mathbf{A}^\omega$-gmod$_0$ *if and only if* $\mathrm{gdim}\,\mathbf{e}(\lambda)M \in \mathbf{N}[v^{\pm 1}]$ *for all* $\lambda \in W_S\lambda_0$*.*

(iii) Every object of $\mathbf{A}^\omega$-gmod *admits a projective cover in the same category.*

(iv) We have $\mathrm{Irr}(\mathbf{A}^\omega\text{-gmod}_0) \cong \mathrm{Irr}(\mathbf{A}^\omega\text{-gmod})$*.*

(v) The map (50) *is an isomorphism* $\mathcal{Z} \cong \mathrm{gEnd}\,(\mathrm{id}_{\mathbf{A}^\omega\text{-gmod}})$*.*

Proof. By the graded Morita equivalence (57), it suffices to show the corresponding statements for A_Σ-gmod.

Since A_Σ is of finite rank over the graded polynomial ring $\mathbf{C}[V]^{W_{\lambda_0}}$, it is laurentian (i.e. its graded dimension is in $\mathbf{N}(\!(v)\!)$) and thus graded semi-perfect. The statements (i)–(iv) result from the laurentian property.

We prove (v). Consider the $\mathbf{A}^\omega$-module $\mathrm{Pol}_{W_S\lambda_0} \in \mathbf{A}^\omega$-gmod. Since each factor $\mathrm{Pol}_\lambda = \mathbf{C}[V]$ is a free $\mathcal{Z}$-module of finite rank, the sum $\mathrm{Pol}_{W_S\lambda_0}$ is a free $\mathcal{Z}$-module of infinite rank. Taking base-change to the rational function field $\mathrm{Frac}\,\mathcal{Z}$, we get a homomorphism

$$\rho : \mathbf{A}^{-\infty} \longrightarrow \bigoplus_{\lambda,\lambda' \in W_S\lambda_0} \mathrm{Hom}_{\mathrm{Frac}\,\mathcal{Z}}\left(\mathrm{Rat}_\lambda, \mathrm{Rat}_{\lambda'}\right),$$

We claim that ρ is an isomorphism. It is injective since $\mathrm{Pol}_{W_S\lambda_0}$ is a faithful $\mathbf{A}^\omega$-module by definition and it remains faithful after localisation. It is easy to see from the definition of $\mathbf{A}^\omega$ that for $\lambda \in W_S\lambda_0$ and $a \in \Delta$, the operator $s_a\mathbf{e}(\lambda) : \mathrm{Rat}_\lambda \longrightarrow \mathrm{Rat}_{s_a\lambda}$ is in the image of ρ. For any $\lambda, \lambda' \in W_S\lambda_0$, let $W_{\lambda,\lambda'} = \{w \in W_S \,;\, w\lambda = \lambda'\}$. The family $\{\mathbf{e}(\lambda')w\mathbf{e}(\lambda)\}_{w \in W_{\lambda,\lambda'}}$ is in the image of ρ. The rational function field Rat_λ is a Galois extension of $\mathrm{Frac}\,\mathcal{Z}$ with Galois group W_λ. It follows from the Galois theory that

$$\mathrm{End}_{\mathrm{Frac}\,\mathcal{Z}}(\mathrm{Rat}_\lambda) \cong \mathrm{Rat}_\lambda \rtimes \mathbf{C}W_\lambda.$$

We have already seen that $\{w\,\mathbf{e}(\lambda)\}_{w \in W_\lambda}$ is in $\mathrm{im}\,\rho$ the and Rat_λ is also in the image of ρ. It follows that $\mathrm{End}_{\mathrm{Frac}\,\mathcal{Z}}(\mathrm{Rat}_\lambda) \subset \mathrm{im}\,\rho$. Let $\lambda, \lambda' \in W_S\lambda_0$ and choose $w \in W_{\lambda,\lambda'}$. Then $w\,\mathbf{e}(\lambda) \in \mathrm{im}\,\rho$ is an isomorphism $w\,\mathbf{e}(\lambda) : \mathrm{Rat}_\lambda \cong \mathrm{Rat}_{\lambda'}$ and the pre-composition yields

$$- \circ w\mathbf{e}(\lambda) : \mathrm{End}_{\mathrm{Frac}\,\mathcal{Z}}(\mathrm{Rat}_\lambda) \cong \mathrm{Hom}_{\mathrm{Frac}\,\mathcal{Z}}\left(\mathrm{Rat}_\lambda, \mathrm{Rat}_{\lambda'}\right).$$

Thus $\mathrm{Hom}_{\mathrm{Frac}\,\mathcal{Z}}\left(\mathrm{Rat}_\lambda, \mathrm{Rat}_{\lambda'}\right) \subset \mathrm{im}\,\rho$. We see that ρ is surjective and the claim is proven.

There is an isomorphism

$$A_\Sigma \otimes_{\mathcal{Z}} \mathrm{Frac}\,\mathcal{Z} = \mathbf{e}_\Sigma \mathbf{A}^{-\infty}\mathbf{e}_\Sigma \cong \mathrm{End}_{\mathrm{Frac}\,\mathcal{Z}}\left(\bigoplus_{w \in \Sigma} \mathrm{Rat}_{w\lambda_0}\right)$$

induced by ρ. Since the right-hand side is a matrix algebra over a field $\mathrm{Frac}\,\mathcal{Z}$, its centre is $\mathrm{Frac}\,\mathcal{Z}$. It follows that $\mathrm{Z}(A_\Sigma) = \mathrm{Frac}\,\mathcal{Z}$. Hence

$$\begin{aligned}\mathrm{gEnd}\,(\mathrm{id}_{\mathbf{A}^\omega\text{-gmod}}) &\cong \mathrm{gEnd}\,(\mathrm{id}_{A_\Sigma\text{-gmod}}) = \mathrm{Z}\,(A_\Sigma) = \mathrm{Z}\,(A_\Sigma \otimes_{\mathcal{Z}} \mathrm{Frac}\,\mathcal{Z}) \cap A_\Sigma \\ &= \mathrm{Frac}\,\mathcal{Z} \cap A_\Sigma = \mathcal{Z},\end{aligned}$$

where the last equation follows from the basis theorem Theorem 40. □

7.5 Basic properties of ungraded $\mathbf{A}^\omega$-modules

Let $U : \mathbf{A}^\omega\text{-gmod}_0 \longrightarrow \mathbf{A}^\omega\text{-mod}_0$ be the grading-forgetting functor. We extend it to $U : \mathbf{A}^\omega\text{-gmod} \longrightarrow \mathrm{Pro}(\mathbf{A}^\omega\text{-mod}_0)$ by requiring U to preserve filtered inverse limits. The extended functor is exact. Define the subcategory $\mathbf{A}^\omega\text{-mod}^\wedge \subset \mathrm{Pro}(\mathbf{A}^\omega\text{-mod}_0)$ to be the essential image of this functor. Let $\mathcal{Z}^\wedge = \varprojlim_{N\to\infty} \mathcal{Z}/\mathfrak{m}_{\mathcal{Z}}^N$.

Proposition 59. *Then the following properties are satisfied:*

(i) The functor forgetting the grading $U : \mathbf{A}^\omega\text{-gmod} \longrightarrow \mathbf{A}^\omega\text{-mod}$ *is exact and it induces* $\operatorname{Irr}(\mathbf{A}^\omega\text{-gmod})/\langle \mathbf{Z}\rangle \cong \operatorname{Irr}(\mathbf{A}^\omega\text{-mod}^\wedge)$. *Moreover, for all* $M, N \in \mathbf{A}^\omega\text{-gmod}$ *and* $n \in \mathbf{N}$ *we have*

$$\prod_{k\in\mathbf{Z}} \operatorname{Ext}^n(M, N\langle k\rangle) \cong \operatorname{Ext}^n(UM, UN).$$

(ii) The category $\mathbf{A}^\omega\text{-mod}^\wedge$ *is noetherian and the subcategory* $\mathbf{A}^\omega\text{-mod}_0$ *consists of objects of finite length.*

(iii) Every object of $\mathbf{A}^\omega\text{-mod}^\wedge$ *admits a projective cover in the same category.*

(iv) We have $\operatorname{Irr}(\mathbf{A}^\omega\text{-mod}_0) \cong \operatorname{Irr}(\mathbf{A}^\omega\text{-mod}^\wedge)$.

(v) The ungraded analogue of the map (50) *induces an isomorphism* $\mathcal{Z}^\wedge \cong \operatorname{End}(\mathrm{id}_{\mathbf{A}^\omega\text{-mod}^\wedge})$.

These statements follow from Proposition 58. □

7.6 Induction and restriction

Let $\mathbf{A}^\omega_R \subset \mathbf{A}^\omega$ be the subalgebra generated by $f\mathbf{e}(\lambda)$ and $\tau^\omega_a\mathbf{e}(\lambda)$ for $\lambda \in W_S\lambda_0$, $f \in \mathbf{C}[V]$ and $a \in \Delta_0$. For $\lambda_1 \in W_S\lambda_0$, denote $\mathbf{e}_{R,\lambda_1} = \sum_{\lambda\in W_R\lambda_1} \mathbf{e}(\lambda)$ and define $\mathbf{A}^\omega_{R,\lambda_1} = \mathbf{e}_{R,\lambda_1}\mathbf{A}^\omega_R\mathbf{e}_{R,\lambda_1}$ to be the idempotent subalgebra. In other words, $\mathbf{A}^\omega_{R,\lambda_1}$ is the subalgebra of $\mathbf{A}^\omega$ generated by $f\mathbf{e}(\lambda)$ and $\tau^\omega_a\mathbf{e}(\lambda)$ for $\lambda \in W_R\lambda_1$, $f \in \mathbf{C}[V]$ and $a \in \Delta_0$.

For each $\lambda_1 \in W_S\lambda_0$, we define the induction, restriction and co-induction functors

$$\begin{aligned}
\operatorname{ind}^S_{R,\lambda_1} &: \mathbf{A}^\omega_{R,\lambda_1}\text{-gmod} \longrightarrow \mathbf{A}^\omega\text{-gmod}, \quad N \mapsto \mathbf{A}^\omega\mathbf{e}_{R,\lambda_1} \otimes_{\mathbf{A}^\omega_{R,\lambda_1}} N\\
\operatorname{res}^S_{R,\lambda_1} &: \mathbf{A}^\omega\text{-gmod} \longrightarrow \mathbf{A}^\omega_{R,\lambda_1}\text{-gmod}, \quad M \mapsto \mathbf{e}_{R,\lambda_1}M \cong \operatorname{gHom}_{\mathbf{A}^\omega}(\mathbf{A}^\omega\mathbf{e}_{R,\lambda_1}, M)\\
\operatorname{coind}^S_{R,\lambda_1} &: \mathbf{A}^\omega_{R,\lambda_1}\text{-gmod} \longrightarrow \mathbf{A}^\omega\text{-gmod}, \quad N \mapsto \bigoplus_{\lambda\in W_S\lambda_0} \operatorname{gHom}_{\mathbf{A}^\omega_{R,\lambda_1}}(\mathbf{e}_{R,\lambda_1}\mathbf{A}^\omega\mathbf{e}(\lambda), N).
\end{aligned}$$

They form a triplet of adjoint functors $\left(\operatorname{ind}^S_{R,\lambda_1}, \operatorname{res}^S_{R,\lambda_1}, \operatorname{coind}^S_{R,\lambda_1}\right)$

Proposition 60. *The functors* $\operatorname{ind}^S_{R,\lambda_1}, \operatorname{res}^S_{R,\lambda_1}$ *and* $\operatorname{coind}^S_{R,\lambda_1}$ *are exact.*

Proof. The functor $\operatorname{res}^S_{R,\lambda_1}$ is clearly exact. By Theorem 40, we have a decomposition of right $\mathbf{A}^\omega_{R,\lambda_1}$-module

$$\mathbf{A}^\omega\mathbf{e}_{R,\lambda_1} \cong \bigoplus_{w\in W^R} \tau^\omega_w \mathbf{A}^\omega_{R,\lambda_1} \tag{61}$$

where $W^R \subset W_S$ is the set of shortest representatives of the elements in W_S/W_R and $\tau^\omega_w = \bigoplus_{\lambda\in W_R\lambda_1} \tau^\omega_{a_l}\cdots\tau^\omega_{a_1}\mathbf{e}(\lambda)$ for any reduced expression $w = s_{a_l}\cdots s_{a_1}$. Therefore $\mathbf{A}^\omega\mathbf{e}_{R,\lambda_1}$ is a free right $\mathbf{A}^\omega_{R,\lambda_1}$-module, so $\operatorname{ind}^S_{R,\lambda_1}$ is exact. Similarly, $\operatorname{coind}^S_{R,\lambda_1}$ is also exact. □

8 Filtered $\mathbf{A}^\omega$-modules

We consider $\mathbf{A}^\omega$-modules equipped with filtrations which are compatible with the filtration by length F on $\mathbf{A}^\omega$. Most results in this section are non-unital version of the classical theory of filtered rings and filtered modules which one can find in [18]. The goal of this section is to introduce (§8.3) the support and the Gelfand–Kirillov dimension of an object $M \in \mathbf{A}^\omega\text{-gmod}_0$ and show (Proposition 69) that "induced modules" have the full support.

8.1 Good filtrations on $\mathbf{A}^\omega$-modules

Let $M \in \mathbf{A}^\omega\text{-gmod}$.

Definition 62. *A good filtration* F *on* M *is a sequence* $\{F_{\leq n}M\}_{n\in\mathbf{Z}}$ *of graded* $\mathbf{C}[V]$*-submodules of* M *satisfying the following properties:*

(i) $F_{\leq n-1} \subseteq F_{\leq n}$ *for all* $n \in \mathbf{Z}$;

(ii) for each $n \in \mathbf{Z}$, there exists a finite subset $\Sigma_n \subset W_S\lambda_0$ such that[§]

$$F_{\leq n}M = \bigoplus_{\lambda \in \Sigma_n} \mathbf{e}(\lambda) F_{\leq n}M;$$

(iii) $F_nM = 0$ *for* $n \ll 0$*;*

(iv) $\bigcup_{n\in\mathbf{Z}} F_nM = M$*;*

(v)

$$(F_{\leq n}\mathbf{A}^\omega)(F_{\leq m}M) \subseteq F_{\leq n+m}M, \quad \forall n, m \in \mathbf{N};$$

(vi) there exists $m_0 \gg 0$ satisfying

$$(F_{\leq n}\mathbf{A}^\omega)(F_{\leq m}M) = F_{\leq n+m}M, \quad \forall n \geq 0, \forall m \geq m_0.$$

The following result is standard, see [18, D.1.3]

Proposition 63. *Good filtrations exist for the objects of* $\mathbf{A}^\omega$-gmod*. If F and F' are two good filtrations on* $M \in \mathbf{A}^\omega$-gmod*, then there exists $i_0 \gg 0$ such that*

$$F'_{\leq n-i_0}M \leq F_{\leq n}M \leq F'_{\leq n+i_0}M, \quad \forall n \in \mathbf{Z}. \qquad \square$$

The following lemma is a direct consequence of Proposition 63.

Corollary 64. *If F and F' are good filtrations on M, then there exist*

(i) a finite filtration of $\mathrm{gr}^F\,\mathbf{A}^\omega$*-submodules F' on* $\mathrm{gr}^F M$*,*

(ii) a finite filtration of $\mathrm{gr}^F\,\mathbf{A}^\omega$*-submodules F on* $\mathrm{gr}^{F'} M$ *and*

(iii) an isomorphism of $\mathrm{gr}^F\,\mathbf{A}^\omega$*-modules* $\mathrm{gr}^{F'}\mathrm{gr}^F M \cong \mathrm{gr}^F \mathrm{gr}^{F'} M$*.*

Proof. By Proposition 63, there exists $i_0 \gg 0$ such that $F_{\leq n-i_0}M \leq F'_{\leq n}M \leq F_{\leq n+i_0}M$ for all $n \in \mathbf{Z}$. For $m \in [-i_0, i_0]$, define $F'_{\leq n,\leq m}M = \left(F'_{\leq n}M \cap F_{\leq n+m}M\right) + F'_{\leq n-1}M$. Then the quotient $\mathrm{gr}^{F'} M$ acquires a filtration

$$F_{\leq m}\,\mathrm{gr}^{F'}_n M = F'_{\leq n,m}M/F'_{\leq n-1}M \subseteq F'_{\leq n}M/F'_{\leq n-1}M = \mathrm{gr}^{F'}_n M,$$

which satisfies $\left(\mathrm{gr}^F_l\,\mathbf{A}^\omega\right)\left(F_{\leq m}\,\mathrm{gr}^{F'}_n M\right) \subseteq F_{\leq m}\,\mathrm{gr}^{F'}_{n+l} M$. Hence for each $m \in [-i_0, i_0]$, the quotient $\mathrm{gr}^F_m \mathrm{gr}^{F'} M = F_{\leq m}\mathrm{gr}^{F'} M/F_{\leq m-1}\mathrm{gr}^{F'} M$ is itself a $\mathrm{gr}\mathbf{A}^\omega$-module. Similarly, we put $F_{\leq m,\leq n}M = \left(F_{\leq m}M \cap F'_{\leq m+n}M\right) + F_{\leq m-1}M$ so that $\mathrm{gr}^F M$ acquires a filtration by $\mathrm{gr}^F\,\mathbf{A}^\omega$-modules. Zassenhaus lemma yields $\mathrm{gr}^F_{m-n}\,\mathrm{gr}^{F'}_n M \cong \mathrm{gr}^{F'}_{n-m}\,\mathrm{gr}^F_m M$. Therefore,

$$\bigoplus_{n=-i_0}^{i_0} \mathrm{gr}^{F'}_n\,\mathrm{gr}^F M \cong \bigoplus_{m=-i_0}^{i_0} \mathrm{gr}^F_m\,\mathrm{gr}^{F'} M.$$

$\square$

8.2 Associated graded of good filtrations

Recall the monoid algebra $\mathbf{C}Q^\vee_+$ from §6.6. Given a good filtration F on an object $M \in \mathbf{A}^\omega$-gmod, the associated graded $\mathrm{gr}^F M = \bigoplus_{k\in\mathbf{Z}} F_{\leq k}M/F_{\leq k-1}M$ is a $\mathrm{gr}^F\mathbf{A}^\omega$-module. The $\mathrm{gr}^F\,\mathbf{A}^\omega$-action on $\mathrm{gr}^F M$ extends to an action of the unital completion introduced in §6.6 via the natural inclusion

$$\mathrm{gr}^F\,\mathbf{A}^\omega \hookrightarrow \mathrm{End}_{(\mathrm{gr}^F\,\mathbf{A}^\omega)^{\mathrm{op}}}(\mathrm{gr}^F\,\mathbf{A}^\omega) \cong \prod_{\lambda\in W_S\lambda_0} \mathrm{gr}^F\,\mathbf{A}^\omega\mathbf{e}(\lambda).$$

We obtain a $\mathbf{C}Q^\vee_+$-module structure on $\mathrm{gr}^F M$ via the map (47).

Proposition 65. *Let* $M \in \mathbf{A}^\omega$-gmod *and F a good filtration on M. Then $\mathrm{gr}^F M$ is a coherent* $\mathbf{C}Q^\vee_+ \otimes \mathbf{C}[V]$*-module. Moreover, if* $M \in \mathbf{A}^\omega$-gmod$_0$*, then $\mathrm{gr}^F M$ is a coherent* $\mathbf{C}Q^\vee_+$*-module.*

[§]We require this condition because we work with a non-unital associative algebra.

Proof. We observe that the coherence for $\mathrm{gr}^F M$ is independent of the choice of the good filtration F. Indeed, if F' is another good filtration on M, then by Corollary 64,

$$\mathrm{gr}^F M \text{ coherent} \Leftrightarrow \bigoplus_{n=-i_0}^{i_0} \mathrm{gr}_n^{F'} \mathrm{gr}^F M \cong \bigoplus_{m=-i_0}^{i_0} \mathrm{gr}_m^F \mathrm{gr}^{F'} M \text{ coherent} \Leftrightarrow \mathrm{gr}^{F'} M \text{ coherent}.$$

We prove the first assertion. By Lemma 49 and the compactness of M, there is a surjection of the form

$$p : \bigoplus_{j=1}^{r} \mathbf{A}^\omega \mathbf{e}(\lambda_j)\langle a_j\rangle \twoheadrightarrow M.$$

Equip the source of p with the length filtration and the target of p with the induced filtration, denoted by F, so that p induces a surjection on the associated graded $\mathrm{gr}^F \mathbf{A}^\omega$-module. The coherence on the source of $\mathrm{gr}^F p$ implies that of $\mathrm{gr}^F M$. Thus we may suppose that M is of the form $M = \mathbf{A}^\omega \mathbf{e}(\lambda_j)$ and equipped with the length filtration. It follows from Proposition 48 that

$$\mathrm{gr}^F \mathbf{A}^\omega \mathbf{e}(\lambda_j) \cong \mathcal{E} \otimes_{\mathbf{C}} \left(\bigoplus_{w \in W_R} \tau_w^\omega \mathbf{C}[V] \right) \mathbf{e}(\lambda_j).$$

Since $\mathcal{E}$ is coherent over $\mathbf{C}Q_+^\vee$ by Lemma 44 (ii) and $\bigoplus_{w\in W_R} \mathbf{C}[V]\tau_w^\omega$ is free of finite rank over $\mathcal{Z}$, it follows that $\mathrm{gr}^F \mathbf{A}^\omega \mathbf{e}(\lambda_j)$ is coherent over $\mathbf{C}Q_+^\vee \otimes \mathcal{Z}$.

Suppose now $M \in \mathbf{A}^\omega\text{-gmod}_0$ so that $\mathcal{Z}$ acts via the quotient $\mathcal{Z}/\mathfrak{m}_{\mathcal{Z}}^n$ for some $n \in \mathbf{N}$. Since $\mathcal{Z}/\mathfrak{m}_{\mathcal{Z}}^n$ is finite-dimensional, M must be coherent over $\mathbf{C}Q_+^\vee$. □

8.3 Support of $\mathbf{A}^\omega$-modules of finite length

Let $M \in \mathbf{A}^\omega\text{-gmod}_0$. In view of Proposition 65, we can make the following definition:

Definition 66. *The support of M, denoted by* $\mathrm{Supp}\, M$*, is defined to be the support of* $\mathrm{gr}^F M$ *as coherent $\mathbf{C}Q_+^\vee$-module, for any choice of good filtration F on M.*

By Corollary 64, the definition of $\mathrm{Supp}\, M$ is independent of the choice of a good filtration.

We define the Gelfand–Kirillov dimension of a weight module M of $\mathbf{A}^\omega$ to be the following number: upon choosing a good filtration F on M,

$$\dim_{\mathrm{GK}} M = \limsup_{n \longrightarrow \infty} \frac{\log \dim F_{\le n} M}{\log n}.$$

By Proposition 63, this number does not depend on the choice of F.

Proposition 67. *Let $M \in \mathbf{A}^\omega\text{-gmod}_0$. Then the Gelfand–Kirillov dimension* $\dim_{\mathrm{GK}} M$ *coincides with the Krull dimension of* $\mathrm{Supp}\, M$.

Proof. Taking the associated graded, we have

$$\dim F_{\le n} M = \dim \bigoplus_{k=-\infty}^{n} \mathrm{gr}_k^F M.$$

Notice that $\mathbf{C}Q_+^\vee$ is finitely generated graded ring, where $\deg X^\mu = \ell(X^\mu)$, and $\mathrm{gr}^F M$ is a finitely generated graded module over it. Hence $\dim_{\mathrm{GK}} M$ is nothing but the degree of the Hilbert polynomial of $\mathrm{gr}^F M$, which is equal to the Krull dimension of $\mathrm{Supp}\, M$. □

8.4 Induction of filtered modules

Recall the subalgebra $\mathbf{A}_{R,\lambda_1}^\omega \subset \mathbf{A}^\omega$ from §7.6. Good filtrations on objects of $\mathbf{A}_{R,\lambda_1}^\omega\text{-gmod}$ are defined in a similar manner.

Suppose $N \in \mathbf{A}_{R,\lambda_1}^\omega\text{-gmod}$ is equipped with a good filtration F which satisfies $F_{\le k} N = \left(F_{\le k} \mathbf{A}_{R,\lambda_1}^\omega \right) (F_{\le 0} N)$ for $k \ge 0$ and $F_{\le -1} N = 0$.

Let $M = \mathrm{ind}_{R,\lambda_1}^S N$. The adjunction unit yields an inclusion of $\mathcal{Z}$-modules $N \hookrightarrow M$. Define a filtration $F_{\le n} M = (F_{\le n} \mathbf{A}^\omega)(F_{\le 0} N)$.

Lemma 68. *The filtration F on M is good and satisfies*

$$\mathrm{gr}^F M \cong \left(\mathrm{gr}^F \mathbf{A}^\omega \mathbf{e}_{R,\lambda_1}\right) \otimes_{\mathrm{gr}^F \mathbf{A}^\omega_{R,\lambda_1}} \left(\mathrm{gr}^F N\right).$$

Proof. By the hypothesis on $F_{\leq n}N$, we have $\mathrm{gr}^F_n N = \left(\mathrm{gr}^F_n \mathbf{A}^\omega_{R,\lambda_1}\right)\left(\mathrm{gr}^F_0 N\right)$ and $\mathrm{gr}^F_n M = \left(\mathrm{gr}^F_n \mathbf{A}^\omega\right)\left(\mathrm{gr}^F_0 N\right)$. By the decomposition (61), we deduce

$$\mathrm{gr}^F_k \mathbf{A}^\omega \mathbf{e}_{R,\lambda_1} = \bigoplus_{j=0}^{k} \bigoplus_{\substack{w\in W^R \\ \ell(w)=j}} \tau^\omega_w \mathrm{gr}^F_{k-j} \mathbf{A}^\omega_{R,\lambda_1},$$

from which

$$\begin{aligned}\left(\left(\mathrm{gr}^F \mathbf{A}^\omega \mathbf{e}_{R,\lambda_1}\right) \otimes_{\mathrm{gr}^F \mathbf{A}^\omega_{R,\lambda_1}} \left(\mathrm{gr}^F N\right)\right)_n &= \bigoplus_{j=0}^{n} \bigoplus_{\substack{w\in W^R \\ \ell(w)=j}} \tau^\omega_w \left(\mathrm{gr}^F_{n-j} \mathbf{A}^\omega_{R,\lambda_1}\right)\left(\mathrm{gr}^F_0 N\right) \\ &= \left(\mathrm{gr}^F_n \mathbf{A}^\omega_n\right)\left(\mathrm{gr}^F_0 N\right) = \mathrm{gr}^F_n M\end{aligned}$$

□

Proposition 69. *For any $N \in \mathbf{A}^\omega\text{-gmod}_0$ and $0 \neq M' \subset \mathrm{ind}^S_{R,\lambda_1} N$, we have* $\mathrm{Supp}\, M' = \mathrm{Spec}\, \mathbf{C}Q^\vee_+$.

Proof. Let F be a good filtration on N as above and denote $M = \mathrm{ind}^S_{R,\lambda_1} N$, so that $\mathrm{gr}^F M \cong \mathrm{gr}^F\left(\mathbf{A}^\omega \mathbf{e}_{R,\lambda_1}\right) \otimes_{\mathrm{gr}^F \mathbf{A}^\omega_{R,\lambda_1}} \left(\mathrm{gr}^F N\right)$ by Lemma 68. By Proposition 48, we have $\mathrm{gr}^F \mathbf{A}^\omega \mathbf{e}_{R,\lambda_1} \cong \mathcal{E} \otimes_{\mathbf{C}} \mathrm{gr}^F \mathbf{A}^\omega_{R,\lambda_1}$; hence

$$\mathrm{gr}^F M \cong \mathcal{E} \otimes_{\mathbf{C}} \mathrm{gr}^F N.$$

By Lemma 44 (ii), $\mathcal{E}$ and thus $\mathrm{gr}^F M$ are a Cohen–Macaulay module of maximal dimension over $\mathbf{C}Q^\vee_+$, so it is torsion-free. For any $0 \neq M' \subset M$, the restriction to M' of F is a good filtration and $\mathrm{gr}^F M' \subset \mathrm{gr}^F M$. Hence $\mathrm{Supp}\, M' = \mathbf{C}Q^\vee_+$. □

Remark 70. *Proposition 69 is an analogue of the following basic property for a double affine Hecke algebra $\mathbb{H}$: the induced module $\mathbb{H} \otimes_{\underline{\mathbb{H}}} M$ is free over the polynomial part $\mathbf{C}[E] \subset \mathbb{H}$ for every module M over the graded affine Hecke algebra $\underline{\mathbb{H}} \subset \mathbb{H}$. A similar property for rational Cherednik algebras was used in [16] in the proof of the double centraliser property of the KZ functor. Our proof of the double centraliser property Theorem 105 also relies on it.*

9 Quiver Hecke algebras

We keep the notation of root systems (E, S, Δ) and (V, R, Δ_0). In this section, we introduce an algebra $\mathbf{B}^\Omega$, which can be viewed as a variant of quiver Hecke algebras. The relation between the quiver Hecke algebras and $\mathbf{B}^\Omega$ in the case where the root system (V, R) is of type A is explained in Remark 73.

9.1 The algebra $\mathbf{B}^\Omega$

Define the torus $T = Q^\vee \otimes \mathbf{C}^\times$ so that the ring of regular functions $\mathbf{C}[T]$ is isomorphic to the group algebra $\mathbf{C}P$. For any $\alpha \in P$, we denote by $Y^\alpha \in \mathbf{C}[T]$ the corresponding element.

Fix $\ell_0 \in T$. Define for each $\ell \in W_R\ell_0$ a polynomial ring $\mathrm{Pol}_\ell = \mathbf{C}[V]$ and let $\mathrm{Pol}_{W_R\ell_0} = \bigoplus_{\ell\in W_R\ell_0} \mathrm{Pol}_\ell$. For each ℓ, define $\mathbf{e}(\ell) : \mathrm{Pol}_{W_R\ell_0} \longrightarrow \mathrm{Pol}_\ell$ to be the idempotent linear endomorphism of projection onto the factor Pol_ℓ. Recall that $R_{\mathrm{red}} = R^+ \setminus 2R$ and $R^+_{\mathrm{red}} = R_{\mathrm{red}} \cap R^+$.

Choose any $\lambda_0 \in \exp^{-1}(\ell_0)$. Then the algebra $\mathcal{Z}$ from §6.2 acts on Pol_ℓ: for any $w \in W_R$, the element $f \in \mathcal{Z} = \mathbf{C}[V]^{W_{\lambda_0}}$ acts on $\mathrm{Pol}_{w\ell_0}$ by multiplication by $w(f)$.

Let $\Omega = \{\Omega_\ell\}_{\ell\in W_R\ell_0}$ be a family of functions $\Omega_\ell : R^+_{\mathrm{red}} \longrightarrow \mathbf{Z}_{\geq -1}$ satisfying the properties:

(i) If $2\alpha \notin R$, then $\Omega_\ell(\alpha) = -1$ implies $Y^\alpha(\ell) = 1$.

(ii) If $2\alpha \in R$, then $\Omega_\ell(\alpha) = -1$ implies $Y^\alpha(\ell) \in \{1, -1\}$.

(iii) For $w \in W_R$ and $\alpha \in R^+_{\mathrm{red}} \cap w^{-1}R^+_{\mathrm{red}}$ we have $\Omega_\ell(\alpha) = \Omega_{w\ell}(w\alpha)$.

For each $\alpha \in \Delta_0$ and $\ell \in W_R\ell_0$, we define an operator $\tau^\Omega_\alpha \mathbf{e}(\ell) : \mathrm{Pol}_\ell \longrightarrow \mathrm{Pol}_{s_\alpha \ell}$ by

$$\tau^\Omega_\alpha \mathbf{e}(\ell) = \begin{cases} \alpha^{-1}(s_\alpha - 1) & \Omega_\ell(\alpha) = -1 \\ \alpha^{\Omega_\ell(\alpha)} s_\alpha & \Omega_\ell(\alpha) \geq 0 \end{cases}.$$

Here $s_\alpha : \mathbf{C}[V] \longrightarrow \mathbf{C}[V]$ is the reflection with respect to α.

Definition 71. *We define $\mathbf{B}^\Omega = \mathbf{B}(R, V, \Delta, W_R\ell_0, \Omega)$ to be the subalgebra of $\mathrm{End}_{\mathcal{Z}}(\mathrm{Pol}_{W_R\ell_0})$ generated by $\mathbf{C}[V]\mathbf{e}(\ell)$ and $\tau^\Omega_\alpha \mathbf{e}(\ell)$.*

All the statements of Proposition 58 for $\mathbf{A}^\omega$ hold equally for $\mathbf{B}^\Omega$. In particular, the centre of $\mathbf{B}^\Omega$ is equal to $\mathcal{Z}$.

Example 72.

(i) If $\ell_0 = 1 \in T$ and $\Omega = \{-1\}_{\ell=\ell_0}$ is the -1 constant function, then $\mathbf{B}^\Omega$ is the affine nil-Hecke algebra of type W_R and is isomorphic to a matrix algebra over its centre.

(ii) If $\ell_0 = 1 \in T$ and $\Omega = \{0\}_{\ell=\ell_0}$ is the zero constant function, then $\mathbf{B}^\Omega = \mathbf{C}[V] \rtimes W_R$ is the skew tensor product.

Remark 73. *In the case where the finite root system (V, R) is of type A_{n-1}, the algebra $\mathbf{B}^\Omega$ recovers the notion of quiver Hecke algebras.*

For any quiver $\Gamma = (I, H)$ with $I \subset \mathbf{C}^\times$ and a dimension vector $\beta \in \mathbf{N}I$ with $|\beta| = n$, the quiver Hecke algebra, denoted by $R_\beta(\Gamma)$ according to [31], is generated by three sets: idempotents $\{\mathbf{e}(\ell)\}_{\ell \in I^\beta}$, Hecke operators $\{\tau_i\}_{i=1}^{n-1}$, polynomial part $\{x_i\}_{i=1}^n$. By translating suitably the set $I \subset \mathbf{C}^\times$, we may assume that $\prod_{r\in I} r^{\beta_r} = 1$, so that each sequence $\nu = (\nu_1, \cdots, \nu_n) \in I^\beta \subset (\mathbf{C}^\times)^n$ lies in the maximal torus $T \subset (\mathbf{C}^\times)^n$ of $\mathrm{SL}_n(\mathbf{C})$. We put $\Omega_\nu(\alpha_{i,j}) = \#\{(h : i \longrightarrow j) \in H\} - \delta_{\nu_i=\nu_j}$. Then there is a surjective homomorphism

$$\begin{aligned} R_\beta(\Gamma) &\longrightarrow \mathbf{B}^\Omega \\ \mathbf{e}(\nu) &\mapsto \mathbf{e}(\nu) \\ \tau_i &\mapsto \tau_{\alpha_i}, \quad i \in \{1, \ldots, n-1\} \\ x_k &\mapsto \frac{1}{n}\left(-\sum_{1\leq j<k} j\alpha_j + \sum_{k\leq j<n}(n-j)\alpha_j\right)\mathbf{e}_R, \quad k \in \{1, \ldots, n\} \end{aligned}$$

whose kernel the ideal generated by $x_1 + \cdots + x_n$. .

9.2 Basis theorem

Theorem 74. *For any $w \in W_R$, choose a reduced expression $w = s_{a_1} \cdots s_{a_l}$ and put $\tau^\Omega_w = \tau^\Omega_{\alpha_l} \cdots \tau^\Omega_{\alpha_1}$. Then there is a decomposition*

$$\mathbf{B}^\Omega = \bigoplus_{\ell \in W_R\ell_0} \bigoplus_{\substack{w \in W_R \\ \ell(w)=n}} \mathbf{C}[V]\tau^\Omega_w \mathbf{e}(\ell).$$

Proof. To prove it, we shall apply the results Theorem 94 and Theorem 82 whose proofs do not rely on this theorem. By Lemma 75 below, we can choose $\omega = \{\omega_\lambda\}_{\lambda \in W_S\lambda_0}$ such that $\int\omega = \Omega$. Then Theorem 82 implies that upon choosing a good $\gamma \in Q^\vee$, there is an isomorphism $\mathbf{B}^\Omega \cong \mathbf{e}_\gamma \mathbf{A}^\omega \mathbf{e}_\gamma$ identifying $\tau^\Omega_\alpha \mathbf{e}(\ell)$ with $\sigma_\alpha \mathbf{e}({}^\gamma\ell)$ and by Theorem 94, the idempotent subalgebra $\mathbf{e}_\gamma \mathbf{A}^\omega \mathbf{e}_\gamma$ has a decomposition in terms of $\sigma_\alpha \mathbf{e}({}^\gamma\ell)$. Hence $\mathbf{B}^\Omega$ also has a decomposition as in the statement. □

Lemma 75. *Given any family of order functions $\Omega = \{\Omega_\ell\}_{\ell \in W_R\ell_0}$ for $\mathbf{B}^\Omega$, there exists a family of order functions $\omega = \{\omega_\lambda\}_{\lambda \in W_S\lambda_0}$ satisfying the conditions from §6.3 such that $\int\omega = \Omega$, where $\int\omega$ is defined in §10.2.*

Proof. We choose a point $\lambda_0 \in \exp^{-1}(\ell_0) \subset V$. Such a family $\omega = \{\omega_\lambda\}_{\lambda \in W_S\lambda_0}$ is determined by a W_{λ_0}-invariant function $\tilde{\omega}_{\lambda_0} : S \longrightarrow \mathbf{Z}_{\geq -1}$ and it suffices to construct it. However, one needs to be careful about the condition (i) from §6.3. We first define a function $\tilde{\Omega}_{\ell_0} : R \longrightarrow \mathbf{Z}_{\geq -1}$ as follows:

(i) For any $\alpha \in R^+_{\mathrm{red}}$ such that $2\alpha \notin R$, we set $\tilde{\Omega}_{\ell_0}(\alpha) = \Omega_{\ell_0}(\alpha)$ and $\tilde{\Omega}_{\ell_0}(-\alpha) = \Omega_{w_0\ell_0}(-w_0\alpha)$.

(ii) For any $\alpha \in R^+_{\mathrm{red}}$ such that $2\alpha \in R$, if $Y^\alpha(\ell_0) = -1$, then we set $\tilde{\Omega}_{\ell_0}(2\alpha) = \Omega_{\ell_0}(\alpha)$, $\tilde{\Omega}_{\ell_0}(-2\alpha) = \Omega_{w_0\ell_0}(-w_0\alpha)$ and $\tilde{\Omega}_{\ell_0}(\alpha) = \tilde{\Omega}_{\ell_0}(-\alpha) = 0$; otherwise, we set $\tilde{\Omega}_{\ell_0}(\alpha) = \Omega_{\ell_0}(\alpha)$, $\tilde{\Omega}_{\ell_0}(-\alpha) = \Omega_{w_0\ell_0}(-w_0\alpha)$ and $\tilde{\Omega}_{\ell_0}(2\alpha) = \tilde{\Omega}_{\ell_0}(-2\alpha) = 0$.

The function $\tilde{\Omega}_{\ell_0}$ is W_{ℓ_0}-invariant by the assumption (iii) from §9.1 and has image in $\mathbf{Z}_{\geq -1}$. We choose a section of the projection $W_{\lambda_0}\backslash S \longrightarrow W_{\ell_0}\backslash R$, denoted $f : W_{\ell_0}\backslash R \longrightarrow W_{\lambda_0}\backslash S$, in such a way that for each $\alpha \in R_{\mathrm{red}}$, the condition $f(\alpha)(\lambda_0) = 0$ holds whenever $\alpha(\ell_0) = 0$. We set $\tilde{\omega}_{\lambda_0} = f_*\tilde{\Omega}_{\ell_0}$ so that $\tilde{\omega}_{\lambda_0} : S \longrightarrow \mathbf{Z}_{\geq -1}$ is a W_{λ_0}-invariant function of finite support. The family $\{\omega_\lambda\}_{\lambda \in W_S\lambda_0}$ is then defined by $\omega_{w\lambda_0}(a) = \tilde{\omega}_{\lambda_0}(w^{-1}a)$ for all $w \in W_S$ and $a \in S^+$.

□

9.3 Frobenius form on $\mathbf{B}^\Omega$

As observed in [4], the basis theorem Theorem 74 implies that the algebra $\mathbf{B}^\Omega$ is Frobenius over its centre $\mathcal{Z}$.

Lemma 76. *$\mathbf{B}^\Omega$ is a Frobenius algebra over $\mathcal{Z}$.*

Proof. Consider the filtration by length

$$F_{\leq n}\mathbf{B}^\Omega = \sum_{\ell \in W_R\ell_0} \sum_{k=0}^{n} \sum_{(\alpha_1,\ldots,\alpha_k)\in\Delta_0^k} \mathbf{C}[V]\tau^\Omega_{\alpha_1}\cdots\tau^\Omega_{\alpha_k}\mathbf{e}(\ell).$$

We set $N = \#R^+ = \ell(w_0)$ and let $w_0 = s_{\alpha_N}\cdots s_{\alpha_1}$ be any reduced expression for the longest element $w_0 \in W_R$ and set $\tau^\Omega_{w_0}\mathbf{e}(\ell) = \tau^\Omega_{\alpha_N}\cdots\tau^\Omega_{\alpha_1}\mathbf{e}(\ell)$. By Theorem 74, we have $F_{\leq N}\mathbf{B}^\Omega = \mathbf{B}^\Omega$ and

$$\mathrm{gr}^F_N\mathbf{B}^\Omega \cong \bigoplus_{\ell\in W_R\ell_0} \mathbf{C}[V]\tau^\Omega_{w_0}\mathbf{e}(\ell).$$

Let $R_{\lambda_0} = \{\alpha \in R\,;\, \alpha(\lambda_0) = 0\}$ be the sub-root system associated with λ_0 and let $\Delta_{\lambda_0} \subset R_{\lambda_0}$ be any basis, which determines a set of positive roots $R^+_{\lambda_0} \subset R_{\lambda_0}$ and a set of Coxeter generators $\{s_a\}_{a\in\Delta_{\lambda_0}} \subset W_{\lambda_0}$. It is well known that $\mathbf{C}[V]$ is a symmetric algebra over $\mathcal{Z}$ with the trace map $f \mapsto \vartheta_{w_0(W_{\lambda_0})}(f)$, where $\vartheta_{w_0(W_{\lambda_0})}$ is a composition of Demazure operators for the longest element $w_0(W_{\lambda_0})$ of the Coxeter group $(W_{\lambda_0}, \Delta_{\lambda_0})$. Let tr be the composition

$$\mathbf{B}^\Omega \longrightarrow \mathrm{gr}^F_N\mathbf{B}^\Omega = \bigoplus_\ell \mathbf{C}[V]\tau_{w_0}\mathbf{e}(\ell) \xrightarrow{\tau_{w_0}\mathbf{e}(\ell)\mapsto 1} \bigoplus_\ell \mathbf{C}[V]$$
$$\xrightarrow{\vartheta_{w_0(W_\ell)}} \bigoplus_\ell \mathbf{C}[V]^{W_\ell} \cong \bigoplus_\ell \mathcal{Z} \xrightarrow{\sum_{\ell\in W_R\ell_0}} \mathcal{Z}.$$

Then tr is a Frobenius form. □

10 Knizhnik–Zamolodchikov functor V

We resume to the assumptions of §6.

In this section, we introduce a functor $\mathbf{V} : \mathbf{A}^\omega\text{-gmod} \longrightarrow \mathbf{B}^\Omega\text{-gmod}$, which is a quotient functor satisfying the double centraliser property. It can be viewed as a generalisation of the monodromy functor of [34] for dDAHAs (which has been reviewed in §4) to the family of algebras $\mathbf{A}^\omega$. It is thus expected to satisfy some properties of the monodromy functor. The main results of this article Theorem 105 and Theorem 108 provide some evidence. We construct $\mathbf{V}$ by choosing an idempotent element $\mathbf{e}_\gamma \in \mathbf{A}^\omega$ and establish an isomorphism $\mathbf{B}^\Omega \cong \mathbf{e}_\gamma\mathbf{A}^\omega\mathbf{e}_\gamma$ in Theorem 82.

10.1 The idempotent construction

Consider the following exponential map

$$E \cong V = Q^\vee \otimes \mathbf{R} \xrightarrow{\exp} Q^\vee \otimes \mathbf{C}^\times = T$$
$$\mu \otimes r \mapsto \mu \otimes e^{2\pi i r}$$

and put $\ell_0 = \exp(\lambda_0) \in T$. Choose an element $\gamma \in Q^\vee$ such that

$$\langle \gamma, \alpha \rangle \ll 0 \quad \text{for all } \alpha \in R^+. \tag{77}$$

We define a section of the projection $\partial^W : W_S \longrightarrow W_S/Q^\vee = W_R$ by

$$\begin{aligned} {}^\gamma\bullet : W_R &\longrightarrow W_S \\ w &\mapsto X^\gamma\, w\, X^{-\gamma} = wX^{w^{-1}\gamma-\gamma} \end{aligned}$$

and a section of the exponential map $W_S\lambda_0 \xrightarrow{\exp} W_R\ell_0$ by

$$\begin{aligned} {}^\gamma\bullet : W_R\ell_0 &\longrightarrow W_S\lambda_0 \\ w\ell_0 &\mapsto X^\gamma\, w\lambda_0. \end{aligned}$$

It is clear that ${}^\gamma w\, {}^\gamma\ell = {}^\gamma(w\ell)$. The choice of γ implies that

$$\alpha({}^\gamma\ell) \ll 0 \quad \text{for all } \alpha \in R^+ \text{ and } \ell \in W_R\ell_0. \tag{78}$$

Given a family of order functions $\{\omega_\lambda : S^+ \longrightarrow \mathbf{Z}_{\geq -1}\}_{\lambda \in W_S\lambda_0}$ satisfying the axioms of §6.3, we can associate a family of order functions $\int\omega = \left\{\int\omega_\ell : R^+_{\mathrm{red}} \longrightarrow \mathbf{Z}_{\geq -1}\right\}_{\ell \in W_R\ell_0}$, called the *integral* of ω along ∂^W, by setting for each $\ell \in W_R\ell_0$

$$\textstyle\int\omega_\ell(\alpha) = \sum_{\substack{a \in S^+ \\ \partial a \in \{\alpha, 2\alpha\}}} \omega_{{}^\gamma\ell}(a). \tag{79}$$

The definition of $\int\omega$ is independent of the choice of γ. Denote $\Omega = \int\omega$. This family of order functions gives rise to an algebra $\mathbf{B}^\Omega$ as defined in §9.1.

For any ℓ and $\alpha \in \Delta_0$, we define an operator $\sigma_\alpha \mathbf{e}({}^\gamma\ell) : \mathrm{Pol}_{{}^\gamma\ell} \longrightarrow \mathrm{Pol}_{{}^\gamma(s_\alpha\ell)}$ by

$$\sigma_\alpha \mathbf{e}({}^\gamma\ell) = \begin{cases} \alpha^{-1}(s_\alpha - 1) & \int\omega_\ell(\alpha) = -1 \\ \alpha^{\Omega_\ell(\alpha)} s_\alpha & \int\omega_\ell(\alpha) \geq 0 \end{cases}. \tag{80}$$

Define the idempotent

$$\mathbf{e}_\gamma = \sum_{\lambda \in {}^\gamma(W_R\ell_0)} \mathbf{e}(\lambda) \in \mathbf{A}^\omega. \tag{81}$$

The main result is the following, which will be proven in §10.5:

Theorem 82. *Upon choosing $\gamma \in Q^\vee$ satisfying (77), there is an isomorphism of graded $\mathcal{Z}$-algebras*

$$\begin{aligned} i_\gamma : \mathbf{B}^\Omega &\cong \mathbf{e}_\gamma \mathbf{A}^\omega \mathbf{e}_\gamma \\ f\mathbf{e}(\ell) &\mapsto f\mathbf{e}({}^\gamma\ell) \\ \tau_\alpha^\Omega \mathbf{e}(\ell) &\mapsto \sigma_\alpha \mathbf{e}({}^\gamma\ell). \end{aligned}$$

Moreover, for any other choice γ', the intertwiner

$$\varphi_{\gamma,\gamma'} := \sum_{w \in W_R/W_{\ell_0}} \mathbf{e}({}^\gamma(w\ell_0))\tau^\omega_{X^{w(\gamma-\gamma')}}\mathbf{e}({}^{\gamma'}(w\ell_0)) \in \mathbf{e}_\gamma \mathbf{A}^\omega \mathbf{e}_{\gamma'}$$

yields a factorisation $i_\gamma(f) = \varphi_{\gamma,\gamma'} \cdot i_{\gamma'}(f) \cdot \varphi_{\gamma',\gamma}$ for each $f \in \mathbf{B}^\Omega$.

Example 83. *Resume to the setting of Example 33 (iii) and Example 55. The coroot lattice is given by $Q^\vee = \mathbf{Z}$, which acts by translation on $E = \mathbf{R}$. Recall that $\lambda_0 = 1/4 \in E$. We may take $\gamma = s_1 s_0 = -1$ so that ${}^\gamma(W_R\ell_0) = \{\lambda_+, \lambda_-\}$, where $\lambda_+ = s_1 s_0 \lambda_0 = -3/4$ and $\lambda_- = s_1 s_0 s_1 \lambda_0 = -5/4$. It follows that $\lambda_- = s_1 s_0 s_1 s_0 s_1 \lambda_+$ and*

$$\mathbf{e}(\lambda_-)\mathbf{A}^\omega\mathbf{e}(\lambda_+) = \mathbf{C}[\epsilon]\tau^\omega_{a_1}\tau^\omega_{a_0}\tau^\omega_{a_1}\tau^\omega_{a_0}\tau^\omega_{a_1}\mathbf{e}(\lambda_+), \quad \mathbf{e}(\lambda_+)\mathbf{A}^\omega\mathbf{e}(\lambda_-) = \mathbf{C}[\epsilon]\tau^\omega_{a_1}\tau^\omega_{a_0}\tau^\omega_{a_1}\tau^\omega_{a_0}\tau^\omega_{a_1}\mathbf{e}(\lambda_-).$$

Denote by $s : \mathbf{C}[\epsilon] \longrightarrow \mathbf{C}[\epsilon]$ the automorphism $\epsilon \mapsto -\epsilon$. Calculate the products:

$$\begin{aligned}\tau^{\omega}_{a_1}\tau^{\omega}_{a_0}\tau^{\omega}_{a_1}\tau^{\omega}_{a_0}\tau^{\omega}_{a_1}\mathbf{e}(\lambda_+) &= \tau^{\omega}_{a_1}\mathbf{e}(5/4)\tau^{\omega}_{a_0}\mathbf{e}(-1/4)\tau^{\omega}_{a_1}\mathbf{e}(1/4)\tau^{\omega}_{a_0}\mathbf{e}(3/4)\tau^{\omega}_{a_1}\mathbf{e}(-3/4),\\ &= s\cdot s\cdot(\epsilon s)\cdot s\cdot s = \epsilon s\\ \tau^{\omega}_{a_1}\tau^{\omega}_{a_0}\tau^{\omega}_{a_1}\tau^{\omega}_{a_0}\tau^{\omega}_{a_1}\mathbf{e}(\lambda_-) &= \tau^{\omega}_{a_1}\mathbf{e}(3/4)\tau^{\omega}_{a_0}\mathbf{e}(1/4)\tau^{\omega}_{a_1}\mathbf{e}(-1/4)\tau^{\omega}_{a_0}\mathbf{e}(5/4)\tau^{\omega}_{a_1}\mathbf{e}(-5/4)\\ &= s\cdot(-\epsilon s)\cdot s\cdot s\cdot s = \epsilon s.\end{aligned}$$

Let $\alpha = \partial a_1 \in \Delta_0$ be the simple root for $(V, R) = A_1$. Denote $\ell_+ = \exp(2\pi i\lambda_+) = i$ and $\ell_- = \exp(2\pi i\lambda_-) = -i$. The family of order functions $\Omega = \int\omega$ for $\mathbf{B}^{\Omega}$ is given by

$$\Omega_{\ell_+}(\alpha) = \sum_{k\in\mathbf{N}}\omega_{\lambda_+}(\alpha+k) = 1, \quad \Omega_{\ell_-}(\alpha) = \sum_{k\in\mathbf{N}}\omega_{\lambda_-}(\alpha+k) = 1.$$

It follows that

$$\sigma_\alpha\mathbf{e}(\lambda_+) = \alpha^{\Omega_{\ell_+}(\alpha)}s = \tau^{\omega}_{a_1}\tau^{\omega}_{a_0}\tau^{\omega}_{a_1}\tau^{\omega}_{a_0}\tau^{\omega}_{a_1}\mathbf{e}(\lambda_+) \quad \sigma_\alpha\mathbf{e}(\lambda_-) = \alpha^{\Omega_{\ell_-}(\alpha)}s = \tau^{\omega}_{a_1}\tau^{\omega}_{a_0}\tau^{\omega}_{a_1}\tau^{\omega}_{a_0}\tau^{\omega}_{a_1}\mathbf{e}(\lambda_-)$$

and therefore there is an isomorphism

$$\begin{aligned}\mathbf{B}^{\Omega} &\xrightarrow{\sim} \mathbf{e}_\gamma\mathbf{A}^{\omega}\mathbf{e}_\gamma\\ \mathbf{e}(\ell_+) &\mapsto \mathbf{e}(\lambda_+)\\ \mathbf{e}(\ell_-) &\mapsto \mathbf{e}(\lambda_-)\\ \tau^{\Omega}_{\alpha} &\mapsto \tau^{\omega}_{a_1}\tau^{\omega}_{a_0}\tau^{\omega}_{a_1}\tau^{\omega}_{a_0}\tau^{\omega}_{a_1}\mathbf{e}_\gamma.\end{aligned}$$

Remark 84. *As we will see in Lemma 90, the idempotent $\mathbf{e}_\gamma$ corresponds to generic clans (§7.3). The choice of $\mathbf{e}_\gamma$ is inspired from the sheaf-theoretic study of extension algebras over a cyclically graded simple Lie algebra $\mathfrak{g}_*$ in [24] and the sheaf-theoretic construction of the KZ functor. In the language of op. cit. and [26], each eigenvalue $\lambda \in W_S\lambda_0$ corresponds to the spiral induction of a cuspidal local system $\mathscr{C}$ through one spiral of $\mathfrak{g}_*$. On the other hand, affine Hecke algebras arise as extension algebra of parabolic inductions of $\mathscr{C}$ through parabolic subalgebras of $\mathfrak{g}_*$, which appear also as spiral induction of $\mathscr{C}$ through "generic spirals". Therefore, the definition of the sheaf-theoretic KZ functor is nothing but picking idempotents of the extension algebra corresponding to those "generic spirals". In the algebraic and combinatorial language, they corresponds to alcoves lying in the generic clans, as introduced in §7.3.*

10.2 A formula for order functions

By the hypothesis of finite support for $\tilde{\omega}_{\lambda_0} : S \longrightarrow \mathbf{Z}_{\geq -1}$, there exists $M \gg 0$ such that $\tilde{\omega}_{\lambda_0}(\alpha + k) = 0$ for all $\alpha \in R$ and $|k| \geq M$. Let $\gamma \in Q^\vee$ be an element satisfying (77). More specifically, we require that

$$\langle\alpha, \gamma\rangle \leq -M, \quad \forall\alpha \in R^+. \tag{85}$$

We prove a relation between the family $\omega = \{\omega_\lambda\}_{\lambda\in W_S\lambda_0}$ for $\mathbf{A}^{\omega}$ and its integral $\Omega = \{\Omega_\ell\}_{\ell\in W_R\ell_0}$ for $\mathbf{B}^{\Omega}$ defined in (79).

Lemma 86. *For any $\ell \in W_R\ell_0$ and $w \in W_R$, following formula holds in $\mathbf{C}(V)$:*

$$\prod_{b\in S^+\cap{}^{\gamma}w^{-1}S^-}(-\partial b)^{\omega_\lambda(b)} = \epsilon\cdot\prod_{\beta\in R^+_{\mathrm{red}}\cap w^{-1}R^-_{\mathrm{red}}}(-\beta)^{\Omega_\ell(\beta)},$$

where $\lambda = {}^{\gamma}\ell$ and $\epsilon \in \mathbf{C}^\times$ is constant (which is a power of 2).

Proof. We divide the index set of the product on the left-hand side into two

$$S^+\cap{}^{\gamma}w^{-1}S^- = \{b \in S^+\cap{}^{\gamma}w^{-1}S^- \ ;\ \partial b \notin R^+\cap w^{-1}R^-\}\sqcup\{b \in S^+\cap{}^{\gamma}w^{-1}S^- \ ;\ \partial b \in R^+\cap w^{-1}R^-\}$$

and treat the two sub-products separately.

Step 1. We prove that $\omega_\lambda(b) = 0$ when $b \in S^+ \cap {}^\gamma w^{-1} S^-$ and $\beta := \partial b \notin R^+ \cap w^{-1}R^-$. Write $b = \beta + k$ for $k \in \mathbf{Z}$. We deduce

$$\omega_\lambda(b) = \omega_\lambda(\beta + k) = \tilde{\omega}_{\lambda_0}(y\beta + k + \langle \beta, \gamma \rangle), \quad \text{where } y \in W_R \text{ is such that } y\ell = \ell_0.$$

Since ${}^\gamma w = wX^{w^{-1}\gamma - \gamma}$, the condition $b \in S^+ \cap {}^\gamma w^{-1} S^-$ implies that $0 \leq k < \langle w\beta - \beta, \gamma \rangle$. There are two cases: $\beta \in R^-$ or $\beta \in w^{-1}R^+$. In the case where $\beta \in w^{-1}R^+$, since $k + \langle \beta, \gamma \rangle < \langle w\beta, \gamma \rangle \leq -M$, the hypothesis (85) implies that $\omega_\lambda(b) = 0$. In the case where $\beta \in R^-$, we have $k + \langle \beta, \gamma \rangle \geq \langle \beta, \gamma \rangle \geq M$, whence $\omega_\lambda(b) = 0$ as well.

Step 2. We prove that

$$\prod_{\substack{b \in S^+ \cap {}^\gamma w^{-1} S^- \\ \partial b \in R^+ \cap w^{-1} R^-}} (-\partial b)^{\omega_\lambda(b)} = \epsilon \cdot \prod_{\beta \in R^+_{\mathrm{red}} \cap w^{-1} R^-_{\mathrm{red}}} (-\beta)^{\Omega_\ell(\beta)}. \tag{87}$$

for some $\epsilon \in \mathbf{C}^\times$ which is a power of 2. We rewrite the left-hand side according to ∂b:

$$\prod_{\substack{b \in S^+ \cap {}^\gamma w^{-1} S^- \\ \partial b \in R^+ \cap w^{-1} R^-}} (-\partial b)^{\omega_\lambda(b)} = \epsilon \cdot \prod_{\beta \in R^+_{\mathrm{red}} \cap w^{-1} R^-_{\mathrm{red}}} \prod_{\substack{b \in S^+ \cap {}^\gamma w^{-1} S^- \\ \partial b \in \{\beta, 2\beta\}}} (-\beta)^{\omega_\lambda(b)} \tag{88}$$

Let $\beta \in R^+_{\mathrm{red}} \cap w^{-1}R^-_{\mathrm{red}}$. Let $N := \langle w\beta - \beta, \gamma \rangle$. It follows by the same arguments as Step 1 that $b = \beta + k \in S^+ \cap {}^\gamma w^{-1} S^-$ for $0 \leq k \leq N$. For $k \geq N$, we obtain $k + \langle \beta, \gamma \rangle \geq \langle w\beta, \gamma \rangle \geq M$, thus $\omega_\lambda(\beta + k) = \tilde{\omega}_{\lambda_0}(y\beta + k + \langle \beta, \gamma \rangle) = 0$ and hence

$$\sum_{\substack{b \in S^+ \cap {}^\gamma w^{-1} S^- \\ \partial b = \beta}} \omega_\lambda(b) = \sum_{k=0}^{N} \omega_\lambda(\beta + k) = \sum_{k \in \mathbf{N}} \omega_\lambda(\beta + k) = \sum_{\substack{b \in S^+ \\ \partial b = \beta}} \omega_\lambda(b).$$

In the case where $2\beta \in R$, we obtain similarly

$$\sum_{\substack{b \in S^+ \cap {}^\gamma w^{-1} S^- \\ \partial b = 2\beta}} \omega_\lambda(b) = \sum_{k=0}^{N-1} \omega_\lambda(2\beta + (2k+1)) = \sum_{k \in \mathbf{N}} \omega_\lambda(2\beta + (2k+1)) = \sum_{\substack{b \in S^+ \\ \partial b = 2\beta}} \omega_\lambda(b).$$

Hence

$$\sum_{\substack{b \in S^+ \cap {}^\gamma w^{-1} S^- \\ \partial b \in \{\beta, 2\beta\}}} \omega_\lambda(b) = \Omega_\ell(\beta). \tag{89}$$

The equation (87) follows from (88) and (89).

Combining the two steps, we obtain the product formula. □

10.3 Preparatory lemmas

Let $\gamma \in Q^\vee$ be an element satisfying (77). Recall the notion of clans and generic clans from §7.3 and the fundamental alcove $\nu_0 \subset E$.

Lemma 90. *For $w \in W_R$, the alcove $w^{-1}X^{-\gamma}\nu_0$ is in a generic clan and every generic clan contains at least one such alcove. Moreover, for a different choice $\gamma' \in Q^\vee$, the alcoves $w^{-1}X^{-\gamma}\nu_0$ and $w^{-1}X^{-\gamma'}\nu_0$ are in the same clan.*

Proof. Since the clans are connected components of the complement $E^\omega_\circ$ of the hyperplanes in $\mathfrak{D}^\omega = \{H_a \subset E \,;\, a \in S,\, \tilde{\omega}_{\lambda_0}(a) \geq 1\}$, any two points $x, y \in E^\omega_\circ$ are in the same clan if $a(x)a(y) > 0$ for all $a \in S$ with $H_a \in \mathfrak{D}^\omega$. Let $\mathfrak{C}_w \subset E^\omega_\circ$ be the clan such that $w^{-1}X^{-\gamma}\nu_0 \subset \mathfrak{C}_w$. Take any point $x \in \nu_0$. Set $x_w(t) = w^{-1}(x - (1+t)\gamma)$ for $t \in \mathbf{R}_{\geq 0}$ so that in particular

$x_w(0) \in \mathfrak{C}_w$. Let $a \in S$ such that $H_a \in \mathfrak{D}^\omega$. Suppose that $w\partial a \in R^+$ (resp. $w\partial a \in R^-$); then $\langle w\partial a, \gamma\rangle \ll 0$ (resp. $\langle w\partial a, \gamma\rangle \gg 0$), so we have

$$a(x_w(t)) = (wa)(x) - \langle w\partial a, (1+t)\gamma\rangle \gg 0, \quad \forall t \in \mathbf{R}_{\geq 0}$$

(resp. $a(x_w(t)) \ll 0$). Hence $x_w(t) \in \mathfrak{C}_w$ for all $t \geq 0$. Moreover, we see that the value $a(x_w(t))$ is unbounded when $t \longrightarrow +\infty$. Hence every affine root is unbounded on the $\mathfrak{C}_w$, from which the genericity of $\mathfrak{C}_w$.

Conversely, let $\mathfrak{C}$ be a generic clan, consider the salient cone κ defined in §7.3. The genericity of $\mathfrak{C}$ means that κ is of full dimension $\dim V$. Let $\mathrm{C}_0 \subset V$ be the fundamental Weyl chamber and let $w^{-1}\mathrm{C}_0 \subset V$ be a Weyl chamber with $w \in W_R$ such that $\mathrm{Int}(\kappa) \cap w^{-1}\mathrm{C}_0 \neq \emptyset$. It is obvious that $w^{-1}X^{-\gamma}\nu_0 \subset \mathfrak{C}$. ☐

Recall the element σ_α from (80).

Lemma 91. *We have $\sigma_\alpha \mathbf{e}({}^\gamma\ell) \in \mathbf{A}^\omega$.*

Proof. Denote $\lambda = {}^\gamma\ell$. Let ${}^\gamma s_\alpha = s_{a_l}\cdots s_{a_1}$ be any reduced decomposition and denote $\sigma'_\alpha\mathbf{e}(\lambda) = \tau^\omega_{a_l}\cdots\tau^\omega_{a_1}\mathbf{e}(\lambda)$. Applying Lemma 37(ii), we see that

$$\sigma'_\alpha\mathbf{e}(\lambda) \equiv s_{a_l}\cdots s_{a_1}\left(\prod_{c\in S^+\cap{}^\gamma s_\alpha S^-}(-\partial c)^{\omega_\lambda(c)}\right)\mathbf{e}(\lambda) \mod F_{\leq l-1}\mathbf{A}^{-\infty} \tag{92}$$

and Lemma 86 yields

$$\prod_{c\in S^+\cap{}^\gamma s_\alpha S^-}(-\partial c)^{\omega_\lambda(c)} = \epsilon\cdot(-\alpha)^{\omega_\ell(\alpha)}, \quad \epsilon \in \mathbf{C}^\times.$$

Thus the right-hand side of (92) is congruent to $\epsilon\sigma_\alpha\mathbf{e}(\lambda)$ modulo $F_{\leq l-1}\mathbf{A}^{-\infty}$. Notice that $\sigma_\alpha\mathbf{e}(\lambda) \in \mathbf{A}^o$ by Proposition 30 (iii) and $\sigma'_\alpha\mathbf{e}(\lambda) \in \mathbf{A}^\omega \subset \mathbf{A}^o$. Hence by the compatibility of the filtrations by length Corollary 41, we have

$$(\sigma'_\alpha - \epsilon\sigma_\alpha)\mathbf{e}(\lambda) \in \mathbf{e}({}^\gamma s_\alpha\lambda)\left(F_{\leq l-1}\mathbf{A}^{-\infty}\cap\mathbf{A}^o\right)\mathbf{e}(\lambda) = \mathbf{e}({}^\gamma s_\alpha\lambda)\left(F_{\leq l-1}\mathbf{A}^o\right)\mathbf{e}(\lambda).$$

We show that in fact $(\sigma'_\alpha - \epsilon\sigma_\alpha)\mathbf{e}(\lambda) \in \mathbf{A}^\omega$. For any different choice γ' satisfying (77), Lemma 90 implies that the intertwiner $\varphi_{\gamma,\gamma'}$ defined in Theorem 82 satisfies $\varphi_{\gamma,\gamma'}\varphi_{\gamma',\gamma} = \mathbf{e}_\gamma$, $\varphi_{\gamma',\gamma}\varphi_{\gamma,\gamma'} = \mathbf{e}_{\gamma'}$ and

$$\varphi_{\gamma,\gamma'}\sigma_\alpha\mathbf{e}({}^{\gamma'}\ell)\varphi_{\gamma',\gamma} = \sigma_\alpha\mathbf{e}({}^\gamma\ell), \quad \varphi_{\gamma,\gamma'}\sigma'_\alpha\mathbf{e}({}^{\gamma'}\ell)\varphi_{\gamma',\gamma} = \sigma'_\alpha\mathbf{e}({}^\gamma\ell);$$

thus the validity of the statement is independent of the choice of γ. We claim that if we choose γ in such a way that $|\langle\alpha,\gamma\rangle| \ll |\langle\beta,\gamma\rangle|$ for all $\beta \in \Delta_0\setminus\{\alpha\}$, then there is an inequality of lengths

$$l = \ell({}^\gamma s_\alpha) \leq \ell({}^\gamma w), \quad \forall w \in W_R\setminus\{1\}. \tag{93}$$

We complete the proof provided (93). Note that the stabilisers satisfy ${}^\gamma W_\ell = W_\lambda$. There are two cases to be discussed:

(i) If $s_\alpha\ell \neq \ell$, then by (93) we have $\ell(w) \geq l$ for all $w \in W_S$ such that $w\lambda = {}^\gamma s_\alpha\lambda$. It follows from Theorem 40 that $\mathbf{e}({}^\gamma s_\alpha\lambda)\left(F_{\leq l-1}\mathbf{A}^o\right)\mathbf{e}(\lambda) = 0$. Hence $\sigma_\alpha\mathbf{e}(\lambda) = \epsilon\sigma'_\alpha\mathbf{e}(\lambda) \in \mathbf{A}^\omega$.

(ii) If $s_\alpha\ell = \ell$, then by (93) we have $\ell(w) \geq l$ for all $1 \neq w \in W_\lambda$ and thus by Theorem 40, we see that $\mathbf{e}(\lambda)\left(F_{\leq l-1}\mathbf{A}^o\right)\mathbf{e}(\lambda) = \mathbf{C}[V]\mathbf{e}(\lambda) = \mathbf{e}(\lambda)\left(F_{\leq l-1}\mathbf{A}^\omega\right)\mathbf{e}(\lambda)$. Thus $(\sigma_\alpha - \epsilon^{-1}\sigma'_\alpha)\mathbf{e}(\lambda) \in \mathbf{A}^\omega$ and consequently $\sigma_\alpha\mathbf{e}(\lambda) \in \mathbf{A}^\omega$. Hence the proof is completed.

We prove (93). Indeed by Proposition 1,

$$l = \ell({}^\gamma s_\alpha) \leq 1 + \ell(X^{-\langle\alpha,\gamma\rangle\alpha^\vee}) \leq 1 + |\langle 2\rho,\alpha^\vee\rangle\langle\alpha,\gamma\rangle| \ll |\langle\beta,\gamma\rangle|,$$

while for any $w \in W_R\setminus\{1, s_\alpha\}$, there exists $\beta \in R^+_{\mathrm{red}}\cap w^{-1}R^-_{\mathrm{red}}$ with $\beta \neq \alpha$, so

$$\ell({}^\gamma w) \geq \left|\langle\beta, w^{-1}\gamma - \gamma\rangle\right| - \ell(w) = |\langle w\beta - \beta, \gamma\rangle| - \ell(w) \geq |\langle\beta,\gamma\rangle| - \ell(w) \gg l;$$

here, the second-to-last inequality is due to (77). ☐

10.4 Basis theorem for generic clans

Let $\gamma \in Q^\vee$ be an element satisfying (77). Recall the idempotent of generic clans $\mathbf{e}_\gamma$ from (81) and the elements $\sigma_\alpha \mathbf{e}({}^\gamma \ell)$ from (80).

Theorem 94. *The idempotent subalgebra $\mathbf{e}_\gamma \mathbf{A}^\omega \mathbf{e}_\gamma$ is generated by $\mathbf{C}[V]\mathbf{e}(\lambda)$ and $\sigma_\alpha \mathbf{e}(\lambda)$ for $\alpha \in \Delta_0$ and $\lambda \in {}^\gamma(W_R \ell_0)$. Moreover, if for any $w \in W_R$ we set $\sigma_w = \sigma_{\alpha_n} \cdots \sigma_{\alpha_1}$ by choosing any reduced expression $w = s_{\alpha_n} \cdots s_{\alpha_1}$, then there is a decomposition*

$$\mathbf{e}_\gamma \mathbf{A}^\omega \mathbf{e}(\lambda) = \bigoplus_{\lambda \in {}^\gamma(W_R \ell_0)} \bigoplus_{w \in W_R} \mathbf{C}[V] \sigma_w \mathbf{e}(\lambda).$$

Proof. Let $\ell \in W_R \ell_0$ and $w \in W_R$. Denote $\lambda = {}^\gamma \ell$. Choose any reduced expressions $w = s_{\alpha_n} \cdots s_{\alpha_1}$ and ${}^\gamma w = s_{a_l} \cdots s_{a_1}$ for $\alpha_1, \cdots, \alpha_n \in \Delta_0$ and $a_1, \cdots, a_l \in \Delta$ and set

$$\sigma'_w \mathbf{e}(\lambda) = \tau^\omega_{a_l} \cdots \tau^\omega_{a_1} \mathbf{e}(\lambda) \in \mathbf{A}^\omega, \quad \sigma_w \mathbf{e}(\lambda) = \sigma_{\alpha_n} \cdots \sigma_{\alpha_1} \mathbf{e}(\lambda).$$

By Lemma 91, we see that $\sigma_w \mathbf{e}(\lambda) \in \mathbf{A}^\omega$. We claim that

$$\sigma'_w \mathbf{e}(\lambda) \equiv \epsilon \sigma_w \mathbf{e}(\lambda) \mod F_{\leq l-1} \mathbf{A}^\omega. \tag{95}$$

for some $\epsilon \in \mathbf{C}^\times$. Recall the rational function matrix algebra $\mathbf{A}^{-\infty} = \operatorname{Frac} \mathcal{Z} \otimes_{\mathcal{Z}} \mathbf{A}^o$. By Lemma 37 (ii) and Lemma 86, we have

$$\begin{aligned} \sigma'_w \mathbf{e}(\lambda) &\equiv s_{\partial a_l} \cdots s_{\partial a_1} \left(\prod_{b \in S^+ \cap {}^\gamma w^{-1} S^-} (-\partial b)^{\omega_\lambda(b)} \right) \mathbf{e}(\lambda) \mod F_{\leq l-1} \mathbf{A}^{-\infty} \\ &\equiv \epsilon \sigma_w \mathbf{e}(\lambda) \mod F_{\leq l-1} \mathbf{A}^{-\infty}. \end{aligned}$$

for some $\epsilon \in \mathbf{C}^\times$. As $n \leq l$, the above congruences yield $(\sigma'_w - \epsilon \sigma_w)\mathbf{e}(\lambda) \in \mathbf{A}^\omega \cap F_{\leq l-1} \mathbf{A}^{-\infty}$. By Corollary 41, we have $\mathbf{A}^\omega \cap F_{\leq l-1} \mathbf{A}^{-\infty} = F_{\leq l-1} \mathbf{A}^\omega$, so the claim (95) is proven.

According to Theorem 40, the family $\{\sigma'_w \mathbf{e}(\lambda)\}_{w \in W_R}$ form a basis for $\mathbf{e}_\gamma \mathbf{A}^\omega \mathbf{e}(\lambda)$. The decomposition of $\mathbf{e}_\gamma \mathbf{A}^\omega \mathbf{e}(\lambda)$ follows from the triangularity (95) of the transition matrix between the basis $\{\sigma'_w \mathbf{e}(\lambda)\}_{w \in W_R}$ and the family $\{\sigma_w \mathbf{e}(\lambda)\}_{w \in W_R}$. □

10.5 Proof of Theorem 82

Proof. We define an isomorphism of $\mathcal{Z}$-modules $\operatorname{Pol}_{W_R \ell_0} \cong \mathbf{e}_\gamma \operatorname{Pol}_{W_S \lambda_0}$ straightforwardly by the identification:

$$\operatorname{Pol}_\ell = \mathbf{C}[V] = \operatorname{Pol}_{{}^\gamma \ell}, \quad \ell \in W_R \ell_0.$$

It yields a faithful representation of $\mathbf{B}^\Omega$ on $\mathbf{e}_\gamma \operatorname{Pol}_{W_S \lambda_0}$, which by definition of $\mathbf{B}^\Omega$ is described by the formula

$$f\mathbf{e}(\ell) \cdot g = f\mathbf{e}({}^\gamma \ell) g, \quad \tau^\Omega_\alpha \mathbf{e}(\ell) \cdot g = \sigma_\alpha \mathbf{e}({}^\gamma \ell) g.$$

By Theorem 94, the image of $\mathbf{B}^\Omega$ in $\operatorname{End}_{\mathcal{Z}} (\mathbf{e}_\gamma \operatorname{Pol}_{W_S \lambda_0})$ coincides with $\mathbf{e}_\gamma \mathbf{A}^\omega \mathbf{e}_\gamma$ and the map $\mathbf{B}^\Omega \longrightarrow \mathbf{e}_\gamma \mathbf{A}^\omega \mathbf{e}_\gamma$ must be an isomorphism since both sides are free $\mathbf{C}[V]$-modules of same rank. Notice that $\deg \tau^\Omega_\alpha \mathbf{e}(\ell) = \Omega_\ell(\alpha) = \deg \sigma_\alpha \mathbf{e}({}^\gamma \ell)$. Hence the map i_γ is an isomorphism of graded $\mathcal{Z}$-algebras.

For any other choice γ', since by Lemma 90, $w^{-1} X^\gamma$ and $w^{-1} X^{\gamma'}$ lie in the same generic clan for each $w \in W_R$, by Proposition 53, the intertwiner $\varphi_{\gamma', \gamma}$ yields isomorphisms of $\mathbf{A}^\omega$-modules $\mathbf{A}^\omega \mathbf{e}_{\gamma'} \cong \mathbf{A}^\omega \mathbf{e}_\gamma$ by right multiplication and hence isomorphisms of algebras

$$\mathbf{e}_{\gamma'} \mathbf{A}^\omega \mathbf{e}_{\gamma'} \cong \operatorname{End}_{\mathbf{A}^\omega} (\mathbf{A}^\omega \mathbf{e}_{\gamma'}) \cong \operatorname{End}_{\mathbf{A}^\omega} (\mathbf{A}^\omega \mathbf{e}_\gamma) \cong \mathbf{e}_\gamma \mathbf{A}^\omega \mathbf{e}_\gamma.$$

The factorisation $i_\gamma = \varphi_{X^{\gamma' - \gamma}} \circ i_{\gamma'}$ follows from the observation that $\partial(X^{\gamma' - \gamma}) = 1 \in W_R$. □

10.6 The functor V

Choose a $\gamma \in Q^\vee$ satisfying (77) as in §10.2. With Theorem 82, we can make the following definition:

Definition 96. *The Knizhnik–Zamolodchikov (KZ) functor* $\mathbf{V}$ *is defined by*

$$\mathbf{V} : \mathbf{A}^\omega\text{-gmod} \longrightarrow \mathbf{e}_\gamma \mathbf{A}^\omega \mathbf{e}_\gamma\text{-gmod} \xrightarrow[\cong]{i_\gamma^*} \mathbf{B}^\Omega\text{-gmod}$$
$$M \mapsto \mathbf{e}_\gamma M.$$

By the second assertion of Theorem 82, the definition of $\mathbf{V}$ is independent of the choice of γ up to canonical isomorphism (provided by the intertwiner $\varphi_{\gamma,\gamma'}$).

Since $\mathbf{V}$ is defined as an idempotent truncation, it admits left and right adjoint functors

$$\mathbf{V}^\top : N \mapsto \bigoplus_{\lambda \in W_S \lambda_0} \mathrm{gHom}_{\mathbf{B}^\Omega}(\mathbf{e}_\gamma \mathbf{A}^\omega \mathbf{e}(\lambda), N) \quad \text{and} \quad {}^\top\mathbf{V} : N \mapsto \mathbf{A}^\omega \mathbf{e}_\gamma \otimes_{\mathbf{B}^\Omega} N$$

and $\mathbf{V}$ is a quotient functor in the sense that the adjoint counit $\mathbf{V} \circ \mathbf{V}^\top \longrightarrow \mathrm{id}_{\mathbf{B}^\Omega}$ is an isomorphism.

10.7 Support characterisation of V

For $M \in \mathbf{A}^\omega$-gmod, define the following subset of E:

$$\mathrm{Spec}_E\, M = \{\lambda \in W_S \lambda_0 \;;\; \mathbf{e}(\lambda) M \neq 0\}.$$

For each alcove $\nu \subset E$, there is a unique $w \in W_S$ such that $\nu = w^{-1}\nu_0$; we denote $\lambda_\nu = w\lambda_0$. Recall the Gelfand–Kirillov dimension $\dim_{\mathrm{GK}} M$ and the support $\mathrm{Supp}\, M$ from §8.3.

Theorem 97. *Let* $M \in \mathbf{A}^\omega$*-gmod*$_0$*. The following conditions are equivalent:*

(i) $\mathbf{V}M = 0$*;*

(ii) for every alcove ν *lying in a generic clan, we have* $\mathbf{e}(\lambda_\nu)M = 0$*;*

(iii) the set $\mathrm{Spec}_E\, M$ *is contained in a finite union of (not-necessarily root) affine hyperplanes of* E*;*

(iv) $\dim_{\mathrm{GK}} M \leq \mathrm{rk}\, R - 1$*;*

(v) $\mathrm{Supp}\, M \neq \mathrm{Spec}\, \mathbf{C}Q_+^\vee$*.*

Proof. Since every object of the category $\mathbf{A}^\omega$-gmod is of finite length and all the conditions (i)–(v) are stable under extensions, we may suppose that M is simple.

(i) ⇔ (ii) follows from the definition $\mathbf{V}M = \mathbf{e}_\gamma M$ and the invariance of dimension of $\mathbf{e}(\lambda)M$ for λ's in the same clan Corollary 54.

We prove (ii) ⇒ (iii). By the finiteness of the clan decomposition, it suffices to show that for each non-generic clan $\mathfrak{C}$, the set $\{\lambda_\nu \;;\; \nu \subseteq \mathfrak{C}\}$ lies in a finite union of affine hyperplanes of E. By the non-genericity of $\mathfrak{C}$, there exists $\alpha \in R$ which is bounded on $\mathfrak{C}$. Let $\Lambda = \ker \alpha \cap Q^\vee$. Notice that $Q^\vee$ is a free $\mathbf{Z}$-module of rank $\mathrm{rk}\, R - 1$. Let $\mathfrak{A}_{\mathfrak{C}}$ be the set of alcoves contained in $\mathfrak{C}$. For $\nu, \nu' \in \mathfrak{A}_{\mathfrak{C}}$, we write $\nu \sim_\Lambda \nu'$ if there exists $\mu \in \Lambda$ such that $\nu + \mu = \nu'$. For any $\nu \in \mathfrak{A}_{\mathfrak{C}}$, since $X^\mu \lambda_\nu = \lambda_\nu + \mu$, the set $\{\lambda_{\nu'} \;;\; \nu' \sim_\Lambda \nu\}$ is contained in the hyperplane $w(\lambda_0 + \Lambda_{\mathbf{R}})$ for any $w \in W_S$ such that $\nu = w^{-1}\nu_0$. Since α is bounded on $\mathfrak{C}$, the quotient $\mathfrak{A}_{\mathfrak{C}}/\sim_\Lambda$ is a finite set and thus the set

$$\{\lambda_\nu \;;\; \nu \subset \mathfrak{C}\} \subset \bigcup_{\nu \in \mathfrak{A}_{\mathfrak{C}}/\sim_\Lambda} \{\lambda_{\nu'} \;;\; \nu' \sim_\Lambda \nu\}$$

is contained in a finite union of hyperplanes, whence (iii).

We prove (iii) ⇒ (iv). Suppose that $\mathrm{Spec}_E\, M$ is contained in a finite number of hyperplanes. Choose any $\lambda_1 \in \mathrm{Spec}_E\, M$. Let $r = \mathrm{rk}\, R = \dim E$. Via the identification $E \cong V$ induced by $\Delta_0 \subset \Delta$, we view E as an euclidean vector space. Since

$$\mathrm{Spec}_E\, M \subset \bigcup_{w \in W_R} (w\lambda_1 + Q^\vee)$$

is contained in a finite union of the intersection of lattices and hyperplanes, we have

$$\lim_{n\longrightarrow\infty} \frac{\#\{\lambda \in \operatorname{Spec}_E M \; ; \; \|\lambda\| < n\}}{n^{r-1+\varepsilon}} = 0, \quad \forall \varepsilon > 0.$$

For every affine simple root $a \in \Delta$, we have $\tau_a^\omega \mathbf{e}(\lambda)M \subseteq \mathbf{e}(s_a\lambda)M$. Moreover, we have $\|s_a\lambda\| \leq \|\lambda\| + \delta$ for some constant δ which depends only on the affine root system (E, S). It follows that if we define for $t \in \mathbf{R}_{\geq 0}$ the subspace

$$M_{\leq t} = \sum_{\substack{\lambda \in \operatorname{Spec}_E M \\ \|\lambda\| \leq t}} \mathbf{e}(\lambda)M,$$

then $\tau_a^\omega M_{\leq t} \subset M_{\leq t+\delta}$, so $F_{\leq 1}\mathbf{A}^\omega M_{\leq t} \subset M_{\leq t+\delta}$. By induction on $n \in \mathbf{N}$, we see that $(F_{\leq n}\mathbf{A}^\omega)\, M_{\leq t} \subset M_{\leq t+n\delta}$. Since there is only a finite number of clans and since the dimension of $\mathbf{e}(w\lambda_0)M_\lambda$ for $w^{-1}\nu_0$ in a fixed clan is constant by Corollary 54, the set $\{\dim \mathbf{e}(\lambda)M \; ; \; \lambda \in W_S\lambda_0\}$ is bounded. Hence for any finite-dimensional subspaces $L \subset M$, we have

$$\lim_{n\longrightarrow\infty} \frac{\dim (F_{\leq n}\mathbf{A}^\omega \cdot L)}{n^{r-1+\varepsilon}} = 0, \quad \forall \varepsilon > 0. \tag{98}$$

Indeed, let $t_0 \in \mathbf{R}$ be such that $L \subset M_{\leq t_0}$, then $\dim (F_{\leq n}\mathbf{A}^\omega \cdot L) \leq \dim M_{\leq t_0+n\delta} = o\left(n^{r-1+\varepsilon}\right)$. The estimate (98) implies (iv). The equivalence (iv) $\Leftrightarrow$ (v) results from Proposition 67.

We prove $\neg$(ii) $\Rightarrow$ $\neg$(iv). Suppose there exists a generic clan $\mathfrak{C}$ and an alcove $\nu \subset \mathfrak{C}$ such that $\mathbf{e}(\lambda_\nu)\, M \neq 0$. Let $\kappa \subset V$ be the salient cone of $\mathfrak{C}$ (*cf.* §7.3). For any $\mu \in \kappa \cap Q^\vee$, we have $X^\mu\nu \in \mathfrak{C}$ and by Proposition 53, $\mathbf{e}(X^{-\mu}\lambda_\nu)M \cong \mathbf{e}(\lambda_\nu)M \neq 0$. It follows that

$$\dim (F_{\leq n}\mathbf{A}^\omega)\, (\mathbf{e}(\lambda_\nu)M) \geq \dim \sum_{\substack{\mu \in \kappa\cap Q^\vee \\ \ell(X^\mu)\leq n}} \mathbf{e}(X^{-\mu}\lambda_\nu)M = \#\{\mu \in \kappa \cap Q^\vee \; ; \; \ell(X^\mu) \leq n\} \dim \mathbf{e}(\lambda_\nu)M.$$

By the genericity of $\mathfrak{C}$, the salient cone κ contains an open subset of V, so its intersection with a full-ranked lattice $Q^\vee$ satisfies

$$\lim_{n\longrightarrow\infty} \frac{\#\{\mu \in \kappa \cap Q^\vee \; ; \; \ell(X^\mu) \leq n\}}{n^r} = c, \quad c > 0.$$

Hence

$$\dim_{\mathrm{GK}} M \geq \lim_{n\longrightarrow\infty} \frac{\log \dim (F_{\leq n}\mathbf{A}^\omega)\mathbf{e}(\lambda_\nu)M}{\log n} \geq \lim_{n\longrightarrow\infty} \frac{\log cn^r}{\log n} = r,$$

whence (iv) is not satisfied. □

10.8 Double centraliser property

Recall the parabolic subalgebra $\mathbf{A}^\omega_{R,\lambda_1}$ from §7.6.

Lemma 99. *Let* $\lambda_1 \in W_S\lambda_0$, $N \in \mathbf{A}^\omega_{R,\lambda_1}$-gmod *and* $L \in \mathbf{A}^\omega$-gmod. *Suppose that* $\mathbf{V}L = 0$, *then* $\operatorname{gHom}\left(L, \operatorname{ind}^S_{R,\lambda_1} N\right) = 0$.

Proof. It follows from Theorem 97 (i)⇒(v) and Proposition 69. □

Remark 100. *We shall establish in Theorem 105 the double centraliser property for the functor* $\mathbf{V}$. *The strategy is close to the case of rational Cherednik algebras in [16, 5.3]: the first step consists of showing that "induced modules" are torsion-free for the KZ functor. In the case of the dDAHA* $\mathbb{H}$ *discussed in Part I, the parabolic subalgebra* $\mathbf{A}^\omega_{R,\lambda_1}$ *plays the rôle of be graded affine Hecke subalgebra* $\underline{\mathbb{H}} = \mathbf{C}W_R \otimes \mathbf{C}[E] \subset \mathbb{H}$, *whereas* $\mathbf{B}^\Omega$ *plays the rôle of the affine Hecke algebra* $\mathbb{K}$. *In this sense, Lemma 99 is an analogue of the first step in the proof of loc. cit.*

Let $(\mathbf{A}^\omega/\mathfrak{m}_\mathcal{Z})$-gmod be the full subcategory of $\mathbf{A}^\omega$-gmod consisting of objects M such that $\mathfrak{m}_\mathcal{Z}M = 0$. The inclusion $(\mathbf{A}^\omega/\mathfrak{m}_\mathcal{Z})$-gmod $\hookrightarrow$ $\mathbf{A}^\omega$-gmod has a left adjoint functor $- \otimes_\mathcal{Z}\mathbf{C}$, which is right exact. We denote by $- \otimes^{\mathrm{L}}_\mathcal{Z}\mathbf{C}$ its derived functor. The next lemma is the method of lifting faithfulness borrowed from [29, 4.42].

Lemma 101. *Let $M \in \mathbf{A}^\omega$-gmod be an object satisfying the following properties:*

(i) M is free over the centre $\mathcal{Z}$;

(ii) there exists $\lambda_1 \in W_S\lambda_0$ and $N \in \mathbf{A}^\omega_{R,\lambda_1}$-gmod$_0$ such that $M/\mathfrak{m}_\mathcal{Z} M \cong \operatorname{ind}^S_{R,\lambda_1} N$.

Then for any $L \in \mathbf{A}^\omega$-gmod such that $\mathbf{V}L = 0$, we have $\operatorname{gHom}(L, M) = 0$ *and* $\operatorname{gExt}^1(L, M) = 0$.

Proof. We suppose that $M \neq 0$. Let $K = \operatorname{RgHom}(L, M)$ be in the derived category $\mathrm{D}^+(\mathcal{Z}\text{-gMod})$. We suppose that K is a minimal projective resolution. Since

$$K \otimes_\mathcal{Z} \mathbf{C} \cong \operatorname{RgHom}\left(L \otimes^{\mathrm{L}}_\mathcal{Z} \mathbf{C}, M \otimes^{\mathrm{L}}_\mathcal{Z} \mathbf{C}\right) \cong \operatorname{RgHom}\left(L \otimes^{\mathrm{L}}_\mathcal{Z} \mathbf{C}, M/\mathfrak{m}_\mathcal{Z} M\right)$$

by the flatness of M over $\mathcal{Z}$, we have $K \otimes_\mathcal{Z} \mathbf{C} \in \mathrm{D}^{\geq 0}(\mathbf{C})$. By the second assumption and Lemma 99, we have

$$\mathrm{H}^0(K \otimes_\mathcal{Z} \mathbf{C}) = \operatorname{Hom}\left(L \otimes_\mathcal{Z} \mathbf{C}, M/\mathfrak{m}_\mathcal{Z} M\right) = 0.$$

Consequently $\mathrm{H}^{\leq 0}(K) = 0$ by Nakayama's lemma.

Suppose that $\mathrm{H}^1(K) \neq 0$. Since the localisation $\operatorname{Frac}\mathcal{Z} \otimes_\mathcal{Z} M$ is a weight module over $\mathbf{A}^{-\infty}$, which is semisimple, $\mathrm{H}^1(K)$ must be a torsion module over $\mathcal{Z}$ so $K^0 \neq 0$. However, the minimality of K would imply $\mathrm{H}^0(K \otimes_\mathcal{Z} \mathbf{C}) \neq 0$, contradiction. Hence $\mathrm{H}^{\leq 1}(K) = 0$ and so $\operatorname{gHom}(L, M) = 0$ and $\operatorname{gExt}^1(L, M) = 0$ as asserted. □

Lemma 102. *Let $M \in \mathbf{A}^\omega$-gmod be an object satisfying Lemma 101. Then the adjoint unit yields an isomorphism $M \cong (\mathbf{V}^\top \circ \mathbf{V})M$.*

Proof. Set $X = \operatorname{Cone}\left(M \longrightarrow (\mathrm{R}\mathbf{V}^\top \circ \mathbf{V})M\right) \in \mathrm{D}^+(\mathbf{A}^\omega\text{-gmod})$, so that there is a distinguished triangle

$$M \longrightarrow (\mathrm{R}\mathbf{V}^\top \circ \mathbf{V})M \longrightarrow X \longrightarrow M[1]. \tag{103}$$

By the adjunction and the exactness of $\mathbf{V}$, we have $\mathbf{V}X \cong \operatorname{Cone}(\mathbf{V}M \longrightarrow (\mathbf{V} \circ \mathrm{R}\mathbf{V}^\top \circ \mathbf{V})M) = 0$ and hence

$$\mathbf{V}\mathrm{H}^k(X) \cong \mathrm{H}^k(\mathbf{V}X) = 0, \quad k \in \mathbf{Z}.$$

Applying Lemma 101 with $L = \mathrm{H}^0(X)$ and $L = \mathrm{H}^{-1}(X)$, we deduce

$$\operatorname{gHom}\left(\mathrm{H}^0(X), M\right) = 0, \quad \operatorname{gHom}\left(\mathrm{H}^0(X)[-1], M\right) = \operatorname{gExt}^1\left(\mathrm{H}^0(X), M\right) = 0$$
$$\operatorname{gHom}\left(\mathrm{H}^{-1}(X), M\right) = 0,$$

whence

$$\operatorname{gHom}(\tau_{\leq 0}X, M) = 0, \quad \operatorname{gHom}(\tau_{\leq 0}X, M[1]) = \operatorname{gHom}(\tau_{\leq 0}X[-1], M) = 0. \tag{104}$$

Applying $\operatorname{RgHom}(\tau_{\leq 0}X, -)$ to the distinguished triangle (103), we obtain the long exact sequence

$$\operatorname{gHom}(\tau_{\leq 0}X, M) \longrightarrow \operatorname{gHom}\left(\tau_{\leq 0}X, (\mathrm{R}\mathbf{V}^\top \circ \mathbf{V})M\right) \longrightarrow \operatorname{gHom}(\tau_{\leq 0}X, X) \longrightarrow \operatorname{gHom}(\tau_{\leq 0}X, M[1]).$$

By (104), the first and the last term of the sequence vanish. Hence,

$$\operatorname{gHom}(\tau_{\leq 0}X, X) \cong \operatorname{gHom}\left(\tau_{\leq 0}X, (\mathrm{R}\mathbf{V}^\top \circ \mathbf{V})M\right) \cong \operatorname{gHom}(\tau_{\leq 0}\mathbf{V}X, \mathbf{V}M) = 0,$$

which implies that $\tau_{\leq 0}X = 0$. Applying H^0 to the distinguished triangle (103), we deduce that the adjunction unit $M \longrightarrow (\mathbf{V}^\top \circ \mathbf{V})M$ is an isomorphism. □

Theorem 105 (Double centraliser property[¶]). *The canonical map*

$$\mathbf{A}^\omega \longrightarrow \bigoplus_{\lambda, \lambda' \in W_S\lambda_0} \operatorname{gHom}_{\mathbf{B}^\Omega}\left(\mathbf{V}\mathbf{A}^\omega \mathbf{e}(\lambda), \mathbf{V}\mathbf{A}^\omega \mathbf{e}(\lambda')\right)$$

is an isomorphism.

[¶]Let A and B be unital associative rings. Usually, one says that an (A, B)-bimodule P satisfies the double centraliser property if the structural maps $A \longrightarrow \operatorname{End}_{B^{\mathrm{op}}}(P)$ and $B \longrightarrow \operatorname{End}_A(P)^{\mathrm{op}}$ are isomorphisms. The above theorem provides a graded, non-unital version of this property for the $(\mathbf{A}^\omega, \mathbf{B}^\Omega)$-bimodule $\mathbf{A}^\omega \mathbf{e}_\gamma$.

Proof. Observe that for each $\lambda \in W_S\lambda_0$, the module $\mathbf{A}^\omega\mathbf{e}(\lambda) \in \mathbf{A}^\omega$-gmod satisfies the conditions of Lemma 101. Indeed, $\mathbf{A}^\omega\mathbf{e}(\lambda)$ is flat over $\mathcal{Z}$ by Theorem 40. For the second condition, we have $\mathbf{A}^\omega\mathbf{e}(\lambda) \cong \mathrm{ind}^S_{R,\lambda_1} \mathbf{A}^\omega_{R,\lambda}\mathbf{e}(\lambda)$, so $\mathbf{A}^\omega\mathbf{e}(\lambda)/\mathfrak{m}_\mathcal{Z} \cong \mathrm{ind}^S_{R,\lambda_1}\left(\mathbf{A}^\omega_{R,\lambda}\mathbf{e}(\lambda)/\mathfrak{m}_\mathcal{Z}\right)$. Applying Lemma 102, we obtain

$$\mathbf{A}^\omega \cong \bigoplus_{\lambda,\lambda'\in W_S\lambda_0} \mathrm{gHom}_{\mathbf{A}^\omega}\left(\mathbf{A}^\omega\mathbf{e}\left(\lambda\right),\mathbf{A}^\omega\mathbf{e}\left(\lambda'\right)\right) \xrightarrow{\cong} \bigoplus_{\lambda,\lambda'\in W_S\lambda_0} \mathrm{gHom}_{\mathbf{A}^\omega}\left(\mathbf{V}\mathbf{A}^\omega\mathbf{e}\left(\lambda\right),\mathbf{V}\mathbf{A}^\omega\mathbf{e}\left(\lambda'\right)\right).$$

□

10.9 Categorical characterisation of V

We shall exploit the Frobenius structure on $\mathbf{B}^\Omega$ introduced in Lemma 76. Consider the anti-involution $\mathbf{A}^\omega \cong (\mathbf{A}^\omega)^{\mathrm{op}}$ which fixes pointwise $\mathbf{C}[V]\mathbf{e}(\lambda)$ for $\lambda \in W_S$ and sends $\tau^\omega_a\mathbf{e}(\lambda) \mapsto \tau^\omega_a\mathbf{e}(s_a\lambda)$. The duality

$$M \mapsto M^* := \bigoplus_{\lambda\in W_S\lambda_0} \mathrm{Hom}_\mathbf{C}(\mathbf{e}(\lambda)M,\mathbf{C}) \tag{106}$$

yields an equivalence

$$\mathbf{A}^\omega\text{-gmod}_0 \cong ((\mathbf{A}^\omega)^{\mathrm{op}}\text{-gmod}_0)^{\mathrm{op}} \cong (\mathbf{A}^\omega\text{-gmod}_0)^{\mathrm{op}}.$$

Similarly, the anti-involution $\mathbf{B}^\Omega \cong (\mathbf{B}^\Omega)^{\mathrm{op}}$ given by $\tau^\Omega_\alpha\mathbf{e}(\ell) \mapsto \tau^\Omega_\alpha\mathbf{e}(s_\alpha\ell)$ yields

$$\mathbf{B}^\Omega\text{-gmod}_0 \cong (\mathbf{B}^\Omega\text{-gmod}_0)^{\mathrm{op}}.$$

Denote $\overline{\mathbf{A}}^\omega = \mathbf{A}^\omega/\mathbf{A}^\omega\mathfrak{m}_\mathcal{Z}$ and $\overline{\mathbf{B}}^\Omega = \mathbf{B}^\Omega/\mathbf{B}^\Omega\mathfrak{m}_\mathcal{Z}$. Notice that the pairing

$$\mathbf{e}_\gamma\overline{\mathbf{A}}^\omega \times \overline{\mathbf{A}}^\omega\mathbf{e}_\gamma \xrightarrow{(a,b)\mapsto ab} \mathbf{e}_\gamma\overline{\mathbf{A}}^\omega\mathbf{e}_\gamma = \overline{\mathbf{B}}^\Omega$$

composed with the Frobenius form $\overline{\mathbf{B}}^\Omega \xrightarrow{\mathrm{tr}} \mathcal{Z}/\mathfrak{m}_\mathcal{Z} = \mathbf{C}$ yields an isomorphism $(\overline{\mathbf{A}}^\omega\mathbf{e}_\gamma)^* \cong \overline{\mathbf{A}}^\omega\mathbf{e}_\gamma$.

Lemma 107. *There are canonical isomorphisms* ${}^\top\mathbf{V}\overline{\mathbf{B}}^\Omega \cong \overline{\mathbf{A}}^\omega\mathbf{e}_\gamma \cong \mathbf{V}^\top\overline{\mathbf{B}}^\Omega$.

Proof. The first isomorphism is obvious: ${}^\top\mathbf{V}\overline{\mathbf{B}}^\Omega = \mathbf{A}^\omega\mathbf{e}_\gamma \otimes_{\mathbf{B}^\Omega} \overline{\mathbf{B}}^\Omega = \overline{\mathbf{A}}^\omega\mathbf{e}_\gamma$.

Observe that $(\overline{\mathbf{A}}^\omega\mathbf{e}_\gamma)^* \cong \overline{\mathbf{A}}^\omega\mathbf{e}_\gamma$ implies $\mathbf{V}M^* \cong (\mathbf{V}M)^*$ for $M \in \overline{\mathbf{A}}^\omega\text{-gmod}_0$ and hence $\mathbf{V}^\top N^* \cong ({}^\top\mathbf{V}N)^*$ for $N \in \overline{\mathbf{B}}^\Omega\text{-gmod}_0$. Therefore

$$\mathbf{V}^\top\overline{\mathbf{B}}^\Omega \cong ({}^\top\mathbf{V}(\overline{\mathbf{B}}^\Omega)^*)^* \cong ({}^\top\mathbf{V}\overline{\mathbf{B}}^\Omega)^* \cong (\overline{\mathbf{A}}^\omega\mathbf{e}_\gamma)^* \cong \overline{\mathbf{A}}^\omega\mathbf{e}_\gamma.$$

□

Theorem 108. *Let $L \in \mathbf{A}^\omega\text{-gmod}_0$ be a simple object. Then the following conditions are equivalent:*

(i) $\mathbf{V}L \neq 0$;

(ii) the injective hull of L in the subcategory $\overline{\mathbf{A}}^\omega$-gmod is projective;

(iii) the projective cover of L in the subcategory $\overline{\mathbf{A}}^\omega$-gmod is injective.

Proof. Since $\mathbf{V}^\top$ preserves injective objects, we see that by Lemma 107, $\overline{\mathbf{A}}^\omega\mathbf{e}_\gamma$ is injective-projective in $\overline{\mathbf{A}}^\omega$-gmod.

We prove (i) ⇔ (ii). Let $L \in \mathbf{A}^\omega\text{-gmod}_0$ be any simple object. If $\mathbf{V}L = 0$, then by Lemma 99, we have $\mathrm{gHom}(L,\overline{\mathbf{A}}^\omega) = 0$; hence (ii) fails for L. If $\mathbf{V}L \neq 0$, since $\mathbf{V}$ is a quotient functor, $\mathbf{V}L \in \mathbf{B}^\Omega\text{-gmod}_0$ must be simple. We have $\mathfrak{m}_\mathcal{Z}\mathbf{V}L = 0$, so we may view L as a $\overline{\mathbf{B}}^\Omega$-module. By the self-injectivity of $\overline{\mathbf{B}}^\Omega$, there exists a non-zero map $\iota : \mathbf{V}L \hookrightarrow \overline{\mathbf{B}}^\Omega$ and the adjunction yields an injective map $L \longrightarrow \mathbf{V}^\top\overline{\mathbf{B}}^\omega \cong \overline{\mathbf{A}}^\omega\mathbf{e}_\gamma$, whence (ii) holds for L.

Finally, since the duality (106) exchanges the projective and injective objects in $\overline{\mathbf{A}}^\omega$-gmod and preserves $\ker\mathbf{V}$, we deduce

$$\text{(iii) for } L \Leftrightarrow \text{(ii) for } L^* \Leftrightarrow \text{(i) for } L^* \Leftrightarrow \text{(i) for } L.$$

□

Example 109. *Resume to the setting of examples Example 33 (iii), Example 55 and Example 83. We have $\mathbf{A}^\omega \mathbf{e}_\gamma = P_+ \oplus P_-$ so $\mathbf{V}L_0 = 0$, while $\mathbf{V}L_+ \neq 0$ and $\mathbf{V}L_- \neq 0$ are simple objects in* $\mathbf{B}^\omega$-gmod. *In regard of Theorem 97, we have* $\dim_{\mathrm{GK}} L_+ = \dim_{\mathrm{GK}} L_- = 1$ *while* $\dim_{\mathrm{GK}} L_0 = 0$. *The cosocle filtration of $\mathbf{V}P_+$, $\mathbf{V}P_0$ and $\mathbf{V}P_-$ are described by the following:*

$$\mathbf{V}P_+ = \begin{bmatrix} \mathbf{V}L_+ & \\ \mathbf{V}L_+\langle -2\rangle & \mathbf{V}L_-\langle -2\rangle \\ \mathbf{V}L_+\langle -4\rangle & \mathbf{V}L_-\langle -4\rangle \\ \mathbf{V}L_+\langle -6\rangle & \mathbf{V}L_-\langle -6\rangle \\ \vdots & \end{bmatrix}, \ \mathbf{V}P_0 = \begin{bmatrix} \mathbf{V}L_+\langle -1\rangle & \mathbf{V}L_-\langle -1\rangle \\ \mathbf{V}L_+\langle -3\rangle & \mathbf{V}L_-\langle -3\rangle \\ \mathbf{V}L_+\langle -5\rangle & \mathbf{V}L_-\langle -5\rangle \\ \vdots & \end{bmatrix}, \ \mathbf{V}P_- = \begin{bmatrix} & \mathbf{V}L_- \\ \mathbf{V}L_+\langle -2\rangle & \mathbf{V}L_-\langle -2\rangle \\ \mathbf{V}L_+\langle -4\rangle & \mathbf{V}L_-\langle -4\rangle \\ \mathbf{V}L_+\langle -6\rangle & \mathbf{V}L_-\langle -6\rangle \\ \vdots & \end{bmatrix}.$$

From this description it is obvious that the functor $\mathbf{V}$ is fully faithful on the projective objects, so $\mathbf{V}$ satisfies the double centraliser property Theorem 105.

Consider the quotients

$$P_+/\mathfrak{m}_{\mathcal{Z}} = \begin{bmatrix} L_+ \\ L_0\langle -1\rangle \\ L_-\langle -2\rangle \end{bmatrix}, \ P_0/\mathfrak{m}_{\mathcal{Z}} = \begin{bmatrix} L_0 \\ L_+\langle -1\rangle \ L_-\langle -1\rangle \end{bmatrix}, \ P_-/\mathfrak{m}_{\mathcal{Z}} = \begin{bmatrix} L_- \\ L_0\langle -1\rangle \\ L_+\langle -2\rangle \end{bmatrix}.$$

It follows that $P_+/\mathfrak{m}_{\mathcal{Z}}$ (resp. $P_-/\mathfrak{m}_{\mathcal{Z}}$) is the injective hull of $L_-\langle -2\rangle$ (resp. $L_+\langle -2\rangle$) in the category $(\mathbf{A}^\omega/\mathfrak{m}_{\mathcal{Z}})$-gmod *while $P_0/\mathfrak{m}_{\mathcal{Z}}$ is not injective. Hence L_+ and L_- satisfy the equivalent conditions of Theorem 108.*

A Category of pro-objects

A.1

Let $\mathcal{A}$ be an abelian category. We denote by $\mathrm{Pro}(\mathcal{A})$ and $\mathrm{Ind}(\mathcal{A})$ the category of pro-objects and ind-objects. The basic reference for these is [19, 8.6]. All the results below are stated for $\mathrm{Pro}(\mathcal{A})$ while they all have a dual version for $\mathrm{Ind}(\mathcal{A})$. An object of $\mathrm{Pro}(\mathcal{A})$ is a filtered "projective limit" of objects of $\mathcal{A}$. If

$$M^{(i)} = \text{“}\varprojlim_{j\in\mathcal{I}^{(i)}}\text{”}\, M_j^{(i)}, \quad M_j^{(i)} \in \mathcal{A}, \quad i \in \{1,2\}$$

are two objects of $\mathrm{Pro}(\mathcal{A})$, where $\mathcal{I}^{(i)}$'s are filtrant diagram categories and $M^{(i)} : \mathcal{I}^{(i)\mathrm{op}} \longrightarrow \mathcal{A}$'s are functors, then the Hom-space between them is given by

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{A})}\left(M^{(1)}, M^{(2)}\right) = \varprojlim_{j\in\mathcal{I}^{(2)}} \varinjlim_{i\in\mathcal{I}^{(1)}} \mathrm{Hom}_{\mathcal{A}}\left(M_i^{(1)}, M_j^{(2)}\right). \tag{110}$$

A.2

For every $M \in \mathrm{Pro}(\mathcal{A})$, let $\mathcal{A}^M$ denotes the category whose objects are pairs (M', a) where $M' \in \mathcal{A}$ and $a \in \mathrm{Hom}_{\mathrm{Pro}(\mathcal{A})}(M, M')$, and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{A}^M}\left((M_1, a_1), (M_2, a_2)\right) = \{b \in \mathrm{Hom}_{\mathcal{A}}(M_1, M_2)\ ;\ a_2 = b \circ a_1\}.$$

Every object $M \in \mathrm{Pro}(\mathcal{A})$ can be expressed as the following filtered limit:

$$M \cong \text{“}\varprojlim_{(M',a)\in\mathcal{A}^M}\text{”}\, M'. \tag{111}$$

Let $\mathcal{A}^M_{\mathrm{epi}} \subset \mathcal{A}^M$ be the full subcategory whose objects are the pairs (M', q) with q being an epimorphism.

Proposition 112. *Let $\mathcal{A}$ be an artinian abelian category. Then the following statements hold:*

(i) $\mathcal{A}$ is a Serre subcategory of $\mathrm{Pro}(\mathcal{A})$.

(ii) Every object $M \in \mathrm{Pro}(\mathcal{A})$ can be written as the following filtered projective limit

$$M \cong \text{“}\varprojlim_{(M',a)\in\mathcal{A}^M_{\mathrm{epi}}}\text{”}\, M'.$$

(iii) $\mathcal{A}$ is the full subcategory of artinian objects in $\mathrm{Pro}(\mathcal{A})$.

(iv) If $\varphi : N \longrightarrow M$ is a morphism in $\mathrm{Pro}(\mathcal{A})$ *such that for every* (M', q) *in* $\mathcal{A}^M_{\mathrm{epi}}$*, the composite* $q \circ \varphi$ *is an epimorphism, then* φ *is an epimorphism.*

Proof. We first prove that $\mathcal{A} \subset \mathrm{Pro}(\mathcal{A})$ is closed under taking sub-objects.

Let $M \in \mathrm{Pro}(\mathcal{A})$. Suppose that there exists $\tilde{M} \in \mathcal{A}$ and a monomorphism $\iota : M \rightarrowtail \tilde{M}$. We can consider the full subcategory $\mathcal{A}^M_1 \subset \mathcal{A}^M$ of pairs (M', a) with a being monomorphism. The subcategory $\mathcal{A}^M_1$ is cofinal. Indeed, if $(M', a) \in \mathcal{A}^M$, then $\left(M' \times \tilde{M}, (a, \iota)\right) \in \mathcal{A}^M_1$. Let $\mathcal{A}^M_2 \subset \mathcal{A}^M_1$ be the full subcategory of objects which are minimal, in the sense that if there is $(M'', b) \in \mathcal{A}^M_1$ with a monomorphism $\varphi \in \mathrm{Hom}_{\mathcal{A}}(M'', M')$ such that $\varphi \circ b = a$, then φ is an isomorphism. By the minimality of the objects of $\mathcal{A}^M_2$, it is easy to see that the Hom-space $\mathrm{Hom}_{\mathcal{A}^M_2}((M', a), (M'', b))$ consists of exactly one element for every $(M', a), (M'', b) \in \mathcal{A}^M_2$. It follows that any object $(M', a) \in \mathcal{A}^M_2$ yields an isomorphism $a : M \cong M'$. As $\mathcal{A}$ is artinian, $\mathcal{A}^M_2$ cannot be empty, whence $M \in \mathcal{A}$.

To prove (ii), in view of (111), it suffices to show that $\mathcal{A}^M_{\mathrm{epi}}$ is cofinal. The previous paragraph shows that for $(M', a) \in \mathcal{A}^M$, the image $\mathrm{im}(a)$ is in $\mathcal{A}$. Consider the factorisation $M \xrightarrow{\pi_a} \mathrm{im}(a) \xrightarrow{\bar{a}} M'$. Then $(\mathrm{im}(a), \pi_a) \in \mathcal{A}^M_{\mathrm{epi}}$ and there is a morphism $\bar{a} : (\mathrm{im}(a), \pi_a) \longrightarrow (M', a)$ in $\mathcal{A}^M$. Thus $\mathcal{A}^M_{\mathrm{epi}}$ is cofinal in $\mathcal{A}^M$.

We prove (iii). Let $M \in \mathcal{A}$. Since $\mathcal{A} \subset \mathrm{Pro}(\mathcal{A})$ is closed under taking sub-objects, every descending chain of sub-objects of M is in the subcategory $\mathcal{A}$, which by assumption must stabilise. Thus M is artinian in $\mathrm{Pro}(\mathcal{A})$. Suppose that $M \in \mathrm{Pro}(\mathcal{A})$ is artinian. There must be a minimal sub-object $M' \subset M$ such that M/M' lies in $\mathcal{A}$, meaning that the category $\mathcal{A}^M_{\mathrm{epi}}$ has an initial object. By (ii), M being the projective limit over $\mathcal{A}^M_{\mathrm{epi}}$ must lie in $\mathcal{A}$, whence (iii). The assertion (i) follows immediately from (iii).

We prove (iv). Let $c : M \longrightarrow \mathrm{coker}\,\varphi = C$ be the cokernel. Suppose that $C \neq 0$. Since $C \in \mathrm{Pro}(\mathcal{A})$, there exists an epimorphism $p : C \longrightarrow C'$ with $0 \neq C' \in \mathcal{A}$. Since $p \circ c : M \longrightarrow C'$ is epimorphism, the composite $p \circ c \circ \varphi$ is also an epimorphism by hypothesis. However, as $c \circ \varphi = 0$, we see that $C' = 0$, contradiction. Thus $C = 0$ and φ is an epimorphism. ☐

A.3

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $F : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor. We define the extension of F:

$$F : \mathrm{Pro}(\mathcal{A}) \longrightarrow \mathrm{Pro}(\mathcal{B}), \quad F(M) = \underset{(M',a)\in\mathcal{A}^M}{\text{“}\varprojlim\text{”}} F(M').$$

According to [19, 8.6.8], if F is exact, then the extended functor $F : \mathrm{Pro}(\mathcal{A}) \longrightarrow \mathrm{Pro}(\mathcal{B})$ is also exact.

A.4

Suppose that $\mathcal{A}$ is noetherian-artinian. We define an endo-functor

$$\mathrm{hd} : \mathrm{Pro}(\mathcal{A}) \longrightarrow \mathrm{Pro}(\mathcal{A}), \quad \mathrm{hd}(M) = \underset{(M',q)\in\mathcal{A}^M_{\mathrm{epi}}}{\text{“}\varprojlim\text{”}} \mathrm{hd}(M')$$

where $\mathrm{hd}(M')$ is the largest semisimple quotient of M' in $\mathcal{A}$. For every $M \in \mathrm{Pro}(\mathcal{A})$, there is a canonical map $\pi_M : M \longrightarrow \mathrm{hd}(M)$.

Proposition 113 (Nakayama's lemma)**.** *Let $\mathcal{A}$ be a noetherian-artinian abelian category. Let $\varphi : N \longrightarrow M$ be a morphism in* $\mathrm{Pro}(\mathcal{A})$*. Suppose that the composite* $N \xrightarrow{\varphi} M \xrightarrow{\pi_M} \mathrm{hd}(M)$ *is an epimorphism. Then φ is an epimorphism.*

Proof. We first prove the statement in the case where $M \in \mathcal{A}$. In this case, since $\mathrm{coker}\,\varphi$ is a quotient of M, we have an epimorphism $\mathrm{hd}(M) \twoheadrightarrow \mathrm{hd}(\mathrm{coker}\,\varphi)$. As the composite $N \longrightarrow \mathrm{hd}(M) \longrightarrow \mathrm{hd}(\mathrm{coker}\,\varphi)$ is zero and is an epimorphism, it implies that $\mathrm{hd}(\mathrm{coker}\,\varphi) = 0$. As $\mathcal{A}$ is noetherian, it follows that $\mathrm{coker}\,\varphi = 0$, so φ is surjective.

In general, let $M \in \mathrm{Pro}(\mathcal{A})$. Let (M', q) be any object of $\mathcal{A}^M_{\mathrm{epi}}$. Then $\pi_{M'} \circ q \circ \varphi$ is an epimorphism. By the previous paragraph, $q \circ \varphi$ is also an epimorphism. Then Proposition 112 (iv) implies that φ is an epimorphism. ☐

A.5

Proposition 114. *Suppose $\mathcal{A}$ is an essentially small noetherian-artinian abelian category. Let $M \in \mathcal{A}$ be a simple object. Then there exists a projective cover $P_M \in \mathrm{Pro}(\mathcal{A})$.*

Proof. We construct an object $P^{(n)} \in \mathrm{Pro}(\mathcal{A})$ for $n \in \mathbf{N}$ by induction. Let $P^{(0)} = M$. For $n > 0$, let

$$0 \longrightarrow \text{“}\prod_{\substack{L \in \mathrm{Irr}(\mathcal{A})/\sim \\ \gamma \in \mathrm{Ext}^1_{\mathcal{A}}\left(P^{(n-1)}, L\right)}}\text{”} L \longrightarrow P^{(n)} \longrightarrow P^{(n-1)} \longrightarrow 0$$

be the short exact sequence corresponding to the tautological class

$$\Delta = (\gamma)_{L,\gamma} \in \prod_{\substack{L \in \mathrm{Irr}(\mathcal{A})/\sim \\ \gamma \in \mathrm{Ext}^1_{\mathcal{A}}\left(P^{(n-1)}, L\right)}} \mathrm{Ext}^1_{\mathcal{A}}\left(P^{(n-1)}, L\right).$$

Put $P = \text{“}\varprojlim_{n \longrightarrow \infty}\text{”}\, P^{(n)}$. Then P is a projective since we have

$$\mathrm{Ext}^1_{\mathrm{Pro}(\mathcal{A})}(P, L) = 0$$

by construction and since $\mathcal{A}$ is noetherian-artinian. Let $p : P \longrightarrow M$ be the obvious epimorphism.

Now, let $\mathcal{A}^P_M$ be the category whose objects are triples (π, Q, π'), where

- $Q \in \mathcal{A}$
- $\pi \in \mathrm{Hom}_{\mathrm{Pro}(\mathcal{A})}(P, Q)$ is an epimorphism and
- $\pi' \in \mathrm{Hom}_{\mathcal{A}}(Q, M)$

such that

- $\pi' \circ \pi = p \in \mathrm{Hom}_{\mathrm{Pro}(\mathcal{A})}(P, M)$ and
- π' induces an isomorphism $\mathrm{hd}(Q) \cong M$.

The morphisms are defined by

$$\mathrm{Hom}_{\mathcal{A}^P_M}\left((\pi_1, Q_1, \pi_1'), (\pi_2, Q_2, \pi_2')\right) = \left\{\varphi \in \mathrm{Hom}_{\mathcal{A}}\left(Q_1, Q_2\right)\ ;\ \varphi \circ \pi_1 = \pi_2\right\}.$$

Put

$$P_M = \text{“}\varprojlim_{(\pi, Q, \pi') \in \mathcal{A}^P_M}\text{”}\ Q.$$

Then the obvious morphism $P_M \longrightarrow M$ is a projective cover. ◻

Index of notation

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