# DYNAMICAL AND qKZ EQUATIONS MODULO $p^{s}$, AN EXAMPLE 

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#### Abstract

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#### Abstract

We consider an example of the joint system of dynamical differential equations and $q K Z$ difference equations with parameters corresponding to equations for elliptic integrals. We solve this system of equations modulo any power $p^{n}$ of a prime integer $p$. We show that the $p$-adic limit of these solutions as $n \rightarrow \infty$ determines a sequence of line bundles, each of which is invariant with respect to the corresponding dynamical connection, and that sequence of line bundles is invariant with respect to the corresponding $q K Z$ difference connection.


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## 1. Introduction

Let $\boldsymbol{z}=\left(z_{1}, z_{2}\right)$,

$$
\Phi(t ; \boldsymbol{z} ; \lambda, \mu)=t^{-\lambda}\left(t-z_{1}\right)^{-\mu}\left(t-z_{2}\right)^{-\mu}
$$

where $\lambda, \mu$ are rational numbers. Consider the column vector

$$
\begin{equation*}
I^{(C)}(\boldsymbol{z} ; \lambda, \mu)=\int_{C}\left(\frac{\Phi}{t-z_{1}}, \frac{\Phi}{t-z_{2}}\right) d t \tag{1.1}
\end{equation*}
$$

where $C \subset \mathbb{C}-\left\{0, z_{1}, z_{2}\right\}$ is a contour on which the integrand takes its initial value when $t$ encircles $C$. As a function of $\boldsymbol{z}$, the vector $I^{(C)}(\boldsymbol{z} ; \lambda, \mu)$ extends to a multi-valued analytic function on $\left\{\boldsymbol{a} \in \mathbb{C}^{2} \mid a_{1} a_{2}\left(a_{1}-a_{2}\right) \neq 0\right\}$.

The function $I^{(C)}(\boldsymbol{z} ; \lambda, \mu)$ satisfies the differential and difference equations,

$$
\begin{align*}
z_{1} \frac{\partial I}{\partial z_{1}}(\boldsymbol{z} ; \lambda, \mu) & =\left(\left[\begin{array}{cc}
-\lambda-\mu & -\mu \\
0 & 0
\end{array}\right]+\frac{\mu z_{1}}{z_{1}-z_{2}}\left[\begin{array}{cc}
-1 & 1 \\
1 & -1
\end{array}\right]\right) I(\boldsymbol{z} ; \lambda, \mu),  \tag{1.2}\\
z_{2} \frac{\partial I}{\partial z_{2}}(\boldsymbol{z} ; \lambda, \mu) & =\left(\left[\begin{array}{cc}
0 & 0 \\
-\mu & -\lambda-\mu
\end{array}\right]+\frac{\mu z_{2}}{z_{2}-z_{1}}\left[\begin{array}{cc}
-1 & 1 \\
1 & -1
\end{array}\right]\right) I(\boldsymbol{z} ; \lambda, \mu), \\
I(\boldsymbol{z} ; \lambda+1, \mu) & =\left[\begin{array}{ll}
\frac{\lambda+\mu}{z_{1} \lambda} & \frac{\mu}{z_{1} \lambda} \\
\frac{\mu}{z_{2} \lambda} & \frac{\lambda_{1}+\mu}{z_{2} \lambda}
\end{array}\right] I(\boldsymbol{z} ; \lambda, \mu) . \tag{1.3}
\end{align*}
$$

If $\lambda, \mu, \lambda+2 \mu \notin \mathbb{Z}_{>0}$, then all solutions of these equations are given by integrals $I^{(C)}(\boldsymbol{z} ; \lambda, \mu)$ (with different choices of $C$ ) up to multiplication by a scalar 1-periodic function of $\lambda$, this fact follows from [V1, Theorem 1.1].

Up to a gauge transformation, equations (1.2), (1.3) are the simplest example of the trigonometric $K Z$ differential equations and dynamical difference equations, see [TV1, MV]. They are also the simplest example of the dynamical differential equations and qKZ difference equations, see [TV2, TV3]. Up to a gauge transformation, they are the equivariant quantum differential equations and $q K Z$ difference equation associated with the cotangent bundle of projective line, see [TV3]. The family of functions $I^{(C)}(\boldsymbol{z} ; \lambda, \mu)$, labeled by contours $C$, are the hypergeometric solutions of these equations constructed in [MV, TV2]. In particular, see the integral $I^{(C)}(\boldsymbol{z} ; \lambda, \mu)$ (gauge transformed) in [TV2, Section 7.4]. We call equations (1.2) the dynamical differential equations and equation (1.3) the $q K Z$ difference equation.

In this paper we discuss polynomial solutions of equations (1.2) and (1.3) modulo $p^{n}$, where $p$ is a prime integer and $n$ is a positive integer. We also discuss the $p$-adic limit of these solutions as $n \rightarrow \infty$.

More precisely, we consider the following problem. For $\lambda_{0} \in \mathbb{Q}$, let $\Lambda\left(\lambda_{0}\right)=\left\{\lambda_{0}+l \mid l \in \mathbb{Z}\right\}$ be the arithmetic sequence with initial term $\lambda_{0}$ and step 1 . For a positive integer $\ell$, let $\Lambda\left(\lambda_{0}, \ell\right)=\left\{\lambda_{0}+l\left|l \in \mathbb{Z},\left|\lambda_{0}+l\right|<\ell\right\}\right.$ be an interval of the sequence $\Lambda$.
Problem. Let $p$ be an prime integer, $\lambda_{0}, \mu_{0} \in \mathbb{Q}, \ell \in \mathbb{Z}_{>0}$. For any $n \in \mathbb{Z}_{>0}$, find a sequence of column vectors

$$
\begin{equation*}
I(\boldsymbol{z} ; \lambda ; \ell ; n)=\left(I_{1}(\boldsymbol{z} ; \lambda ; \ell ; n), I_{2}(\boldsymbol{z} ; \lambda, \ell ; n)\right), \quad \lambda \in \Lambda\left(\lambda_{0}, \ell\right) \tag{1.4}
\end{equation*}
$$

such that
(i) the coordinates of these vectors are polynomials in $\boldsymbol{z}$ with integer coefficients,
(ii) each of the vectors $I(\boldsymbol{z} ; \lambda ; \ell ; n)$ satisfies modulo $p^{n}$ differential equations (1.2) with parameter $\mu=\mu_{0}$;
(iii) this sequence of vectors satisfies modulo $p^{n}$ the difference equation (1.3) with parameter $\mu=\mu_{0}$.
We may also require that the vectors are functorial in the following sense. If $\lambda \in \Lambda\left(\lambda_{0}, \ell_{0}\right)$ for some $l_{0}$, then the vector $I(\boldsymbol{z} ; \lambda ; \ell ; n)$ does not depend on $\ell$ for $\ell \geqslant \ell_{0}$. Having a solution $\{I(\boldsymbol{z} ; \lambda ; \ell ; n)\}$ of this problem we may study the $p$-adic limit of the vectors as $n \rightarrow \infty$.

In this paper we construct a solution of this problem for

$$
\begin{equation*}
\left(\lambda_{0}, \mu_{0}\right)=\left(\frac{1}{2}, \frac{1}{2}\right) \tag{1.5}
\end{equation*}
$$

We also describe the $p$-adic limit of our solution. It turns out that the limit is not a solution of equations (1.2) and (1.3) over a $p$-adic field, as one may naively think, but a line bundle invariant with respect to the dynamical connection, defined by equations (1.2), and invariant with respect to the discrete $q K Z$ connection, defined by equation (1.3), see Theorem 5.9. Notice that there is no such a line bundle if we consider the same differential and difference equations over the field of complex numbers, see Section 5.8.

The choice of parameters in (1.5) corresponds to elliptic integrals in (1.1). This choice is technically, arithmetically easier than the choice of an arbitrary pair ( $\lambda_{0}, \mu_{0}$ ) of rational numbers, although a similar construction can be performed for a wide class of parameters $\left(\lambda_{0}, \mu_{0}\right)$.

Quantum differential equations and associated $q K Z$ difference equations, as well as their solutions, is a mathematical structure with applications in representation theory, algebraic geometry, theory of special function, to name a few. It would be interesting to study how the properties of these equations and their solutions are reflected in the solutions of the same equations modulo powers of a prime integer and in their $p$-adic limits.

In Section 2 we reformulate equations (1.2) and (1.3) for $\left(\lambda_{0}, \mu_{0}\right)=\left(\frac{1}{2}, \frac{1}{2}\right)$, see equations (2.2), (2.3). In Section 3 we solve equations (2.2), (2.3) modulo a power $p^{s}$ of an odd prime integer $p$. The construction of these solutions is a variant of the constructions in [SV, V3]. The constructed solutions are called the $p^{s}$-hypergeometric solutions in [V3]. On the $p^{s}$ hypergeometric solutions see also [V2, V4, V5, VZ1, VZ2, RV1, RV2].

We prove Dwork-type congruences for these solutions in Section 4. Using these congruences we descibe the $p$-adic limit of our solutions as $s \rightarrow \infty$ in Section 5 .

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$$
\text { 2. EQUATIONS FOR }\left(\lambda_{0}, \mu_{0}\right)=\left(\frac{1}{2}, \frac{1}{2}\right)
$$

Denote

$$
\begin{align*}
& H_{1}(\boldsymbol{z} ; \lambda)=\left[\begin{array}{cc}
-\lambda-1 & -1 \\
0 & 0
\end{array}\right]+\frac{z_{1}}{z_{1}-z_{2}}\left[\begin{array}{cc}
-1 & 1 \\
1 & -1
\end{array}\right],  \tag{2.1}\\
& H_{2}(\boldsymbol{z} ; \lambda)=\left[\begin{array}{cc}
0 & 0 \\
-1 & -\lambda-1
\end{array}\right]+\frac{z_{2}}{z_{2}-z_{1}}\left[\begin{array}{cc}
-1 & 1 \\
1 & -1
\end{array}\right], \\
& K(\boldsymbol{z}, \lambda)=\left[\begin{array}{ll}
\frac{\lambda+1}{z_{1} \lambda} & \frac{1}{z_{1} \lambda} \\
\frac{1}{z_{2} \lambda} & \frac{\lambda+1}{z_{2} \lambda}
\end{array}\right] .
\end{align*}
$$

The substitution $(\lambda, \mu) \rightarrow\left(\frac{\lambda}{2}, \frac{1}{2}\right)$ transforms the system of equations (1.2), (1.3) to the following system of equations for a column vector $I(\boldsymbol{z} ; \lambda)$,

$$
\begin{align*}
2 z_{i} \frac{\partial I}{\partial z_{i}}(\boldsymbol{z} ; \lambda) & =H_{i}(\boldsymbol{z} ; \lambda) I(\boldsymbol{z} ; \lambda), \quad i=1,2  \tag{2.2}\\
I(\boldsymbol{z} ; \lambda+2) & =K(z, \lambda) I(\boldsymbol{z} ; \lambda) \tag{2.3}
\end{align*}
$$

Denote

$$
\begin{equation*}
\mathcal{D}_{i}(\lambda):=2 z_{i} \frac{\partial}{\partial z_{i}}-H_{i}(\boldsymbol{z} ; \lambda), \quad i=1,2 \tag{2.4}
\end{equation*}
$$

## 3. Solutions modulo powers of $p$

3.1. Notations. In this paper $p$ is an odd prime integer.

In this paper we consider the system of equations (2.2) and (2.3) for the values of $\lambda$ from the arithmetic sequence of odd integers, $\Lambda:=1+2 \mathbb{Z}$.

Given a positive integer $s$, denote

$$
\begin{equation*}
\Lambda_{s}=\left\{\lambda \in 1+2 \mathbb{Z} \mid-p^{s}<\lambda<p^{s}\right\} \tag{3.1}
\end{equation*}
$$

an interval of the arithmetic sequence $\Lambda$.

- For $\lambda \in \Lambda_{s}$, we have $0<\frac{p^{s}-\lambda}{2}<p^{s}$.
- For a positive integer $e$, if $s>e, \lambda \in \Lambda_{e}$, then $\left|\frac{p^{s}-\lambda}{2}\right|_{p}>p^{-e}$, where $|x|_{p}$ denotes the $p$-adic norm of a rational number $x$.
For a polynomial $f(t)$, denote by $\{f(t)\}_{s}$ the coefficient of $t^{p^{s}-1}$ in $f(t)$. For a function $g(\boldsymbol{z})$, denote by $\operatorname{grad}_{z} g$ the column gradient vector $\left(\frac{\partial g}{\partial z_{1}}, \frac{\partial g}{\partial z_{2}}\right)$.
3.2. Solutions. For $\lambda \in \Lambda_{s}$, define the master polynomial,

$$
\begin{equation*}
\Phi_{s}(t ; \boldsymbol{z} ; \lambda):=t^{\left(p^{s}-\lambda\right) / 2}\left(t-z_{1}\right)^{\left(p^{s}-1\right) / 2}\left(t-z_{2}\right)^{\left(p^{s}-1\right) / 2} . \tag{3.2}
\end{equation*}
$$

Define the column vector

$$
\Psi_{s}(t ; \boldsymbol{z} ; \lambda)=\left(\Psi_{s, 1}(t ; \boldsymbol{z} ; \lambda), \Psi_{s, 2}(t ; \boldsymbol{z} ; \lambda)\right):=\Phi_{s}(t ; \boldsymbol{z} ; \lambda)\left(\frac{1}{t-z_{1}}, \frac{1}{t-z_{2}}\right)
$$

The coordinates of $\Psi_{s}$ are polynomials in $t, \boldsymbol{z}$ with integer coefficients. We denote

$$
\begin{equation*}
I_{s}(\boldsymbol{z} ; \lambda)=\left(I_{s, 1}(\boldsymbol{z} ; \lambda), I_{s, 2}(\boldsymbol{z} ; \lambda)\right):=\left\{\Psi_{s}(t ; \boldsymbol{z} ; \lambda)\right\}_{s} \tag{3.3}
\end{equation*}
$$

the coefficient of $t^{p^{s}-1}$ in $\Psi_{s}(t ; \boldsymbol{z} ; \lambda)$, and

$$
\begin{equation*}
T_{s}(\boldsymbol{z} ; \lambda):=\left\{\Phi_{s}(t ; \boldsymbol{z} ; \lambda)\right\}_{s} \tag{3.4}
\end{equation*}
$$

the coefficient of $t^{p^{s}-1}$ in $\Phi_{s}(t ; \boldsymbol{z} ; \lambda)$. For every $\lambda \in \Lambda_{s}$, the functions $I_{s, 1}, I_{s, 2}, T_{s}$ are polynomials in $z_{1}, z_{2}$ with integer coefficients.

We have

$$
\begin{equation*}
\frac{1-p^{s}}{2} I_{s}(\boldsymbol{z} ; \lambda)=\operatorname{grad}_{\boldsymbol{z}} T_{s}(\boldsymbol{z} ; \lambda) \tag{3.5}
\end{equation*}
$$

Theorem 3.1. Let $s \in \mathbb{Z}_{>0}$, $\lambda \in \Lambda_{s}$, $i=1,2$. Then the vector $I_{s}(\boldsymbol{z} ; \lambda)$ satisfies the congruence

$$
\begin{equation*}
\mathcal{D}_{i}(\lambda) I_{s}(\boldsymbol{z} ; \lambda) \equiv 0 \quad\left(\bmod p^{s}\right) \tag{3.6}
\end{equation*}
$$

Proof. We have

$$
\begin{equation*}
\frac{\partial \Phi_{s}}{\partial t}=\left(\frac{p^{s}-\lambda}{2 t}+\frac{p^{s}-1}{2\left(t-z_{1}\right)}+\frac{p^{s}-1}{2\left(t-z_{2}\right)}\right) \Phi_{s} \tag{3.7}
\end{equation*}
$$

Also

$$
\begin{aligned}
& -\frac{\partial \Psi_{s, 1}}{\partial t}=\Phi_{s}\left(-\frac{p^{s}-\lambda}{2} \frac{1}{t\left(t-z_{1}\right)}-\frac{p^{s}-3}{2} \frac{1}{\left(t-z_{1}\right)^{2}}-\frac{p^{s}-1}{2} \frac{1}{\left(t-z_{1}\right)\left(t-z_{2}\right)}\right) \\
& =\Phi_{s}\left(\frac{p^{s}-\lambda}{2 z_{1}}\left[\frac{1}{t}-\frac{1}{t-z_{1}}\right]-\frac{p^{s}-3}{2} \frac{1}{\left(t-z_{1}\right)^{2}}-\frac{p^{s}-1}{2} \frac{1}{z_{1}-z_{2}}\left[\frac{1}{t-z_{1}}-\frac{1}{t-z_{2}}\right]\right) .
\end{aligned}
$$

Hence

$$
\begin{aligned}
& -\Phi_{s} \frac{p^{s}-3}{2} \frac{1}{\left(t-z_{1}\right)^{2}}=-\frac{\partial}{\partial t}\left(\frac{\Phi}{t-z_{1}}\right)+ \\
& +\Phi_{s}\left(\frac{p^{s}-\lambda}{2 z_{1}}\left[\frac{1}{t-z_{1}}-\frac{1}{t}\right]+\frac{p^{s}-1}{2} \frac{1}{z_{1}-z_{2}}\left[\frac{1}{t-z_{1}}-\frac{1}{t-z_{2}}\right]\right) \\
& =-\frac{\partial}{\partial t}\left(\frac{\Phi_{s}}{t-z_{1}}\right)+\Phi_{s}\left(\frac{p^{s}-\lambda}{2 z_{1}} \frac{1}{t-z_{1}}+\frac{p^{s}-1}{2} \frac{1}{z_{1}-z_{2}}\left[\frac{1}{t-z_{1}}-\frac{1}{t-z_{2}}\right]\right) \\
& +\Phi_{s} \frac{1}{z_{1}}\left(\frac{p^{s}-1}{2} \frac{1}{t-z_{1}}+\frac{p^{s}-1}{2} \frac{1}{t-z_{2}}\right)-\frac{1}{z_{1}} \frac{\partial \Phi_{s}}{\partial t} \\
& =-\frac{\partial}{\partial t}\left(\frac{\Phi_{s}}{t-z_{1}}\right)-\frac{1}{z_{1}} \frac{\partial \Phi_{s}}{\partial t} \\
& +\Phi_{s}\left(\frac{2 p^{s}-\lambda-1}{2 z_{1}} \frac{1}{t-z_{1}}+\frac{p^{s}-1}{2 z_{1}} \frac{1}{t-z_{2}}+\frac{p^{s}-1}{2} \frac{1}{z_{1}-z_{2}}\left[\frac{1}{t-z_{1}}-\frac{1}{t-z_{2}}\right]\right) .
\end{aligned}
$$

We have

$$
\begin{equation*}
\left\{\frac{\partial}{\partial t}\left(\frac{\Phi_{s}}{t-z_{1}}\right)\right\}_{s} \equiv 0, \quad\left\{\frac{\partial \Phi_{s}}{\partial t}\right\}_{s} \equiv 0, \quad \frac{2 p^{s}-\lambda-1}{2} \equiv-\frac{\lambda+1}{2}, \quad \frac{2 p^{s}-1}{2} \equiv-\frac{1}{2} \tag{3.8}
\end{equation*}
$$

modulo $p^{s}$. Hence

$$
2 z_{1} \frac{\partial I_{s, 1}}{\partial z_{1}} \equiv-(\lambda+1) I_{s, 1}-I_{s, 2}+\frac{z_{1}}{z_{1}-z_{2}}\left(-I_{s, 1}+I_{s, 2}\right) \quad\left(\bmod p^{s}\right)
$$

We also have

$$
\frac{\partial \Psi_{s, 2}}{\partial z_{1}}=-\frac{p^{s}-1}{2} \frac{\Phi_{s}}{\left(t-z_{1}\right)\left(t-z_{2}\right)}=-\frac{p^{s}-1}{2} \frac{\Phi_{s}}{z_{1}-z_{2}}\left[\frac{1}{t-z_{1}}-\frac{1}{t-z_{2}}\right]
$$

Hence

$$
2 z_{1} \frac{\partial I_{s, 2}}{\partial z_{1}} \equiv \frac{z_{1}}{z_{1}-z_{2}}\left(I_{s, 1}-I_{s, 2}\right) \quad\left(\bmod p^{s}\right)
$$

Equation (3.6) for $i=1$ is proved. Equation (3.6) for $i=2$ is proved similarly.
Theorem 3.2. Let $s>e$ be positive integers, and $\lambda, \lambda+2 \in \Lambda_{e}$. Then the vector $I_{s}(\boldsymbol{z} ; \lambda)$ satisfies the congruence:

$$
\begin{equation*}
I(\boldsymbol{z} ; \lambda+2) \equiv K(z, \lambda) I(\boldsymbol{z} ; \lambda) \quad\left(\bmod p^{s-e}\right) \tag{3.9}
\end{equation*}
$$

Proof. Equation (3.7) can be written as

$$
-\frac{\Phi_{s}}{t}=-\frac{2}{p^{s}-\lambda} \frac{\partial \Phi_{s}}{\partial t}+\frac{p^{s}-1}{p^{s}-\lambda}\left(\frac{1}{t-z_{1}}+\frac{1}{t-z_{2}}\right) \Phi_{s}
$$

Hence

$$
\begin{align*}
& \Psi_{s, 1}(\boldsymbol{z} ; \lambda+2)=\frac{\Phi_{s}(\boldsymbol{z} ; \lambda)}{t\left(t-z_{1}\right)}=-\frac{\Phi_{s}(\boldsymbol{z} ; \lambda)}{z_{1}}\left[\frac{1}{t}-\frac{1}{t-z_{1}}\right]  \tag{3.10}\\
& =\frac{\Phi_{s}(\boldsymbol{z} ; \lambda)}{z_{1}\left(t-z_{1}\right)}-\frac{2}{\left(p^{s}-\lambda\right) z_{1}} \frac{\partial \Phi_{s}}{\partial t}+\frac{p^{s}-1}{p^{s}-\lambda}\left(\frac{1}{z_{1}\left(t-z_{1}\right)}+\frac{1}{z_{1}\left(t-z_{2}\right)}\right) \Phi_{s}(\boldsymbol{z} ; \lambda)
\end{align*}
$$

By (3.8) the term $\left\{\frac{2}{\left(p^{s}-\lambda\right) z_{1}} \frac{\partial \Phi_{s}}{\partial t}\right\}_{s}$ is divisible at least by $p^{s-e}$. Hence (3.10) implies

$$
I_{s, 1}(\boldsymbol{z} ; \lambda+2) \equiv \frac{\lambda+1}{z_{1} \lambda} I_{s, 1}(\boldsymbol{z} ; \lambda)+\frac{1}{z_{1} \lambda} I_{s, 2}(\boldsymbol{z} ; \lambda) \quad\left(\bmod p^{s-e}\right)
$$

Similarly we obtain

$$
I_{s, 2}(\boldsymbol{z} ; \lambda+2) \equiv \frac{\lambda+1}{z_{2} \lambda} I_{s, 2}(\boldsymbol{z} ; \lambda)+\frac{1}{z_{2} \lambda} I_{s, 1}(\boldsymbol{z} ; \lambda) \quad\left(\bmod p^{s-e}\right)
$$

Theorem 3.2 is proved.
3.3. Formulas for $I_{s, 1}, I_{s, 2}, T_{s}$.

Lemma 3.3. For $\lambda \in \Lambda_{s}$ we have

$$
\begin{align*}
& T_{s}(\boldsymbol{z} ; \lambda)=(-1)^{\frac{p^{s}-\lambda}{2}} \sum_{k+\ell=\frac{p^{s}-\lambda}{2}}\binom{\frac{p^{s}-1}{2}}{k}\binom{\frac{p^{s}-1}{2}}{\ell} z_{1}^{k} z_{2}^{\ell},  \tag{3.11}\\
& I_{s, 1}(\boldsymbol{z} ; \lambda)=(-1)^{\frac{p^{s}-\lambda}{2}-1} \sum_{k+\ell=\frac{p^{s}-\lambda}{2}-1}\binom{\frac{p^{s}-1}{2}-1}{k}\binom{\frac{p^{s}-1}{2}}{\ell} z_{1}^{k} z_{2}^{\ell},  \tag{3.12}\\
& I_{s, 2}(\boldsymbol{z} ; \lambda)=(-1)^{\frac{p^{s}-\lambda}{2}-1} \sum_{k+\ell=\frac{p^{s}-\lambda}{2}-1}\binom{\frac{p^{s}-1}{2}}{k}\binom{\frac{p^{s}-1}{2}-1}{\ell} z_{1}^{k} z_{2}^{\ell} . \tag{3.13}
\end{align*}
$$

3.4. $p$-ary representations. For $\lambda \in \Lambda_{s}$ we have the following $p$-ary representations

$$
\begin{align*}
\frac{p^{s}-\lambda}{2} & =w_{0}(\lambda)+w_{1}(\lambda) p+\cdots+w_{s-1}(\lambda) p^{s-1}  \tag{3.14}\\
\frac{-\lambda}{2} & =w_{0}(\lambda)+w_{1}(\lambda) p+\cdots+w_{s-1}(\lambda) p^{s-1}+\frac{p-1}{2} p^{s}+\frac{p-1}{2} p^{s+1}+\ldots \tag{3.15}
\end{align*}
$$

for some integers $w_{i}(\lambda), \quad 0 \leqslant w_{i}(\lambda) \leqslant p-1$. Denote $w_{i}(\lambda)=\frac{p-1}{2}$ for $i \geqslant s$.
For $\lambda \in \Lambda$, denote by $W(\lambda)$ the set of all distinct integers $w_{i}(\lambda)$ in the $p$-ary representation of $\frac{-\lambda}{2}$. The set $W(\lambda)$ has at most $p$ elements.

For example, $\frac{-1}{2}=\frac{p-1}{2}(1+p+\ldots)$ and $W(1)=\left\{\frac{p-1}{2}\right\}$, while $\frac{1}{2}=\frac{p+1}{2}+\frac{p-1}{2}\left(p+p^{2}+\ldots\right)$, and $W(-1)=\left\{\frac{p+1}{2}, \frac{p-1}{2}\right\}$.

For $w=0,1, \ldots, p-1$, let $h(\boldsymbol{z} ; w)$ (resp. $g_{1}(\boldsymbol{z} ; w)$, resp. $g_{2}(\boldsymbol{z} ; w)$ ) be the coefficient of $t^{p-1}$ in $t^{w}\left(t-z_{1}\right)^{(p-1) / 2}\left(t-z_{2}\right)^{(p-1) / 2}\left(\right.$ resp. in $t^{w}\left(t-z_{1}\right)^{(p-1) / 2-1}\left(t-z_{2}\right)^{(p-1) / 2}$, resp. in $\left.t^{w}\left(t-z_{1}\right)^{(p-1) / 2}\left(t-z_{2}\right)^{(p-1) / 2-1}\right)$.
Lemma 3.4. For $\lambda \in \Lambda_{s}$ we have

$$
\begin{equation*}
T_{s}(\boldsymbol{z} ; \lambda) \equiv \prod_{i=0}^{s-1} h\left(\boldsymbol{z}^{p^{i}} ; w_{i}(\lambda)\right) \quad(\bmod p) \tag{3.16}
\end{equation*}
$$

The polynomial $T_{s}(\boldsymbol{z} ; \lambda)$ is nonzero modulo $p$.
Proof. We have $\frac{p^{s}-1}{2}=\frac{p-1}{2}\left(1+p+\cdots+p^{s-1}\right)$. Then

$$
\begin{equation*}
\Phi_{s}(t ; \boldsymbol{z} ; \lambda) \equiv \prod_{i=0}^{s-1}\left(t^{p^{i}}\right)^{w_{i}(\lambda)}\left(t^{p^{i}}-z_{1}^{p^{i}}\right)^{(p-1) / 2}\left(t^{p^{i}}-z_{2}^{p^{i}}\right)^{(p-1) / 2} \quad(\bmod p) \tag{3.17}
\end{equation*}
$$

This implies (3.16). To prove the second statement of the lemma it is enough to check that the polynomial $h(\boldsymbol{z} ; w)$ is nonzero modulo $p$ for $w=0,1, \ldots, p-1$. Indeed, there exist nonnegative integers $k, \ell$ such that $w=k+\ell$ and $k, \ell \leqslant(p-1) / 2$. Then the coefficient of $z_{1}^{k} z_{2}^{\ell}$ in $h(\boldsymbol{z} ; w)$ equals $(-1)^{w}\left(\frac{p-1}{k}\right)\binom{\frac{p-1}{2}}{\ell}$ and is nonzero modulo $p$ by Lucas Theorem.
Lemma 3.5. Let $\lambda \in \Lambda_{s}$ and $j=1,2$. Then

$$
\begin{equation*}
I_{s, j}(\boldsymbol{z} ; \lambda) \equiv g_{j}\left(\boldsymbol{z} ; w_{0}(\lambda)\right) \prod_{i=1}^{s-1} h\left(\boldsymbol{z}^{p^{i}} ; w_{i}(\lambda)\right) \quad(\bmod p) \tag{3.18}
\end{equation*}
$$

If $\lambda$ is not divisible by $p$, then $I_{s, j}(\boldsymbol{z} ; \lambda)$ is nonzero modulo $p$.
Proof. If $\lambda$ is not divisible by $p$, then $w_{0}(\lambda)>0$. Then $g_{j}\left(\boldsymbol{z} ; w_{0}(\lambda)\right)$ is nonzero modulo $p$. Also, the polynomials $h\left(\boldsymbol{z} ; w_{i}(\lambda)\right)$ are nonzero modulo $p$.

## 4. DWORK-TYPE CONGRUENCES

In this section we apply results from [V5] to obtain congruences relating the functions $I_{s}(\boldsymbol{z} ; \lambda), T_{s}(\boldsymbol{z} ; \lambda)$ in $\boldsymbol{z}$ for different $s$. This type of congruences was originated by B. Dwork in $[\mathrm{Dw}]$, see also, for example, $[\mathrm{Me}, \mathrm{MeV}$, VZ1, VZ2].

A congruence $F(x) \equiv G(x)\left(\bmod p^{s}\right)$ for two polynomials in some variables $x$ with integer coefficients is understood as the divisibility by $p^{s}$ of all coefficients of $F(x)-G(x)$.

Let $F_{1}(x), F_{2}(x), G_{1}(x), G_{2}(x)$ be polynomials such that $F_{2}(x), G_{2}(x)$ are both nonzero modulo $p$. Then the congruence $F_{1}(x) / F_{2}(x) \equiv G_{1}(x) / G_{2}(x)$ modulo $p^{s}$ is understood as the congruence

$$
F_{1}(x) G_{2}(x) \equiv G_{1}(x) F_{2}(x) \quad\left(\bmod p^{s}\right)
$$

Recall the master polynomial

$$
\Phi_{s}(t ; \boldsymbol{z} ; \lambda)=t^{\left(p^{s}-\lambda\right) / 2}\left(t-z_{1}\right)^{\left(p^{s}-1\right) / 2}\left(t-z_{2}\right)^{\left(p^{s}-1\right) / 2}
$$

in particular,

$$
\Phi_{1}(t ; \boldsymbol{z} ; 1)=t^{(p-1) / 2}\left(t-z_{1}\right)^{(p-1) / 2}\left(t-z_{2}\right)^{(p-1) / 2}
$$

For $\lambda \in \Lambda_{e}$ and $s>e$ we have

$$
\begin{equation*}
\Phi_{s}(t ; \boldsymbol{z} ; \lambda)=\Phi_{e}(t ; \boldsymbol{z} ; \lambda) \Phi_{1}(t ; \boldsymbol{z} ; 1)^{p^{e}+p^{e+1}+\cdots+p^{s-e-1}} \tag{4.1}
\end{equation*}
$$

In particular, we have

$$
\Phi_{s-e}(t ; \boldsymbol{z} ; 1)=\Phi_{1}(t ; \boldsymbol{z} ; 1)^{1+p+\cdots+p^{s-e-1}}
$$

For $\lambda \in \Lambda_{e}$ and $s \geqslant e$, the Newton polytope of $\Phi_{s}(t ; \boldsymbol{z} ; \lambda)$ with respect to the variable $t$ is the interval $\left[\frac{p^{s}-\lambda}{2}, \frac{p^{s}-\lambda}{2}+p^{s}-1\right]$.
(4.2) For a positive integer $k$, the point $k p^{s}-1$ lies in this interval only if $k=1$.

Recall that $T_{s}(\boldsymbol{z} ; \lambda)=\left\{\Phi_{s}(t ; \boldsymbol{z} ; \lambda)\right\}_{s}$ is the coefficient of $t^{p^{s}-1}$ in $\Psi_{s}(t ; \boldsymbol{z} ; \lambda)$.
For $\lambda \in \Lambda_{e}$, the polynomial $T_{s}(\boldsymbol{z} ; \lambda)$ is nonzero modulo $p$,
by Lemma 5.7.
In [V5], certain congruences were proved for a sequence of polynomials like $\Phi_{s}(\boldsymbol{z} ; \lambda), s \geqslant e$, with properties like (4.1), (4.2), (4.3). In the case of the polynomials $\Phi_{s}(\boldsymbol{z} ; \lambda), s \geqslant e$, the congruences in [V5] say the following.

Theorem 4.1. Let $e \in \mathbb{Z}_{>0}, \lambda \in \Lambda_{e}$.
(i) For $j \in\{1,2\}$ denote $D_{j}=\frac{\partial}{\partial z_{j}}$. Then for $s>e$ we have

$$
\begin{equation*}
\frac{D_{j}\left(T_{s}(\boldsymbol{z} ; \lambda)\right)}{T_{s}(\boldsymbol{z} ; \lambda)} \equiv \frac{D_{j}\left(T_{s-1}(\boldsymbol{z} ; \lambda)\right)}{T_{s-1}(\boldsymbol{z} ; \lambda)} \quad\left(\bmod p^{s-e}\right) \tag{4.4}
\end{equation*}
$$

(ii) For $i, j \in\{1,2\}$ and $s>e$ we have

$$
\begin{equation*}
\frac{D_{i}\left(D_{j}\left(T_{s}(\boldsymbol{z} ; \lambda)\right)\right)}{T_{s}(\boldsymbol{z} ; \lambda)} \equiv \frac{D_{i}\left(D_{j}\left(T_{s-1}(\boldsymbol{z} ; \lambda)\right)\right)}{T_{s-1}(\boldsymbol{z} ; \lambda)} \quad\left(\bmod p^{s-e}\right) . \tag{4.5}
\end{equation*}
$$

Statements (i-ii) are special cases of [V5, Theorems 2.8 and 2.9].
Corollary 4.2. Let $e \in \mathbb{Z}_{>0}, \lambda \in \Lambda_{e}, s>e$, and $i, j \in\{1,2\}$. Then

$$
\begin{align*}
& \frac{I_{s, j}(\boldsymbol{z} ; \lambda)}{T_{s}(\boldsymbol{z} ; \lambda)} \equiv \frac{I_{s-1, j}(\boldsymbol{z} ; \lambda)}{T_{s-1}(\boldsymbol{z} ; \lambda)} \quad\left(\bmod p^{s-e}\right)  \tag{4.6}\\
& \frac{\partial I_{s, j}}{\partial z_{i}}(\boldsymbol{z} ; \lambda)  \tag{4.7}\\
& T_{s}(\boldsymbol{z} ; \lambda) \equiv \frac{\frac{\partial I_{s-1, j}}{\partial z_{i}}(\boldsymbol{z} ; \lambda)}{T_{s-1}(\boldsymbol{z} ; \lambda)} \quad\left(\bmod p^{s-e}\right)
\end{align*}
$$

The corollary follows from formula (3.5) and Theorem 4.1.
Theorem 4.3. Let $e \in \mathbb{Z}_{>0}$. Let $\lambda, \lambda+2 \in \Lambda_{e}$ and $s>2 e$. Then

$$
\begin{equation*}
\frac{I_{s}(\boldsymbol{z} ; \lambda+2)}{T_{s}(\boldsymbol{z} ; \lambda)} \equiv \frac{I_{s-1}(\boldsymbol{z} ; \lambda+2)}{T_{s-1}(\boldsymbol{z} ; \lambda)} \quad\left(\bmod p^{s-2 e}\right) . \tag{4.8}
\end{equation*}
$$

Proof. Using formulas (4.6), (4.7) and the fact that $\lambda \in \Lambda_{e}$, we obtain

$$
\frac{1}{\lambda}\left[\begin{array}{cc}
\frac{\lambda+1}{z_{1}} & \frac{1}{z_{1}} \\
\frac{1}{z_{2}} & \frac{\lambda+1}{z_{2}}
\end{array}\right] \frac{I_{s}(z ; \lambda)}{T_{s}(z ; \lambda)} \equiv \frac{1}{\lambda}\left[\begin{array}{cc}
\frac{\lambda+1}{z_{1}} & \frac{1}{z_{1}} \\
\frac{1}{z_{2}} & \frac{\lambda+1}{z_{2}}
\end{array}\right] \frac{I_{s-1}(z ; \lambda)}{T_{s-1}(z ; \lambda)} \quad\left(\bmod p^{s-2 e}\right) .
$$

Congruence (3.9) implies the congruences:

$$
\begin{aligned}
\frac{I_{s}(\boldsymbol{z} ; \lambda+2)}{T_{s}(\boldsymbol{z} ; \lambda)} & \equiv \frac{1}{\lambda}\left[\begin{array}{cc}
\frac{\lambda+1}{z_{1}} & \frac{1}{z_{1}} \\
\frac{1}{z_{2}} & \frac{\lambda+1}{z_{2}}
\end{array}\right] \frac{I_{s}(\boldsymbol{z} ; \lambda)}{T_{s}(\boldsymbol{z} ; \lambda)} \quad\left(\bmod p^{s-e}\right) \\
\frac{I_{s-1}(\boldsymbol{z} ; \lambda+2)}{T_{s-1}(\boldsymbol{z} ; \lambda)} & \equiv \frac{1}{\lambda}\left[\begin{array}{cc}
\frac{\lambda+1}{z_{1}} & \frac{1}{z_{1}} \\
\frac{1}{z_{2}} & \frac{\lambda+1}{z_{2}}
\end{array}\right] \frac{I_{s-1}(\boldsymbol{z} ; \lambda)}{T_{s-1}(\boldsymbol{z} ; \lambda)} \quad\left(\bmod p^{s-e-1}\right) .
\end{aligned}
$$

These three congruences imply congruence (4.8).

## 5. Convergence

5.1. Unramified extensions of $\mathbb{Q}_{p}$. We fix an algebraic closure $\overline{\mathbb{Q}_{p}}$ of $\mathbb{Q}_{p}$. For every $m$, there is a unique unramified extension of $\mathbb{Q}_{p}$ in $\overline{\mathbb{Q}_{p}}$ of degree $m$, denoted by $\mathbb{Q}_{p}^{(m)}$. This can be obtained by attaching to $\mathbb{Q}_{p}$ a primitive root of 1 of order $p^{m}-1$. The norm $|\cdot|_{p}$ on $\mathbb{Q}_{p}$ extends to a norm $|\cdot|_{p}$ on $\mathbb{Q}_{p}^{(m)}$. Let

$$
\mathbb{Z}_{p}^{(m)}=\left\{\left.a \in \mathbb{Q}_{p}^{(m)}| | a\right|_{p} \leqslant 1\right\}
$$

denote the ring of integers in $\mathbb{Q}_{p}^{(m)}$. The ring $\mathbb{Z}_{p}^{(m)}$ has the unique maximal ideal

$$
\mathbb{M}_{p}^{(m)}=\left\{\left.a \in \mathbb{Q}_{p}^{(m)}| | a\right|_{p}<1\right\},
$$

such that $\mathbb{Z}_{p}^{(m)} / \mathbb{M}_{p}^{(m)}$ is isomorphic to the finite field $\mathbb{F}_{p^{m}}$.
For every $t \in \mathbb{F}_{p^{m}}$ there is a unique $\tilde{t} \in \mathbb{Z}_{p}^{(m)}$ that is a lift of $t$ and such that $\tilde{t}^{p^{m}}=\tilde{t}$. The element $\tilde{t}$ is called the Teichmuller lift of $t$.
5.2. Domain $\mathfrak{D}_{B}$. For $t \in \mathbb{F}_{p^{m}}$ and $r>0$ denote

$$
D_{t, r}=\left\{a \in \mathbb{Z}_{p}^{(m)}| | a-\left.\tilde{t}\right|_{p}<r\right\} .
$$

We have the partition

$$
\mathbb{Z}_{p}^{(m)}=\bigcup_{t \in \mathbb{F}_{p^{m}}} D_{t, 1}
$$

Recall $\boldsymbol{z}=\left(z_{1}, z_{2}\right)$. For $B(\boldsymbol{z}) \in \mathbb{Z}[\boldsymbol{z}]$, define

$$
\mathfrak{D}_{B}=\left\{\left.\boldsymbol{a} \in\left(\mathbb{Z}_{p}^{(m)}\right)^{2}| | B(\boldsymbol{a})\right|_{p}=1\right\} .
$$

Let $\bar{B}(\boldsymbol{z})$ be the projection of $B(\boldsymbol{z})$ to $\mathbb{F}_{p}[\boldsymbol{z}] \subset \mathbb{F}_{p^{m}}[\boldsymbol{z}]$. Then $\mathfrak{D}_{B}$ is the union of unit polydiscs,

$$
\mathfrak{D}_{B}=\bigcup_{\substack{t_{1}, t_{2} \in \mathbb{F}_{p} m \\ \bar{B}\left(t_{1}, t_{2}\right) \neq 0}} D_{t_{1}, 1} \times D_{t_{2}, 1}
$$

Lemma 5.1. For any nonnegative integer $k$ we have

$$
\begin{aligned}
\left\{\left.\boldsymbol{a} \in\left(\mathbb{Z}_{p}^{(m)}\right)^{2}| | B\left(\boldsymbol{a}^{p^{k}}\right)\right|_{p}=1\right\} & =\bigcup_{\substack{t_{1}, t_{2} \in \mathbb{F}_{p^{m}} \\
\bar{B}\left(t_{1}^{k}, t t_{2}^{k^{k}}\right) \neq 0}} D_{t_{1}, 1} \times D_{t_{2}, 1}= \\
& =\bigcup_{\substack{t_{1}, t_{2} \in F_{p^{m}} \\
\bar{B}\left(t_{1}, t_{2}\right) \neq 0}} D_{t_{1}, 1} \times D_{t_{2}, 1}=\mathfrak{D}_{B}
\end{aligned}
$$

Lemma 5.2 ([VZ2, Lemma 6.1]). Let $\bar{B}(\boldsymbol{z}) \in \mathbb{F}_{p}[\boldsymbol{z}]$ be a nonzero polynomial of degree $d$, and $d+1<p^{m}$. Then the set $\left\{\boldsymbol{a} \in\left(\mathbb{F}_{p^{m}}\right)^{2} \mid \bar{B}(\boldsymbol{a}) \neq 0\right\}$ is nonempty. Moreover, there are at least $\frac{p^{2 m}-1}{p^{m}-1}\left(p^{m}-1-d\right)+1$ points of $\left(\mathbb{F}_{p^{m}}\right)^{2}$ where $\bar{B}(\boldsymbol{z})$ is nonzero.
5.3. Domains of convergence. Recall the polynomials $T_{s}(\boldsymbol{z} ; \lambda), I_{s, 1}(\boldsymbol{z} ; \lambda), I_{s, 2}(\boldsymbol{z} ; \lambda)$ as well as the polynomials $h(\boldsymbol{z} ; w), g_{1}(\boldsymbol{z} ; w), g_{2}(\boldsymbol{z} ; w)$. For $\lambda \in \Lambda$, denote

$$
\begin{aligned}
H(\boldsymbol{z} ; \lambda) & =\prod_{w \in W(\lambda)} h(\boldsymbol{z} ; w) \\
\mathfrak{D}^{(m)}(\lambda) & =\left\{\left.\boldsymbol{a} \in\left(\mathbb{Z}_{p}^{(m)}\right)^{2}| | H(\boldsymbol{a} ; \lambda)\right|_{p}=1\right\}
\end{aligned}
$$

Let $\lambda \in \Lambda$ be not divisible by $p$ (that is, $w_{0}(\lambda)>0$ ). For $j=1,2$, denote

$$
\begin{aligned}
G_{j}(\boldsymbol{z} ; \lambda) & =g_{j}\left(\boldsymbol{z} ; w_{0}(\lambda)\right) \prod_{w \in W(\lambda)} h(\boldsymbol{z} ; w) \\
\mathfrak{D}_{*}^{(m)}(\lambda) & =\left\{\left.\boldsymbol{a} \in\left(\mathbb{Z}_{p}^{(m)}\right)^{2}| | G_{1}(\boldsymbol{a} ; \lambda)\right|_{p}=1 \text { or }\left|G_{2}(\boldsymbol{a} ; \lambda)\right|_{p}=1\right\} .
\end{aligned}
$$

Denote $\mathfrak{C}^{(m)}=\left\{\boldsymbol{a} \in\left(\mathbb{Z}_{p}^{(m)}\right)^{2} \mid a_{1} a_{2} \neq 0\right\}$. For $\lambda \in \Lambda$ divisible by $p$, denote

$$
\mathfrak{D}_{*}^{(m)}(\lambda)=\mathfrak{D}^{(m)}(\lambda) \cap \mathfrak{D}_{*}^{(m)}(\lambda+2) \cap \mathfrak{C}^{(m)} .
$$

Clearly, for any $\lambda \in \Lambda$, we have

$$
\mathfrak{D}_{*}^{(m)}(\lambda) \subset \mathfrak{D}^{(m)}(\lambda)
$$

and

$$
\begin{equation*}
\bigcap_{\lambda \in \Lambda} \mathfrak{D}_{*}^{(m)}(\lambda) \supset\left\{\left.\boldsymbol{a} \in\left(\mathbb{Z}_{p}^{(m)}\right)^{2}| | a_{1} a_{2} h(\boldsymbol{a} ; 0) \prod_{w=1}^{p-1} h(\boldsymbol{a} ; w) g_{1}(\boldsymbol{a} ; w) g_{2}(\boldsymbol{a} ; w)\right|_{p}=1\right\} . \tag{5.1}
\end{equation*}
$$

Lemma 5.3. If $m \geqslant 3$, then $\bigcap_{\lambda \in \Lambda} \mathfrak{D}_{*}^{(m)}(\lambda)$ is nonempty.
Proof. We have $\operatorname{deg}_{\boldsymbol{z}} h(\boldsymbol{z} ; w)=w$ and $\operatorname{deg}_{\boldsymbol{z}} g_{1}(\boldsymbol{z} ; w)=\operatorname{deg}_{\boldsymbol{z}} g_{2}(\boldsymbol{z} ; w)=w-1$. Hence the polynomial in (5.1) has degree $\frac{3 p^{2}-7 p+8}{2}<p^{3}-1$. The lemma follows from Lemma 5.2.

Theorem 5.4. For $e \in \mathbb{Z}_{>0}$ and $\lambda \in \Lambda_{e}$, we have the following statements.
(i) The sequence of column vectors, $\left(\frac{I_{s}(z ; \lambda)}{T_{s}(z ; \lambda)}\right)_{s \geqslant e}$, whose entries are rational functions in $\boldsymbol{z}$ regular on $\mathfrak{D}^{(m)}(\lambda)$, uniformly converges on $\mathfrak{D}^{(m)}(\lambda)$ as $s \rightarrow \infty$ to a a vector, whose entries are analytic functions on $\mathfrak{D}^{(m)}(\lambda)$. The vector will be denoted by $\mathcal{I}(\boldsymbol{z} ; \lambda)=$ $\left(\mathcal{I}_{1}(\boldsymbol{z} ; \lambda), \mathcal{I}_{2}(\boldsymbol{z} ; \lambda)\right)$.
(ii) The sequence of column vectors $\left(\frac{I_{s}(z ; \lambda+2)}{T_{s}(z ; \lambda)}\right)_{s>2 e}$, whose entries are rational functions in $\boldsymbol{z}$ regular on $\mathfrak{D}^{(m)}(\lambda)$, uniformly converges on $\mathfrak{D}^{(m)}(\lambda)$ as $s \rightarrow \infty$ to a vector, whose entries are analytic functions on $\mathfrak{D}^{(m)}(\lambda)$. The vector will be denoted by $\tilde{\mathcal{I}}(\boldsymbol{z} ; \lambda+2)=\left(\tilde{\mathcal{I}}_{1}(\boldsymbol{z} ; \lambda+2), \tilde{\mathcal{I}}_{2}(\boldsymbol{z} ; \lambda+2)\right)$.
(iii) For $j=1,2$, the sequence of column vectors, $\left(\frac{\frac{\partial I_{s}}{z_{j}}(\boldsymbol{z} ; \lambda)}{T_{s}(\boldsymbol{z} ; \lambda)}\right)_{s \geqslant e}$, whose entries are rational functions in $\boldsymbol{z}$ regular on $\mathfrak{D}^{(m)}(\lambda)$, uniformly converges on $\mathfrak{D}^{(m)}(\lambda)$ as $s \rightarrow \infty$ to a vector, whose entries are analytic functions on $\mathfrak{D}^{(m)}(\lambda)$. The vector will be denoted by $\mathcal{I}^{(i)}(\boldsymbol{z} ; \lambda)=\left(\mathcal{I}_{1}^{(i)}(\boldsymbol{z} ; \lambda), \mathcal{I}_{2}^{(i)}(\boldsymbol{z} ; \lambda)\right)$.
Proof. Parts (i), (ii), (iii) follow from the congruences of (4.6), (4.7), (4.8), respectively.

### 5.4. Relations between limiting vectors.

Lemma 5.5. Let $\lambda \in \Lambda$. We have the following equations on $\mathfrak{D}^{(m)}(\lambda)$ :

$$
\begin{align*}
\frac{\partial \mathcal{I}}{\partial z_{i}}(\boldsymbol{z} ; \lambda) & =\mathcal{I}^{(i)}(\boldsymbol{z} ; \lambda)-\frac{1}{2} \mathcal{I}_{i}(\boldsymbol{z} ; \lambda) \mathcal{I}(\boldsymbol{z} ; \lambda)  \tag{5.2}\\
\mathcal{I}^{(i)}(\boldsymbol{z} ; \lambda) & =H_{i}(\boldsymbol{z} ; \lambda) \mathcal{I}(\boldsymbol{z} ; \lambda)  \tag{5.3}\\
\frac{\partial \mathcal{I}}{\partial z_{i}}(\boldsymbol{z} ; \lambda) & =\left(H_{i}(\boldsymbol{z} ; \lambda)-\frac{1}{2} \mathcal{I}_{i}(\boldsymbol{z} ; \lambda)\right) \mathcal{I}(\boldsymbol{z} ; \lambda)  \tag{5.4}\\
\tilde{\mathcal{I}}(\boldsymbol{z} ; \lambda+2) & =K(\boldsymbol{z} ; \lambda) \mathcal{I}(\boldsymbol{z} ; \lambda) \tag{5.5}
\end{align*}
$$

Proof. Differentiating the congruence in (4.6) with respect to $z_{i}$ we obtain

$$
\frac{\frac{\partial I_{s, j}}{\partial z_{i}}}{T_{s}}-\frac{\frac{\partial T_{s}}{\partial z_{i}}}{T_{s}} \frac{I_{s, j}}{T_{s}} \equiv \frac{\frac{\partial I_{s-1, j}}{\partial z_{i}}}{T_{s-1}}-\frac{\frac{\partial T_{s-1}}{\partial z_{i}}}{T_{s-1}} \frac{I_{s-1, j}}{T_{s-1}} \quad\left(\bmod p^{s-e}\right)
$$

This congruence gives equation (5.2) as $s \rightarrow \infty$. Equation (5.3) follows from the congruences in (3.6) after dividing by $T_{s}(\boldsymbol{z} ; \lambda)$ and taking the limit $s \rightarrow \infty$. Equation (5.4) follows from equations (5.2) and (5.3).

Equation (5.5) follows from congruence (3.9) after dividing by $T_{s}(\boldsymbol{z} ; \lambda)$ and taking the limit $s \rightarrow \infty$.
5.5. Limiting vectors. The vector-function $\tilde{\mathcal{I}}(\boldsymbol{z} ; \lambda+2)=\left(\tilde{\mathcal{I}}_{1}(\boldsymbol{z} ; \lambda+2), \tilde{\mathcal{I}}_{2}(\boldsymbol{z} ; \lambda+2)\right)$ is defined on $\mathfrak{D}^{(m)}(\lambda)$, and the vector-function $\mathcal{I}(\boldsymbol{z} ; \lambda+2)=\left(\mathcal{I}_{1}(\boldsymbol{z} ; \lambda+2), \mathcal{I}_{2}(\boldsymbol{z} ; \lambda+2)\right)$ is defined on $\mathfrak{D}^{(m)}(\lambda+2)$, see Theorem 5.4.
Lemma 5.6. The vector-functions $\tilde{\mathcal{I}}(\boldsymbol{z} ; \lambda+2)$ and $\mathcal{I}(\boldsymbol{z} ; \lambda+2)$ are proportional on $\mathfrak{D}^{(m)}(\lambda) \cap \mathfrak{D}^{(m)}(\lambda+2)$, that is,

$$
\begin{equation*}
\tilde{\mathcal{I}}_{1}(\boldsymbol{z} ; \lambda+2) \mathcal{I}_{2}(\boldsymbol{z} ; \lambda+2)-\tilde{\mathcal{I}}_{2}(\boldsymbol{z} ; \lambda+2) \mathcal{I}_{1}(\boldsymbol{z} ; \lambda+2)=0 . \tag{5.6}
\end{equation*}
$$

Proof. To obtain $\tilde{\mathcal{I}}(\boldsymbol{z} ; \lambda+2)$ we divide the vector $\mathrm{t} I_{s}(\boldsymbol{z} ; \lambda+2)$ by $T_{s}(\boldsymbol{z} ; \lambda)$ and take the limit as $s \rightarrow \infty$. To obtain $\mathcal{I}(\boldsymbol{z} ; \lambda+2)$ we divide the same vector $I_{s}(\boldsymbol{z} ; \lambda+2)$ by $T_{s}(\boldsymbol{z} ; \lambda+2)$ and take the limit as $s \rightarrow \infty$. Hence the limits are proportional.

Lemma 5.7. Let $e \in \mathbb{Z}_{>0}, \lambda \in \Lambda_{e}$.
(i) Assume that $\boldsymbol{a} \in \mathfrak{D}^{(m)}(\lambda)$. Then $\left|T_{s}(\boldsymbol{a} ; \lambda)\right|_{p}=1$ for any $s \geqslant e$.
(ii) Assume that $\lambda$ is not divisible by $p$ and $\boldsymbol{a} \in \mathfrak{D}_{*}^{(m)}(\lambda)$. Then there exists $j \in\{1,2\}$ such that $\left|I_{s, j}(\boldsymbol{a} ; \lambda)\right|_{p}=1$ for any $s \geqslant e$.

Proof. The lemma follows from Lemmas 3.4 and 3.5.

## Lemma 5.8.

(i) Let $\lambda \in \Lambda$ be not divisible by $p$ and $\boldsymbol{a} \in \mathfrak{D}_{*}^{(m)}(\lambda)$. Then there exists $j \in\{1,2\}$ such that $\left|\mathcal{I}_{j}(\boldsymbol{a} ; \lambda)\right|_{p}=1$.
(ii) Let $\lambda+2 \in \Lambda$ be not divisible by $p$ and $\boldsymbol{a} \in \mathfrak{D}^{(m)}(\lambda) \cap \mathfrak{D}_{*}^{(m)}(\lambda+2)$. Then there exists $j \in\{1,2\}$ such that $\left|\tilde{\mathcal{I}}_{j}(\boldsymbol{a} ; \lambda+2)\right|_{p}=1$.
(iii) Let $\lambda \in \Lambda$ be divisible by $p$ and $\boldsymbol{a} \in \mathfrak{D}_{*}^{(m)}(\lambda)$. Then the vector $\mathcal{I}(\boldsymbol{a} ; \lambda)$ is nonzero.

Proof. Under assumptions of part (i), there exists $j \in\{1,2\}$ such that $\left|G_{j}(\boldsymbol{a} ; \lambda)\right|_{p}=1$. Then $\left|I_{s, j}(\boldsymbol{a} ; \lambda)\right|_{p}=\left|T_{s}(\boldsymbol{a} ; \lambda)\right|_{p}=1$ for all large $s$ by Lemmas 3.4, 3.5, 5.1, 5.7. Part (i) is proved.

Under assumptions of part (ii), there exists $j \in\{1,2\}$ such that $\left|G_{j}(\boldsymbol{a} ; \lambda+2)\right|_{p}=1$. Then $\left|I_{s, j}(\boldsymbol{a} ; \lambda+2)\right|_{p}=\left|T_{s}(\boldsymbol{a} ; \lambda)\right|_{p}=1$ for all large $s$ by Lemmas 3.4, 3.5, 5.1, 5.7. Part (ii) is proved.

To prove part (iii) consider equation (5.5),

$$
\tilde{\mathcal{I}}(\boldsymbol{a} ; \lambda+2)=\frac{1}{\lambda}\left[\begin{array}{cc}
\frac{\lambda+1}{a_{1}} & \frac{1}{a_{1}}  \tag{5.7}\\
\frac{1}{a_{2}} & \frac{\lambda+1}{a_{2}}
\end{array}\right] \mathcal{I}(\boldsymbol{a}, \lambda),
$$

which holds for $\boldsymbol{a} \in \mathfrak{D}^{(m)}(\lambda)$. By part (ii), the vector $\tilde{\mathcal{I}}(\boldsymbol{a} ; \lambda+2)$ is nonzero for $\boldsymbol{a} \in$ $\mathfrak{D}^{(m)}(\lambda) \cap \mathfrak{D}_{*}^{(m)}(\lambda+2)$. Since $\boldsymbol{a} \in \mathfrak{D}^{(m)}(\lambda) \cap \mathfrak{D}_{*}^{(m)}(\lambda+2) \cap \mathfrak{C}^{(m)}$ we have $a_{1} a_{2} \neq 0$. Hence the matrix in (5.7) is well defined, and therefore $\mathcal{I}(\boldsymbol{a} ; \lambda)$ is nonzero.
5.6. Invariant line bundle. Denote $\mathcal{W}=\left(\mathbb{Q}_{p}^{(m)}\right)^{2}$, The differential operators

$$
\mathcal{D}_{i}(\lambda)=\frac{\partial}{\partial z_{i}}-H_{i}(\boldsymbol{z} ; \lambda), \quad i=1,2
$$

define a connection on the trivial bundle $\mathcal{W} \times\left(\mathbb{Z}_{p}^{(m)}\right)^{2} \times \Lambda \rightarrow\left(\mathbb{Z}_{p}^{(m)}\right)^{2} \times \Lambda$ called the dynamical connection.

For any $\lambda \in \Lambda$, we have a map of local sections of the bundle $\mathcal{W} \times\left(\mathbb{Z}_{p}^{(m)}\right)^{2} \times\{\lambda\} \rightarrow$ $\left(\mathbb{Z}_{p}^{(m)}\right)^{2} \times\{\lambda\}$ to local sections of the bundle $\mathcal{W} \times\left(\mathbb{Z}_{p}^{(m)}\right)^{2} \times\{\lambda+2\} \rightarrow\left(\mathbb{Z}_{p}^{(m)}\right)^{2} \times\{\lambda+2\}$ defined by the formula,

$$
\begin{equation*}
\tau: s(\boldsymbol{z}) \mapsto K(\boldsymbol{z} ; \lambda) s(\boldsymbol{z}) \tag{5.8}
\end{equation*}
$$

We call the operator $\tau$ the $q K Z$ discrete connection on the trivial bundle $\mathcal{W} \times\left(\mathbb{Z}_{p}^{(m)}\right)^{2} \times \Lambda \rightarrow$ $\left(\mathbb{Z}_{p}^{(m)}\right)^{2} \times \Lambda$.

The dynamical and $q K Z$ connections are compatible. Namely, for $\lambda \in \Lambda$ and a local section $s(\boldsymbol{z})$ of $\mathcal{W} \times\left(\mathbb{Z}_{p}^{(m)}\right)^{2} \times\{\lambda\} \rightarrow\left(\mathbb{Z}_{p}^{(m)}\right)^{2} \times\{\lambda\}$ we have

$$
\begin{aligned}
\mathcal{D}_{1}(\lambda)\left(\mathcal{D}_{2}(\lambda) s(\boldsymbol{z})\right) & =\mathcal{D}_{2}(\lambda)\left(\mathcal{D}_{1}(\lambda) s(\boldsymbol{z})\right), \\
\tau\left(\mathcal{D}_{i}(\lambda) s(\boldsymbol{z})\right) & =\mathcal{D}_{i}(\lambda+2)(\tau s(\boldsymbol{z})), \quad i=1,2
\end{aligned}
$$

Denote

$$
\mathfrak{D}^{(m)}[\Lambda]:=\bigcup_{\lambda \in \Lambda} \mathfrak{D}_{*}^{(m)}(\lambda) \times\{\lambda\} \subset\left(\mathbb{Z}_{p}^{(m)}\right)^{2} \times \Lambda .
$$

For any $(\boldsymbol{a}, \lambda) \in \mathfrak{D}^{(m)}[\Lambda]$, the vector $\mathcal{I}(\boldsymbol{a}, \lambda)$ is nonzero by Lemma 5.8.
For any $(\boldsymbol{a}, \lambda) \in \mathfrak{D}^{(m)}[\Lambda]$, let $\mathcal{L}_{(\boldsymbol{a}, \lambda)} \subset \mathcal{W}$ be the one-dimensional vector subspace generated by the vector $\mathcal{I}(\boldsymbol{a}, \lambda)$. Then

$$
\mathcal{L}:=\bigcup_{(\boldsymbol{a}, \lambda) \in \mathfrak{D}^{(m)}[\Lambda]} \mathcal{L}_{(\boldsymbol{a}, \lambda)} \times\{(\boldsymbol{a}, \lambda)\} \rightarrow \mathfrak{D}^{(m)}[\Lambda]
$$

is an analytic line subbundle of the trivial bundle $\mathcal{W} \times \mathfrak{D}^{(m)}[\Lambda] \rightarrow \mathfrak{D}^{(m)}[\Lambda]$
Theorem 5.9. The line bundle $\mathcal{L} \rightarrow \mathfrak{D}^{(m)}[\Lambda]$ is invariant with respect to the dynamical and qKZ connections. More precisely,
(i) if $s(\boldsymbol{z})$ is a local section of $\mathcal{L}$ over $\mathfrak{D}_{*}^{(m)}(\lambda) \times\{\lambda\}$, then $\mathcal{D}_{i}(\lambda) s(\boldsymbol{z}), i=1,2$, also are local sections of $\mathcal{L}$ over $\mathfrak{D}_{*}^{(m)}(\lambda) \times\{\lambda\}$;
(ii) if $s(\boldsymbol{z})$ is a local section of $\mathcal{L}$ over $\left(\mathfrak{D}_{*}^{(m)}(\lambda) \cap \mathfrak{D}_{*}^{(m)}(\lambda+2)\right) \times\{\lambda\}$, then $\tau s(\boldsymbol{z})$, is a local section of $\mathcal{L}$ over $\left(\mathfrak{D}_{*}^{(m)}(\lambda) \cap \mathfrak{D}_{*}^{(m)}(\lambda+2)\right) \times\{\lambda+2\}$.
Proof. Let $(\boldsymbol{a}, \lambda) \in \mathfrak{D}_{*}^{(m)}(\lambda) \times\{\lambda\}$. Let $c(\boldsymbol{z})$ be a scalar analytic function at $\boldsymbol{a}$. Consider the local section $c(\boldsymbol{z}) \mathcal{I}(\boldsymbol{z} ; \lambda)$ of $\mathcal{L}$ at $(\boldsymbol{a}, \lambda)$. Then

$$
\begin{aligned}
\mathcal{D}_{i}(\lambda)(c(\boldsymbol{z}) \mathcal{I}(\boldsymbol{z} ; \lambda)) & =-c H_{i} \mathcal{I}+c \frac{\partial \mathcal{I}}{\partial z_{i}}+\frac{\partial c}{\partial z_{i}} \mathcal{I} \\
& =-c H_{i} \mathcal{I}+c\left(\mathcal{I}^{(i)}-\frac{1}{2} \mathcal{I}_{i} \mathcal{I}\right)+\frac{\partial c}{\partial z_{i}} \mathcal{I} \\
& =-c H_{i} \mathcal{I}+c\left(H_{i} \mathcal{I}-\frac{1}{2} \mathcal{I}_{i} \mathcal{I}\right)+\frac{\partial c}{\partial z_{i}} \mathcal{I} \\
& =\left(-\frac{c}{2} \mathcal{I}_{i}+\frac{\partial c}{\partial z_{i}}\right) \mathcal{I} .
\end{aligned}
$$

Here we used Lemma 5.5. Clearly, the last expression is a local section of $\mathcal{L}$ at $(\boldsymbol{a}, \lambda)$. Part (i) is proved.

By definition of $\tau$, we have $\tau(c(\boldsymbol{z}) \mathcal{I}(\boldsymbol{z} ; \lambda))=c(\boldsymbol{z}) K(\boldsymbol{z} ; \lambda) \mathcal{I}(\boldsymbol{z} ; \lambda)$. We also have the equality

$$
c(\boldsymbol{z}) K(\boldsymbol{z} ; \lambda) \mathcal{I}(\boldsymbol{z} ; \lambda)=c(\boldsymbol{z}) \tilde{\mathcal{I}}(\boldsymbol{z} ; \lambda+2),
$$

which holds on $\mathfrak{D}^{(m)}(\lambda)$, by Lemma 5.5. The vectors $\tilde{\mathcal{I}}(\boldsymbol{z} ; \lambda+2)$ and $\mathcal{I}(\boldsymbol{z} ; \lambda+2)$ are proportional on $\mathfrak{D}^{(m)}(\lambda) \cap \mathfrak{D}^{(m)}(\lambda+2)$, by Lemma 5.6. For the smaller set $\mathfrak{D}_{*}^{(m)}(\lambda) \cap \mathfrak{D}_{*}^{(m)}(\lambda+2)$, the initial vector vector $\mathcal{I}(\boldsymbol{z} ; \lambda)$ and the resulting vector $\mathcal{I}(\boldsymbol{z} ; \lambda+2)$ are both nonzero, by Lemmas 5.7 and 5.8. This proves part (ii).

### 5.7. Special points.

Lemma 5.10. The points $(0,1 ; 1),(1,0 ; 1),(1,1 ; 1)$ belong to $\mathfrak{D}_{*}^{(m)}(1) \times\{1\} \subset \mathfrak{D}^{(m)}[\Lambda]$.
Proof. Straightforward calculation shows that $T_{s}(0,1 ; 1)=(-1)^{\left(p^{s}-1\right) / 2}, I_{s, 1}(0,1 ; 1)=$ $(-1)^{\left(p^{s}-3\right) / 2}, I_{s, 2}(0,1 ; 1)=(-1)^{\left(p^{s}-3\right) / 2} \frac{p^{s}-1}{2}$. Hence $\mathcal{I}(0,1 ; 1)=\left(-1, \frac{1}{2}\right)$. Similarly we obtain $\mathcal{I}(1,0 ; 1)=\left(\frac{1}{2},-1\right), \mathcal{I}(1,1 ; 1)=\left(-\frac{1}{2},-\frac{1}{2}\right)$. These vectors are nonzero modulo $p$.

By Lemma 5.10 , the analytic vector-function $\mathcal{I}(\boldsymbol{z} ; 1)$ is nonzero at $(0,1 ; 1)$, and its values generate the line bundle $\mathcal{L}$ over a neighborhood of $(0,1)$ in $\mathfrak{D}_{*}^{(m)}(1)$. Over a neighborhood of $(0,1)$, the same line bundle can be defined differently. Consider the family of elliptic curves $X(\boldsymbol{z})$ defined by the equation $y^{2}=t\left(t-z_{1}\right)\left(t-z_{2}\right)$. If the parameter $\left(z_{1}, z_{2}\right)$ is close to $(0,1)$ the curve $X(\boldsymbol{z})$ has a vanishing cycle denoted by $C_{0,1}$. The vector-function

$$
\begin{equation*}
I^{\left(C_{0,1}\right)}(\boldsymbol{z} ; 1):=\int_{C_{0,1}}\left(\frac{1}{\left(t-z_{1}\right) y}, \frac{1}{\left(t-z_{2}\right) y}\right) d t \tag{5.9}
\end{equation*}
$$

is holomorphic at $(0,1)$, solves dynamical equations $(2.2)$ for $\lambda=1$, and $I^{(C)}(0,1 ; 1) \neq 0$. The values of $I^{(C)}(\boldsymbol{z} ; 1)$ generate a line bundle denoted by $\mathcal{L}_{0,1}$ over a neighborhood of $(0,1)$. The line bundle $\mathcal{L}_{0,1}$ is invariant with respect to the dynamical connection. The dynamical connection for $\lambda=1$ does not have other invariant proper nontrivial subbundles near $(0,1)$ since other solutions of equations (2.2) for $\lambda=1$ at $(0,1)$ include $\log z_{1}$. Hence our line bundle $\mathcal{L}$ coincides with the line bundle $\mathcal{L}_{0,1}$ over a neighborhood of $(0,1) \subset \mathfrak{D}_{*}^{(m)}(1)$.

Similarly, the elliptic curve $X(\boldsymbol{z})$ has a vanishing cycle denoted by $C_{1,0}$ if the parameter $\left(z_{1}, z_{2}\right)$ is close to $(1,0)$. The values of the nonzero vector-function

$$
I^{\left(C_{1,0}\right)}(\boldsymbol{z} ; 1):=\int_{C_{1,0}}\left(\frac{1}{\left(t-z_{1}\right) y}, \frac{1}{\left(t-z_{2}\right) y}\right) d t
$$

generate a line bundle denoted by $\mathcal{L}_{1,0}$ over a neighborhood of $(1,0)$. Our line bundle $\mathcal{L}$ coincides with the line bundle $\mathcal{L}_{1,0}$ over a neighborhood of $(1,0)$ in $\mathfrak{D}_{*}^{(m)}(1)$.

Also, the elliptic curve $X(\boldsymbol{z})$ has a vanishing cycle denoted by $C_{1,1}$ if the parameter $\left(z_{1}, z_{2}\right)$ is close to $(1,1)$. Then the values of the nonzero vector-function

$$
I^{\left(C_{1,1}\right)}(\boldsymbol{z} ; 1):=\int_{C_{1,1}}\left(\frac{1}{\left(t-z_{1}\right) y}, \frac{1}{\left(t-z_{2}\right) y}\right) d t
$$

generate a line bundle $\mathcal{L}_{1,1}$ over a neighborhood of $(1,1)$. Our line bundle $\mathcal{L}$ coincides with the line bundle $\mathcal{L}_{1,1}$ over a neighborhood of $(1,1)$ in $\mathfrak{D}_{*}^{(m)}(1)$.

Thus our global line bundle $\mathcal{L}$ extends over the field $\mathbb{Q}^{(m)}$ the three local line bundles $\mathcal{L}_{0,1}, \mathcal{L}_{1,0}, \mathcal{L}_{1,1}$, each defined by integrals over the cycles vanishing at different places. This $p$-adic phenomenon was observed by B. Dwork in a different context, see [Dw] and also [V2, Appendix], [VZ1]. The corresponding global line bundle was called a $p$-cycle in [Dw].

The operator $\tau$ of the $q K Z$ difference connection identifies solutions of the dynamical equations (2.2) with parameter $\lambda$ with solutions of dynamical equations with parameter $\lambda+2$. Hence our line bundle $\mathcal{L}$ over a neighborhood of the point $(0,1) \in \mathfrak{D}_{*}^{(m)}(1+2 k)$,
$k \in \mathbb{Z}$, corresponds to the line bundle generated by the vector-function

$$
\begin{equation*}
I^{\left(C_{0,1}\right)}(\boldsymbol{z} ; 1+2 k)=\int_{C_{0,1}}\left(\frac{1}{t^{k}\left(t-z_{1}\right) y}, \frac{1}{t^{k}\left(t-z_{2}\right) y}\right) d t \tag{5.10}
\end{equation*}
$$

The vector-valued functions

$$
\begin{aligned}
I^{\left(C_{1,0}\right)}(\boldsymbol{z} ; 1+2 k) & =\int_{C_{1,0}}\left(\frac{1}{t^{k}\left(t-z_{1}\right) y}, \frac{1}{t^{k}\left(t-z_{2}\right) y}\right) d t \\
I^{\left(C_{1,1}\right)}(\boldsymbol{z} ; 1+2 k) & =\int_{C_{1,1}}\left(\frac{1}{t^{k}\left(t-z_{1}\right) y}, \frac{1}{t^{k}\left(t-z_{2}\right) y}\right) d t
\end{aligned}
$$

play similar roles in neighborhoods of points $(1,0)$ and $(1,1)$, respectively.
5.8. Monodromy. For an odd integer $\lambda$, the differential operators $\mathcal{D}_{i}(\lambda), i=1,2$, define a flat dynamical connection on the trivial bundle $\mathbb{C}^{2} \times \mathbb{C}^{2} \rightarrow \mathbb{C}^{2}$. The flat sections of the connection have the form

$$
I^{(C)}(\boldsymbol{z} ; \lambda, \mu)=\int_{C} t^{-\lambda / 2}\left(t-z_{1}\right)^{-1 / 2}\left(t-z_{2}\right)^{-1 / 2}\left(\frac{1}{t-z_{1}}, \frac{1}{t-z_{2}}\right) d t
$$

see (1.1). The monodromy of this connection does not depend on the choice of the odd integer $\lambda$ and is isomorphic to the monodromy of the Gauss-Manin connection on the bundle with fibers being the first homology groups of elliptic curves $X(\boldsymbol{z})$ of the family defined by the equation $y^{2}=t\left(t-z_{1}\right)\left(t-z_{2}\right)$. It is classically known that this monodromy is irreducible. Hence the dynamical connection defined by $\mathcal{D}_{i}(\lambda), i=1,2$, over the field of complex numbers has no invariant line subbundles. Thus the presence of our line subbundle $\mathcal{L} \rightarrow \mathfrak{D}^{(m)}[\Lambda]$, invariant with respect to the dynamical and $q K Z$ connections, is a specific $p$-adic feature.

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