

LANDAU-GINZBURG MIRROR, QUANTUM DIFFERENTIAL EQUATIONS AND qKZ DIFFERENCE EQUATIONS FOR A PARTIAL FLAG VARIETY

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ABSTRACT. We consider the system of quantum differential equations for a partial flag variety and construct a basis of solutions in the form of multidimensional hypergeometric functions, that is, we construct a Landau-Ginzburg mirror for that partial flag variety. In our construction, the solutions are labeled by elements of the K -theory algebra of the partial flag variety.

To establish these facts we consider the equivariant quantum differential equations for a partial flag variety and introduce a compatible system of difference equations, which we call the qKZ equations. We construct a basis of solutions of the joint system of the equivariant quantum differential equations and qKZ difference equations in the form of multidimensional hypergeometric functions. Then the facts about the non-equivariant quantum differential equations are obtained from the facts about the equivariant quantum differential equations by a suitable limit.

Analyzing these constructions we obtain a formula for the fundamental Levelt solution of the quantum differential equations for a partial flag variety.

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1. INTRODUCTION

The genus zero Gromov-Witten invariants of a partial flag variety V answer enumerative questions about rational curves in V and are put together in various ways to define rich mathematical structures such as the quantum cohomology algebra and the system of quantum differential equations. Those structures are part of the so-called ‘ A -model’ of V . Landau-Ginzburg mirror symmetry seeks to describe these structures in terms a mirror dual ‘ B -model’ associated to V . The way originated by A. Givental in [Gi1, Gi2] seeks to encode the data from the A -model by oscillating integrals of a superpotential function W , which is defined on a ‘mirror dual’ affine variety V° . In particular, these oscillating integrals are supposed to give all solutions of the system of quantum differential equations.

This Landau-Ginzburg mirror symmetry has been established by A. Givental for full flag varieties and projective spaces in [Gi1, Gi2, GK1]. The construction for Grassmannians has been performed by R. J. Marsh and K. Rietsch in [MR]. A standard problem, related to generalizations of Givental’s approach, is to check that the number of critical points of the superpotential equals the dimension of the cohomology algebra of V , and then check that the oscillating integrals, which are intrinsically labeled by the critical points, indeed generate a basis of solutions of the system of quantum differential equations. While the first part is of algebraic nature, the second part is of global topological nature (to determine integration cycles exiting critical points). For example, in [MR] a superpotential for a Grassmannian was constructed, and it was shown that the number of its critical points is the correct one, but the part that the oscillating integrals indeed give a basis of solutions was not addressed.

In this paper we suggest a new ‘hypergeometric Landau-Ginzburg mirror symmetry model’ for a partial flag variety V and construct the full set of solutions of the system of quantum differential equation of V in the form of multidimensional hypergeometric functions. In these functions the role of Givental’s exponential of the superpotential is played by what we call the master function, which is the product of Gamma functions multiplied by the exponential of a linear function, and the role of the mirror dual V° is played by the complement in an affine space to the set of all poles of the product of Gamma functions composing the master function. In particular, our V° is *not* an algebraic variety, but a complex analytic variety.

It is interesting to note that in our construction, the solutions of the system of quantum differential equations of V are naturally labeled by elements of the K -theory algebra of V . In particular, that observation suggests that the monodromy and Stokes phenomenon of the system of quantum differential equations of V may be described in terms of the K -theory algebra of V , in accordance with the philosophy of B. Dubrovin, see [D1, D2, CDG, TV6, TV7, TV2, CV].

Consider the example of the m -dimensional complex projective space $V = \mathbb{C}P^m$. In the body of the paper, this example corresponds to the case $n = m + 1$, $N = 2$, $\lambda = (1, m)$. The cohomology algebra is $H^*(V; \mathbb{C}) = \mathbb{C}[x]/\langle x^{m+1} \rangle$, where x is the first Chern class of the tautological line bundle. The quantum multiplication $*_p$ on $H^*(V; \mathbb{C})$ depends on a parameter p ,

$$x^i *_p x^j = x^{i+j} \quad \text{for } i + j \leq m, \quad x^i *_p x^j = p x^{i+j-m-1} \quad \text{for } i + j > m,$$

$i, j = 0, \dots, m$. The parameter p corresponds to p_2/p_1 in the body of the paper. The quantum differential equation is

$$(1.1) \quad -\kappa p \frac{\partial I}{\partial p} = x *_p I,$$

where κ is the parameter of the differential equation and I is the unknown $H^*(V; \mathbb{C})$ -valued function. The K -theory algebra is $K(V; \mathbb{C}) = \mathbb{C}[X, X^{-1}]/\langle (X - 1)^{m+1} \rangle$, where X is the class of the tautological line bundle.

The $H^*(V; \mathbb{C})$ -valued weight function is $W(t, x) = \sum_{i=0}^m t^{m-i} x^i$. The master function is

$$\Phi(t, p, \kappa) = (\kappa^{m+1}/p)^{t/\kappa} (\Gamma(t/\kappa))^{m+1}.$$

For a univariate Laurent polynomial $P(X)$, define

$$\Psi_P(p, \kappa) = \kappa^{-1} \sum_{r=0}^{\infty} \operatorname{Res}_{t=-r\kappa} (\Phi(t, p, \kappa) P(e^{2\pi\sqrt{-1}t/\kappa}) W(t, x)).$$

The function $\Psi_P(p, \kappa)$ depends only on the class of P in $K(V; \mathbb{C})$. Alternatively, $\Psi_P(p, \kappa)$ can be written as an integral over an appropriate contour C encircling the poles of $\Phi(t, p, \kappa)$ in the positive direction,

$$\Psi_P(p, \kappa) = \frac{1}{2\pi\sqrt{-1}} \int_C \Phi(t, p, \kappa) P(e^{2\pi\sqrt{-1}t/\kappa}) W(t, x) dt.$$

For instance, one can take the parabola $C = \{ \kappa(1 - s^2 + s\sqrt{-1}) \mid s \in \mathbb{R} \}$.

Theorem 1.1. *For any Laurent polynomial P , the $H^*(V)$ -valued function $\Psi_P(p, \kappa)$ is an entire function of $\log p$ that solves the quantum differential equation (1.1). If the classes of Laurent polynomials P_0, \dots, P_m give a basis of $K(V; \mathbb{C})$, then the functions $\Psi_{P_0}(p, \kappa), \dots, \Psi_{P_m}(p, \kappa)$ give a basis of solutions of the quantum differential equation.*

Theorem 1.2. *The following determinant formula holds*

$$\det \left(\sum_{r=0}^{\infty} \operatorname{Res}_{t=-r\kappa} (t^i e^{-2\pi\sqrt{-1}jt/\kappa} \Phi(t, p, \kappa)) \right)_{i,j=0}^m = (2\pi\sqrt{-1})^{m(m+1)/2} \kappa^{(m+1)(m+2)/2}.$$

Theorem 1.1 follows from Proposition 6.22, see also Proposition 5.14. Theorem 1.2 follows from Theorem 5.15, see also formula (5.60).

Notice that in [Gu], D. Guzzetti considers a scalar linear differential equation of order $m+1$ equivalent to the quantum differential equation (1.1) and constructs a basis of solutions in the form of similar integrals.

Our construction of the basis of solutions of the system of quantum differential equations for a partial flag variety is done in several steps and is based on constructions from representation theory.

We consider the joint compatible system of rational qKZ difference equations and dynamical differential equations for sections of the trivial bundle $\pi : (\mathbb{C}^N)^{\otimes n} \times \mathbb{C}^n \times \mathbb{C}^N \rightarrow \mathbb{C}^n \times \mathbb{C}^N$. This system is defined in terms of the Yangian $Y_h(\mathfrak{gl}_N)$ action on $(\mathbb{C}^N)^{\otimes n}$ and depends on the Yangian deformation parameter h . In this paper we describe the $h \rightarrow \infty$ limit of this system of difference and differential equations. This is our first main result, see Section 3. We call the obtained equations the limiting qKZ difference equations and limiting dynamical differential equations.

In [TV1, TV6] we constructed solutions of the joint system of initial qKZ difference equations and dynamical differential equations in the form of multidimensional hypergeometric functions, see Theorem 4.7. In this paper we calculate the $h \rightarrow \infty$ limit of these solutions and obtain solutions of the joint system of limiting equations, see Theorem 5.11. This is our second main result.

According to the general theory in [MO] the initial system of qKZ difference equations and dynamical differential equations is identified with the system of qKZ difference equations

and equivariant quantum differential equations in the equivariant cohomology of cotangent bundles of partial flag varieties. The precise formulae for that identification can be found in [GRTV, RTV1]. It was expected that a suitable $h \rightarrow \infty$ limit of these equations gives a system of difference and differential equations related to the equivariant quantum cohomology of partial flag varieties themselves instead of their cotangent bundles. In particular, it was shown in [BMO] that the $h \rightarrow \infty$ limit of the equivariant quantum differential equations for the cotangent bundle of the full flag variety gives the equivariant quantum differential equations of the corresponding full flag variety. Also, the appropriate $h \rightarrow \infty$ limit of the Yangian R -matrix associated with the quantum cohomology of the cotangent bundle of a Grassmannian, was used in [GK, GKS] to calculate the quantum multiplication in the cohomology of the Grassmannian itself.

In this paper we identify our limiting dynamical differential equations for sections of the bundle π with the equivariant quantum differential equations for partial flag varieties. Under this identification our limiting qKZ difference equations become a system of difference equations in the equivariant cohomology of partial flag varieties, hence, a new system of difference equations compatible with the equivariant quantum differential equations. Under this identification the multidimensional hypergeometric solutions of the limiting equations for sections of the bundle π become solutions of the equivariant quantum differential equation and qKZ difference equations for partial flag varieties. This is our third main result, see Theorem 6.14.

The particular case of a projective space was considered in [TV7, CV], where the joint system of the equivariant quantum differential equation and compatible qKZ difference equations, together with hypergeometric solutions, was used to analyze the Stokes phenomenon for the equivariant quantum differential equation of a projective space. We expect similar applications of our compatible qKZ equations for partial flag varieties.

The paper is organized as follows. In Section 2 we define the initial joint system of qKZ and dynamical equations. In Section 3 we describe the $h \rightarrow \infty$ limit of these equations. In Section 4 we describe multidimensional hypergeometric solutions of the initial qKZ and dynamical equations. In Section 5 we obtain solutions of the limiting equations by taking the $h \rightarrow \infty$ limit of the solutions of the initial equations. In Section 6 we identify the limiting equations with the equation in equivariant cohomology of partial flag varieties. Appendices A–D contain technical information.

In Appendix E we show that the monodromy of the system of dynamical differential equations (2.6) is abelian under a certain resonance condition for the equivariant parameters. This is analogous to the corresponding property of the quantum differential equation of the Hilbert scheme of points in the plane, see [OP].

Notice that Proposition 6.20 provides an equivariant Gamma theorem for a partial flag variety \mathcal{F}_λ , cf. Theorem B.2 and formula (11.19) in [TV6]. Notice also that in Theorems 4.8, 4.14, 5.12, 5.15 we prove different determinant formulas which imply that the functions entering the determinants form bases in the spaces of solutions of the corresponding differential and difference equations. The Levelt fundamental solutions are discussed in Sections 4.7 and 5.5.

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2. DYNAMICAL AND qKZ EQUATIONS

2.1. Notations. Fix $N, n \in \mathbb{Z}_{>0}$. Let $\boldsymbol{\lambda} \in \mathbb{Z}_{\geq 0}^N$, $|\boldsymbol{\lambda}| = \lambda_1 + \dots + \lambda_N = n$. Let $I = (I_1, \dots, I_N)$ be a partition of $\{1, \dots, n\}$ into disjoint subsets I_1, \dots, I_N . Denote by $\mathcal{I}_{\boldsymbol{\lambda}}$ the set of all partitions I with $|I_j| = \lambda_j$, $j = 1, \dots, N$.

Consider the space \mathbb{C}^N with a basis $v_i = (0, \dots, 0, 1_i, 0, \dots, 0)$, $i = 1, \dots, N$, and the tensor product $(\mathbb{C}^N)^{\otimes n}$ with basis $(v_I)_I$,

$$v_I = v_{i_1} \otimes \dots \otimes v_{i_n},$$

where $i_j = m$ if $j \in I_m$.

The space $(\mathbb{C}^N)^{\otimes n}$ is a module over the Lie algebra \mathfrak{gl}_N with basis $e_{i,j}$, $i, j = 1, \dots, N$. The \mathfrak{gl}_N -module $(\mathbb{C}^N)^{\otimes n}$ has weight decomposition $(\mathbb{C}^N)^{\otimes n} = \sum_{|\boldsymbol{\lambda}|=n} (\mathbb{C}^N)_{\boldsymbol{\lambda}}^{\otimes n}$, where $(\mathbb{C}^N)_{\boldsymbol{\lambda}}^{\otimes n}$ is the subspace with basis $(v_I)_{I \in \mathcal{I}_{\boldsymbol{\lambda}}}$.

2.2. Difference qKZ equations. Define the R -matrix acting on $(\mathbb{C}^N)^{\otimes 2}$,

$$R(u; h) = \frac{u - hP}{u - h},$$

where P is the permutation of factors of $(\mathbb{C}^N)^{\otimes 2}$ and $u, h \in \mathbb{C}$. The R -matrix satisfies the Yang-Baxter and unitarity equations,

$$(2.1) \quad R^{(12)}(u - v; h) R^{(13)}(u; h) R^{(23)}(v; h) = R^{(23)}(v; h) R^{(13)}(u; h) R^{(12)}(u - v; h),$$

$$(2.2) \quad R^{(12)}(u; h) R^{(21)}(-u; h) = 1.$$

The first equation is an equation in $\text{End}((\mathbb{C}^N)^{\otimes 3})$. The superscript indicates the factors of $(\mathbb{C}^N)^{\otimes 3}$ on which the corresponding operators act.

Let $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$, $\mathbf{q} = (q_1, \dots, q_N) \in \mathbb{C}^N$, and $\kappa \in \mathbb{C}$. Define the qKZ operators K_1, \dots, K_n acting on $(\mathbb{C}^N)^{\otimes n}$:

$$(2.3) \quad K_a(\mathbf{z}; h; \mathbf{q}; \kappa) = R^{(a, a-1)}(z_a - z_{a-1} + \kappa; h) \dots R^{(a, 1)}(z_a - z_1 + \kappa; h) \times \\ \times q_1^{e_{1,1}^{(a)}} \dots q_N^{e_{N,N}^{(a)}} R^{(a, n)}(z_a - z_n; h) \dots R^{(a, a+1)}(z_i - z_{i+1}; h).$$

The qKZ operators preserve the weight decomposition of $(\mathbb{C}^N)^{\otimes n}$ and form a discrete flat connection with parameter κ ,

$$K_a(z_1, \dots, z_b + \kappa, \dots, z_n; h; \mathbf{q}; \kappa) K_b(\mathbf{z}; h; \mathbf{q}; \kappa) = \\ = K_b(z_1, \dots, z_a + \kappa, \dots, z_n; h; \mathbf{q}; \kappa) K_a(\mathbf{z}; h; \mathbf{q}; \kappa)$$

for all $a, b = 1, \dots, n$, see [FR]. The system of difference equations with step κ ,

$$(2.4) \quad f(z_1, \dots, z_a + \kappa, \dots, z_n; h; \mathbf{q}; \kappa) = K_a(\mathbf{z}; h; \mathbf{q}; \kappa) f(\mathbf{z}; h; \mathbf{q}; \kappa), \quad a = 1, \dots, n,$$

for a $(\mathbb{C}^N)^{\otimes n}$ -valued function $f(\mathbf{z}, h, \mathbf{q}; \kappa)$ is called the *qKZ equations*.

2.3. Differential dynamical equations. Define the linear operators X_1, \dots, X_N acting on $(\mathbb{C}^N)^{\otimes n}$, called the dynamical Hamiltonians:

$$X_i(\mathbf{z}; h; \mathbf{q}) = \sum_{a=1}^n z_a e_{i,i}^{(a)} - h \left(\frac{\tilde{e}_{i,i}(1 - \tilde{e}_{i,i})}{2} + \sum_{1 \leq a < b \leq n} \sum_{k=1}^N e_{i,k}^{(a)} e_{k,i}^{(b)} + \sum_{\substack{j=1 \\ j \neq i}}^N \frac{q_j}{q_i - q_j} (\tilde{e}_{i,j} \tilde{e}_{j,i} - \tilde{e}_{i,i}) \right),$$

where $\tilde{e}_{s,t} = \sum_{a=1}^n e_{s,t}^{(a)}$. The differential operators

$$(2.5) \quad \nabla_{\mathbf{q}, \kappa, i} = \kappa q_i \frac{\partial}{\partial q_i} - X_i(\mathbf{z}; h; \mathbf{q}), \quad i = 1, \dots, N,$$

preserve the weight decomposition of $(\mathbb{C}^N)^{\otimes n}$ and pairwise commute, see [TV3], also [GRTV, Section 3.4], [RTV1, Section 7.1], [MTV]. The operators $\nabla_{\mathbf{q}, \kappa, i}$ define the $(\mathbb{C}^N)^{\otimes n}$ -valued *dynamical connection*.

The system of differential equations

$$(2.6) \quad \kappa q_i \frac{\partial f}{\partial q_i} = X_i(\mathbf{z}; h; \mathbf{q}) f, \quad i = 1, \dots, N,$$

for a $(\mathbb{C}^N)^{\otimes n}$ -valued function $f(\mathbf{z}; h; \mathbf{q}; \kappa)$ is called the *dynamical equations*.

Theorem 2.1 ([TV3]). *The joint system of dynamical and qKZ equations with the same parameter κ is compatible.*

Lemma 2.2. *The dynamical Hamiltonians X_1, \dots, X_N acting on $(\mathbb{C}^N)^{\otimes n}$ can be written in the form*

$$(2.7) \quad X_i(\mathbf{z}; h; \mathbf{q}) = \sum_{a=1}^n z_a e_{i,i}^{(a)} - h \sum_{1 \leq a < b \leq n} \sum_{\substack{j=1 \\ j \neq i}}^N \left(\frac{e_{i,j}^{(a)} e_{j,i}^{(b)}}{1 - q_j/q_i} - \frac{e_{j,i}^{(a)} e_{i,j}^{(b)}}{1 - q_i/q_j} \right).$$

Proof. The proof is by direct verification. □

3. LIMIT $h \rightarrow \infty$

3.1. Limit of qKZ operators. Introduce the R -matrix $R^\circ(u)$ acting on $(\mathbb{C}^N)^{\otimes 2}$:

$$R^\circ(u) = P - u \sum_{1 \leq i < j \leq N} e_{i,i} \otimes e_{j,j}.$$

For example, for $N = 2$ the matrix is

$$R^\circ(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -u & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Lemma 3.1. *We have*

$$(3.1) \quad R^\circ(u) = \lim_{h \rightarrow \infty} \left((-h)^{\sum_{i < j} e_{i,i}^{(1)} e_{j,j}^{(2)}} R(u; h) (-h)^{-\sum_{i < j} e_{j,j}^{(1)} e_{i,i}^{(2)}} \right).$$

Moreover, $R^\circ(u)$ satisfies the Yang-Baxter and unitarity equations,

$$(3.2) \quad (R^\circ(u-v))^{(12)} (R^\circ(u))^{(13)} (R^\circ(v))^{(23)} = (R^\circ(v))^{(23)} (R^\circ(u))^{(13)} (R^\circ(u-v))^{(12)},$$

$$(3.3) \quad (R^\circ(u))^{(12)} (R^\circ(-u))^{(21)} = 1.$$

Proof. The proof of formula (3.1) is straightforward. Equations (3.2), (3.3) follow from (3.1) and (2.1), (2.2), respectively. \square

Let $\mathbf{p} = (p_1, \dots, p_N) \in \mathbb{C}^N$. Define the q KZ operators $K_1^\circ, \dots, K_n^\circ$ acting on $(\mathbb{C}^N)^{\otimes n}$:

$$(3.4) \quad K_a^\circ(\mathbf{z}; \mathbf{p}; \kappa) = (R^\circ(z_a - z_{a-1} + \kappa))^{(a, a-1)} \dots (R^\circ(z_a - z_1 + \kappa))^{(a, 1)} \times \\ \times p_1^{e_{1,1}^{(a)}} \dots p_N^{e_{N,N}^{(a)}} (R^\circ(z_i - z_n))^{(a, n)} \dots (R^\circ(z_a - z_{a+1}))^{(a, a+1)}.$$

Lemma 3.2. *For $a = 1, \dots, n$, we have*

$$K_a^\circ(\mathbf{z}; \mathbf{p}; \kappa) = \lim_{h \rightarrow \infty} \left((-h)^{\sum_{b < c, i < j} e_{j,j}^{(b)} e_{i,i}^{(c)}} \tilde{K}_i(\mathbf{z}; h; \mathbf{p}; \kappa) (-h)^{-\sum_{b < c, i < j} e_{j,j}^{(b)} e_{i,i}^{(c)}} \right),$$

where the operators $\tilde{K}_1, \dots, \tilde{K}_n$ are obtained from the q KZ operators K_1, \dots, K_n , see (2.3), by the substitution

$$(3.5) \quad q_i \mapsto p_i (-h)^{\sum_{b=1}^n (\sum_{j>i} e_{j,j}^{(b)} - \sum_{j<i} e_{j,j}^{(b)})}, \quad i = 1, \dots, N.$$

Proof. The lemma is proved by direct verification using Lemma 3.1. \square

Corollary 3.3. *The q KZ operators $K_1^\circ, \dots, K_n^\circ$ preserve the weight decomposition of $(\mathbb{C}^N)^{\otimes n}$ and form a discrete flat connection,*

$$K_a^\circ(z_1, \dots, z_b + \kappa, \dots, z_n; \mathbf{p}; \kappa) K_b^\circ(\mathbf{z}; \mathbf{p}; \kappa) = K_b^\circ(z_1, \dots, z_a + \kappa, \dots, z_n; \mathbf{p}; \kappa) K_a^\circ(\mathbf{z}; \mathbf{p}; \kappa)$$

for all $a, b = 1, \dots, n$.

Remark. The R -matrix $R^\circ(u)$ for $N = 2$ is analogous to the R -matrix of the five-vertex model and the q -boson model, and to the R -matrices considered in [GK, GKS] in relation to the generalized quantum cohomology theory for Grassmannians.

Remark. Since $\det R^\circ(u) = 1$, the q KZ operators $K_1^\circ, \dots, K_n^\circ$ are invertible for all $\mathbf{z}, \mathbf{p}, \kappa$ provided $p_i \neq 0$ for all $i = 1, \dots, N$. That is, the discrete flat connection on $(\mathbb{C}^N)^{\otimes n}$ defined by $K_1^\circ, \dots, K_n^\circ$ is regular for all \mathbf{z} , unlike the discrete flat connection defined by the q KZ operators K_1, \dots, K_n , see (2.3).

3.2. Dynamical operators $X_1^\circ, \dots, X_N^\circ$. Recall $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)$. Let $\lambda^{(i)} = \sum_{j=1}^i \lambda_j$, $i = 1, \dots, N$, so that $\lambda^{(N)} = n$. For $1 \leq i < j \leq n$, set $\lambda_{\langle i, j \rangle} = \sum_{k=i}^{j-1} (\lambda_k + \lambda_{k+1}) = \lambda_i + \lambda_j + 2 \sum_{k=i+1}^{j-1} \lambda_k$.

Recall $I = (I_1, \dots, I_N)$ and the vectors $v_I \in (\mathbb{C}^N)^{\otimes n}$. For $\sigma \in S_n$ and $I = (I_1, \dots, I_N)$, set $\sigma(I) = (\sigma(I_1), \dots, \sigma(I_N))$. For $a, b = 1, \dots, n$, let $s_{a,b}$ be the transposition of a, b .

Given $I \in \mathcal{I}_\lambda$, let $a \in I_i$ and $b \in I_j$. A pair (a, b) is called *I-disordered* if either $a < b$, $i > j$, or $a > b$, $i < j$. A pair (a, b) is called *I-ordered* if either $a < b$, $i < j$, or $a > b$, $i > j$. A pair (a, b) is called *I-flat* if $i = j$.

Denote $M = \{\min(a, b) + 1, \dots, \max(a, b) - 1\}$, $k = \min(i, j)$, $l = \max(i, j)$. If $i \neq j$, so that $k < l$, set $m_{a,b,I} = \lambda_{\langle k,l \rangle}$ and

$$r_{a,b,I} = |M \cap I_i| + |M \cap I_j| + 2 \left| M \cap \bigcup_{r=k+1}^{l-1} I_r \right|.$$

Clearly, $m_{a,b,I} \geq 2$ and $0 \leq r_{a,b,I} \leq m_{a,b,I} - 2$ if $i \neq j$. Set $I_{[a,b]} = \bigcup_{r=k}^l I_r$.

Set

$$(3.6) \quad I_\lambda^{\min} = ((\lambda_1, \dots, \lambda^{(1)}), (\lambda^{(1)} + 1, \dots, \lambda^{(2)}), \dots, (\lambda^{(N-1)} + 1, \dots, n)).$$

For $I \in \mathcal{I}_\lambda$, let $\sigma_I \in S_n$ be the element of minimal length such that $\sigma_I(I_\lambda^{\min}) = I$. Notice that

$$(3.7) \quad |\sigma_I| = |\{(a, b) \mid a \in I_i, b \in I_j, a < b, i > j\}|.$$

Lemma 3.4. *We have*

- a) *If the pair (a, b) is I-disordered, then $|\sigma_{s_{a,b}(I)}| = |\sigma_I| - r_{a,b,I} - 1$.*
- b) *If the pair (a, b) is I-ordered, then $|\sigma_{s_{a,b}(I)}| = |\sigma_I| + r_{a,b,I} + 1$.*
- c) *If the pair (a, b) is I-flat, then $s_{a,b}(I) = I$.*

Corollary 3.5. *If $|\sigma_{s_{a,b}(I)}| < |\sigma_I|$, then the pair (a, b) is I-disordered. If $|\sigma_{s_{a,b}(I)}| > |\sigma_I|$, then the pair (a, b) is I-ordered.*

A pair (a, b) is called *I-admissible of the first kind* if $|\sigma_{s_{a,b}(I)}| = |\sigma_I| - 1$, and *I-admissible of the second kind* if $|\sigma_{s_{a,b}(I)}| = |\sigma_I| + m_{a,b,I} - 1$.

Lemma 3.6. *A pair (a, b) is I-admissible of the first kind if and only if it is I-disordered and the intersection $\{\min(a, b) + 1, \dots, \max(a, b) - 1\} \cap I_{[a,b]}$ is empty.*

Lemma 3.7. *A pair (a, b) is I-admissible of the second kind if and only if it is I-ordered and the intersection $\{1, \dots, \min(a, b) - 1, \max(a, b) + 1, \dots, n\} \cap I_{[a,b]}$ is empty.*

Example. Let $N = 4$, $n = 5$, $\lambda = (2, 1, 1, 1)$, $I = (\{1, 3\}, \{4\}, \{2\}, \{5\})$. Then the *I*-admissible pairs of the first kind are $(3, 2)$, $(2, 3)$, $(4, 2)$, $(2, 4)$, and the *I*-admissible pairs of the second kind are $(1, 4)$, $(4, 1)$, $(1, 5)$, $(5, 1)$, $(2, 5)$, $(5, 2)$.

For all i, j, a, b , define linear operators $Q_{i,j}^{a,b}$ acting on $(\mathbb{C}^N)^{\otimes n}$ by the rule

$$\begin{aligned} Q_{i,j}^{a,b} v_I &= v_{s_{a,b}(I)}, & \text{if } a \in I_i, b \in I_j, \text{ and the pair } (a, b) \text{ is } I\text{-admissible,} \\ Q_{i,j}^{a,b} v_I &= 0, & \text{otherwise.} \end{aligned}$$

Recall the linear operators $e_{i,i}^{(a)}$ acting on $(\mathbb{C}^N)^{\otimes n}$.

The dynamical operators $X_1^\circ, \dots, X_N^\circ$, acting on $(\mathbb{C}^N)^{\otimes n}$ are given by the formula

$$(3.8) \quad X_i^\circ(\mathbf{z}; \mathbf{p}) = \sum_{a=1}^n z_a e_{i,i}^{(a)} + \sum_{1 \leq b < a \leq n} \left(\sum_{j=i+1}^N Q_{i,j}^{a,b} - \sum_{j=1}^{i-1} \frac{p_i}{p_j} Q_{i,j}^{a,b} \right) + \\ + \sum_{1 \leq a < b \leq n} \left(\sum_{j=i+1}^N \frac{p_j}{p_i} Q_{i,j}^{a,b} - \sum_{j=1}^{i-1} Q_{i,j}^{a,b} \right).$$

Notice that the operators $X_1^\circ, \dots, X_N^\circ$ preserve the weight decomposition of $(\mathbb{C}^N)^{\otimes n}$.

Example. Let $N = 2$, $n = 3$. The operators X_1°, X_2° preserve the subspace spanned by the vectors

$$v_{\{\{1\}, \{2,3\}\}} = v_1 \otimes v_2 \otimes v_2, \quad v_{\{\{2\}, \{1,3\}\}} = v_2 \otimes v_1 \otimes v_2, \quad v_{\{\{3\}, \{1,2\}\}} = v_2 \otimes v_2 \otimes v_1.$$

of weight $\lambda = (1, 2)$. The matrices of X_1°, X_2° in this basis are

$$X_1^\circ = \begin{pmatrix} z_1 & 1 & 0 \\ 0 & z_2 & 1 \\ p_2/p_1 & 0 & z_3 \end{pmatrix}, \quad X_2^\circ = \begin{pmatrix} z_2 + z_3 & -1 & 0 \\ 0 & z_1 + z_3 & -1 \\ -p_2/p_1 & 0 & z_1 + z_2 \end{pmatrix}.$$

Example. Let $N = 5$, $n = 6$, $I = (\{2, 5\}, \{6\}, \{3\}, \{1\}, \{4\})$. Then

$$X_1^\circ(\mathbf{z}; \mathbf{p}) v_I = (z_2 + z_5) v_I + v_{\{\{1,5\}, \{6\}, \{3\}, \{2\}, \{4\}\}} + v_{\{\{2,4\}, \{6\}, \{3\}, \{1\}, \{5\}\}} + \\ + v_{\{\{2,3\}, \{6\}, \{5\}, \{1\}, \{4\}\}} + \frac{p_2}{p_1} v_{\{\{5,6\}, \{2\}, \{3\}, \{1\}, \{4\}\}}, \\ X_2^\circ(\mathbf{z}; \mathbf{p}) v_I = z_5 v_I + v_{\{\{2,5\}, \{4\}, \{3\}, \{1\}, \{6\}\}} + v_{\{\{2,5\}, \{3\}, \{6\}, \{1\}, \{4\}\}} - \frac{p_2}{p_1} v_{\{\{5,6\}, \{2\}, \{3\}, \{1\}, \{4\}\}}.$$

3.3. Limit of dynamical Hamiltonians. Let the operators $\tilde{X}_1, \dots, \tilde{X}_N$ be obtained from the dynamical Hamiltonians X_1, \dots, X_N , see (2.7), by substitution (3.5). In more detail, the operators $\tilde{X}_1, \dots, \tilde{X}_N$ preserve the weight decomposition of $(\mathbb{C}^N)^{\otimes n}$ and the action of $\tilde{X}_i(\mathbf{z}; h; \mathbf{p})$ on $(\mathbb{C}^N)_\lambda^{\otimes n}$ coincides with the action of $X_i(\mathbf{z}; h; \mathbf{q})$ for $\mathbf{q} = (q_1, \dots, q_N)$,

$$(3.9) \quad q_j = p_j (-h)^{\sum_{k>j} \lambda_k - \sum_{k<j} \lambda_k}, \quad j = 1, \dots, N.$$

Lemma 3.8. For $i = 1, \dots, N$, we have

$$(3.10) \quad X_i^\circ(\mathbf{z}; \mathbf{p}) = \lim_{h \rightarrow \infty} \left((-h)^{\sum_{b<c, j<k} e_{k,k}^{(b)} e_{j,j}^{(c)}} \tilde{X}_i(\mathbf{z}; h; \mathbf{p}) (-h)^{-\sum_{b<c, j<k} e_{k,k}^{(b)} e_{j,j}^{(c)}} \right).$$

Proof. The lemma is proved by direct verification using Lemma 2.2, the equality

$$\sum_{b<c, j<k} e_{k,k}^{(b)} e_{j,j}^{(c)} v_I = |\sigma_I| v_I,$$

following from (3.7), and Lemma 3.4. □

Corollary 3.9. *The differential operators*

$$(3.11) \quad \nabla_{\mathbf{p}, \kappa, i}^\circ = \kappa p_i \frac{\partial}{\partial p_i} - X_i^\circ(\mathbf{z}; \mathbf{p}), \quad i = 1, \dots, N,$$

preserve the weight decomposition of $(\mathbb{C}^N)^{\otimes n}$ and pairwise commute.

3.4. Limiting equations. The system of difference equations with step κ ,

$$(3.12) \quad f(z_1, \dots, z_a + \kappa, \dots, z_n; \mathbf{p}) = K_a^\circ(\mathbf{z}; \mathbf{p}; \kappa) f(\mathbf{z}; \mathbf{p}), \quad a = 1, \dots, n,$$

for a $(\mathbb{C}^N)^{\otimes n}$ -valued function $f(\mathbf{z}, \mathbf{p})$ will be called the *limiting qKZ equations*.

The system of differential equations with parameter κ ,

$$(3.13) \quad \kappa p_i \frac{\partial f}{\partial p_i} = X_i^\circ(\mathbf{z}; \mathbf{p}) f, \quad i = 1, \dots, N,$$

for a $(\mathbb{C}^N)^{\otimes n}$ -valued function $f(\mathbf{z}; \mathbf{p})$ will be called the *limiting dynamical equations*.

Theorem 3.10. *The joint systems of limiting dynamical and limiting qKZ equations with the same parameter κ is compatible.*

Proof. The theorem follows from Theorem 2.1 and Lemmas 3.2, 3.8. □

4. HYPERGEOMETRIC SOLUTIONS OF THE DYNAMICAL AND qKZ EQUATIONS

4.1. Weight functions $W_I(\mathbf{t}; \mathbf{z}; h)$. For $I \in \mathcal{I}_\lambda$, we define the weight functions $W_I(\mathbf{t}; \mathbf{z}; h)$, cf. [TV1, TV4, RTV1, TV6]. The functions $W_I(\mathbf{t}; \mathbf{z}; h)$ here coincide with those defined in [TV6, Section 3.4].

Recall $\lambda = (\lambda_1, \dots, \lambda_N)$ and $I = (I_1, \dots, I_N)$. Set $\bigcup_{k=1}^j I_k = \{i_1^{(j)} < \dots < i_{\lambda^{(j)}}^{(j)}\}$. Consider the variables $t_a^{(j)}$, $j = 1, \dots, N-1$, $a = 1, \dots, \lambda^{(j)}$. Set $t_a^{(N)} = z_a$, $a = 1, \dots, n$. Denote $\mathbf{t}^{(j)} = (t_1^{(j)}, \dots, t_{\lambda^{(j)}}^{(j)})$ and $\mathbf{t} = (\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(N-1)})$.

For $I \in \mathcal{I}_\lambda$, define

$$(4.1) \quad \Sigma_I = (z_{i_1^{(1)}}^{(1)}, \dots, z_{i_{\lambda^{(1)}}^{(1)}}^{(1)}, z_{i_1^{(2)}}^{(2)}, \dots, z_{i_{\lambda^{(2)}}^{(2)}}^{(2)}, \dots, z_{i_1^{(N-1)}}^{(N-1)}, \dots, z_{i_{\lambda^{(N-1)}}^{(N-1)}}^{(N-1)}),$$

so that, $\mathbf{t} = \Sigma_I$ reads in detail as

$$t_a^{(k)} = z_{i_a^{(k)}}, \quad k = 1, \dots, N-1, \quad i = a, \dots, \lambda^{(k)}.$$

For a function $f(x_1, \dots, x_k)$ of some variables, denote

$$\text{Sym}_{x_1, \dots, x_k} f(x_1, \dots, x_k) = \sum_{\sigma \in S_k} f(x_{\sigma(1)}, \dots, x_{\sigma(k)}).$$

The weight functions are

$$(4.2) \quad W_I(\mathbf{t}; \mathbf{z}; h) = \text{Sym}_{t_1^{(1)}, \dots, t_{\lambda^{(1)}}^{(1)}} \dots \text{Sym}_{t_1^{(N-1)}, \dots, t_{\lambda^{(N-1)}}^{(N-1)}} U_I(\mathbf{t}; \mathbf{z}; h),$$

$$\begin{aligned}
U_I(\mathbf{t}; \mathbf{z}; h) &= \\
&= \prod_{j=1}^{N-1} \prod_{a=1}^{\lambda^{(j)}} \left(\prod_{\substack{c=1 \\ i_c^{(j+1)} < i_a^{(j)}}}^{\lambda^{(j+1)}} (t_a^{(j)} - t_c^{(j+1)}) \prod_{\substack{d=1 \\ i_d^{(j+1)} > i_a^{(j)}}}^{\lambda^{(j+1)}} (t_a^{(j)} - t_d^{(j+1)} - h) \prod_{b=a+1}^{\lambda^{(j)}} \frac{t_b^{(j)} - t_a^{(j)} - h}{t_b^{(j)} - t_a^{(j)}} \right).
\end{aligned}$$

Example. Let $N = 2$, $n = 2$, $\boldsymbol{\lambda} = (1, 1)$, $I = (\{1\}, \{2\})$, $J = (\{2\}, \{1\})$. Then

$$W_I(\mathbf{t}; \mathbf{z}; h) = t_1^{(1)} - z_2 - h, \quad W_J(\mathbf{t}; \mathbf{z}; h) = t_1^{(1)} - z_1.$$

Lemma 4.1 ([RTV1, TV6]). *For any $I \in \mathcal{I}_\lambda$, $i = 1, \dots, N-1$, and $a = 1, \dots, n-1$, we have*

$$\begin{aligned}
(4.3) \quad W_{s_{a,a+1}(I)}(\mathbf{t}; \mathbf{z}; h) &= \\
&= \frac{z_a - z_{a+1} - h}{z_a - z_{a+1}} W_I(\mathbf{t}; z_1, \dots, z_{a+1}, z_a, \dots, z_n; h) + \frac{h}{z_a - z_{a+1}} W_I(\mathbf{t}; \mathbf{z}; h).
\end{aligned}$$

Remark. Define the operators $\hat{s}_1, \dots, \hat{s}_{n-1}$ acting on functions of z_1, \dots, z_n :

$$(4.4) \quad \hat{s}_a f(\mathbf{z}) = \frac{z_a - z_{a+1} - h}{z_a - z_{a+1}} f(z_1, \dots, z_{a+1}, z_a, \dots, z_n) + \frac{h}{z_a - z_{a+1}} f(\mathbf{z}).$$

The assignment $s_{a,a+1} \mapsto \hat{s}_a$, $a = 1, \dots, n-1$, yields a representation of the symmetric group S_n .

$$\begin{aligned}
\text{Set} \\
(4.5) \quad c_\lambda(\mathbf{t}; h) &= \prod_{i=1}^{N-1} \prod_{\substack{a=1 \\ b \neq a}}^{\lambda^{(i)}} \prod_{b=1}^{\lambda^{(i)}} (t_a^{(i)} - t_b^{(i)} - h).
\end{aligned}$$

Recall the permutations σ_I , $I \in \mathcal{I}_\lambda$, defined in Section 3.2.

Lemma 4.2 ([RTV1]). *We have $W_J(\Sigma_I; \mathbf{z}; h) = 0$ unless $I = J$ or $|\sigma_I| > |\sigma_J|$, and*

$$(4.6) \quad W_I(\Sigma_I; \mathbf{z}; h) = c_\lambda(\Sigma_I; h) \prod_{j=1}^{N-1} \prod_{k=j+1}^N \prod_{a \in I_j} \left(\prod_{\substack{b \in I_k \\ b < a}} (z_a - z_b) \prod_{\substack{b \in I_k \\ b > a}} (z_a - z_b - h) \right).$$

Lemma 4.3. *For any $J \in \mathcal{I}_\lambda$, the polynomial $W_J(\Sigma_I; \mathbf{z}; h)$ is divisible by*

$$c_\lambda(\Sigma_I; h) \prod_{j=1}^{N-1} \prod_{k=j+1}^N \prod_{a \in I_j} \prod_{\substack{b \in I_k \\ b > a}} (z_a - z_b - h).$$

Lemma 4.3 is a version of [RTV2, Lemma 3.1, item (I)]. A stronger more technical analogue of Lemma 4.3 is given by Lemma 4.11 in Section 4.5.

Let σ_0 be the longest permutation, $\sigma_0(i) = n+1-i$, $i = 1, \dots, n$. For $I \in \mathcal{I}_\lambda$, define

$$(4.7) \quad \check{W}_I(\mathbf{t}; \mathbf{z}; h) = W_{\sigma_0(I)}(\mathbf{t}; z_n, \dots, z_1; h).$$

Set

$$(4.8) \quad R_\lambda(\mathbf{z}) = \prod_{i=1}^{N-1} \prod_{a=1}^{\lambda^{(i)}} \prod_{b=\lambda^{(i)}+1}^{\lambda^{(i+1)}} (z_a - z_b), \quad Q_\lambda(\mathbf{z}; h) = \prod_{i=1}^{N-1} \prod_{a=1}^{\lambda^{(i)}} \prod_{b=\lambda^{(i)}+1}^{\lambda^{(i+1)}} (z_a - z_b - h),$$

For $\sigma \in S_n$, denote $\mathbf{z}_\sigma = (z_{\sigma(1)}, \dots, z_{\sigma(n)})$.

Proposition 4.4 ([RTV1, Lemma 3.4]). *The functions $W_I(\mathbf{t}; \mathbf{z}; h)$ and $\check{W}_J(\mathbf{t}; \mathbf{z}; h)$ are biorthogonal,*

$$(4.9) \quad \sum_{I \in \mathcal{I}_\lambda} \frac{W_J(\Sigma_I; \mathbf{z}; h) \check{W}_K(\Sigma_I; \mathbf{z}; h)}{c_\lambda^2(\Sigma_I; h) R_\lambda(\mathbf{z}_{\sigma_I}) Q_\lambda(\mathbf{z}_{\sigma_I}; h)} = \delta_{J,K}.$$

Corollary 4.5. *We have*

$$(4.10) \quad \sum_{I \in \mathcal{I}_\lambda} W_I(\Sigma_J; \mathbf{z}; h) \check{W}_I(\Sigma_K; \mathbf{z}; h) = \delta_{J,K} c_\lambda^2(\Sigma_J; h) R_\lambda(\mathbf{z}_{\sigma_J}) Q_\lambda(\mathbf{z}_{\sigma_J}; h).$$

4.2. Master function. Let $\phi(x, h, \kappa) = \Gamma(x/\kappa) \Gamma((h-x)/\kappa)$. Define the *master function*:

$$(4.11) \quad \Phi_\lambda(\mathbf{t}; \mathbf{z}; h; \mathbf{q}; \kappa) = (e^{\pi\sqrt{-1}(n-\lambda_N)} q_N)^{\sum_{a=1}^n z_a/\kappa} \prod_{i=1}^{N-1} \left(e^{\pi\sqrt{-1}(\lambda_{i+1}-\lambda_i)} \frac{q_i}{q_{i+1}} \right)^{\sum_{a=1}^{\lambda^{(i)}} t_a^{(i)}/\kappa} \times \\ \times \prod_{i=1}^{N-1} \prod_{a=1}^{\lambda^{(i)}} \left(\prod_{\substack{b=1 \\ b \neq a}}^{\lambda^{(i)}} \frac{1}{(t_a^{(i)} - t_b^{(i)} - h) \phi(t_a^{(i)} - t_b^{(i)}, h, \kappa)} \prod_{c=1}^{\lambda^{(i+1)}} \phi(t_a^{(i)} - t_c^{(i+1)}, h, \kappa) \right),$$

where $\lambda^{(N)} = n$ and $t_a^{(N)} = z_a$, $a = 1, \dots, n$. The function $\Phi_\lambda(\mathbf{t}; \mathbf{z}; h; \mathbf{q}; \kappa)$ is symmetric in the variables $t_1^{(i)}, \dots, t_{\lambda^{(i)}}^{(i)}$ for each $i = 1, \dots, N-1$.

4.3. Jackson integrals. Let $\lambda^{\{1\}} = \sum_{i=1}^{N-1} \lambda^{(i)}$. Consider the space $\mathbb{C}^{\lambda^{\{1\}}} \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^N$ with coordinates $\mathbf{t}, \mathbf{z}, h, \mathbf{q}$. The lattice $\kappa \mathbb{Z}^{\lambda^{\{1\}}}$ naturally acts on this space by shifting the \mathbf{t} -coordinates.

For a function of $f(\mathbf{t})$ and a point $\mathbf{s} \in \mathbb{C}^{\lambda^{\{1\}}}$, define $\text{Res}_{\mathbf{t}=\mathbf{s}} f(\mathbf{t})$ to be the iterated residue,

$$\text{Res}_{\mathbf{t}=\mathbf{s}} f(\mathbf{t}) = \text{Res}_{t_1^{(1)}=s_1^{(1)}} \dots \text{Res}_{t_{\lambda^{(1)}}^{(1)}=s_{\lambda^{(1)}}^{(1)}} \dots \text{Res}_{t_1^{(N-1)}=s_1^{(N-1)}} \dots \text{Res}_{t_{\lambda^{(N-1)}}^{(N-1)}=s_{\lambda^{(N-1)}}^{(N-1)}} f(\mathbf{t}).$$

Let L' be the complement in $\mathbb{C}^n \times \mathbb{C}$ of the union of the hyperplanes

$$(4.12) \quad h = m\kappa, \quad z_a - z_b = m\kappa, \quad z_a - z_b + h = m\kappa,$$

for all $a, b = 1, \dots, n$, $a \neq b$, and all $m \in \mathbb{Z}_{\leq 0}$. Let $L'' \subset \mathbb{C}^N$ be the domain

$$(4.13) \quad |q_{i+1}/q_i| < 1, \quad i = 1, \dots, N-1,$$

with additional cuts fixing a branch of $\log q_i$ for all $i = 1, \dots, N$. Set $L = L' \times L'' \subset \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^N$.

Let $\mathbf{l} = (l_1^{(1)}, \dots, l_{\lambda^{(1)}}^{(1)}, \dots, l_1^{(N-1)}, \dots, l_{\lambda^{(N-1)}}^{(N-1)}) \in \mathbb{Z}^{\lambda^{\{1\}}}$. By convention, set $l_a^{(i)} = 0$ for $a > \lambda^{(i)}$ or $i = N$.

Let $f(\mathbf{t}; \mathbf{z}; h; \mathbf{q})$ be a polynomial in \mathbf{t} and a holomorphic function of $\mathbf{z}, h, \mathbf{q}$ in L . For $(\mathbf{z}; h; \mathbf{q}) \in L$, define

$$(4.14) \quad \mathcal{M}_J(\Phi_\lambda f)(\mathbf{z}; h; \mathbf{q}; \kappa) = \sum_{\mathbf{l} \in \mathbb{Z}^{\lambda^{\{1\}}}} \text{Res}_{\mathbf{t}=\Sigma_J - \mathbf{l}\kappa} (\Phi_\lambda(\mathbf{t}; \mathbf{z}; h; \mathbf{q}; \kappa) f(\mathbf{t}; \mathbf{z}; h; \mathbf{q})).$$

This sum is called the *Jackson integral over the discrete cycle* $\widehat{\Sigma}_J \subset \mathbb{C}^{\lambda^{\{1\}}} \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^N$,

$$\widehat{\Sigma}_J = \{(\Sigma_J - \mathbf{l}\kappa; \mathbf{z}; h; \mathbf{q}) \mid \mathbf{l} \in \mathbb{Z}^{\lambda^{\{1\}}}, (\mathbf{z}; h; \mathbf{q}) \in L\}.$$

Notice that the master function $\Phi_\lambda(\mathbf{t}; \mathbf{z}; h; \mathbf{q}; \kappa)$ has only simple poles, and

$$\begin{aligned} \text{Res}_{\mathbf{t}=\Sigma_J - \mathbf{l}\kappa} (\Phi_\lambda(\mathbf{t}; \mathbf{z}; h; \mathbf{q}; \kappa) f(\mathbf{t}; \mathbf{z}; h; \mathbf{q})) &= \\ &= f(\Sigma_J - \mathbf{l}\kappa; \mathbf{z}; h; \mathbf{q}) \text{Res}_{\mathbf{t}=\Sigma_J - \mathbf{l}\kappa} \Phi_\lambda(\mathbf{t}; \mathbf{z}; h; \mathbf{q}; \kappa). \end{aligned}$$

A closed expression for the residue $\text{Res}_{\mathbf{t}=\Sigma_J - \mathbf{l}\kappa} \Phi_\lambda(\mathbf{t}; \mathbf{z}; h; \mathbf{q}; \kappa)$ is given by Lemma 4.6 below.

Recall $c_\lambda(\mathbf{t}; h)$, see (4.5). Set

$$(4.15) \quad M_\lambda(\mathbf{z}; \kappa) = \prod_{i=1}^{N-1} \prod_{a=1}^{\lambda^{(i)}} \prod_{b=\lambda^{(i)}+1}^{\lambda^{(i+1)}} \frac{\sin(\pi(z_a - z_b)/\kappa)}{\pi e^{\pi\sqrt{-1}(z_a+z_b)/\kappa}},$$

and

$$(4.16) \quad A_\lambda(\mathbf{t}; \mathbf{z}; h; \kappa) = \prod_{i=1}^{N-1} \prod_{a=1}^{\lambda^{(i)}} \left(\prod_{\substack{b=1 \\ b \neq a}}^{\lambda^{(i)}} \frac{\Gamma(1 + (t_b^{(i)} - t_a^{(i)})/\kappa)}{\Gamma((t_b^{(i)} - t_a^{(i)} + h)/\kappa)} \prod_{c=1}^{\lambda^{(i+1)}} \frac{\Gamma((t_c^{(i+1)} - t_a^{(i)} + h)/\kappa)}{\Gamma(1 + (t_c^{(i+1)} - t_a^{(i)})/\kappa)} \right).$$

Lemma 4.6. *If $\mathbf{l} \notin \mathbb{Z}_{\geq 0}^{\lambda^{\{1\}}}$, then $\text{Res}_{\mathbf{t}=\Sigma_J - \mathbf{l}\kappa} \Phi_\lambda(\mathbf{t}; \mathbf{z}; h; \mathbf{q}; \kappa) = 0$. For $\mathbf{l} \in \mathbb{Z}_{\geq 0}^{\lambda^{\{1\}}}$,*

$$(4.17) \quad \begin{aligned} \text{Res}_{\mathbf{t}=\Sigma_J - \mathbf{l}\kappa} \Phi_\lambda(\mathbf{t}; \mathbf{z}; h; \mathbf{q}; \kappa) &= \\ &= \frac{\kappa^{\lambda^{\{1\}}} A_\lambda(\Sigma_J - \mathbf{l}\kappa; \mathbf{z}; h; \kappa)}{M_\lambda(\mathbf{z}_{\sigma_J}; \kappa) c_\lambda(\Sigma_J - \mathbf{l}\kappa; h)} \prod_{i=1}^N q_i^{\sum_{a \in J_i} z_a/\kappa} \prod_{i=1}^{N-1} (q_{i+1}/q_i)^{\sum_{a=1}^{\lambda^{(i)}} l_a^{(i)}}. \end{aligned}$$

In particular,

$$(4.18) \quad \begin{aligned} \text{Res}_{\mathbf{t}=\Sigma_J} \Phi_\lambda(\mathbf{t}; \mathbf{z}; h; \mathbf{q}; \kappa) &= \frac{(\kappa \Gamma(h/\kappa))^{\lambda^{\{1\}}}}{c_\lambda(\Sigma_J; h)} \prod_{i=1}^N (e^{\pi\sqrt{-1}(n-\lambda_i)} q_i)^{\sum_{a \in J_i} z_a/\kappa} \times \\ &\times \prod_{i=1}^{N-1} \prod_{j=i+1}^N \prod_{a \in J_i} \prod_{b \in J_j} \Gamma((z_a - z_b)/\kappa) \Gamma((z_b - z_a + h)/\kappa). \end{aligned}$$

By Lemma 4.6, the actual summation in formula (4.14) is only over the positive cone of the lattice,

$$(4.19) \quad \mathcal{M}_J(\Phi_\lambda f)(\mathbf{z}; h; \mathbf{q}; \kappa) = \sum_{\mathbf{l} \in \mathbb{Z}_{\geq 0}^{\lambda^{\{1\}}}} f(\Sigma_J - \mathbf{l}\kappa; \mathbf{z}; h; \mathbf{q}) \text{Res}_{\mathbf{t}=\Sigma_J - \mathbf{l}\kappa} \Phi_\lambda(\mathbf{t}; \mathbf{z}; h; \mathbf{q}; \kappa).$$

4.4. **Solutions of the dynamical and qKZ equations.** For $J \in \mathcal{I}_\lambda$, define

$$(4.20) \quad \Psi_J(\mathbf{z}; h; \mathbf{q}; \kappa) = (\kappa \Gamma(h/\kappa))^{-\lambda^{(1)}} \Omega_\lambda(h; \mathbf{q}; \kappa) \sum_{I \in \mathcal{I}_\lambda} \mathcal{M}_J(\Phi_\lambda W_I)(\mathbf{z}; h; \mathbf{q}; \kappa) v_I,$$

where

$$(4.21) \quad \Omega_\lambda(h; \mathbf{q}; \kappa) = \prod_{i=1}^{N-1} \prod_{j=i+1}^N (1 - q_j/q_i)^{h\lambda_i/\kappa}.$$

Say that a function $f(\mathbf{q})$ is *holomorphic in the unit polydisk around $\mathbf{q} = \mathbf{0}$* if $f(\mathbf{q}) = g(q_2/q_1, \dots, q_N/q_{N-1})$ for a function $g(s_1, \dots, s_{N-1})$ holomorphic in s_1, \dots, s_{N-1} , provided $|s_i| < 1$ for all $i = 1, \dots, N-1$. Denote $f(\mathbf{0}) = g(0, \dots, 0)$.

Theorem 4.7. *The $(\mathbb{C}^N)_\lambda^{\otimes n}$ -valued function $\Psi_J(\mathbf{z}; h; \mathbf{q}; \kappa)$ is a solution of the joint system of dynamical differential equations (2.6) and qKZ difference equations (2.4). It has the form*

$$(4.22) \quad \Psi_J(\mathbf{z}; h; \mathbf{q}; \kappa) = \Psi_J^\diamond(\mathbf{z}; h; \mathbf{q}; \kappa) \prod_{i=1}^N (e^{\pi\sqrt{-1}(n-\lambda_i)} q_i)^{\sum_{a \in J_i} z_a/\kappa} \times \\ \times \prod_{i=1}^{N-1} \prod_{j=i+1}^N \prod_{a \in J_i} \left(\prod_{b \in J_j} \frac{\Gamma(1 + (z_b - z_a + h)/\kappa)}{\sin(\pi(z_a - z_b)/\kappa)} \prod_{\substack{c \in J_j \\ c < a}} \frac{1}{z_a - z_c - h} \right),$$

where the function $\Psi_J^\diamond(\mathbf{z}; h; \mathbf{q}; \kappa)$ is entire in \mathbf{z}, h and holomorphic in \mathbf{q} in the unit polydisk around $\mathbf{q} = \mathbf{0}$. In more detail,

$$(4.23) \quad \Psi_J^\diamond(\mathbf{z}; h; \mathbf{q}; \kappa) = \prod_{i=1}^{N-1} \prod_{j=i+1}^N \prod_{a \in J_i} \prod_{b \in J_j} \frac{-\pi \kappa}{\Gamma(1 + (z_b - z_a)/\kappa)} \times \\ \times \left(\Psi_{J,0}^\diamond(\mathbf{z}; h) + \sum_{\substack{\mathbf{m} \in \mathbb{Z}_{\geq 0}^{N-1} \\ \mathbf{m} \neq \mathbf{0}}} \Psi_{J,\mathbf{m}}^\diamond(\mathbf{z}; h; \kappa) \prod_{i=1}^{N-1} (q_{i+1}/q_i)^{m_i} \right),$$

where

$$(4.24) \quad \Psi_{J,0}^\diamond(\mathbf{z}; h) = \frac{1}{c_\lambda(\Sigma_J; h)} \prod_{i=1}^{N-1} \prod_{j=i+1}^N \prod_{a \in J_i} \prod_{\substack{b \in J_j \\ b > a}} \frac{1}{z_a - z_b - h} \times \\ \times \left(W_J(\Sigma_J; \mathbf{z}; h) v_J + \sum_{\substack{I \in \mathcal{I}_\lambda \\ |\sigma_I| < |\sigma_J|}} W_I(\Sigma_J; \mathbf{z}; h) v_I \right)$$

is a polynomial in \mathbf{z}, h , and $\Psi_{J,\mathbf{m}}^\diamond(\mathbf{z}; h; \kappa)$ for $\mathbf{m} \neq \mathbf{0}$ are rational functions of \mathbf{z}, h, κ with at most simple poles on the hyperplanes $z_a - z_b \in \kappa \mathbb{Z}_{>0}$ for $a \in J_i, b \in J_j, 1 \leq i < j \leq N$. Furthermore, for any transposition $s_{a,b} \in S_n$,

$$(4.25) \quad \Psi_J^\diamond(\mathbf{z}; h; \mathbf{q}; \kappa) \Big|_{z_a=z_b} = \Psi_{s_{a,b}(J)}^\diamond(\mathbf{z}; h; \mathbf{q}; \kappa) \Big|_{z_a=z_b}.$$

Proof. The fact that $\Psi_J(\mathbf{z}; h; \mathbf{q}; \kappa)$ solves the dynamical equations (2.6) is proved in [TV6, Theorem 8.4], and the fact that $\Psi_J(\mathbf{z}; h; \mathbf{q}; \kappa)$ solves the qKZ equations (2.4) is proved in [TV1, Theorem 1.5.2], cf. [TV4].

Analytic properties of $\Psi_J^\circ(\mathbf{z}; h; \mathbf{q}; \kappa)$ are proved in Section 4.5. The fact that the right-hand side of formula (4.24) is a polynomial in \mathbf{z}, h follows from Lemmas 4.2, 4.3. \square

The functions $\Psi_J(\mathbf{z}; h; \mathbf{q}; \kappa)$ are called the *multidimensional hypergeometric solutions* of the dynamical equations. In [TV5], we constructed another type of solutions of the dynamical equations using multidimensional hypergeometric integrals.

The next theorem computes the determinant of coordinates of solutions $\Psi_J(\mathbf{z}; h; \mathbf{q}; \kappa)$ and is analogous to [TV6, Theorem 11.3].

Theorem 4.8. *Let $n \geq 2$. Then*

$$(4.26) \quad \det(\Omega_\lambda(\mathbf{q}, \kappa) \mathcal{M}_J(\Phi_\lambda W_I)(\mathbf{z}; h; \mathbf{q}; \kappa))_{I, J \in \mathcal{I}_\lambda} =$$

$$= (\kappa \Gamma(h/\kappa))^{\lambda^{\{1\}} d_\lambda} \left(e^{2\pi\sqrt{-1}(n-1)d_\lambda^{(2)}} \prod_{i=1}^N q_i^{d_{\lambda,i}^{(1)}} \right)^{\sum_{a=1}^n z_a/\kappa} \times$$

$$\times \prod_{a=1}^{n-1} \prod_{b=a+1}^n \left(\frac{\pi \kappa^2 \Gamma((z_a - z_b + h)/\kappa) \Gamma(1 + (z_b - z_a + h)/\kappa)}{\sin(\pi(z_a - z_b)/\kappa)} \right)^{d_\lambda^{(2)}},$$

where $\lambda^{\{1\}} = \sum_{i=1}^{N-1} (N-i) \lambda_i$,

$$(4.27) \quad d_\lambda = \frac{n!}{\lambda_1! \dots \lambda_N!}, \quad d_{\lambda,i}^{(1)} = \frac{\lambda_i(n-1)!}{\lambda_1! \dots \lambda_N!}, \quad d_\lambda^{(2)} = \frac{(n-2)!}{\lambda_1! \dots \lambda_N!} \sum_{i=1}^{N-1} \sum_{j=i+1}^N \lambda_i \lambda_j.$$

Proof. Denote by $F(\mathbf{z}; \mathbf{q})$ the determinant in the left-hand side of formula (4.26). By Theorem 4.7, it solves the differential equations

$$\left(\kappa q_i \frac{\partial}{\partial q_i} - \text{tr } X_i(\mathbf{z}; h; \mathbf{q})|_{(\mathbb{C}^N)_\lambda^{\otimes n}} \right) F(\mathbf{z}; \mathbf{q}) = 0, \quad i = 1, \dots, N,$$

where $X_i(\mathbf{z}; h; \mathbf{q})|_{(\mathbb{C}^N)_\lambda^{\otimes n}}$ are the restrictions of dynamical Hamiltonians (2.7) to the invariant subspace $(\mathbb{C}^N)_\lambda^{\otimes n}$. Since $\text{tr } X_i(\mathbf{z}; h; \mathbf{q})|_{(\mathbb{C}^N)_\lambda^{\otimes n}} = d_{\lambda,i}^{(1)} \sum_{a=1}^n z_a$, the function $F(\mathbf{z}; \mathbf{q})$ equals the product of powers of q_1, \dots, q_n in the right-hand side of formula (4.26) multiplied by a factor that does not depend on \mathbf{q} . This factor can be found by taking the limit $q_{i+1}/q_i \rightarrow 0$ for all $i = 1, \dots, N-1$, using Theorem 4.7. \square

Remark. By Theorem 4.7, the determinant $F(\mathbf{z}; \mathbf{q})$ in Theorem 4.8 solves the difference equations

$$(4.28) \quad F(z_1, \dots, z_a + \kappa, \dots, z_n; \mathbf{q}) = \det K_a(\mathbf{z}; h; \mathbf{q}; \kappa)|_{(\mathbb{C}^N)_\lambda^{\otimes n}} F(\mathbf{z}; \mathbf{q}), \quad a = 1, \dots, n,$$

where $K_a(\mathbf{z}; h; \mathbf{q}; \kappa)|_{(\mathbb{C}^N)_\lambda^{\otimes n}}$ are the restrictions of the qKZ operators (2.3) to the invariant subspace $(\mathbb{C}^N)_\lambda^{\otimes n}$. Equations (4.28) determine the product of Gamma functions in the right-hand side of formula (4.26) up to a κ -periodic function of z_1, \dots, z_n .

4.5. Proof of Theorem 4.7. Recall $\lambda^{\{1\}} = \sum_{i=1}^{N-1} \lambda^{(i)}$. Set $\lambda^{\{2\}} = \sum_{i=1}^{N-1} (\lambda^{(i)})^2$ and $\lambda_{\{2\}} = \sum_{1 \leq i < j \leq N} \lambda_i \lambda_j$. Notice the homogeneity properties

$$(4.29) \quad \begin{aligned} \Phi_\lambda(\mathbf{t}; \mathbf{z}; h; \mathbf{q}; \kappa) &= \kappa^{\lambda^{\{1\}} - \lambda^{\{2\}}} \Phi_\lambda(\mathbf{t}/\kappa; \mathbf{z}/\kappa; h/\kappa; \mathbf{q}; 1), \\ \Psi_J(\mathbf{z}; h; \mathbf{q}; \kappa) &= \kappa^{\lambda^{\{1\}} + \lambda_{\{2\}}} \Psi_J(\mathbf{z}/\kappa; h/\kappa; \mathbf{q}; 1). \end{aligned}$$

To simplify writing, we assume in this section that $\kappa = 1$ and omit the corresponding argument in all functions. The general case can be recovered by the homogeneity.

Recall I_λ^{\min} , $A_\lambda(\mathbf{t}; \mathbf{z}; h)$, see (3.6), (4.16). For $\mathbf{l} \in \mathbb{Z}_{\geq 0}^{\lambda^{\{1\}}}$, define

$$(4.30) \quad B_\mathbf{l}(\mathbf{z}; h) = \frac{A_\lambda(\sum_{I_\lambda^{\min}} - \mathbf{l}; \mathbf{z}; h)}{c_\lambda(\sum_{I_\lambda^{\min}} - \mathbf{l}; h)}.$$

Set

$$(4.31) \quad \mathcal{Z}_\lambda = \{\mathbf{l} \in \mathbb{Z}_{\geq 0}^{\lambda^{\{1\}}} \mid l_a^{(i)} \geq l_a^{(i+1)}, \quad i = 1, \dots, N-1, \quad a = 1, \dots, \lambda^{(i)}\}.$$

Lemma 4.9. *If $\mathbf{l} \notin \mathcal{Z}_\lambda$, then $B_\mathbf{l}(\mathbf{z}; h) = 0$. For $\mathbf{l} \in \mathcal{Z}_\lambda$,*

$$(4.32) \quad B_\mathbf{l}(\mathbf{z}; h) = \prod_{i=1}^{N-1} \prod_{a=1}^{\lambda^{(i)}} \left(\prod_{\substack{b=1 \\ b \neq a}}^{\lambda^{(i)}} \frac{\Gamma(z_b - z_a + l_a^{(i)} - l_b^{(i)} + 1) \Gamma(z_b - z_a + l_a^{(i)} - l_b^{(i+1)} + h)}{\Gamma(z_b - z_a + l_a^{(i)} - l_b^{(i)} + h + 1) \Gamma(z_b - z_a + l_a^{(i)} - l_b^{(i+1)} + 1)} \times \right. \\ \left. \times \frac{\Gamma(l_a^{(i)} - l_a^{(i+1)} + h)}{(l_a^{(i)} - l_a^{(i+1)})!} \prod_{c=\lambda^{(i)+1}}^{\lambda^{(i+1)}} \frac{\Gamma(z_c - z_a + l_a^{(i)} - l_c^{(i+1)} + h)}{\Gamma(z_c - z_a + l_a^{(i)} - l_c^{(i+1)} + 1)} \right).$$

Recall $\mathbf{z}_\sigma = (z_{\sigma(1)}, \dots, z_{\sigma(n)})$ and the permutations σ_I , $I \in \mathcal{I}_\lambda$, defined in Section 3.2. Let $\mathbf{r} = (r_1, \dots, r_{N-1})$. For $I, J \in \mathcal{I}_\lambda$, define

$$(4.33) \quad \mathcal{B}_{I,J}(\mathbf{z}; h; \mathbf{r}) = \sum_{\mathbf{l} \in \mathcal{Z}_\lambda} B_\mathbf{l}(\mathbf{z}; h) W_I(\sum_{I_\lambda^{\min}} - \mathbf{l}; \mathbf{z}_{\sigma_J^{-1}}; h) \prod_{i=1}^{N-1} r_i^{\sum_{a=1}^{\lambda^{(i)}} l_a^{(i)}}.$$

Set

$$(4.34) \quad \tilde{M}_\lambda(\mathbf{z}; h) = \prod_{i=1}^{N-1} \prod_{a=1}^{\lambda^{(i)}} \left(\sin(\pi h) \prod_{b=a+1}^{\lambda^{(i)}} \frac{\sin(\pi(z_a - z_b))}{z_a - z_b} \prod_{c=\lambda^{(i)+1}}^{\lambda^{(i+1)}} \sin(\pi(z_a - z_c - h)) \right),$$

Proposition 4.10. *For any $I, J \in \mathcal{I}_\lambda$, the function $\tilde{M}_\lambda(\mathbf{z}; h) \mathcal{B}_{I,J}(\mathbf{z}; h; \mathbf{r})$ is entire in \mathbf{z}, h and holomorphic in \mathbf{r} provided $|r_i| < 1$ for all $i = 1, \dots, N-1$.*

Proof. By Stirling's formula, ratios of Gamma functions appearing in formula (4.32) have the following asymptotics as $k \rightarrow +\infty$ over integers,

$$(4.35) \quad \frac{\Gamma(\alpha + k)}{\Gamma(\beta + k)} = k^{\alpha-\beta} (1 + o(1)), \quad \frac{\Gamma(\alpha - k)}{\Gamma(\beta - k)} = k^{\alpha-\beta} \frac{\sin(\pi\alpha)}{\sin(\pi\beta)} (1 + o(1)),$$

provided α and β are not integers in the second case, and these asymptotics can be differentiated with respect to α and β .

The right-hand side of formula (4.33) allows one to present $\tilde{M}_\lambda(\mathbf{z}; h) \mathcal{B}_{I,J}(\mathbf{z}; h; \mathbf{r})$ as a power series in \mathbf{r} :

$$\tilde{M}_\lambda(\mathbf{z}; h) \mathcal{B}_{I,J}(\mathbf{z}; h; \mathbf{r}) = \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^{N-1}} \tilde{B}_{I,J,\mathbf{k}}(\mathbf{z}; h) \prod_{i=1}^{N-1} r_i^{k_i}.$$

Formulae (4.32), (4.35) show that for given \mathbf{z}, h , this series and its formal derivatives with respect to \mathbf{z}, h converge provided $|r_i| < 1$ for all $i = 1, \dots, N-1$. Therefore, the function $\tilde{M}_\lambda(\mathbf{z}; h) \mathcal{B}_{I,J}(\mathbf{z}; h; \mathbf{r})$ is holomorphic in \mathbf{r} provided $|r_i| < 1$ for all $i = 1, \dots, N-1$, and is holomorphic in \mathbf{z}, h whenever all the coefficients $\tilde{B}_{I,J,\mathbf{k}}(\mathbf{z}; h)$ are holomorphic in \mathbf{z}, h .

It remains to show that all the coefficients $\tilde{B}_{I,J,\mathbf{k}}(\mathbf{z}; h)$ are entire in \mathbf{z}, h . To this end, it suffices to show that for any $\mathbf{l} \in \mathcal{Z}_\lambda$, the product $\tilde{M}_\lambda(\mathbf{z}; h) B_{\mathbf{l}}(\mathbf{z}; h) W_I(\Sigma_{I_\lambda^{\min}} - \mathbf{l}; \mathbf{z}_{\sigma_J^{-1}}; h)$ is an entire function of \mathbf{z}, h .

Take $\mathbf{l} \in \mathcal{Z}_\lambda$ and $a \neq b$. Since $l_a^{(i)} \geq l_a^{(i+1)}$ and $l_b^{(i)} \geq l_b^{(i+1)}$, the ratio

$$\frac{\sin(\pi(z_a - z_b)) \Gamma(z_a - z_b + l_b^{(i)} - l_a^{(i)} + 1) \Gamma(z_b - z_a + l_a^{(i)} - l_b^{(i)} + 1)}{(z_a - z_b) \Gamma(z_a - z_b + l_b^{(i)} - l_a^{(i+1)} + 1) \Gamma(z_b - z_a + l_a^{(i)} - l_b^{(i+1)} + 1)}$$

is an entire function of z_a, z_b , and if $l_b^{(i)} > l_b^{(i+1)}$, then the ratio

$$\frac{\Gamma(z_b - z_a + l_a^{(i)} - l_b^{(i+1)} + h)}{\Gamma(z_b - z_a + l_a^{(i)} - l_b^{(i)} + h + 1)}$$

is a polynomial in z_a, z_b, h . Set

$$(4.36) \quad F_{\mathbf{l}}(\mathbf{z}; h) = \prod_{i=1}^{N-1} \prod_{a=1}^{\lambda^{(i)}} \prod_{\substack{b=1 \\ b \neq a, l_b^{(i)} = l_b^{(i+1)}}}^{\lambda^{(i)}} (z_b - z_a + l_a^{(i)} - l_b^{(i)} + h).$$

Then by formulae (4.32), (4.34), the product $\tilde{M}_\lambda(\mathbf{z}; h) B_{\mathbf{l}}(\mathbf{z}; h) F_{\mathbf{l}}(\mathbf{z}; h)$ is an entire function of \mathbf{z}, h . Since by Lemma 4.11 below, the ratio $W_I(\Sigma_{I_\lambda^{\min}} - \mathbf{l}; \mathbf{z}_{\sigma_J^{-1}}; h) / F_{\mathbf{l}}(\mathbf{z}; h)$ is a polynomial in \mathbf{z}, h , the product $\tilde{M}_\lambda(\mathbf{z}; h) B_{\mathbf{l}}(\mathbf{z}; h) W_I(\Sigma_{I_\lambda^{\min}} - \mathbf{l}; \mathbf{z}_{\sigma_J^{-1}}; h)$ is an entire function of \mathbf{z}, h too. Proposition 4.10 is proved. \square

Lemma 4.11. *Let $F_{\mathbf{l}}(\mathbf{z}; h)$ be given by (4.36). Then the ratio $W_I(\Sigma_{I_\lambda^{\min}} - \mathbf{l}; \mathbf{z}_{\sigma_J^{-1}}; h) / F_{\mathbf{l}}(\mathbf{z}; h)$ is a polynomial in \mathbf{z}, h .*

Proof. The proof is by inspection, similarly to the proof of Lemma 4.3. \square

Proof of Theorem 4.7. Recall that we assume $\kappa = 1$ and skip the corresponding argument in all functions. By formulae (4.19), (4.30)–(4.33), and Lemmas 4.6, 4.9,

$$(4.37) \quad \mathcal{M}_J(\Phi_\lambda W_I)(\mathbf{z}; h; \mathbf{q}) = \mathcal{B}_{I,J}(\mathbf{z}_{\sigma_J}; h; q_2/q_1, \dots, q_N/q_{N-1}) \times \\ \times \prod_{i=1}^N \left(e^{\pi\sqrt{-1}(n-\lambda_i)} q_i \right)^{\sum_{a \in J_i} z_a} \prod_{i=1}^{N-1} \prod_{j=i+1}^N \prod_{a \in J_i} \prod_{b \in J_j} \frac{\pi}{\sin(\pi(z_a - z_b))}.$$

Then by formulae (4.20), (4.22), (4.37), we have

$$(4.38) \quad \Psi_J^\diamond(\mathbf{z}; h; \mathbf{q}) = \sum_{I \in \mathcal{I}_\lambda} \mathcal{B}_{I,J}(\mathbf{z}_{\sigma_J}; h; q_2/q_1, \dots, q_N/q_{N-1}) v_I \times \\ \times \frac{\Omega_\lambda(\mathbf{q})}{(\Gamma(h))^{\lambda^{(1)}}} \prod_{i=1}^{N-1} \prod_{j=i+1}^N \prod_{a \in J_i} \left(\prod_{b \in J_j} \frac{\pi}{\Gamma(z_b - z_a + h + 1)} \prod_{\substack{c \in J_j \\ c < a}} (z_a - z_c - h) \right).$$

By Proposition 4.10, the function in the right-hand side of formula (4.38) is holomorphic in \mathbf{q} in the unit polydisk around $\mathbf{0}$, and may have poles in \mathbf{z}, h at most at the hyperplanes $z_a - z_b \in \mathbb{Z}_{\neq 0}$ for $a \neq b$, $z_a - z_b - h \in \mathbb{Z}_{\geq 0}$ for $a \in J_i, b \in J_j, i < j$, and $h \in \mathbb{Z}_{> 0}$. To complete the proof of Theorem 4.7, it remains to show that those poles do not actually occur.

For the hyperplanes $z_a - z_b \in \mathbb{Z}_{\neq 0}, a \neq b$, we will show the regularity of the function $M_\lambda(\mathbf{z}_{\sigma_J}) \Psi_J(\mathbf{z}; h; \mathbf{q})$. Observe that $M_\lambda(\mathbf{z}_{\sigma_J}) \Psi_J(\mathbf{z}; h; \mathbf{q})$ is regular at the hyperplanes $z_a = z_b$ for all $a \neq b$, and solves qKZ difference equations (2.4). Since all the qKZ operators $K_1(\mathbf{z}; h; \mathbf{q}), \dots, K_n(\mathbf{z}; h; \mathbf{q})$ and their inverses are regular at the hyperplanes $z_a - z_b \in \mathbb{Z}$ for $a \neq b$, the function $M_\lambda(\mathbf{z}_{\sigma_J}) \Psi_J(\mathbf{z}; h; \mathbf{q})$ is regular at all hyperplanes $z_a - z_b \in \mathbb{Z}$ for $a \neq b$.

To deal with the hyperplanes $z_a - z_b - h \in \mathbb{Z}_{\geq 0}$ and $h \in \mathbb{Z}_{> 0}$, we will show that for given \mathbf{z} , the function $\Psi_J^\diamond(\mathbf{z}; h; \mathbf{q})$ is entire in h , and it suffices to do it for generic \mathbf{z} .

To simplify writing, we will omit the argument \mathbf{z} in all functions for a while. Let $\mathbf{r} = (r_1, \dots, r_{N-1})$ and $\mathbf{r}_* = (1, r_1, r_1 r_2, \dots, r_1 \dots r_{N-1})$. Denote $F(h; \mathbf{r}) = \Psi_J^\diamond(h; \mathbf{r}_*)$, so that $\Psi_J^\diamond(h; \mathbf{q}) = F(h; q_2/q_1, \dots, q_N/q_{N-1})$.

Recall the dynamical operators X_1, \dots, X_N , see (2.7). Set

$$X_{(i)}(h; \mathbf{r}) = \sum_{j=i+1}^N \left(X_j(h; \mathbf{r}_*) - \sum_{a \in J_j} z_a \right) \Big|_{(\mathbb{C}^N)_\lambda^{\otimes n}}, \quad i = 1, \dots, N-1.$$

The dynamical differential equations (2.6) for $\Psi_J(h; \mathbf{q})$ are equivalent to the following equations for $F(h; \mathbf{r})$,

$$(4.39) \quad r_i \frac{\partial}{\partial r_i} F(h; \mathbf{r}) = X_{(i)}(h; \mathbf{r}) F(h; \mathbf{r}), \quad i = 1, \dots, N-1.$$

The operators $X_{(1)}(h; \mathbf{r}), \dots, X_{(N-1)}(h; \mathbf{r})$ are linear functions in h and rational functions in \mathbf{r} , regular provided $|r_i| < 1$ for all $i = 1, \dots, N-1$. The eigenvalues of the operator

$X_{(i)}(h; 0)$ for a given i are $\sum_{j=i+1}^N (\sum_{a \in I_j} z_a - \sum_{a \in J_j} z_a)$, $I \in \mathcal{I}_\lambda$. Hence, one of the eigenvalues of $X_{(i)}(h; 0)$ equals zero and all other eigenvalues are not integers for generic \mathbf{z} . Therefore, a solution $F(h; \mathbf{r})$ of equations (4.39) holomorphic $|r_i| < 1$ for all i , is uniquely determined by the value $F(h; 0)$, and $F(h, \mathbf{r})$ is holomorphic in h whenever $F(h; 0)$ is. The value $F(h; 0) = \Psi^\circ(h; \mathbf{0})$ can be found from formulae (4.20), (4.18), (4.32), (4.33), (4.37), (4.38). It is a polynomial in h by Lemmas 4.2, 4.3. Therefore, $F(h, \mathbf{r})$ is an entire function of h , and so is $\Psi^\circ(h; \mathbf{q})$.

Formulae (4.23), (4.24), (4.25), and the rationality of $\Psi_{J, \mathbf{m}}^\circ(\mathbf{z}; h)$ follow from formulae (4.20), (4.30)–(4.33), (4.37), (4.38), and Lemma 4.2. The properties of poles of $\Psi_{J, \mathbf{m}}^\circ(\mathbf{z}; h)$ are determined by the analytic properties of $\Psi_J^\circ(\mathbf{z}; h; \mathbf{q})$. The fact that $\Psi_{J, 0}^\circ(\mathbf{z}; h)$ is a polynomial in \mathbf{z}, h follows from Lemma 4.3. \square

4.6. Solutions of the dynamical and qKZ equations parametrized by Laurent polynomials. In the sequel, we will use the following notation. For any variable x , there is the companion \acute{x} , and for any function $f(\acute{x})$, we set $\acute{f}(x; \kappa) = f(e^{2\pi\sqrt{-1}x/\kappa})$. The convention for collections of variables is similar. For instance, $\acute{\mathbf{z}} = (\acute{z}_1, \dots, \acute{z}_n)$ and $\acute{f}(\mathbf{z}; \kappa) = f(e^{2\pi\sqrt{-1}z_1/\kappa}, \dots, e^{2\pi\sqrt{-1}z_n/\kappa})$.

Introduce the variables $\gamma_{i,j}$, $i = 1, \dots, N$, $j = 1, \dots, \lambda_i$. Denote

$$(4.40) \quad \mathbf{\Gamma} = (\gamma_{1,1}, \dots, \gamma_{1,\lambda_1}, \dots, \gamma_{N,1}, \dots, \gamma_{N,\lambda_N}), \quad \acute{\mathbf{\Gamma}}^{\pm 1} = (\acute{\gamma}_{1,1}^{\pm 1}, \dots, \acute{\gamma}_{N,\lambda_N}^{\pm 1}).$$

Recall $\mathbf{z}_\sigma = (z_{\sigma(1)}, \dots, z_{\sigma(n)})$, and the permutations σ_I , $I \in \mathcal{I}_\lambda$, defined in Section 3.2. For a Laurent polynomial $P(\acute{\mathbf{\Gamma}}; \acute{\mathbf{z}}; \acute{h})$, set

$$(4.41) \quad \Psi_P(\mathbf{z}; h; \mathbf{q}; \kappa) = \sum_{J \in \mathcal{I}_\lambda} \acute{P}(\mathbf{z}_{\sigma_J}; \mathbf{z}; h; \kappa) \Psi_J(\mathbf{z}; h; \mathbf{q}; \kappa),$$

where the functions $\Psi_J(\mathbf{z}; h; \mathbf{q}; \kappa)$, are given by (4.20).

Proposition 4.12. *The function $\Psi_P(\mathbf{z}; h; \mathbf{q}; \kappa)$ is a solution of the joint system of dynamical differential equations (2.6) and qKZ difference equations (2.4).*

Proof. By Theorem 4.7, the function $\Psi_P(\mathbf{z}; h; \mathbf{q}; \kappa)$ solves the system of equations (2.6) and (2.4) since $\acute{P}(\mathbf{z}_{\sigma_J}; \mathbf{z}; h; \kappa)$ does not depend on \mathbf{q} and is κ -periodic function in each of the variables z_1, \dots, z_n . \square

Let $\mathbb{C}[\acute{\mathbf{\Gamma}}^{\pm 1}]^{S_\lambda}$ be the space of Laurent polynomials in $\acute{\mathbf{\Gamma}}$ symmetric in $\acute{\gamma}_{i,1}, \dots, \acute{\gamma}_{i,\lambda_i}$ for each $i = 1, \dots, N$.

Proposition 4.13. *For any $P \in \mathbb{C}[\acute{\mathbf{\Gamma}}^{\pm 1}]^{S_\lambda} \otimes \mathbb{C}[\acute{\mathbf{z}}^{\pm 1}, \acute{h}^{\pm 1}]$, the function $\Psi_P(\mathbf{z}; h; \mathbf{q}; \kappa)$ is holomorphic in $\mathbf{z}, h, \mathbf{q}$ provided $z_a - z_b + h \notin \kappa\mathbb{Z}_{\leq 0}$ for all $a, b = 1, \dots, n$, $a \neq b$, and $|q_{i+1}/q_i| < 1$ for all $i = 1, \dots, N-1$, with a branch of $\log q_i$ fixed for each $i = 1, \dots, N$. The singularities of $\Psi_P(\mathbf{z}; h; \mathbf{q}; \kappa)$ at the hyperplanes $z_a - z_b + h \in \kappa\mathbb{Z}_{\leq 0}$ are simple poles.*

Proof. By Theorem 4.7, we need only to show that $\Psi_P(\mathbf{z}; h; \mathbf{q}; \kappa)$ is regular at the hyperplanes $z_a - z_b \in \kappa\mathbb{Z}$, $a \neq b$, where it might have simple poles.

The regularity of $\Psi_P(\mathbf{z}; h; \mathbf{q}; \kappa)$ at the hyperplanes $z_a = z_b$ for all $a \neq b$ follows from formula (4.25). Since $\Psi_P(\mathbf{z}; h; \mathbf{q}; \kappa)$ solves qKZ difference equations (2.4) and all the qKZ

operators $K_1(\mathbf{z}; h; \mathbf{q}; \kappa), \dots, K_n(\mathbf{z}; h; \mathbf{q}; \kappa)$ and their inverses are regular at the hyperplanes $z_a - z_b \in \kappa\mathbb{Z}$, the function $\Psi_P(\mathbf{z}; h; \mathbf{q}; \kappa)$ is regular at all hyperplanes $z_a - z_b \in \kappa\mathbb{Z}$, $a \neq b$. \square

Denote by \mathcal{S}_λ the space of solutions of the system of dynamical differential equations (2.6) and qKZ difference equations (2.4) spanned over \mathbb{C} by the functions $\Psi_P(\mathbf{z}; h; \mathbf{q}; \kappa)$, $P \in \mathbb{C}[\dot{\Gamma}^{\pm 1}]^{S_\lambda} \otimes \mathbb{C}[\dot{z}^{\pm 1}, \dot{h}^{\pm 1}]$. The space \mathcal{S}_λ is a $\mathbb{C}[\dot{z}^{\pm 1}, \dot{h}^{\pm 1}]$ -module with $f(\dot{z}; \dot{h})$ acting as multiplication by $\dot{f}(\mathbf{z}; h; \kappa)$.

Define the algebra

$$(4.42) \quad \mathcal{K}_\lambda = \mathbb{C}[\dot{\Gamma}^{\pm 1}]^{S_\lambda} \otimes \mathbb{C}[\dot{z}^{\pm 1}, \dot{h}^{\pm 1}] \left/ \left\langle \prod_{i=1}^N \prod_{j=1}^{\lambda_i} (u - \dot{\gamma}_{i,j}) = \prod_{a=1}^n (u - \dot{z}_a) \right\rangle \right.,$$

where u is a formal variable. By (4.41), the assignment $P \mapsto \Psi_P$ defines a homomorphism

$$(4.43) \quad \mu_\lambda^\kappa : \mathcal{K}_\lambda \rightarrow \mathcal{S}_\lambda, \quad Y \mapsto \Psi_Y,$$

of $\mathbb{C}[\dot{z}^{\pm 1}, \dot{h}^{\pm 1}]$ -modules.

By Propositions A.2, A.3, the algebra \mathcal{K}_λ is a free $\mathbb{C}[\dot{z}^{\pm 1}, \dot{h}^{\pm 1}]$ -module generated by the classes

$$(4.44) \quad Y_I(\Gamma) = V_I(\dot{\gamma}_{1,1}^{-1}, \dots, \dot{\gamma}_{1,\lambda_1}^{-1}, \dots, \dot{\gamma}_{N,1}^{-1}, \dots, \dot{\gamma}_{N,\lambda_N}^{-1}), \quad I \in \mathcal{I}_\lambda,$$

where the polynomials V_I are defined by formula (A.6). Introduce the coordinates of solutions Ψ_{Y_I} :

$$(4.45) \quad \Psi_{Y_I}(\mathbf{z}; h; \mathbf{q}; \kappa) = \sum_{J \in \mathcal{I}_\lambda} \bar{\Psi}_{I,J}(\mathbf{z}; h; \mathbf{q}; \kappa) v_J.$$

Theorem 4.14. *Let $n \geq 2$. Then*

$$(4.46) \quad \det(\bar{\Psi}_{I,J}(\mathbf{z}; h; \mathbf{q}; \kappa))_{I,J \in \mathcal{I}_\lambda} = \left(e^{\pi\sqrt{-1}(n-1)d_\lambda^{(2)}} \prod_{i=1}^N q_i^{d_{\lambda,i}^{(1)}} \right)^{\sum_{a=1}^n z_a/\kappa} \prod_{j=2}^{n-1} j^{(n-j)d_\lambda^{(2)}} \times \\ \times \prod_{a=1}^{n-1} \prod_{b=a+1}^n (2\pi\sqrt{-1} \kappa^2 \Gamma((z_a - z_b + h)/\kappa) \Gamma(1 + (z_b - z_a + h)/\kappa))^{d_\lambda^{(2)}},$$

where $d_{\lambda,i}^{(1)}$, $d_\lambda^{(2)}$ are given by formulae (4.27).

Proof. The statement follows from Theorem 4.8 and formula (A.7). \square

Corollary 4.15. *The map $\mu_\lambda^\kappa : \mathcal{K}_\lambda \rightarrow \mathcal{S}_\lambda$ is an isomorphism of $\mathbb{C}[\dot{z}^{\pm 1}, \dot{h}^{\pm 1}]$ -modules.*

Remark. The algebra \mathcal{K}_λ is the equivariant K -theory algebra $K_{T \times \mathbb{C}^\times}(T^*\mathcal{F}_\lambda; \mathbb{C})$ of the cotangent bundle of the partial flag variety \mathcal{F}_λ , see the notation in Section 6.

4.7. Levelt fundamental solution. Recall that a function $f(\mathbf{q})$ is called holomorphic in the unit polydisk around $\mathbf{0}$ if $f(\mathbf{q}) = g(q_2/q_1, \dots, q_N/q_{N-1})$ for a function $g(s_1, \dots, s_{N-1})$ holomorphic in the polydisk $\{|s_1| < 1, \dots, |s_{N-1}| < 1\}$, and $f(\mathbf{0}) = g(0, \dots, 0)$.

The dynamical Hamiltonians $X_1(\mathbf{z}; h; \mathbf{q}), \dots, X_n(\mathbf{z}; h; \mathbf{q})$ given by (2.7) are holomorphic in \mathbf{q} in the unit polydisk around $\mathbf{0}$ and

$$(4.47) \quad X_i(\mathbf{z}; h; \mathbf{0}) = \sum_{a=1}^n z_a e_{i,i}^{(a)} - h \sum_{1 \leq a < b \leq n} \left(\sum_{j=1}^{i-1} e_{j,i}^{(a)} e_{i,j}^{(b)} - \sum_{j=i+1}^N e_{i,j}^{(a)} e_{j,i}^{(b)} \right).$$

Notice that for $I \in \mathcal{I}_\lambda$,

$$X_i(\mathbf{z}; h; \mathbf{0}) v_I = \sum_{a \in I_i} z_a v_I + \sum_{\substack{J \in \mathcal{I}_\lambda \\ |\sigma_J| < |\sigma_I|}} \xi_{i,I,J} v_J,$$

where the coefficients $\xi_{i,I,J}$ take values $0, \pm h$. Therefore, the eigenvalues of the restriction of the operator $X_i(\mathbf{z}; h; \mathbf{0})$ on $(\mathbb{C}^N)_\lambda^{\otimes n}$ are $\sum_{a \in I_i} z_a$, $I \in \mathcal{I}_\lambda$. A more detailed statement is given by Proposition 4.16 below.

Recall the function $\Psi_{I,0}^\circ(\mathbf{z}; h)$, $I \in \mathcal{I}_\lambda$, given by (4.24).

Proposition 4.16. *Given $I \in \mathcal{I}_\lambda$, we have $X_i(\mathbf{z}; h; \mathbf{0}) \Psi_{I,0}^\circ(\mathbf{z}; h) = \sum_{a \in I_i} z_a \Psi_{I,0}^\circ(\mathbf{z}; h)$, and $\Psi_{I,0}^\circ(\mathbf{z}; h) \neq 0$ provided $z_a \neq z_b$ for all pairs a, b such that $a < b$ and $\sigma_I^{-1}(a) > \sigma_I^{-1}(b)$.*

Proof. The first part of the statement follows from Theorem 4.7. The nonvanishing of $\Psi_{I,0}^\circ(\mathbf{z}; h)$ is implied by formula (4.24) and Lemmas 4.2, 4.3. \square

For $I \in \mathcal{I}_\lambda$, set $\mathbf{E}_I(\mathbf{z}) = (E_I^{(1)}(\mathbf{z}), \dots, E_I^{(N-1)}(\mathbf{z}))$, where $E_I^{(i)}(\mathbf{z}) = \sum_{j=1}^i \sum_{a \in I_j} z_a$ is the eigenvalue of $X_1(\mathbf{z}; h; \mathbf{0}) + \dots + X_i(\mathbf{z}; h; \mathbf{0})$ on $\Psi_{I,0}^\circ(\mathbf{z}; h)$. For $I, J \in \mathcal{I}_\lambda$, denote by $D_{I,J}$ the set of points \mathbf{z} such that $\mathbf{E}_I(\mathbf{z}) - \mathbf{E}_J(\mathbf{z}) \in \mathbb{Z}_{\geq 0}^{N-1}$ and $\mathbf{E}_I(\mathbf{z}) \neq \mathbf{E}_J(\mathbf{z})$. Set $D_\lambda = \bigcup_{I, J \in \mathcal{I}_\lambda} D_{I,J}$.

Theorem 4.17. (i) *For any \mathbf{z}, h such that $z_a - z_b \notin \kappa \mathbb{Z}_{\neq 0}$ for all $1 \leq a < b \leq n$, there exists an $\text{End}((\mathbb{C}^N)_\lambda^{\otimes n})$ -valued function $\Psi^\#(\mathbf{z}; h; \mathbf{q})$, holomorphic in \mathbf{q} in the unit polydisk around $\mathbf{0}$, such that $\Psi^\#(\mathbf{z}; h; \mathbf{0})$ is the identity operator and the function*

$$(4.48) \quad \widehat{\Psi}(\mathbf{z}; h; \mathbf{q}; \kappa) = \Psi^\#(\mathbf{z}/\kappa; h/\kappa; \mathbf{q}) \prod_{i=1}^N q_i^{X_i(\mathbf{z}/\kappa; h/\kappa; \mathbf{0})},$$

solves dynamical differential equations (2.6). For given \mathbf{z}, h , the function $\Psi^\#(\mathbf{z}; h; \mathbf{q})$ with the specified properties is unique if and only if $\mathbf{z} \notin \kappa D_\lambda$. Furthermore, $\det \Psi^\#(\mathbf{z}; h; \mathbf{q}) = 1$ and

$$(4.49) \quad \det \widehat{\Psi}(\mathbf{z}; h; \mathbf{q}; \kappa) = \prod_{i=1}^N q_i^{d_{\lambda,i}^{(1)} \sum_{a=1}^n z_a / \kappa},$$

where $d_{\lambda,1}^{(1)}, \dots, d_{\lambda,N}^{(1)}$ are given by formula (4.27).

(ii) Define the function $\Psi^\#(\mathbf{z}; h; \mathbf{q})$ for generic \mathbf{z}, h as in item (i). Then $\Psi^\#(\mathbf{z}; h; \mathbf{q})$ is holomorphic in \mathbf{z} if $z_a - z_b \notin \mathbb{Z}_{\neq 0}$ for all $1 \leq a < b \leq n$, and is entire in h . The singularities of $\Psi^\#(\mathbf{z}; h; \mathbf{q})$ at the hyperplanes $z_a - z_b \in \mathbb{Z}_{\neq 0}$ are simple poles.

The theorem is proved in Section 4.8. An explicit expression for the function $\Psi^\#(\mathbf{z}; h; \mathbf{q})$ is given by formula (4.60).

Following [AB, Chapter 2], we will call $\widehat{\Psi}(\mathbf{z}; h; \mathbf{q}; \kappa)$ the *Levelt fundamental solution* of dynamical differential equations (2.6) on $(\mathbb{C}^N)_\lambda^{\otimes n}$, see also [CV, Section 6.2].

For a $(\mathbb{C}^N)_\lambda^{\otimes n}$ -valued solution $\Psi(\mathbf{z}; h; \mathbf{q}; \kappa)$ of dynamical differential equations (2.6), define its *principal term*

$$(4.50) \quad \Psi^0(\mathbf{z}; h; \kappa) = \widehat{\Psi}(\mathbf{z}; h; \mathbf{q}; \kappa)^{-1} \Psi(\mathbf{z}; h; \mathbf{q}; \kappa).$$

By Theorem 4.17, the principal term does not depend on \mathbf{q} .

Set

$$(4.51) \quad C_\lambda(\mathbf{z}; \kappa) = \prod_{i=1}^N e^{\pi\sqrt{-1}(n-\lambda_i) \sum_{a=\lambda^{(i-1)+1}}^{\lambda^{(i)}} z_a/\kappa}$$

and

$$(4.52) \quad G_\lambda(\mathbf{z}; h; \kappa) = \prod_{i=1}^{N-1} \prod_{a=1}^{\lambda^{(i)}} \prod_{b=\lambda^{(i)+1}}^{\lambda^{(i+1)}} \Gamma((z_a - z_b)/\kappa) \Gamma((z_b - z_a + h)/\kappa).$$

Proposition 4.18. *For a Laurent polynomial $P(\mathring{\Gamma}; \mathring{z}; \mathring{h})$, the principal term of the solution $\Psi_P(\mathbf{z}; h; \mathbf{q}; \kappa)$, given by (4.41), equals*

$$(4.53) \quad \Psi_P^0(\mathbf{z}; h; \kappa) = \sum_{I, J \in \mathcal{I}_\lambda} \acute{P}(\mathbf{z}_{\sigma_J}; \mathbf{z}; h; \kappa) C_\lambda(\mathbf{z}_{\sigma_J}; \kappa) G_\lambda(\mathbf{z}_{\sigma_J}; h; \kappa) \frac{W_I(\Sigma_J; \mathbf{z}; h)}{c_\lambda(\Sigma_J; h)} v_I.$$

Here $c_\lambda(\Sigma_J; h)$ is given by formula (4.5).

The proposition is proved in Section 4.8.

4.8. Proofs of Theorem 4.17 and Proposition 4.18. To simplify writing, we assume in this section that $\kappa = 1$, similarly to Section 4.5, and omit the corresponding argument in all functions. The general case can be recovered by the homogeneity properties (4.29).

Lemma 4.19. *Given $A \in \text{End}((\mathbb{C}^N)_\lambda^{\otimes n})$, assume that the function*

$$F_A(\mathbf{z}; h; \mathbf{q}) = \prod_{i=1}^N q_i^{X_i(\mathbf{z}; h; \mathbf{0})} A \prod_{i=1}^N q_i^{-X_i(\mathbf{z}; h; \mathbf{0})}$$

is holomorphic in \mathbf{q} in the unit polydisk around $\mathbf{0}$, and $F_A(\mathbf{z}; h; \mathbf{0}) = 1$, the identity operator. Then $A = 1$ and $F_A(\mathbf{z}; h; \mathbf{q}) = 1$, provided $\mathbf{z} \notin D_\lambda$. On the other hand, if $\mathbf{z} \in D_\lambda$, then there exists $A \neq 1$ such that the function $F_A(\mathbf{z}; h; \mathbf{q})$ has the requested properties and is not a constant function of \mathbf{q} .

Proof. The statement follows from a basic linear algebra reasoning. □

Proof of Theorem 4.17. For the uniqueness statement, assume that the function $\Psi^\#(\mathbf{z}; h; \mathbf{q})$ is as required in Theorem 4.17. Then the function $\Psi_1^\#(\mathbf{z}; h; \mathbf{q})$ has the same requested properties if and only if the product

$$E(\mathbf{z}; h) = \prod_{i=1}^N q_i^{-X_i(\mathbf{z}; h; \emptyset)} \Psi^\#(\mathbf{z}; h; \mathbf{q})^{-1} \Psi_1^\#(\mathbf{z}; h; \mathbf{q}) \prod_{i=1}^N q_i^{X_i(\mathbf{z}; h; \emptyset)}$$

does not depend on \mathbf{q} , whilst the function

$$F(\mathbf{z}; h; \mathbf{q}) = \prod_{i=1}^N q_i^{X_i(\mathbf{z}; h; \emptyset)} E(\mathbf{z}; h) \prod_{i=1}^N q_i^{-X_i(\mathbf{z}; h; \emptyset)} = \Psi^\#(\mathbf{z}; h; \mathbf{q})^{-1} \Psi_1^\#(\mathbf{z}; h; \mathbf{q}).$$

is holomorphic in \mathbf{q} in the unit polydisk around \emptyset , and $F(\mathbf{z}; h; \emptyset)$ is the identity operator. Thus the uniqueness statement follows from Lemma 4.19.

Recall the functions $\Psi_J^\diamond(\mathbf{z}; h; \mathbf{q})$ and $\Psi_{J,0}^\diamond(\mathbf{z}; h)$, see (4.23), (4.24). By Lemma 4.2,

$$(4.54) \quad \Psi_{J,0}^\diamond(\mathbf{z}; h) = \prod_{i=1}^{N-1} \prod_{j=i+1}^N \prod_{\substack{a \in J_i \\ b \in J_j \\ b > a}} \frac{1}{z_a - z_b - h} \sum_{I \in \mathcal{I}_\lambda} \frac{W_I(\Sigma_J; \mathbf{z}; h)}{c_\lambda(\Sigma_J; h)} v_I.$$

Let $\Psi^\#(\mathbf{z}; h; \mathbf{q})$ to be the operator such that for any $I \in \mathcal{I}_\lambda$,

$$(4.55) \quad \begin{aligned} \Psi^\#(\mathbf{z}; h; \mathbf{q}) : v_I \mapsto & \sum_{J \in \mathcal{I}_\lambda} \Psi_J^\diamond(\mathbf{z}; h; \mathbf{q}) \prod_{i=1}^{N-1} \prod_{j=i+1}^N \prod_{a \in J_i} \prod_{b \in J_j} \frac{\Gamma(z_b - z_a)}{\pi} \times \\ & \times \frac{\check{W}_I(\Sigma_J; \mathbf{z}; h)}{c_\lambda(\Sigma_J; h)} \prod_{i=1}^{N-1} \prod_{j=i+1}^N \prod_{\substack{a \in J_i \\ b \in J_j \\ b < a}} \frac{1}{z_a - z_b - h} \end{aligned}$$

We will verify below that the function $\Psi^\#(\mathbf{z}; h; \mathbf{q})$ is as required in Theorem 4.17.

By Theorem 4.7, the functions $\Psi_J^\diamond(\mathbf{z}; h; \mathbf{q})$ are holomorphic in \mathbf{q} in the unit polydisk around \emptyset , hence so does $\Psi^\#(\mathbf{z}; h; \mathbf{q})$. Then $\Psi^\#(\mathbf{z}; h; \emptyset)$ is the identity operator by formulae (4.23), (4.54), and orthogonality relation (4.9).

By formulae (4.9), (4.54), and Proposition 4.16,

$$\prod_{i=1}^N q_i^{X_i(\mathbf{z}; h; \emptyset)} v_I = \sum_{J, K \in \mathcal{I}_\lambda} \frac{\check{W}_I(\Sigma_J; \mathbf{z}; h) W_K(\Sigma_J; \mathbf{z}; h)}{c_\lambda^2(\Sigma_J; h) R_\lambda(\mathbf{z}_{\sigma_J}) Q_\lambda(\mathbf{z}_{\sigma_J}; h)} \prod_{i=1}^N q_i^{\sum_{a \in J_i} z_a} v_K.$$

Then by formulae (4.55), (4.10), (4.22), the functions

$$\begin{aligned} \Psi^\#(\mathbf{z}; h; \mathbf{q}) \prod_{i=1}^N q_i^{X_i(\mathbf{z}; h; \emptyset)} v_I = & \sum_{J \in \mathcal{I}_\lambda} \Psi_J(\mathbf{z}; h; \mathbf{q}) \frac{\check{W}_I(\Sigma_J; \mathbf{z}; h)}{c_\lambda(\Sigma_J; h)} \prod_{i=1}^N e^{\pi \sqrt{-1} (\lambda_i - n) \sum_{a \in J_i} z_a} \times \\ & \times \prod_{i=1}^{N-1} \prod_{j=i+1}^N \prod_{a \in J_i} \prod_{b \in J_j} \frac{-1}{\Gamma(1 + z_a - z_b) \Gamma(1 - z_a + z_b + h)} \end{aligned}$$

are linear combinations of the solutions $\Psi_J(\mathbf{z}; h; \mathbf{q})$ of dynamical differential equations (2.6) with coefficients independent of \mathbf{q} . Hence, $\widehat{\Psi}(\mathbf{z}; h; \mathbf{q}) = \Psi^\#(\mathbf{z}; h; \mathbf{q}) \prod_{i=1}^N q_i^{X_i(\mathbf{z}; h; \mathbf{0})}$ solves dynamical differential equations (2.6).

The determinant $\det \widehat{\Psi}(\mathbf{z}; h; \mathbf{q})$ satisfies the equations

$$\left(q_i \frac{\partial}{\partial q_i} - \operatorname{tr} X_i(\mathbf{z}; h; \mathbf{q})|_{(\mathbb{C}^N)_\lambda^{\otimes n}} \right) \det \widehat{\Psi}(\mathbf{z}; h; \mathbf{q}) = 0, \quad i = 1, \dots, N,$$

Since $\operatorname{tr} X_i(\mathbf{z}; h; \mathbf{q})|_{(\mathbb{C}^N)_\lambda^{\otimes n}} = \operatorname{tr} X_i(\mathbf{z}; h; \mathbf{0})|_{(\mathbb{C}^N)_\lambda^{\otimes n}} = d_{\lambda, i}^{(1)}(z_1 + \dots + z_n)$, where $d_{\lambda, i}^{(1)}$ are given by (4.27), the determinant $\det \Psi^\#(\mathbf{z}; h; \mathbf{q})$ does not depend on \mathbf{q} . Therefore,

$$\det \Psi^\#(\mathbf{z}; h; \mathbf{q}) = \det \Psi^\#(\mathbf{z}; h; \mathbf{0}) = 1 \quad \text{and} \quad \det \widehat{\Psi}(\mathbf{z}; h; \mathbf{q}) = \prod_{i=1}^N q_i^{d_{\lambda, i}^{(1)} \sum_{a=1}^n z_a}.$$

The functions $\Psi_J^\circ(\mathbf{z}; h; \mathbf{q})$ in formula (4.55) are entire in \mathbf{z}, h by Theorem 4.7, and the expressions

$$\frac{\check{W}_I(\Sigma_J; \mathbf{z}; h)}{c_\lambda(\Sigma_J; h)} \prod_{i=1}^{N-1} \prod_{j=i+1}^N \prod_{a \in J_i} \prod_{\substack{b \in J_j \\ b < a}} \frac{1}{z_a - z_b - h}$$

are polynomials by Lemma 4.3 applied to the functions $\check{W}_I(\mathbf{t}; \mathbf{z}; h)$. Hence, $\Psi^\#(\mathbf{z}; h; \mathbf{q})$ is holomorphic in \mathbf{z}, h provided $z_a - z_b \notin \mathbb{Z}$ for all $a \neq b$. Moreover, $\Psi^\#(\mathbf{z}; h; \mathbf{q})$ is regular for $z_a = z_b$ by formula (4.25), and the singularities at the hyperplanes $z_a - z_b \in \mathbb{Z}_{\neq 0}$ are simple poles. Theorem 4.17 is proved. \square

Proof of Proposition 4.18. Denote by $\bar{\Psi}_P(\mathbf{z}; h)$ the right-hand side of formula (4.53). Then formula (4.41), Proposition 4.16, and Theorem 4.7 yield $\widehat{\Psi}(\mathbf{z}; h; \mathbf{q}) \bar{\Psi}_P(\mathbf{z}; h) = \Psi_P(\mathbf{z}; h; \mathbf{q})$. Hence by definition (4.50) of the principal term, $\Psi_P^\circ(\mathbf{z}; h) = \bar{\Psi}_P(\mathbf{z}; h)$. \square

Consider the entries of $\Psi^\#(\mathbf{z}; h; \mathbf{q})$ in the standard basis $\{v_I, I \in \mathcal{I}_\lambda\}$ of $(\mathbb{C}^N)_\lambda^{\otimes n}$,

$$(4.56) \quad \Psi^\#(\mathbf{z}; h; \mathbf{q}) : v_J \mapsto \sum_{I \in \mathcal{I}_\lambda} \Psi_{I, J}^\#(\mathbf{z}; h; \mathbf{q}) v_I.$$

Recall the function $A(\mathbf{t}; \mathbf{z}; h; \kappa)$ at $\kappa = 1$, see (4.16),

$$(4.57) \quad A(\mathbf{t}; \mathbf{z}; h) = \prod_{i=1}^{N-1} \prod_{a=1}^{\lambda^{(i)}} \left(\prod_{\substack{b=1 \\ b \neq a}}^{\lambda^{(i)}} \frac{\Gamma(t_b^{(i)} - t_a^{(i)} + 1)}{\Gamma(t_b^{(i)} - t_a^{(i)} + h)} \prod_{c=1}^{\lambda^{(i+1)}} \frac{\Gamma(t_c^{(i+1)} - t_a^{(i)} + h)}{\Gamma(t_c^{(i+1)} - t_a^{(i)} + 1)} \right),$$

where $\lambda^{(N)} = n$ and $t_a^{(N)} = z_a$, $a = 1, \dots, n$. Notice that

$$A(\Sigma_I; \mathbf{z}; h) = (\Gamma(h))^{\lambda^{(1)}} \prod_{i=1}^{N-1} \prod_{j=i+1}^N \prod_{a \in I_i} \prod_{b \in I_j} \frac{\Gamma(z_b - z_a + h)}{\Gamma(z_b - z_a + 1)},$$

where $\lambda^{\{1\}} = \sum_{i=1}^{N-1} \lambda^{(i)}$. Recall the function $\Omega_\lambda(h; \mathbf{q}; \kappa)$ at $\kappa = 1$, see (4.21),

$$(4.58) \quad \Omega_\lambda(h; \mathbf{q}) = \prod_{i=1}^{N-1} \prod_{j=i+1}^N (1 - q_j/q_i)^{h\lambda_i}.$$

For $\mathbf{l} \in \mathbb{Z}^{\lambda^{\{1\}}}$, set

$$(4.59) \quad \mathcal{J}_{I,J,\mathbf{l}}(\mathbf{z}; h) = \sum_{K \in \mathcal{I}_\lambda} \frac{A(\Sigma_K - \mathbf{l}; \mathbf{z}; h) W_I(\Sigma_K - \mathbf{l}; \mathbf{z}; h) \check{W}_J(\Sigma_K; \mathbf{z}; h)}{R_\lambda(\mathbf{z}_{\sigma_K}) Q_\lambda(\mathbf{z}_{\sigma_K}; h) A(\Sigma_K; \mathbf{z}; h) c_\lambda(\Sigma_K - \mathbf{l}; h) c_\lambda(\Sigma_K; h)}.$$

Proposition 4.20. *We have*

$$(4.60) \quad \Psi_{I,J}^\#(\mathbf{z}; h; \mathbf{q}) = \Omega_\lambda(h; \mathbf{q}) \sum_{\mathbf{l} \in \mathbb{Z}_{\geq 0}^{\lambda^{\{1\}}}} \mathcal{J}_{I,J,\mathbf{l}}(\mathbf{z}; h) \prod_{i=1}^{N-1} (q_{i+1}/q_i)^{\sum_{a=1}^{\lambda^{(i)}} l_a^{(i)}}.$$

Proof. The statement follows from formula (4.55) and Theorem 4.7. \square

4.9. The map \mathbb{B}_λ . In Section 4.6, we introduced the space \mathcal{S}_λ of $(\mathbb{C}^N)_\lambda^{\otimes n}$ -valued solutions of the joint system of dynamical differential equations (2.6) and qKZ difference equations (2.4) spanned over \mathbb{C} by the functions $\Psi_P(\mathbf{z}; h; \mathbf{q}; \kappa)$ labeled by Laurent polynomials in $\hat{\Gamma}, \hat{z}, \hat{h}$; we also defined the map

$$\mu_\lambda^\kappa : \mathcal{K}_\lambda \rightarrow \mathcal{S}_\lambda, \quad Y \mapsto \Psi_Y,$$

see (4.43). In Section 4.7 we introduced the Levelt fundamental solution $\widehat{\Psi}(\mathbf{z}; h; \mathbf{q}; \kappa)$ of dynamical differential equations (2.6), see (4.48), (4.55). Denote by $\mathcal{S}_\lambda^\wedge$ the space of $(\mathbb{C}^N)_\lambda^{\otimes n}$ -valued solutions of dynamical differential equations (2.6) spanned over \mathbb{C} by the functions $\widehat{\Psi}(\mathbf{z}; h; \mathbf{q}; \kappa)v$, $v \in (\mathbb{C}^N)_\lambda^{\otimes n}$. Since $\det \widehat{\Psi}(\mathbf{z}; h; \mathbf{q}; \kappa) \neq 0$, see (4.49), there is an isomorphism

$$\mu_\lambda^\wedge : (\mathbb{C}^N)_\lambda^{\otimes n} \rightarrow \mathcal{S}_\lambda^\wedge, \quad v \mapsto \widehat{\Psi}(\mathbf{z}; h; \mathbf{q}; \kappa)v.$$

Let $L \subset \mathbb{C}^n \times \mathbb{C}$ be the complement of the union of the hyperplanes

$$(4.61) \quad z_a - z_b + h \in \kappa \mathbb{Z}_{\leq 0}, \quad z_a - z_b \in \kappa \mathbb{Z}_{\neq 0}, \quad a, b = 1, \dots, n, \quad a \neq b.$$

Denote by \mathcal{O}_L the ring of functions of \mathbf{z}, h holomorphic in L . Let $\mathcal{S}_\lambda^{\mathcal{O}_L}$ be the space of $(\mathbb{C}^N)_\lambda^{\otimes n}$ -valued solutions of dynamical differential equations (2.6) holomorphic in \mathbf{q} provided $|q_{i+1}/q_i| < 1$ for all $i = 1, \dots, N-1$, with a branch of $\log q_i$ fixed for each $i = 1, \dots, N$, and holomorphic in \mathbf{z}, h in L . Both spaces \mathcal{S}_λ and $\mathcal{S}_\lambda^\wedge$ are subspaces of $\mathcal{S}_\lambda^{\mathcal{O}_L}$, see Proposition 4.13 and Theorem 4.17. Let

$$(4.62) \quad \mu_\lambda^{\mathcal{O}_L} : (\mathbb{C}^N)_\lambda^{\otimes n} \otimes_{\mathbb{C}} \mathcal{O}_L \rightarrow \mathcal{S}_\lambda^{\mathcal{O}_L}, \quad v \mapsto \widehat{\Psi}(\mathbf{z}; h; \mathbf{q}; \kappa)v,$$

be the \mathcal{O}_L -linear extension of the map μ_λ^\wedge .

Recall the functions

$$C_\lambda(\mathbf{z}; \kappa) = \prod_{i=1}^N e^{\pi \sqrt{-1} (n - \lambda_i) \sum_{a=\lambda^{(i-1)+1}}^{\lambda_i} z_a / \kappa}$$

and

$$G_{\lambda}(\mathbf{z}; h; \kappa) = \prod_{i=1}^{N-1} \prod_{a=1}^{\lambda^{(i)}} \prod_{b=\lambda^{(i)}+1}^{\lambda^{(i+1)}} \Gamma((z_a - z_b)/\kappa) \Gamma((z_b - z_a + h)/\kappa),$$

see (4.51), (4.52). Define a map

$$(4.63) \quad [P] \mapsto \sum_{I, J \in \mathcal{I}_{\lambda}} \acute{P}(\mathbf{z}_{\sigma_J}; \mathbf{z}; h; \kappa) C_{\lambda}(\mathbf{z}_{\sigma_J}; \kappa) G_{\lambda}(\mathbf{z}_{\sigma_J}; h; \kappa) \frac{W_I(\Sigma_J; \mathbf{z}; h)}{c_{\lambda}(\Sigma_J; h)} v_I,$$

where $[P] \in \mathcal{K}_{\lambda}$ stands for the class of the Laurent polynomial $P(\acute{\Gamma}; \acute{\mathbf{z}}; \acute{h})$. By Proposition 4.18, the map \mathbb{B}_{λ} sends the class $Y \in \mathcal{K}_{\lambda}$ to the principal term of the solution Ψ_Y of the joint system of dynamical differential equations (2.6) and qKZ difference equations (2.4).

Proposition 4.21. *The map $\mathbb{B}_{\lambda} : \mathcal{K}_{\lambda} \rightarrow (\mathbb{C}^N)_{\lambda}^{\otimes n} \otimes_{\mathbb{C}} \mathcal{O}_L$ is well-defined and the following diagram is commutative,*

$$(4.64) \quad \begin{array}{ccc} \mathcal{K}_{\lambda} & \xrightarrow{\mathbb{B}_{\lambda}} & (\mathbb{C}^N)_{\lambda}^{\otimes n} \otimes_{\mathbb{C}} \mathcal{O}_L \\ & \searrow \mu_{\lambda}^{\kappa} & \swarrow \mu_{\lambda}^{\mathcal{O}_L} \\ & \mathcal{S}_{\lambda}^{\mathcal{O}_L} & \end{array}$$

Proof. By Lemma 4.3, poles of the sum in the right-hand side of (4.63) are at most those of the function $G_{\lambda}(\mathbf{z}; h; \kappa)$ and, therefore, in addition to hyperplanes (4.61) can occur only at the hyperplanes $z_a = z_b$, $a \neq b$. However, the sums

$$\sum_{J \in \mathcal{I}_{\lambda}} \acute{P}(\mathbf{z}_{\sigma_J}; \mathbf{z}; h; \kappa) C_{\lambda}(\mathbf{z}_{\sigma_J}; \kappa) G_{\lambda}(\mathbf{z}_{\sigma_J}; h; \kappa) \frac{W_I(\Sigma_J; \mathbf{z}; h)}{c_{\lambda}(\Sigma_J; h)}$$

are regular at the hyperplanes $z_a = z_b$ for all a, b , by the standard reasoning. Hence, the map \mathbb{B}_{λ} is well-defined.

The commutativity of diagram (4.64) follows from Proposition 4.18. \square

Remark. Since $\widehat{\Psi}(\mathbf{z}; h; \mathbf{q}; \kappa)$ is holomorphic in \mathbf{z}, h in L , and $(\det \widehat{\Psi}(\mathbf{z}; h; \mathbf{q}; \kappa))^{-1}$, is entire in \mathbf{z}, h , see (4.49), the inverse matrix $\widehat{\Psi}(\mathbf{z}; h; \mathbf{q}; \kappa)^{-1}$ is holomorphic in \mathbf{z}, h in L . Therefore, for every $\Psi \in \mathcal{S}_{\lambda}^{\mathcal{O}_L}$, its principal term $\Psi^{\theta} = \widehat{\Psi}^{-1} \Psi$, defined by (4.50), belongs to $(\mathbb{C}^N)_{\lambda}^{\otimes n} \otimes_{\mathbb{C}} \mathcal{O}_L$, and there is an isomorphism $\mathcal{S}_{\lambda}^{\mathcal{O}_L} \rightarrow (\mathbb{C}^N)_{\lambda}^{\otimes n} \otimes_{\mathbb{C}} \mathcal{O}_L$, $\Psi \mapsto \Psi^{\theta}$. The inverse map equals $\mu_{\lambda}^{\mathcal{O}_L}$, see (4.62), so that, $\mathcal{S}_{\lambda}^{\mathcal{O}_L} = \widehat{\mathcal{S}}_{\lambda} \otimes_{\mathbb{C}} \mathcal{O}_L$.

4.10. Example $\lambda = (1, n-1)$. Throughout this section, let $N = 2$ and $\lambda = (1, n-1)$. Denote by $[a]$ the element $(\{a\}, \{1, \dots, a-1, a+1, \dots, n\}) \in \mathcal{I}_{\lambda}$. The space $(\mathbb{C}^2)_{\lambda}^{\otimes n}$ has a basis $v_{[1]}, \dots, v_{[n]}$, where $v_{[a]} = v_2^{\otimes(a-1)} \otimes v_1 \otimes v_2^{\otimes(n-a)}$. Clearly $e_{1,1}^{(a)} v_{[b]} = \delta_{a,b} v_{[b]}$ and $e_{2,2}^{(a)} v_{[b]} = (1 - \delta_{a,b}) v_{[b]}$.

The qKZ operators K_1, \dots, K_n , see (2.3), are

$$(4.65) \quad K_a(\mathbf{z}; h; \mathbf{q}; \kappa) = R^{(a,a-1)}(z_a - z_{a-1} + \kappa; h) \dots R^{(a,1)}(z_a - z_1 + \kappa; h) \times \\ \times q_1^{e_{1,1}^{(a)}} q_2^{e_{2,2}^{(a)}} R^{(a,n)}(z_a - z_n; h) \dots R^{(a,a+1)}(z_i - z_{i+1}; h).$$

The R -matrices in the right-hand side preserve the subspace $(\mathbb{C}^2)_\lambda^{\otimes n} \subset (\mathbb{C}^2)^{\otimes n}$, acting there as follows,

$$R^{(a,b)}(z; h) v_{[a]} = \frac{z}{z-h} v_{[a]} - \frac{h}{z-h} v_{[b]}, \quad R^{(a,b)}(z; h) v_{[b]} = \frac{z}{z-h} v_{[b]} - \frac{h}{z-h} v_{[a]}, \\ R^{(a,b)}(z; h) v_{[c]} = v_{[c]}, \quad c \neq a, b.$$

The qKZ difference equations (2.4) are

$$(4.66) \quad f(z_1, \dots, z_a + \kappa, \dots, z_n; h; \mathbf{q}; \kappa) = K_a(\mathbf{z}; h; \mathbf{q}; \kappa) f(\mathbf{z}; h; \mathbf{q}; \kappa), \quad a = 1, \dots, n.$$

The dynamical Hamiltonians X_1, X_2 , see (2.7), act on $(\mathbb{C}^2)_\lambda^{\otimes n}$ as follows

$$(4.67) \quad X_1(\mathbf{z}; h; \mathbf{q}) v_{[a]} = z_a v_{[a]} - h \sum_{b=1}^{a-1} v_{[b]} - \frac{h q_2}{q_1 - q_2} \sum_{\substack{b=1 \\ b \neq a}}^n v_{[b]}, \\ X_2(\mathbf{z}; h; \mathbf{q}) v_{[a]} = \left(-X_1(\mathbf{z}; h; \mathbf{q}) + \sum_{b=1}^n z_b \right) v_{[a]},$$

and the dynamical differential equations (2.6) are

$$(4.68) \quad \kappa q_1 \frac{\partial}{\partial q_1} \Psi(\mathbf{z}; h; \mathbf{q}; \kappa) = X_1(\mathbf{z}; h; \mathbf{q}) \Psi(\mathbf{z}; h; \mathbf{q}; \kappa), \\ \kappa q_2 \frac{\partial}{\partial q_2} \Psi(\mathbf{z}; h; \mathbf{q}; \kappa) = X_2(\mathbf{z}; h; \mathbf{q}) \Psi(\mathbf{z}; h; \mathbf{q}; \kappa).$$

In this section, we use the variable $t = t_1^{(1)}$. The substitution $\mathbf{t} = \Sigma_{[a]}$ reads as $t = z_a$. The weight functions are

$$(4.69) \quad W_{[a]}(t; \mathbf{z}; h) = \prod_{b=1}^{a-1} (t - z_b) \prod_{b=a+1}^n (t - z_b - h), \quad a = 1, \dots, n.$$

We have $c_\lambda(t; h) = 1$, see (4.5). The permutations $\sigma_{[1]}, \dots, \sigma_{[n]}$ are

$$\sigma_{[a]}(1) = a, \quad \sigma_{[a]}(b) = b - 1, \quad b = 1, \dots, a - 1, \quad \sigma_{[a]}(b) = b, \quad b = a + 1, \dots, n,$$

and $|\sigma_{[a]}| = a - 1$. We have

$$(4.70) \quad W_{[a]}(z_a; \mathbf{z}; h) = \prod_{b=1}^{a-1} (z_a - z_b) \prod_{b=a+1}^n (z_a - z_b - h),$$

$$W_{[a]}(z_b; \mathbf{z}; h) = 0, \quad b = 1, \dots, a-1,$$

and $W_{[a]}(z_b; \mathbf{z}; h)$ is divisible by the product $\prod_{b=a+1}^n (z_a - z_b - h)$ for any $b = 1, \dots, n$, cf. Lemmas 4.2, 4.3

The functions $\check{W}_{[a]}(t; \mathbf{z}; h)$, see (4.7), are

$$(4.71) \quad \check{W}_{[a]}(t; \mathbf{z}; h) = \prod_{b=1}^{a-1} (t - z_b - h) \prod_{c=a+1}^n (t - z_c), \quad a = 1, \dots, n.$$

Set

$$\bar{R}_a(\mathbf{z}) = \prod_{\substack{b=1 \\ b \neq a}}^n (z_a - z_b), \quad \bar{Q}_a(\mathbf{z}; h) = \prod_{\substack{b=1 \\ b \neq a}}^n (z_a - z_b - h), \quad a = 1, \dots, n.$$

Then $\bar{R}_a(\mathbf{z}) = R_\lambda(\mathbf{z}_{\sigma_{[a]}})$, and $\bar{Q}_a(\mathbf{z}; h) = Q_\lambda(\mathbf{z}_{\sigma_{[a]}}; h)$, where the functions $R_\lambda(\mathbf{z})$, $Q_\lambda(\mathbf{z}; h)$, are given by (4.8). Biorthogonality relations (4.9), (4.10) become

$$\sum_{c=1}^n \frac{W_{[a]}(z_c; \mathbf{z}; h) \check{W}_{[b]}(z_c; \mathbf{z}; h)}{\bar{R}_c(\mathbf{z}) \bar{Q}_c(\mathbf{z}; h)} = \delta_{a,b},$$

$$\sum_{c=1}^n W_{[c]}(z_a; \mathbf{z}; h) \check{W}_{[c]}(z_b; \mathbf{z}; h) = \delta_{a,b} \bar{R}_a(\mathbf{z}) \bar{Q}_a(\mathbf{z}; h).$$

The master function, see (4.11), is

$$\begin{aligned} \Phi_\lambda(t; \mathbf{z}; h; \mathbf{q}; \kappa) &= \\ &= (e^{\pi\sqrt{-1}} q_2)^{\sum_{a=1}^n z_a/\kappa} (e^{\pi\sqrt{-1}(n-2)} q_1/q_2)^{t/\kappa} \prod_{a=1}^n \Gamma((t - z_a)/\kappa) \Gamma((z_a - t + h)/\kappa). \end{aligned}$$

The hypergeometric solutions (4.20) of the joint system of differential equations (4.68) and difference equations (4.66) have the form

$$\Psi_{[a]}(\mathbf{z}; h; \mathbf{q}; \kappa) = \frac{(1 - q_2/q_1)^{h/\kappa}}{\kappa \Gamma(h/\kappa)} \sum_{b=1}^n \sum_{l=0}^{\infty} \text{Res}_{t=z_a-l\kappa} \Phi_\lambda(t; \mathbf{z}; h; \mathbf{q}; \kappa) W_{[b]}(z_a - l\kappa; h) v_{[b]},$$

where the residues of the master function are

$$\begin{aligned} \text{Res}_{t=z_a-l\kappa} \Phi_\lambda(t; \mathbf{z}; h; \mathbf{q}; \kappa) &= q_1^{z_a/\kappa} q_2^{\sum_{c=1, c \neq a}^n z_c/\kappa} \prod_{\substack{c=1 \\ c \neq a}}^n \frac{\pi e^{\pi\sqrt{-1}(z_a+z_c)/\kappa}}{\sin(\pi(z_a - z_c)/\kappa)} \times \\ &\times \frac{\kappa (q_2/q_1)^l \Gamma(l + h/\kappa)}{l!} \prod_{\substack{c=1 \\ c \neq a}}^n \frac{\Gamma(l + (z_c - z_a + h)/\kappa)}{\Gamma(l + 1 + (z_c - z_a)/\kappa)}. \end{aligned}$$

Determinant formula for coordinates of the hypergeometric solutions, see (4.26), is

$$(4.72) \quad \det \left(\sum_{l=0}^{\infty} \operatorname{Res}_{t=z_a-l\kappa} \Phi_{\lambda}(t; \mathbf{z}; h; \mathbf{q}; \kappa) W_{[b]}(z_a - l\kappa; h) \right)_{a,b=1}^n =$$

$$= \frac{(q_1 q_2^{n-1})^{\sum_{a=1}^n z_a/\kappa} (\kappa \Gamma(h/\kappa))^n}{(1 - q_2/q_1)^{nh/\kappa}} \prod_{a=1}^{n-1} \prod_{b=a+1}^n \frac{\pi \kappa^2 \Gamma((z_a - z_b + h)/\kappa) \Gamma(1 + (z_b - z_a + h)/\kappa)}{e^{-\pi\sqrt{-1}(z_a+z_b)/\kappa} \sin(\pi(z_a - z_b)/\kappa)}.$$

By formulae (4.69), (4.70), and the Vandermonde determinant formula, equality (4.72) transforms to

$$(4.73) \quad \det \left(\sum_{l=0}^{\infty} \sum_{c=1}^n \operatorname{Res}_{t=z_c-l\kappa} (t^{a-1} e^{-2\pi\sqrt{-1}(b-1)t/\kappa} \Phi_{\lambda}(t; \mathbf{z}; h; \mathbf{q}; \kappa)) \right)_{a,b=1}^n =$$

$$= \frac{(q_1 q_2^{n-1})^{\sum_{a=1}^n z_a/\kappa} (-2\pi\sqrt{-1})^{n(n-1)/2} \kappa^{n(n+1)/2} (\Gamma(h/\kappa))^n}{(1 - q_2/q_1)^{nh/\kappa}} \times$$

$$\times \prod_{a=1}^{n-1} \prod_{b=a+1}^n \Gamma((z_a - z_b + h)/\kappa) \Gamma((z_b - z_a + h)/\kappa).$$

Let $\gamma = \gamma_{1,1}$. For a Laurent polynomial $P(\gamma; \mathbf{z}; h)$, we have

$$\acute{P}(\gamma; \mathbf{z}; h; \kappa) = P(e^{2\pi\sqrt{-1}\gamma/\kappa}; e^{2\pi\sqrt{-1}z_1/\kappa}, \dots, e^{2\pi\sqrt{-1}z_n/\kappa}; e^{2\pi\sqrt{-1}h/\kappa}).$$

The solution Ψ_P of differential equations (4.68) and difference equations (4.66) corresponding to $P(\gamma; \mathbf{z}; h)$ is

$$(4.74) \quad \Psi_P(\mathbf{z}; h; \mathbf{q}; \kappa) = \sum_{a=1}^n \acute{P}(z_a; \mathbf{z}; h; \kappa) \Psi_{[a]}(\mathbf{z}; h; \mathbf{q}; \kappa).$$

By Proposition 4.13, $\Psi_P(\mathbf{z}; h; \mathbf{q}; \kappa)$ is holomorphic in $\mathbf{z}, h, \mathbf{q}$ provided $z_a - z_b + h \notin \kappa\mathbb{Z}_{\leq 0}$ for all $a, b = 1, \dots, n$, $a \neq b$, and $|q_2/q_1| < 1$ with branches of $\log q_1$ and $\log q_2$ fixed. The singularities of $\Psi_P(\mathbf{z}; h; \mathbf{q}; \kappa)$ at the hyperplanes $z_a - z_b + h \in \kappa\mathbb{Z}_{\leq 0}$ are simple poles.

The solution $\Psi_P(\mathbf{z}; h; \mathbf{q}; \kappa)$ can be written as an integral over a suitable contour C encircling the poles of the product $\prod_{a=1}^n \Gamma((t - z_a)/\kappa)$ counterclockwise and separating them from the poles of the product $\prod_{a=1}^n \Gamma((z_a - t + h)/\kappa)$,

$$(4.75) \quad \Psi_P(\mathbf{z}; h; \mathbf{q}; \kappa) =$$

$$= \frac{(1 - q_2/q_1)^{h/\kappa}}{2\pi\sqrt{-1} \kappa \Gamma(h/\kappa)} \int_C \acute{P}(t; \mathbf{z}; h; \kappa) \Phi_{\lambda}(t; \mathbf{z}; h; \mathbf{q}; \kappa) \sum_{a=1}^n W_{[a]}(t; \mathbf{z}; h) v_{[a]} dt.$$

For instance, if h/κ is sufficiently large positive real, the integral can be taken over the parabola

$$C = \{ h/2 - \kappa(s^2 - s\sqrt{-1}) \mid s \in \mathbb{R} \}.$$

Formula (4.75) can be used to give an alternative proof of analytic properties of the function $\Psi_P(\mathbf{z}; h; \mathbf{q}; \kappa)$.

Let \mathcal{S}_λ be the space of solutions of the joint system of dynamical differential equations (4.68) and qKZ difference equations (4.66) spanned over \mathbb{C} by the functions $\Psi_P(\mathbf{z}; h; \mathbf{q}; \kappa)$ corresponding to Laurent polynomials $P(\gamma; \hat{\mathbf{z}}; \hat{h})$. The space \mathcal{S}_λ is a $\mathbb{C}[\hat{\mathbf{z}}^{\pm 1}, \hat{h}^{\pm 1}]$ -module with $f(\hat{\mathbf{z}}; \hat{h})$ acting as multiplication by $f(e^{2\pi\sqrt{-1}z_1/\kappa}, \dots, e^{2\pi\sqrt{-1}z_n/\kappa}; e^{2\pi\sqrt{-1}h/\kappa})$.

For $\lambda = (1, n-1)$, the algebra \mathcal{K}_λ , see (4.42), can be presented as follows

$$(4.76) \quad \mathcal{K}_\lambda = \mathbb{C}[\hat{\gamma}^{\pm 1}, \hat{\mathbf{z}}^{\pm 1}, \hat{h}^{\pm 1}] \left/ \left\langle \prod_{a=1}^n (\hat{\gamma} - \hat{z}_a) = 0 \right\rangle \right.$$

The function $\Psi_P(\mathbf{z}; h; \mathbf{q}; \kappa)$ depends only on the class of the Laurent polynomial P in \mathcal{K}_λ . The assignment $P \mapsto \Psi_P$ defines the homomorphism

$$\mu_\lambda^\kappa : \mathcal{K}_\lambda \rightarrow \mathcal{S}_\lambda, \quad Y \mapsto \Psi_Y,$$

of $\mathbb{C}[\hat{\mathbf{z}}^{\pm 1}, \hat{h}^{\pm 1}]$ -modules. Formula (4.73) implies that the homomorphism μ_λ^κ is an isomorphism.

The algebra \mathcal{K}_λ is the equivariant K -theory algebra $K_{T \times \mathbb{C}^\times}(T^*\mathbb{C}\mathbb{P}^{n-1}; \mathbb{C})$ of the cotangent bundle of the projective space $\mathbb{C}\mathbb{P}^{n-1}$, see the notation in Section 6.

Recall that a function $f(\mathbf{q})$ is called holomorphic in the unit polydisk around $\mathbf{q} = \mathbf{0}$, if $f(\mathbf{q}) = g(q_2/q_1)$ for a function $g(s)$ holomorphic provided $|s| < 1$, and $f(\mathbf{0}) = g(0)$. For example, the dynamical Hamiltonians X_1, X_2 , see (4.67), are holomorphic in the unit polydisk around $\mathbf{q} = \mathbf{0}$, and $X_1(\mathbf{z}; h; \mathbf{0}), X_2(\mathbf{z}; h; \mathbf{0})$ act on $(\mathbb{C}^2)_\lambda^{\otimes n}$ as follows

$$X_1(\mathbf{z}; h; \mathbf{0})v_{[a]} = z_a v_{[a]} - h \sum_{b=1}^{a-1} v_{[b]}, \quad X_2(\mathbf{z}; h; \mathbf{0})v_{[a]} = \left(-X_1(\mathbf{z}; h; \mathbf{0}) + \sum_{b=1}^n z_b \right) v_{[a]},$$

The Levelt fundamental solution $\widehat{\Psi}(\mathbf{z}; h; \mathbf{q}; \kappa)$, see (4.48), of differential equations (4.68) is

$$\widehat{\Psi}(\mathbf{z}; h; \mathbf{q}; \kappa) = \Psi^\#(\mathbf{z}/\kappa; h/\kappa; \mathbf{q}) q_1^{X_1(\mathbf{z}; h; \mathbf{0})/\kappa} q_2^{X_2(\mathbf{z}; h; \mathbf{0})/\kappa},$$

where the $\text{End}((\mathbb{C}^N)_\lambda^{\otimes n})$ -valued function $\Psi^\#(\mathbf{z}; h; \mathbf{q})$ is as follows,

$$\begin{aligned} \Psi^\#(\mathbf{z}; h; \mathbf{q}) : v_{[b]} &\mapsto \sum_{a=1}^n \Psi_{a,b}^\#(\mathbf{z}; h; \mathbf{q}) v_{[a]}, \\ \Psi_{a,b}^\#(\mathbf{z}; h; \mathbf{q}) &= \\ &= (1 - q_2/q_1)^h \sum_{l=0}^{\infty} (q_2/q_1)^l \sum_{c=1}^n \frac{W_{[a]}(z_c - l; \mathbf{z}; h) \check{W}_{[b]}(z_c; \mathbf{z}; h)}{\bar{R}_c(\mathbf{z}) \bar{Q}_c(\mathbf{z}; h)} \prod_{d=1}^n \prod_{m=0}^{l-1} \frac{z_d - z_c + h + m}{z_d - z_c + 1 + m}. \end{aligned}$$

Furthermore, one has $\det \Psi^\#(\mathbf{z}; h; \mathbf{q}) = 1$ and $\det \widehat{\Psi}(\mathbf{z}; h; \mathbf{q}; \kappa) = (q_1 q_2^{n-1})^{\sum_{a=1}^n z_a/\kappa}$.

For a solution $\Psi(\mathbf{z}; h; \mathbf{q}; \kappa)$ of dynamical differential equations (4.68), its principal term, see (4.50), is

$$\Psi^0(\mathbf{z}; h; \kappa) = \widehat{\Psi}(\mathbf{z}; h; \mathbf{q}; \kappa)^{-1} \Psi(\mathbf{z}; h; \mathbf{q}; \kappa).$$

The principal term of the solution $\Psi_P(\mathbf{z}; h; \mathbf{q}; \kappa)$, corresponding to a Laurent polynomial $P(\gamma; \dot{\mathbf{z}}; \dot{h})$, see (4.53), equals

$$(4.77) \quad \Psi_P^0(\mathbf{z}; h; \kappa) = \sum_{a=1}^n v_{[a]} \sum_{b=1}^n W_{[a]}(z_b; \mathbf{z}; h) \dot{P}(z_b; \mathbf{z}; h; \kappa) e^{\pi\sqrt{-1}((n-2)z_b + \sum_{c=1}^n z_c)} \times \\ \times \prod_{\substack{c=1 \\ c \neq b}}^n \Gamma((z_b - z_c)/\kappa) \Gamma((z_c - z_b + h)/\kappa).$$

Let L be the complement of the union of the hyperplanes

$$z_a - z_b \in \kappa \mathbb{Z}_{\neq 0}, \quad z_a - z_b + h \in \kappa \mathbb{Z}_{\leq 0}, \quad a, b = 1, \dots, n, \quad a \neq b,$$

cf. (4.61). Denote by \mathcal{O}_L the ring of functions of \mathbf{z}, h holomorphic in L . The map

$$(4.78) \quad \mathbb{B}_\lambda : \mathcal{K}_\lambda \rightarrow (\mathbb{C}^N)_\lambda^{\otimes n} \otimes_{\mathbb{C}} \mathcal{O}_L, \\ [P] \mapsto \sum_{a=1}^n v_{[a]} \sum_{b=1}^n W_{[a]}(z_b; \mathbf{z}; h) \dot{P}(z_b; \mathbf{z}; h; \kappa) e^{\pi\sqrt{-1}((n-2)z_b + \sum_{c=1}^n z_c)/\kappa} \times \\ \times \prod_{\substack{c=1 \\ c \neq b}}^n \Gamma((z_b - z_c)/\kappa) \Gamma((z_c - z_b + h)/\kappa),$$

sends the class $[P] \in \mathcal{K}_\lambda$ of the Laurent polynomial $P(\gamma; \dot{\mathbf{z}}; \dot{h})$ to the principal term of the solution Ψ_P of the joint system of differential equations (4.68) and difference equations (4.66).

Let $\mathcal{S}_\lambda^{\mathcal{O}_L}$ be the space of $(\mathbb{C}^N)_\lambda^{\otimes n}$ -valued solutions of dynamical differential equations (4.68) holomorphic in \mathbf{q} provided $|q_2/q_1| < 1$ with branches of $\log q_1$ and $\log q_2$ fixed, and holomorphic in \mathbf{z}, h in L . The space \mathcal{S}_λ is a subspace of $\mathcal{S}_\lambda^{\mathcal{O}_L}$.

Since the matrix $\widehat{\Psi}(\mathbf{z}; h; \mathbf{q}; \kappa)$ is holomorphic in \mathbf{z}, h in L , and $(\det \widehat{\Psi}(\mathbf{z}; h; \mathbf{q}; \kappa))^{-1}$ is entire in \mathbf{z}, h , the inverse matrix $\widehat{\Psi}(\mathbf{z}; h; \mathbf{q}; \kappa)^{-1}$ is also holomorphic in \mathbf{z}, h in L . Thus the map

$$\mu_\lambda^{\mathcal{O}_L} : (\mathbb{C}^N)_\lambda^{\otimes n} \otimes_{\mathbb{C}} \mathcal{O}_L \rightarrow \mathcal{S}_\lambda^{\mathcal{O}_L}, \quad v \mapsto \widehat{\Psi}(\mathbf{z}; h; \mathbf{q}; \kappa)v,$$

gives an isomorphism of \mathcal{O}_L -modules. Furthermore, the following diagram is commutative,

$$\begin{array}{ccc} \mathcal{K}_\lambda & \xrightarrow{\mathbb{B}_\lambda} & (\mathbb{C}^N)_\lambda^{\otimes n} \otimes_{\mathbb{C}} \mathcal{O}_L \\ & \searrow \mu_\lambda^\kappa & \swarrow \mu_\lambda^{\mathcal{O}_L} \\ & \mathcal{S}_\lambda^{\mathcal{O}_L} & \end{array}$$

see Proposition 4.21.

5. LIMIT $h \rightarrow \infty$ FOR SOLUTIONS OF THE DYNAMICAL AND qKZ EQUATIONS

5.1. **Limiting weight functions $W_I^\circ(\mathbf{t}; \mathbf{z})$.** For $I \in \mathcal{I}_\lambda$, define the *limiting weight functions*

$$(5.1) \quad W_I^\circ(\mathbf{t}; \mathbf{z}) = \text{Sym}_{t_1^{(1)}, \dots, t_{\lambda(1)}^{(1)}} \dots \text{Sym}_{t_1^{(N-1)}, \dots, t_{\lambda(N-1)}^{(N-1)}} U_I^\circ(\mathbf{t}; \mathbf{z}),$$

$$U_I^\circ(\mathbf{t}; \mathbf{z}) = \prod_{j=1}^{N-1} \prod_{a=1}^{\lambda^{(j)}} \left(\prod_{\substack{c=1 \\ i_c^{(j+1)} < i_a^{(j)}}}^{\lambda^{(j+1)}} (t_a^{(j)} - t_c^{(j+1)}) \prod_{b=a+1}^{\lambda^{(j)}} \frac{1}{t_b^{(j)} - t_a^{(j)}} \right).$$

Recall the element I_λ^{\min} and the permutation σ_I . Set

$$d_I = \sum_{j=1}^{N-1} \lambda^{(j)} (\lambda^{(j+1)} - 1) - |\sigma_I|.$$

Lemma 5.1. *For $I \in \mathcal{I}_\lambda$, we have*

$$W_I^\circ(\mathbf{t}; \mathbf{z}) = \lim_{h \rightarrow \infty} (-h)^{-d_I} W_I(\mathbf{t}; \mathbf{z}; h).$$

Proof. The statement follows from formulae (4.2), (5.1) by induction on the length of σ_I . \square

Example. Let $N = 2$, $n = 2$, $\lambda = (1, 1)$, $I = (\{1\}, \{2\})$, $J = (\{2\}, \{1\})$. Then

$$W_I^\circ(\mathbf{t}; \mathbf{z}) = 1, \quad W_J^\circ(\mathbf{t}; \mathbf{z}) = t_1^{(1)} - z_1.$$

Lemma 5.2. *We have $W_{I_\lambda^{\min}}^\circ(\mathbf{t}; \mathbf{z}) = 1$ and*

$$(5.2) \quad W_{\sigma_0(I_\lambda^{\min})}^\circ(\mathbf{t}; \mathbf{z}) = \prod_{j=1}^{N-1} \prod_{a=1}^{\lambda^{(j)}} \prod_{b=\lambda^{(j)}+1}^{\lambda^{(j+1)}} (t_a^{(j)} - z_{n-b+1}),$$

where $\sigma_0 \in S_n$ is the longest permutation, $\sigma_0(a) = n + 1 - a$, $a = 1, \dots, n$.

Let $\Upsilon_\lambda = \{i \mid \lambda_{i+1} \neq 0\} \subset \{1, \dots, N-1\}$.

Lemma 5.3. *For the transpositions $s_{\lambda^{(i)}, \lambda^{(i+1)}}$, $i \in \Upsilon_\lambda$, we have*

$$(5.3) \quad W_{s_{\lambda^{(i)}, \lambda^{(i+1)}}(I_\lambda^{\min})}^\circ(\mathbf{t}; \mathbf{z}) = \sum_{j=1}^{\lambda^{(i)}} (t_j^{(i)} - z_j).$$

Lemmas 5.2 and 5.3 are proved in Appendix C

The following lemma describes the $h \rightarrow \infty$ limit of the three-term relations of Lemma 4.1 for functions $W_I^\circ(\mathbf{t}; \mathbf{z}; h)$.

Lemma 5.4. *For any $I \in \mathcal{I}_\lambda$, $i = 1, \dots, N-1$, and $a = 1, \dots, n-1$, we have*

$$W_{s_{a, a+1}(I)}^\circ(\mathbf{t}; \mathbf{z}) = \frac{W_I^\circ(\mathbf{t}; \mathbf{z}) - W_I^\circ(\mathbf{t}; z_1, \dots, z_{a+1}, z_a, \dots, z_n)}{z_{a+1} - z_a},$$

if $|s_{a, a+1} \sigma_I| < |\sigma_I|$, and $W_I^\circ(\mathbf{t}; z_1, \dots, z_{a+1}, z_a, \dots, z_n) = W_I^\circ(\mathbf{t}; \mathbf{z})$, otherwise.

Proof. Notice that $\sigma_{s_{a,a+1}(I)} = s_{a,a+1}\sigma_I$ unless $s_{a,a+1}(I) = I$. Now the statement follows from Lemmas 5.1 and 4.1. \square

Remark. Define the operators $\Delta_1, \dots, \Delta_{n-1}$ acting on functions of z_1, \dots, z_n :

$$\Delta_a f(\mathbf{z}) = \frac{f(\mathbf{z}) - f(z_1, \dots, z_{a+1}, z_a, \dots, z_n)}{z_a - z_{a+1}}.$$

They satisfy the nil-Coxeter relations,

$$\Delta_a^2 = 0, \quad \Delta_a \Delta_b = \Delta_b \Delta_a, \quad |a - b| > 1, \quad \Delta_a \Delta_{a+1} \Delta_a = \Delta_{a+1} \Delta_a \Delta_{a+1},$$

for any a, b .

Denote $\mathbf{y} = (y_1, \dots, y_n)$. Let the functions $\tilde{W}_I^\circ(\mathbf{y}; \mathbf{z})$ be obtained from $W_I^\circ(\mathbf{t}; \mathbf{z})$ by the substitution $t_j^{(i)} = y_j$, $i = 1, \dots, N-1$, $j = 1, \dots, \lambda^{(i)}$. By Lemmas 5.2, 5.4, B.4, and Proposition B.3, the functions $\tilde{W}_I^\circ(\mathbf{y}; \mathbf{z})$ coincide with the A -type double Schubert polynomials $\mathfrak{S}_\sigma(\mathbf{y}; \mathbf{z})$,

$$(5.4) \quad \tilde{W}_I^\circ(\mathbf{y}; \mathbf{z}) = \mathfrak{S}_{\sigma_I}(\mathbf{y}; \mathbf{z}).$$

For $I \in \mathcal{I}_\lambda$, define

$$(5.5) \quad \check{W}_I^\circ(\mathbf{t}; \mathbf{z}) = W_{\sigma_0(I)}^\circ(\mathbf{t}; z_n, \dots, z_1),$$

where σ_0 is the longest permutation. Recall $R_\lambda(\mathbf{z})$, see (4.8), and $\mathbf{z}_\sigma = (z_{\sigma(1)}, \dots, z_{\sigma(n)})$.

Lemmas 5.5 and 5.6 below follow by Lemma 5.1 from Lemma 4.2, Proposition 4.4, and Corollary 4.5, respectively. Lemmas 5.5 and 5.6 are equivalent to the vanishing and orthogonality properties of the double Schubert polynomials.

Lemma 5.5. *For $I, J \in \mathcal{I}_\lambda$, we have $W_J^\circ(\Sigma_I; \mathbf{z}; h) = 0$ unless $I = J$ or $|\sigma_I| > |\sigma_J|$, and*

$$(5.6) \quad W_I^\circ(\Sigma_I; \mathbf{z}) = \prod_{j=1}^{N-1} \prod_{k=j+1}^N \prod_{\substack{a \in I_j \\ b \in I_k \\ b < a}} (z_a - z_b).$$

Lemma 5.6. *The functions $W_I^\circ(\mathbf{t}; \mathbf{z})$ and $\check{W}_J^\circ(\mathbf{t}; \mathbf{z})$ are biorthogonal,*

$$(5.7) \quad \sum_{I \in \mathcal{I}_\lambda} \frac{W_J^\circ(\Sigma_I; \mathbf{z}) \check{W}_K^\circ(\Sigma_I; \mathbf{z})}{R_\lambda(\mathbf{z}_{\sigma_I})} = \delta_{J,K}, \quad \sum_{I \in \mathcal{I}_\lambda} W_I^\circ(\Sigma_J; \mathbf{z}) \check{W}_I^\circ(\Sigma_K; \mathbf{z}) = \delta_{J,K} R_\lambda(\mathbf{z}_{\sigma_J}).$$

5.2. Limiting master function. Define the *limiting master function*:

$$(5.8) \quad \Phi_\lambda^\circ(\mathbf{t}; \mathbf{z}; \mathbf{p}; \kappa) = (\kappa^{\lambda_{N-n}} p_N)^{\sum_{a=1}^n z_a / \kappa} \prod_{i=1}^{N-1} (\kappa^{\lambda_i + \lambda_{i+1}} p_i / p_{i+1})^{\sum_{a=1}^{\lambda^{(i)}} t_a^{(i)} / \kappa} \times \\ \times \prod_{i=1}^{N-1} \prod_{\substack{a=1 \\ b \neq a}}^{\lambda^{(i)}} \left(\prod_{b=1}^{\lambda^{(i)}} \frac{1}{\Gamma((t_a^{(i)} - t_b^{(i)}) / \kappa)} \prod_{c=1}^{\lambda^{(i+1)}} \Gamma((t_a^{(i)} - t_c^{(i+1)}) / \kappa) \right),$$

where $\lambda^{(N)} = n$ and $t_a^{(N)} = z_a$, $a = 1, \dots, n$. The last formula can be also written as

$$\begin{aligned} \Phi_\lambda^\circ(\mathbf{t}; \mathbf{z}; \mathbf{p}; \kappa) &= (\kappa^{\lambda_N - n} p_N)^{\sum_{a=1}^n z_a / \kappa} \prod_{i=1}^{N-1} (\kappa^{\lambda_i + \lambda_{i+1}} p_i / p_{i+1})^{\sum_{a=1}^{\lambda^{(i)}} t_a^{(i)} / \kappa} \times \\ &\times \prod_{i=1}^{N-1} \prod_{a=1}^{\lambda^{(i)}} \left(\prod_{b=1}^{a-1} \frac{(t_a^{(i)} - t_b^{(i)}) \sin(\pi(t_b^{(i)} - t_a^{(i)}) / \kappa)}{\pi \kappa} \prod_{c=1}^{\lambda^{(i+1)}} \Gamma((t_a^{(i)} - t_c^{(i+1)}) / \kappa) \right), \end{aligned}$$

Denote by $\tilde{\Phi}_\lambda(\mathbf{t}; \mathbf{z}; h; \mathbf{p}; \kappa)$ the function obtained from the master function $\Phi_\lambda(\mathbf{t}; \mathbf{z}; h; \mathbf{q}; \kappa)$ by the substitution

$$(5.9) \quad q_i = p_i h^{\sum_{j=i+1}^N \lambda_j - \sum_{j=1}^{i-1} \lambda_j} e^{\pi \sqrt{-1} (\lambda_i - n)}, \quad i = 1, \dots, N,$$

cf. (3.9).

Recall $\lambda^{\{1\}} = \sum_{i=1}^{N-1} \lambda^{(i)}$, $\lambda^{\{2\}} = \sum_{i=1}^{N-1} (\lambda^{(i)})^2$, $\lambda_{\{2\}} = \sum_{1 \leq i < j \leq N} \lambda_i \lambda_j$.

Lemma 5.7. *For $|\arg(h/\kappa)| < \pi$, we have*

$$\Phi_\lambda^\circ(\mathbf{t}; \mathbf{z}; \mathbf{p}; \kappa) = \lim_{h \rightarrow \infty} (-h)^{\lambda^{\{2\}} - \lambda^{\{1\}}} (\Gamma(h/\kappa))^{-\lambda^{\{1\}} - \lambda_{\{2\}}} \tilde{\Phi}_\lambda(\mathbf{t}; \mathbf{z}; h; \mathbf{p}; \kappa).$$

Proof. The statement follows from formulae (4.11), (5.8) by Stirling's formula

$$(5.10) \quad \frac{\Gamma(\alpha + h/\kappa)}{\Gamma(\beta + h/\kappa)} \sim (h/\kappa)^{\alpha - \beta}, \quad h \rightarrow \infty, \quad |\arg(h/\kappa)| < \pi. \quad \square$$

5.3. Solutions of the limiting dynamical and qKZ equations. For a polynomial $f(\mathbf{t}; \mathbf{z})$, define the Jackson integral

$$(5.11) \quad \mathcal{M}_J(\Phi_\lambda^\circ f)(\mathbf{z}; \mathbf{p}; \kappa) = \sum_{l \in \mathbb{Z}^{\lambda^{\{1\}}}} \text{Res}_{\mathbf{t} = \Sigma_J - l\kappa} (\Phi_\lambda^\circ(\mathbf{t}; \mathbf{z}; \mathbf{p}; \kappa) f(\mathbf{t}; \mathbf{z})), \quad J \in \mathcal{I}_\lambda.$$

Notice that the master function $\Phi_\lambda^\circ(\mathbf{t}; \mathbf{z}; \mathbf{p}; \kappa)$ has only simple poles, and

$$\text{Res}_{\mathbf{t} = \Sigma_J - l\kappa} (\Phi_\lambda^\circ(\mathbf{t}; \mathbf{z}; \mathbf{p}; \kappa) f(\mathbf{t}; \mathbf{z})) = f(\Sigma_J - l\kappa; \mathbf{z}) \text{Res}_{\mathbf{t} = \Sigma_J - l\kappa} \Phi_\lambda^\circ(\mathbf{t}; \mathbf{z}; \mathbf{p}; \kappa).$$

A closed expression for the residue $\text{Res}_{\mathbf{t} = \Sigma_J - l\kappa} \Phi_\lambda^\circ(\mathbf{t}; \mathbf{z}; \mathbf{p}; \kappa)$ is given by Lemma 5.8 below.

Set

$$(5.12) \quad M_\lambda^\circ(\mathbf{z}; \kappa) = \pi^{-\lambda_{\{2\}}} \prod_{i=1}^{N-1} \prod_{a=1}^{\lambda^{(i)}} \prod_{b=\lambda^{(i)}+1}^{\lambda^{(i+1)}} \sin(\pi(z_a - z_b) / \kappa),$$

cf. (4.15), and

$$(5.13) \quad A^\circ(\mathbf{t}; \mathbf{z}; \kappa) = \prod_{i=1}^{N-1} \prod_{a=1}^{\lambda^{(i)}} \left(\prod_{\substack{b=1 \\ b \neq a}}^{\lambda^{(i)}} \Gamma(1 + (t_b^{(i)} - t_a^{(i)}) / \kappa) \prod_{c=1}^{\lambda^{(i+1)}} \frac{1}{\Gamma(1 + (t_c^{(i+1)} - t_a^{(i)}) / \kappa)} \right).$$

Lemma 5.8. *If $\mathbf{l} \notin \mathbb{Z}_{\geq 0}^{\lambda\{1\}}$, then $\text{Res}_{t=\Sigma_J - \mathbf{l}\kappa} \Phi_\lambda^\circ(\mathbf{t}; \mathbf{z}; \mathbf{p}; \kappa) = 0$. For $\mathbf{l} \in \mathbb{Z}_{\geq 0}^{\lambda\{1\}}$,*

$$(5.14) \quad \text{Res}_{t=\Sigma_J - \mathbf{l}\kappa} \Phi_\lambda^\circ(\mathbf{t}; \mathbf{z}; \mathbf{p}; \kappa) = \frac{A_\lambda^\circ(\Sigma_J - \mathbf{l}\kappa; \mathbf{z}; \kappa)}{M_\lambda^\circ(\mathbf{z}_{\sigma_J}; \kappa)} \times \\ \times \kappa^{\lambda\{1\}} \prod_{i=1}^N (\kappa^{\sum_{j=i+1}^N \lambda_j - \sum_{j=1}^{i-1} \lambda_j} p_i)^{\sum_{a \in J_i} z_a / \kappa} \prod_{i=1}^{N-1} ((-\kappa)^{-\lambda_i - \lambda_{i+1}} p_{i+1} / p_i)^{\sum_{a=1}^{\lambda(i)} l_a^{(i)}}.$$

In particular,

$$(5.15) \quad \text{Res}_{t=\Sigma_J} \Phi_\lambda^\circ(\mathbf{t}; \mathbf{z}; \mathbf{p}; \kappa) = \\ = \kappa^{\lambda\{1\}} \prod_{i=1}^N (\kappa^{\sum_{j=i+1}^N \lambda_j - \sum_{j=1}^{i-1} \lambda_j} p_i)^{\sum_{a \in J_i} z_a / \kappa} \prod_{i=1}^{N-1} \prod_{j=i+1}^N \prod_{a \in J_i} \prod_{b \in J_j} \Gamma((z_a - z_b) / \kappa).$$

By Lemma 5.8, the actual summation in formula (5.11) is only over the positive cone of the lattice,

$$(5.16) \quad \mathcal{M}_J(\Phi_\lambda f)(\mathbf{z}; \mathbf{p}; \kappa) = \sum_{\mathbf{l} \in \mathbb{Z}_{\geq 0}^{\lambda\{1\}}} f(\Sigma_J - \mathbf{l}\kappa; \mathbf{z}; \mathbf{p}) \text{Res}_{t=\Sigma_J - \mathbf{l}\kappa} \Phi_\lambda^\circ(\mathbf{t}; \mathbf{z}; \mathbf{p}; \kappa).$$

Lemma 5.9. *For $|\arg(h/\kappa)| < \pi$, we have*

$$\text{Res}_{t=\Sigma_J - \mathbf{l}\kappa} \Phi_\lambda^\circ(\mathbf{t}; \mathbf{z}; \mathbf{p}; \kappa) = \\ = \lim_{h \rightarrow \infty} (-h)^{\lambda\{2\} - \lambda\{1\}} (\Gamma(h/\kappa))^{-\lambda\{1\} - \lambda\{2\}} \text{Res}_{t=\Sigma_J - \mathbf{l}\kappa} \widetilde{\Phi}_\lambda(\mathbf{t}; \mathbf{z}; h; \mathbf{p}; \kappa).$$

The convergence is uniform in \mathbf{l} and locally uniform in \mathbf{z}, \mathbf{p} .

Proof. The claim follows from formulae (4.17), (5.9), (5.14) by Lemmas D.1, D.2 that are detalization of Stirling's formula (5.10). \square

Define

$$(5.17) \quad \Psi_J^\circ(\mathbf{z}; \mathbf{p}; \kappa) = \kappa^{-\lambda\{1\}} \Omega_\lambda^\circ(\mathbf{p}; \kappa) \sum_{I \in \mathcal{I}_\lambda} \mathcal{M}_J(\Phi_\lambda^\circ W_I^\circ)(\mathbf{z}; \mathbf{p}; \kappa) v_I,$$

where

$$(5.18) \quad \Omega_\lambda^\circ(\mathbf{p}; \kappa) = e^{\sum_{i < j} p_j / (\kappa p_i)}$$

the sum taken over all pairs $1 \leq i < j \leq N$ such that $\lambda_i = 1$ and $\lambda_s = 0$ for all $s = i + 1, \dots, j$.

Recall the function $\Psi_J(\mathbf{z}; h; \mathbf{q}; \kappa)$, see (4.20). Let $\widetilde{\Psi}_J(\mathbf{z}; h; \mathbf{p}; \kappa)$ be the function obtained from $\Psi_J(\mathbf{z}; h; \mathbf{q}; \kappa)$ by substitution (5.9).

Proposition 5.10. *For $|\arg(h/\kappa)| < \pi$, we have*

$$(5.19) \quad \Psi_J^\circ(\mathbf{z}; \mathbf{p}; \kappa) = \lim_{h \rightarrow \infty} (-h \Gamma(h/\kappa))^{-\lambda\{2\}} (-h)^{\sum_{b < c, j < k} e_{k,k}^{(b)} e_{j,j}^{(c)}} \widetilde{\Psi}_J(\mathbf{z}; h; \mathbf{p}; \kappa).$$

The convergence is locally uniform in \mathbf{z}, \mathbf{p} .

Proof. The statement follows from formulae (4.19), (4.20), (4.21), (5.16), (5.17), (5.18), and Lemmas 5.1, 5.9. \square

Say that a function $f(\mathbf{p})$ is *entire in \mathbf{p}_\pm* if $f(\mathbf{p}) = g(p_2/p_1, \dots, p_N/p_{N-1})$ for an entire function $g(s_1, \dots, s_{N-1})$. Denote $f(\mathbf{0}) = g(0, \dots, 0)$.

Theorem 5.11. *The $(\mathbb{C}^N)_\lambda^{\otimes n}$ -valued function $\Psi_J^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ is a solution of the joint system of dynamical differential equations (3.13) and qKZ difference equations (3.12). It has the form*

$$(5.20) \quad \Psi_J^\circ(\mathbf{z}; \mathbf{p}; \kappa) = \Psi_J^\Delta(\mathbf{z}; \mathbf{p}; \kappa) \prod_{i=1}^N (\kappa^{\sum_{j=i+1}^N \lambda_j - \sum_{j=1}^{i-1} \lambda_j} p_i)^{\sum_{a \in J_i} z_a / \kappa} \times \\ \times \prod_{i=1}^{N-1} \prod_{j=i+1}^N \prod_{a \in J_i} \prod_{b \in J_j} \frac{1}{\sin(\pi(z_a - z_b) / \kappa)},$$

where the function $\Psi_J^\Delta(\mathbf{z}; \mathbf{p}; \kappa)$ is entire in \mathbf{z} and is entire in \mathbf{p}_\pm . In more detail,

$$(5.21) \quad \Psi_J^\Delta(\mathbf{z}; \mathbf{p}; \kappa) = \prod_{i=1}^{N-1} \prod_{j=i+1}^N \prod_{a \in J_i} \prod_{b \in J_j} \frac{\pi}{\Gamma(1 + (z_b - z_a) / \kappa)} \times \\ \times \left(\Psi_{J,0}^\Delta(\mathbf{z}) + \sum_{\substack{\mathbf{m} \in \mathbb{Z}_{\geq 0}^{N-1} \\ \mathbf{m} \neq \mathbf{0}}} \Psi_{J,\mathbf{m}}^\Delta(\mathbf{z}; \kappa) \prod_{i=1}^{N-1} (p_{i+1}/p_i)^{m_i} \right),$$

where

$$(5.22) \quad \Psi_{J,0}^\Delta(\mathbf{z}) = W_J^\circ(\Sigma_J; \mathbf{z}) v_J + \sum_{\substack{I \in \mathcal{I}_\lambda \\ |\sigma_I| < |\sigma_J|}} W_I^\circ(\Sigma_J; \mathbf{z}) v_I$$

is a polynomial in \mathbf{z} , and $\Psi_{J,\mathbf{m}}^\Delta(\mathbf{z}; \kappa)$ for $\mathbf{m} \neq \mathbf{0}$ are rational functions of \mathbf{z}, κ with at most simple poles on the hyperplanes $z_a - z_b \in \kappa \mathbb{Z}_{>0}$ for $a \in J_i, b \in J_j, 1 \leq i < j \leq N$. Furthermore, for any transposition $s_{a,b} \in S_n$,

$$(5.23) \quad \Psi_J^\Delta(\mathbf{z}; \mathbf{p}; \kappa) \Big|_{z_a=z_b} = \Psi_{s_{a,b}(J)}^\Delta(\mathbf{z}; \mathbf{p}; \kappa) \Big|_{z_a=z_b}.$$

Proof. The statement follows from Theorem 4.7, Lemmas 3.2, 3.8, 5.1, 5.8, 5.9, and Proposition 5.10. See also [TV6, Section 11]. \square

The functions $\Psi_J^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ are called the *multidimensional hypergeometric solutions* of the dynamical equations 3.13 and qKZ difference equations (3.12).

The next theorem computes the determinant of coordinates of solutions $\Psi_J^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ and is analogous to [TV6, formula (11.23)].

Theorem 5.12. *Let $n \geq 2$. Then*

$$(5.24) \quad \det \left(\Omega_\lambda^\circ(\mathbf{p}, \kappa) \mathcal{M}_J(\Phi_\lambda^\circ W_I^\circ)(\mathbf{z}; \mathbf{p}; \kappa) \right)_{I, J \in \mathcal{I}_\lambda} = \\ = \kappa^{\lambda^{\{1\}} d_\lambda} \prod_{i=1}^N p_i^{d_{\lambda, i}^{(1)} \sum_{a=1}^n z_a / \kappa} \prod_{a=1}^{n-1} \prod_{b=a+1}^n \left(\frac{\pi \kappa}{\sin(\pi(z_a - z_b) / \kappa)} \right)^{d_\lambda^{(2)}},$$

where $\lambda^{\{1\}} = \sum_{i=1}^{N-1} (N-i) \lambda_i$ and $d_\lambda, d_{\lambda, i}^{(1)}, d_\lambda^{(2)}$ are given by formulae (4.27).

Proof. The statement follows from Lemma 5.1, Proposition 5.10, and formula (4.26).

Alternatively, denote by $F(\mathbf{z}; \mathbf{p})$ the determinant in the left-hand side of formula (5.24). By Theorem 5.11, it solves the differential equations

$$\left(\kappa p_i \frac{\partial}{\partial p_i} - \operatorname{tr} X_i^\circ(\mathbf{z}; \mathbf{p})|_{(\mathbb{C}^N)_\lambda^{\otimes n}} \right) F(\mathbf{z}; \mathbf{p}) = 0, \quad i = 1, \dots, N,$$

where $X_i^\circ(\mathbf{z}; \mathbf{p})|_{(\mathbb{C}^N)_\lambda^{\otimes n}}$ are the restrictions of dynamical Hamiltonians (3.8) to the invariant subspace $(\mathbb{C}^N)_\lambda^{\otimes n}$. Since $\operatorname{tr} X_i^\circ(\mathbf{z}; h; \mathbf{q})|_{(\mathbb{C}^N)_\lambda^{\otimes n}} = d_{\lambda, i}^{(1)} \sum_{a=1}^n z_a$, the function $F(\mathbf{z}; \mathbf{p})$ equals the product of powers of p_1, \dots, p_n in the right-hand side of formula (4.26) multiplied by a factor that does not depend on \mathbf{p} . This factor can be found by taking the limit $p_{i+1}/p_i \rightarrow 0$ for all $i = 1, \dots, N-1$, using Theorem 5.11. \square

Remark. By Theorem 5.11, the determinant $F(\mathbf{z}; \mathbf{p})$ in Theorem 5.12 solves the difference equations

$$(5.25) \quad F(z_1, \dots, z_a + \kappa, \dots, z_n; \mathbf{p}) = \det K_a^\circ(\mathbf{z}; \mathbf{p}; \kappa)|_{(\mathbb{C}^N)_\lambda^{\otimes n}} F(\mathbf{z}; \mathbf{p}), \quad a = 1, \dots, n,$$

where $K_a^\circ(\mathbf{z}; h; \mathbf{q}; \kappa)|_{(\mathbb{C}^N)_\lambda^{\otimes n}}$ are the restrictions of qKZ operators (3.4) to the invariant subspace $(\mathbb{C}^N)_\lambda^{\otimes n}$. Since $\det K_a^\circ(\mathbf{z}; \mathbf{p}; \kappa) = (-1)^{d_\lambda^{(2)}}$, equations (5.25) determine the product of sines in the right-hand side of formula (5.24) up to a κ -periodic function of z_1, \dots, z_n .

5.4. Solutions parametrized by Laurent polynomials. Recall the notation from Section 4.6. For a Laurent polynomial $P(\hat{\Gamma}; \hat{\mathbf{z}})$, set

$$(5.26) \quad \Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa) = \sum_{J \in \mathcal{I}_\lambda} \hat{P}(\mathbf{z}_{\sigma_J}; \mathbf{z}; \kappa) \Psi_J^\circ(\mathbf{z}; \mathbf{p}; \kappa).$$

Let $\tilde{\Psi}_P(\mathbf{z}; h; \mathbf{p}; \kappa)$ be the function obtained from $\Psi_P(\mathbf{z}; h; \mathbf{q}; \kappa)$, see (4.41), by substitution (5.9).

Lemma 5.13. *For $|\arg(h/\kappa)| < \pi$, we have*

$$(5.27) \quad \Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa) = \lim_{h \rightarrow \infty} \left(-h \Gamma(h/\kappa) \right)^{-\lambda_{\{2\}}} (-h)^{\sum_{b < c, j < k} e_{k, k}^{(b)} e_{j, j}^{(c)}} \tilde{\Psi}_P(\mathbf{z}; h; \mathbf{p}; \kappa).$$

The convergence is locally uniform in \mathbf{z}, \mathbf{p} .

Proof. The statement follows from formulae (4.41), (5.26), and Proposition 5.10. \square

Proposition 5.14. *The function $\Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ is a solution of the joint system of limiting dynamical differential equations (3.13) and qKZ difference equations (3.12). Furthermore, for $P \in \mathbb{C}[\Gamma^{\pm 1}]^{S_\lambda} \otimes \mathbb{C}[\mathbf{z}^{\pm 1}]$, the function $\Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ is entire in \mathbf{z} and is holomorphic in \mathbf{p} provided a branch of $\log p_i$ is fixed for each $i = 1, \dots, N$.*

Proof. The statement follows from Propositions 4.12, 4.13, and Lemma 5.13. \square

Denote by $\mathcal{S}_\lambda^\circ$ the space of solutions of the system of dynamical differential equations (3.13) and qKZ difference equations (3.12) spanned over \mathbb{C} by the functions $\Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa)$, $P \in \mathbb{C}[\Gamma^{\pm 1}]^{S_\lambda} \otimes \mathbb{C}[\mathbf{z}^{\pm 1}]$. The space $\mathcal{S}_\lambda^\circ$ is a $\mathbb{C}[\mathbf{z}^{\pm 1}]$ -module with $f(\mathbf{z})$ acting as multiplication by $\acute{f}(\mathbf{z})$.

Define the algebra

$$(5.28) \quad \mathcal{K}_\lambda^\circ = \mathbb{C}[\Gamma^{\pm 1}]^{S_\lambda} \otimes \mathbb{C}[\mathbf{z}^{\pm 1}] \left/ \left\langle \prod_{i=1}^N \prod_{j=1}^{\lambda_i} (u - \gamma_{i,j}) = \prod_{a=1}^n (u - z_a) \right\rangle \right.,$$

where u is a formal variable. By (5.26), the assignment $P \mapsto \Psi_P^\circ$ defines a homomorphism

$$(5.29) \quad \mu_\lambda^{\mathcal{K}^\circ} : \mathcal{K}_\lambda^\circ \rightarrow \mathcal{S}_\lambda^\circ, \quad Y \mapsto \Psi_Y^\circ,$$

of $\mathbb{C}[\mathbf{z}^{\pm 1}]$ -modules.

By Proposition A.2, A.3, the algebra $\mathcal{K}_\lambda^\circ$ is a free $\mathbb{C}[\mathbf{z}^{\pm 1}]$ -module generated by the classes

$$(5.30) \quad Y_I(\Gamma) = V_I(\gamma_{1,1}^{-1}, \dots, \gamma_{1,\lambda_1}^{-1}, \dots, \gamma_{N,1}^{-1}, \dots, \gamma_{N,\lambda_N}^{-1}), \quad I \in \mathcal{I}_\lambda,$$

where the polynomials V_I are defined by formula (A.6). Introduce the coordinates of solutions $\Psi_{Y_I}^\circ$:

$$\Psi_{Y_I}^\circ(\mathbf{z}; \mathbf{p}; \kappa) = \sum_{J \in \mathcal{I}_\lambda} \bar{\Psi}_{I,J}^\circ(\mathbf{z}; \mathbf{p}; \kappa) v_J.$$

Theorem 5.15. *Let $n \geq 2$. Then*

$$(5.31) \quad \det \left(\bar{\Psi}_{I,J}^\circ(\mathbf{z}; \mathbf{p}; \kappa) \right)_{I,J \in \mathcal{I}_\lambda} = \\ = (2\pi\sqrt{-1}\kappa)^{n(n-1)d_\lambda^{(2)}/2} \kappa^{\lambda^{(1)}d_\lambda} \left(e^{-\pi\sqrt{-1}(n-1)d_\lambda^{(2)}} \prod_{i=1}^N p_i^{d_{\lambda,i}^{(1)}} \right)^{\sum_{a=1}^n z_a/\kappa} \prod_{j=2}^{n-1} j^{(n-j)d_\lambda^{(2)}},$$

where d_λ , $d_{\lambda,i}^{(1)}$, $d_\lambda^{(2)}$ are given by formulae (4.27).

Proof. The statement follows from Theorem 5.12 and formula (A.7). Alternatively, the statement follows from Theorem 4.14 and Lemma 5.13. \square

Corollary 5.16. *The map $\mu_\lambda^{\mathcal{K}^\circ} : \mathcal{K}_\lambda^\circ \rightarrow \mathcal{S}_\lambda^\circ$ is an isomorphism of $\mathbb{C}[\mathbf{z}^{\pm 1}]$ -modules.*

Remark. The algebra $\mathcal{K}_\lambda^\circ$ is the equivariant K -theory algebra $K_T(\mathcal{F}_\lambda; \mathbb{C})$ of the partial flag variety \mathcal{F}_λ , see Section 6.5.

5.5. Levelt fundamental solution. Recall that a function $f(\mathbf{p})$ is called entire in \mathbf{p}_\pm if $f(\mathbf{p}) = g(p_2/p_1, \dots, p_N/p_{N-1})$ for an entire function $g(s_1, \dots, s_{N-1})$, and $f(\mathbf{0}) = g(0, \dots, 0)$.

The dynamical Hamiltonians $X_1^\circ(\mathbf{z}; \mathbf{p}), \dots, X_n^\circ(\mathbf{z}; \mathbf{p})$ given by (3.8) are entire in \mathbf{p}_\pm and

$$(5.32) \quad X_i^\circ(\mathbf{z}; \mathbf{0}) = \sum_{a=1}^n z_a e_{i,i}^{(a)} - \sum_{1 \leq a < b \leq n} \left(\sum_{j=1}^{i-1} Q_{i,j}^{a,b} - \sum_{j=i+1}^N Q_{i,j}^{b,a} \right).$$

Notice that for $I \in \mathcal{I}_\lambda$,

$$X_i^\circ(\mathbf{z}; \mathbf{0}) v_I = \sum_{a \in I_i} z_a v_I + \sum_{\substack{J \in \mathcal{I}_\lambda \\ |\sigma_J| < |\sigma_I|}} \xi_{i,I,J} v_J,$$

where the coefficients $\xi_{i,I,J}$ take values $0, \pm 1$. Therefore, the eigenvalues of the restriction of the operator $X_i^\circ(\mathbf{z}; \mathbf{0})$ on $(\mathbb{C}^N)_\lambda^{\otimes n}$ are $\sum_{a \in I_i} z_a$, $I \in \mathcal{I}_\lambda$. A more detailed statement is given by Proposition 5.17 below.

Recall the function $\Psi_{I,0}^\Delta(\mathbf{z})$, $I \in \mathcal{I}_\lambda$, given by (5.22).

Proposition 5.17. *Given $I \in \mathcal{I}_\lambda$, we have $X_i^\circ(\mathbf{z}; \mathbf{0}) \Psi_{I,0}^\Delta(\mathbf{z}) = \sum_{a \in I_i} z_a \Psi_{I,0}^\Delta(\mathbf{z})$, and $\Psi_{I,0}^\Delta(\mathbf{z}) \neq 0$ provided $z_a \neq z_b$ for all pairs a, b such that $a < b$ and $\sigma_I^{-1}(a) > \sigma_I^{-1}(b)$.*

Proof. The first part of the statement follows from Theorem 5.11. The nonvanishing of $\Psi_{I,0}^\Delta(\mathbf{z})$ is implied by formula (5.22) and Lemma 5.5. \square

For $I \in \mathcal{I}_\lambda$, set $\mathbf{E}_I(\mathbf{z}) = (E_I^{(1)}(\mathbf{z}), \dots, E_I^{(N-1)}(\mathbf{z}))$, where $E_I^{(i)}(\mathbf{z}) = \sum_{j=1}^i \sum_{a \in I_j} z_a$ is the eigenvalue of $X_1^\circ(\mathbf{z}; \mathbf{0}) + \dots + X_i^\circ(\mathbf{z}; \mathbf{0})$ on $\Psi_{I,0}^\Delta(\mathbf{z})$. For $I, J \in \mathcal{I}_\lambda$, denote by $D_{I,J}$ the set of points \mathbf{z} such that $\mathbf{E}_I(\mathbf{z}) - \mathbf{E}_J(\mathbf{z}) \in \mathbb{Z}_{\geq 0}^{N-1}$ and $\mathbf{E}_I(\mathbf{z}) \neq \mathbf{E}_J(\mathbf{z})$. Set $D_\lambda = \bigcup_{I, J \in \mathcal{I}_\lambda} D_{I,J}$.

Theorem 5.18. (i) *For any \mathbf{z} such that $z_a - z_b \notin \kappa \mathbb{Z}_{\neq 0}$ for all $1 \leq a < b \leq n$, there exists an $\text{End}((\mathbb{C}^N)_\lambda^{\otimes n})$ -valued function $\Psi^\bullet(\mathbf{z}; \mathbf{p}; \kappa)$, entire in \mathbf{p}_\pm such that $\Psi^\bullet(\mathbf{z}; \mathbf{0}; \kappa)$ is the identity operator and the function*

$$(5.33) \quad \widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa) = \Psi^\bullet(\mathbf{z}; \mathbf{p}; \kappa) \prod_{i=1}^N p_i^{X_i^\circ(\mathbf{z}; \mathbf{0})/\kappa},$$

solves dynamical differential equations (3.13). For given \mathbf{z} , the function $\Psi^\bullet(\mathbf{z}; \mathbf{p}; \kappa)$ with the specified properties is unique if and only if $\mathbf{z} \notin \kappa D_\lambda$. Furthermore, $\det \Psi^\#(\mathbf{z}; \mathbf{p}; \kappa) = 1$ and

$$(5.34) \quad \det \widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa) = \prod_{i=1}^N p_i^{d_{\lambda,i}^{(1)} \sum_{a=1}^n z_a/\kappa},$$

where $d_{\lambda,1}^{(1)}, \dots, d_{\lambda,N}^{(1)}$ are given by formula (4.27).

(ii) *Define the function $\Psi^\bullet(\mathbf{z}; \mathbf{p}; \kappa)$ for generic \mathbf{z} as in item (i). Then $\Psi^\bullet(\mathbf{z}; \mathbf{p}; \kappa)$ is holomorphic in \mathbf{z} if $z_a - z_b \notin \kappa \mathbb{Z}_{\neq 0}$ for all $1 \leq a < b \leq n$. The singularities of $\Psi^\bullet(\mathbf{z}; \mathbf{p}; \kappa)$ at the hyperplanes $z_a - z_b \in \kappa \mathbb{Z}_{\neq 0}$ are simple poles.*

Proof. The proof of the uniqueness statement is similar to that in Theorem 4.17. To prove the existence part, we give an explicit expression for the function $\Psi^\bullet(\mathbf{z}; \mathbf{p}; \kappa)$, see formulae (5.37), (5.38).

Recall the function $A^\circ(\mathbf{t}; \mathbf{z}; \kappa)$ at $\kappa = 1$, see (5.13),

$$(5.35) \quad A^\circ(\mathbf{t}; \mathbf{z}) = \prod_{i=1}^{N-1} \prod_{a=1}^{\lambda^{(i)}} \left(\prod_{\substack{b=1 \\ b \neq a}}^{\lambda^{(i)}} \Gamma(1 + t_b^{(i)} - t_a^{(i)}) \prod_{c=1}^{\lambda^{(i+1)}} \frac{1}{\Gamma(1 + t_c^{(i+1)} - t_a^{(i)})} \right),$$

where $\lambda^{(N)} = n$ and $t_a^{(N)} = z_a$, $a = 1, \dots, n$. Notice that

$$A^\circ(\Sigma_I; \mathbf{z}) = \prod_{i=1}^{N-1} \prod_{j=i+1}^N \prod_{a \in I_i} \prod_{b \in I_j} \frac{1}{\Gamma(z_b - z_a + 1)}.$$

For $\mathbf{l} \in \mathbb{Z}^{\lambda^{\{1\}}}$, set

$$(5.36) \quad \mathcal{J}_{I,J,\mathbf{l}}^\circ(\mathbf{z}) = \sum_{K \in \mathcal{I}_\lambda} \frac{A^\circ(\Sigma_K - \mathbf{l}; \mathbf{z}) W_I^\circ(\Sigma_K - \mathbf{l}; \mathbf{z}) \check{W}_J^\circ(\Sigma_K; \mathbf{z})}{A^\circ(\Sigma_K; \mathbf{z}) R_\lambda(\mathbf{z}_{\sigma_K})}.$$

Recall the function $\Omega_\lambda^\circ(\mathbf{p}; \kappa) = e^{\sum_{i < j} p_j / (\kappa p_i)}$, where the sum is taken over all pairs $1 \leq i < j \leq N$ such that $\lambda_i = 1$ and $\lambda_s = 0$ for all $s = i + 1, \dots, j$, see (5.18). Set

$$(5.37) \quad \Psi_{I,J}^\bullet(\mathbf{z}; \mathbf{p}; \kappa) = \Omega_\lambda^\circ(\mathbf{p}; \kappa) \sum_{\mathbf{l} \in \mathbb{Z}_{\geq 0}^{\lambda^{\{1\}}}} \kappa^{|\sigma_I| - |\sigma_J|} \mathcal{J}_{I,J,\mathbf{l}}^\circ(\mathbf{z}/\kappa) \prod_{i=1}^{N-1} ((-\kappa)^{-\lambda_i - \lambda_{i+1}} p_{i+1}/p_i)^{\sum_{a=1}^{\lambda^{(i)}} l_a^{(i)}}.$$

Let $\Psi^\bullet(\mathbf{z}; \mathbf{p}; \kappa)$ be the linear operator with the entries $\Psi_{I,J}^\bullet(\mathbf{z}; \mathbf{p}; \kappa)$ in the standard basis $\{v_I, I \in \mathcal{I}_\lambda\}$ of $(\mathbb{C}^N)_\lambda^{\otimes n}$,

$$(5.38) \quad \Psi^\bullet(\mathbf{z}; \mathbf{p}; \kappa) : v_J \mapsto \sum_{I \in \mathcal{I}_\lambda} \Psi_{I,J}^\bullet(\mathbf{z}; \mathbf{p}; \kappa) v_I.$$

To verify that the function $\Psi^\bullet(\mathbf{z}; \mathbf{p}; \kappa)$ is as required in Theorem 5.18, recall the function $\Psi^\#(\mathbf{z}; h; \mathbf{q})$, introduced in Theorem 4.17 and its entries $\Psi_{I,J}^\#(\mathbf{z}; h; \mathbf{q})$. Let the functions $\tilde{\Psi}_{I,J}^\#(\mathbf{z}; h; \mathbf{p}; \kappa)$ be obtained from $\Psi_{I,J}^\#(\mathbf{z}/\kappa; h/\kappa; \mathbf{q})$ by substitution (5.9). Formulae (4.59), (4.60), (5.36), (5.37), and Lemma D.2 imply that

$$(5.39) \quad \Psi_{I,J}^\bullet(\mathbf{z}; \mathbf{p}; \kappa) = \lim_{h \rightarrow \infty} \tilde{\Psi}_{I,J}^\#(\mathbf{z}; h; \mathbf{p}; \kappa)$$

locally uniformly in \mathbf{z}, \mathbf{p} . Therefore, the properties of $\Psi^\#(\mathbf{z}; h; \mathbf{q})$ established in Theorem 4.17 yield the properties of $\Psi^\bullet(\mathbf{z}; \mathbf{p}; \kappa)$ required in Theorem 5.18. \square

Following [AB, Chapter 2], we will call $\widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ the *Levelt fundamental solution* of dynamical differential equations (3.13) on $(\mathbb{C}^N)_\lambda^{\otimes n}$, see also [CV, Section 6.2].

For a solution $\Psi^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ of dynamical differential equations (3.13), define its *principal term*

$$(5.40) \quad \Psi^\circ(\mathbf{z}; \kappa) = \widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa)^{-1} \Psi^\circ(\mathbf{z}; \mathbf{p}; \kappa).$$

By Theorem 5.18, the principal term does not depend on \mathbf{p} .

Set

$$(5.41) \quad C_\lambda^\circ(\mathbf{z}; \kappa) = \prod_{i=1}^N \kappa^{(\sum_{j=i+1}^N \lambda_j - \sum_{j=1}^{i-1} \lambda_j) \sum_{a=\lambda^{(i-1)+1}}^{\lambda^{(i)}} z_a / \kappa}$$

and

$$(5.42) \quad G_\lambda^\circ(\mathbf{z}; \kappa) = \prod_{i=1}^{N-1} \prod_{a=1}^{\lambda^{(i)}} \prod_{b=\lambda^{(i)+1}}^{\lambda^{(i+1)}} \Gamma((z_a - z_b) / \kappa).$$

Proposition 5.19. *For a Laurent polynomial $P(\mathbf{\Gamma}; \mathbf{z})$, the principal term of the solution $\Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa)$, given by (5.26), equals*

$$(5.43) \quad \Psi_P^\circ(\mathbf{z}; \kappa) = \sum_{I, J \in \mathcal{I}_\lambda} \hat{P}(\mathbf{z}_{\sigma_J}; \mathbf{z}; \kappa) C_\lambda^\circ(\mathbf{z}_{\sigma_J}; \kappa) G_\lambda^\circ(\mathbf{z}_{\sigma_J}; \kappa) W_I^\circ(\Sigma_J; \mathbf{z}) v_I.$$

Proof. Denote by $\bar{\Psi}_P^\circ(\mathbf{z}; \kappa)$ the right-hand side of formula (5.43). Then formula (5.26), Proposition 5.17, and Theorem 5.11 yield $\hat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa) \bar{\Psi}_P^\circ(\mathbf{z}; \kappa) = \Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa)$. Hence by definition (5.40) of the principal term, $\Psi_P^\circ(\mathbf{z}; \kappa) = \bar{\Psi}_P^\circ(\mathbf{z}; \kappa)$. \square

5.6. The map \mathbb{B}_λ° . In Section 5.4, we introduced the space $\mathcal{S}_\lambda^\circ$ of $(\mathbb{C}^N)_\lambda^{\otimes n}$ -valued solutions of the joint system of dynamical differential equations (3.13) and qKZ difference equations (3.12) spanned over \mathbb{C} by the functions $\Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ labeled by Laurent polynomials in $\mathbf{\Gamma}, \mathbf{z}$; we also defined a map

$$(5.44) \quad \mu_\lambda^{\mathcal{K}^\circ} : \mathcal{K}_\lambda^\circ \rightarrow \mathcal{S}_\lambda^\circ, \quad Y \mapsto \Psi_Y^\circ,$$

see (5.29). In Section 5.5 we introduced the Levelt fundamental solution $\hat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ of dynamical differential equations (3.13), see (5.33), (5.38). Denote by $\mathcal{S}_\lambda^{\hat{\circ}}$ the space of $(\mathbb{C}^N)_\lambda^{\otimes n}$ -valued solutions of dynamical differential equations (3.13) spanned over \mathbb{C} by the functions $\hat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa)v$, $v \in (\mathbb{C}^N)_\lambda^{\otimes n}$. Since $\det \hat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa) \neq 0$, see (5.34), there is an isomorphism

$$(5.45) \quad \mu_\lambda^{\hat{\circ}} : (\mathbb{C}^N)_\lambda^{\otimes n} \rightarrow \mathcal{S}_\lambda^{\hat{\circ}}, \quad v \mapsto \hat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa)v.$$

Let $L^\circ \subset \mathbb{C}^n$ be the complement of the union of the hyperplanes

$$(5.46) \quad z_a - z_b \in \kappa \mathbb{Z}_{\neq 0}, \quad a, b = 1, \dots, n, \quad a \neq b.$$

Denote by \mathcal{O} the ring of functions of \mathbf{z} holomorphic in L° . Let $\mathcal{S}_\lambda^{\circ}$ be the space of $(\mathbb{C}^N)_\lambda^{\otimes n}$ -valued solutions of dynamical differential equations (3.13) holomorphic in \mathbf{z} in L° . Both spaces $\mathcal{S}_\lambda^\circ$ and $\mathcal{S}_\lambda^{\hat{\circ}}$ are subspaces of $\mathcal{S}_\lambda^{\circ}$, see Proposition 5.14 and Theorem 5.18. Let

$$(5.47) \quad \mu_\lambda^{\circ} : (\mathbb{C}^N)_\lambda^{\otimes n} \otimes_{\mathbb{C}} \mathcal{O} \rightarrow \mathcal{S}_\lambda^{\circ}, \quad v \mapsto \hat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa)v,$$

be the \mathcal{O} -linear extension of the map $\mu_\lambda^{\hat{\circ}}$.

Define a map

$$(5.48) \quad \mathbb{B}_\lambda^\circ : \mathcal{K}_\lambda^\circ \rightarrow (\mathbb{C}^N)_\lambda^{\otimes n} \otimes_{\mathbb{C}} \mathcal{O},$$

$$(5.49) \quad [P] \mapsto \sum_{I, J \in \mathcal{I}_\lambda} \dot{P}(\mathbf{z}_{\sigma_J}; \mathbf{z}; \kappa) C_\lambda^\circ(\mathbf{z}_{\sigma_J}; \kappa) G_\lambda^\circ(\mathbf{z}_{\sigma_J}; \kappa) W_I(\Sigma_J; \mathbf{z}) v_I,$$

where $[P] \in \mathcal{K}_\lambda^\circ$ stands for the class of the Laurent polynomial $P(\dot{\mathbf{T}}; \dot{\mathbf{z}})$ and the functions $C_\lambda^\circ(\mathbf{z}_{\sigma_J}; \kappa)$, $G_\lambda^\circ(\mathbf{z}_{\sigma_J}; \kappa)$ are given by (5.41), (5.42), respectively. By Proposition 5.19, the map \mathbb{B}_λ° sends the class $Y \in \mathcal{K}_\lambda^\circ$ to the principal term of the solution Ψ_Y° of the joint system of dynamical differential equations (3.13) and qKZ difference equations (3.12).

Proposition 5.20. *The map $\mathbb{B}_\lambda^\circ : \mathcal{K}_\lambda^\circ \rightarrow (\mathbb{C}^N)_\lambda^{\otimes n} \otimes_{\mathbb{C}} \mathcal{O}$ is well-defined and the following diagram is commutative,*

$$(5.50) \quad \begin{array}{ccc} \mathcal{K}_\lambda^\circ & \xrightarrow{\mathbb{B}_\lambda^\circ} & (\mathbb{C}^N)_\lambda^{\otimes n} \otimes_{\mathbb{C}} \mathcal{O} \\ & \searrow \mu_{H_\lambda}^K & \swarrow \mu_{H_\lambda}^\circ \\ & \mathcal{S}_{H_\lambda}^\circ & \end{array}$$

Proof. Poles of the sum in the right-hand side of (5.49) are at most those of the function $G_\lambda^\circ(\mathbf{z}; h; \kappa)$ and, therefore, in addition to hyperplanes (5.46) can occur only at the hyperplanes $z_a = z_b$, $a \neq b$. However, the sums

$$\sum_{I, J \in \mathcal{I}_\lambda} \dot{P}(\mathbf{z}_{\sigma_J}; \mathbf{z}; \kappa) C_\lambda^\circ(\mathbf{z}_{\sigma_J}; \kappa) G_\lambda^\circ(\mathbf{z}_{\sigma_J}; \kappa) W_I(\Sigma_J; \mathbf{z})$$

are regular at the hyperplanes $z_a = z_b$ for all $a \neq b$ by the standard reasoning. Hence, the map \mathbb{B}_λ° is well-defined.

The commutativity of diagram (5.50) follows from Proposition 5.19. \square

Remark. Since $\widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ is holomorphic in \mathbf{z} in L° , and $(\det \widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa))^{-1}$ is entire in \mathbf{z} , see (5.34), the inverse matrix $\widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa)^{-1}$ is holomorphic in \mathbf{z} in L° . Therefore, for every $\Psi \in \mathcal{S}_\lambda^\circ$, its principal term $\Psi^\circ = \widehat{\Psi}^{\circ^{-1}} \Psi$, defined by (5.40), belongs to $(\mathbb{C}^N)_\lambda^{\otimes n} \otimes_{\mathbb{C}} \mathcal{O}$, and we have an isomorphism $\mathcal{S}_\lambda^\circ \rightarrow (\mathbb{C}^N)_\lambda^{\otimes n} \otimes_{\mathbb{C}} \mathcal{O}$, $\Psi \mapsto \Psi^\circ$. The inverse map, equals μ_λ° , see (5.47), so that, $\mathcal{S}_\lambda^\circ = \mathcal{S}_\lambda^\circ \otimes_{\mathbb{C}} \mathcal{O}$.

5.7. Example $\lambda = (1, n-1)$. Throughout this section, let $N = 2$, $n \geq 2$, $\lambda = (1, n-1)$. Like in Section 4.10, denote by $[a]$ the element $(\{a\}, \{1, \dots, a-1, a+1, \dots, n\}) \in \mathcal{I}_\lambda$. The space $(\mathbb{C}^2)_\lambda^{\otimes n}$ has a basis $v_{[1]}, \dots, v_{[n]}$, where $v_{[a]} = v_2^{\otimes(a-1)} \otimes v_1 \otimes v_2^{\otimes(n-a)}$. Clearly $e_{1,1}^{(a)} v_{[b]} = \delta_{a,b} v_{[b]}$ and $e_{2,2}^{(a)} v_{[b]} = (1 - \delta_{a,b}) v_{[b]}$.

The qKZ operators $K_1^\circ, \dots, K_n^\circ$, see (3.4), are

$$(5.51) \quad \begin{aligned} K_a^\circ(\mathbf{z}; \mathbf{p}; \kappa) &= (R^\circ(z_a - z_{a-1} + \kappa))^{(a, a-1)} \dots (R^\circ(z_a - z_1 + \kappa))^{(a, 1)} \times \\ &\times p_1^{e_{1,1}^{(a)}} p_2^{e_{2,2}^{(a)}} (R^\circ(z_i - z_n))^{(a, n)} \dots (R^\circ(z_a - z_{a+1}))^{(a, a+1)}. \end{aligned}$$

The R -matrices in the right-hand side preserve the subspace $(\mathbb{C}^2)_{\lambda}^{\otimes n} \subset (\mathbb{C}^2)^{\otimes n}$, acting there as follows,

$$\begin{aligned} (R^{\circ}(z))^{(a,b)} v_{[a]} &= v_{[b]} - z v_{[a]}, & (R^{\circ}(z))^{(a,b)} v_{[b]} &= v_{[a]}, \\ (R^{\circ}(z))^{(a,b)} v_{[c]} &= v_{[c]}, & c &\neq a, b. \end{aligned}$$

The qKZ difference equations (3.12) are

$$(5.52) \quad f(z_1, \dots, z_a + \kappa, \dots, z_n; \mathbf{p}; \kappa) = K_a^{\circ}(\mathbf{z}; \mathbf{p}; \kappa) f(\mathbf{z}; \mathbf{p}; \kappa), \quad a = 1, \dots, n.$$

The dynamical Hamiltonians X_1°, X_2° , see (3.8), act on $(\mathbb{C}^2)_{\lambda}^{\otimes n}$ as follows

$$(5.53) \quad \begin{aligned} X_1^{\circ}(\mathbf{z}; \mathbf{p}) v_{[1]} &= z_1 v_{[1]} + \frac{p_2}{p_1} v_{[n]}, \\ X_1^{\circ}(\mathbf{z}; \mathbf{p}) v_{[a]} &= z_a v_{[a]} + v_{[a-1]}, & a &= 2, \dots, n, \\ X_2^{\circ}(\mathbf{z}; \mathbf{p}) v_{[b]} &= \left(-X_1^{\circ}(\mathbf{z}; \mathbf{p}) + \sum_{c=1}^n z_c \right) v_{[b]}, & b &= 1, \dots, n, \end{aligned}$$

and the dynamical differential equations (2.6) are

$$(5.54) \quad \begin{aligned} \kappa p_1 \frac{\partial}{\partial p_1} \Psi(\mathbf{z}; \mathbf{p}; \kappa) &= X_1^{\circ}(\mathbf{z}; \mathbf{p}) \Psi(\mathbf{z}; \mathbf{p}; \kappa), \\ \kappa p_2 \frac{\partial}{\partial p_2} \Psi(\mathbf{z}; \mathbf{p}; \kappa) &= X_2^{\circ}(\mathbf{z}; \mathbf{p}) \Psi(\mathbf{z}; \mathbf{p}; \kappa). \end{aligned}$$

In this section, we use the variable $t = t_1^{(1)}$. The substitution $\mathbf{t} = \Sigma_{[a]}$ reads as $t = z_a$. The weight functions are

$$(5.55) \quad W_{[a]}^{\circ}(t; \mathbf{z}) = \prod_{b=1}^{a-1} (t - z_b), \quad a = 1, \dots, n.$$

The permutations $\sigma_{[1]}, \dots, \sigma_{[n]}$ are

$$\sigma_{[a]}(1) = a, \quad \sigma_{[a]}(b) = b - 1, \quad b = 1, \dots, a - 1, \quad \sigma_{[a]}(b) = b, \quad b = a + 1, \dots, n,$$

and $|\sigma_{[a]}| = a - 1$. We have

$$(5.56) \quad W_{[a]}^{\circ}(z_a; \mathbf{z}) = \prod_{b=1}^{a-1} (z_a - z_b),$$

$$W_{[a]}^{\circ}(z_b; \mathbf{z}) = 0, \quad b = 1, \dots, a - 1,$$

cf. Lemma 5.5.

The functions $\check{W}_{[a]}^{\circ}(t; \mathbf{z})$, see (5.5), are

$$(5.57) \quad \check{W}_{[a]}^{\circ}(t; \mathbf{z}) = \prod_{c=a+1}^n (t - z_c), \quad a = 1, \dots, n.$$

Set

$$\bar{R}_a(\mathbf{z}) = \prod_{\substack{b=1 \\ b \neq a}}^n (z_a - z_b), \quad a = 1, \dots, n.$$

Then $\bar{R}_a(\mathbf{z}) = R_\lambda(\mathbf{z}_{\sigma[a]})$, where the function $R_\lambda(\mathbf{z})$ is given by (4.8). Biorthogonality relations (5.7) become

$$\sum_{c=1}^n \frac{W_{[a]}^\circ(z_c; \mathbf{z}) \check{W}_{[b]}^\circ(z_c; \mathbf{z})}{\bar{R}_c(\mathbf{z})} = \delta_{a,b}, \quad \sum_{c=1}^n W_{[c]}^\circ(z_a; \mathbf{z}) \check{W}_{[c]}^\circ(z_b; \mathbf{z}) = \delta_{a,b} \bar{R}_a(\mathbf{z}).$$

The master function, see (5.8), is

$$(5.58) \quad \Phi_\lambda^\circ(t; \mathbf{z}; \mathbf{p}; \kappa) = (p_2/\kappa)^{\sum_{a=1}^n z_a/\kappa} (\kappa^n p_1/p_2)^{t/\kappa} \prod_{a=1}^n \Gamma((t - z_a)/\kappa).$$

The hypergeometric solutions (4.20) of the joint system of differential equations (4.68) and difference equations (4.66) have the form

$$\Psi_{[a]}^\circ(\mathbf{z}; \mathbf{p}; \kappa) = \frac{1}{\kappa} \sum_{b=1}^n \sum_{l=0}^{\infty} \text{Res}_{t=z_a-l\kappa} \Phi_\lambda^\circ(t; \mathbf{z}; \mathbf{p}; \kappa) W_{[b]}^\circ(z_a - l\kappa) v_{[b]},$$

where the residues of the master function are

$$\begin{aligned} \text{Res}_{t=z_a-l\kappa} \Phi_\lambda^\circ(t; \mathbf{z}; \mathbf{p}; \kappa) &= \\ &= \kappa (\kappa^{n-1} p_1)^{z_a/\kappa} (p_2/\kappa)^{\sum_{c=1, c \neq a}^n z_c/\kappa} \frac{(-\kappa^{-n} p_2/p_1)^l}{l!} \prod_{\substack{c=1 \\ c \neq a}}^n \Gamma(-l + (z_a - z_c)/\kappa). \end{aligned}$$

Determinant formula for coordinates of the hypergeometric solutions, see (5.24), is

$$(5.59) \quad \det \left(\sum_{l=0}^{\infty} \text{Res}_{t=z_a-l\kappa} \Phi_\lambda^\circ(t; \mathbf{z}; \mathbf{p}; \kappa) W_{[b]}^\circ(z_a - l\kappa) \right)_{a,b=1}^n = \\ = \pi^{n(n-1)/2} \kappa^{n(n+1)/2} (p_1 p_2^{n-1})^{\sum_{a=1}^n z_a/\kappa} \prod_{a=1}^{n-1} \prod_{b=a+1}^n \frac{1}{\sin(\pi(z_a - z_b)/\kappa)}.$$

By formula (5.55) and the Vandermonde determinant formula, equality (4.72) transforms to

$$(5.60) \quad \det \left(\sum_{l=0}^{\infty} \sum_{c=1}^n \text{Res}_{t=z_c-l\kappa} (t^{a-1} e^{-2\pi\sqrt{-1}(b-1)t/\kappa} \Phi_\lambda^\circ(t; \mathbf{z}; \mathbf{p}; \kappa)) \right)_{a,b=1}^n = \\ = (2\pi\sqrt{-1})^{n(n-1)/2} \kappa^{n(n+1)/2} (e^{-\pi\sqrt{-1}(n-1)} p_1 p_2^{n-1})^{\sum_{a=1}^n z_a/\kappa}.$$

Let $\gamma = \gamma_{1,1}$. For a Laurent polynomial $P(\gamma; \mathbf{z})$, we have

$$\dot{P}(\gamma; \mathbf{z}; \kappa) = P(e^{2\pi\sqrt{-1}\gamma/\kappa}; e^{2\pi\sqrt{-1}z_1/\kappa}, \dots, e^{2\pi\sqrt{-1}z_n/\kappa}).$$

The solution Ψ_P° of differential equations (5.54) and difference equations (5.52) corresponding to $P(\gamma; \mathbf{z})$ is

$$(5.61) \quad \Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa) = \sum_{a=1}^n \dot{P}(z_a; \mathbf{z}; \kappa) \Psi_{[a]}^\circ(\mathbf{z}; \mathbf{p}; \kappa).$$

By Proposition 5.14, it is entire in \mathbf{z} and is holomorphic in p_1, p_2 provided branches of $\log p_1$ and $\log p_2$ are fixed.

The solution $\Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ can be written as an integral over a suitable contour C encircling the poles of the product $\prod_{a=1}^n \Gamma((t - z_a)/\kappa)$ counterclockwise,

$$(5.62) \quad \Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa) = \frac{1}{2\pi\sqrt{-1}\kappa} \int_C \dot{P}(t; \mathbf{z}; \kappa) \Phi_\lambda^\circ(t; \mathbf{z}; \mathbf{p}; \kappa) \sum_{a=1}^n W_{[a]}^\circ(t; \mathbf{z}) v_{[a]} dt.$$

For instance, the integral can be taken over the parabola

$$C = \{ \kappa(A - s^2 + s\sqrt{-1}) \mid s \in \mathbb{R} \},$$

where A is a sufficiently large positive real number. Formula (4.75) can be used to give an alternative proof of analytic properties of the function $\Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa)$.

Let $\mathcal{S}_\lambda^\circ$ be the space of solutions of the joint system of dynamical differential equations (5.54) and qKZ difference equations (5.52) spanned over \mathbb{C} by the functions $\Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ corresponding to Laurent polynomials $P(\gamma; \mathbf{z})$. The space $\mathcal{S}_\lambda^\circ$ is a $\mathbb{C}[\mathbf{z}^{\pm 1}]$ -module with $f(\mathbf{z})$ acting as multiplication by $f(e^{2\pi\sqrt{-1}z_1/\kappa}, \dots, e^{2\pi\sqrt{-1}z_n/\kappa})$.

For $\lambda = (1, n-1)$, the algebra $\mathcal{K}_\lambda^\circ$, see (5.28), can be presented as follows

$$(5.63) \quad \mathcal{K}_\lambda^\circ = \mathbb{C}[\gamma^{\pm 1}, \mathbf{z}^{\pm 1}] / \left\langle \prod_{a=1}^n (\gamma - z_a) = 0 \right\rangle.$$

The function $\Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ depends only on the class of the Laurent polynomial P in $\mathcal{K}_\lambda^\circ$. The assignment $P \mapsto \Psi_P^\circ$ defines the homomorphism

$$\mu_\lambda^{\kappa^\circ} : \mathcal{K}_\lambda^\circ \rightarrow \mathcal{S}_\lambda^\circ, \quad Y \mapsto \Psi_Y^\circ,$$

of $\mathbb{C}[\mathbf{z}^{\pm 1}]$ -modules. Formula (5.60) implies that the homomorphism $\mu_\lambda^{\kappa^\circ}$ is an isomorphism.

The algebra $\mathcal{K}_\lambda^\circ$ is the equivariant K -theory algebra $K_T(\mathbb{C}\mathbb{P}^{n-1}; \mathbb{C})$ of the projective space $\mathbb{C}\mathbb{P}^{n-1}$, see the notation in Section 6.

Recall that a function $f(\mathbf{p})$ is called entire in \mathbf{p}_\pm if $f(\mathbf{p}) = g(p_2/p_1)$ for an entire function $g(s)$, and $f(\mathbf{0}) = g(0)$. For example, the dynamical Hamiltonians X_1°, X_2° , see (5.53), are entire in \mathbf{p}_\pm and $X_1^\circ(\mathbf{z}; \mathbf{0}), X_2^\circ(\mathbf{z}; \mathbf{0})$ act on $(\mathbb{C}^2)_\lambda^{\otimes n}$ as follows

$$X_1^\circ(\mathbf{z}; \mathbf{0}) v_{[a]} = z_a v_{[a]} + v_{[a-1]}, \quad X_2^\circ(\mathbf{z}; \mathbf{0}) v_{[a]} = \left(-X_1^\circ(\mathbf{z}; \mathbf{0}) + \sum_{b=1}^n z_b \right) v_{[a]},$$

where $v_{[0]} = 0$.

The Levelt fundamental solution $\widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa)$, see (5.33), of differential equations (5.54) is

$$\widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa) = \Psi^\bullet(\mathbf{z}; \mathbf{p}; \kappa) p_1^{X_1^\circ(\mathbf{z}; \mathbf{0})/\kappa} p_2^{X_2^\circ(\mathbf{z}; \mathbf{0})/\kappa},$$

where the $\text{End}((\mathbb{C}^N)_\lambda^{\otimes n})$ -valued function $\Psi^\bullet(\mathbf{z}; \mathbf{p}; \kappa)$ is as follows,

$$(5.64) \quad \Psi^\bullet(\mathbf{z}; \mathbf{p}; \kappa) : v_{[b]} \mapsto \sum_{a=1}^n \Psi_{a,b}^\bullet(\mathbf{z}; \mathbf{p}; \kappa) v_{[a]},$$

$$\Psi_{a,b}^\bullet(\mathbf{z}; \mathbf{p}; \kappa) = \sum_{l=0}^{\infty} \kappa^{a-b} \mathcal{J}_{a,b,l}^\circ(\mathbf{z}/\kappa) ((-\kappa)^{-n} p_2/p_1)^l,$$

$$\mathcal{J}_{a,b,l}^\circ(\mathbf{z}) = \sum_{c=1}^n W_{[a]}^\circ(z_c - l; \mathbf{z}) \check{W}_{[b]}^\circ(z_c; \mathbf{z}) \frac{(-1)^{n-1}}{l!} \prod_{\substack{d=1 \\ d \neq c}}^n \prod_{m=0}^l \frac{1}{z_d - z_c + m}.$$

Furthermore, one has $\det \Psi^\bullet(\mathbf{z}; \mathbf{p}; \kappa) = 1$ and $\det \widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa) = (p_1 p_2^{n-1})^{\sum_{a=1}^n z_a/\kappa}$.

For a solution $\Psi^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ of dynamical differential equations (5.54), its principal term, see (5.40), is

$$\Psi^\circ(\mathbf{z}; \kappa) = \widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa)^{-1} \Psi^\circ(\mathbf{z}; \mathbf{p}; \kappa).$$

The principal term of the solution $\Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa)$, corresponding to a Laurent polynomial $P(\gamma; \mathbf{z})$, see (5.43), equals

$$(5.65) \quad \Psi_P^\circ(\mathbf{z}; \kappa) = \sum_{a=1}^n v_{[a]} \sum_{b=1}^n W_{[a]}^\circ(z_b; \mathbf{z}) \dot{P}(z_b; \mathbf{z}; \kappa) \kappa^{(nz_b - \sum_{c=1}^n z_c)/\kappa} \prod_{\substack{c=1 \\ c \neq b}}^n \Gamma((z_b - z_c)/\kappa).$$

Let L° be the complement of the union of the hyperplanes $z_a - z_b \in \kappa \mathbb{Z}_{\neq 0}$, $a, b = 1, \dots, n$, $a \neq b$, cf. (5.46). Denote by \mathcal{O} the ring of functions of \mathbf{z} holomorphic in L° . The map

$$(5.66) \quad \begin{aligned} \mathbb{B}_\lambda^\circ : \mathcal{K}_\lambda^\circ &\rightarrow (\mathbb{C}^N)_\lambda^{\otimes n} \otimes_{\mathbb{C}} \mathcal{O}, \\ [P] &\mapsto \sum_{a=1}^n v_{[a]} \sum_{b=1}^n W_{[a]}^\circ(z_b; \mathbf{z}) \dot{P}(z_b; \mathbf{z}; \kappa) \kappa^{(nz_b - \sum_{c=1}^n z_c)/\kappa} \prod_{\substack{c=1 \\ c \neq b}}^n \Gamma((z_b - z_c)/\kappa) \end{aligned}$$

sends the class $[P] \in \mathcal{K}_\lambda^\circ$ of the Laurent polynomial $P(\gamma; \mathbf{z})$ to the principal term of the solution Ψ_P° of the joint system of differential equations (5.54) and difference equations (5.52).

Let $\mathcal{S}_\lambda^\circ$ be the space of $(\mathbb{C}^N)_\lambda^{\otimes n}$ -valued solutions of dynamical differential equations (5.54) holomorphic in \mathbf{z} in L° . The space $\mathcal{S}_\lambda^\circ$ is a subspace of $\mathcal{K}_\lambda^\circ$.

Since the matrix $\widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ is holomorphic in \mathbf{z} in L° , and $(\det \widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa))^{-1}$ is entire in \mathbf{z} , the inverse matrix $\widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa)^{-1}$ is also holomorphic in \mathbf{z} in L° . Thus the map

$$\mu_\lambda^\circ : (\mathbb{C}^N)_\lambda^{\otimes n} \otimes_{\mathbb{C}} \mathcal{O} \rightarrow \mathcal{S}_\lambda^\circ, \quad v \mapsto \widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa) v,$$

gives an isomorphism of \mathcal{O} -modules. Furthermore, the following diagram is commutative,

$$\begin{array}{ccc} \mathcal{K}_\lambda^\circ & \xrightarrow{B_\lambda^\circ} & (\mathbb{C}^N)_\lambda^{\otimes n} \otimes_{\mathbb{C}} \mathcal{O} \\ & \searrow \mu_\lambda^{\mathcal{K}^\circ} & \swarrow \mu_\lambda^{\mathcal{O}} \\ & \mathcal{F}_\lambda^\circ & \end{array}$$

see Proposition 5.20.

6. EQUATIONS FOR PARTIAL FLAG VARIETIES

6.1. Equivariant cohomology of partial flag varieties. Consider the partial flag variety \mathcal{F}_λ parametrizing chains of subspaces

$$0 = F_0 \subset F_1 \subset \dots \subset F_N = \mathbb{C}^n$$

with $\dim F_i/F_{i-1} = \lambda_i$, $i = 1, \dots, N$.

Given a basis of \mathbb{C}^n , the group $GL_n(\mathbb{C})$ acts on \mathbb{C}^n . Let $T \subset GL_n(\mathbb{C})$ be the torus of diagonal matrices. Let z_1, \dots, z_n the Chern roots corresponding to the factors of T , and for $i = 1, \dots, N$, let $\gamma_{i,1}, \dots, \gamma_{i,\lambda_i}$ be the Chern roots of the bundle over \mathcal{F}_λ with fiber F_i/F_{i-1} . Denote $\mathbf{\Gamma} = (\gamma_{1,1}, \dots, \gamma_{1,\lambda_1}, \dots, \gamma_{N,1}, \dots, \gamma_{N,\lambda_N})$ and $\mathbf{z} = (z_1, \dots, z_n)$, cf. (4.40). Let $\mathbb{C}[\mathbf{\Gamma}]^{S_\lambda}$ be the space of polynomials in $\mathbf{\Gamma}$ symmetric in $\gamma_{i,1}, \dots, \gamma_{i,\lambda_i}$ for each $i = 1, \dots, N$.

Consider the equivariant cohomology algebra $H_T^*(\mathcal{F}_\lambda; \mathbb{C})$. Then

$$(6.1) \quad H_T^*(\mathcal{F}_\lambda; \mathbb{C}) = \mathbb{C}[\mathbf{\Gamma}]^{S_\lambda} \otimes \mathbb{C}[\mathbf{z}] / \left\langle \prod_{i=1}^N \prod_{j=1}^{\lambda_i} (u - \gamma_{i,j}) = \prod_{a=1}^n (u - z_a) \right\rangle,$$

where u is a formal variable. For a polynomial $f(\mathbf{\Gamma}; \mathbf{z}) \in \mathbb{C}[\mathbf{\Gamma}]^{S_\lambda} \otimes \mathbb{C}[\mathbf{z}]$, denote by $[f]$ its class in $H_T^*(\mathcal{F}_\lambda; \mathbb{C})$. Notice that for each $i = 1, \dots, N$,

$$(6.2) \quad c_1(E_i) = [\gamma_{i,1} + \dots + \gamma_{i,\lambda_i}]$$

is the equivariant first Chern class of the vector bundle E_i over \mathcal{F}_λ with fiber F_i/F_{i-1} .

Given a point $\mathbf{z}^0 \in \mathbb{C}^n$, consider the algebra

$$(6.3) \quad H_T^*(\mathcal{F}_\lambda; \mathbb{C})_{\mathbf{z}^0} = \mathbb{C}[\mathbf{\Gamma}]^{S_\lambda} / \left\langle \prod_{i=1}^N \prod_{j=1}^{\lambda_i} (u - \gamma_{i,j}) = \prod_{a=1}^n (u - z_a^0) \right\rangle.$$

The evaluation map $\mathbb{C}[\mathbf{\Gamma}]^{S_\lambda} \otimes \mathbb{C}[\mathbf{z}] \rightarrow \mathbb{C}[\mathbf{\Gamma}]^{S_\lambda}$, $f(\mathbf{\Gamma}; \mathbf{z}) \mapsto f(\mathbf{\Gamma}; \mathbf{z}^0)$, identifies $H_T^*(\mathcal{F}_\lambda; \mathbb{C})_{\mathbf{z}^0}$ with the quotient $H_T^*(\mathcal{F}_\lambda; \mathbb{C}) / \langle \mathbf{z} = \mathbf{z}^0 \rangle$. Denote by $[f]_{\mathbf{z}^0}$ the class of $f(\mathbf{\Gamma}) \in \mathbb{C}[\mathbf{\Gamma}]^{S_\lambda}$ in $H_T^*(\mathcal{F}_\lambda; \mathbb{C})_{\mathbf{z}^0}$.

Recall the polynomials $V_I(\mathbf{\Gamma})$, $I \in \mathcal{I}_\lambda$, given by formula (A.6).

Lemma 6.1. *The algebra $H_T^*(\mathcal{F}_\lambda; \mathbb{C})$ is a free module over $H_T^*(pt; \mathbb{C}) = \mathbb{C}[\mathbf{z}]$ generated by the classes $[V_I]$, $I \in \mathcal{I}_\lambda$. For every $\mathbf{z}^0 \in \mathbb{C}^n$, the classes $[V_I]_{\mathbf{z}^0}$, $I \in \mathcal{I}_\lambda$, give a basis of $H_T^*(\mathcal{F}_\lambda; \mathbb{C})_{\mathbf{z}^0}$. In particular, $\dim H_T^*(\mathcal{F}_\lambda; \mathbb{C})_{\mathbf{z}^0} = n! / (\lambda_1! \dots \lambda_N!)$ for every \mathbf{z}^0 .*

Proof. The statement follows from Propositions A.2 and A.3. \square

Recall the polynomial $R_\lambda(\mathbf{z})$, see (4.8). Let

$$(6.4) \quad V_\lambda(\Gamma; \mathbf{z}) = \prod_{i=1}^{N-1} \prod_{j=1}^{\lambda_i} \prod_{a=\lambda^{(i)}+1}^n (\gamma_{i,j} - z_a),$$

cf. (A.1). Then $V_\lambda(\mathbf{z}; \mathbf{z}) = R_\lambda(\mathbf{z})$ and $V_\lambda(\mathbf{z}_{\sigma_I}; \mathbf{z}) = 0$ for $I \neq I_\lambda^{\min}$.

For a polynomial $f(\Gamma; \mathbf{z})$, consider the restrictions $f(\mathbf{z}_{\sigma_I}; \mathbf{z})$, $I \in \mathcal{I}_\lambda$.

Lemma 6.2. *The map $H_T^*(\mathcal{F}_\lambda; \mathbb{C}) \rightarrow \bigoplus_{I \in \mathcal{I}_\lambda} \mathbb{C}[z]$, $[f] \mapsto (f(\mathbf{z}_{\sigma_I}; \mathbf{z}), I \in \mathcal{I}_\lambda)$ is well-defined and injective. A collection $(F_I(\mathbf{z}), I \in \mathcal{I}_\lambda)$ belongs to the image of this map if and only if for any $I \in \mathcal{I}_\lambda$ and any transposition $s_{a,b}$,*

$$(6.5) \quad F_I(\mathbf{z})|_{z_a=z_b} = F_{s_{a,b}(I)}(\mathbf{z})|_{z_a=z_b}.$$

For a collection $(F_I(\mathbf{z}), I \in \mathcal{I}_\lambda)$ obeying (6.5), the function

$$(6.6) \quad f(\Gamma; \mathbf{z}) = \sum_{I \in \mathcal{I}_\lambda} \frac{F_I(\mathbf{z}) V_\lambda(\Gamma; \mathbf{z}_{\sigma_I})}{R_\lambda(\mathbf{z}_{\sigma_I})}.$$

is a polynomial and $f(\mathbf{z}_{\sigma_I}; \mathbf{z}) = F_I(\mathbf{z})$ for every $I \in \mathcal{I}_\lambda$.

Proof. Straightforward. \square

Lemma 6.3. *For any $f(\Gamma; \mathbf{z}) \in \mathbb{C}[\Gamma]^{S_\lambda} \otimes \mathbb{C}[\mathbf{z}]$, the function*

$$(6.7) \quad \mathcal{E}\langle f \rangle(\mathbf{z}) = \sum_{I \in \mathcal{I}_\lambda} \frac{f(\mathbf{z}_{\sigma_I}, \mathbf{z})}{R_\lambda(\mathbf{z}_{\sigma_I})}$$

is a polynomial depending only on the class of f in $H_T^*(\mathcal{F}_\lambda; \mathbb{C})$.

Proof. Straightforward. \square

The induced map

$$(6.8) \quad \mathcal{E} : H_T^*(\mathcal{F}_\lambda; \mathbb{C}) \rightarrow \mathbb{C}[\mathbf{z}], \quad [f] \mapsto \mathcal{E}\langle f \rangle,$$

is the equivariant integration map on $H_T^*(\mathcal{F}_\lambda; \mathbb{C})$.

Identify $\mathbb{C}[\Gamma]^{S_\lambda}$ with the subspace of $\mathbb{C}[\Gamma]^{S_\lambda} \otimes \mathbb{C}[\mathbf{z}]$ of polynomials not depending on \mathbf{z} . For $f(\Gamma) \in \mathbb{C}[\Gamma]^{S_\lambda}$, denote $\mathcal{E}_{z^0}\langle f \rangle = \mathcal{E}\langle f \rangle(\mathbf{z}^0)$.

Lemma 6.4. *For any $f(\Gamma; \mathbf{z}) \in \mathbb{C}[\Gamma]^{S_\lambda} \otimes \mathbb{C}[\mathbf{z}]$ and $\mathbf{z}^0 \in \mathbb{C}^n$, we have*

$$\mathcal{E}\langle f(\Gamma; \mathbf{z}) \rangle(\mathbf{z}^0) = \mathcal{E}_{z^0}\langle f(\Gamma; \mathbf{z}^0) \rangle.$$

Proof. Straightforward. \square

By Lemmas 6.3, 6.4, for every $\mathbf{z}^0 \in \mathbb{C}^n$, there is a well-defined map

$$(6.9) \quad \mathcal{E}_{\mathbf{z}^0} : H_T^*(\mathcal{F}_\lambda; \mathbb{C})_{\mathbf{z}^0} \rightarrow \mathbb{C}, \quad \mathcal{E}_{\mathbf{z}^0} : [f]_{\mathbf{z}^0} \mapsto \mathcal{E}_{\mathbf{z}^0}\langle f \rangle,$$

called the *integration map on $H_T^*(\mathcal{F}_\lambda; \mathbb{C})_{\mathbf{z}^0}$* .

6.2. Stable envelope map. Recall the functions $W_I^\circ(\mathbf{t}; \mathbf{z})$, $\check{W}_I^\circ(\mathbf{t}; \mathbf{z})$, see (5.1), (5.5). Let $\text{Stab}_I(\mathbf{\Gamma}; \mathbf{z})$, $\text{Stab}_I^{\text{op}}(\mathbf{\Gamma}; \mathbf{z})$ be the polynomials respectively obtained from $\check{W}_I^\circ(\mathbf{t}; \mathbf{z})$, $W_I^\circ(\mathbf{t}; \mathbf{z})$ by the substitution

$$(6.10) \quad (t_1^{(i)}, \dots, t_{\lambda^{(i)}}^{(i)}) = (\gamma_{1,1}, \dots, \gamma_{1,\lambda_1}, \dots, \gamma_{i,1}, \dots, \gamma_{i,\lambda_i}), \quad i = 1, \dots, N-1.$$

Notice that

$$(6.11) \quad \text{Stab}_I(\mathbf{z}_{\sigma_J}; \mathbf{z}) = \check{W}_I^\circ(\Sigma_J; \mathbf{z}), \quad \text{Stab}_I^{\text{op}}(\mathbf{z}_{\sigma_J}; \mathbf{z}) = W_I^\circ(\Sigma_J; \mathbf{z}).$$

By Lemma 5.2 and formula (6.4), we have

$$(6.12) \quad \begin{aligned} \text{Stab}_{I_\lambda^{\min}}(\mathbf{\Gamma}; \mathbf{z}) &= V_\lambda(\mathbf{\Gamma}; \mathbf{z}), & \text{Stab}_{\sigma_0(I_\lambda^{\min})}(\mathbf{\Gamma}; \mathbf{z}) &= 1, \\ \text{Stab}_{I_\lambda^{\min}}^{\text{op}}(\mathbf{\Gamma}; \mathbf{z}) &= 1, & \text{Stab}_{\sigma_0(I_\lambda^{\min})}^{\text{op}}(\mathbf{\Gamma}; \mathbf{z}) &= V_\lambda(\mathbf{\Gamma}; \mathbf{z}_{\sigma_0}), \end{aligned}$$

where σ_0 is the longest permutation, $\sigma_0(a) = n+1-a$, $a = 1, \dots, n$. By Proposition B.5, the polynomials $\text{Stab}_I(\mathbf{\Gamma}; \mathbf{z})$, $\text{Stab}_I^{\text{op}}(\mathbf{\Gamma}; \mathbf{z})$ coincide with the A -type double Schubert polynomials $\mathfrak{S}_\sigma(\mathbf{\Gamma}; \mathbf{z})$,

$$(6.13) \quad \text{Stab}_I(\mathbf{\Gamma}; \mathbf{z}) = \mathfrak{S}_{\sigma_{\sigma_0(I)}}(\mathbf{\Gamma}; \mathbf{z}_{\sigma_0}), \quad \text{Stab}_I^{\text{op}}(\mathbf{\Gamma}; \mathbf{z}) = \mathfrak{S}_{\sigma_I}(\mathbf{\Gamma}; \mathbf{z}).$$

Lemma 6.5. *For any $I \in \mathcal{I}_\lambda$,*

$$\text{Stab}_I(\mathbf{\Gamma}; \mathbf{z}) = \sum_{J \in \mathcal{I}_\lambda} \frac{\check{W}_I^\circ(\Sigma_J; \mathbf{z}) V_\lambda(\mathbf{\Gamma}; \mathbf{z}_{\sigma_J})}{R_\lambda(\mathbf{z}_{\sigma_J})},$$

and

$$(6.14) \quad \text{Stab}_I^{\text{op}}(\mathbf{\Gamma}; \mathbf{z}) = \sum_{J \in \mathcal{I}_\lambda} \frac{W_I^\circ(\Sigma_J; \mathbf{z}) V_\lambda(\mathbf{\Gamma}; \mathbf{z}_{\sigma_J})}{R_\lambda(\mathbf{z}_{\sigma_J})}.$$

Proof. Fix a generic \mathbf{z} and consider both sides of each formula as functions of $\mathbf{\Gamma}$. Formulae (6.12) and Lemma 5.4 imply that $\text{Stab}_I(\mathbf{\Gamma}; \mathbf{z})$, $\text{Stab}_I^{\text{op}}(\mathbf{\Gamma}; \mathbf{z})$ are in the span of the polynomials $V_\lambda(\mathbf{\Gamma}; \mathbf{z}_{\sigma_J})$, $J \in \mathcal{I}_\lambda$. Then the statement follows from formulae (6.6), (6.13). \square

Lemma 6.6. *The polynomials $\text{Stab}_I(\mathbf{\Gamma}; \mathbf{z})$ and $\text{Stab}_I^{\text{op}}(\mathbf{\Gamma}; \mathbf{z})$ are biorthogonal,*

$$(6.15) \quad \mathcal{E}\langle \text{Stab}_I(\mathbf{\Gamma}; \mathbf{z}) \text{Stab}_J^{\text{op}}(\mathbf{\Gamma}; \mathbf{z}) \rangle = \delta_{I,J}.$$

Proof. The statement follows from formulae (6.7), (6.13), and Lemma 5.6. \square

Lemma 6.15 is equivalent to the orthogonality relation for the Schubert polynomials.

Let Stab_I , $\text{Stab}_I^{\text{op}}$ be the classes of $\text{Stab}_I(\mathbf{\Gamma}; \mathbf{z})$, $\text{Stab}_I^{\text{op}}(\mathbf{\Gamma}; \mathbf{z})$ in $H_T^*(\mathcal{F}_\lambda; \mathbb{C})$. Define the *stable envelope map* by the rule

$$(6.16) \quad \text{Stab}_\lambda : (\mathbb{C}^N)_\lambda^{\otimes n} \otimes \mathbb{C}[\mathbf{z}] \rightarrow H_T^*(\mathcal{F}_\lambda; \mathbb{C}), \quad v_I \mapsto \text{Stab}_I, \quad I \in \mathcal{I}_\lambda.$$

Lemma 6.7. *The map Stab_λ is an isomorphism of free $\mathbb{C}[\mathbf{z}]$ -modules.*

Proof. The statement follows from Lemma 6.6. For any $f \in \mathbb{C}[\Gamma]^{S_\lambda} \otimes \mathbb{C}[\mathbf{z}]$, one has

$$(\text{Stab}_\lambda)^{-1} : [f] \mapsto \sum_{J \in \mathcal{I}_\lambda} \mathcal{E} \langle f(\Gamma; \mathbf{z}) \text{Stab}_J^{\text{op}}(\Gamma; \mathbf{z}) \rangle v_J. \quad \square$$

Let $\text{Stab}_{I, \mathbf{z}^0}$, $\text{Stab}_{I, \mathbf{z}^0}^{\text{op}}$ be the classes of $\text{Stab}_I(\Gamma; \mathbf{z}^0)$, $\text{Stab}_I^{\text{op}}(\Gamma; \mathbf{z}^0)$ in $H_T^*(\mathcal{F}_\lambda; \mathbb{C})_{\mathbf{z}^0}$. Consider the map

$$(6.17) \quad \text{Stab}_{\lambda, \mathbf{z}^0} : (\mathbb{C}^N)_\lambda^{\otimes n} \rightarrow H_T^*(\mathcal{F}_\lambda; \mathbb{C})_{\mathbf{z}^0}, \quad v_I \mapsto \text{Stab}_{I, \mathbf{z}^0}, \quad I \in \mathcal{I}_\lambda.$$

Lemma 6.8. *For every $\mathbf{z}^0 \in \mathbb{C}^n$, the classes $\text{Stab}_{I, \mathbf{z}^0}$, $I \in \mathcal{I}_\lambda$, give a basis of $H_T^*(\mathcal{F}_\lambda; \mathbb{C})_{\mathbf{z}^0}$. That is, the map $\text{Stab}_{\lambda, \mathbf{z}^0}$ is an isomorphism.*

Proof. The statement follows from Lemmas 6.6, 6.4. For any $f \in \mathbb{C}[\Gamma]^{S_\lambda}$, one has

$$(\text{Stab}_{\lambda, \mathbf{z}^0})^{-1} : [f]_{\mathbf{z}^0} \mapsto \sum_{J \in \mathcal{I}_\lambda} \mathcal{E}_{\mathbf{z}^0} \langle f(\Gamma) \text{Stab}_J^{\text{op}}(\Gamma; \mathbf{z}^0) \rangle v_J. \quad \square$$

Consider a polynomial

$$(6.18) \quad \widehat{W}_\lambda^\circ(\mathbf{t}; \Gamma) = \prod_{i=1}^{N-1} \prod_{j=1}^{\lambda^{(i)}} \prod_{k=1}^{\lambda_{i+1}} (t_j^{(i)} - \gamma_{i+1, k}).$$

and its image $[\widehat{W}_\lambda^\circ(\mathbf{t}; \Gamma)]$ in $\mathbb{C}[\mathbf{t}] \otimes H_T^*(\mathcal{F}_\lambda; \mathbb{C})$. Clearly, for any $I, J \in \mathcal{I}_\lambda$,

$$(6.19) \quad \widehat{W}_\lambda^\circ(\Sigma_I; \mathbf{z}_J) = \delta_{I, J} R_\lambda(\mathbf{z}_{\sigma_I}).$$

Theorem 6.9. *We have*

$$[\widehat{W}_\lambda^\circ(\mathbf{t}; \Gamma)] = \sum_{I \in \mathcal{I}_\lambda} W_I^\circ(\mathbf{t}; \mathbf{z}) \text{Stab}_I.$$

Proof. The statement follows from [TV6, Proposition 9.2] and Lemmas 5.1, 5.2. \square

Remark. According to formulae (6.13), the classes Stab_I , $\text{Stab}_I^{\text{op}}$, $I \in \mathcal{I}_\lambda$, are the equivariant fundamental classes of the Schubert subvarieties in \mathcal{F}_λ defined for the opposite orderings of the chosen basis of \mathbb{C}^n in the standard way, see [O, Section 4.1], cf. [RV, Section 6]. On the equivariant fundamental classes of Schubert varieties see, in particular, [R, FRW].

Remark. Stable envelope maps for Nakajima quiver varieties were introduced in [MO]. They were defined in [MO] geometrically in terms of the associated torus action. The map Stab_λ given by formula (6.16) is the limit as $h \rightarrow \infty$ of the stable envelope map of [MO] associated with the cotangent bundle $T^*\mathcal{F}_\lambda$ of the partial flag variety. In [RTV1], the stable envelope map for $T^*\mathcal{F}_\lambda$ is described in terms of the Chern roots of the bundles $\mathbf{F}_1, \dots, \mathbf{F}_{N-1}$ over \mathcal{F}_λ with fibers F_1, \dots, F_{N-1} , respectively.

Remark. Unlike the stable envelope map Stab_λ defined by (6.16), the stable envelope map for the cotangent bundle $T^*\mathcal{F}_\lambda$ is not an isomorphism of free $\mathbb{C}[\mathbf{z}]$ -modules $(\mathbb{C}^N)_\lambda^{\otimes n} \otimes \mathbb{C}[\mathbf{z}]$ and $H_{T \times \mathbb{C}^\times}^*(T^*\mathcal{F}_\lambda; \mathbb{C})$, but only an embedding.

6.3. Quantum multiplication. The quantum multiplication in $H_T^*(\mathcal{F}_\lambda; \mathbb{C})$, see for example [M, BM], is a deformation of the multiplication in $H_T^*(\mathcal{F}_\lambda; \mathbb{C})$ depending on quantum parameters. In the notation of [M], the quantum parameters are q^{α_i} , $i = \dots, N-1$. In the notation of this paper, we have $q^{\alpha_i} = p_{i+1}/p_i$.

For $Y \in H_T^*(\mathcal{F}_\lambda; \mathbb{C})$, denote by $Y *_{\mathbf{p}}$ the operator of quantum multiplication by Y . Recall that the operators $Y *_{\mathbf{p}}$ are $\mathbb{C}[\mathbf{z}]$ -module endomorphisms of $H_T^*(\mathcal{F}_\lambda; \mathbb{C})$.

The dynamical operators $X_1^\circ(\mathbf{z}; \mathbf{p}), \dots, X_N^\circ(\mathbf{z}; \mathbf{p})$, see (3.8), are linear functions of \mathbf{z} . Thus their action on $(\mathbb{C}^N)_\lambda^{\otimes n}$ extends to the $\mathbb{C}[\mathbf{z}]$ -linear action on $(\mathbb{C}^N)_\lambda^{\otimes n} \otimes \mathbb{C}[\mathbf{z}]$.

For $i = 1, \dots, N$, denote $D_i = \gamma_{i,1} + \dots + \gamma_{i,\lambda_i}$, cf. (6.2).

Theorem 6.10. *The isomorphism Stab_λ intertwines the dynamical operators $X_1^\circ(\mathbf{z}; \mathbf{p}), \dots, X_N^\circ(\mathbf{z}; \mathbf{p})$ acting on $(\mathbb{C}^N)_\lambda^{\otimes n} \otimes \mathbb{C}[\mathbf{z}]$ and the operators of quantum multiplication $[D_1] *_{\mathbf{p}}, \dots, [D_N] *_{\mathbf{p}}$ acting on $H_T^*(\mathcal{F}_\lambda; \mathbb{C})$,*

$$(6.20) \quad \text{Stab}_\lambda \circ X_i^\circ(\mathbf{z}; \mathbf{p}) = [D_i] *_{\mathbf{p}} \circ \text{Stab}_\lambda.$$

Proof. Theorem 6.4 in [M] describes the quantum multiplication in $H_T^*(\mathcal{F}_\lambda; \mathbb{C})$ by the equivariant divisor classes. In the notation of this paper those classes equal $\text{Stab}_{\sigma_0 s_{\lambda^{(i)}, \lambda^{(i)+1}}(I_\lambda^{\min})}$, $i = 1, \dots, N-1$, where σ_0 is the longest permutation. By formulae (6.16), (5.5), (5.3),

$$(6.21) \quad \text{Stab}_{\sigma_0 s_{\lambda^{(i)}, \lambda^{(i)+1}}(I_\lambda^{\min})}(\mathbf{\Gamma}; \mathbf{z}) = D_1 + \dots + D_i - z_{n-\lambda^{(i)+1}} - \dots - z_n.$$

Comparing formula (3.8) term by term with formula (6.1) in [M] yields formula (6.20) for $i = 1, \dots, N-1$. Formula (6.20) for $i = N$ holds since $[D_1 + \dots + D_N] = z_1 + \dots + z_n$ in $H_T^*(\mathcal{F}_\lambda; \mathbb{C})$ and $X_1(\mathbf{z}; \mathbf{p}) + \dots + X_N(\mathbf{z}; \mathbf{p}) = z_1 + \dots + z_n$. Recall that for each $i = 1, \dots, N$, the operator $z_i *_{\mathbf{p}}$ is the ordinary multiplication by z_i , \square

Since the operators of quantum multiplication act $\mathbb{C}[\mathbf{z}]$ -linearly on $H_T^*(\mathcal{F}_\lambda; \mathbb{C})$, the quantum multiplication in $H_T^*(\mathcal{F}_\lambda; \mathbb{C})$ can be restricted to $H_T^*(\mathcal{F}_\lambda; \mathbb{C})_{\mathbf{z}^0}$. Denote by $Y *_{\mathbf{p}, \mathbf{z}^0}$ the operator of quantum multiplication by $Y \in H_T^*(\mathcal{F}_\lambda; \mathbb{C})_{\mathbf{z}^0}$.

Corollary 6.11. *For every $\mathbf{z}^0 \in \mathbb{C}^n$, the isomorphism $\text{Stab}_{\lambda, \mathbf{z}^0}$ intertwines the dynamical operators $X_1^\circ(\mathbf{z}^0; \mathbf{p}), \dots, X_N^\circ(\mathbf{z}^0; \mathbf{p})$ acting on $(\mathbb{C}^N)_\lambda^{\otimes n}$ and the operators of quantum multiplication $[D_1]_{\mathbf{z}^0} *_{\mathbf{p}, \mathbf{z}^0}, \dots, [D_N]_{\mathbf{z}^0} *_{\mathbf{p}, \mathbf{z}^0}$ acting on $H_T^*(\mathcal{F}_\lambda; \mathbb{C})_{\mathbf{z}^0}$,*

$$(6.22) \quad \text{Stab}_{\lambda, \mathbf{z}^0} \circ X_i^\circ(\mathbf{z}^0; \mathbf{p}) = [D_i]_{\mathbf{z}^0} *_{\mathbf{p}, \mathbf{z}^0} \circ \text{Stab}_{\lambda, \mathbf{z}^0}.$$

6.4. Differential and difference equations. Consider the space $\mathbb{C}^n \times \mathbb{C}^N$ with coordinates \mathbf{z}, \mathbf{p} . By Lemma 6.1, we have a trivial vector bundle $H_\lambda \rightarrow \mathbb{C}^n \times \mathbb{C}^N$ with fiber over a point $(\mathbf{z}^0, \mathbf{p}^0)$ given by $H_T^*(\mathcal{F}_\lambda; \mathbb{C})_{\mathbf{z}^0}$. Let $U_\lambda \rightarrow \mathbb{C}^n \times \mathbb{C}^N$ be the trivial vector bundle with fiber $(\mathbb{C}^N)_\lambda^{\otimes n}$.

Lemma 6.12. *The map $\text{Stab}_\lambda^\diamond : U_\lambda \rightarrow H_\lambda$, $(\mathbf{z}^0, \mathbf{p}^0, v) \mapsto (\mathbf{z}^0, \mathbf{p}^0, \text{Stab}_{\lambda, \mathbf{z}^0} v)$, is an isomorphism of vector bundles.*

Proof. The statement follows from Lemma 6.8. \square

The *equivariant quantum differential equations* for sections of H_λ is a system of differential equations

$$(6.23) \quad \kappa p_i \frac{\partial f}{\partial p_i} = [D_i]_{\mathbf{z}} *_{\mathbf{p}, \mathbf{z}} f, \quad i = 1, \dots, N,$$

where κ is the parameter of the equations. By Corollary 6.11, the isomorphism $\text{Stab}_\lambda^\diamond$ identifies equations (6.23) with the limiting dynamical differential equation (3.13) for sections of U_λ ,

$$(6.24) \quad \kappa p_i \frac{\partial f}{\partial p_i} = X_i^\circ(\mathbf{z}; \mathbf{p}) f, \quad i = 1, \dots, N.$$

Furthermore, the isomorphism $\text{Stab}_\lambda^\diamond$ and the limiting qKZ equations (3.12) for sections of U_λ define the *qKZ difference equations in cohomology*

$$(6.25) \quad f(z_1, \dots, z_a + \kappa, \dots, z_n; \mathbf{p}; \kappa) = K_a^H(\mathbf{z}; \mathbf{p}; \kappa) f(\mathbf{z}; \mathbf{p}; \kappa), \quad a = 1, \dots, n,$$

where for each $a = 1, \dots, n$, and fixed \mathbf{z}, \mathbf{p} , the operator $K_a^H(\mathbf{z}; \mathbf{p}; \kappa)$ is a map of fibers,

$$K_a^H(\mathbf{z}; \mathbf{p}; \kappa) : H_T^*(\mathcal{F}_\lambda; \mathbb{C})_{\mathbf{z}} \rightarrow H_T^*(\mathcal{F}_\lambda; \mathbb{C})_{(z_1, \dots, z_a + \kappa, \dots, z_n)},$$

$$K_a^H(\mathbf{z}; \mathbf{p}; \kappa) = \text{Stab}_{\lambda, (z_1, \dots, z_a + \kappa, \dots, z_n)} \circ K_a^\circ(\mathbf{z}; \mathbf{p}; \kappa) \circ (\text{Stab}_{\lambda, \mathbf{z}})^{-1},$$

and the operator $K_a^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ is given by (3.4).

Theorem 6.13. *Quantum differential equations (6.23) and qKZ difference equations (6.25) is a compatible system of differential and difference equations for sections of H_λ .*

Proof. The statement follows from Theorem 3.10. \square

6.5. Solutions with values in cohomology. Recall the isomorphism $\text{Stab}_\lambda^\diamond : U_\lambda \rightarrow H_\lambda$ of vector bundles, see Lemma 6.12. For any function $f(\mathbf{z}; \mathbf{p}; \kappa)$ with values in $(\mathbb{C}^N)_\lambda^{\otimes n}$, that is, a section of U_λ , denote by $\text{Stab}_\lambda f$ the corresponding section of H_λ with values

$$\text{Stab}_\lambda f(\mathbf{z}; \mathbf{p}; \kappa) = \text{Stab}_{\lambda, \mathbf{z}}(f(\mathbf{z}; \mathbf{p}; \kappa)).$$

Recall the solutions $\Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ of the joint system of limiting dynamical differential equations (3.13) and qKZ difference equations (3.12) labeled by Laurent polynomials $P(\hat{\Gamma}; \hat{\mathbf{z}})$, see formula (5.26) and Proposition 5.14.

Theorem 6.14. *The isomorphism $\text{Stab}_\lambda^\diamond$ transforms the solutions $\Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ of equations (3.13), (3.12) to solutions of the joint system of quantum differential equations and qKZ difference equations (6.23), (6.25). Namely, for each $P \in \mathbb{C}[\hat{\Gamma}^{\pm 1}]^{S_\lambda} \otimes \mathbb{C}[\hat{\mathbf{z}}^{\pm 1}]$, the function $\text{Stab}_\lambda \Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ is a solution of equations (3.13), (3.12).*

Proof. The theorem follows from Theorem 6.10 and the definition of the qKZ difference equations in (6.25). \square

In this section we work out the geometric interpretation of the obtained solutions. Notice that it would be interesting to relate our solutions with constructions in [C], where Mellin-Barnes integral representations of solutions of the quantum differential equations were constructed for a class of smooth projective varieties.

For $i = 1, \dots, N$, let $\gamma_{1,1}, \dots, \gamma_{1,\lambda_i}$ be virtual line bundles such that $\bigoplus_{j=1}^{\lambda_i} \gamma_{i,j} = F_i/F_{i-1}$, and z_1, \dots, z_n correspond to the factors of the torus T . Denote $\mathbf{\Gamma}^{\pm 1} = (\gamma_{1,1}^{\pm 1}, \dots, \gamma_{N,\lambda_N}^{\pm 1})$ and $\mathbf{z}^{\pm 1} = (z_1^{\pm 1}, \dots, z_n^{\pm 1})$, cf. (4.40). Let $\mathbb{C}[\mathbf{\Gamma}^{\pm 1}]^{S_\lambda}$ be the space of Laurent polynomials in $\mathbf{\Gamma}$ symmetric in $\gamma_{i,1}, \dots, \gamma_{i,\lambda_i}$ for each $i = 1, \dots, N$.

Consider the equivariant K -theory algebra $K_T(\mathcal{F}_\lambda; \mathbb{C})$. Then

$$(6.26) \quad K_T(\mathcal{F}_\lambda; \mathbb{C}) = \mathbb{C}[\mathbf{\Gamma}^{\pm 1}]^{S_\lambda} \otimes \mathbb{C}[\mathbf{z}^{\pm 1}] / \left\langle \prod_{i=1}^N \prod_{j=1}^{\lambda_i} (u - \gamma_{i,j}) = \prod_{a=1}^n (u - z_a) \right\rangle,$$

where u is a formal variable. That is, $K_T(\mathcal{F}_\lambda; \mathbb{C})$ coincides with the algebra $\mathcal{K}_\lambda^\circ$, see (5.28). For a Laurent polynomial $P(\mathbf{\Gamma}^{\pm 1}; \mathbf{z}^{\pm 1}) \in \mathbb{C}[\mathbf{\Gamma}^{\pm 1}]^{S_\lambda} \otimes \mathbb{C}[\mathbf{z}^{\pm 1}]$, denote by $[P]$ its class in $K_T(\mathcal{F}_\lambda; \mathbb{C})$.

Lemma 6.15. *The algebra $K_T(\mathcal{F}_\lambda; \mathbb{C})$ is a free module over $\mathbb{C}[\mathbf{z}^{\pm 1}]$.*

Proof. The statement follows from Propositions A.2 and A.3. \square

Denote by $\mathcal{S}_{H_\lambda}^K$ the space spanned over \mathbb{C} by the solutions $\text{Stab}_\lambda \Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ of equations (6.23), (6.25). By Proposition 5.14, each element of $\mathcal{S}_{H_\lambda}^K$ is a section of H_λ holomorphic in \mathbf{p} provided a branch of $\log p_i$ is fixed for each $i = 1, \dots, N$, and entire in \mathbf{z} .

Since the function $\Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ depends only on the class of P in $K_T(\mathcal{F}_\lambda; \mathbb{C})$, then so does the section $\text{Stab}_\lambda \Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ of H_λ . Thus the map

$$(6.27) \quad \mu_{H_\lambda}^K : K_T(\mathcal{F}_\lambda; \mathbb{C}) \rightarrow \mathcal{S}_{H_\lambda}^K, \quad [P] \mapsto \text{Stab}_\lambda \Psi_P^\circ,$$

is well-defined, cf. (5.29). By Corollary 5.16, the map $\mu_{H_\lambda}^K$ is an isomorphism of $\mathbb{C}[\mathbf{z}^{\pm 1}]$ -modules.

In Section 5.5 we introduced the Levelt fundamental solution $\widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ of dynamical differential equations (3.13). Consider the vector bundle $EH_\lambda \rightarrow \mathbb{C}^n \times \mathbb{C}^N$ with fiber over a point $(\mathbf{z}^0, \mathbf{p}^0)$ given by $\text{End}(H_T^*(\mathcal{F}_\lambda; \mathbb{C})_{\mathbf{z}^0})$, and its section $\text{Stab}_\lambda \widehat{\Psi}^\circ$ with values

$$(6.28) \quad \text{Stab}_\lambda \widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa) = \text{Stab}_{\lambda, \mathbf{z}} \circ \widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa) \circ (\text{Stab}_{\lambda, \mathbf{z}})^{-1}.$$

By Theorem 4.17 and Corollary 6.11, for any section f of H_λ not depending on \mathbf{p} , the section $\text{Stab}_\lambda \widehat{\Psi}^\circ f$ of H_λ with values $\text{Stab}_\lambda \widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa) f(\mathbf{z})$ is a solution of quantum differential equations (6.23). We will call the section $\text{Stab}_\lambda \widehat{\Psi}^\circ$ of EH_λ the *Levelt fundamental solution* of quantum differential equations (6.23).

Recall the function $\Psi^\bullet(\mathbf{z}; \mathbf{p}; \kappa)$ with values in $\text{End}((\mathbb{C}^N)_\lambda^{\otimes n})$, see (5.33). Consider the section and $\text{Stab}_\lambda \Psi^\bullet$ of EH_λ with values

$$\text{Stab}_\lambda \Psi^\bullet(\mathbf{z}; \mathbf{p}; \kappa) = \text{Stab}_{\lambda, \mathbf{z}} \circ \Psi^\bullet(\mathbf{z}; \mathbf{p}; \kappa) \circ (\text{Stab}_{\lambda, \mathbf{z}})^{-1}.$$

Then by formulae (5.33) and (6.22),

$$(6.29) \quad \text{Stab}_\lambda \widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa) = (\text{Stab}_\lambda \Psi^\bullet(\mathbf{z}; \mathbf{p}; \kappa)) \prod_{i=1}^N p_i^{[D_i]_{\mathbf{z}}/\kappa},$$

where $\prod_{i=1}^N p_i^{[D_i]z/\kappa}$ acts on the fiber $H_T^*(\mathcal{F}_\lambda; \mathbb{C})_z$ as multiplication by itself. The expression for $\text{Stab}_\lambda \Psi^\bullet(\mathbf{z}; \mathbf{p}; \kappa)$ is given by formula (6.33) below.

Recall the function $A^\circ(\mathbf{t}; \mathbf{z}; \kappa)$, see (5.13), (5.35),

$$A^\circ(\mathbf{t}; \mathbf{z}; \kappa) = \prod_{i=1}^{N-1} \prod_{a=1}^{\lambda^{(i)}} \left(\prod_{\substack{b=1 \\ b \neq a}}^{\lambda^{(i)}} \Gamma(1 + (t_b^{(i)} - t_a^{(i)})/\kappa) \prod_{c=1}^{\lambda^{(i+1)}} \frac{1}{\Gamma(1 + (t_c^{(i+1)} - t_a^{(i)})/\kappa)} \right),$$

where $\lambda^{(N)} = n$ and $t_a^{(N)} = z_a$, $a = 1, \dots, n$. For $I \in \mathcal{I}_\lambda$ and $\mathbf{m} = (m_1, \dots, m_{N-1}) \in \mathbb{Z}_{\geq 0}^{N-1}$, set

$$(6.30) \quad \hat{\mathcal{J}}_{I, \mathbf{m}}^H(\mathbf{\Gamma}; \mathbf{z}; \kappa) = \sum_{i=1}^{N-1} \sum_{\substack{\mathbf{l} \in \mathbb{Z}^{\lambda^{(i)}} \\ |\mathbf{l}^{(i)}| = m_i}} \sum_{J \in \mathcal{I}_\lambda} \frac{A^\circ(\Sigma_J - \mathbf{l}\kappa; \mathbf{z}; \kappa) W_I^\circ(\Sigma_J - \mathbf{l}\kappa; \mathbf{z}) V_\lambda(\mathbf{\Gamma}; \mathbf{z}_{\sigma_J})}{A^\circ(\Sigma_J; \mathbf{z}) R_\lambda(\mathbf{z}_{\sigma_J})},$$

where $|\mathbf{l}^{(i)}| = l_j^{(i)} + \dots + l_{\lambda^{(i)}}^{(i)}$, $i = 1, \dots, N-1$. cf. (5.36), (6.6). The functions $\mathcal{J}_{I, \mathbf{l}}^H(\mathbf{\Gamma}; \mathbf{z}; \kappa)$ are polynomials in $\mathbf{\Gamma}$ and rational functions in \mathbf{z}, κ .

Proposition 6.16. *The functions $\hat{\mathcal{J}}_{I, \mathbf{m}}^H(\mathbf{\Gamma}; \mathbf{z}; \kappa)$ are regular if $z_a - z_b \notin \kappa \mathbb{Z}_{\neq 0}$ for all $a \neq b$, and have at most simple poles at the hyperplanes $z_a - z_b \in \kappa \mathbb{Z}_{\neq 0}$.*

Proof. Recall the functions $\Psi_{I, J}^\bullet(\mathbf{z}; \mathbf{p}; \kappa)$ and $\Omega_\lambda^\circ(\mathbf{p}; \kappa)$, see (5.37), (5.18), respectively. By formulae (5.36), (5.37), (6.14), (5.7), (6.30), we have

$$(6.31) \quad \sum_{J \in \mathcal{I}_\lambda} \Psi_{I, J}^\bullet(\mathbf{z}; \mathbf{p}; \kappa) \text{Stab}_J^{\text{op}}(\mathbf{\Gamma}; \mathbf{z}) = \Omega_\lambda^\circ(\mathbf{p}; \kappa) \sum_{\mathbf{m} \in \mathbb{Z}_{\geq 0}^{N-1}} \hat{\mathcal{J}}_{I, \mathbf{m}}^H(\mathbf{\Gamma}; \mathbf{z}; \kappa) \prod_{i=1}^{N-1} ((-\kappa)^{-\lambda_i - \lambda_{i+1}} p_{i+1}/p_i)^{m_i}.$$

Formula (5.38) and Theorem 5.18 imply that the functions $\Psi_{I, J}^\bullet(\mathbf{z}; \mathbf{p}; \kappa)$ are regular if $z_a - z_b \notin \kappa \mathbb{Z}_{\neq 0}$ for all $a \neq b$, and have at most simple poles at the hyperplanes $z_a - z_b \in \kappa \mathbb{Z}_{\neq 0}$. Thus the statement of Proposition 6.16 follows from formula (6.31). \square

Define the section $\hat{\mathcal{J}}_{\mathbf{m}}^H$ of the bundle EH_λ with values

$$(6.32) \quad \hat{\mathcal{J}}_{\mathbf{m}}^H(\mathbf{z}; \kappa) : H_T^*(\mathcal{F}_\lambda; \mathbb{C})_z \rightarrow H_T^*(\mathcal{F}_\lambda; \mathbb{C})_z,$$

$$\hat{\mathcal{J}}_{\mathbf{m}}^H(\mathbf{z}; \kappa) : Y \mapsto \sum_{I \in \mathcal{I}_\lambda} \mathcal{E}_z \langle Y [\hat{\mathcal{J}}_{I, \mathbf{m}}^H(\mathbf{\Gamma}; \mathbf{z}; \kappa)]_z \rangle \text{Stab}_{I, z},$$

where \mathcal{E}_z is the integration map on $H_T^*(\mathcal{F}_\lambda; \mathbb{C})_z$, see (6.9).

Proposition 6.17. *We have*

$$(6.33) \quad \text{Stab}_\lambda \Psi^\bullet(\mathbf{z}; \mathbf{p}; \kappa) = \Omega_\lambda^\circ(\mathbf{p}; \kappa) \sum_{\mathbf{m} \in \mathbb{Z}_{\geq 0}^{N-1}} \hat{\mathcal{J}}_{\mathbf{m}}^H(\mathbf{z}; \kappa) \prod_{i=1}^{N-1} ((-\kappa)^{-\lambda_i - \lambda_{i+1}} p_{i+1}/p_i)^{m_i}.$$

Proof. The statement follows from formulae (5.38), (6.15), (6.31), (6.32). \square

Proposition 6.18. *For any $Y \in H_T^*(\mathcal{F}_\lambda; \mathbb{C})_z$,*

$$(6.34) \quad \mathcal{E}_z \langle (\hat{\mathcal{J}}_m^H(\mathbf{z}; \kappa) Y) \text{Stab}_{I,z}^{\text{op}} \rangle = \mathcal{E}_z \langle Y [\hat{\mathcal{J}}_{I,m}^H(\mathbf{\Gamma}; \mathbf{z}; \kappa)]_z \rangle.$$

In particular,

$$(6.35) \quad \mathcal{E}_z \langle \hat{\mathcal{J}}_m^H(\mathbf{z}; \kappa) Y \rangle = \mathcal{E}_z \langle Y [\hat{\mathcal{J}}_{I^{\min}, m}^H(\mathbf{\Gamma}; \mathbf{z}; \kappa)]_z \rangle.$$

Proof. The statement follows from formulae (6.32), (6.15), (6.12). \square

6.6. The map B_λ^H . Let $L^\circ \subset \mathbb{C}^n$ be the complement of the union of the hyperplanes

$$(6.36) \quad z_a - z_b \in \kappa \mathbb{Z}_{\neq 0}, \quad a, b = 1, \dots, n, \quad a \neq b.$$

Denote by \mathcal{O} the ring of functions of \mathbf{z} holomorphic in L° , and by \mathcal{O}_{H_λ} the space of sections of the bundle H_λ holomorphic in \mathbf{z} for $\mathbf{z} \in L^\circ$ and not depending on \mathbf{p} . The stable envelope map Stab_λ induces an isomorphism $(\mathbb{C}^N)_\lambda^{\otimes n} \otimes_{\mathbb{C}} \mathcal{O} \rightarrow \mathcal{O}_{H_\lambda}$, $f \mapsto \text{Stab}_\lambda f$.

Let $\mathcal{S}_{H_\lambda}^\circ$ be the space of sections of H_λ that are solutions of quantum differential equations (6.24) holomorphic in \mathbf{z} for $\mathbf{z} \in L^\circ$. The space $\mathcal{S}_{H_\lambda}^\circ$ is a counterpart of the space $\mathcal{S}_\lambda^\circ$ of solutions of limiting dynamical differential equations (3.13) introduced in Section 5.6. The map Stab_λ induces an isomorphism $\mathcal{S}_\lambda^\circ \rightarrow \mathcal{S}_{H_\lambda}^\circ$.

Recall the Levelt solution $\text{Stab}_\lambda \widehat{\Psi}^\circ$ of quantum differential equations (6.23).

Proposition 6.19. *For any $f \in \mathcal{O}_{H_\lambda}$, the section $\text{Stab}_\lambda \widehat{\Psi}^\circ f$ with values*

$$(\text{Stab}_\lambda \widehat{\Psi}^\circ f)(\mathbf{z}; \mathbf{p}; \kappa) = (\text{Stab}_\lambda \widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa)) f(\mathbf{z})$$

belongs to $\mathcal{S}_{H_\lambda}^\circ$. Moreover, the map

$$(6.37) \quad \mu_{H_\lambda}^\circ : \mathcal{O}_{H_\lambda} \rightarrow \mathcal{S}_{H_\lambda}^\circ, \quad f \mapsto \text{Stab}_\lambda \widehat{\Psi}^\circ f,$$

is an isomorphism.

Proof. For any $f \in \mathcal{O}_{H_\lambda}$, the section $\text{Stab}_\lambda \widehat{\Psi}^\circ f$ solves quantum differential equations (6.23). Since by formula (6.28), Lemma 6.12, and Theorem 5.18, the section $\text{Stab}_\lambda \widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ is holomorphic in \mathbf{z} for $\mathbf{z} \in L^\circ$, the section $\text{Stab}_\lambda \widehat{\Psi}^\circ f$ belongs to $\mathcal{S}_{H_\lambda}^\circ$. Backwards, for any $f \in \mathcal{S}_{H_\lambda}^\circ$, the section f° with values

$$(6.38) \quad f^\circ(\mathbf{z}) = (\text{Stab}_\lambda \widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa))^{-1} f(\mathbf{z}; \mathbf{p})$$

does not depend on \mathbf{p} because $f(\mathbf{z}; \mathbf{p})$ is a solution of quantum differential equations (6.24). Moreover, $(\text{Stab}_\lambda \widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa))^{-1}$ is holomorphic in \mathbf{z} for $\mathbf{z} \in L^\circ$ since $(\det \text{Stab}_\lambda \widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa))^{-1}$ is entire in \mathbf{z} . Thus $f^\circ \in \mathcal{O}_{H_\lambda}$ and $f = \mu_{H_\lambda}^\circ f^\circ$. Hence, the map $\mu_{H_\lambda}^\circ$ is an isomorphism. \square

For a solution $f \in \mathcal{S}_{H_\lambda}^\circ$ of quantum differential equations (6.23), we call the section f° defined by (6.38) the *principal term* of f . For the solution $\text{Stab}_\lambda \Psi_p^\circ$ corresponding to a Laurent polynomial $P(\mathbf{\Gamma}; \mathbf{z})$, see (6.27), the principal term is described in Proposition 6.20 below.

Recall the functions $C_\lambda^\circ(\mathbf{z}; \kappa)$ and $G_\lambda^\circ(\mathbf{z}; \kappa)$, see (5.41), (5.42),

$$C_\lambda^\circ(\mathbf{z}; \kappa) = \prod_{i=1}^N \kappa^{(\sum_{j=i+1}^N \lambda_j - \sum_{j=1}^{i-1} \lambda_j) \sum_{a=\lambda^{(i-1)+1}}^{\lambda^{(i)}} z_a / \kappa},$$

$$G_\lambda^\circ(\mathbf{z}; \kappa) = \prod_{i=1}^{N-1} \prod_{a=1}^{\lambda^{(i)}} \prod_{b=\lambda^{(i)+1}}^{\lambda^{(i+1)}} \Gamma((z_a - z_b) / \kappa),$$

and the function $\acute{P}(\mathbf{\Gamma}; \mathbf{z}; \kappa)$ obtained from a Laurent polynomial $P(\acute{\mathbf{\Gamma}}; \acute{\mathbf{z}})$ by substituting the variables $\acute{\gamma}_{i,j}$ and \acute{z}_a with the exponentials $e^{2\pi\sqrt{-1}\gamma_{i,j}/\kappa}$ and $e^{2\pi\sqrt{-1}z_a/\kappa}$, respectively. Set

$$C_\lambda^H(\mathbf{\Gamma}; \mathbf{z}; \kappa) = \sum_{I \in \mathcal{I}_\lambda} \frac{C_\lambda^\circ(\mathbf{z}_{\sigma_I}; \kappa) V_\lambda(\mathbf{\Gamma}; \mathbf{z}_{\sigma_I})}{R_\lambda(\mathbf{z}_{\sigma_I})}, \quad G_\lambda^H(\mathbf{\Gamma}; \mathbf{z}; \kappa) = \sum_{I \in \mathcal{I}_\lambda} G_\lambda^\circ(\mathbf{z}_{\sigma_I}; \kappa) V_\lambda(\mathbf{\Gamma}; \mathbf{z}_{\sigma_I}),$$

and

$$\acute{P}^H(\mathbf{\Gamma}; \mathbf{z}; \kappa) = \sum_{I \in \mathcal{I}_\lambda} \frac{\acute{P}(\mathbf{z}_{\sigma_I}; \mathbf{z}; \kappa) V_\lambda(\mathbf{\Gamma}; \mathbf{z}_{\sigma_I})}{R_\lambda(\mathbf{z}_{\sigma_I})},$$

cf. (6.6).

Proposition 6.20. *For a Laurent polynomial $P(\acute{\mathbf{\Gamma}}; \acute{\mathbf{z}})$, the values of the principal term Ψ_P° of the solution $\text{Stab}_\lambda \Psi_P^\circ$ are*

$$(6.39) \quad \Psi_P^\circ(\mathbf{z}; \kappa) = [\acute{P}^H(\mathbf{\Gamma}; \mathbf{z}; \kappa) C_\lambda^H(\mathbf{\Gamma}; \mathbf{z}; \kappa) G_\lambda^H(\mathbf{\Gamma}; \mathbf{z}; \kappa)]_{\mathbf{z}}.$$

Proof. By Proposition 5.19,

$$(6.40) \quad \Psi_P^\circ(\mathbf{z}; \kappa) = \sum_{I \in \mathcal{I}_\lambda} \mathcal{E}_{\mathbf{z}} \langle \acute{P}^H(\mathbf{\Gamma}; \mathbf{z}; \kappa) C_\lambda^H(\mathbf{\Gamma}; \mathbf{z}; \kappa) G_\lambda^H(\mathbf{\Gamma}; \mathbf{z}; \kappa) \text{Stab}_I^{\text{op}}(\mathbf{\Gamma}; \mathbf{z}) \rangle \text{Stab}_{I, \mathbf{z}}.$$

By Lemmas 6.6, 6.8, for any section f of H_λ ,

$$[f(\mathbf{\Gamma}; \mathbf{z})]_{\mathbf{z}} = \sum_{I \in \mathcal{I}_\lambda} \mathcal{E}_{\mathbf{z}} \langle f(\mathbf{\Gamma}; \mathbf{z}) \text{Stab}_I^{\text{op}}(\mathbf{\Gamma}; \mathbf{z}) \rangle \text{Stab}_{I, \mathbf{z}},$$

Hence, the right-hand sides of formulae (6.39) and (6.40) coincide. \square

The functions $C_\lambda^H(\mathbf{\Gamma}; \mathbf{z}; \kappa)$, $G_\lambda^H(\mathbf{\Gamma}; \mathbf{z}; \kappa)$, and $\acute{P}^H(\mathbf{\Gamma}; \mathbf{z}; \kappa)$ are polynomials in $\mathbf{\Gamma}$ analytically depending on \mathbf{z} . The sections of H_λ defined by their cohomology classes can be thought of as the cohomology classes of the respective analytic functions of $\mathbf{\Gamma}$,

$$(6.41) \quad C_\lambda^\circ(\mathbf{\Gamma}; \kappa) = \prod_{i=1}^N \kappa^{(\sum_{j=i+1}^N \lambda_j - \sum_{j=1}^{i-1} \lambda_j) \sum_{j=1}^{\lambda_i} \gamma_{i,j} / \kappa},$$

$$G_\lambda^+(\mathbf{\Gamma}; \kappa) = \kappa^{\lambda_{\{2\}}} \prod_{i=1}^{N-1} \prod_{j=i+1}^N \prod_{k=1}^{\lambda_i} \prod_{l=1}^{\lambda_j} \Gamma(1 + (\gamma_{i,k} - \gamma_{j,l}) / \kappa),$$

and $\acute{P}(\mathbf{\Gamma}; \mathbf{z}; \kappa)$. The class of $C_\lambda^\circ(\mathbf{\Gamma}; \kappa)$ is a product of exponentials of the equivariant first Chern classes $c_1(E_i) = [\gamma_{i,1} + \dots + \gamma_{i,\lambda_i}]$ of the vector bundles E_i over \mathcal{F}_λ with

fibers F_i/F_{i-1} , the class of $G_\lambda^+(\Gamma; \kappa)$ is the equivariant Gamma-class of \mathcal{F}_λ , and the class of $\acute{P}(\Gamma; \mathbf{z}; \kappa)$ is the equivariant Chern character of the class of the Laurent polynomial $P(\acute{\Gamma}; \acute{\mathbf{z}})$ in $K_T(\mathcal{F}_\lambda; \mathbb{C})$.

Define a map

$$(6.42) \quad \mathbb{B}_\lambda^H : K_T(\mathcal{F}_\lambda; \mathbb{C}) \rightarrow \mathcal{O}_{H_\lambda}$$

that sends the class $[P]$ of the Laurent polynomial $P(\acute{\Gamma}; \acute{\mathbf{z}})$ in $K_T(\mathcal{F}_\lambda; \mathbb{C})$ to the section of H_λ with values $[\acute{P}^H(\Gamma; \mathbf{z}; \kappa) C_\lambda^H(\Gamma; \mathbf{z}; \kappa) G_\lambda^H(\Gamma; \mathbf{z}; \kappa)]_z$. By Proposition 6.20, the map \mathbb{B}_λ^H sends the class $[P]$ to the principal term of the solution $\text{Stab}_\lambda \Psi_P^\circ$ of the joint system of quantum differential equations (6.23) and qKZ difference equations in cohomology (6.25).

Theorem 6.21. *Recall the space \mathcal{O}_{H_λ} of sections of H_λ holomorphic in \mathbf{z} for $\mathbf{z} \in L^\circ$ and not depending on \mathbf{p} , and the space $\mathcal{S}_{H_\lambda}^\circ$ of solutions of quantum differential equations (6.23) holomorphic in \mathbf{z} for $\mathbf{z} \in L^\circ$. Then the map $\mathbb{B}_\lambda^H : K_T(\mathcal{F}_\lambda; \mathbb{C}) \rightarrow \mathcal{O}_{H_\lambda}$ is well-defined and the following diagram is commutative,*

$$(6.43) \quad \begin{array}{ccc} K_T(\mathcal{F}_\lambda; \mathbb{C}) & \xrightarrow{\mathbb{B}_\lambda^H} & \mathcal{O}_{H_\lambda} \\ & \searrow \mu_{H_\lambda}^K & \swarrow \mu_{H_\lambda}^\circ \\ & & \mathcal{S}_{H_\lambda}^\circ \end{array}$$

Proof. The statement follows from Proposition 5.20 by applying the stable envelope map Stab_λ . In more detail, by the standard reasoning the functions $C_\lambda^H(\Gamma; \mathbf{z}; \kappa)$, $G_\lambda^H(\Gamma; \mathbf{z}; \kappa)$, $\acute{P}(\Gamma; \mathbf{z}; \kappa)$, and $\mathcal{E}_z \langle \acute{P}^H(\Gamma; \mathbf{z}; \kappa) C_\lambda^H(\Gamma; \mathbf{z}; \kappa) G_\lambda^H(\Gamma; \mathbf{z}; \kappa) \text{Stab}_\lambda^\circ(\Gamma; \mathbf{z}) \rangle$ are holomorphic in \mathbf{z} for $\mathbf{z} \in L^\circ$. Hence, the map \mathbb{B}_λ^H is well-defined. The commutativity of diagram (5.50) follows from Proposition 6.20. \square

6.7. The nonequivariant case $\mathbf{z} = 0$. In this section, we will discuss solutions of the quantum differential equations for the cohomology algebra $H^*(\mathcal{F}_\lambda; \mathbb{C})$ of the partial flag variety \mathcal{F}_λ by specializing the results obtained for the equivariant case at $\mathbf{z} = 0$.

The cohomology algebra $H^*(\mathcal{F}_\lambda; \mathbb{C})$ of the partial flag variety \mathcal{F}_λ has the form

$$(6.44) \quad H^*(\mathcal{F}_\lambda; \mathbb{C}) = \mathbb{C}[\Gamma]^{S_\lambda} / \left\langle \prod_{i=1}^N \prod_{j=1}^{\lambda_i} (u - \gamma_{i,j}) = u^n \right\rangle,$$

where u is a formal variable. It is isomorphic to the algebra $H^*(\mathcal{F}_\lambda; \mathbb{C})_{z^0}$ at $\mathbf{z}^0 = 0$, see (6.3). We denote the class of a polynomial $f(\Gamma) \in \mathbb{C}[\Gamma]^{S_\lambda}$ in $H^*(\mathcal{F}_\lambda; \mathbb{C})$ by $[f]_0$.

The integration map on $H^*(\mathcal{F}_\lambda; \mathbb{C})$ coincides with the map \mathcal{E}_{z^0} at $\mathbf{z}^0 = 0$, see (6.9),

$$(6.45) \quad \mathcal{E}_0 : H^*(\mathcal{F}_\lambda; \mathbb{C}) \rightarrow \mathbb{C}, \quad [f]_0 \mapsto \mathcal{E}_0 \langle f \rangle,$$

where

$$\mathcal{E}_0 \langle f \rangle = \left(\sum_{I \in \mathcal{I}_\lambda} \frac{f(\mathbf{z}_{\sigma_I})}{R_\lambda(\mathbf{z}_{\sigma_I})} \right) \Big|_{z=0},$$

see Lemmas 6.3, 6.4.

Consider the polynomials $\text{Stab}_{I,0}(\Gamma) = \text{Stab}_I(\Gamma; 0)$ and $\text{Stab}_{I,0}^{\text{op}}(\Gamma) = \text{Stab}_I^{\text{op}}(\Gamma; 0)$. By formulae (6.12),

$$\text{Stab}_{\sigma_0(I_{\lambda}^{\min}),0}(\Gamma) = \text{Stab}_{I_{\lambda}^{\min},0}^{\text{op}}(\Gamma) = 1,$$

$$\text{Stab}_{I_{\lambda}^{\min},0}(\Gamma) = \text{Stab}_{\sigma_0(I_{\lambda}^{\min}),0}^{\text{op}}(\Gamma) = \prod_{i=1}^{N-1} \prod_{j=1}^{\lambda_i} \gamma_{i,j}^{n-\lambda^{(i)}}.$$

In general, the polynomials $\text{Stab}_{I,0}(\Gamma)$, $\text{Stab}_{I,0}^{\text{op}}(\Gamma)$ coincide with the A -type Schubert polynomials, see (6.13),

$$(6.46) \quad \text{Stab}_{I,0}(\Gamma) = \mathfrak{S}_{\sigma_{\sigma_0(I)}}(\Gamma; 0), \quad \text{Stab}_{I,0}^{\text{op}}(\Gamma) = \mathfrak{S}_{\sigma_I}(\Gamma; 0).$$

By Lemma 6.6, the polynomials $\text{Stab}_{I,0}(\Gamma)$ and $\text{Stab}_{I,0}^{\text{op}}(\Gamma)$ are biorthogonal,

$$(6.47) \quad \mathcal{E}_0 \langle \text{Stab}_{I,0}(\Gamma) \text{Stab}_{J,0}^{\text{op}}(\Gamma) \rangle = \delta_{I,J}.$$

Denote by $\text{Stab}_{I,0}$, $\text{Stab}_{I,0}^{\text{op}}$ the classes of $\text{Stab}_{I,0}(\Gamma)$, $\text{Stab}_{I,0}^{\text{op}}(\Gamma)$ in $H^*(\mathcal{F}_{\lambda}; \mathbb{C})$. They are the fundamental classes of the Schubert subvarieties in \mathcal{F}_{λ} defined for the opposite orderings of the chosen basis of \mathbb{C}^n in the standard way,

The quantum multiplication in $H^*(\mathcal{F}_{\lambda}; \mathbb{C})$, is a deformation of the multiplication in $H^*(\mathcal{F}_{\lambda}; \mathbb{C})$ depending on quantum parameters \mathbf{p} . For $Y \in H^*(\mathcal{F}_{\lambda}; \mathbb{C})$, let $Y *_{\mathbf{p}}$ be the operator of quantum multiplication by Y depending on \mathbf{p} .

For $i = 1, \dots, N$, denote $D_i = \gamma_{i,1} + \dots + \gamma_{i,\lambda_i}$, cf. (6.2). The quantum differential equations for $H^*(\mathcal{F}_{\lambda}; \mathbb{C})$ -valued functions of \mathbf{p} is the following system of compatible differential equations,

$$(6.48) \quad \kappa p_i \frac{\partial f}{\partial p_i} = [D_i]_0 *_{\mathbf{p}} f, \quad i = 1, \dots, N,$$

where κ is a parameter of the equations. Equations (6.48) coincide with the restriction of equivariant quantum differential equations (6.23) to the subbundle $H^*(\mathcal{F}_{\lambda}; \mathbb{C}) \times \mathbb{C}^N \rightarrow \mathbb{C}^N$ of the bundle $H_{\lambda} \rightarrow \mathbb{C}^n \times \mathbb{C}^N$ located over the points $(0, \mathbf{p}) \in \mathbb{C}^n \times \mathbb{C}^N$ of the base.

Consider the K -theory algebra $K(\mathcal{F}_{\lambda}; \mathbb{C})$. Then

$$(6.49) \quad K(\mathcal{F}_{\lambda}; \mathbb{C}) = \mathbb{C}[\dot{\Gamma}^{\pm 1}]^{S_{\lambda}} \otimes \mathbb{C}[\dot{z}^{\pm 1}] / \left\langle \prod_{i=1}^N \prod_{j=1}^{\lambda_i} (u - \dot{\gamma}_{i,j}) = \prod_{a=1}^n (u - 1) \right\rangle,$$

where u is a formal variable. The evaluation map

$$\mathbb{C}[\dot{\Gamma}^{\pm 1}]^{S_{\lambda}} \otimes \mathbb{C}[\dot{z}^{\pm 1}] \rightarrow \mathbb{C}[\dot{\Gamma}^{\pm 1}]^{S_{\lambda}}, \quad P(\dot{\Gamma}; \dot{z}) \mapsto P(\dot{\Gamma}; (1, \dots, 1)),$$

induces the isomorphism of $K(\mathcal{F}_{\lambda}; \mathbb{C}) \rightarrow K_T(\mathcal{F}_{\lambda}; \mathbb{C}) / \langle z = (1, \dots, 1) \rangle$. Denote by $[P]_1$ the class of $P(\dot{\Gamma}) \in \mathbb{C}[\dot{\Gamma}^{\pm 1}]^{S_{\lambda}}$ in $K(\mathcal{F}_{\lambda}; \mathbb{C})$.

Identify $\mathbb{C}[\dot{\Gamma}^{\pm 1}]^{S_{\lambda}}$ with the subspace of $\mathbb{C}[\dot{\Gamma}^{\pm 1}]^{S_{\lambda}} \otimes \mathbb{C}[\dot{z}^{\pm 1}]$ of Laurent polynomials not depending on \dot{z} . For each $P \in \mathbb{C}[\dot{\Gamma}^{\pm 1}]^{S_{\lambda}}$, the solution $\text{Stab}_{\lambda} \Psi_P^{\circ}(\mathbf{z}; \mathbf{p}; \kappa)$ of equivariant

quantum differential equations (6.23), see Section 6.5, is regular at $\mathbf{z} = 0$, and the solution $\text{Stab}_\lambda \Psi_P^\circ(0; \mathbf{p}; \kappa)$ of equations (6.48) depends only on the class of P in $K(\mathcal{F}_\lambda; \mathbb{C})$. That is, there is a well-defined map

$$(6.50) \quad \mu_{\mathcal{H}_\lambda}^K : K(\mathcal{F}_\lambda; \mathbb{C}) \rightarrow \mathcal{H}_\lambda, \quad [P]_1 \mapsto \text{Stab}_\lambda \Psi_P^\circ(0; \mathbf{p}; \kappa),$$

where \mathcal{H}_λ is the space of solutions of equations (6.48).

Proposition 6.22. *The map $\mu_{\mathcal{H}_\lambda}^K$, see (6.50), is an isomorphism.*

Proof. The statement follows from Theorem 5.15 at $\mathbf{z} = 0$. \square

The Levelt fundamental solution $\text{Stab}_\lambda \widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ of equivariant quantum differential equations (6.23), see Section 6.5, is regular at $\mathbf{z} = 0$ and the function $\text{Stab}_\lambda \widehat{\Psi}^\circ(0; \mathbf{p}; \kappa)$ is an $\text{End}(H^*(\mathcal{F}_\lambda; \mathbb{C}))$ -valued solution of quantum differential equations (6.48). We call $\text{Stab}_\lambda \widehat{\Psi}^\circ(0; \mathbf{p}; \kappa)$ the Levelt fundamental solution of quantum differential equations (6.48). It induces the map

$$(6.51) \quad \mu_{\mathcal{H}_\lambda}^H : H^*(\mathcal{F}_\lambda; \mathbb{C}) \rightarrow \mathcal{H}_\lambda, \quad f \mapsto \text{Stab}_\lambda \widehat{\Psi}^\circ(0; \mathbf{p}; \kappa) f,$$

to the space \mathcal{H}_λ of solutions of equations (6.48).

Proposition 6.23. *The map $\mu_{\mathcal{H}_\lambda}^H$, see (6.51), is an isomorphism.*

Proof. The statement follows from Proposition 6.19 at $\mathbf{z} = 0$. \square

For a solution $f(\mathbf{p}) \in \mathcal{H}_\lambda$ of quantum differential equations (6.48), we call the element

$$(6.52) \quad f^{\circ,0} = (\text{Stab}_\lambda \widehat{\Psi}^\circ(0; \mathbf{p}; \kappa))^{-1} f(\mathbf{p})$$

the *principal term* of $f(\mathbf{p})$. The principal term of the solution $\text{Stab}_\lambda \Psi_P^\circ(0; \mathbf{p}; \kappa)$ corresponding to a Laurent polynomial $P(\mathbf{\Gamma})$ is given by Proposition 6.25 below.

For a function $F(\mathbf{\Gamma})$ holomorphic in a neighborhood of $\mathbf{\Gamma} = 0$ and symmetric in $\gamma_{i,1}, \dots, \gamma_{i,\lambda_i}$ for each $i = 1, \dots, N$, define its class $[F]_0$ in $H^*(\mathcal{F}_\lambda; \mathbb{C})$ by expanding $F(\mathbf{\Gamma})$ in the power series about $\mathbf{\Gamma} = 0$ and replacing each term by the corresponding class in $H^*(\mathcal{F}_\lambda; \mathbb{C})$. The resulting sum contains only finitely many nonzero terms and, hence, is well-defined. Alternatively, the class $[F]_0$ can be evaluated as follows. Set

$$(6.53) \quad F^H(\mathbf{\Gamma}; \mathbf{z}) = \sum_{I \in \mathcal{I}_\lambda} \frac{F(\mathbf{z}_{\sigma_I}) V_\lambda(\mathbf{\Gamma}; \mathbf{z}_{\sigma_I})}{R_\lambda(\mathbf{z}_{\sigma_I})}.$$

Lemma 6.24. *The function $F^H(\mathbf{\Gamma}; \mathbf{z})$ is regular at $\mathbf{z} = 0$ and $[F]_0 = [F^H(\mathbf{\Gamma}; 0)]_0$.*

Proof. By the standard facts on power series, it suffices to prove the statement for a polynomial $F(\mathbf{\Gamma})$. In the polynomial case, formulae (6.53), (6.4) yield $F(\mathbf{\Gamma}) = F^H(\mathbf{\Gamma}; \mathbf{\Gamma})$. Then $[F(\mathbf{\Gamma})]_0 = [F^H(\mathbf{\Gamma}; \mathbf{\Gamma})]_0 = [F^H(\mathbf{\Gamma}; 0)]_0$, because $F^H(\mathbf{\Gamma}; \mathbf{z})$ is symmetric in \mathbf{z} . \square

Recall the functions $C_\lambda^\circ(\mathbf{\Gamma}; \kappa)$, $G_\lambda^+(\mathbf{\Gamma}; \kappa)$, see (6.41), and the function $\acute{P}(\mathbf{\Gamma}; \kappa)$ obtained from a Laurent polynomial $P(\acute{\mathbf{\Gamma}})$ by substituting the variables $\gamma_{i,j}$ with the exponentials $e^{2\pi\sqrt{-1}\gamma_{i,j}/\kappa}$. The class of $C_\lambda^\circ(\mathbf{\Gamma}; \kappa)$ is a product of exponentials of the first Chern classes $c_1(E_i) = [\gamma_{i,1} + \dots + \gamma_{i,\lambda_i}]$ of the vector bundles E_i over \mathcal{F}_λ with fibers F_i/F_{i-1} , the class of $G_\lambda^+(\mathbf{\Gamma}; \kappa)$ is the Gamma-class of \mathcal{F}_λ , and the class of $\acute{P}(\mathbf{\Gamma}; \kappa)$ is the Chern character of the class of the Laurent polynomial $P(\acute{\mathbf{\Gamma}})$ in $K(\mathcal{F}_\lambda; \mathbb{C})$.

Proposition 6.25. *For a Laurent polynomial $P(\acute{\mathbf{\Gamma}})$, the principal term $\Psi_P^{\circ,0}$ of the solution $\text{Stab}_\lambda \Psi_P^\circ(0; \mathbf{p}; \kappa)$ equals*

$$(6.54) \quad \Psi_P^{\circ,0}(\kappa) = [\acute{P}(\mathbf{\Gamma}; \kappa) C_\lambda^\circ(\mathbf{\Gamma}; \kappa) G_\lambda^+(\mathbf{\Gamma}; \kappa)]_0.$$

Proof. The statement follows from Proposition 6.25 at $z = 0$ and Lemma 6.24. \square

Define a map

$$(6.55) \quad \mathbb{B}_\lambda^{H,0} : K(\mathcal{F}_\lambda; \mathbb{C}) \rightarrow H^*(\mathcal{F}_\lambda; \mathbb{C}), \quad [P]_1 \rightarrow [\acute{P}(\mathbf{\Gamma}; \kappa) C_\lambda^\circ(\mathbf{\Gamma}; \kappa) G_\lambda^+(\mathbf{\Gamma}; \kappa)]_0,$$

By Proposition 6.25, the map $\mathbb{B}_\lambda^{H,0}$ sends the class $[P]_1$ to the principal term of the solution $\text{Stab}_\lambda \Psi_P^\circ(0; \mathbf{p}; \kappa)$ of quantum differential equations (6.48).

Theorem 6.26. *Recall the space \mathcal{H}_λ of solutions of quantum differential equations (6.48). The following diagram is commutative,*

$$(6.56) \quad \begin{array}{ccc} K(\mathcal{F}_\lambda; \mathbb{C}) & \xrightarrow{\mathbb{B}_\lambda^{H,0}} & H^*(\mathcal{F}_\lambda; \mathbb{C}) \\ & \searrow \mu_{\mathcal{H}_\lambda}^K & \swarrow \mu_{\mathcal{H}_\lambda}^H \\ & \mathcal{H}_\lambda & \end{array}$$

Proof. The statement is equivalent to Proposition 6.25. \square

6.8. Topological-enumerative morphism and J -function. Recall the Levelt fundamental solution $\text{Stab}_\lambda \widehat{\Psi}^\circ(0; \mathbf{p}; \kappa)$ of quantum differential equations (6.48). By Lemma 6.8 and biorthogonality relation (6.6), for any $I \in \mathcal{I}_\lambda$, there exists a unique $H^*(\mathcal{F}_\lambda; \mathbb{C})$ -valued function $\hat{\mathcal{J}}_I^H(\mathbf{p}; \kappa)$ such that for any $f \in H^*(\mathcal{F}_\lambda; \mathbb{C})$,

$$(6.57) \quad \mathcal{E}_0 \langle (\text{Stab}_\lambda \widehat{\Psi}^\circ(0; \mathbf{p}; \kappa) f) \text{Stab}_{I,0}^{\text{op}} \rangle = \mathcal{E}_0 \langle \hat{\mathcal{J}}_I^H(\mathbf{p}; \kappa) f \rangle.$$

Since $\text{Stab}_{I_\lambda^{\min},0}^{\text{op}} = 1$, see (6.12), formula (6.57) for $I = I_\lambda^{\min}$ takes the form

$$(6.58) \quad \mathcal{E}_0 \langle \text{Stab}_\lambda \widehat{\Psi}^\circ(0; \mathbf{p}; \kappa) f \rangle = \mathcal{E}_0 \langle \hat{\mathcal{J}}_{I_\lambda^{\min}}^H(\mathbf{p}; \kappa) f \rangle.$$

Proposition 6.27. *For any $I \in \mathcal{I}_\lambda$,*

$$(6.59) \quad \hat{\mathcal{J}}_I^H(\mathbf{p}; \kappa) = \Omega_\lambda^\circ(\mathbf{p}; \kappa) \sum_{\mathbf{m} \in \mathbb{Z}_{\geq 0}^{N-1}} [\hat{\mathcal{J}}_{I,\mathbf{m}}^H(\mathbf{\Gamma}; 0; \kappa)]_0 \prod_{i=1}^{N-1} ((-\kappa)^{-\lambda_i - \lambda_{i+1}} p_{i+1}/p_i)^{m_i} \prod_{i=1}^N p_i^{[D_i]_0/\kappa},$$

where the functions $\hat{\mathcal{J}}_{I,\mathbf{m}}^H$ are given by (6.30), $\Omega_\lambda^\circ = e^{\sum_{i<j} p_j/(\kappa p_i)}$ in which the sum is taken over all pairs $i < j$ such that $\lambda_i = 1$ and $\lambda_{i+1} = \dots = \lambda_j = 0$, see (5.18), and $D_i = \gamma_{i,1} + \dots + \gamma_{i,\lambda_i}$.

Proof. By Proposition 6.16, the functions $\hat{\mathcal{J}}_{I,\mathbf{m}}^H(\Gamma; \mathbf{z}; \kappa)$ are regular at $\mathbf{z} = 0$. Thus the classes $[\hat{\mathcal{J}}_{I,\mathbf{m}}^H(\Gamma; 0; \kappa)]_0$ in formula (6.59) are well defined. Then the statement follows from Proposition 6.18 and formulae (6.57), (6.33), (6.29). \square

By inspection, the function $\hat{\mathcal{J}}_{I_{\min}}^H(\mathbf{p}; \kappa)$ in this paper coincides with the nonequivariant J -function of \mathcal{F}_λ obtained in [BCK] up to a change of notation.

In more detail, formula (6.30) for the function $\hat{\mathcal{J}}_{I_{\min},\mathbf{m}}^H(\Gamma; \mathbf{z}; \kappa)$ takes the form

$$(6.60) \quad \hat{\mathcal{J}}_{I_{\min},\mathbf{m}}^H(\Gamma; \mathbf{z}; \kappa) = \sum_{i=1}^{N-1} \sum_{\substack{\mathbf{l} \in \mathbb{Z}^{\lambda^{(1)}} \\ |\mathbf{l}^{(i)}| = m_i}} \sum_{J \in \mathcal{I}_\lambda} \frac{A^\circ(\Sigma_J - \mathbf{l}; \mathbf{z}; \kappa) V_\lambda(\Gamma; \mathbf{z}_{\sigma_J})}{A^\circ(\Sigma_J; \mathbf{z}) R_\lambda(\mathbf{z}_{\sigma_J})},$$

because $W_{I_{\min}}^\circ(\Sigma_J - \mathbf{l}\kappa; \mathbf{z}) = 1$, see Lemma 5.2. In (6.60), $\mathbf{m} = (m_1, \dots, m_{N-1}) \in \mathbb{Z}_{\geq 0}^{N-1}$, $\mathbf{l} = (l_1^{(1)}, \dots, l_{\lambda^{(1)}}^{(1)}, \dots, l_1^{(N-1)}, \dots, l_{\lambda^{(N-1)}}^{(N-1)}) \in \mathbb{Z}^{\lambda^{(1)}}$, $|\mathbf{l}^{(i)}| = l_j^{(i)} + \dots + l_{\lambda^{(i)}}^{(i)}$,

$$A^\circ(\mathbf{t}; \mathbf{z}; \kappa) = \prod_{i=1}^{N-1} \prod_{a=1}^{\lambda^{(i)}} \left(\prod_{\substack{b=1 \\ b \neq a}}^{\lambda^{(i)}} \Gamma(1 + (t_b^{(i)} - t_a^{(i)})/\kappa) \prod_{c=1}^{\lambda^{(i+1)}} \frac{1}{\Gamma(1 + (t_c^{(i+1)} - t_a^{(i)})/\kappa)} \right),$$

where $\lambda^{(N)} = n$, $t_a^{(N)} = z_a$, and $l_a^{(i)} = 0$ for $a > \lambda^{(i)}$ or $i = N$, and Σ_J is given by (4.1). Then the expression in [BCK, Theorem 1.3] for $J_d^F(\hbar)$ with $l = N-1$, $d_i = m_i$, $d_{i,j} = l_j^{(i)}$, $\hbar = \kappa$, and $H_{i,j} = \gamma_{a,j-\lambda^{(a-1)}}$ for $\lambda^{(a-1)} < j \leq \lambda^{(a)}$, agrees term by term with the expression for the class $[\hat{\mathcal{J}}_{I,\mathbf{m}}^H(\Gamma; 0; \kappa)]_0$ obtained from (6.60). Also, formula (6.59) for $\hat{\mathcal{J}}_{I_{\min}}^H(\mathbf{p}; \kappa)$ is the same as the series expansion of the nonequivariant J -function of \mathcal{F}_λ in [BCK].

Observe that relation (6.58) between the J -function $\hat{\mathcal{J}}_{I_{\min}}^H(\mathbf{p}; \kappa)$ and the Levelt fundamental solution $\text{Stab}_\lambda \widehat{\Psi}^\circ(0; \mathbf{p}; \kappa)$ matches the general relation between the J -function and the topological-enumerative morphism, see Definition 5.3 and formula (5.8) in [CV], where the J -function is defined in terms of the topological-enumerative morphism. This naturally suggests the following conjecture.

Conjecture 6.28. *The Levelt fundamental solution $\text{Stab}_\lambda \widehat{\Psi}^\circ(0; \mathbf{p}; \kappa)$ of quantum differential equations (6.48) is the topological-enumerative morphism of \mathcal{F}_λ .*

Observe that formulae (6.57)–(6.59) are specializations of respective formulae for the Levelt fundamental solution $\text{Stab}_\lambda \widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ of equivariant quantum differential equations (6.23). By Lemma 6.8 and biorthogonality relation (6.6), for any $I \in \mathcal{I}_\lambda$, there exists a unique section $\hat{\mathcal{J}}_I^H(\mathbf{z}; \mathbf{p}; \kappa)$ of the bundle H_λ such that for any $f \in H^*(\mathcal{F}_\lambda; \mathbb{C})$ considered as a section of H_λ not depending on \mathbf{p} ,

$$(6.61) \quad \mathcal{E}_z \langle (\text{Stab}_\lambda \widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa) [f]_z) \text{Stab}_{I,z}^{\text{op}} \rangle = \mathcal{E}_z \langle \hat{\mathcal{J}}_I^H(\mathbf{z}; \mathbf{p}; \kappa) [f]_z \rangle.$$

Since $\text{Stab}_{I_{\lambda}^{\min}, \mathbf{z}}^{\text{op}} = 1$, see (6.12), formula (6.57) for $I = I_{\lambda}^{\min}$ takes the form

$$(6.62) \quad \mathcal{E}_{\mathbf{z}} \langle \text{Stab}_{\lambda} \widehat{\Psi}^{\circ}(\mathbf{z}; \mathbf{p}; \kappa) [f]_{\mathbf{z}} \rangle = \mathcal{E}_{\mathbf{z}} \langle \widehat{\mathcal{J}}_{I_{\lambda}^{\min}}^H(\mathbf{z}; \mathbf{p}; \kappa) [f]_{\mathbf{z}} \rangle.$$

Furthermore, by Proposition 6.18 and formulae (6.61), (6.33), (6.29),

$$(6.63) \quad \begin{aligned} \widehat{\mathcal{J}}_I^H(\mathbf{z}; \mathbf{p}; \kappa) &= \\ &= \Omega_{\lambda}^{\circ}(\mathbf{p}; \kappa) \sum_{\mathbf{m} \in \mathbb{Z}_{\geq 0}^{N-1}} [\widehat{\mathcal{J}}_{I, \mathbf{m}}^H(\mathbf{\Gamma}; \mathbf{z}; \kappa)]_{\mathbf{z}} \prod_{i=1}^{N-1} ((-\kappa)^{-\lambda_i - \lambda_{i+1}} p_{i+1}/p_i)^{m_i} \prod_{i=1}^N p_i^{[D_i]_{\mathbf{z}}/\kappa}, \end{aligned}$$

where the notation is the same as in formula (6.59). This suggests the equivariant version of Conjecture 6.28.

Conjecture 6.29. *The Levelt fundamental solution $\text{Stab}_{\lambda} \widehat{\Psi}^{\circ}(\mathbf{z}; \mathbf{p}; \kappa)$ of equivariant quantum differential equations (6.23) is the equivariant topological-enumerative morphism of \mathcal{F}_{λ} .*

6.9. Example $\lambda = (1, n-1)$, $\mathcal{F}_{\lambda} = \mathbb{C}\mathbb{P}^{n-1}$. Throughout this section, let $N = 2$, $n \geq 2$, $\lambda = (1, n-1)$, $\mathcal{F}_{\lambda} = \mathbb{C}\mathbb{P}^{n-1}$.

Denote $\gamma = \gamma_{1,1}$. Identify $\mathbb{C}[\gamma]$ with the subspace of $\mathbb{C}[\mathbf{\Gamma}]^{S_{\lambda}}$ consisting of polynomials not depending on $\gamma_{2,1}, \dots, \gamma_{2,n-1}$, and $\mathbb{C}[\mathbf{\Gamma}]^{S_{\lambda}}$ with the subspace of $\mathbb{C}[\mathbf{\Gamma}]^{S_{\lambda}} \otimes \mathbb{C}[\mathbf{z}]$ consisting of polynomials not depending on \mathbf{z} . Then the equivariant cohomology algebra $H_T^*(\mathbb{C}\mathbb{P}^{n-1}; \mathbb{C})$, see (6.1), can be presented as follows

$$(6.64) \quad H_T^*(\mathbb{C}\mathbb{P}^{n-1}; \mathbb{C}) = \mathbb{C}[\gamma] \otimes \mathbb{C}[\mathbf{z}] \left/ \left\langle \prod_{a=1}^n (\gamma - z_a) = 0 \right\rangle \right.$$

Similarly, for any $\mathbf{z}^0 \in \mathbb{C}^n$, the algebra $H_T^*(\mathbb{C}\mathbb{P}^{n-1}; \mathbb{C})_{\mathbf{z}^0}$ can be presented as follows

$$(6.65) \quad H_T^*(\mathbb{C}\mathbb{P}^{n-1}; \mathbb{C})_{\mathbf{z}^0} = \mathbb{C}[\gamma] \left/ \left\langle \prod_{a=1}^n (\gamma - z_a^0) = 0 \right\rangle \right.$$

For any polynomial $f(\gamma; \mathbf{z})$, the polynomial $\mathcal{E}\langle f \rangle(\mathbf{z})$, see (6.7), can be evaluated as an integral

$$(6.66) \quad \mathcal{E}\langle f \rangle(\mathbf{z}) = \frac{1}{2\pi\sqrt{-1}} \int_C \frac{f(\gamma; \mathbf{z})}{(\gamma - z_1) \dots (\gamma - z_n)} d\gamma$$

over a contour C encircling z_1, \dots, z_n counterclockwise. Similarly, for any polynomial $f(\gamma)$ and $\mathbf{z}^0 \in \mathbb{C}^n$,

$$(6.67) \quad \mathcal{E}_{\mathbf{z}^0}\langle f \rangle = \frac{1}{2\pi\sqrt{-1}} \int_C \frac{f(\gamma)}{(\gamma - z_1^0) \dots (\gamma - z_n^0)} d\gamma$$

for a contour C encircling z_1^0, \dots, z_n^0 counterclockwise.

Following Section 5.7, denote by $[a]$ the element $(\{a\}, \{1, \dots, a-1, a+1, \dots, n\}) \in \mathcal{I}_{\lambda}$.

The polynomials $\text{Stab}_{[a]}(\gamma; \mathbf{z})$, $\text{Stab}_{[a]}^{\text{op}}(\gamma; \mathbf{z})$ are respectively obtained from the functions $\check{W}_{[a]}^{\circ}(t; \mathbf{z})$, $W_{[a]}^{\circ}(t; \mathbf{z})$, see (5.57), (5.55), by the substitution $t = \gamma$,

$$(6.68) \quad \text{Stab}_{[a]}(\gamma; \mathbf{z}) = \prod_{c=a+1}^n (\gamma - z_b), \quad \text{Stab}_{[a]}^{\text{op}}(\gamma; \mathbf{z}) = \prod_{c=1}^{a-1} (\gamma - z_b), \quad a = 1, \dots, n.$$

The biorthogonality relation, see Lemma 6.6,

$$(6.69) \quad \mathcal{E} \langle \text{Stab}_{[a]}(\gamma; \mathbf{z}) \text{Stab}_{[b]}^{\text{op}}(\gamma; \mathbf{z}) \rangle = \delta_{a,b}$$

is clear from formula (6.66).

Consider a polynomial

$$(6.70) \quad \bar{W}_{\lambda}(t; \gamma; \mathbf{z}) = \frac{1}{t - \gamma} \left(\prod_{a=1}^n (t - z_a) - \prod_{a=1}^n (\gamma - z_a) \right),$$

and its image $[\bar{W}_{\lambda}(t; \gamma; \mathbf{z})]$ in $\mathbb{C}[t] \otimes H_T^*(\mathbb{C}\mathbb{P}^{n-1}; \mathbb{C})$. Recall the polynomial $\widehat{W}_{\lambda}^{\circ}(t; \Gamma)$, see (6.18), and its image $[\widehat{W}_{\lambda}^{\circ}(t; \Gamma)]$ in $\mathbb{C}[t] \otimes H_T^*(\mathbb{C}\mathbb{P}^{n-1}; \mathbb{C})$. By relations (6.1), (6.64),

$$[\widehat{W}_{\lambda}^{\circ}(t; \Gamma)] = [\bar{W}_{\lambda}(t; \gamma; \mathbf{z})].$$

Thus Theorem 6.9 for $\mathbb{C}\mathbb{P}^{n-1}$ reads as follows.

$$(6.71) \quad [\bar{W}_{\lambda}(t; \gamma; \mathbf{z})] = \sum_{a=1}^n W_{[a]}^{\circ}(t; \mathbf{z}) [\text{Stab}_{[a]}(\gamma; \mathbf{z})].$$

In fact, formula (6.71) is the image $\mathbb{C}[t] \otimes H_T^*(\mathbb{C}\mathbb{P}^{n-1}; \mathbb{C})$ of the equality of polynomials

$$(6.72) \quad \bar{W}_{\lambda}(t; \gamma; \mathbf{z}) = \sum_{a=1}^n W_{[a]}^{\circ}(t; \mathbf{z}) \text{Stab}_{[a]}(\gamma; \mathbf{z}).$$

Recall that for $\mathbf{z} \in \mathbb{C}^n$ and $a = 1, \dots, n$, the classes of the polynomials $\text{Stab}_{[a]}(\gamma; \mathbf{z})$, $\text{Stab}_{[a]}^{\text{op}}(\gamma; \mathbf{z})$ in $H_T^*(\mathbb{C}\mathbb{P}^{n-1}; \mathbb{C})_{\mathbf{z}}$ are denoted by $\text{Stab}_{[a], \mathbf{z}}$, $\text{Stab}_{[a], \mathbf{z}}^{\text{op}}$. Consider a basis $v_{[1]}, \dots, v_{[n]}$ of $(\mathbb{C}^2)_{\lambda}^{\otimes n}$, where $v_{[a]} = v_2^{\otimes(a-1)} \otimes v_1 \otimes v_2^{\otimes(n-a)}$. For every $\mathbf{z} \in \mathbb{C}^n$, the map

$$(6.73) \quad \text{Stab}_{\lambda, \mathbf{z}} : (\mathbb{C}^N)_{\lambda}^{\otimes n} \rightarrow H_T^*(\mathcal{F}_{\lambda}; \mathbb{C})_{\mathbf{z}}, \quad v_{[a]} \mapsto \text{Stab}_{[a], \mathbf{z}}, \quad a = 1, \dots, n,$$

gives an isomorphism of $(\mathbb{C}^N)_{\lambda}^{\otimes n}$ and $H_T^*(\mathbb{C}\mathbb{P}^{n-1}; \mathbb{C})_{\mathbf{z}}$.

Set $D_1 = \gamma$, $D_2 = z_1 + \dots + z_n - \gamma$. The operators of quantum multiplication $[D_1]_{\mathbf{z}} *_{\mathbf{p}, \mathbf{z}}$, $[D_2]_{\mathbf{z}} *_{\mathbf{p}, \mathbf{z}}$ act on $H_T^*(\mathbb{C}\mathbb{P}^{n-1}; \mathbb{C})_{\mathbf{z}}$ as follows

$$(6.74) \quad \begin{aligned} [D_1]_{\mathbf{z}} *_{\mathbf{p}, \mathbf{z}} \text{Stab}_{[1], \mathbf{z}} &= z_1 \text{Stab}_{[1], \mathbf{z}} + \frac{p_2}{p_1} \text{Stab}_{[n], \mathbf{z}}, \\ [D_1]_{\mathbf{z}} *_{\mathbf{p}, \mathbf{z}} \text{Stab}_{[a], \mathbf{z}} &= z_a \text{Stab}_{[a], \mathbf{z}} + \text{Stab}_{[a-1], \mathbf{z}}, \quad a = 2, \dots, n, \\ [D_2]_{\mathbf{z}} *_{\mathbf{p}, \mathbf{z}} \text{Stab}_{[b], \mathbf{z}} &= \left(-[D_1]_{\mathbf{z}} *_{\mathbf{p}, \mathbf{z}} + \sum_{c=1}^n z_c \right) \text{Stab}_{[b], \mathbf{z}}, \quad b = 1, \dots, n, \end{aligned}$$

see for instance [TV7]. As Corollary 6.11 states, the map $\text{Stab}_{\lambda, z}$ intertwines the dynamical operators $X_1^\circ(\mathbf{z}; \mathbf{p})$, $X_2^\circ(\mathbf{z}; \mathbf{p})$, see (5.53), acting on $(\mathbb{C}^N)_\lambda^{\otimes n}$ and the operators of quantum multiplication, $[D_1]_{\mathbf{z}} *_{\mathbf{p}, z}$, $[D_2]_{\mathbf{z}} *_{\mathbf{p}, z}$,

$$(6.75) \quad \text{Stab}_{\lambda, z} \circ X_i^\circ(\mathbf{z}; \mathbf{p}) = [D_i]_{\mathbf{z}} *_{\mathbf{p}, z} \circ \text{Stab}_{\lambda, z}, \quad i = 1, 2.$$

Consider the trivial vector bundle $H_\lambda \rightarrow \mathbb{C}^n \times \mathbb{C}^2$ with fiber over a point $(\mathbf{z}^0, \mathbf{p}^0)$ given by $H_T^*(\mathbb{C}\mathbb{P}^{n-1}; \mathbb{C})_{\mathbf{z}^0}$. The equivariant quantum differential equations for sections of H_λ is the system of differential equations

$$(6.76) \quad \kappa p_i \frac{\partial f}{\partial p_i} = [D_i]_{\mathbf{z}} *_{\mathbf{p}, z} f, \quad i = 1, 2,$$

where κ is the parameter of the equations. The isomorphisms $\text{Stab}_{\lambda, z}$ identify equations (6.76) with the limiting dynamical differential equation (5.54). Furthermore, the isomorphisms $\text{Stab}_{\lambda, z}$ and the limiting qKZ equations (5.52) define the qKZ difference equations in cohomology

$$(6.77) \quad f(z_1, \dots, z_a + \kappa, \dots, z_n; \mathbf{p}; \kappa) = K_a^H(\mathbf{z}; \mathbf{p}; \kappa) f(\mathbf{z}; \mathbf{p}; \kappa), \quad a = 1, \dots, n,$$

where for each $a = 1, \dots, n$, and fixed \mathbf{z}, \mathbf{p} , the operator $K_a^H(\mathbf{z}; \mathbf{p}; \kappa)$ is a map of fibers,

$$K_a^H(\mathbf{z}; \mathbf{p}; \kappa) : H_T^*(\mathcal{F}_\lambda; \mathbb{C})_{\mathbf{z}} \rightarrow H_T^*(\mathcal{F}_\lambda; \mathbb{C})_{(z_1, \dots, z_a + \kappa, \dots, z_n)},$$

$$K_a^H(\mathbf{z}; \mathbf{p}; \kappa) = \text{Stab}_{\lambda, (z_1, \dots, z_a + \kappa, \dots, z_n)} \circ K_a^\circ(\mathbf{z}; \mathbf{p}; \kappa) \circ (\text{Stab}_{\lambda, \mathbf{z}})^{-1},$$

and the operator $K_a^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ is given by (5.51). By Theorem 6.13, quantum differential equations (6.76) and qKZ difference equations (6.77) is a compatible system of differential and difference equations for sections of H_λ .

Denote $\dot{\gamma} = \dot{\gamma}_{1,1}$. Identify $\mathbb{C}[\dot{\gamma}^{\pm 1}]$ with the subspace of $\mathbb{C}[\dot{\Gamma}^{\pm 1}]^{S_\lambda}$ consisting of Laurent polynomials not depending on $\dot{\gamma}_{2,1}, \dots, \dot{\gamma}_{2,n-1}$, and $\mathbb{C}[\dot{\Gamma}^{\pm 1}]^{S_\lambda}$ with the subspace of $\mathbb{C}[\dot{\Gamma}^{\pm 1}]^{S_\lambda} \otimes \mathbb{C}[\dot{z}^{\pm 1}]$ consisting of polynomials not depending on \dot{z} . Then the equivariant K -theory algebra $K_T(\mathbb{C}\mathbb{P}^{n-1}; \mathbb{C})$, see (6.26), can be presented as in (5.63),

$$(6.78) \quad K_T(\mathbb{C}\mathbb{P}^{n-1}; \mathbb{C}) = \mathbb{C}[\dot{\gamma}^{\pm 1}, \dot{z}^{\pm 1}] \left/ \left\langle \prod_{a=1}^n (\dot{\gamma} - \dot{z}_a) = 0 \right\rangle \right.$$

Recall that for a Laurent polynomial $P(\dot{\gamma}; \dot{z})$, we denote

$$\dot{P}(\gamma; \mathbf{z}; \kappa) = P(e^{2\pi\sqrt{-1}\gamma/\kappa}; e^{2\pi\sqrt{-1}z_1/\kappa}, \dots, e^{2\pi\sqrt{-1}z_n/\kappa}).$$

In Section 5.7, we described solutions $\Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ of equations (5.54), (5.52) labeled by Laurent polynomials $P(\dot{\gamma}; \dot{z})$, see (5.61). Denote by $\mathcal{S}_{H_\lambda}^K$ the space spanned over \mathbb{C} by the solutions $\text{Stab}_\lambda \Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ of equations (5.54), (5.52). Each element of $\mathcal{S}_{H_\lambda}^K$ is a section of H_λ holomorphic in p_1, p_2 provided branches of $\log p_1$ and $\log p_2$ are fixed, and entire in \mathbf{z} , see Proposition 5.14. The space $\mathcal{S}_{H_\lambda}^K$ is a $\mathbb{C}[\dot{z}^{\pm 1}]$ -module, with a Laurent polynomial $P(\dot{z})$ acting as multiplication by $\dot{P}(\mathbf{z}; \kappa)$.

Since the function $\Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ depends only on the class of P in $K_T(\mathcal{F}_\lambda; \mathbb{C})$, then so does the section $\text{Stab}_\lambda \Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ of H_λ . Thus the map

$$(6.79) \quad \mu_{H_\lambda}^K : K_T(\mathcal{F}_\lambda; \mathbb{C}) \rightarrow \mathcal{S}_{H_\lambda}^K, \quad [P] \mapsto \text{Stab}_\lambda \Psi_P^\circ,$$

is well-defined, cf. (5.29). By Corollary 5.16, the map $\mu_{H_\lambda}^K$ is an isomorphism of $\mathbb{C}[\mathbf{z}^{\pm 1}]$ -modules.

There is an integral formula for the function $\Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa)$, see (5.62). Taking into account formula (6.71), we get an integral presentation for the section $\text{Stab}_\lambda \Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa)$,

$$(6.80) \quad \text{Stab}_\lambda \Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa) = \frac{1}{2\pi\sqrt{-1}\kappa} \int_C \dot{P}(t; \mathbf{z}; \kappa) \Phi_\lambda^\circ(t; \mathbf{z}; \mathbf{p}; \kappa) [\bar{W}_\lambda(t; \gamma; \mathbf{z})]_z dt,$$

where

$$(6.81) \quad \Phi_\lambda^\circ(t; \mathbf{z}; \mathbf{p}; \kappa) = (p_2/\kappa)^{\sum_{a=1}^n z_a/\kappa} (\kappa^n p_1/p_2)^{t/\kappa} \prod_{a=1}^n \Gamma((t - z_a)/\kappa)$$

is the master function, see (5.58), the polynomial $\bar{W}_\lambda(t; \gamma; \mathbf{z})$ is given by (6.70), a contour C encircles the poles of the product $\prod_{a=1}^n \Gamma((t - z_a)/\kappa)$ counterclockwise. For instance, C can be the parabola

$$C = \{ \kappa(A - s^2 + s\sqrt{-1}) \mid s \in \mathbb{R} \},$$

where A is a sufficiently large positive real number.

In Section 6.5, we introduced the Levelt fundamental solution $\text{Stab}_\lambda \widehat{\Psi}^\circ$ of quantum differential equations. For the example of equations (6.76), $\text{Stab}_\lambda \widehat{\Psi}^\circ$ is the section of the vector bundle $EH_\lambda \rightarrow \mathbb{C}^n \times \mathbb{C}^2$ with fiber over a point $(\mathbf{z}^0, \mathbf{p}^0)$ given by $\text{End}(H_T^*(\mathbb{CP}^{n-1}; \mathbb{C})_{\mathbf{z}^0})$.

By (6.29), the values of $\text{Stab}_\lambda \widehat{\Psi}^\circ$ have the form

$$(6.82) \quad \text{Stab}_\lambda \widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa) = (\text{Stab}_\lambda \Psi^\bullet(\mathbf{z}; \mathbf{p}; \kappa)) (p_1/p_2)^{[\gamma]_z/\kappa} p_2^{\sum_{c=1}^n z_c/\kappa},$$

where $(p_1/p_2)^{[\gamma]_z/\kappa} p_2^{\sum_{c=1}^n z_c/\kappa}$ acts on the fiber $H_T^*(\mathbb{CP}^{n-1}; \mathbb{C})_z$ as multiplication by itself and the section $\text{Stab}_\lambda \Psi^\bullet(\mathbf{z}; \mathbf{p}; \kappa)$ of EH_λ is an entire function of p_2/p_1 equal to the identity map at $p_2 = 0$. Formulae (6.30)–(6.33), (6.67), (6.71) yield an integral expression for $\text{Stab}_\lambda \Psi^\bullet(\mathbf{z}; \mathbf{p}; \kappa)$, see formulae (6.83), (6.84) below.

Recall the complement $L^\circ \subset \mathbb{C}^n$ of the union of hyperplanes (6.36),

$$z_a - z_b \in \kappa \mathbb{Z}_{\neq 0}, \quad a, b = 1, \dots, n, \quad a \neq b.$$

For $l \in \mathbb{Z}_{\geq 0}$, let $\tilde{\mathcal{J}}_l^H$ be the section of the bundle EH_λ with values

$$(6.83) \quad \tilde{\mathcal{J}}_l^H(\mathbf{z}; \kappa) : H_T^*(\mathbb{CP}^{n-1}; \mathbb{C})_z \rightarrow H_T^*(\mathbb{CP}^{n-1}; \mathbb{C})_z,$$

$$\tilde{\mathcal{J}}_l^H(\mathbf{z}; \kappa) : [f]_z \mapsto \frac{1}{2\pi\sqrt{-1}} \int_{C_z} f(t) [\bar{W}_\lambda(t - l\kappa; \gamma; \mathbf{z})]_z \prod_{a=1}^n \prod_{m=0}^l \frac{1}{t - z_a - m\kappa} dt,$$

where the integral is over a contour $C_{\mathbf{z}}$ encircling the points z_1, \dots, z_n counterclockwise and separating them from the sets $z_a + \kappa \mathbb{Z}_{>0}$, $a = 1, \dots, n$. It is assumed in (6.83) that $\mathbf{z} \in L^\circ$. By Proposition 6.17,

$$(6.84) \quad \text{Stab}_\lambda \Psi^\bullet(\mathbf{z}; \mathbf{p}; \kappa) = \sum_{l=0}^{\infty} \tilde{\mathcal{J}}_l^H(\mathbf{z}; \kappa) (p_2/p_1)^l.$$

Notice that $\tilde{\mathcal{J}}_0^H(\mathbf{z}; \kappa)$ is the identity map because the classes in $H_T^*(\mathbb{C}\mathbb{P}^{n-1}; \mathbb{C})_{\mathbf{z}}$ of a polynomial f and the polynomial

$$(6.85) \quad f_*(\gamma) = \frac{1}{2\pi\sqrt{-1}} \int_{C_{\mathbf{z}}} \frac{f(t) \bar{W}_\lambda(t; \gamma; \mathbf{z})}{(t - z_1) \dots (t - z_n)} dt$$

coincide, $[f_*]_{\mathbf{z}} = [f]_{\mathbf{z}} \in H_T^*(\mathbb{C}\mathbb{P}^{n-1}; \mathbb{C})_{\mathbf{z}}$.

Formulae (6.83), (6.84) are counterparts of formulae (5.64) for dynamical differential equations (5.54).

Recall the space \mathcal{O}_{H_λ} of sections of H_λ holomorphic in \mathbf{z} for $\mathbf{z} \in L^\circ$ and not depending on \mathbf{p} , and the space $\mathcal{S}_{H_\lambda}^\circ$ of sections of H_λ that are solutions of quantum differential equations (6.76) holomorphic in \mathbf{z} for $\mathbf{z} \in L^\circ$. By Proposition 6.19, the Levelt solution $\text{Stab}_\lambda \widehat{\Psi}^\circ$ defines an isomorphism

$$(6.86) \quad \mu_{H_\lambda}^\circ : \mathcal{O}_{H_\lambda} \rightarrow \mathcal{S}_{H_\lambda}^\circ, \quad f \mapsto \text{Stab}_\lambda \widehat{\Psi}^\circ f.$$

of \mathcal{O}_{H_λ} and $\mathcal{S}_{H_\lambda}^\circ$. The section f is called the principal term of $\text{Stab}_\lambda \widehat{\Psi}^\circ f$. The principal term of the solution $\text{Stab}_\lambda \Psi_P^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ corresponding to a Laurent polynomial $P(\gamma; \mathbf{z})$ is described below, see (6.89).

For a function $f(t)$ holomorphic in a neighbourhood of the points z_1, \dots, z_n , define the polynomial $f_*(\gamma)$ by the rule

$$(6.87) \quad f_*(\gamma) = \frac{1}{2\pi\sqrt{-1}} \int_C \frac{f(t) \bar{W}_\lambda(t; \gamma; \mathbf{z})}{(t - z_1) \dots (t - z_n)} dt,$$

where a contour C encircles the points z_1, \dots, z_n counterclockwise and $f(t)$ is holomorphic inside C , cf. (6.85). Define the class $[f]_{\mathbf{z}} \in H_T^*(\mathbb{C}\mathbb{P}^{n-1}; \mathbb{C})_{\mathbf{z}}$ by the rule $[f]_{\mathbf{z}} = [f_*]_{\mathbf{z}}$.

Set

$$(6.88) \quad \tilde{C}_\lambda^\circ(\gamma; \mathbf{z}; \kappa) = \kappa^{(n\gamma - \sum_{c=1}^n z_c)/\kappa}, \quad \tilde{G}_\lambda^+(\gamma; \mathbf{z}; \kappa) = \kappa^{n-1} \prod_{a=1}^n \Gamma(1 + (\gamma - z_a)/\kappa),$$

cf. (6.41). By Proposition 6.20, the principal term Ψ_P° of the solution $\text{Stab}_\lambda \Psi_P^\circ$ corresponding to a Laurent polynomial $P(\gamma; \mathbf{z})$ is

$$(6.89) \quad \Psi_P^\circ(\mathbf{z}; \kappa) = [\dot{P}(\gamma; \mathbf{z}; \kappa) \tilde{C}_\lambda^\circ(\gamma; \mathbf{z}; \kappa) \tilde{G}_\lambda^+(\gamma; \mathbf{z}; \kappa)]_{\mathbf{z}}.$$

The right-hand side of the last formula equals the product of the equivariant Chern character $[\dot{P}(\gamma; \mathbf{z}; \kappa)]_{\mathbf{z}}$ of the class of the Laurent polynomial $P(\gamma; \mathbf{z})$ in $K_T(\mathbb{C}\mathbb{P}^{n-1}; \mathbb{C})$, the exponential $[\tilde{C}_\lambda^\circ(\gamma; \mathbf{z}; \kappa)]_{\mathbf{z}}$ of the equivariant first Chern class $[\gamma]_{\mathbf{z}}$ of the tangent bundle of $\mathbb{C}\mathbb{P}^{n-1}$, and is the equivariant Gamma-class $[\tilde{G}_\lambda^+(\gamma; \mathbf{z}; \kappa)]_{\mathbf{z}}$ of $\mathbb{C}\mathbb{P}^{n-1}$.

Recall the map (6.42),

$$\mathbb{B}_\lambda^H : K_T(\mathbb{C}\mathbb{P}^{n-1}; \mathbb{C}) \rightarrow \mathcal{O}_{H_\lambda},$$

that sends the class $[P]$ of the Laurent polynomial $P(\gamma; \mathbf{z})$ in $K_T(\mathbb{C}\mathbb{P}^{n-1}; \mathbb{C})$ to the section of H_λ with values $[\dot{P}(\gamma; \mathbf{z}; \kappa) \tilde{C}_\lambda^\circ(\gamma; \mathbf{z}; \kappa) \tilde{G}_\lambda^+(\gamma; \mathbf{z}; \kappa)]_{\mathbf{z}}$. By (6.89), the map \mathbb{B}_λ^H sends the class $[P]$ to the principal term of the solution $\text{Stab}_\lambda \Psi_P^\circ$ of the joint system of differential equations (6.76) and difference equations (6.77). Furthermore, by Theorem 6.21, the following diagram is commutative,

$$(6.90) \quad \begin{array}{ccc} K_T(\mathbb{C}\mathbb{P}^{n-1}; \mathbb{C}) & \xrightarrow{\mathbb{B}_\lambda^H} & \mathcal{O}_{H_\lambda} \\ & \searrow \mu_{H_\lambda}^K & \swarrow \mu_{H_\lambda}^\circ \\ & \mathcal{S}_{H_\lambda}^\circ & \end{array}$$

where $\mu_{H_\lambda}^K$ and $\mu_{H_\lambda}^\circ$ are the maps (6.79) and (6.86), respectively.

The topological-enumerative morphism for $\mathbb{C}\mathbb{P}^{n-1}$ was studied in [CV]. To refer to formulae and statements in [CV], we will use the superscript ^{CV}. To compare formulae in this paper with their counterparts in [CV], one should make the following substitution,

$$[\gamma] = x, \quad p_1 = q^{-1}, \quad p_2 = 1, \quad \kappa = -1.$$

Then the classes of polynomials $\text{Stab}_{[1]}(\gamma; \mathbf{z}), \dots, \text{Stab}_{[n]}(\gamma; \mathbf{z})$ in $H_T^*(\mathbb{C}\mathbb{P}^{n-1}; \mathbb{C})$ coincide with the classes g_1, \dots, g_n given by (4.4)^{CV}, formulae (6.75) for quantum multiplication agree with formulae (5.13)^{CV}, (5.14)^{CV}, and quantum differential equations (6.76) and (5.19)^{CV} match.

By the results of [CV], Conjecture 6.29 holds true for $\mathbb{C}\mathbb{P}^{n-1}$.

Theorem 6.30. *The Levelt fundamental solution $\text{Stab}_\lambda \widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa)$ of equivariant quantum differential equations (6.76) is the equivariant topological-enumerative morphism of $\mathbb{C}\mathbb{P}^{n-1}$ restricted to the small equivariant quantum locus.*

Proof. The statement follows from formulae (6.82)–(6.84) and Theorem 6.4^{CV}. \square

According to Definition 5.3^{CV}, the equivariant J -function for $\mathbb{C}\mathbb{P}^{n-1}$ is a unique section $J(\mathbf{z}; \mathbf{p}; \kappa)$ of H_λ such that for any $f \in H^*(\mathbb{C}\mathbb{P}^{n-1}; \mathbb{C})$ considered as a section of H_λ not depending on \mathbf{p} ,

$$(6.91) \quad \mathcal{E}_z \langle (\text{Stab}_\lambda \widehat{\Psi}^\circ(\mathbf{z}; \mathbf{p}; \kappa)) [f]_{\mathbf{z}} \rangle = \mathcal{E}_z \langle J(\mathbf{z}; \mathbf{p}; \kappa) [f]_{\mathbf{z}} \rangle.$$

Formulae (6.67), (6.82)–(6.84), and

$$\frac{1}{2\pi\sqrt{-1}} \int_C \frac{\bar{W}_\lambda(t; \gamma; \mathbf{z})}{(\gamma - z_1) \dots (\gamma - z_n)} d\gamma = 1$$

for a contour C encircling z_1, \dots, z_n counterclockwise, yield the following expression for the

equivariant J -function for $\mathbb{C}\mathbb{P}^{n-1}$,

$$(6.92) \quad J(\mathbf{z}; \mathbf{p}; \kappa) = p_2^{\sum_{c=1}^n z_c/\kappa} (p_1/p_2)^{[\gamma]_{\mathbf{z}/\kappa}} \left[\sum_{l=0}^{\infty} (p_2/p_1)^l \prod_{a=1}^n \prod_{m=1}^l \frac{1}{\gamma - z_a - m\kappa} \right]_{\mathbf{z}}.$$

This expression matches the formula for the equivariant J -function for $\mathbb{C}\mathbb{P}^{n-1}$ obtained in [Gi1], [LLY].

All the results of this section admit straightforward specialization to the nonequivariant case by setting $\mathbf{z} = 0$.

APPENDIX A. POLYNOMIAL IDENTITIES

Recall $\lambda = (\lambda_1, \dots, \lambda_N)$, $|\lambda| = n$, the set \mathcal{I}_λ , $I_\lambda^{\min} \in \mathcal{I}_\lambda$, and the permutation $\sigma_I \in S_n$, $I \in \mathcal{I}_\lambda$, such that $\sigma_I(I_\lambda^{\min}) = I$. Let $S_\lambda = S_{\lambda_1} \times \dots \times S_{\lambda_N} \subset S_n$ be the isotropy subgroup of I_λ^{\min} . Each coset in S_n/S_λ contains exactly one permutation of the form σ_I , $I \in \mathcal{I}_\lambda$.

Consider the variables $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n)$. Define

$$(A.1) \quad V_\lambda(\mathbf{x}; \mathbf{y}) = \prod_{a=1}^{\lambda^{(i)}} \prod_{b=\lambda^{(i)}+1}^n (x_a - y_b).$$

Clearly, $V_\lambda(\mathbf{x}; \mathbf{y}) \in \mathbb{C}[\mathbf{x}]^{S_\lambda} \otimes \mathbb{C}[\mathbf{y}]^{S_\lambda}$. For $\sigma \in S_n$, denote $\mathbf{x}_\sigma = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$.

Lemma A.1. *We have $V_\lambda(\mathbf{x}; \mathbf{x}_\sigma) = 0$ unless $\sigma \in S_\lambda$, and $V_\lambda(\mathbf{x}_{\sigma_I}; \mathbf{x}_{\sigma_I}) = (-1)^{|\sigma_I|} V_\lambda(\mathbf{x}; \mathbf{x})$.*

Proof. Straightforward. \square

For any function $f(\mathbf{x})$, define

$$\text{Sym}_{\mathbf{x}}^\lambda f(\mathbf{x}) = \sum_{I \in \mathcal{I}_\lambda} f(\mathbf{x}_{\sigma_I}).$$

Proposition A.2. *We have*

$$(A.2) \quad V_\lambda(\mathbf{x}; \mathbf{z}) = \text{Sym}_{\mathbf{y}}^\lambda \left(\frac{V_\lambda(\mathbf{x}; \mathbf{y}) V_\lambda(\mathbf{y}; \mathbf{z})}{V_\lambda(\mathbf{y}; \mathbf{y})} \right).$$

Proof. By formula (A.1), the expression $V_\lambda(\mathbf{x}; \mathbf{y}) V_\lambda(\mathbf{y}; \mathbf{z}) / V_\lambda(\mathbf{y}; \mathbf{y})$ is invariant under permutations of y_1, \dots, y_n by elements of the subgroup S_λ . Thus the right-hand side of formula (A.2) is a symmetric polynomial in y_1, \dots, y_n of degree zero in each of the variables y_1, \dots, y_n , hence a constant. The constant can be found by evaluating this polynomial at $\mathbf{y} = \mathbf{x}$ or $\mathbf{y} = \mathbf{z}$. \square

Proposition A.3. *For any polynomial $f \in \mathbb{C}[\mathbf{x}]^{S_\lambda}$, we have*

$$(A.3) \quad f(\mathbf{x}) = \text{Sym}_{\mathbf{y}}^\lambda \left(\frac{V_\lambda(\mathbf{x}; \mathbf{y})}{V_\lambda(\mathbf{y}; \mathbf{y})} \text{Sym}_{\mathbf{x}}^\lambda \left(\frac{f(\mathbf{x}) V_\lambda(\mathbf{y}; \mathbf{x})}{V_\lambda(\mathbf{x}; \mathbf{x})} \right) \right).$$

Proof. Changing the order of symmetrizations in the right-hand side of formula (A.3) and applying formula (A.2) yields

$$\begin{aligned} & \text{Sym}_{\mathbf{y}}^{\lambda} \left(\frac{V_{\lambda}(\mathbf{x}; \mathbf{y})}{V_{\lambda}(\mathbf{y}; \mathbf{y})} \left(\text{Sym}_{\mathbf{z}}^{\lambda} \left(\frac{f(\mathbf{z}) V_{\lambda}(\mathbf{y}; \mathbf{z})}{V_{\lambda}(\mathbf{z}; \mathbf{z})} \right) \right) \Big|_{\mathbf{z}=\mathbf{x}} \right) = \\ & = \left(\text{Sym}_{\mathbf{z}}^{\lambda} \left(\frac{f(\mathbf{z})}{V_{\lambda}(\mathbf{z}; \mathbf{z})} \text{Sym}_{\mathbf{y}}^{\lambda} \left(\frac{V_{\lambda}(\mathbf{x}; \mathbf{y}) V_{\lambda}(\mathbf{y}; \mathbf{z})}{V_{\lambda}(\mathbf{y}; \mathbf{y})} \right) \right) \right) \Big|_{\mathbf{z}=\mathbf{x}} = \\ & = \left(\text{Sym}_{\mathbf{z}}^{\lambda} \left(\frac{f(\mathbf{z}) V_{\lambda}(\mathbf{x}; \mathbf{z})}{V_{\lambda}(\mathbf{z}; \mathbf{z})} \right) \right) \Big|_{\mathbf{z}=\mathbf{x}} = f(\mathbf{x}). \end{aligned}$$

□

The expression $\text{Sym}_{\mathbf{x}}^{\lambda}(f(\mathbf{x}) V_{\lambda}(\mathbf{y}; \mathbf{x})/V_{\lambda}(\mathbf{x}; \mathbf{x}))$ is a symmetric polynomial in x_1, \dots, x_n . Thus Lemma A.1 and Proposition A.3 show that the polynomials $V_{\lambda}(\mathbf{x}; \mathbf{y}_{\sigma_I})$, $I \in \mathcal{I}_{\lambda}$, give a basis of the space $\mathbb{C}[\mathbf{x}]^{S_{\lambda}}$ as a free module over the ring of symmetric polynomials $\mathbb{C}[\mathbf{x}]^{S_n}$.

Proposition A.4. *We have*

$$(A.4) \quad \det(V_{\lambda}(\mathbf{x}_{\sigma_I}; \mathbf{y}_{\sigma_J}))_{I, J \in \mathcal{I}_{\lambda}} = \prod_{a=1}^{n-1} \prod_{b=a+1}^n ((x_a - x_b)(y_b - y_a))^{d_{\lambda}^{(2)}},$$

where

$$d_{\lambda}^{(2)} = \frac{(n-2)!}{\lambda_1! \dots \lambda_N!} \sum_{i=1}^{N-1} \sum_{j=i+1}^N \lambda_i \lambda_j.$$

Proof. Consider the matrix $M(\mathbf{x}; \mathbf{y}) = (V_{\lambda}(\mathbf{x}_{\sigma_I}; \mathbf{y}_{\sigma_J}))_{I, J \in \mathcal{I}_{\lambda}}$. Let $F(\mathbf{x}; \mathbf{y}) = \det M(\mathbf{x}; \mathbf{y})$. Formula (A.2) reads $M(\mathbf{x}; \mathbf{z}) = M(\mathbf{x}; \mathbf{y}) (M(\mathbf{y}; \mathbf{y})^{-1} M(\mathbf{y}; \mathbf{z}))$, thus

$$(A.5) \quad F(\mathbf{x}; \mathbf{z}) = \frac{F(\mathbf{x}; \mathbf{y}) F(\mathbf{y}; \mathbf{z})}{F(\mathbf{y}; \mathbf{y})},$$

so that $F(\mathbf{x}; \mathbf{y}) = G(\mathbf{x}) \tilde{G}(\mathbf{y})$ for some functions G, \tilde{G} . Also, $F(\mathbf{x}; \mathbf{y}) = F(\mathbf{y}; \mathbf{x})$, since the matrix $M(\mathbf{y}; \mathbf{x})$ is the transpose of $M(\mathbf{x}; \mathbf{y})$. Hence the functions G and \tilde{G} are proportional and one can take $\tilde{G} = G$. Finally, the matrix $M(\mathbf{x}; \mathbf{x})$ is diagonal and

$$(G(\mathbf{x}))^2 = F(\mathbf{x}; \mathbf{x}) = \prod_{I \in \mathcal{I}_{\lambda}} V_{\lambda}(\mathbf{x}_{\sigma_I}; \mathbf{x}_{\sigma_I}) = \prod_{a=1}^n \prod_{\substack{b=1 \\ b \neq a}}^n (x_a - x_b)^{d_{\lambda}^{(2)}},$$

which implies formula (A.4). □

For $I \in \mathcal{I}_{\lambda}$, set

$$(A.6) \quad V_I(\mathbf{x}) = V_{\lambda}(x_1, \dots, x_n; \sigma_I(n) - 1, \dots, \sigma_I(1) - 1).$$

We use the polynomials $V_I(\mathbf{x})$ in Section 4.6 to give a basis of the algebra \mathcal{K}_λ . By formula (A.4),

$$(A.7) \quad \det(V_I(\mathbf{x}_{\sigma_J}))_{I, J \in \mathcal{I}_\lambda} = \prod_{k=2}^{n-1} j^{(n-j)d_\lambda^{(2)}} \prod_{a=1}^{n-1} \prod_{b=a+1}^n (x_b - x_a)^{d_\lambda^{(2)}}.$$

APPENDIX B. SCHUBERT POLYNOMIALS

Consider the operators $\Delta_1^{\mathbf{x}}, \dots, \Delta_{n-1}^{\mathbf{x}}$ acting on functions of x_1, \dots, x_n :

$$\Delta_a^{\mathbf{x}} f(\mathbf{x}) = \frac{f(\mathbf{x}) - f(x_1, \dots, x_{a+1}, x_a, \dots, x_n)}{x_a - x_{a+1}}.$$

They satisfy the nil-Coxeter relations,

$$(B.1) \quad (\Delta_a^{\mathbf{x}})^2 = 0, \quad \Delta_a^{\mathbf{x}} \Delta_b^{\mathbf{x}} = \Delta_b^{\mathbf{x}} \Delta_a^{\mathbf{x}}, \quad |a - b| > 1, \quad \Delta_a^{\mathbf{x}} \Delta_{a+1}^{\mathbf{x}} \Delta_a^{\mathbf{x}} = \Delta_{a+1}^{\mathbf{x}} \Delta_a^{\mathbf{x}} \Delta_{a+1}^{\mathbf{x}},$$

for any a, b .

Given a permutation $\sigma \in S_n$, define the operator $\Delta_\sigma^{\mathbf{x}}$ as follows. For $a = 1, \dots, n-1$, let s_a be the transposition of a and $a+1$. If $\sigma = s_{a_1} \dots s_{a_k}$ is the reduced presentation, then $\Delta_\sigma^{\mathbf{x}} = \Delta_{a_1}^{\mathbf{x}} \dots \Delta_{a_k}^{\mathbf{x}}$. In particular, $\Delta_{s_a}^{\mathbf{x}} = \Delta_a^{\mathbf{x}}$. The operators $\Delta_\sigma^{\mathbf{x}}$ are well-defined due to relations (B.1).

Lemma B.1. *For $\sigma, \tau \in S_n$, we have $\Delta_\sigma^{\mathbf{x}} \Delta_\tau^{\mathbf{x}} = \Delta_{\sigma\tau}^{\mathbf{x}}$ if $|\sigma\tau| = |\sigma| + |\tau|$, and $\Delta_\sigma^{\mathbf{x}} \Delta_\tau^{\mathbf{x}} = 0$, otherwise.*

Proof. The statement follows from relations (B.1). \square

Define the operators $\Delta_\sigma^{\mathbf{y}}$, $\sigma \in S_n$, acting on functions of y_1, \dots, y_n similarly.

The A -type double Schubert polynomials \mathfrak{S}_σ , $\sigma \in S_n$, see [L, Chapter 2], are defined as follows. For the longest permutation σ_0 , $\sigma_0(i) = n+1-i$, $i = 1, \dots, n$, set

$$\mathfrak{S}_{\sigma_0}(\mathbf{x}; \mathbf{y}) = \prod_{i=1}^{n-1} \prod_{j=1}^{n-i} (x_i - y_j),$$

For any $\sigma \in S_n$, set $\mathfrak{S}_\sigma(\mathbf{x}; \mathbf{y}) = \Delta_{\sigma^{-1}\sigma_0}^{\mathbf{x}} \mathfrak{S}_{\sigma_0}(\mathbf{x}; \mathbf{y})$.

Lemma B.2. *For any $\sigma \in S_n$, we have $\Delta_\sigma^{\mathbf{x}} \mathfrak{S}_{\sigma_0}(\mathbf{x}; \mathbf{y}) = (-1)^{|\sigma|} \Delta_{\sigma_0\sigma^{-1}\sigma_0}^{\mathbf{y}} \mathfrak{S}_{\sigma_0}(\mathbf{x}; \mathbf{y})$.*

Proof. The statement follow by induction on the length of σ from the equalities

$$\Delta_{s_a}^{\mathbf{x}} \mathfrak{S}_{\sigma_0}(\mathbf{x}; \mathbf{y}) = \frac{\mathfrak{S}_{\sigma_0}(\mathbf{x}; \mathbf{y})}{x_a - y_{n-a}} = -\Delta_{s_{n-a-1}}^{\mathbf{y}} \mathfrak{S}_{\sigma_0}(\mathbf{x}; \mathbf{y})$$

and $s_{n-a-1} = \sigma_0 s_a \sigma_0$ for every $a = 1, \dots, n-1$. \square

Proposition B.3. *For any $\sigma \in S_n$, we have*

$$\mathfrak{S}_\sigma(\mathbf{x}; \mathbf{y}) = (-1)^{|\sigma_0| - |\sigma|} \Delta_{\sigma\sigma_0}^{\mathbf{y}} \mathfrak{S}_{\sigma_0}(\mathbf{x}; \mathbf{y}) = (-1)^{|\sigma|} \mathfrak{S}_{\sigma^{-1}}(\mathbf{y}; \mathbf{x}).$$

Proof. The statement follows from the equality $\mathfrak{S}_{\sigma_0}(\mathbf{y}; \mathbf{x}) = (-1)^{|\sigma_0|} \mathfrak{S}_{\sigma_0}(\mathbf{x}; \mathbf{y})$ and Lemma B.2. \square

Denote by σ_λ the longest element of the subgroup $S_\lambda = S_{\lambda_1} \times \dots \times S_{\lambda_N} \subset S_n$,

$$\sigma_\lambda(a) = \lambda^{(i-1)} + \lambda^{(i)} + 1 - a, \quad \lambda^{(i-1)} + 1 < a \leq \lambda^{(i)}, \quad i = 1, \dots, N.$$

Lemma B.4. *We have*

$$\Delta_{\sigma_\lambda}^{\mathbf{x}} \mathfrak{S}_{\sigma_0}(\mathbf{x}; \mathbf{y}) = \prod_{a=1}^{\lambda^{(i)}} \prod_{b=\lambda^{(i)}+1}^n (x_a - y_{n-b+1}).$$

Proof. Straightforward. \square

Recall the polynomials $\text{Stab}_I(\Gamma; \mathbf{z})$, $\text{Stab}_I^{\text{op}}(\Gamma; \mathbf{z})$, see Section 6.2.

Proposition B.5. *For any $I \in \mathcal{I}_\lambda$,*

$$(B.2) \quad \text{Stab}_I(\Gamma; \mathbf{z}) = \mathfrak{S}_{\sigma_{\sigma_0(I)}}(\Gamma; \mathbf{z}_{\sigma_0}), \quad \text{Stab}_I^{\text{op}}(\Gamma; \mathbf{z}) = \mathfrak{S}_{\sigma_I}(\Gamma; \mathbf{z}).$$

Proof. By formula (6.12) and Lemmas 5.4,

$$\text{Stab}_I^{\text{op}}(\Gamma; \mathbf{z}) = (-1)^{|\sigma_{\sigma_0(I)}| - |\sigma_I|} \Delta_{\sigma_I \sigma_{\sigma_0(I)}^{-1}}^{\mathbf{z}} V_\lambda(\Gamma; \mathbf{z}_{\sigma_0}),$$

where the operator $\Delta_\sigma^{\mathbf{z}}$ acts on functions of \mathbf{z} . By formula (6.4) and Lemmas B.4, B.2,

$$V_\lambda(\Gamma; \mathbf{z}_{\sigma_0}) = (-1)^{|\sigma_\lambda|} \Delta_{\sigma_0 \sigma_\lambda \sigma_0}^{\mathbf{z}} \mathfrak{S}_{\sigma_0}(\mathbf{x}; \mathbf{y})$$

because $\sigma_\lambda^{-1} = \sigma_\lambda$. Since $\sigma_{\sigma_0(I)} = \sigma_0 \sigma_\lambda$ and $|\sigma_I \sigma_{\sigma_0(I)}^{-1}| + |\sigma_{\sigma_0(I)} \sigma_0| = |\sigma_I \sigma_0|$,

$$\text{Stab}_I^{\text{op}}(\Gamma; \mathbf{z}) = (-1)^{|\sigma_0| - |\sigma_I|} \Delta_{\sigma_0 \sigma_0}^{\mathbf{z}} \mathfrak{S}_{\sigma_0}(\Gamma; \mathbf{z}) = \mathfrak{S}_{\sigma_I}(\Gamma; \mathbf{z}).$$

by Lemma B.1 and Proposition B.3, that proves the second equality in (B.2). The first equality in (B.2) follows from the relation $\text{Stab}_I(\Gamma; \mathbf{z}) = \text{Stab}_{\sigma_0(I)}^{\text{op}}(\Gamma; \mathbf{z}_{\sigma_0})$. \square

APPENDIX C. PROOFS OF LEMMAS 5.2 AND 5.3

Proof of Lemma 5.2. The proof is by induction on N . To show that $W_{I_\lambda^{\min}}^\circ(\mathbf{t}; \mathbf{z}) = 1$, set $f_1 = 1$, and for $N \geq 2$, write

$$f_N(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(N-1)}; \mathbf{z}) = W_{I_\lambda^{\min}}^\circ(\mathbf{t}; \mathbf{z}),$$

indicating the dependence on N explicitly. By formula (5.1), the function f_N is a polynomial in $t_1^{(N-1)}, \dots, t_{\lambda^{(N-1)}}^{(N-1)}$ of degree zero in each of these variables, hence a constant. Evaluating

$f_N(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(N-1)}; \mathbf{z})$ at $\mathbf{t}^{(N-1)} = (z_1, \dots, z_{\lambda(N-1)})$ yields

$$f_N(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(N-1)}; \mathbf{z}) = f_{N-1}(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(N-2)}; z_1, \dots, z_{\lambda(N-1)}).$$

Thus $f_N(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(N-1)}; \mathbf{z}) = 1$ by the induction assumption.

The proof of formula (5.2) is similar. Set $f_1(\mathbf{z}) = 1$, and for $N \geq 2$, write

$$f_N(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(N-1)}; \mathbf{z}) = W_{\sigma_0(I_{\lambda}^{\min})}^{\circ}(\mathbf{t}; \mathbf{z}),$$

Define polynomials g_1, \dots, g_{N-1} ,

$$g_i(\mathbf{t}^{(i)}; \mathbf{z}) = \prod_{a=1}^{\lambda^{(i)}} \prod_{b=\lambda^{(i)}+1}^{\lambda^{(i+1)}} (t_a^{(i)} - z_{n-b+1}).$$

By formula (5.1), the function f_N is a polynomial in $t_1^{(N-1)}, \dots, t_{\lambda(N-1)}^{(N-1)}$ of degree λ_N in each of these variables, and f_N is divisible by $g_{N-1}(\mathbf{t}^{(N-1)}; \mathbf{z})$. Hence,

$$f_N(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(N-1)}; \mathbf{z}) = r_N(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(N-2)}; \mathbf{z}) g_{N-1}(\mathbf{t}^{(N-1)}; \mathbf{z})$$

for some polynomial r_N . Evaluating both sides at $\mathbf{t}^{(N-1)} = (z_{\lambda_N+1}, \dots, z_n)$

$$r_N(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(N-2)}; \mathbf{z}) = f_{N-1}(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(N-2)}; z_{\lambda_N+1}, \dots, z_n).$$

Thus $f_N(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(N-1)}; \mathbf{z}) = g_1(\mathbf{t}^{(1)}; \mathbf{z}) \dots g_{N-1}(\mathbf{t}^{(N-1)}; \mathbf{z})$, which proves formula (5.2). \square

Proof of Lemma 5.3. The proof is by induction on N , similarly to that of Lemma 5.2. Set $f_1 = 1$, and for $N \geq 2$, write

$$f_N(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(N-1)}; \mathbf{z}) = W_{I_{\lambda}^{\min}}^{\circ}(\mathbf{t}; \mathbf{z}), \quad f_N^{(i)}(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(N-1)}; \mathbf{z}) = W_{s_{\lambda^{(i)}, \lambda^{(i)}+1}(I_{\lambda}^{\min})}^{\circ}(\mathbf{t}; \mathbf{z}),$$

indicating the dependence on N explicitly. By Lemma 5.2, $f_N(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(N-1)}; \mathbf{z}) = 1$.

By formula (5.1), the function $f_N^{(N-1)}$ is a symmetric linear function in $t_1^{(N-1)}, \dots, t_{\lambda(N-1)}^{(N-1)}$ vanishing at $\mathbf{t}^{(N-1)} = (z_1, \dots, z_{\lambda(N-1)})$. Hence,

$$f_N^{(N-1)}(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(N-1)}; \mathbf{z}) = r_N(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(N-2)}; \mathbf{z}) \left(\sum_{j=1}^{\lambda(N-1)} (t_j^{(N-1)} - z_j) \right)$$

for some polynomial r_N . Evaluating both sides at $\mathbf{t}^{(N-1)} = (z_1, \dots, z_{\lambda(N-1)-1}, 0)$ yields

$$r_N(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(N-2)}; \mathbf{z}) = f_{N-1}(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(N-2)}; z_1, \dots, z_{\lambda(N-1)-1}, 0) = 1$$

by Lemma 5.2, which proves Lemma 5.3 for $i = N - 1$.

For $i \neq N - 1$, formula (5.1) implies that the function $f_N^{(i)}$ is a polynomial in $t_1^{(N-1)}, \dots, t_{\lambda(N-1)}^{(N-1)}$ of degree zero in each of these variables, hence a constant. Evaluating $f_N^{(i)}(\mathbf{t}^{(1)}, \dots,$

$\mathbf{t}^{(N-1)}; \mathbf{z}$ at $\mathbf{t}^{(N-1)} = (z_1, \dots, z_{\lambda(N-1)})$ yields

$$f_N^{(i)}(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(N-1)}; \mathbf{z}) = f_{N-1}^{(i)}(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(N-2)}; z_1, \dots, z_{\lambda(N-1)}), = \sum_{j=1}^{\lambda^{(i)}} (t_j^{(i)} - z_j),$$

where the second equality holds by the induction assumption. Lemma 5.3 is proved. \square

APPENDIX D. STIRLING'S FORMULA

The next two lemmas are used in the proof of Lemma 5.9. The first lemma is well known.

Lemma D.1. *For $|\arg(h/\kappa)| < \pi$ and any $\alpha \in \mathbb{C}$, we have*

$$\lim_{h \rightarrow \infty} \frac{\Gamma(\alpha + h/\kappa)}{(h/\kappa)^\alpha \Gamma(h/\kappa)} = 1$$

locally uniformly in α .

Lemma D.2. *For any $\alpha, r \in \mathbb{C}$, $l \in \mathbb{Z}$, we have*

$$\lim_{h \rightarrow \infty} \frac{r^l \Gamma(\alpha + l + h/\kappa)}{(h/\kappa)^l \Gamma(\alpha + h/\kappa) \Gamma(l + 1)} = \frac{r^l}{\Gamma(l + 1)}$$

uniformly in l and locally uniformly in α, r .

Proof. Let $y = h/\kappa$ and $f(x, y, \alpha) = (1 - xy^{-1})^{-\alpha-y}$. For $r < |y|$ by Cauchy's formula,

$$\frac{r^l \Gamma(\alpha + l + y)}{y^l \Gamma(\alpha + y) \Gamma(l + 1)} = \frac{r^l}{2\pi\sqrt{-1}} \int_{|x|=r} \frac{f(x, y, \alpha)}{x^{l+1}} dx.$$

Since $\lim_{y \rightarrow \infty} f(x, y, \alpha) = e^x$ locally uniformly in x, α , and

$$\frac{r^l}{\Gamma(l + 1)} = \frac{r^l}{2\pi\sqrt{-1}} \int_{|x|=r} \frac{e^x}{x^{l+1}} dx,$$

Lemma D.2 follows. \square

APPENDIX E. POLYNOMIALITY OF SOLUTIONS

Consider dynamical differential equations (2.6) for nonpositive integer values of h/κ . Recall the $\text{End}((\mathbb{C}^N)_\lambda^{\otimes n})$ -valued Levelt fundamental solution $\widehat{\Psi}(\mathbf{z}; h; \mathbf{q}; \kappa)$, see Theorem 4.17, Proposition 4.20, and formula (4.47) for the dynamical Hamiltonians X_1, \dots, X_n at $\mathbf{q} = \emptyset$.

Say that a rational function $f(\mathbf{z})$ is over integers if $f(\mathbf{z})$ is a ratio of polynomials in \mathbf{z} with integer coefficients.

Theorem E.1. *Let $m \in \mathbb{Z}_{\geq 0}$ and $h = -m\kappa$. Then the Levelt fundamental solution has the form*

$$(E.1) \quad \widehat{\Psi}(\mathbf{z}; -m\kappa; \mathbf{q}; \kappa) = P(\mathbf{z}/\kappa; m; \mathbf{q}) \prod_{i=1}^{N-1} \prod_{j=i+1}^N (1 - q_j/q_i)^{-m\lambda_i} \prod_{i=1}^N q_i^{X_i(\mathbf{z}/\kappa; -m; \emptyset)},$$

where $P(\mathbf{z}; m; \mathbf{q})$ is a polynomial in the variables $q_2/q_1, \dots, q_n/q_{n-1}$ with coefficients being rational functions of \mathbf{z} over integers. The degree of $P(\mathbf{z}; m; \mathbf{q})$ in the variable q_{i+1}/q_i is at most $m(n-i)\lambda^{(i)}$. The function $P(\mathbf{z}; m; \mathbf{q})$ is holomorphic in \mathbf{z} if $z_a - z_b \notin \mathbb{Z}_{\neq 0}$ for all $1 \leq a < b \leq n$. The singularities of $P(\mathbf{z}; m; \mathbf{q})$ at the hyperplanes $z_a - z_b \in \mathbb{Z}_{\neq 0}$ are simple poles. For any \mathbf{z} such that $z_a - z_b \notin \kappa \mathbb{Z}_{\neq 0}$ for all $1 \leq a < b \leq n$, the columns of $\widehat{\Psi}(\mathbf{z}; -m\kappa; \mathbf{q}; \kappa)$ form a basis of the space of $(\mathbb{C}^N)_{\lambda}^{\otimes n}$ -valued solutions of dynamical differential equations (2.6).

Proof. By formulae (4.48), (4.60), the function $\widehat{\Psi}(\mathbf{z}; -m\kappa; \mathbf{q}; \kappa)$ has the form (E.1) with $P(\mathbf{z}; m; \mathbf{q})$ being a power series in $q_2/q_1, \dots, q_n/q_{n-1}$. The first step of the proof is to show that the power series terminate so that $P(\mathbf{z}; m; \mathbf{q})$ is a polynomial in $q_2/q_1, \dots, q_n/q_{n-1}$.

Denote by $P_{I,J}(\mathbf{z}; m; \mathbf{q})$ the entries of $P(\mathbf{z}; m; \mathbf{q})$ in the standard basis $\{v_I, I \in \mathcal{I}_{\lambda}\}$ of $(\mathbb{C}^N)_{\lambda}^{\otimes n}$. By formula (4.60),

$$(E.2) \quad P_{I,J}(\mathbf{z}; m; \mathbf{q}) = \sum_{\mathbf{l} \in \mathbb{Z}_{\geq 0}^{\lambda^{(1)}}} \mathcal{J}_{I,J,\mathbf{l}}(\mathbf{z}; -m) \prod_{i=1}^{N-1} (q_{i+1}/q_i)^{\sum_{a=1}^{\lambda^{(i)}} l_a^{(i)}},$$

where the coefficients $\mathcal{J}_{I,J,\mathbf{l}}(\mathbf{z}; -m)$ are given by formula (E.4) below.

Denote $[x, j] = x(x-1)\dots(x-j+1)$. Recall

$$\mathcal{Z}_{\lambda} = \{\mathbf{l} \in \mathbb{Z}_{\geq 0}^{\lambda^{(1)}} \mid l_a^{(i)} \geq l_a^{(i+1)}, i = 1, \dots, N-1, a = 1, \dots, \lambda^{(i)}\},$$

see (4.31). For $\mathbf{l} \in \mathbb{Z}_{\geq 0}^{\lambda^{(1)}}$, set $F_{\mathbf{l}}(\mathbf{z}; m) = 0$ if $\mathbf{l} \notin \mathcal{Z}_{\lambda}$, and

$$(E.3) \quad F_{\mathbf{l}}(\mathbf{z}; m) = \prod_{i=1}^{N-1} \prod_{a=1}^{\lambda^{(i)}} \left(\frac{(-1)^{l_a^{(i)} - l_a^{(i+1)}} [m, l_a^{(i)} - l_a^{(i+1)}]}{(l_a^{(i)} - l_a^{(i+1)})!} \times \right. \\ \left. \times \prod_{\substack{b=1 \\ b \neq a}}^{\lambda^{(i)}} \frac{[z_b - z_a + l_a^{(i)} - l_b^{(i)}, m+1]}{[z_b - z_a, m+1]} \prod_{\substack{c=1 \\ c \neq a}}^{\lambda^{(i+1)}} \frac{[z_c - z_a, m+1]}{[z_c - z_a + l_a^{(i)} - l_c^{(i+1)}, m+1]} \right)$$

if $\mathbf{l} \in \mathcal{Z}_{\lambda}$. Here $\lambda^{(N)} = n$ and $l_a^{(N)} = 0$ for all $a = 1, \dots, n$.

Notice that the factor $[m, l_a^{(i)} - l_a^{(i+1)}]$ in formula (E.3) equals zero if $l_a^{(i)} - l_a^{(i+1)} > m$. Therefore, $F_{\mathbf{l}}(\mathbf{z}; m) = 0$ unless $l_a^{(i)} - l_a^{(i+1)} \leq m$ for all a, i , and in particular, $F_{\mathbf{l}}(\mathbf{z}; m) = 0$ unless $l_a^{(i)} \leq m(n-i)$ for all a, i .

Denote

$$\mathcal{Z}_{\lambda, m} = \{\mathbf{l} \in \mathbb{Z}_{\geq 0}^{\lambda^{(1)}} \mid 0 \leq l_a^{(i)} - l_a^{(i+1)} \leq m, i = 1, \dots, N-1, a = 1, \dots, \lambda^{(i)}\},$$

By formulae (4.59), (4.57), (E.3),

$$(E.4) \quad \mathcal{J}_{I,J,\mathbf{l}}(\mathbf{z}; -m) = \sum_{K \in \mathcal{I}_{\lambda}} \frac{F_{\mathbf{l}}(\mathbf{z}_{\sigma_K}; m) W_I(\Sigma_K - \mathbf{l}; \mathbf{z}; -m) \check{W}_J(\Sigma_K; \mathbf{z}; -m)}{R_{\lambda}(\mathbf{z}_{\sigma_K}) Q_{\lambda}(\mathbf{z}_{\sigma_K}; -m) c_{\lambda}(\Sigma_K - \mathbf{l}; -m) c_{\lambda}(\Sigma_K; -m)}$$

if $\mathbf{l} \in \mathcal{Z}_{\lambda, m}$, and $\mathcal{J}_{I, J, \mathbf{l}}(\mathbf{z}; -m) = 0$ if $\mathbf{l} \notin \mathcal{Z}_{\lambda, m}$. Therefore,

$$(E.5) \quad P_{I, J}(\mathbf{z}; m; \mathbf{q}) = \sum_{\mathbf{l} \in \mathcal{Z}_{\lambda, m}} \mathcal{J}_{I, J, \mathbf{l}}(\mathbf{z}; -m) \prod_{i=1}^{N-1} (q_{i+1}/q_i)^{\sum_{a=1}^{\lambda^{(i)}} l_a^{(i)}},$$

so that $P(\mathbf{z}; m; \mathbf{q})$ is a polynomial in $q_2/q_1, \dots, q_n/q_{n-1}$ of degree at most $m(n-i)\lambda^{(i)}$ in the variable q_{i+1}/q_i , $i = 1, \dots, N$. Moreover, by formulae (E.3), (4.2), (4.5), (4.8), all the coefficients $\mathcal{J}_{I, J, \mathbf{l}}(\mathbf{z}; -m)$ are rational functions of \mathbf{z} over integers. The regularity properties of $P(\mathbf{z}; m; \mathbf{q})$ follow from Theorem 4.17, item (ii).

The columns of $\widehat{\Psi}(\mathbf{z}; -m\kappa; \mathbf{q}; \kappa)$ form a basis of the space of $(\mathbb{C}^N)_{\lambda}^{\otimes n}$ -valued solutions of dynamical differential equations (2.6) by Theorem 4.17, see formula (4.49). \square

Theorem E.1 allows us to describe the monodromy of the system of dynamical differential equations (2.6) if h/κ is a nonpositive integer.

Corollary E.2. *Let $h \in \kappa\mathbb{Z}_{\leq 0}$. Then the monodromy of the system of dynamical differential equations (2.6) is abelian. The monodromy representation is generated by the matrices $e^{2\pi\sqrt{-1}X_1(\mathbf{z}/\kappa; h/\kappa; \emptyset)}, \dots, e^{2\pi\sqrt{-1}X_N(\mathbf{z}/\kappa; h/\kappa; \emptyset)}$. For any \mathbf{z} such that $z_a - z_b \notin \kappa\mathbb{Z}$ for all $1 \leq a < b \leq n$, the monodromy representation is the direct sum of one-dimensional representations generated by the solutions $\Psi_I(\mathbf{z}; -m\kappa; \mathbf{q}; \kappa)$, $I \in \mathcal{I}_{\lambda}$, defined by (4.20).*

Notice that for integer values of h/κ , the factors $(1 - q_j/q_i)^{\lambda_i h/\kappa}$ in formula (4.21) are rational functions and the only nontrivial monodromy of the solutions $\Psi_J(\mathbf{z}; -m\kappa; \mathbf{q}; \kappa)$ comes from the factors $q_i^{\sum_{a \in J_i} z_a/\kappa}$ in formula (4.22).

The results of this appendix were motivated by the corresponding statements in [OP] on the quantum differential equation of the Hilbert scheme of points in the plane.

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