BOUNDEDNESS OF SEMISTABLE SHEAVES

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ABSTRACT. In this expository article, we follow Langer's work in [Lan04b] to prove the boundedness of the moduli space of semistable torsion-free sheaves over a projective variety, in any characteristic.

CONTENTS

1. INTRODUCTION

The study of moduli is one of the oldest branches of algebraic geometry and forms one of the central pillars of our modern understanding. Historically, such questions date back to Riemann, who, in his pursuit to understand what we now call Riemann surfaces, determined that a complex projective curve of genus g depends on exactly $3g - 3$ parameters (which he referred to as "moduli" - hence the name). The central question of moduli theory has not evolved much; we are interested in whether or not various algebraic objects can be "parametrized" (in some sense) by some other algebraic object. For example, one might be interested in forming a moduli space of algebraic curves with some fixed genus, or in our case, a moduli space of torsion-free sheaves with fixed Hilbert polynomial on a fixed variety X.

To construct the moduli space of torsion-free sheaves $\{E_\alpha\}$ with fixed Hilbert polynomial P over a given variety X, one of the first and the most fundamental problems is the *boundedness* of the collection $\{E_\alpha\}$. This property is equivalent to the moduli space, if exists, being a finite type scheme over the base field, and thus a reasonable geometric object that one can work with. However it was quickly discovered that even with fixing the Hilbert polynomial, certain pathological examples prevent such a moduli space from being well-behaved. Mumford, in the case of curves, introduced the notion of *semistability* for a torsion-free sheaf as a solution for this problem, and to demonstrate its efficacy, he showed that the collection of semistable vector bundles of a fixed rank and degree on a fixed curve is bounded. As the notion of semistability can be generalized to torsion-free sheaves over general projective varieties, it is natural to ask about the boundedness for semistable sheaves on higher dimensional varieties. Following Langer's work in [Lan04b], the goal of our article is to give a positive answer to the boundedness problem. Precisely, we prove the following:

Theorem 1. *Let $\mathcal{M}^P(X)$ be the moduli space of semistable torsion-free sheaves with fixed Hilbert polynomial P. Then it is of finite-type.*

Let us now discuss the idea of the proof. Given any (semistable) torsion-free sheaf E in our collection, we write $E|_H$ for the restriction of E to a general hypersurface H, and $\mu_{\max}(E|_H)$ for the maximal slope in the Harder-Narasimhan polygon of $E|_H$.

Our most important tool in the proof is Kleiman's criterion (Theorem 14) and an accompanying inequality (Lemma 15), which, when combined, say that to prove the boundedness of the family of torsion-free sheaves $\mathcal{M}^P(X)$, it suffices to give a uniform bound of $\mu_{\max}(E|_H)$. Because of this fact, most of the work in proving boundedness is producing such a uniform bound, which we obtain in Corollary 44 and Theorem 45. Explicitly, it can be understood as follows:

Theorem 2. *Let E be a torsion-free sheaf over X, with Hilbert polynomial P, and denote the maximal slope of its Harder-Narasimhan polygon be μ_0. Then we have*

$$\mu_{\max}(E|_H) \leq C_1(P) + C_2(P)\mu_0,$$

where H is a general hypersurface of X, and $C_i(P)$ are constants determined only by P and X.

The key is that this result allows us to bound the slopes $\mu_{\max}(E|_H)$ of the restrictions $E|_H$ by the slopes of the original E, plus extra terms and coefficients that are controlled only by the Hilbert polynomial of E.

The major contents of our article are then devoted to obtaining the inequality above, following [Lan04b, Section 3]. The strategy can be understood as a double induction on two different results. The first one, Theorem $\text{Res}(r)$, is a special case of Theorem 2 above, considering the invariants before and after the restriction of E to a hypersurface, where E is torsion-free of rank $\leq r$ (c.f. [Lan04b, Theorem 3.1]). It relates the slopes of the Harder-Narasimhan filtration of the restriction $E|_H$ to the discriminant and Harder-Narasimhan filtration of E itself. The second result is a collection of several various numerical inequalities, which we refer to as *Bogomolov's inequalities* (Theorem $\text{BI}(r)$, c.f. [Lan04b, Theorem 3.2-3.4, and $T^5(r)$]), due to their similarity with the well-known inequality in characteristic zero of the same name.

Up to an extra term, it says that the discriminant $\Delta(E)$ of E is non-negative when the torsion-free sheaf E is (strongly) semistable, where E is of rank $\leq r$. Using the notations above, Langer's induction schema can be summarized as the following two implications (c.f. Section 4):

$$(1.1) \qquad \begin{cases} BI(r) \Rightarrow Res(r); \\ Res(r) + BI(r) \Rightarrow BI(r+1). \end{cases}$$

We note that both results are true automatically for rank $r = 1$. In this way, the two technical results are proved together, thus so is the inequality in Theorem 2 and the boundedness in Theorem 1.

We also mention that in the positive characteristic case, a Frobenius pullback of a semistable sheaf may no longer be semistable. So we define semistable sheaves whose Frobenius pullbacks are all semistable as being *strongly semistable*. A priori, for a given sheaf, the Harder-Narasimhan filtrations of its Frobenius pullbacks could be unrelated to each other. One of Langer's key observations is that their differences are controllable: the Harder-Narasimhan filtrations eventually stabilize with respect to the Frobenius (Theorem 29, [Lan04b, Theorem 2.7]). This allows us to pass between semistable and strongly semistable sheaves in the proof.

Lastly, we briefly mention the history preceding Langer's result; a more detailed introduction can be found in [Lan04b]. Theorem 1 was known in characteristic zero, proven by Barth, Spindler, Maruyama, Forster, Hirschowitz and Schneider. In positive characteristic however, only the cases for curves and surfaces were known. At that time, it was known that understanding the numerical quantities associated to E before and after restricting E to a hypersurface (often called Grauet–Mülich

type results) should allow one to prove boundedness for a given family, but prior to [Lan04b] only a coarser result by Mehta and Ramanathan was available. On the other hand, Bogomolov showed the non-negativity of the discriminant of E whenever E is semistable, assuming X is in characteristic zero. It was unknown as well whether the result is completely true in positive characteristic, but it was clear that some version would be needed. Thus Langer's key contributions were in developing positive characteristic versions of the above, and also in combining them in a fruitful way.

Leitfaden of the article. We start with Section 2 on necessary preliminaries. The section contains three subsections, including basics on the stability of coherent sheaves and additional results in positive characteristic. Moreover, as the proof of main theorems requires working with polarizations consisting of nef divisors, we also include a subsection on how to approximate the nef polarizations by ample polarizations. In Section 3 we introduce several forms of Bogomolov's inequalities in positive characteristic, and show that they are equivalent in Theorem BI(r). For the reader's convenience, this section corresponds to the implications $T^5(r) \Rightarrow T^3(r) \Rightarrow T^4(r) \Rightarrow T^2(r) \Rightarrow T^5(r)$ in [Lan04b, §3.6 - §3.8]. In Section 4, we prove Theorem 41, and complete the major technical part of the article by proving the induction schema as in (1). This corresponds to [Lan04b, §3.5, §3.9] and [Lan04a]. At last, in Section 5, we combine the main technical results above to show the boundedness of the moduli space $\mathcal{M}^P(X)$, thus finishing the proof of Theorem 1.

2. Preliminaries

In this section, we provide the preliminaries for the article.

Stability of Coherent Sheaves. In this section we recall some useful facts on the (semi)stability of coherent sheaves. We refer to [HL10] for details and proofs. Let X be a smooth projective variety of dimension n over an algebraically closed field. Fix $n-1$ ample divisors $D_1, \ldots, D_{n-1}$ on X. Let E be a coherent sheaf on X. Recall that a sheaf E is said to be pure (of dimension $d \le \dim X$) if for all nontrivial subsheaves $f \subset E$, $\dim(F) := \dim \operatorname{supp}(F) = d$.

Definition 3. The *slope* of E, with respect to the polarization $(D_1, \ldots, D_{n-1})$, is defined by

$$\mu(E) := \frac{\deg(E)}{\operatorname{rk}(E)} := \frac{c_1(E) \cdot D_1 \cdots D_{n-1}}{\operatorname{rk}(E)}$$

where $\operatorname{rk}(E)$ is the rank of E at the generic point. If $\operatorname{rk}(E) = 0$, then the $\mu(E) := \infty$.

As indicated, the slope of a coherent sheaf does depend on the choice of ample (or nef) divisors defining a polarization. For the purpose of clarity, we will not reference this choice of polarization when discussing slope unless there is a serious risk of confusion.

Definition 4. We say that E is *(semi)stable* if E is pure and for any nonzero proper subsheaf $F \subset E$, $\mu(F) < \mu(E)$ (resp. $\mu(F) \le \mu(E)$).

This notion of (semi)stability is more generally known as slope-(semi)stability or μ-(semi)stability. An equivalent formulation of the definition can also be stated in terms of quotient sheaves.

Lemma 5. *[HL10, Proposition 1.2.6] Let E be a coherent sheaf. Then E is semistable if and only if for all the quotient sheaves $E \to G$, $\mu(E) \leq \mu(G)$.*

This formulation of the definition gives a quick proof of the following very useful fact.

Lemma 6. *[HL10, Proposition 1.2.7] Let E_1 and E_2 be semistable sheaves with slopes μ_1 and μ_2 such that $\mu_1 > \mu_2$. Then, $\mathrm{Hom}(E_1, E_2) = 0$.*

The definition of semistability also behaves well with regards to exact sequences, as the following two lemmas indicate.

Lemma 7. *Let $E' \to E \to E''$ be a short exact sequence of torsion-free coherent sheaves, and let μ', μ, μ'' and r', r, r'' be the slopes and the ranks of E', E, E'' respectively. Then, $r\mu = r'\mu' + r''\mu''$. In particular if the slopes of two of E', E, E'' are the same, then so is the case for the third.*

Lemma 8. *If $E' \to E \to E''$ is a short exact sequence and let μ', μ, μ'' denote the slopes of E', E, E'' respectively, then*

- *If E is semistable and either $\mu' = \mu$ or $\mu'' = \mu$, then E' and E'' are also semistable.*
- *If E' and E'' are semistable and $\mu' = \mu''$, then E is semistable with $\mu = \mu' = \mu''$.*

One of the more important technical tools is the Harder-Narasimhan filtration, which is defined below.

Definition 9. A *Harder-Narasimhan filtration* for E is an increasing filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_d = E$$

such that the factors $F_i := E_i/E_{i-1}, i = 1, \cdots, d$ are semistable sheaves with slopes μ_i satisfying

$$\mu_{\max}(E) := \mu_1 > \mu_2 > \cdots > \mu_d =: \mu_{\min}(E).$$

Proposition 10. *[HL10, Theorem 1.3.4] Every torsion-free sheaf E has a unique Harder-Narasimhan filtration.*

Remark 11. Just like the slope, the Harder-Narasimhan filtration is also dependent on the choice of the polarization $(D_1, \ldots, D_{n-1})$, and could be different with respect to polarizations.

Remark 12. We should also remark here that in the construction of the Harder-Narasimhan filtration for a torsion-free sheaf, an important step is establishing the existence and uniqueness of a *maximal destabilizing subsheaf*. We use this notion a few times, so we give its definition here. Given a torsion-free sheaf E, then the *maximal destabilizing subsheaf* $F \subset E$ is a semistable coherent subsheaf such that $\mu(F) \geq \mu(G)$ for all other subsheaves $G \subset E$, and moreover if $\mu(F) = \mu(G)$, then $F \supset G$.

Next, we wish to introduce the key tools of the paper, Kleiman's criterion and a related inequality. For the conclusion of Kleiman's criterion to make sense however, we should remind the reader of the technical property of boundedness.

Definition 13. Let M be a set of coherent sheaves on X. Then M is said to be *bounded* if there is a scheme B of finite-type and a coherent sheaf F on $X \times B$ with $M \subset \{F_b \mid b \in B \text{ closed }\}$. Here F_b is the pullback of F along $X \times \{b\} \to X \times B$.

Kleiman's criterion provides a very convenient way of determining whether or not a given family is bounded.

Theorem 14 (Kleiman's criterion)**.** *[HL10, Theorem 1.7.8] Let $\{E_\alpha\}$ be a family of coherent sheaves over X with the same Hilbert polynomial P. Then the family is bounded if and only if there are constants C_i, for $i = 0, \ldots, \deg(P)$, such that for every E_α there exists an E_α-regular sequence of hyperplanes $H_1, \ldots, H_{\deg(P)}$, satisfying*

$$h^0(E_\alpha|_{\cap_{j \le i} H_j}) \le C_i, \forall i.$$

Here we recall that a hyperplane $s \in H^0(X, \mathcal{O}_X(1))$ is said to be *E-regular* if the map

$$E(-1) \xrightarrow{\cdot s} E$$

is injective. A sequence $\{s_1, ..., s_l\} \subset H^0(X, \mathcal{O}_X(1))$ is E-regular if s_i is $E/(s_1, .., s_{i-1})E(-i)$-regular for all $1 \le i \le l$.

Lemma 15. *[HL10, Lemma 3.3.2] Let E be a torsion-free sheaf of rank r. Then for any E-regular sequence of hyperplane sections $H_1, \cdots, H_n$, the following inequality holds for $i = 1, \cdots, n$:*

$$\frac{h^0(X_i, E_i)}{r \deg(X)} \le \frac{1}{i!} \left[\frac{\mu_{\max}(E_1)}{\deg(X)} + i \right]_+^i,$$

where $X_i \in |H_1| \cap \cdots \cap |H_{n-i}|$, $E_i = E|_{X_i}$, and $[x]_+ = \max\{0, x\}$ for any real number x.

Combining the previous two results, provided we can get a uniform bound on $\mu_{\max}$, will prove the boundedness for semistable sheaves with a fixed Hilbert polynomial. To get such estimates, we will often use the following lemma.

Lemma 16. *If $0 \to A \to B \to C \to 0$ is an exact sequence of torsion-free sheaves, then $\mu_{\max}(B) \le \max\{\mu_{\max}(A), \mu_{\max}(C)\}$.*

At the end of the subsection, we introduce the following definition that helps visualize the Harder-Narasimhan filtration.

Definition 17 (Harder-Narasimhan polygon)**.** Consider a torsion-free sheaf E on X with its Harder-Narasimhan filtration $0 = E_0 \subset \cdots \subset E_m = E$. Let $p(E_i) = (\mathrm{rk}(E_i), \deg(E_i))$. We define the *Harder-Narasimhan polygon* of E, denoted as $\mathrm{HNP}(E)$, to be the convex hull of the points $p(E_0), \ldots, p(E_m)$ in $\mathbb{R}^2$. See Figure 1.
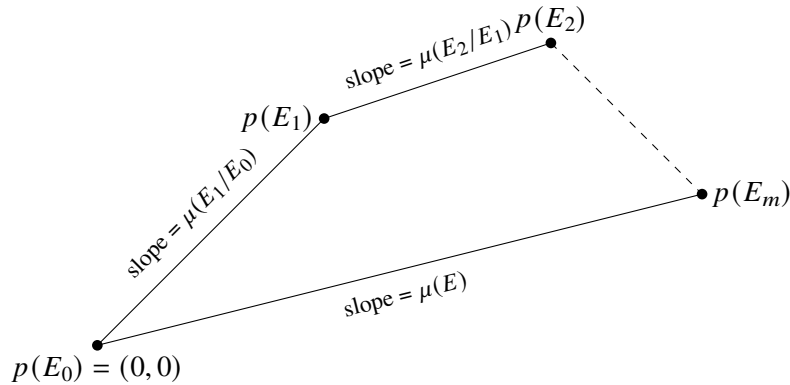

FIGURE 1. An illustration of the Harder Narasimhan Polygon

In fact, the Harder-Narasimhan polygon is a convex polygon with vertices $p(E_0), \ldots, p(E_m)$, and the slope of the line segment $\overline{p(E_i)p(E_{i+1})}$ is $\mu(E_i/E_{i+1})$.

Remark 18. If $F \subset E$ is a torsion-free subsheaf F inside of a semistable sheaf E, then the point $p(F) = (\mathrm{rk}(F), \deg(F))$ lies below the polygon $\mathrm{HNP}(E)$ i.e. for any $x \in [0, \mathrm{rk}(F)]$, we have

$$\sup\{y \mid (x, y) \in \mathrm{HNP}(F)\} \leq \sup\{y \mid (x, y) \in \mathrm{HNP}(E)\}.$$

Approximation of nef polarizations. In this subsection, we approximate the Harder-Narasimhan filtration for a polarization consisting of *nef divisors*, that will be used later. As this is a nonstandard setup and does not appear in most of the literature, we provide the full details of the proof. We assume the existence and uniqueness of the (absolute) Harder-Narasimhan filtration with respect to an *ample polarization* as in last subsection.

Our first main result this subsection is the following approximation result.

Theorem 19. *Let $(D_1, \ldots, D_{n-1})$ be a set of nef divisors, and H an ample divisor. Assume E is a torsion-free coherent sheaf over X. Then there exists a positive number ϵ, and a (unique) filtration of saturated subsheaves*

$$0 = E_0 \subset E_1 \subset \cdots E_l = E,$$

such that for any $t \in (0, \epsilon)$, the above is the Harder-Narasimhan filtration for the ample polarization $(D_1 + tH, \ldots, D_{n-1} + tH)$.

Proof. Denote $H_i(t)$ to be the $\mathbb{R}$-divisor $D_i + tH$, which is ample [Laz17, Corollary 1.4.10]. For a torsion-free sheaf F, we denote $\deg_t(F)$ and $\mu_t(F)$ to be the degree and slope of F with respect to the polarization $(H_1(t), \ldots, H_{n-1}(t))$. Note that when $t = 0$, we get exactly the degree and slope with respect to the polarization $(D_1, \cdots, D_{n-1})$, which we write in short as $\deg(F)$ and $\mu(F)$ separately.

We first notice that it suffices to show the following: there exists $\epsilon > 0$ and a saturated subsheaf $E_1 \subset E$, such that E_1 is the maximal destabilizing subsheaf of E for $(H_1(t), \ldots, H_{n-1}(t))$ and any $t \in (0, \epsilon)$. The uniqueness of E_1 follows from the uniqueness of the Harder-Narasimhan filtration for the ample polarization $(H_1(t), \cdots H_{n-1}(t))$ as in Proposition 10.

To find such E_1, let C denote the set of torsion-free subsheaves of E and consider the following map of sets

$$\mu_t(-) : C \longrightarrow \{\text{Degree } n - 1 \text{ polynomials in } t\};$$

$$E' \longmapsto \mu_t(E') = \frac{c_1(E')H_1(t) \cdots H_{n-1}(t)}{\mathrm{rk}(E')}.$$

For each $E' \subset E$, the image $\mu_t(E')$ is the polynomial $a_0 + a_1 t + \cdots + a_{n-1} t^{n-1}$, where we have

$$a_0(E') = \mu(E');$$

$$a_1(E') = \sum_{1 \leq i \leq n-1} \mu_{(H, D_1, \ldots, \hat{D}_i, \ldots, D_{n-1})}(E').$$

For general $j \leq n$, the coefficient a_j is the sum of slopes of E' with respect to all possible choice of polarizations $(H, \ldots, H, D_{i_1}, \ldots, D_{i_{n-1-j}})$, such that each of the first j entries of the polarization are all equal to H. Moreover, each polynomial $\mu_t(E')$ has coefficients in $\frac{1}{r!}\mathbb{Z}$, whose coefficients are finite linear combinations of slopes of E' for a fixed, finite number of choices of polarization. In particular, for each $0 \leq i \leq n$, the collection of coefficients $\{a_i(E') \mid E' \subset E\}$ is bounded above. [1] Here we denote the subset of polynomials consisting of the image of the map $\mu_t(-)$ by $\mathcal{D}$.

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[1]To see this, we first note that since $a_i(E')$ is a finite positive linear combination of $\mu_{H^i, D_{j_1}, \ldots, D_{j_{n-1-i}}}(E')$, it suffices to bound each slope for the given nef polarization. The latter can be proved by induction on ranks of E as in the classical case: To check the case when E is a line bundle, it suffices to show the set of slopes of sub line bundles is bounded by slope of E itself, which is true again by approximation via adding each D_{j_l} by tH and making t approach to zero. Here

We then define a lexicographic order on elements in $\mathcal{D}$ as follows:

$$W_1(t) = \sum_{}^{d} a_i t^i < W_2(t) = \sum_{}^{d} b_i t^i \text{ if}$$

$$a_0 = b_0, \cdots, a_{i-1} = b_{i-1}, a_i < b_i \text{ for some } i.$$

Since the coefficients of polynomials in $\mathcal{D}$ are bounded above, we can find the maximum polynomial $P(t)$ in $\mathcal{D}$. Moreover, since the degree of a sheaf is a discrete quantity, we can find a subsheaf E_1 in the preimage of $\mu_t^{-1}(P(t))$ whose rank is maximal.

Finally, we prove that E_1 is exactly the maximal rank destabilizing subsheaf of E with respect to $(H_1(t), \ldots, H_{n-1}(t))$ for t small enough. To show this, it suffices to show the following, whose complete details are left to the reader:

Claim 20. *Let $\mathcal{D}$ be a set of degree n polynomials with coefficients bounded above, and whose coefficients are in $\frac{1}{N}\mathbb{Z}$ for some positive integer N. Then a polynomial $P(t) \in \mathcal{D}$ is maximum with respect to the lexicographic order if and only if for small enough $t > 0$ $P(t) > Q(t)$ (in $\mathbb{R}$) for all other $Q(t) \in \mathcal{D}$.*

The idea of the claim is the following: as t approaches zero, higher power terms are dominated by those with lower powers, so the polynomial that has the largest first several terms under the lexicographic order will also have maximum value in $\mathbb{R}$ for t small enough. ∎

The above filtration of subsheaves $\{E_i\}$ in general fails to be the Harder-Narasimhan filtration with respect to $(D_1, \ldots, D_{n-1})$ for $t = 0$. However, the filtration is in fact a *weak Harder-Narasimhan filtration* with respect to $(D_1, \ldots, D_{n-1})$, in the sense that each E_i/E_{i-1} is semistable with respect to $(D_1, \ldots, D_{n-1})$, and we have inequalities

$$\mu(E_{i+1}/E_i) \geq \mu(E_i/E_{i-1}), \; 1 \leq i \leq l - 1.$$

Note that we do not have strict inequalities here, which is part of the definition of the Harder-Narasimhan filtration.

The semistability of the subsheaves E_i can be seen as follows:

Lemma 21. *Let E' be a subsheaf of the torsion-free sheaf E such that for $t > 0$ small enough, E' is semistable with respect to $(H_1(t), \ldots, H_{n-1}(t))$. Then E' is semistable with respect to $(D_1, \ldots, D_{n-1})$.*

Proof. Assume E' admits a subsheaf E_1' whose slope with respect to $(D_1, \ldots, D_{n-1})$ is strictly larger than that of E'. By the continuity of the polynomial μ_t in t, we get the inequality $\mu_t(E_1') > \mu_t(E')$ for $t > 0$ small enough, which contradicts our assumption. ∎

The same strategy above in fact implies the existence and uniqueness of the Harder-Narasimhan filtration for nef divisors, generalizing the result for ample ones.

Corollary 22. *Let $(D_1, \ldots, D_{n-1})$ be a set of nef divisors of X. Then any torsion-free coherent sheaf E over X admits a unique Harder-Narasimhan filtration with respect to the polarization $(D_1, \ldots, D_{n-1})$.*

Proof. Let H be a fixed ample divisor over X. By Theorem 19, there exists a unique filtration of saturated subsheaves E_i of E, such that when $t > 0$ is very small, E_i is the Harder-Narasimhan filtration of E with respect to $(H_1(t), \ldots, H_{n-1}(t))$. Each E_i is semistable with respect to $(D_1, \ldots, D_{n-1})$

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we observe that the inequality holds for any $t > 0$, so by the continuity, we get the upper bound. The general case of E follows from the induction hypothesis on E' and E'' in a short exact sequence of vector bundles $E' \to E \to E''$.

by Lemma 21, and the slopes of E_i/E_{i-1} is non-strictly decreasing. Thus by picking the sub-sheaves in this filtration so that the slopes of the factors become strictly decreasing, we get the Harder-Narasimhan filtration of E with respect to $(D_1, \ldots, D_{n-1})$. At last, the uniqueness of the Harder-Narasimhan filtration follows from the uniqueness of the maximal destabilizing subsheaf with respect to $(D_1, \ldots, D_{n-1})$, where the latter can be checked via the continuity of μ_t. So we are done. ∎

At last, we give a simple example illustrating the degeneration of the Harder-Narasimhan filtration.

Example:

Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ be the product of two projective lines over a field k. Let D be the trivial divisor, H be the ample divisor of bidegree $(1, 2)$ over X, and let $H(t) := D + tH$ be the sum. Denote $L_1 = (0, 1)$ and $L_2 = (1, 0)$ to be the two line bundles over X, and let E be the the direct sum $L_1 \oplus L_2$. Then for any $t > 0$, the divisor $H(t)$ is ample in X as it can be written as a sum of the pullback of ample divisors on two separate factors of X. Moreover, for each $t > 0$, we have $L_1 \cdot H(t) = 2t > t = L_2 \cdot H(t)$, and thus

$$(*) \qquad\qquad 0 \subset L_1 \subset E$$

is the Harder-Narasimhan filtration of E for the ample polarization $H(t)$, for any $t > 0$. However, when $t = 0$, as $H(0) = D = 0$ is the trivial divisor, any subsheaf of E has the same slope 0. In particular, the Harder-Narasimhan filtration of E is the trivial filtration, and the filtration $(*)$ becomes a filtration of subsheaves whose graded pieces have the same slopes.

Positive characteristic. The main difference in characteristic p is that we need to work with the notion of *strong semistability*, as the Frobenius pullbacks of semistable sheaves are not necessarily semistable.

First, we recall a few basic notions from algebraic geometry in positive characteristic. Let k be an algebraically closed field of characteristic $p > 0$. Let X be a smooth projective k-variety. The *absolute Frobenius* morphism $F_X : X \to X$ is the map on X given locally on an open subset $\operatorname{Spec} R \subset X$ by $a \mapsto a^p$. For simplicity of notation, we just denote F_X by F. Note that F is not a map of k-schemes.

We also fix nef divisors $D_1, \ldots, D_{n-1}$ on X and we will compute slope with respect to the polarization $(D_1, \ldots, D_{n-1})$.

Definition 23 (Strong semistability). A coherent sheaf E on X is said to be *strongly semistable* if $(F^e)^*E$ is a semistable sheaf on X for all $e \geq 0$.

Remark 24. A coherent sheaf that is semistable but not strongly semistable can be found for example in [Bre05, Corollary 2].

In the positive characteristic, instead of just keeping track of $\mu_{\min}$ and $\mu_{\max}$ of a coherent sheaf, we also keep track of a few other invariants related to the Frobenius pullbacks. We note first that Frobenius pullback alters the slope of a sheaf, indeed we have that $\mu((F^e)^*E) = p^e\mu(E)$. For this reason, we define the related quantity

$$L_{\max}(E) := \lim_{e \to \infty} \frac{\mu_{\max}((F^e)^*E)}{p^e}.$$

Note that the sequence $\frac{\mu_{\max}((F^e)^*E)}{p^e}$ is increasing in e, thus the limit $L_{\max}(E)$ exists in $\mathbb{R} \cup \{\infty\}$. We will show later that this limit is indeed finite. Similarly, we can define $L_{\min}$. By definition, we

have $L_{\max}(E) \geq \mu_{\max}(E)$ and $L_{\min}(E) \leq \mu_{\min}(E)$. It immediately follows from the definition that if E is strongly semistable, then

$$L_{\max} = \mu_{\max} = \mu(E) = \mu_{\min}(E) = L_{\min}(E).$$

Let us set

$$\alpha(E) := \max\{L_{\max}(E) - \mu_{\max}(E), \mu_{\min}(E) - L_{\min}(E)\}.$$

We would like to find an upper estimate for $\alpha(E)$. To do this, we first state a theorem on how to detect instability of the Frobenius pullback of a semistable sheaf. For details, refer to [Lan04b, Section 2]. The idea is to make use of the canonical connection $F^*E \to F^*E \otimes \Omega_X$ on the Frobenius pullback.

Theorem 25. *Let E be a semistable sheaf on X such that F^*E is not a semistable sheaf on X. Let $0 = E_0 \subset E_1 \subset \cdots \subset E_m = F^*E$ be the Harder-Narasimhan filtration of F^*E. Then, the natural O_X homomorphisms $E_i \to (E/E_i) \otimes \Omega_X$ induced by the canonical connection are non-zero.*

We now estimate the the minimum and maximum slopes of the Frobenius pullback of a semistable sheaf. The idea is to use the non-zero maps above along with Theorem 25 to get an estimate of how far apart the slopes can be.

Lemma 26. *Let A be a nef divisor such that $T_X(A)$ is globally generated (or equivalently if $\Omega \hookrightarrow O_X(A)^{\oplus l}$ for some l) and let E be a torsion-free semistable sheaf on X. Then,*

$$\mu_{\max}(F^*E) - \mu_{\min}(F^*E) \leq (\mathrm{rk}(E) - 1)A \cdot D_1 \cdot \cdots \cdot D_{n-1}.$$

Proof. Let $0 = E_0 \subset E_1 \subset \cdots \subset E_m = F^*E$ be the Harder-Narasimhan filtration of F^*E. Using Theorem 25, we have that the O_X-homomorphism $E_i \to F^*E/E_i \otimes \Omega_X$ is non-zero. Thus, we get that

$$\mu(E_i/E_{i-1}) = \mu_{\min}(E_i) \leq \mu_{\max}(F^*E/E_i \otimes \Omega_X).$$

Since $\Omega \hookrightarrow O_X(A)^{\oplus l}$, we get that $\mu_{\max}(F^*E/E_i \otimes \Omega_X) \leq \mu_{\max}(F^*E/E_i \otimes O_X(A)) = \mu(E_{i+1}/E_i) + A \cdot D_1 \cdot \cdots \cdot D_{n-1}$.

Thus, we get that

$$\mu(E_i/E_{i-1}) - \mu(E_{i+1}/E_i) \leq A \cdot D_1 \cdot \cdots \cdot D_{n-1}$$

Summing this inequality, we get the result. ∎

Proposition 27. *If A is a nef divisor such that $T_X(A)$ is globally generated and E is a torsion-free sheaf on X, then*

$$\frac{\mu_{\max}(F^*E)}{p} \leq \mu_{\max}(E) + \frac{(\mathrm{rk}(E) - 1)}{p}A \cdot D_1 \ldots D_{n-1}$$

Proof. Let $0 = E_0 \subset E_1 \subset \cdots \subset E_m = E$ be the Harder-Narasimhan filtration of E. Applying the previous proposition to $F^*(E_i/E_{i-1})$, we get that

$$\frac{\mu_{\max}(F^*(E_i/E_{i-1}))}{p} \leq \mu(E_i/E_{i-1}) + \frac{(\mathrm{rk}(E) - 1)}{p}A \cdot D_1 \ldots D_{n-1}.$$

Note that F^*E_i form a filtration of F^*E and thus by Lemma 16 $\mu_{\max}(F^*E) \leq \max_i\{\mu_{\max}(F^*(E_i/E_{i-1}))\}$. Using this, we get the required result. ∎

We have the following bound on $\alpha(E)$ (which in particular shows that $L_{\max}(E)$ and $L_{\min}(E)$ are finite).

Proposition 28. *If A is a nef divisor such that $T_X(A)$ is globally generated, then*

$$\alpha(E) \leq \frac{(\mathrm{rk}(E) - 1)}{p - 1}A \cdot D_1 \ldots D_{n-1}.$$

Proof. Applying induction to the previous proposition, we see that

$$\frac{\mu_{\max}((F_X^e)^*E)}{p^e} \leq \mu_{\max}(E) + (\mathrm{rk}(E) - 1)\left(\frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^e}\right) A \cdot D_1 \ldots D_{n-1}.$$

Letting $e \to \infty$, we get the required result. ∎

Finite determinancy of the Harder-Narasimhan filtration. We will now prove the following theorem.

Theorem 29. *For every torsion-free sheaf E, there exists a non-negative integer e_0 such that all the factors in the Harder-Narasimhan filtration of $(F^{e_0})^*E$ are strongly semistable.*

The proof uses the Harder-Narasimhan polygon (see Definition 17). For any sheaf G over a smooth projective variety X, let us define $p(G) := (\mathrm{rk} G, \deg G) \in \mathbb{R}^2$. We also define

$$\mathrm{HNP}_e(E) := \{(x, y) \in \mathbb{R}^2 \mid (x, p^e y) \in \mathrm{HNP}((F^e)^*E)\}.$$

Note that $\mathrm{HNP}_e(E)$ forms an increasing sequence of convex subsets of $\mathbb{R}^2$ and we define $\mathrm{HNP}_\infty(E) := \overline{\bigcup_e \mathrm{HNP}_e(E)}$.

Proposition 30. *$HNP_\infty(E)$ is a bounded convex polygon.*

Proof. It is clear that $\mathrm{HNP}_\infty(E)$ is convex as each of the $\mathrm{HNP}_e(E)$ are convex. To see that it is bounded, note that the rank coordinates of $\mathrm{HNP}_e(E)$ lie in in the interval $[0, \mathrm{rk}(E)]$. The fact that $\alpha(E)$ is finite also tells us that there is a uniform bound on the degree coordinates of $\mathrm{HNP}_e(E)$. Thus we see that $\mathrm{HNP}_\infty(E)$ is a bounded subset of $\mathbb{R}^2$.

To show $\mathrm{HNP}_\infty(E)$ is a polygon, we claim that it is a convex hull of the set $\{(r, q_{\infty,r}) \mid r \in \{0, \ldots, \mathrm{rk}(E)\}$, where $q_{\infty,r} = \sup\{d \mid (r, d) \in \mathrm{HNP}_\infty(E)\}$. Note that $q_{\infty,0} = p(0) = (0, 0)$ and $q_{\infty,\mathrm{rk}(E)} = p(E) = (\mathrm{rk}(E), \deg(E))$.

It is clear that the convex hull of these points is contained in $\mathrm{HNP}_\infty(E)$. To see the converse, first note that all $\mathrm{HNP}_k(E)$ and thus $\mathrm{HNP}_\infty(E)$ lie above the line segment joining $p(E_0) = (0, 0)$ and $p(E) = (\mathrm{rk}(E), \deg(E))$. It is thus enough to show that every vertex of $\mathrm{HNP}_e(E)$ lies in the convex hull for all e. Pick any vertex (r, d) of $\overline{\mathrm{HNP}_e(E)}$ for some e and some $r \in \{0, \ldots, \mathrm{rk}(E)\}$. Then, since (r, d) lies above the line segment $\overline{p(0)p(E)}$ and below the point (r, q_r), one can easily see that (r, d) lies in the convex hull of $p(0), p(E)$ and (r, q_r). ∎

Before proving the finite determinancy of the Harder-Narasimhan filtration, let us introduce some notation. Let $0 = E_{0,e} \subset \cdots \subset E_{m_e,e}$ denote the Harder-Narasimhan filtration of $(F^e)^*E$. Let $p_{i,e} = (\mathrm{rk}(E_{i,e}),$
$\deg(E_{i,e})/p^e)$ denote the vertices of $\mathrm{HNP}_e(E)$.

Let $(0, 0) = p_{0,\infty}, \ldots, p_{s,\infty} = p(E)$ denote the vertices of the $\mathrm{HNP}_\infty(E)$ and denote the coordinates of $p_{j,\infty}$ as $p_{j,\infty} = (r_{j,\infty}, d_{j,\infty})$. Let us denote the slope of the line segment $\overline{p_{(j-1),\infty}p_{j,\infty}}$ as $\mu_{j,\infty}$. For every j, there exists a sequence $p_{i_j,e} \to p_{j,\infty}$ for some sequence i_j as $e \to \infty$.

Proof of Theorem 29. We prove that there exists some e_0 such that $HNP_{e_0}(E) = HNP_\infty(E)$. We show this by induction on rank. In the rank one case, there is nothing to show.

Pick $0 < \epsilon \ll 1$. Then we have the Euclidean distance $\|p_{i_j,e} - p_{j,\infty}\| < \epsilon$ for all j and all $e \gg 0$. Replacing E by $(F^e)^*E$ for some large e, we may assume that $\|p_{i_j,e} - p_{j,\infty}\| < \epsilon$ holds for all $e \geq 0$. Since the rank coordinates of $p_{i_j,e}$ and $p_{j,\infty}$ can only take integer values, it must be the case that $r_{j,\infty} = \mathrm{rk}(E_{i_j,e})$.

We first show that there exists an integer e_0 such that $E_{i_1,e} = (F^{e-e_0})^*E_{i_1,e_0}$ for all $e \geq e_0$. Consider the first line segment s of the polygon $\mathrm{HNP}_\infty(E_{i_1,0})$. Note that s cannot lie above the line

segment $\overline{p_{0,\infty}p_{1,\infty}}$ (see Remark 18). We will show that s actually lies on the line segment $\overline{p_{0,\infty}p_{1,\infty}}$. Assuming that is the case, we apply the induction hypothesis to $E_{i_1,0}$ to get: there exists a subsheaf G of $(F^l)^*E_{i_1,0}$ such that $(G, \frac{\deg G}{p^l}) \in s \subset \overline{p_{0,\infty}p_{1,\infty}}$ i.e. G is strongly semistable. Once again, we may replace E by $(F^l)^*E$ to assume that $l = 0$. By induction hypothesis, since the theorem holds for E/G and since $\mu(G) = \mu_{\max}(E) = \frac{d_{1,\infty}}{r_{1,\infty}}$, we also get the theorem for E.

Now suppose that s lies strictly below the line segment $\overline{p_{0,\infty}p_{1,\infty}}$. We will deduce a contradiction from this. Since $p_{i_1,l} \to p_{i_1,\infty}$ as $l \to \infty$, we can find an integer l such that $\overline{p_{0,l}p_{i_1,l}}$ lies above s. Thus, $\mu_{\max}((F^l)^*E) > \mu_{\max}((F^l)^*E_{i_1,0})$ for such an integer l.

Since $\{(F^l)^*E_{i_1,0}, (F^l)^*E_{i_1+1,0}, \ldots, (F^l)^*E\}$ form a filtration of $(F^l)^*E$, using Lemma 16, we get that there exists an integer $j > i_1$ such that

$$\mu_{\max}((F^l)^*(E_{j,0}/E_{j-1,0})) > \mu_{\max}((F^l)^*E_{i_1,0}).$$

Consider a saturated subsheaf $G \subset (F^l)^*(E_{j,0})$ such that $\mu(G/(F^l)^*(E_{j-1,0})) = \mu_{\max}((F^l)^*(E_{j,0}/E_{j-1,0}))$. Then, $(\mathrm{rk}(G), \frac{\deg(G)}{p^l}) \in \mathrm{HNP}_l(E)$. Let us try to estimate the difference in the areas of $\mathrm{HNP}_l(E)$ and $\mathrm{HNP}(E)$. There is a lower bound on the difference in areas by considering the triangle W joining the points $p_{j-1,0}$, $p_{j,0}$ and $(\mathrm{rk}(G), \frac{\deg(G)}{p^l})$. Let $\tilde{\mu}_1 := \frac{\mu((F^l)^*(E_{j,0}/E_{j-1,0}))}{p^l}$ and $\tilde{\mu}_2 := \frac{\mu(G/(F^l)^*E_{j-1,0})}{p^l}$ denote the slopes of the sides of W that contain $p_{j-1,0}$. Using the fact that the rk-coordinates of these points can only take integer values, we can get a get a lower bound on the area of the triangle W by considering the triangle W' joining $p_{j-1,0}$, $p_{j-1,0} + (1, \tilde{\mu}_1)$, $p_{j-1,0} + (1, \tilde{\mu}_2)$ (see Figure 2).
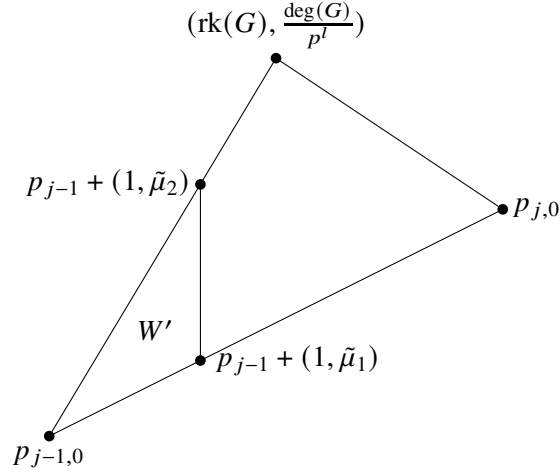

FIGURE 2. Triangles W and W' appearing in the proof of Theorem 29. The larger triangle is W and the smaller triangle inside it is W'.

Thus it follows:

$$\mathrm{Area}(\mathrm{HNP}_l(E)) - \mathrm{Area}(\mathrm{HNP}(E))$$
$$\geq \mathrm{Area}(W')$$
$$\geq \frac{1}{2}(\tilde{\mu}_2 - \tilde{\mu}_1)$$
$$= \frac{1}{2}\left(\frac{\mu(G/(F_X^l)^*(E_{j-1,0}))}{p^l} - \frac{\mu((F_X^l)^*(E_{j,0}/E_{j-1,0}))}{p^l}\right)$$

$$\geq \frac{1}{2}\left(\mu(E_{i_1,0}) - \mu(E_{j,0}/E_{j-1,0})\right)$$

$$\geq \frac{1}{2}((\mu_{1\infty} - \epsilon) - \mu(E_{j,0}/E_{j-1,0})).$$

To finish the estimate, we need to get an upper bound on $\mu(E_{j,0}/E_{j-1,0})$. To do this, let u be such that $i_u < j \leq i_{u+1}$. Then the line segment $\overline{p_{j-1,0}p_{j,0}}$ is sandwiched between the line segments $\overline{p_{i_u,0}p_{i_{u+1},0}}$ and $\overline{p_{i_u,\infty}p_{i_{u+1},\infty}}$ (see Figure 3). Using the fact that $\|p_{i_u,\infty} - p_{i_u 0}\| \leq \epsilon$ and estimating the slope of the line segment $\overline{p_{j-1,0}p_{j,0}}$, we get that $|\mu(E_{j,0}/E_{j-1,0}) - \mu_{u+1,\infty}| \leq 3\epsilon$. Thus, we have
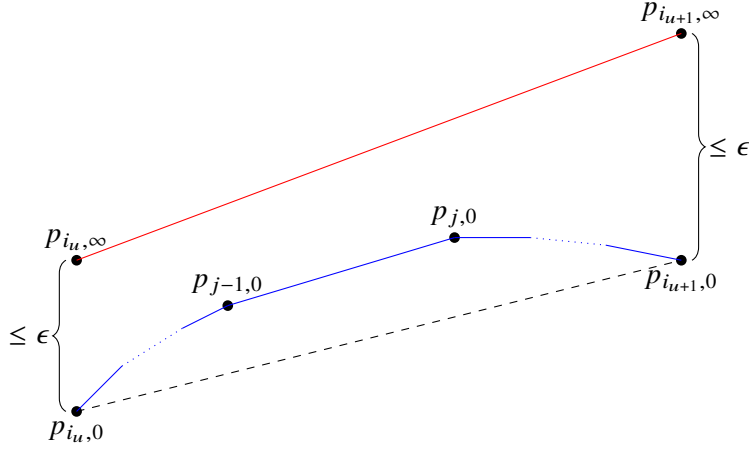

FIGURE 3. Estimating $\mu(E_{j,0}/E_{j-1,0})$. The red line shows the boundary of $\mathrm{HNP}_\infty(E)$ and the blue line shows the boundary of $\mathrm{HNP}(E)$.

$$\mathrm{Area}(\mathrm{HNP}_l(E)) - \mathrm{Area}(\mathrm{HNP}(E)) \geq \frac{1}{2}(\mu_{1\infty} - \mu_{2\infty} - 4\epsilon).$$

But the difference in areas between $\mathrm{HNP}(E)$ and $\mathrm{HNP}_\infty(E)$ is at most $r\epsilon$, which gives us a contradiction for a small enough choice of ϵ. ∎

3. Bogomolov's inequality

In this section, we give several equivalent statements on Bogomolov's inequality, about the positivity of $\Delta(E) \cdot D_2 \cdots D_{n-1}$.

Fix a positive integer r. Let X be a smooth projective variety of dimension $n \geq 2$ over an algebraically closed field k in any characteristic. Note that in the characteristic zero case, we define strong semistability to be the same as the semistability. We fix a nef divisor A over X such that $T_X(A)$ is globally generated. Define the constant $\beta_r = \beta_r(A; D_1, \ldots, D_{n-1})$ for a choice of divisors $(D_1, \ldots, D_{n-1})$ as below

$$\beta_r(A; D_1, \ldots, D_{n-1}) := \begin{cases} 0, & \text{if } \mathrm{char}(k) = 0; \\ \left(\frac{r(r-1)}{p-1}AD_1 \cdots D_{n-1}\right)^2, & \text{if } \mathrm{char}(k) = p. \end{cases}$$

We also reference the *discriminant* of a rank r torsion-free sheaf E, denoted as $\Delta(E)$, is defined as

$$\Delta(E) := 2r \cdot c_2(E) - (r-1)c_1(E)^2.$$

When G is a non-trivial subsheaf of rank s in E, we set

$$\xi_{G,E} = \frac{c_1(G)}{s} - \frac{c_1(E)}{r}.$$

We also use K^+ to denote an open cone in $\text{Num}(X)$, defined as in the beginning of Subsection 3.

The main theorem is the following.

Theorem 31 (BI(r)). *Let $(D_1, \ldots, D_{n-1})$ be a collection of nef line bundles over X, and let $d = D_1^2 \cdot D_2 \cdots D_{n-1} \geq 0$. Fix a positive integer r. Then the following statements are equivalent.*

(i) *Assume each D_i is ample, and E is a strongly $(D_1, \ldots, D_{n-1})$-semistable torsion-free sheaf of rank $r' \leq r$. Then we have*

$$\Delta(E)D_2 \ldots D_{n-1} \geq 0.$$

(ii) *Let E be a $(D_1, \ldots, D_{n-1})$-semistable torsion-free sheaf of rank r' for some $r' \leq r$. Then we have*

$$d \cdot \Delta(E)D_2 \cdots D_{n-1} + \beta_{r'} \geq 0.$$

(iii) *Let E be a torsion-free sheaf of rank r' for some $r' \leq r$ over X, and assume $d \cdot \Delta(E)D_2 \cdots D_{n-1} + \beta_{r'} < 0$. Then there exists a saturated subsheaf E' of E such that $\xi_{E',E} \in K^+$.*

(iv) *Let E be a strongly $(D_1, \ldots, D_{n-1})$-semistable torsion-free sheaf of rank $r' \leq r$ over X. Then*

$$\Delta(E)D_2 \cdots D_{n-1} \geq 0.$$

Remark 32. In terms of the notations as in [Lan04b], the above corresponds to $T^5(r) \Rightarrow T^3(r) \Rightarrow T^4(r) \Rightarrow T^2(r) \Rightarrow T^5(r)$. Note that the last implication from $T^2(r)$ to $T^5(r)$ is trivial, as the statement (i) is the same as (iv) but with a stronger assumption.

Remark 33. In characteristic zero, since $\beta_r = 0$ and the strong semistability is the same as the semistability, the statement above can be more or less combined into one single statement about the non-negativity of $\Delta(E)D_2 \cdots D_{n-1} \geq 0$. This was first proved by Bogomolov in [Bog78], thus the name.

$(i) \Rightarrow (ii)$. In this subsection, our goal is to prove the implication $(i) \Rightarrow (ii)$ in Theorem BI(r) for torsion-free sheaf E of rank r, and we follow [Lan04b, §3.6].

To prove the statement, we start with the following preparation. Let H be an ample line bundle over X, and let $(D_1, \ldots, D_{n-1})$ be as in Theorem BI(r) (ii). Then $(H_1(t), \ldots, H_{n-1}(t)) := (D_1 + tH, \ldots, D_{n-1} + tH)$ is a polarization of ample line bundles.

Lemma 34. *Assume Theorem BI(r) (i). Then for $t > 0$ we have*

$$H_1(t)^2 H_2(t) \cdots H_{n-1}(t) \cdot \Delta(E)H_2(t) \ldots H_{n-1}(t) +$$
$$r^2(L_{max,t}(E) - \mu_t(E))(\mu_t(E) - L_{min,t}(E)) \geq 0.$$

Here we follow the notations L_{max} and L_{min} as in Subsection 2, and denote $L_{max,t}$, $L_{min,t}$ and μ_t as the ones defined using the polarization $(H_1(t), \ldots, H_{n-1}(t))$.

Remark 35. Before we prove the lemma, it is worth noting that the above inequality together with its proof, which only uses Theorem BI(r) (i), applies also to a general polarization $(D_1, \ldots, D_{n-1})$ for ample D_i, in place of $(H_1(t), \ldots, H_{n-1}(t))$. More explicitly, Theorem BI(r), (i) will imply the inequality

$$D_1^2 D_2 \cdots D_{n-1} \cdot \Delta(E)D_2 \cdots D_{n-1} + r^2(L_{max} - \mu)(L_{min} - \mu) \geq 0,$$

where D_i are all ample.

Proof. We first notice that by Theorem 29, there exists $k \in \mathbb{N}$ such that all of the factors in the Harder-Narasimhan filtration of $(F^k)^*E$ are strongly semistable. Denote $0 = E_0 \subset \cdots E_m = (F^k)^*E$ to be the corresponding filtration of $(F^k)^*E$, and let F_i be the quotient E_i/E_{i-1}, $r_i = \mathrm{rk}(F_i)$, and $\mu_{i,t} = \mu_t(F_i)$. Then by the Hodge index theorem, [2] we have

$$\frac{\Delta((F^k)^*E)H_2(t)\cdots H_{n-1}(t)}{r}$$
$$= \sum_i \frac{\Delta(F_i)H_2(t)\cdots H_{n-1}(t)}{r_i} - \frac{1}{r}\sum_{i<j} r_i r_j \left(\frac{c_1(F_i)}{r_i} - \frac{c_1(F_j)}{r_j}\right)^2 H_2(t)\cdots H_{n-1}(t)$$
$$\geq \sum_i \frac{\Delta(F_i)H_2(t)\cdots H_{n-1}(t)}{r_i} - \frac{1}{rd}\sum_{i<j} r_i r_j (\mu_{i,t} - \mu_{j,t})^2.$$

The assumption of Theorem BI(r) (i) and the ampleness of $H_i(t)$ provides us with the inequality

$$\Delta(F_i)H_2(t)\cdots H_{n-1}(t) \geq 0.$$

On the other hand, we have the following elementary inequality on r_i, μ_i (with $r = \sum_i r_i$ and $r\mu = \sum_i r_i \mu_i$) that

(3.1) $$\sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 \leq r^2 (\mu_1 - \mu)(\mu - \mu_m).$$

Combine with the inequality above, we get the inequality in the statement of lemma. ∎

With the above lemma together with the approximation technique as in Subsection 2, we are ready to prove the aforementioned implication.

Proof of Theorem BI(r), (i) ⟹ (ii). Let H be a fixed ample line bundle. By Theorem 19, we can find a filtration of torsion-free coherent subsheaves $0 = E_0 \subset E_1 \subset \cdots \subset E_m = E$ of E such that it is the Harder-Narasimhan filtration with respect to the polarizations $(H_1(t), \ldots, H_{n-1}(t))$, where $t > 0$ is very close to 0. As E is $(D_1, \ldots, D_{n-1})$-semistable, we have

$$\mu(E) \geq \mu(E_1)$$
$$= \lim_{t\to 0^+} \mu_t(E_1)$$
$$\geq \lim_{t\to 0^+} \mu_t(E)$$
$$= \mu(E).$$

By the choice of E_1, the above implies the equality

$$\mu(E) = \lim_{t\to 0^+} \mu_{max,t}(E).$$

Similarly, we have

$$\mu(E) = \lim_{t\to 0^+} \mu_{min,t}(E).$$

On the other hand, Proposition 28 states that for a nef divisor A such that $T_X(A)$ is globally generated, we have

$$L_{max,t}(E) - \mu_{max,t}(E), \ \mu_{min,t}(E) - L_{min,t}(E) \leq \frac{r-1}{p-1} A H_1(t)\cdots H_{n-1}(t).$$

---

[2]Here we are using the form of the Hodge index theorem that $(D^2 D_2 \ldots D_{n-1})\cdot(D_1^2 D_2 \ldots D_{n-1}) \leq (DD_1 D_2 \ldots D_{n-1})^2$, where D_i are nef divisors satisfying $D_1^2 D_2 \cdots D_{n-1} > 0$. This can be proved using the standard Hodge index theorem for ample divisors together with the approximation technique as in Subsection 2.

In particular, the limit of the right hand side of the inequality above is equal to $\frac{r-1}{p-1}AD_1\cdots D_{n-1}$, whose square is $\frac{1}{r^2}\beta_r$. In this way, apply the inequality of Lemma 34 for each $t > 0$, we get

$$D_1^2 D_2 \cdots D_{n-1} \cdot \Delta(E) D_2 \ldots D_{n-1} + \beta_r$$

$$= \lim_{t\to 0^+}\left(D_1^2 D_2 \cdots D_{n-1} \cdot \Delta(E) D_2 \ldots D_{n-1} + r^2\left(\frac{r-1}{p-1}AH_1(t)\cdots H_{n-1}(t)\right)^2\right)$$

$$\geq \lim_{t\to 0^+}\left(H_1(t)^2 H_2(t)\cdots H_{n-1}(t)\cdot\Delta(E)H_2(t)\cdots H_{n-1}(t)\right)$$

$$+ r^2(L_{max,t}(E) - \mu_t(E))(\mu_t(E) - L_{min,t}(E))$$

$$\geq 0.$$

∎

$(ii) \Rightarrow (iii)$. In this subsection, we show the implication $(ii) \Rightarrow (iii)$, following [Lan04b, §3.7]
As a preparation, we define an open cone in $\text{Num}(X)$ as follows.

Definition 36. Let K^+ be the following open cone in $\text{Num}(X)$:

$$K^+ = \{D \in \text{Num}(X) : D^2 D_2 \cdots D_{n-1} > 0, \text{ and } DD'D_2 \cdots D_{n-1} \geq 0$$

$$\text{for all nef } D'\}.$$

The following description is used in the proof.

Lemma 37. *A divisor* $D \in K^+$ *if and only if it satisfies the inequalities* $DLD_2 \cdots D_n > 0$ *for all* $L \in \bar{K}^+ \setminus \{0\}$.

Before we start, we also mention several elementary computational results whose proofs are left as exercises to the reader.

Lemma 38. *Let* $0 \to E' \to E \to E'' \to 0$ *be a short exact sequence of coherent sheaves. Let* r' *(resp.* r''*) be the rank of* E' *(resp.* E''*), and assume they are positive. We have*

$$\frac{\Delta(E)D_2 \cdots D_{n-1}}{r} + \frac{rr'}{r''}\xi_{E',E}^2 D_2 \cdots D_{n-1} = \frac{\Delta(E')D_2 \cdots D_{n-1}}{r'} +$$

$$\frac{\Delta(E'')D_2 \cdots D_{n-1}}{r''}$$

Lemma 39. *Let* $0 \to E' \to E \to E'' \to 0$ *be a short exact sequence of coherent sheaves. Let* r' *(r'') be the rank of* E' *(E'') respectively.*

- *If* G *is a non-trivial subsheaf of* E'*, then*

$$\xi_{G,E} = \xi_{E',E} + \xi_{G,E'}.$$

- *If* $G'' \subset E''$ *is a proper subsheaf of rank* s*, and denote by* G *the kernel of the map* $E \to E''/G''$*, then*

$$\xi_{G,E} = \frac{r'(r''-s)}{(r'+s)r''}\xi_{E',E} + \frac{s}{r'+s}\xi_{G'',E''}.$$

Proof of Theorem BI(r), $(ii) \Rightarrow (iii)$. We prove (iii) inductively on $\text{rk}(E) = r$, assuming the statement (ii). As a starting point, we notice that when E is of rank one, as $\Delta(E) = 0$ and $\beta_1 \geq 0$, the statement (iii) is automatically true. In general, the assumption of Theorem BI(r) (ii) tells us that E is not semistable with respect to $(D_1, \ldots, D_{n-1})$. Thus we have the maximal destabilizing

subsheaf $E' \subset E$ with respect to the polarization. Let $E'' = E/E'$, and r', r'' be the ranks of E', E'' respectively. Then by Lemma 38 we have

$$\frac{\Delta(E)D_2 \cdots D_{n-1}}{r} + \frac{rr'}{r''}\xi^2_{E',E}D_2 \cdots D_{n-1} =$$

$$\frac{\Delta(E')D_2 \cdots D_{n-1}}{r'} + \frac{\Delta(E'')D_2 \cdots D_{n-1}}{r''}.$$

We also have

$$\frac{\beta_r}{r} \geq \frac{\beta_{r'}}{r'} + \frac{\beta_{r''}}{r''}.$$

So by multiplying $d = D_1^2 D_2 \cdots D_{n-1}$, either $\xi^2_{E',E}D_2 \cdots D_{n-1} > 0$, or one of $d\Delta(E')D_2 \cdots D_{n-1} + \beta_{r'}$ or $d\Delta(E'')D_2 \cdots D_{n-1} + \beta_{r''}$ is negative.

In the first case above, we claim that $\xi_{E',E} \in K^+$. To see this, by the definition of K^+ and the assumption on $\xi_{E',E}$ it suffices to show that for any $L \in K^+\backslash\{0\}$ we have $\xi_{E',E}LD_2 \cdots D_{n-1} > 0$. Note first the assumption of E' and E implies that the inequality is true for $L = D_1$, and it reduces to show the sign of the function $L \mapsto \xi_{E',E}LD_2 \cdots D_{n-1}$ is not changing on K^+. This then follows from the continuity of the function, the connectivity of K^+, and the Hodge Index Theorem that

$$(\xi_{E',E}LD_2 \cdots D_{n-1})^2 \geq \xi^2_{E',E}D_2 \cdots D_{n-1} \cdot L^2 D_2 \cdots D_{n-1} > 0,$$

for $L \in K^+\backslash\{0\}$.

Suppose now $\xi^2_{E',E}D_2 \cdots D_{n-1} \leq 0$ and we are in the second case above. Note that since both E' and E'' are of smaller ranks, we can apply the induction hypotheses of ranks to get the following dichotomy:

- There is a saturated subsheaf $G \subset E'$ such that $\xi_{G,E'} \in K^+$, or
- There is a saturated subsheaf $G'' \subset E''$ such that $\xi_{G'',E''} \in K^+$.

Applying Lemma 39, we get in either case that $\xi_{G,E}$ is a positive linear combination of $\xi_{E',E}$ and some element L in K^+.

To proceed, define the open subcone $C(\xi) := \{D \in \overline{K^+} \backslash \{0\} : \xi \cdot DD_2 \cdots D_{n-1} > 0\} \subset \overline{K^+}$ for a given $\xi \in \mathrm{Num}(X)$. We observe that by replacing E' with G in either case, we get strictly larger cones than before: $C(\xi_{E',E}) \subsetneq C(\xi_{G,E})$. Here the inclusion part is clear, as $\xi_{G,E}$ is the sum of $\xi_{E',E}$ with some $L \in K^+$, where the second term only contributes positive values in the product $\xi_{G,E}DD_2 \cdots D_{n-1}$. To see the inclusion is strict, as $\xi_{E',E}$ is not in K^+, by Lemma 37 there exists some $L' \in \overline{K^+}\backslash\{0\}$ such that $\xi_{E',E}L'D_2 \cdots D_{n-1} \leq 0$ but $LL'D_2 \cdots D_{n-1} > 0$. On the other hand, as E' is the maximal destabilizing subsheaf of E with respect to the polarization $(D_1, \ldots, D_{n-1})$, we have $\xi_{E',E}D_1 \cdots D_{n-1} > 0$ and $LD_1 \cdots D_{n-1} > 0$, where the latter follows as $L \in K^+$. In this way, there exists some real number $t \in [0, 1)$ such that $\xi_{E',E}(tD_1 + (1-t)L')D_2 \cdots D_{n-1} = 0$, and thus

$$\xi_{G,E}(tD_1 + (1-t)L')D_2 \cdots D_{n-1}$$
$$= \xi_{E',E}(tD_1 + (1-t)L')D_2 \cdots D_{n-1}$$
$$+ L(tD_1 + (1-t)L')D_2 \cdots D_{n-1}$$
$$= 0 + L(tD_1 + (1-t)L')D_2 \cdots D_{n-1}$$
$$> 0 = \xi_{E',E}(tD_1 + (1-t)L')D_2 \cdots D_{n-1}.$$

So the element $tD_1 + (1-t)L'$ is in $C(\xi_{G,E})$ but not in $C(\xi_{E',E})$. In this way, we get a sequence of strictly increasing subcones of $\overline{K^+}$ by replacing E' by G, until we reach to the situation where $\xi^2_{E',E}D_2 \cdots D_{n-1} > 0$.

It remains to show that this process terminates within finite replacements. First note that by choosing ample $\mathbb{R}$-basis $H_1, \cdots, H_\rho$, which are contained in $C(\xi_{E',E})$ for $\text{Num}(X)$, [3] we have

$$\xi_{G,E} \in \frac{1}{r!}(\mathbb{Z}H_1 + \cdots + \mathbb{Z}H_\rho).$$

Furthermore, we also have

$$0 < \xi_{G,E}H_jD_2 \cdots D_{n-1} < \mu_{\max}^j(E) - \mu^j(E),$$

for all $j = 1, 2, \cdots, \rho$, where μ^j denote the slopes with respect to $(H_j, D_2, \cdots, D_{n-1})$. Thus $\xi_{G,E}$ is in fact contained in a bounded discrete hence finite subset of $\text{Num}(X)$. ∎

$(iii) \Rightarrow (iv)$. In this section, we follow [Lan04b, §3.8] to show the implication $(iii) \Rightarrow (iv)$.

We follow the definition of β_r as in the beginning of this section, and the definition of the cone K^+ with respect to the polarization $(D_1, \ldots, D_{n-1})$ as in the last subsection.

By the definition of $\xi_{E',E}$ and K^+, the statement (iii) implies that E is not semistable with respect to $(D_1, \ldots, D_{n-1})$.

Remark 40. The implication $(iii) \implies (iv)$ is easy in $\text{char}(k) = 0$, by the definition of K^+ applying at $\xi_{E',E}D_1 \cdots D_{n-1}$.

Proof of Theorem BI(r), $(iii) \Rightarrow (iv)$. Assume E is a torsion-free sheaf such that $\Delta(E)D_2 \cdots D_{n-1} < 0$. We will deduce that E is not strongly semistable. We would like to apply Theorem BI(r), (iii) to the Frobenius pullback $(F^l)^*E$ of E. Since $\Delta((F^l)^*E) = p^{2l}\Delta(E)$, we get that

$$D_1^2D_2 \cdots D_{n-1} \cdot \Delta((F^l)^*E)D_2 \cdots D_{n-1} + \beta_r =$$
$$D_1^2D_2 \cdots D_{n-1} \cdot p^{2l} \cdot \Delta(E)D_2 \cdots D_{n-1} + \beta_r$$

is negative when l is large enough. By the assumption of Theorem BI(r), (iii) we can find a saturated subsheaf E' of $(F^l)^*E$ with $\xi_{E',(F^l)^*E} \in K^+$. By the definition of K^+ as in Definition 36, we have

$$\xi_{E',(F^l)^*E}D_1 \cdots D_{n-1} > 0,$$

in particular, the pullback $(F^l)^*E$ is not semistable. Hence the torsion-free sheaf E itself is not strongly semistable. ∎

4. Restriction to hypersurfaces and Bogomolov's inequality

In this section, we prove one of the main technical ingredients Theorem 41, which controls the change of various numerical invariants as one passes to hypersurfaces. Following this, we complete the induction schema sketched in the introduction. We follow the proof as in [Lan04b, §3.5, §3.9] and [Lan04a]. Combining the section with the equivalent statements in Theorem BI(r), we finish the proof of all theorems in [Lan04b, Section 3].

Fix a positive integer r, and let X be a smooth projective variety over k. Consider the following statement.

Theorem 41 (Res(r)). *Let E be a torsion-free sheaf of rank r over X, and let $(D_1, \ldots, D_{n-1})$ be a collection of nef divisors over X with $d := D_1^2D_2 \cdots D_{n-1} \geq 0$.*

---

[3]Note that since $C(\xi_{E'E})$ is an open subcone in $K^+ \subset \text{Num}(X)$ that is non-empty (it contains D_1), such a basis of H_i exists.

Assume that D_1 is very ample and the restriction of E to a very general divisor $D \in |D_1|$ is not semistable (with respect to $(D_2|_D, ..., D_{n-1}|_D)$). Let μ_i (r_i) denote the slopes (ranks) of the Harder-Narasimhan filtration of $E|_D$. Then

$$\sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 \le d\Delta(E)D_2...D_{n-1} + 2r^2(L_{max} - \mu)(\mu - L_{min}).$$

Our goal this section is to show the following two implications

$$Res(r) + BI(r-1) \implies BI(r) \implies Res(r+1).$$

Note that since both Theorem Res(r) and Theorem BI(r) are empty and thus automatically true for $r = 1$, the above induction process proves that both Res(r) and BI(r) hold true for all $r \ge 1$.

$Res(r) + BI(r-1) \Rightarrow BI(r)$. In this subsection, we consider the implication $Res(r) + BI(r-1) \Rightarrow BI(r)$, following [Lan04b, §3.5] and [Lan04a]. This is achieved via an inductive argument on the dimension of X. We first prove the case when X is a surface. Further, we do not need to assume $BI(r-1)$ for the base case.

Proposition 42. *Let X be a surface. Assume Theorem Res(r) holds for all torsion-free sheaves of rank $\le r$, then Theorem BI(r) holds for all torsion-free sheaves of rank $\le r$ and all ample divisor D_1.*

Proof. By the equivalence of statements in Theorem BI(r), it suffices to show the assumption implies the statement in Theorem BI(r), (i), i.e. the inequality $\Delta(E) \ge 0$, for a strongly D_1-semistable torsion-free sheaf E of rank $\le r$, where D_1 is ample.

Let E be a rank r vector bundle that is strongly D_1-semistable. Suppose by way of contradiction that $\Delta(E) < 0$. We may further assume that E is locally free, by the inequality

$$\Delta(E^{**}) = \Delta(E) - 2r \operatorname{length}(E^{**}/E),$$

where the reflection E^{**} is locally free on X. We first claim that under the assumption, the restriction to a general curve $C \in |D_1|$ is also strongly semistable. To see this, since E is strongly semistable, $\mu = L_{max} = L_{min}$, and since (by assumption) $\Delta(E) < 0$, from the assumption we would get a contradiction

$$0 \le \sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 < 0,$$

where we use the notation in Res(r). Similarly, as $(F^k)^*E$ is semistable, its restriction to a general curve C is also semistable. Thus, the restriction of E to a very general curve C is strongly semistable. For the remainder of the proof, fix such a very general C.

Now consider the symmetric power $S^{kr}E|_C$. The strategy is to compute $\chi(S^{kr}E)$ in two different ways and use them to deduce a contradiction. Since symmetric powers of strongly semistable sheaves on curves are strongly semistable (see [HL10, Corollary 3.2.10] for the characteristic zero case and [Miy87, Sections 3 and 5] for the positive characteristic case), it follows that $S^{kr}E|_C$ is strongly semistable. Consider the short exact sequence arising from C:

$$0 \to S^{kr}E(-kc_1(E) - C) \to S^{kr}E(-kc_1(E)) \to S^{kr}E(-kc_1(E))|_C \to 0.$$

This allows us to estimate

$$h^0(S^{kr}E(-kc_1(E))) \le h^0(S^{kr}E(-kc_1(E) - C)) + h^0(S^{kr}E(-kc_1(E))|_C).$$

Consider the first term on the right hand side. This quantity is equal to $\dim\big(\mathrm{Hom}(O_X(kc_1(E)+C), S^{kr}E)\big)$, and we claim that this number is zero. Note that the slope of $O_X(kc_1(E)+C)$ is given by $kr\mu(E)+D_1^2$, and by the splitting principle, the slope of the symmetric power can be seen to be

$$\mu(S^{kr}(E)) = \frac{\binom{r+kr-1}{kr}kc_1(E)D_1}{\binom{r+kr-1}{kr}} = kr\mu(E).$$

Since D_1 is ample, $D_1^2 > 0$, so the claim follows as there are no morphisms from semistable sheaves of higher slope to those of lower slope. Thus we get $h^0(S^{kr}E(-kc_1(E)) \le h^0(S^{kr}E(-kc_1(E))|_C)$.

Given any semistable vector bundle G on a curve, we have the further estimate $h^0(G) \le \max\{0, \deg G + \mathrm{rk}\, G\}$. Using the sequence

$$0 \longrightarrow G(-lP) \longrightarrow G \longrightarrow G|_{lP} \longrightarrow 0,$$

the above inequality can be seen from the estimation $h^0(G) \le h^0(G \otimes O_C(-lP)) + \mathrm{rk}(G)l = 0 + \mathrm{rk}(G)l \le \max\{0, \mathrm{rk}G + \deg G\}$, where P is a point on C, $l = \max\{0, \lceil \mu(G) \rceil\}$, and $h^0(G \otimes O_C(-lP)) = 0$ as $\mu(O_C(lP)) > \mu(G)$.

Thus, we have $h^0(S^{kr}E(-kc_1(E)) = O(k^r)$. By Serre duality, we have the same order of magnitude estimate for $h^2(S^{kr}E(-kc_1(E)))$. On the other hand, the splitting principle for Chern classes and Hirzebruch-Riemann-Roch theorem can be used to get the following estimate (for example, see [Bog78, Section 10])

$$\chi(X, S^{kr}E(-kc_1(E))) = -\frac{r^r\Delta(E)}{2(r+1)!}k^{r+1} + O(k^r).$$

Since $\Delta(E) < 0$, this polynomial is eventually positive and is of order k^{r+1}, but this is an obvious contradiction as h^0 and h^2 are of order k^r. ∎

To show the implication for general X, we induct on the dimension of X, using the above surface case as the base case. However, different from the assumption in Proposition 42, for this to work we require the additional assumption of $BI(r-1)$ (i.e. Theorem $BI(r-1)$). Precisely, we want to show that assuming

- $BI(r-1)$ and $Res(r)$ for X of any dimension;
- $BI(r)$ for varieties of dimension $< n$,

then we have $BI(r)$ holds for any variety X of dimension n.

Proof of Theorem $BI(r)$. Similar to the proposition above, we aim to prove Theorem $BI(r)$, (i) for X of dimension n, namely the inequality $\Delta(E)D_2 \cdots D_{n-1} \ge 0$ for a strongly $(D_1, \ldots, D_{n-1})$-semistable torsion-free sheaf E over X, where D_i are all ample.

Assume to the contrary that $\Delta(E)D_2 \ldots D_{n-1} < 0$. For a general divisor in $|D_2|$ (which we also denote as D_2), we have $\Delta(E|_{D_2})D_3 \ldots D_{n-1} = \Delta(E)D_2 \ldots D_{n-1} < 0$. Consider the polarization $(D_2^2, D_3, \ldots, D_{n-1}) := (D_2, D_2, D_3, \ldots, D_{n-1})$. We first assume that E is strongly $(D_2^2, D_3, \ldots D_{n-1})$-semistable. Applying $Res(r)$ to the Frobenius twists $(F^k)^*E$ with respect to $(D_2^2, \ldots, D_{n-1})$, then if $(F^k)^*E|_{D_2}$ is not semistable we would get the inequality

$$\sum_{i<j} r_ir_j(\mu_i - \mu_j)^2 \le dp^k\Delta(E)D_2 \cdots D_{n-1} + 0 < 0,$$

which is impossible. Thus $E|_{D_2}$ is strongly semistable for ample divisors $(D_2, \ldots D_{n-1})$, and by the induction hypothesis, since $E|_{D_2}$ is defined over a hypersurface of X we have $\Delta(E|_{D_2})D_3 \ldots D_{n-1} = \Delta(E)D_2 \ldots D_{n-1} \ge 0$, a contradiction.

For the rest, we assume E is not strongly $(D_2^2, D_3, \ldots, D_{n-1})$-semistable. The trick is to interpolate between the polarizations $(D_1, D_2, \ldots, D_{n-1})$ and $(D_2^2, D_3, \ldots, D_{n-1})$. Let B_t be the product $((1-t)D_1 + tD_2)D_2 \ldots D_{n-1}$. By assumption, E is strongly B_0-semistable while it is not strongly B_1-semistable. Since not being strongly semistable is an open condition with respect to t, there exists a number $t_k \in [0, 1)$ such that E is strongly B_{t_k}-semistable but not strongly B_t-semistable for all $t_k < t \le 1$.

Let k be a positive integer such that $(F^k)^*E$ is not semistable for B_1. Using Theorem 19, we see that the Harder-Narasimhan filtration of E with respect to B_t remains constant for all $t_k < t < t_k + \epsilon$ for some $\epsilon > 0$ small enough. We let E' denote the maximal destabilizing sheaf of $(F^k)^*E$ with respect to the polarization B_t for $t_k < t < t_k + \epsilon$. By the continuity of the slope with respect to t, it follows that E' and $(F^k)^*E$ have the same B_{t_k}-slope (see Lemma 21). Since E' is a subsheaf of a semistable sheaf with the same slope, E' is also semistable with respect to B_{t_k}. It also follows that the quotient $E'' = (F^k)^*E/E'$ is B_{t_k}-semistable as well. Let r, r' and r'' denote the ranks of E, E' and E'' respectively.

Now we apply Lemma 38 to the short exact sequence $E' \to E \to E''$ to get that

$$\frac{\Delta((F^k)^*E)D_2 \ldots D_{n-1}}{r} = \frac{\Delta(E')D_2 \ldots D_{n-1}}{r'} + \frac{\Delta(E'')D_2 \ldots D_{n-1}}{r''} -$$
$$\frac{r'r''}{r}\xi_{E',E''}^2 D_2 \ldots D_{n-1}.$$

Note that by the Hodge index theorem, we have

$$\xi_{E',E''}^2 D_2 \ldots D_{n-1} \cdot (t_k D_1 + (1 - t_k)D_2)^2 D_2 \ldots D_{n-1} \le (\xi_{E',E''} B_{t_k})^2.$$

Since E', E'' have the same B_{t_k}-slope, $(\xi_{E',E''} B_{t_k})^2 = 0$ and by ampleness of $D_1, \ldots, D_{n-1}$, we have that

$$d(t_k) := (t_k D_1 + (1 - t_k)D_2)^2 D_2 \ldots D_{n-1} > 0.$$

Moreover by Theorem BI(r), (ii) but for $(r - 1)$ in place of r, we get that the inequalities as below

$$\frac{\Delta((F^k)^*E)D_2 \ldots D_{n-1}}{r} \ge -\frac{1}{d(t_k)}\left(\frac{\beta_{r'}(t_k)}{r'} + \frac{\beta_{r''}(t_k)}{r''}\right) \ge -\frac{\beta_r(t_k)}{rd(t_k)},$$

where we denote $\beta_r(t)$ to be $\beta_r((1 - t)D_1, +tD_2, D_2, \ldots, D_{n-1})$. In this way, we get

$$\Delta(E)D_2 \cdots D_{n-1} \ge -\frac{\beta_r(t_k)}{d(t_k)p^{2k}}.$$

Notice that the function $\frac{\beta_r(t)}{d(t)}$ for $t \in [0, 1]$ is defined and continuous, thus bounded. Taking the limit as k approaches infinity, we get the inequality

$$\Delta(E)D_2 \cdots D_{n-1} \ge 0,$$

which is a contradiction, so we are done.   ∎

Remark 43. Pointed out by the referee, the last paragraph of the proof above can be argued as follows. As E' and E'' are of the same B_{t_k}-slopes as the strongly B_{t_k}-semistable sheaf E, both of them are strongly B_{t_k}-semistable. Thus applying the induction hypothesis of Theorem BI(r), (i) to E', E'' for the polarization B_{t_k}, we see

$$\frac{\Delta(E')D_2 \ldots D_{n-1}}{r'} + \frac{\Delta(E'')D_2 \ldots D_{n-1}}{r''} \ge 0.$$

In this way, by the same vanishing of $\xi_{E',E''}D_2 \cdots D_{n-1}$ proved above, we get the positivity of $\Delta(E)D_2 \cdots D_{n-1}$.

$BI(r) \Rightarrow Res(r + 1)$. We then consider the implication $BI(r) \Rightarrow Res(r+1)$, following [Lan04b, §3.9].

Let r be a non-negative integer, and $(D_1, \ldots, D_{n-1})$ be a nef polarization over X with $d = D_1^2 D_2 \cdots D_{n-1} \geq 0$.

Proof. Let Π be the projective space associated to the linear system $|D_1|$. Let $Z \subset \Pi \times X$ be the incident scheme, defined as the locus $\{(D, x) \in \Pi \times X \mid x \in D\}$, where $p : Z \to \Pi$ and $q : Z \to X$ are the two natural maps induced by the projections. For each $s \in \Pi$, we let Z_s be its fiber along the map q, which is a divisor in X. Let $0 = E_0 \subset E_1 \subset \cdots \subset E_m = q^*E$ be the Harder–Narasimhan filtration of q^*E associated to the polarizaion

$$(p^*O_\Pi(1)^{\dim \Pi}, q^*D_2, \ldots, q^*D_{n-1}) =$$
$$(p^*O_\Pi(1), \ldots, p^*O_\Pi(1), q^*D_2, \ldots, q^*D_{n-1})$$

over Z. Denote F_i to be the subquotient E_i/E_{i-1}. Here we observe that the pull back of the filtration $\{E_i\}$ at a general fiber Z_s for $s \in \Pi$ is the Harder–Narasimhan filtration of $E|_{Z_s}$ at the hypersurface $Z_s \in |D|$. Indeed, the slope of q^*E with respect to $(p^*O_\Pi(1)^{\dim \Pi}, q^*D_2, \ldots, q^*D_{n-1})$ is the same as the slope of $q^*E|_{Z_s}$. This shows that the filtration $\{E_i\}$ is also the relative Harder-Narasimhan filtraion for q^*E with respect to the map p. In particular, we have the equalities $r_i = \mathrm{rk}F_i$ and $\mu_i = \mu(F_i)$, where the latter is with respect to the polarization given by $(p^*O_\Pi(1)^{\dim \Pi}, q^*D_2, \ldots, q^*D_{n-1})$.

Now let $\mathbb{P}^1 \cong \Lambda \subset \Pi$ be a sufficiently general pencil, corresponding to a linear subsystem of $|D|$ in X parametrized by a line $\mathbb{P}^1 \subset \Pi$. Let $B \subset X$ be the base of Λ, which is the intersection of Z_s for any two (hence all) $s \in \Lambda$, and is of codimension 2 in X. Denote by Y the incident scheme for Λ, which is equal to the closed subscheme $p^{-1}\Lambda$ inside of Z. Here by the construction of the incident scheme for a pencil, the projection map $Y = p^{-1}\Lambda \to X$ coincides with the blowup of X at the base locus B. Moreover, depending on the dimension of X, we can write the first Chern class of the restriction $F_i|_Y$ in terms of the following:

- If $\dim(X) = 2$, then B consists of a union of a finite amount of points, and we may write each $c_1(F_i|_Y)$ as $M_i + \sum_j b_{ij}N_j$, with $M_i \in \mathrm{Pic}(X)$ and N_j being the j-th exceptional divisor for $1 \leq j \leq l$. We define b_i to be the number $\frac{\sum_j b_{ij}}{l}$.
- If $\dim(X) \geq 3$, then by Bertini's theorem the blowup center B is a smooth connected closed subscheme of codimension two in X, and we may write each $c_1(F_i|_Y)$ as $M_i + b_iN$, for $M_i \in \mathrm{Pic}(X)$ and N being the exceptional divisor.

In any of the above cases, with respect to the polarization

$$(p^*O_\Lambda(1), q^*D_2, \ldots, q^*D_{n-1})$$

over Y, we can write the slope μ_i as the following

$$(0) \qquad \mu_i = \frac{M_i D_1 \cdots D_{n-1} + b_i d}{r_i}.$$

Here as the collection of torsion-free sheaves $\{F_i\}$ forms the graded pieces of a filtration of q^*E, we have

$$(1) \qquad \sum_i b_i = c_1(q^*E)p^*O_\Lambda(1)q^*D_2 \cdots q^*D_{n-1} = 0.$$

Furthermore, since $(q|_Y)_*(E_i|_Y) \subset E$, we have the inequalities

$$\frac{\sum_{j \leq i} M_j D_1 \cdots D_{n-1}}{\sum_{j \leq i} r_j} \leq \mu_{\max}, \text{ for all } i,$$

which is equivalent to the following inequalities

$$(2) \qquad \sum_{j \leq i} b_j d \geq \sum_{j \leq i} r_j (\mu_j - \mu_{\max}).$$

Finally, we are ready to prove the implication. As each F_i is of rank $\leq r - 1$, by Theorem BI(r), (i) for the polarization $(p^* O_\Pi(1)^{\dim \Pi}, q^* D_2, \ldots, q^* D_{n-1})$ over Z, we have

$$\Delta(F_i) p^* O_\Pi(1)^{\dim \Pi} q^* D_2 \cdots q^* D_{n-1} =$$

$$\Delta(F_i|_Y) p^* O_\Lambda(1) q^* D_2 \cdots q^* D_{n-1} \geq 0, \text{ for all } i.$$

Applying this, we get

$$\frac{d \Delta(E) D_2 \cdots D_{n-1}}{r}$$

$$= \sum_i \frac{d \Delta(F_i|_Y) p^* O_\Lambda(1) q^* D_2 \cdots q^* D_{n-1}}{r_i}$$

$$- \frac{d}{r} \sum_{i<j} r_i r_j \left(\frac{c_1(F_i|_Y)}{r_i} - \frac{c_1(F_j|_Y)}{r_j} \right)^2 (q|_Y)^* (D_2 \cdots D_{n-1})$$

$$\geq -\frac{d}{r} \sum_{i<j} r_i r_j \left(\frac{c_1(F_i|_Y)}{r_i} - \frac{c_1(F_j|_Y)}{r_j} \right)^2 (q|_Y)^* (D_2 \cdots D_{n-1})$$

$$= \frac{d}{r} \sum_{i<j} r_i r_j \left(d(\frac{b_i}{r_i} - \frac{b_j}{r_j})^2 - (\frac{M_i}{r_i} - \frac{M_j}{r_j})^2 D_2 \cdots D_{n-1} \right)$$

$$\geq \frac{1}{r} \sum_{i<j} r_i r_j \left(d^2 (\frac{b_i}{r_i} - \frac{b_j}{r_j})^2 - (\frac{M_i D_1 \ldots D_{n-1}}{r_i} - \frac{M_j D_1 \cdots D_{n-1}}{r_j})^2 \right),$$

where the first inequality is the application of the statement (i) as above, the equality after is a rearrangement using the formula (0), and the last inequality follows from the Hodge index theorem applied to $\frac{M_i}{r_i} - \frac{M_j}{r_j}$ and the polarization $D_1, \cdots, D_{n-1}$. Moreover, using the formula (0), we may rewrite the last expression of the inequalities above as

$$\frac{2d}{r} \sum_{i<j} (\mu_i - \mu_j)(b_i r_j - b_j r_i) - \frac{1}{r} \sum_{i<j} r_i r_j (\mu_i - \mu_j)^2$$

Using (1), one further simplifies the above to

$$2 \sum_i d b_i \mu_i - \frac{1}{r} \sum_{i<j} r_i r_j (\mu_i - \mu_j)^2.$$

Here more concretely, the simplification above can be seen via an induction on m. If $m = 1$, this is obvious. Suppose this is true for $m - 1$. Then

$$\frac{2d}{r} \sum_{i<j} (\mu_i - \mu_j)(b_i r_j - b_j r_i)$$

$$= 2 \sum_{i<j \leq m-1} (\mu_i - \mu_j)((b_i + \frac{b_m}{m}) r_j - (b_j + \frac{b_m}{m}) r_i)$$

$$+ \frac{b_m}{m} \sum_{i<j \leq m-1} (\mu_i - \mu_j)(r_i - r_j) + \frac{2d}{r} \sum_{i \leq m-1} (\mu_i - \mu_m)(b_i r_m - b_m r_i)$$

$$= \frac{2(r - r_m)}{r} \sum_{i \leq m-1} d(b_i + \frac{b_m}{m})\mu_i$$

$$+ \frac{2d}{r} \left(r_m \sum_{i \leq m-1} b_i \mu_i - b_m(\mu r - \mu_m r_m) + b_m \mu_m r_m + b_m \mu_m (r - r_m) \right)$$

$$+ \frac{b_m}{m} \sum_{i < j \leq m-1} (\mu_i - \mu_j)(r_i - r_j)$$

$$= 2 \sum_i db_i \mu_i$$

where the second equality follows from induction hypothesis. Furthermore, using the formula (2) together with an elementary equality

$$\sum_i a_i b_i = \sum_i (\sum_{j \leq i} a_j)(b_i - b_{i+1}),$$

we get

$$\sum_i db_i \mu_i = \sum_i (\sum_{j \leq i} db_j)(\mu_i - \mu_j)$$

$$\geq \sum_i \left(\sum_{j \leq i} r_j(\mu_j - \mu_{\max})(\mu_i - \mu_{i+1}) \right)$$

$$= \sum_i (r_i \mu_i^2 - r\mu\mu_{\max})$$

$$\geq \sum_i r_i \mu_i^2 - r\mu^2 + r(\mu - \mu_{\max})(\mu - \mu_{\min})$$

$$= \sum_{i < j} \frac{r_i r_j}{r}(\mu_i - \mu_j)^2 + r(\mu - \mu_{\max})(\mu - \mu_{\min}).$$

As a consequence, we get

$$\frac{d\Delta(E)D_2 \cdots D_{n-1}}{r} \geq \sum_{i < j} \frac{r_i r_j}{r}(\mu_i - \mu_j)^2 + 2r(\mu - \mu_{\max})(\mu - \mu_{\min}).$$

At last, by moving the second term in the right hand side above and the inequalities $L_{\max} \geq \mu_{\max}$, $\mu_{\min} \geq L_{\min}$, we get the one as in Theorem Res(r). ∎

5. BOUNDEDNESS OF TORSION-FREE SHEAVES

At last, we use the ingredients in the last two sections to show the boundedness, following [Lan04b, Section 4]

We start with the following result combining the inequalities in the last two sections into a uniform one, following [Lan04b, Corollary 3.11].

Corollary 44. *Let E be a torsion-free sheaf on X and D_1 be very ample, $D_2, \ldots, D_{n-1}$ be ample, and let $D \in |D_1|$ be a general divisor. Then, we have that*

$$\frac{r}{2}(L_{\max}(E|_D) - L_{\min}(E|_D))^2 \leq d\Delta(E)D_2 \ldots D_{n-1} + 2r^2(L_{\max} - \mu)(\mu - L_{\min})$$

Proof. First consider the case when $E|_D$ is not strongly semistable. Then, the inequality follows from Theorem $\text{Res}(r)$ as well as an elementary inequality [Lan04b, Lemma 1.3] as below:

$$\sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 \geq \frac{r_1 r_m}{r_1 + r_m} r (\mu_1 - \mu_m)^2 \geq \frac{r}{2} (\mu_1 - \mu_m)^2$$

where $r_1, \ldots, r_m$ are positive real numbers and $\mu_1, \ldots, \mu_m$ are real numbers and $r = r_1 + \cdots + r_m$.

Now consider the case when $E|_D$ is strongly semistable. Then, the left-hand side of the inequality is just zero and the result follows from the inequality in Remark 35 (where the latter is proved using Theorem $\text{BI}(r)$, (i)). ∎

The boundedness is deduced from the following result. A more precise version is given in [Lan04b].

Theorem 45. *Let $H_1, \ldots, H_{n-1}$ be very ample divisors on X and let $X_l = |H_1| \cap \cdots \cap |H_l|$, $1 \leq l \leq n-1$ be very general complete intersections. Pick a nef divisor A such that $T_{X_l}(A)$ is globally generated for all $0 \leq l \leq n-1$. Set $\beta_r = \beta(r; A, H_1, \ldots, H_{n-1})$. Let $\mu_{\max,l}, \mu_{\min,l}$ denote the maximal and minimal slopes of the Harder-Narasimhan filtration of $E|_{X_l}$. Then we have the following inequality,*

$$\mu_{\max,l} - \mu_{\min,l} \leq$$

$$C(r,n) \left(\sqrt{\max\{d\Delta(E)H_2 \ldots H_{n-1}, 0\}} + \sqrt{\beta_r} + (\mu_{\max} - \mu_{\min}) \right),$$

where $C(r,n)$ is a constant depending only on r and n.

Remark 46. Before we move on to the proof, it is worth mentioning that except for $\mu_{\max} - \mu_{\min}$, the rest of terms in the right hand side together with $\mu_{\min,l}$ can all be bounded using constants depending only on X and the Hilbert polynomial of E.

Proof. For $l = 1$, we use Corollary 44, and get

$$(L_{\max,1} - L_{\min,1})^2 \leq \frac{2}{r} d\Delta(E)H_2 \ldots H_{n-1} + 4r(L_{\max} - \mu)(\mu - L_{\min})$$

$$\leq \frac{2}{r} \max\{d\Delta(E)H_2 \ldots H_{n-1}, 0\} + 4r(L_{\max} - L_{\min})^2$$

$$\leq 4r \left(\max\{d\Delta(E)H_2 \ldots H_{n-1}, 0\} + (L_{\max} - L_{\min})^2 \right)$$

Now using the definition and Proposition 28 that $\mu_{\max} - \mu_{\min} \leq L_{\max} - L_{\min} \leq \mu_{\max} - \mu_{\min} + \frac{2\sqrt{\beta_r}}{r}$, we get

$$(\mu_{\max,1} - \mu_{\min,1})^2$$

$$\leq 4r \left(\max\{d\Delta(E)H_2 \ldots H_{n-1}, 0\} + (\mu_{\max} - \mu_{\min} + 2\sqrt{\beta_r})^2 \right)$$

Using $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b \geq 0$, we get

$$\mu_{\max,1} - \mu_{\min,1}$$

$$\leq 4\sqrt{r} \left(\sqrt{\max\{d\Delta(E)H_2 \ldots H_{n-1}, 0\}} + (\mu_{\max} - \mu_{\min}) + \sqrt{\beta_r} \right),$$

which gives the required inequality for $l = 1$. For higher l, we use induction. ∎

Now we are ready to prove the main theorem of the article, which states that the moduli space, parametrizing torsion-free sheaves with a fixed Hilbert polynomial, and with a upper-bound for the slopes of their Harder-Narasimhan filtrations, is bounded.

Theorem 47. *Let X be a projective variety over an algebraically closed field with an ample line bundle $O_X(1)$. Let P be a polynomial of degree d and let $\mu_0 \in \mathbb{R}$. Then, the family of torsion-free sheaves whose Hilbert polynomial are P and whose $\mu_{\max}$ are at most μ_0 is bounded.*

Proof. We use the same notation as in Theorem 45. According to Kleiman's criteria (Theorem 14) and Lemma 15, to prove boundedness, it is enough to give an upper bound of $\mu_{\max,l}$ (or more specifically for $l = 1$). Note that $\mu_l = \mu(E|_{X_l})$ is independent of E and depends only on P for all $0 \le l \le n$. To get such an upper bound, we use Theorem 45 to get

$$\mu_{\max,l}(E) \le \mu_l + (\mu_{\max,l}(E) - \mu_{\min,l}(E))$$
$$\le C_1 + C_2(\mu_{\max}(E) - \mu_{\min}(E))$$
$$\le C_1 + C_2 r(\mu_0 - \mu),$$

where C_1, C_2 are constants independent of E. Here the last inequality follows from $\mu_{\max} \le \mu_0$ and the observation that $r\mu \le \mu_{\min}(E) + (r - 1)\mu_{\max}(E)$. ∎

A more general version of the above theorem for pure-dimensional sheaves can be proved by imitating the proof in [Sim94, Theorem 1.1].

REFERENCES

[Bog78] F. A. Bogomolov. Holomorphic tensors and vector bundles on projective manifolds. *Izv. Akad. Nauk SSSR Ser. Mat.*, 42(6):1227–1287, 1439, 1978.

[Bre05] Holger Brenner. On a problem of Miyaoka. In *Number fields and function fields—two parallel worlds*, volume 239 of *Progr. Math.*, pages 51–59. Birkhäuser Boston, Boston, MA, 2005.

[HL10] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010.

[Lan04a] Adrian Langer. Addendum to: "Semistable sheaves in positive characteristic" [Ann. of Math. (2) **159** (2004), no. 1, 251–276; mr2051393]. *Ann. of Math. (2)*, 160(3):1211–1213, 2004.

[Lan04b] Adrian Langer. Semistable sheaves in positive characteristic. *Ann. of Math. (2)*, 159(1):251–276, 2004.

[Laz17] R.K. Lazarsfeld. *Positivity in Algebraic Geometry I: Classical Setting: Line Bundles and Linear Series.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer Berlin Heidelberg, 2017.

[Miy87] Yoichi Miyaoka. The Chern classes and Kodaira dimension of a minimal variety. In *Algebraic geometry, Sendai, 1985*, volume 10 of *Adv. Stud. Pure Math.*, pages 449–476. North-Holland, Amsterdam, 1987.

[Sim94] Carlos T. Simpson. Moduli of representations of the fundamental group of a smooth projective variety. I. *Inst. Hautes Études Sci. Publ. Math.*, (79):47–129, 1994.

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