Topology of Toric Gravitational Instantons

Gustav Nilsson
Max Planck Institute for Gravitational Physics (Albert Einstein Institute)
Am Mühlenberg 1, D-14476 Potsdam, Germany

Abstract

For an asymptotically locally Euclidean (ALE) or asymptotically locally flat (ALF) gravitational instanton (M, g) with toric symmetry, we express the signature of (M, g) directly in terms of its rod structure. Applying Hitchin–Thorpe-type inequalities for Ricci-flat ALE/ALF manifolds, we formulate necessary conditions that the rod structures of toric ALE/ALF instantons must satisfy, as a step toward a classification of such spaces. Finally, we apply these results to the study of rod structures with three turning points.

1 Introduction

Ricci-flat Riemannian 4-manifolds are the vacuum solutions to the Riemannian analog of the Einstein equations. Such manifolds occur in the study of quantum gravity, commonly under the name *gravitational instantons* [13], and in this and many other contexts it is common to restrict attention to such manifolds which are complete, non-compact, and with curvature decaying sufficiently fast. The volume growth is then either quartic (ALE), cubic (AF/ALF), quadratic (ALG), or linear (ALH). An ALE manifold has a metric that is asymptotic to $\mathbb{R}^4/\Gamma$, where Γ is a finite subgroup of SO(4) which acts freely on S^3. An ALF manifold, on the other hand, is asymptotic to a circle bundle over $\mathbb{R}^3$.

The classification problem for gravitational instantons is an interesting problem, in which many conjectures have been made. For instance, Gibbons conjectured in 1979 that the only AF gravitational instantons are the Riemannian Schwarzschild and Riemannian Kerr spaces [9], a conjecture which was proven wrong by Chen and Teo in the 2011 paper [7], where they presented a new AF gravitational instanton.

The examples mentioned so far are all *toric*, i.e. admit an effective action of the torus $T^2 = \mathrm{U}(1) \times \mathrm{U}(1)$ by isometries. Toric gravitational instantons were studied in the paper [11]. In that paper, the author introduced the formalism of so-called *rod structures* (defined in detail in Section 2.1), which are combinatorial objects associated to toric gravitational instantons, containing information about the T^2-action. The rod structure formalism was later used in [8, 15].

The rod structure of a toric gravitational instanton (M, g) with n fixed points consists of a sequence of 2-vectors $(v_0, \ldots, v_n)$ with integer entries, satisfying the determinant condition $\det \begin{pmatrix} v_{i-1} & v_i \end{pmatrix} = \pm 1$, which follows from the regularity of the instanton. Without loss of generality, we can assume that

$$\det \begin{pmatrix} v_{i-1} & v_i \end{pmatrix} = 1. \tag{1}$$

Much effort has been devoted to constructing solutions of the Einstein equations with toric symmetry. One example of such a method is the *soliton method* [2, 6], which was used to construct the Chen–Teo instanton (the AF gravitational instanton discussed in [7]). Another method is given in the more recent paper [17]. In spite of much progress, it is in general a hard problem to determine whether a rod structure can be realized as the rod structure of a toric gravitational instanton.

The rod structure of M can be shown to determine its topology. In particular, the Euler characteristic is well-known to equal the number of fixed points, n. In this paper we determine the signature in terms of the rod structure. We do so by explicitly constructing a model manifold with a T^2-action, having the same rod structure as M. On this manifold, we determine a basis for $H_2(M)$, and then compute the intersection form in terms of this basis. This leads to the following result.

Theorem 1. *Let (M, g) be a toric gravitational instanton with rod structure $(v_0, \ldots, v_n)$ satisfying (1). Then there exists a basis for $H_2(M, \mathbb{R})$ in which the intersection form on M is represented by the matrix*

$$\begin{pmatrix} d_1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & d_2 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & d_3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & d_{n-3} & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & d_{n-2} & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & d_{n-1} \end{pmatrix}, \tag{2}$$

where $d_i = -\det\begin{pmatrix} v_{i-1} & v_{i+1} \end{pmatrix}$. In particular, the signature $\tau(M)$ is the signature of this matrix.

Theorem 1 is then used together with Hitchin–Thorpe-type inequalities for ALE or ALF instantons in order to give necessary conditions on the possible rod structures. The Hitchin–Thorpe inequality for Ricci-flat ALE 4-manifolds states that

$$2\left(\chi(M) - \frac{1}{|\Gamma|}\right) \geq 3|\tau(M) + \eta_S(S^3/\Gamma)|, \tag{3}$$

and for Ricci-flat ALF 4-manifolds, the Hitchin–Thorpe inequality reads

$$2\chi(M) \geq 3\left|\tau(M) - \frac{e}{3} + \operatorname{sgn}(e)\right|, \tag{4}$$

where e is the Euler number of the asymptotic circle bundle. Both $|\Gamma|$ and e can be expressed in terms of the rod structure, and as a consequence, these inequalities and Theorem 1 leads to restrictions on the possible rod structures.

The special case of rod structures with three turning points were discussed in [8], where the question was raised whether topological or other constraints could perhaps rule out certain rod structures. We address this question with the following result.

Theorem 2. *If a toric gravitational instanton (M, g) has the rod structure shown in Figure 1 and is ALE, then (a, b) must be one of the pairs in Figure 2a. If instead M is ALF, then either $a = 0$, $b = 0$, or (a, b) is one of the pairs in Figure 2b.*

In this paper we have only used the assumption of Ricci-flatness in a minimal way, through the Hitchin–Thorpe inequality. Using the full field equations, for example as in [17], one could potentially use knowledge about the topology in order to prove certain uniqueness results. For instance, we expect that the Riemannian black hole uniqueness conjecture in the toric case could be approached in this way.

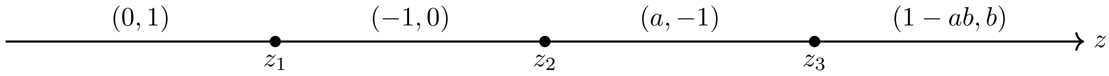
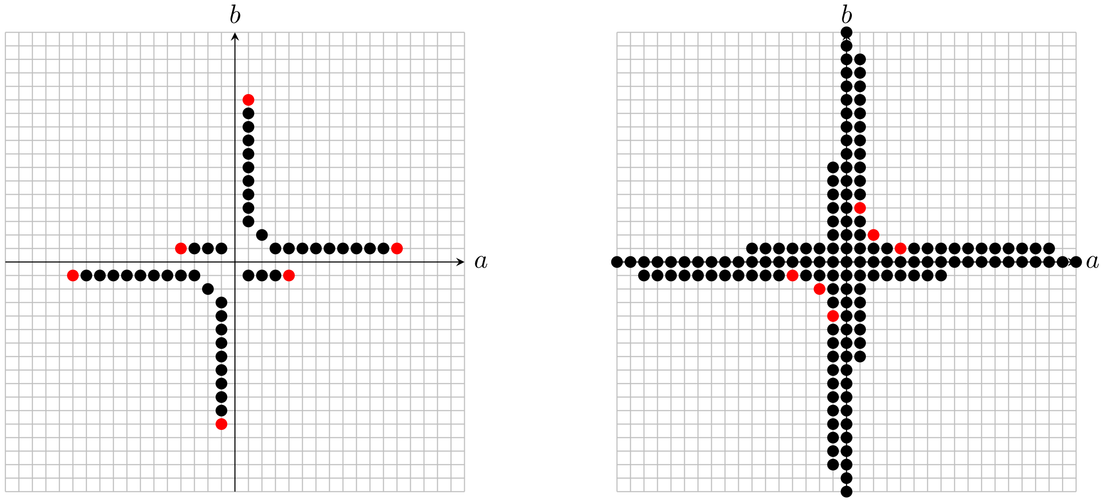


Figure 1: A rod structure with three turning points.



(a) ALE

(b) ALF

Figure 2: The remaining possibilities for (a, b).

Overview of this paper

In Section 2 we give the definition of toric gravitational instanton, discuss the possible asymptotic geometries, and introduce the concept of rod structures. In Section 3 we take a look at some examples of known toric gravitational instantons, and list their rod structures, among other properties. In Section 4, we compute the signature of a toric gravitational instanton in terms of its rod structure, and then use this together with Hitchin–Thorpe-type inequalities in order to formulate necessary conditions on possible rod structures. Finally, in Section 4.2 we apply this to the case of rod structures with 3 fixed points, and briefly discuss the case of an AF rod structure with 4 fixed points.

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2 Toric Gravitational Instantons

In this section we introduce the concept of a gravitational instanton, the main concept of study in this paper. We also introduce the various possibilities for the asymptotic behavior of gravitational instantons. Then we define toric gravitational instantons, and introduce the concept of a rod structure.

Table 1: Possible asymptotic geometries of gravitational instantons.

	Boundary at infinity
ALE[1]	S^3/Γ
ALF-A_{-1}[2]	$S^2 \times S^1$
ALF-A_k, $k \neq -1$	$L(\lvert k+1\rvert, 1)$
ALF-D_2[3]	$(S^2 \times S^1)/\mathbb{Z}_2$
ALF-D_k, $k \neq 2$	$S^3/D_{4\lvert k-2\rvert}$
ALG	T^2-bundle over S^1
ALH-splitting	$\{\pm 1\} \times T^3$
ALH-non-splitting	T^3

Definition 1. A *gravitational instanton* is a Riemannian 4-manifold (M, g) which is complete and Ricci-flat, satisfying the following.

(a) There exists a nonincreasing function $K : [0, \infty) \to [0, \infty)$ such that

$$\int_1^\infty \frac{K(s)}{s}\, ds < \infty, \tag{5}$$

along with a point $p \in M$ such that

$$|\mathrm{Rm}_q|_g \leq \frac{K(d(p,q))}{d(p,q)^2} \tag{6}$$

for all $q \in M$, where $d(\cdot, \cdot)$ is the Riemannian distance function.

(b) There exists a constant $\kappa > 0$, a function $\epsilon : (0, \infty) \to (0, \infty)$ satisfying $\lim_{r \to 0} \epsilon(r) = 0$, along with a compact set $K \subseteq M$ such that for any $r > 0$ and for any geodesic loop γ whose length is less than κr, and which is based at a point in $M \setminus K$, the holonomy of γ rotates any vector by at most the angle $\epsilon(r)$.

Remark 1. We do not require that (M, g) is hyper-Kähler (see Definition 4), an assumption which is sometimes included in the definition of gravitational instanton.

From the conditions (a) and (b) in Definition 1, it follows (see [5]) that (M, g) has must have one out of several special geometries near infinity. The different cases are divided into ALE, ALF, ALG and ALH depending on the volume growth, and the possible subcases [5] are shown in Table 1. Here, we say that the *boundary at infinity* of M is N, if there exists a compact set $K \subseteq M$ such that $\overline{M \setminus K}$ is homeomorphic to $[0, \infty) \times N$, and if the homeomorphism can be chosen to map ∂K onto $\{0\} \times N$. By [5, Lemma 5.27], the boundary at infinity is well-defined up to h-cobordism, and in particular up homotopy equivalence.

Definition 2. A *toric gravitational instanton* is a simply connected gravitational instanton (M, g) together with an effective action of the torus $T^2 = \mathrm{U}(1) \times \mathrm{U}(1)$ by isometries. We assume that the fixed point set[4] is finite and non-empty, and that there are no points whose isotropy group is non-trivial and finite.

Remark 2. The requirement that there are no points which non-trivial and finite isotropy group is equivalent to requiring M to be *locally standard*, see [22].

[1] Here, Γ is any finite subgroup of SO(4) which acts freely on S^3.
[2] ALF-A_{-1} is also called AF.
[3] Here, the action of $\mathbb{Z}_2$ on $S^2 \times S^1$ is given by $(x, y, z, \theta) \mapsto (-x, -y, -z, -\theta)$.
[4] That is, the set of points in M whose isotropy group is the entire group T^2.

2.1 Rod Structure

Lemma 1. *Let (M, g) be a toric gravitational instanton. Then the orbit space $\widehat{M} = M/T^2$ is a 2-dimensional (smooth) manifold with corners (see [18, p. 415]). Furthermore, $\widehat{M}$ is homeomorphic to the closed upper half plane $\{z + i\rho \mid z \in \mathbb{R}, \rho \geq 0\}$, and under this homeomorphism, the set where $\rho > 0$ corresponds to orbits whose points have trivial isotropy group. There exist points $z = z_1, \dots, z_n$ on the axis $\rho = 0$, corresponding to the fixed points, dividing this axis into segments $z_i < z < z_{i+1}$ (where we take $z_0 = -\infty$ and $z_{n+1} = \infty$). For each such segment, the points in the corresponding orbits all have the same isotropy group, and this isotropy group is a circle subgroup of the torus, given as*

$$T^2(v_i^1, v_i^2) := \{(e^{iv_i^1\theta}, e^{iv_i^2\theta}) \mid \theta \in \mathbb{R}\} \tag{7}$$

for some pair $v_i = (v_i^1, v_i^2)$, where v_i^1 and v_i^2 are coprime integers. The vectors $(v_0, \dots, v_n)$ satisfy the determinant condition,

$$\det \begin{pmatrix} v_{i-1} & v_i \end{pmatrix} = \pm 1. \tag{8}$$

Proof. Since M has no points non-trivial finite isotropy group, the fact that $\widehat{M}$ is a 2-manifold with corners follows from the non-compact analogue of [14, Theorem 1]. From the discussion following the theorem, it is also clear that the interior of $\widehat{M}$ corresponds to points with trivial isotropy and that the corners correspond to fixed points. It can also be seen from this discussion that for a boundary segment, the isotropy group is constant on this segment, and it is of the form $T^2(v)$ for some pair v of coprime integers, and that if $T^2(v)$ and $T^2(w)$ are the isotropy groups corresponding to two segments meeting in a corner, then $\det \begin{pmatrix} v & w \end{pmatrix} = \pm 1$. Our statement will thus follow if we show that $\widehat{M}$ is homeomorphic to the closed upper half plane.

To this end, first note that the orbit space $\widehat{M}$ is the quotient of a non-compact manifold by a compact Lie group, and is therefore non-compact. We claim that $\widehat{M}$ is simply connected. To see this, take any loop $\widehat{\gamma}$ in $\widehat{M}$. Since M has a fixed point, we can assume that the basepoint of $\widehat{\gamma}$ is the image of such a point. We can lift $\widehat{\gamma}$ to a curve γ in M, and because the fiber of the basepoint of $\widehat{\gamma}$ consists of a single point, γ must also be a loop. Using the assumption that M is simply connected, we thus see that γ, and therefore $\widehat{\gamma}$ is null-homotopic.

Of the asymptotic geometries in Table 1, all except ALH-splitting have only one end. Furthermore, M cannot be ALH-splitting, since by [4, Theorem 3.7], M would then be homeomorphic to $\mathbb{R} \times T^3$, which contradicts the assumption that M is simply connected. Thus M has only one end, and one can see from this that also $\widehat{M}$ has only one end. By the classification of simply connected surfaces with boundary, $\widehat{M}$ is homeomorphic to the closed upper half plane. □

Definition 3. The sequence $(v_0, \dots, v_n)$, where the vectors $v_i = (v_i^1, v_i^2)$ are as in Lemma 1, is called the *rod structure* of M.

Remark 3. Some papers (e.g. [17]) define the rod structure to also include the lengths $z_i - z_{i-1}$ of the boundary segments. Although these lengths are not well-defined in the context of Lemma 1, they are well-defined if one requires the homeomorphism from $\widehat{M}$ to the closed upper half plane to be given by a canonical set of coordinates, the *Weyl–Papapetrou coordinates.*

Theorem 3. *Let (M, g) be a toric gravitational instanton. Then (M, g) is either ALE with a cyclic group Γ, or ALF-A_k.*

Proof. Fix an identification of the orbit space $\widehat{M}$ with the closed upper half plane $\{z + i\rho \mid \rho \geq 0\}$ as in Lemma 1; such an identification can be chosen to be a diffeomorphism away from the corners. With $v_0, \dots, v_n$ and $z_1, \dots, z_n$ as in Lemma 1, set $R = \max(|z_1|, \dots, |z_n|)$, and consider the subset

$\widehat{A} = \{z + i\rho \mid \rho \geq 0, z^2 + \rho^2 > R^2\} \subseteq \widehat{M}$. In the case where $v_0 = v_n$, the set $\widehat{A}$ can be realized as the orbit space of $(R, \infty) \times S^2 \times S^1$. Here, the latter is endowed with the T^2-action given by

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot (r, t, z_1, z_2) = (r, t, e^{i(a_{11}\theta_1 + a_{12}\theta_2)} z_1, e^{i(a_{21}\theta_1 + a_{22}\theta_2)} z_2), \tag{9}$$

where

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} v_0 & v_1 \end{pmatrix}^{-1}, \tag{10}$$

and where we are identifying S^2 as a subset of $\mathbb{R} \times \mathbb{C}$. By [22, Theorem 1.1], this means that there exists a diffeomorphism $\overline{M \setminus K} \to [R+1, \infty) \times S^2 \times S^1$ which maps ∂K onto $\{R+1\} \times S^2 \times S^1$, where K is the compact set given by $K = \{z + i\rho \mid \rho \geq 0, z^2 + \rho^2 \leq (R+1)^2\}$. Thus, $S^2 \times S^1$ is the boundary at infinity of M.

Now consider the case where $v_0 \neq v_n$. Then similar reasoning shows that the boundary at infinity of M is a lens space $L(p, q)$, where $p = |\det \begin{pmatrix} v_0 & v_n \end{pmatrix}|$ and $q = |\det \begin{pmatrix} v_1 & v_n \end{pmatrix}|$.

We can now rule out any asymptotic geometry for which the boundary at infinity is not homotopy equivalent to $S^2 \times S^1$ or to a lens space. Looking at Table 1, this immediately rules out ALH-splitting, since the end is not even path-connected in that case. Using the fact that $\pi_1(L(p,q))$ is cyclic, we can also rule out any end of the form S^3/Γ where Γ is not cyclic, since $\pi_1(S^3/\Gamma) = \Gamma$. Thus, M is neither ALF-D_k with $k \neq 2$, ALE with non-cyclic group Γ, nor ALH-non-splitting.

It remains to rule out the cases ALF-D_2 and ALG. For ALF-D_2, note that $(S^2 \times S^1)/\mathbb{Z}_2$ is homeomorphic to the quotient $(S^2 \times \mathbb{R})/G$, where G is the subgroup

$$\left\{ \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \middle| b = \pm 1, a \in \mathbb{Z} \right\} \subseteq \mathrm{GL}(2, \mathbb{Z}), \tag{11}$$

acting on $S^2 \times \mathbb{R}$ by

$$\left(\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}, (x_1, x_2, x_3, y) \right) \mapsto (bx_1, bx_2, bx_3, by + a). \tag{12}$$

The action of G on $S^2 \times \mathbb{R}$ is easily seen to be a covering space action (see [12, p. 72]), and since $S^2 \times \mathbb{R}$ is simply connected, G is the fundamental group of $(S^2 \times S^1)/\mathbb{Z}_2$. But G is not cyclic, since it is infinite and contains the element $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, whose order is 2.

Finally, to rule out ALG, we will investigate the fundamental group of a fiber bundle $E \to S^1$ with fiber T^2. By [12, Theorem 4.41, Proposition 4.48], there is an exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_2(S^1) \longrightarrow \pi_1(T^2) \longrightarrow \pi_1(E) \longrightarrow \cdots. \tag{13}$$

But the homotopy group $\pi_2(S^1)$ is trivial, since any continuous map $S^2 \to S^1$ lifts to the universal cover $\mathbb{R}$, which is contractible. Since also $\pi_1(T^2) = \mathbb{Z}^2$, this means that there is an injective group homomorphism $\mathbb{Z}^2 \to \pi_1(E)$, which cannot happen if $\pi_1(E)$ is cyclic. $\square$

Remark 4. Consider $M = \mathbb{R}^2 \times S^1 \times S^1$, with a T^2-action where the first circle factor acts by rotating $\mathbb{R}^2$ around the origin, and the second circle factor acts by rotating one of the circle factors. Then the quotient space $\widehat{M} = M/T^2$ can be identified with $S^1 \times [0, \infty) \cong \mathbb{R}^2 \setminus B_1(0)$. Here, the inner circle represents points fixed by the subgroup corresponding to the first factor, while the exterior points correspond to points with trivial isotropy group. This gives an analogue of a rod structure for an ALH space, where there is one rod joined to itself at both ends, and no turning points.

Remark 5. It is an interesting question to ask whether or not there exist ALG or ALH gravitational instantons with toric symmetry, where the assumption of simple connectivity is omitted. All of the currently known examples are hyper-Kähler, and none of them admit even an S^1 symmetry.[5]

3 Examples

In this chapter we consider some known examples of toric gravitational instantons. The examples are listed in Section 3, and their rod structures are shown in Figure 3. Further details about these instantons can be found in [7, 8, 10].

3.1 Euclidean Space

An elementary example is Euclidean space $\mathbb{R}^4$, which is of course connected, non-compact, complete and Ricci-flat. By identifying $\mathbb{R}^4$ with $\mathbb{C}^2$, we can define an isometric T^2-action on $\mathbb{R}^4$ by $(e^{i\theta_1}, e^{i\theta_2}) \cdot (z_1, z_2) = (e^{i\theta_1} z_1, e^{i\theta_2} z_2)$. Under this action, the points of the form $(z_1, 0)$, where $z_1 \neq 0$, have isotropy group $T^2(0,1)$, and points of the form $(0, z_2)$, where $z_2 \neq 0$, have isotropy group $T^2(1,0)$. The origin has isotropy group T^2, i.e. is fixed under the whole action, all other points have trivial isotropy group.

3.2 Riemannian Schwarzschild Space

An interesting example is provided by the so-called Riemannian Schwarzschild metric, which is a Riemannian metric on $\mathbb{R}^2 \times S^2$. For $m > 0$, we define a function r on $\mathbb{R}^2$ implicitly by the relationship

$$\widetilde{r}^2 = \Big(\frac{r}{2m} - 1\Big) e^{r/2m}, \tag{14}$$

where $\widetilde{r}$ is the radial coordinate on $\mathbb{R}^2$; since the expression on the right is zero when $r = 2m$, is strictly increasing, and goes to ∞ as $r \to \infty$, this makes r well-defined and implies $r \geq 2m$. Since

$$\frac{d}{dr}\Big(\Big(\frac{r}{2m} - 1\Big) e^{r/2m}\Big) = \frac{r}{4m^2} e^{r/2m} > 0, \tag{15}$$

r is smooth as a function of $\widetilde{r}^2$, and therefore smooth as a function on $\mathbb{R}^2$. Using this, we can define a metric on $\mathbb{R}^2 \times S^2$ by

$$g = \frac{32m^3}{r} e^{-r/2m} g_{\mathbb{R}^2} + r^2 g_{S^2}, \tag{16}$$

where $g_{\mathbb{R}^2}$ and g_{S^2} are the standard metrics on $\mathbb{R}^2$ and S^2. This is clearly well-defined and non-degenerate, since $r \geq 2m$. The metric g is called the *Riemannian Schwarzschild metric of mass m*, and when $\mathbb{R}^2 \times S^2$ is equipped with the metric g, the resulting Riemannian manifold M is called the *Riemannian Schwarzschild space of mass m*.

In polar coordinates $(\widetilde{r}, \widetilde{\tau})$ on $\mathbb{R}^2$, the expression of the metric becomes

$$g = \frac{32m^3}{r} e^{-r/2m} (d\widetilde{r}^2 + \widetilde{r}^2 \, d\widetilde{\tau}^2) + r^2 g_{S^2}. \tag{17}$$

[5] Olivier Biquard, private communication.

Table 2: Some known toric gravitational instantons.

	Topology	Euler characteristic	Signature
Euclidean space	$\mathbb{R}^4$	1	0
Schwarzschild	$\mathbb{R}^2 \times S^2$	2	0
Kerr	$\mathbb{R}^2 \times S^2$	2	0
Taub–NUT	$\mathbb{R}^4$	1	0
Taub-bolt	$\mathbb{C}P^2 \setminus \{*\}$	2	1
Eguchi–Hanson	TS^2	2	1
Chen–Teo	$\mathbb{C}P^2 \setminus S^1$	3	1

	Isometry group	Asymptotic geometry	Hyper-Kähler?
Euclidean space	$\mathbb{R}^4 \rtimes \mathrm{O}(4)$	ALE	Yes
Schwarzschild	$\mathrm{O}(2) \times \mathrm{O}(3)$	ALF-A_{-1}	No
Kerr	$\mathrm{O}(2) \times \mathrm{O}(2)$	ALF-A_{-1}	No
Taub–NUT	$(\mathrm{U}(1) \times \mathrm{SU}(2))/\mathbb{Z}_2$	ALF-A_0	Yes
Taub-bolt	$(\mathrm{U}(1) \times \mathrm{SU}(2))/\mathbb{Z}_2$	ALF-A_0	No
Eguchi–Hanson	$(\mathrm{U}(1) \times \mathrm{SU}(2))/\mathbb{Z}_2$	ALE	Yes
Chen–Teo	T^2	ALF-A_{-1}	No

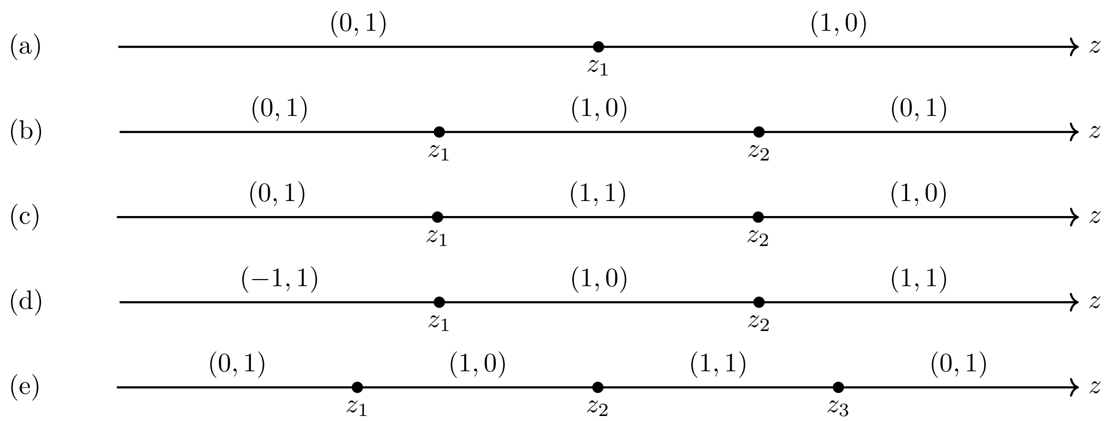

Figure 3: Rod structures of (a) $\mathbb{R}^4$ and Taub–NUT; (b) Schwarzschild and Kerr; (c) Taub-bolt; (d) Eguchi–Hanson; (e) Chen–Teo.

Differentiating (14), squaring both sides and rearranging, we get

$$d\widetilde{r}^2 = \frac{r^2 e^{r/m}}{64m^4\widetilde{r}^2}\,dr^2 = \frac{re^{r/2m}}{32m^4}\left(1-\frac{2m}{r}\right)^{-1} dr^2\,. \tag{18}$$

Introducing the coordinate $\tau = \widetilde{\tau}/4m$, a quick computation also shows that

$$\widetilde{r}^2\,d\widetilde{\tau}^2 = \frac{re^{r/2m}}{32m^3}\,d\tau^2\,. \tag{19}$$

In the coordinates (r,τ), the metric therefore takes the more widely recognized form

$$g = \left(1-\frac{2m}{r}\right)d\tau^2 + \left(1-\frac{2m}{r}\right)^{-1} dr^2 + r^2 g_{S^2}. \tag{20}$$

The Riemannian Schwarzschild space is clearly non-compact, since it has the topology $\mathbb{R}^2\times S^2$. From (20) one can also see that balls in the $\mathbb{R}^2$ factor, centered at the origin, are also balls with respect to the standard metric on $\mathbb{R}^2$. Although the radius of such a ball is not necessarily the same with respect to the two metrics, this shows that a subset of $\mathbb{R}^2\times S^2$ is bounded with respect to the standard metric if and only if it is bounded with respect to g. In particular closed and bounded subsets of M are compact, so according to one of the equivalent conditions for completeness (see [21, Theorem 5.7.1]), M is complete. Taking spherical coordinates (θ,ϕ) for S^2, one can also use the coordinates (r,τ,θ,ϕ) to see that it is Ricci-flat, using the form (20) to calculate the Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2}\sum_l g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \tag{21}$$

and then calculating the Ricci tensor using the coordinate formula

$$R_{ij} = \sum_k(\partial_k\Gamma^k_{ij} - \partial_i\Gamma^k_{kj}) + \sum_{k,l}(\Gamma^l_{ij}\Gamma^k_{kl} - \Gamma^l_{kj}\Gamma^k_{il}). \tag{22}$$

We introduce on M an isometric T^2-action defined by $(e^{i\theta_1},e^{i\theta_2})\cdot(z_1,z_2,t) = (e^{i\theta_1}z_1, e^{i\theta_2}z_2, t)$, where S^2 has been identified with a subset of $\mathbb{C}\times\mathbb{R}$. It can then be seen that points $(z_1,0,t)$ with $z_1\neq 0$ have isotropy group $T^2(0,1)$, points $(0,z_2,t)$ with $z_2\neq 0$ have isotropy group $T^2(1,0)$. The points $(0,0,\pm1)$ are fixed points, while all other points have trivial isotropy.

3.3 Riemannian Kerr Space

Let $m>0$ and $a\in\mathbb{R}$, and consider the manifold $M=\mathbb{R}^2\times S^2$. Letting $(\widetilde{r},\widetilde{\tau})$ be polar coordinates in the plane, and letting $(\theta,\widetilde{\phi})$ be spherical coordinates on the sphere, $(\widetilde{r},\widetilde{\tau},\theta,\widetilde{\phi})$ is a set of coordinates on the subset $(\mathbb{R}^2\setminus\{0\})\times(S^2\setminus\{N,S\})$, and $\widetilde{r}^2$ and $\widetilde{\phi}^2$ are smooth functions on all of M. Now define $r_\pm = \sqrt{m^2+a^2}\pm m$ and $\kappa_+ = (2m(1+m/\sqrt{m^2+a^2}))^{-1}$, and let r be defined by the relation

$$\widetilde{r}^2 = \frac{r-r_+}{2m}\left(\frac{r+r_-}{2m}\right)^{r_-/r_+} e^{2\kappa_+ r}. \tag{23}$$

Because

$$\frac{d}{dr}\left(\frac{r-r_+}{2m}\left(\frac{r+r_-}{2m}\right)^{r_-/r_+} e^{2\kappa_+ r}\right) > 0 \tag{24}$$

whenever $r > 0$, this defines r uniquely, as a smooth function of $\widetilde{r}^2$, and one can check that $r \geq r_+$, with equality if and only if $\widetilde{r} = 0$. Now let $f = 4m^2((r+r_-)/2m)^{2m/r_+}e^{-2\kappa_+ r}$, $\Sigma = r^2 - a^2\cos^2\widetilde{\theta}$ and $\Sigma_+ = r_+^2 - a^2\cos^2\widetilde{\theta}$, and consider the covariant 2-tensor[6]

$$g = \frac{f\Sigma}{\kappa_+^2(r^2-a^2)^2}(d\widetilde{r}^2 + \widetilde{r}^2\,d\widetilde{\tau}^2) + \Sigma\,d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left((r^2-a^2)\,d\widetilde{\phi} - \frac{af\widetilde{r}^2}{\sqrt{m^2+a^2}}\left(\frac{r+r_+}{r+r_-}\right)d\widetilde{\tau}\right)^2 + \frac{f\widetilde{r}^2}{\Sigma}\left(\left(\frac{\Sigma_+}{\kappa_+(r_+^2-a^2)}\,d\widetilde{\tau} - a\sin^2\theta\,d\widetilde{\phi}\right)^2 - \frac{\Sigma^2}{\kappa_+^2(r^2-a^2)^2}\,d\widetilde{\tau}^2\right). \quad (25)$$

By introducing the coordinates $\tau = \widetilde{\tau}/\kappa_+$ and $\phi = \widetilde{\phi} - (a/\sqrt{m^2+a^2})\widetilde{\tau}$, we have the representation

$$g = \Sigma(d\widetilde{r}^2 + d\theta^2) + \frac{\Delta(d\tau + a\sin^2\theta\,d\phi)^2 + \sin^2\theta((r^2-a^2)\,d\phi - a\,d\tau)^2}{\Sigma}, \quad (26)$$

where $\Delta = f\widetilde{r}^2$, which makes it clear that g is positive definite, and therefore is a Riemannian metric. This metric is defined on $(\mathbb{R}^2 \setminus \{0\}) \times (S^2 \setminus \{N, S\})$, and we claim it extends smoothly to a Riemannian metric on all of $\mathbb{R}^2 \times S^2$.

First of all, note that $d\widetilde{r}^2 + \widetilde{r}^2\,d\widetilde{\tau}^2$ is just the standard metric on $\mathbb{R}^2$, and thus the first term in (25) can be viewed as defined on all of $\mathbb{R}^2 \times S^2$. Furthermore, the 1-form $\sin^2\theta\,d\widetilde{\phi}$ also extends smoothly to the entire space, and the extension vanishes at $\theta = 0$ and $\theta = \pi$. Likewise, $\widetilde{r}^2\,d\widetilde{\tau}$ is smoothly extendible, and its extension vanishes at $\widetilde{r} = 0$. The expression $\Sigma\,d\theta^2 + \Sigma^{-1}\sin^2\theta(r^2 - a^2)^2\,d\widetilde{\phi}^2$ is equal to $\Sigma(d\theta^2 + \psi^2(\sin^2\theta)\,d\phi^2)$, where $\psi(t) = t(r-a^2)/(r^2 - a^2(1-t^2))$. Since ψ is odd and satisfies $\dot{\psi}(0) = 1$, this means that $\Sigma\,d\theta^2 + \Sigma^{-1}\sin^2\theta(r^2-a^2)^2\,d\widetilde{\phi}^2$ extends smoothly to the entire space.

Most terms in (25) are smoothly extendible by the previous paragraph, and it only remains to prove smooth extendibility of $\chi\,d\widetilde{\tau}^2$, where

$$\begin{aligned} \chi &= \frac{f\widetilde{r}^2}{\Sigma}\left(\left(\frac{\Sigma_+}{\kappa_+(r_+^2-a^2)}\right)^2 - \frac{\Sigma^2}{\kappa_+^2(r^2-a^2)^2}\right) \\ &= \frac{af\widetilde{r}^2\sin^2\theta(\Sigma_+(r^2-a^2) + \Sigma(r_+^2-a^2))(r+r_+)(r-r_+)}{\kappa_+^2\Sigma(r^2-a^2)^2(r_+^2-a^2)^2}. \end{aligned} \quad (27)$$

But this follows easily: since r is smooth as a function of $\widetilde{r}^2$, we have $r - r_+ = O(\widetilde{r}^2)$, and so $\chi = O(\widetilde{r}^4)$. This proves that g extends smoothly to all of $\mathbb{R}^2 \times S^2$.

We also need to check that g is positive definite at the new points, i.e. points where $\widetilde{r} = 0$ or $\theta = 0$ or $\theta = \pi$. But at these points, the metric reduces to

$$g = \frac{f\Sigma}{\kappa_+^2(r^2-a^2)^2}(d\widetilde{r}^2 + \widetilde{r}^2\,d\widetilde{\tau}^2) + \Sigma\,d\theta^2 + \frac{\sin^2\theta(r^2-a^2)^2}{\Sigma}\,d\widetilde{\phi}^2 + \chi\,d\widetilde{\tau}^2\,, \quad (28)$$

which is clearly positive definite, since $\chi > 0$. Thus, g is (extendible to) a Riemannian metric on $\mathbb{R}^2 \times S^2$.

As for the Riemannian Schwarzschild space, the topology $\mathbb{R}^2 \times S^2$ is non-compact, and one can see from (26) that it is also complete. Straightforward (but tedious) computations in the coordinates $(\widetilde{r}, \tau, \theta, \phi)$ also show that g is Ricci-flat. To show that Riemannian Kerr space is a toric manifold, we give it the same T^2-action as Riemannian Schwarzschild space. This is an action by isometries, as can be seen from the fact that the expression (26) does not depend explicitly on τ and ϕ. The action has the same isotropy groups as that of Riemannian Schwarzschild space.

[6] The choice of coordinates was inspired by [3], in which an analogous procedure was used to extend the Lorentzian form of the Kerr metric.

3.4 Riemannian Taub–NUT Space

In contrast to the two previous sections, we will now consider a metric on $\mathbb{R}^4$, on which we introduce the coordinates $(r, \theta, \widetilde{\psi}, \widetilde{\phi})$ by

$$\begin{cases} x_1 &= \sqrt{r}\sin(\theta/2)\cos\widetilde{\psi}, \\ x_2 &= \sqrt{r}\sin(\theta/2)\sin\widetilde{\psi}, \\ x_3 &= \sqrt{r}\cos(\theta/2)\cos\widetilde{\phi}, \\ x_4 &= \sqrt{r}\cos(\theta/2)\sin\widetilde{\phi}. \end{cases} \tag{29}$$

The 1-forms $\sin^2(\theta/2)\, d\widetilde{\psi}^2 = -x_2\, dx_1 + x_1\, dx_2$ and $\cos^2(\theta/2)\, d\widetilde{\psi}^2 = -x_4\, dx_3 + x_3\, dx_4$ are then smooth on all of $\mathbb{R}^4$, and we may define a metric by

$$\begin{aligned} g = (r+2m)\Big(\frac{dr^2}{r} + r\, d\theta^2 + 4r(\sin^2(\theta/2)\, d\widetilde{\psi}^2 + \cos^2(\theta/2)\, d\widetilde{\phi}^2)\Big) \\ - \frac{r^2(r+4m)}{r+2m}(\sin^2(\theta/2)\, d\widetilde{\psi} + \cos^2(\theta/2)\, d\widetilde{\phi})^2. \end{aligned} \tag{30}$$

This is indeed smooth, since as one can check, $r^{-1}\, dr^2 + r\, d\theta^2 + 4r(\sin^2(\theta/2)\, d\widetilde{\psi}^2 + \cos^2(\theta/2)\, d\widetilde{\phi}^2)$ is just the standard metric on $\mathbb{R}^4$ in these coordinates. Since

$$r(r+2m) - \frac{r^2(r+4m)}{r+2m} = \frac{4m^2 r}{r+2m} \geq 0, \tag{31}$$

and since

$$2(\sin^2(\theta/2)\, d\widetilde{\psi}^2 + \cos^2(\theta/2)\, d\widetilde{\phi}^2) \geq 2(\sin^4(\theta/2)\, d\widetilde{\psi}^2 + \cos^4(\theta/2)\, d\widetilde{\phi}^2) \tag{32}$$

$$\begin{aligned} &= (\sin^2(\theta/2)\, d\widetilde{\psi} + \cos^2(\theta/2)\, d\widetilde{\phi})^2 \\ &\quad + (\sin^2(\theta/2)\, d\widetilde{\psi} - \cos^2(\theta/2)\, d\widetilde{\phi})^2 \end{aligned} \tag{33}$$

$$\geq (\sin^2(\theta/2)\, d\widetilde{\psi} + \cos^2(\theta/2)\, d\widetilde{\phi})^2, \tag{34}$$

we have

$$g \geq \frac{r+2m}{2}\Big(\frac{dr^2}{r} + r\, d\theta^2 + 4r(\sin^2(\theta/2)\, d\widetilde{\psi}^2 + \cos^2(\theta/2)\, d\widetilde{\phi}^2)\Big), \tag{35}$$

from which it follows that g is indeed positive-definite. The metric g is called the *Riemannian Taub–NUT metric*, and the resulting manifold is the *Riemannian Taub–NUT space*.

A perhaps more well-known form of this metric can written down by switching coordinates. Introducing the coordinates $\psi = 2m(\widetilde{\psi} - \widetilde{\psi})$ and $\phi = \widetilde{\psi} + \widetilde{\phi}$, the metric g takes the form

$$g = \Big(1 + \frac{2m}{r}\Big)(dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2) + \Big(1 + \frac{2m}{r}\Big)^{-1}(d\psi + 2m\cos\theta\, d\phi)^2. \tag{36}$$

In this case, it is apparent from (36) that balls around the origin (with respect to g) are also such balls with respect to the standard metric on $\mathbb{R}^4$. As for the previous cases, g is thus complete. One can also see that g is Ricci-flat from the form (36) by calculating the components of the Ricci tensor in the coordinates (r, θ, ψ, ϕ).

Since the expression in (36) does not depend explicitly on ψ or ϕ, the same T^2-action as defined for Euclidean space $\mathbb{R}^4$ is thus isometric with respect to g.

As a remark, the construction also works when $m < 0$. In this case however, the metric is not defined on all of $\mathbb{R}^4$, and is instead a metric on $\mathbb{R}^4 \setminus \overline{B}_{2|m|}(0)$.

3.5 Taub-bolt Space

Let $n \in \mathbb{R}$, and consider the manifold $M = \mathbb{C}P^2 \setminus \{[0:0:1]\}$. On the set $U_i = \{[Z_0 : Z_1 : Z_2] \in M \mid Z_i \neq 0\}$ we have standard coordinates $z_{ij} : U_i \to \mathbb{C}$ (for $i \neq j$) given by $z_{ij}([Z_0 : Z_1 : Z_2]) = Z_j/Z_i$. On the intersection $U_0 \cap U_1 \cap U_2$ we also have another coordinate system, provided by the inverse of the map

$$\begin{aligned} &(0,\infty) \times (0,\pi) \times S^1 \times S^1 \to M, \\ &(\widetilde{r}, \theta, e^{i\widetilde{\psi}}, e^{i\widetilde{\phi}}) \mapsto [\widetilde{r} : e^{i\widetilde{\psi}} \sin(\theta/2) : e^{i\widetilde{\phi}} \cos(\theta/2)]. \end{aligned} \tag{37}$$

Now define r implicitly by the relation $\widetilde{r}^2 = (r - 2|n|)(2r - |n|)^{1/4} e^{r/2|n|}$. Note that $r \geq 2|n|$, and that since $\frac{d}{dr}\big((r-2|n|)(2r-|n|)^{1/4}e^{r/2|n|}\big) > 0$ for (say) $r > 0$, r is smooth as a function of $\widetilde{r}^2$. Now, in terms of the standard coordinates on the sets U_i, we have

$$\begin{cases} z_{01} &= \widetilde{r}^{-1} e^{i\widetilde{\psi}} \sin(\theta/2), \\ z_{02} &= \widetilde{r}^{-1} e^{i\widetilde{\phi}} \cos(\theta/2), \end{cases} \quad \begin{cases} z_{10} &= \widetilde{r} e^{-i\widetilde{\psi}} \csc(\theta/2), \\ z_{12} &= e^{i(\widetilde{\phi}-\widetilde{\psi})} \cot(\theta/2), \end{cases} \quad \begin{cases} z_{20} &= \widetilde{r} e^{-i\widetilde{\phi}} \sec(\theta/2), \\ z_{21} &= e^{-i(\widetilde{\phi}-\widetilde{\psi})} \tan(\theta/2). \end{cases} \tag{38}$$

Since $\widetilde{r}^2 = (|z_{01}|^2 + |z_{02}|^2)^{-1} = |z_{10}|^2(|z_{12}|^2 + 1) = |z_{20}|^2(|z_{21}|^2 + 1)$, and since U_0 does not contain the point $z_{01} = z_{02} = 0$, this means that $\widetilde{r}^2$, and therefore r, extends smoothly to all of M. We define a covariant 2-tensor g by

$$g = \frac{16n^2 f\, d\widetilde{r}^2}{\widetilde{r}^2} + 16n^2 f(\sin^2(\theta/2)\, d\widetilde{\psi} + \cos^2(\theta/2)\, d\widetilde{\phi})^2 + (r^2 - n^2)(d\theta^2 + \sin^2\theta(d\widetilde{\phi} - d\widetilde{\psi})^2), \tag{39}$$

where $f = (r - 2|n|)(r - |n|/2)/(r^2 - n^2)$. This is smooth and positive definite, so it is a Riemannian metric on $U_0 \cap U_1 \cap U_2$.

We claim that g can be extended to a Riemannian metric on all of M. To simplify the discussion, we say that a tensor on $U_0 \cap U_1 \cap U_2$ is *regular* on U_i if it admits a smooth extension to U_i. Consider the set U_0. In this set $\widetilde{r}$ is smooth and does not vanish anywhere, so the first term in (39) is regular on U_0. The 1-form $\sin^2(\theta/2)\, d\widetilde{\psi}$ is regular on U_0, as can be seen by writing

$$\sin^2(\theta/2)\, d\widetilde{\psi} = |\widetilde{r} z_{01}|^2\, d(\arg(\widetilde{r} z_{01}))\,, \tag{40}$$

and similarly, $\cos^2(\theta/2)\, d\widetilde{\phi}$ is regular on U_0. It remains to show regularity of the last term in (39). But

$$\begin{aligned} d\theta^2 + \sin^2\theta(d\widetilde{\phi} - d\widetilde{\psi})^2 =\ & d\theta^2 + 4(\sin^2(\theta/2)\, d\widetilde{\psi}^2 + \cos^2(\theta/2)\, d\widetilde{\phi}^2) \qquad (41)\\ & - 4(\sin^2(\theta/2)\, d\widetilde{\psi} + \cos^2(\theta/2)\, d\widetilde{\phi})^2 \\ =\ & 4(|dw_1|^2 + |dw_2|^2 - (\sin^2(\theta/2)\, d\widetilde{\psi} + \cos^2(\theta/2)\, d\widetilde{\phi})^2), \qquad (42) \end{aligned}$$

where $w_i = z_{0i}/\sqrt{|z_{01}|^2 + |z_{02}|^2}$. Thus g is regular on U_0, and it remains to prove positive definiteness everywhere on this set.

This is already established for $0 < \theta < \pi$, so we consider the cases $\theta = 0$ and $\theta = \pi$. First, note that

$$\begin{aligned} g &= \frac{16n^2 f}{\widetilde{r}^2} d\widetilde{r}^2 + 4n^2(4+f)\, d\widetilde{\phi}^2 + 4(r^2 - n^2)(|dw_1|^2 + |dw_2|^2) \qquad (43)\\ &= \frac{16n^2 f}{\widetilde{r}^2} d\widetilde{r}^2 + 4n^2(4+f)\, d\widetilde{\phi}^2 + 4(r^2-n^2)\left(\frac{|dz_{01}|^2}{|z_{02}|^2} + |dw_2|^2\right) \qquad (44)\\ &\geq 4n^2(4+f)\left(\frac{d\widetilde{r}^2}{\widetilde{r}^2} + d\widetilde{\phi}^2\right) + \frac{4(r^2-n^2)|dz_{01}|^2}{|z_{02}|^2}. \qquad (45) \end{aligned}$$

At points of U_0 where $\theta = 0$, we have $\widetilde{r}^2|dz_{01}|^2 = \widetilde{r}^{-2}\, d\widetilde{r}^2 + d\widetilde{\phi}^2$. Therefore, at any such point we have $g \geq 4n^2(4+f)\widetilde{r}^2|dz_{01}|^2 + 4(r^2-n^2)|z_{02}|^{-2}|dz_{01}|^2$, which is clearly positive definite. For points with $\theta = \pi$ one reasons similarly to see that $g \geq 4n^2(4+f)\widetilde{r}^2|dz_{02}|^2 + 4(r^2-n^2)|z_{01}|^{-2}|dz_{02}|^2$. Thus g is positive definite on U_0.

Next we will prove regularity in U_1, and for this purpose it will be useful to write the metric in the form

$$g = \frac{16n^2 f}{\widetilde{r}^2}(d\widetilde{r}^2 + \widetilde{r}^2(d\widetilde{\psi} + \cos^2(\theta/2)(d\widetilde{\phi} - d\widetilde{\psi}))^2) + (r^2 - n^2)(d\theta^2 + \sin^2\theta(d\widetilde{\phi} - d\widetilde{\psi})^2). \tag{46}$$

From (38) we see that

$$\begin{aligned} d\widetilde{r}^2 + \widetilde{r}^2\, d\widetilde{\psi}^2 &= |d(z_{10}\sin(\theta/2))|^2, \\ d\theta^2 + \sin^2\theta(d\widetilde{\phi} - d\widetilde{\psi})^2 &= 4\sin^4(\theta/2)|dz_{12}|^2, \\ \widetilde{r}^2\, d\widetilde{\psi} &= |z_{10}\sin(\theta/2)|^2\, d(\arg(z_{10}\sin(\theta/2)))\,, \\ \cos^2(\theta/2)(d\widetilde{\phi} - d\widetilde{\psi}) &= |z_{12}\sin(\theta/2)|^2\, d(\arg(z_{12}\sin(\theta/2)))\,. \end{aligned} \tag{47}$$

Since $\sin(\theta/2) = (|z_{12}|^2 + 1)^{-1/2}$, $\sin(\theta/2)$ is regular in U_1, and this shows that all the tensors above are regular on U_1. Looking at (46), it now follows immediately from (47) that g is regular in U_1.

To see that g is positive definite in U_1 it suffices to check points with $\widetilde{r} = 0$. At such a point, the metric becomes

$$g = \frac{8|n|^{3/4}}{3^{1/4}e}|d(z_{10}\sin(\theta/2))|^2 + 12n^2\sin^4(\theta/2)|dz_{12}|^2 \tag{48}$$

$$= \frac{8|n|^{3/4}\sin(\theta/2)}{3^{1/4}e}|dz_{10}|^2 + 12n^2\sin^4(\theta/2)|dz_{12}|^2, \tag{49}$$

where the second equality holds because z_{10} vanishes at such points. This is clearly positive definite. The argument for U_2 is analogous, and we omit it.

The Riemannian manifold M is called the *Taub-bolt space*, and the metric g is the *Taub-bolt metric*. We can view it in another way if we use r as the radial coordinate, and introduce the new angular coordinates $\psi = 2n(\widetilde{\psi} + \widetilde{\phi})$ and $\phi = \widetilde{\phi} - \widetilde{\psi}$. The metric then takes the form

$$g = \frac{dr^2}{f} + f(d\psi + 2n\cos\theta\, d\phi)^2 + (r^2 - n^2)(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{50}$$

Let $p = [Z_0 : Z_1 : Z_2] \in M$ be arbitrary. Since $[1 : 0 : 0] \notin M$, we must have $(Z_1, Z_2) \neq (0, 0)$; without loss of generality, assume that $|Z_1|^2 + |Z_2|^2 = 1$. Then we can write $|Z_1| = \sin(\theta_p/2)$ and $|Z_2| = \cos(\theta_p/2)$ for some θ_p with $0 \leq \theta_p < 4\pi$, and because $|Z_1|, |Z_2| \geq 0$, we must in fact have $0 \leq \theta_p < \pi$. Furthermore, we can find angles $\widetilde{\psi}_p$ and $\widetilde{\phi}_p$ such that $Z_1 = e^{i\widetilde{\psi}_p}\sin(\theta_p/2)$ and $Z_2 = e^{i\widetilde{\phi}_p}\cos(\theta_p/2)$. Letting $\widetilde{r}_p = |Z_0|$, we see that p is the image, under the map (37), of the point $(\widetilde{r}_p, \theta_p, e^{i\widetilde{\psi}_p}, e^{i\widetilde{\phi}_p})$.[7] We also define r_p to be the r-coordinate corresponding to the $\widetilde{r}$-coordinate $\widetilde{r}_p$.

The point $o = [0 : 0 : 1]$ has coordinates $r_o = 2|n|$, $\theta_o = 0$, and the coordinates $\widetilde{\psi}_o$ and $\widetilde{\phi}_o$ chosen arbitrarily to be any two real numbers. If $p \in M$ is an arbitrary point with coordinates

[7]Note that $p \in M$ was arbitrary. Even though the map (37) does not provide a global coordinate system for M, this shows that it is surjective. It is however not injective: the coordinates $(\widetilde{r}_p, \theta_p, e^{i\widetilde{\psi}_p}, e^{i\widetilde{\phi}_p})$ is not necessarily uniquely determined.

$(r_p, \theta_p, e^{i\widetilde{\psi}_p}, e^{i\widetilde{\phi}_p})$, then any curve from o to p must increase the r-coordinate from $2n$ to r_p. The length of such a curve is thus at least

$$\int_{2n}^{r_p} \sqrt{\frac{(r-2|n|)(r-|n|/2)}{r^2-n^2}}\, dr \geq \int_{2n}^{r_p} \frac{r-2|n|}{r-|n|}\, dr \geq r_p - 2|n| - |n|(\log(r_p - |n|)). \tag{51}$$

On the other hand, we can construct a piecewise smooth curve from o to p as follows. Since $\widetilde{\psi}_o$ and $\widetilde{\phi}_o$ can be chosen arbitrarily, let them be the same values as for p, for simplicity. Start at o, and then increase θ from 0 to θ_p. Then, increase r from $2|n|$ to r_p, arriving at p. The length of this curve is

$$\begin{aligned}\int_0^{\theta_p} 3n^2\, d\theta + \int_{2|n|}^{r_p} \sqrt{\frac{(r-2|n|)(r-|n|/2)}{r^2-n^2}}\, dr &\leq 6\pi n^2 + \int_{2|n|}^{r_p} \frac{r-|n|/2}{r+|n|}\, dr \\ &\leq r_p - 2|n| + \frac{3}{2}|n|\log(3|n|) + 6\pi n^2.\end{aligned} \tag{52}$$

Let A is a closed and bounded subset of M; then there exists an $R > 0$ such that $d(o,a) < R$ for all $a \in A$. But for any such a with radial coordinate r_a, we know from (51) that

$$r_a - |n|(\log(r_a - |n|)) \leq d(o,a) + 2|n| < R + 2|n|. \tag{53}$$

The expression on the left is increasing as a function of r_a, and goes to ∞ as $r_a \to \infty$. In other words, A is bounded with respect to the coordinate r, and this means that when viewed as a subset of $\mathbb{C}P^2$, A is contained in the complement of some neighborhood of $[1:0:0]$. But A is then a closed subset of the compact space $\mathbb{C}P^2$, which means that A is compact. Since A was an arbitrary closed and bounded subset, M is complete.

In M, we introduce an isometric T^2-action by letting $(e^{i\theta_1}, e^{i\theta_2}) \cdot [Z_0 : Z_1 : Z_2] = [Z_0 : e^{i\theta_1} Z_1 : e^{i\theta_2} Z_2]$. Then clearly, points of the form $[1:0:Z_2]$ have isotropy group $T^2(1,0)$, and points of the form $[1:Z_1:0]$ have isotropy group $T^2(0,1)$. The points $[0:1:0]$ and $[0:0:1]$ are fixed, while all other points of M have trivial isotropy group.

3.6 Eguchi–Hanson Space

Consider TS^2, the tangent bundle of the 2-sphere. This is a subset of $T\mathbb{R}^3$, and under the identification of the latter with $\mathbb{R}^3 \times \mathbb{R}^3$, the former corresponds to the subset

$$\{(x,v) \in S^2 \times \mathbb{R}^3 \mid x \perp v\} \subseteq \mathbb{R}^3 \times \mathbb{R}^3. \tag{54}$$

Define r implicitly by the relation $|v|^2 = r^4 - a^4$, a relation from which one can easily check that r is a smooth function on TS^2 with $r \geq a$. We can then define a metric on TS^2 by

$$g = \frac{(-x_2\, dv_1 + x_1\, dv_2)^2 + (-x_1\, dv_3 + x_3\, dv_1)^2 + (-x_3\, dv_2 + x_2\, dv_3)^2}{4r^2} + \frac{r^2}{4} g_{S^2}, \tag{55}$$

where g_{S^2} is the standard metric on S^2. The numerator on the left can be viewed as the squared norm of the cross product of x and $dv = (dv_1, dv_2, dv_3)$. Using the fact that a tangent vector is orthogonal to x, together with the fact that g_{S^2} is non-degenerate, it follows that g is non-degenerate. In order to write down a different form of this metric, we define coordinates $(\widetilde{r}, \psi, \theta, \phi)$

by[8]

$$\begin{cases} x_1 &= \sin\theta\cos\phi, \\ x_2 &= \sin\theta\sin\phi, \\ x_3 &= \cos\theta, \\ v_1 &= \widetilde{r}(\cos\phi\cos\psi\cos\theta - \sin\phi\sin\psi), \\ v_2 &= \widetilde{r}(\sin\phi\cos\psi\cos\theta + \cos\phi\sin\psi), \\ v_3 &= -\widetilde{r}\cos\psi\sin\theta. \end{cases} \tag{56}$$

By a lengthy calculation, the numerator on the left in (55) is equal to $d\widetilde{r}^2 + \widetilde{r}^2(d\psi + \cos\theta\, d\phi)^2$. Switching back from $\widetilde{r}$ to r again, in the coordinates (r, ψ, θ, ψ) the metric takes the more widely known form

$$g = \left(1 - \frac{a^4}{r^4}\right)\frac{r^2}{4}(d\psi + \cos\theta\, d\phi)^2 + \left(1 - \frac{a^4}{r^4}\right)^{-1} dr^2 + \frac{r^2}{4}(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{57}$$

When equipping TS^2 with this so-called *Eguchi–Hanson metric*, the resulting manifold M is called the *Eguchi–Hanson space*.

The topology of TS^2 is clearly non-compact, since it contains the tangent spaces T_pS^2, which are closed subsets homeomorphic to $\mathbb{R}^2$. In a single tangent space, that is, for fixed θ and ϕ, it is clear from (57) that balls in this tangent spaces are also balls with respect to the submanifold metric induced from the standard metric of $\mathbb{R}^3 \times \mathbb{R}^3$. Moreover, the correspondence between the radii is independent of θ and ϕ, and so it follows that the set described by $r < R$, for some R, is bounded with respect to the standard metric on $\mathbb{R}^3 \times \mathbb{R}^3$. Subsets which are closed and bounded with respect to g are thus compact, so that g is a complete metric. As with the previous examples, it is a straightforward computation to verify that g is Ricci-flat, for example by considering the form (57).

From (57) it is also apparent that any rotations of the angles ψ and ϕ are isometries. Letting $(e^{i\theta_1}, e^{i\theta_2}) \in T^2$ act on M by $\psi \mapsto \psi + \theta_1$ and $\phi \mapsto \phi + \theta_2$, we thus have a smooth T^2-action by isometries on M. For this action, the isotropy group of points with $\widetilde{r} = 0$ and $0 < \theta < \pi$ is $T^2(1, 0)$. At points where $\theta = 0$ and $\widetilde{r} > 0$, the isotropy group is $T^2(1, -1)$, since at such points we have $x_1 = x_2 = v_3 = 0$, $v_1 = \widetilde{r}\cos(\psi + \phi)$ and $v_2 = \widetilde{r}\sin(\psi + \phi)$, and likewise, points with $\theta = \pi$ and $\widetilde{r} > 0$ have isotropy group $T^2(1, 1)$. Finally, the poles (where $\widetilde{r} = 0$ and θ is either 0 or π) are fixed points, and all other points have trivial isotropy group.

3.7 Chen–Teo Space

Consider the manifold $(-\infty, -1) \times (1, \infty) \times T^2$ with coordinates $(x, y, \widetilde{\psi}, \widetilde{\phi})$, with standard coordinates (x, y) on the first two factors and angular coordinates $(\widetilde{\psi}, \widetilde{\phi})$ on the two circle factors of T^2. For parameters $\varkappa \in (0, \infty)$ and $\lambda \in (-1, 1)$, we can let $\gamma = \sqrt{2 - \lambda^2}$ and introduce the quantities

$$\kappa_E = \frac{2(\gamma - \lambda^2)}{\varkappa^2(1 - \lambda^2)^2(1 + \gamma)^2} \tag{58}$$

and

$$\Omega_{1E} = -\frac{2(\gamma - \lambda^2)}{\varkappa^2(\gamma - \lambda)^2(1 + \gamma)^2}, \tag{59}$$

[8]Note that (θ, ϕ) are just spherical coordinates on S^2.

and then define $(\psi, \phi) = (\widetilde{\psi}/\kappa_E, \Omega_{1E}\widetilde{\psi}/\kappa_E + \widetilde{\phi})$, giving coordinates (x, y, ψ, ϕ) for $(-\infty, -1) \times (1, \infty) \times T^2$. Defining the one-form

$$\Omega = \frac{\varkappa^2(1-\lambda^2)(1+\gamma)(x+\lambda)(y+\lambda)}{2(2+\gamma)(x-y)F(x,y)} \Big(2(x+1)(y-1)(\lambda(3+\gamma)(2(x+\lambda)(y+\lambda) + (1+\gamma)(\gamma x - \gamma y - 2)) + 3(1-\lambda^2)(x+y)) + (1+\lambda)^3(\gamma-\lambda+2)^2(x^2-1) - (1-\lambda)^3(\gamma+\lambda+2)^2(y^2-1) \Big) d\phi\,, \quad (60)$$

along with the functions $G(x) = (1-x^2)(x+\lambda)$,

$$H(x,y) = \frac{(\lambda+\gamma)(x+\lambda y) + (\lambda-\gamma)(y-\lambda x) + 2xy + 2\gamma\lambda^2}{4(2+\gamma)} \Big((2+\gamma)(x+\lambda)(y+\lambda)(\lambda(1+\gamma)(x+y) + (\lambda^2-\gamma)(x-y) + 4 + 2\gamma\lambda^2 - 2xy) + (1-\lambda^2)((\gamma+\lambda+2)(x+\lambda)(x+\lambda\gamma) + (\gamma-\lambda+2)(y+\lambda)(y+\lambda\gamma)) \Big), \quad (61)$$

and

$$F(x,y) = (x+\lambda)(y+\lambda)((1+\lambda\gamma)^2(xy+\lambda x+\lambda y+1) - 2\lambda\gamma(1-\lambda)(x-1)(y-1) - (x^2-1)(y^2-1)), \quad (62)$$

we have a metric g on $(-\infty, -1) \times (1, \infty) \times T^2$, given by

$$g = \frac{F(x,y)}{H(x,y)}(d\psi + \Omega)^2 - \frac{\varkappa^4 H(x,y)}{(x-y)^3}\left(\frac{dx^2}{G(x)} - \frac{dy^2}{G(y)} + \frac{4G(x)G(y)}{(x-y)F(x,y)}\,d\phi^2\right). \quad (63)$$

Letting T^2 act by multiplication on the T^2 factor, we get an action of T^2 on $(-\infty, -1) \times (1, \infty) \times T^2$ by isometries (with respect to g), whose isotropy groups are all trivial.

The coordinates and the metric can be seen as defined on the manifold with corners $([-\infty, -1] \times [1, \infty] \setminus \{(-\infty, \infty)\}) \times T^2$, of which $(-\infty, -1) \times (1, \infty) \times T^2$ is an open subset. In the paper [7], Chen and Teo claim[9] that, with the right identifications of the angles $\widetilde{\psi}, \widetilde{\phi}$ on the set

$$(([-\infty, -1] \times [1, \infty] \setminus \{(-\infty, \infty)\}) \setminus (-\infty, -1) \times (1, \infty)) \times T^2 \times T^2, \quad (64)$$

the following hold:

- The resulting quotient of $([-\infty, -1] \times [1, \infty] \setminus \{(-\infty, \infty)\}) \times T^2$ is a smooth manifold diffeomorphic to $\mathbb{C}P^2 \setminus S^1$, where the latter denotes $\mathbb{C}P^2$ with a subset diffeomorphic to S^1 removed.
- The metric g is Ricci-flat, and extends to a complete metric on $\mathbb{C}P^2 \setminus S^1$.
- The T^2-action on $(-\infty, -1) \times (1, \infty) \times T^2$ extends to a (smooth) action on $\mathbb{C}P^2 \setminus S^1$.
- For this T^2-action, points where $x = -\infty$ and $y > 1$ have isotropy group $T^2(0, 1)$, and the same is true for points where $y = \infty$ and $x < -1$. Points where $y = 1$ and $-\infty < x < -1$ have isotropy group $T^2(1, 0)$, while points where $x = -1$ and $1 < y < \infty$.
- Points with $(x, y) \in \{(-\infty, 1), (-1, 1), (-1, \infty)\}$ are fixed under the whole action.

By continuity, the T^2-action on $\mathbb{C}P^2 \setminus S^1$ is then automatically isometric. Also by continuity, the metric g is Ricci-flat on all of $\mathbb{C}P^2 \setminus S^1$. Since $\mathbb{C}P^2 \setminus S^1$ is a proper and non-empty open subset of the connected space $\mathbb{C}P^2$, it is not closed, so in particular it is non-compact. Thus, $(\mathbb{C}P^2 \setminus S^1, g)$ is a toric gravitational instanton. We will refer to this manifold as the *Chen–Teo space*, and to its metric g as the *Chen–Teo metric*.

[9] We will not give a proof of this fact.

4 Conditions on the Topology

In this section we prove Theorem 1 by explicitly reconstructing M as a smooth manifold with a T^2-action, and calculating the intersection form for the reconstructed model.

Proof of Theorem 1. Let $M_1, \dots, M_n$ denote distinct copies of $\mathbb{R}^4 \cong \mathbb{C}^2$, and for each i, define a T^2-action on M_i by

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot (z_1, z_2) = (e^{i(a_{11}\theta_1 + a_{12}\theta_2)} z_1, e^{i(a_{21}\theta_1 + a_{22}\theta_2)} z_2), \tag{65}$$

where

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} v_{i-1} & v_i \end{pmatrix}^{-1}. \tag{66}$$

Now let $M_i^+ = \{(z_1, z_2) \in M_i \mid |z_1| > |z_2|^2 + 1\}$ and $M_i^- = \{(z_1, z_2) \in M_i \mid |z_2| > |z_1|^2 + 1\}$. Since

$$\begin{pmatrix} v_i & v_{i+1} \end{pmatrix}^{-1} v_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{67}$$

and since $\det\Big(\begin{pmatrix} v_i & v_{i+1} \end{pmatrix}^{-1} \begin{pmatrix} v_{i-1} & v_i \end{pmatrix}\Big) = 1$, we can write

$$\begin{pmatrix} v_i & v_{i+1} \end{pmatrix}^{-1} \begin{pmatrix} v_{i-1} & v_i \end{pmatrix} = \begin{pmatrix} -d_i & 1 \\ -1 & 0 \end{pmatrix} \tag{68}$$

for some $d_i \in \mathbb{Z}$, and since $v_{i-1} = -d_i v_i - v_{i+1}$, we have $\det \begin{pmatrix} v_{i-1} & v_{i+1} \end{pmatrix} = -d_i \det \begin{pmatrix} v_i & v_{i-1} \end{pmatrix} = -d_i$. We have diffeomorphisms $F_i : M_i^+ \to M_{i+1}^-$ given by

$$F_i(z_1, z_2) = \left(\left(\frac{z_1}{|z_1|} \right)^{-d_i} z_2, \left(\frac{z_1}{|z_1|} \right)^{-1} \left(|z_2|^2 + 1 + \frac{1}{|z_1| - |z_2|^2 - 1} \right) \right), \tag{69}$$

and gluing the sets M_i together along these diffeomorphisms, we obtain a manifold M'. It can be checked that the diffeomorphisms are equivariant with respect to the T^2-actions defined on the sets M_i, and that they are orientation-preserving, where each M_i is given the standard orientation of $\mathbb{R}^4$. This makes M' into an oriented manifold with a T^2-action. By construction, we can map the orbit space M'/T^2 diffeomorphically onto the orbit space M/T^2 in such a way that the isotropy groups match, and by [22, Theorem 1.1], this implies that M and M' are equivariantly diffeomorphic. For simplicity, we therefore identify M with M'.

The sets M_i are all simply connected, and since $(M_1 \cup M_{i-1}) \cap M_i = M_{i-1} \cap M_i = M_{i-1}^-$, which is connected, we can apply Seifert–van Kampen's theorem repeatedly to see that unions of the form $M_1 \cup \dots \cup M_i$ are simply connected. Since M_1 is homeomorphic to $\mathbb{R}^4$, which is contractible, we have $H_2(M_1) = 0$. Now assume that $i > 1$ and put $A = M_1 \cup \dots \cup M_{i-1}$ and $B = M_i$, and consider the Mayer-Vietoris sequence,

$$H_2(A \cap B) \to H_2(A) \oplus H_2(B) \to H_2(A \cup B) \to H_1(A \cap B) \to H_1(A) \oplus H_1(B). \tag{70}$$

The first term vanishes because $A \cap B$ is homotopy equivalent to a circle, while the last term vanishes because B is contractible. Thus, the alternating sum

$$\begin{aligned}(\dim(H_2(A, \mathbb{R})) + \dim(H_2(B, \mathbb{R}))) - \dim(H_2(A \cup B, \mathbb{R})) + \dim(H_1(A \cap B, \mathbb{R})) \\ = \dim(H_2(A, \mathbb{R})) - \dim(H_2(A \cup B, \mathbb{R})) + 1\end{aligned} \tag{71}$$

vanishes. Inducting on i, it follows that $\dim(H_2(M,\mathbb{R})) = n-1$.

For $1 \le i \le n-1$, define the set

$$R_i = \{(z_1, z_2) \in M_i \mid z_2 = 0\} \cup \{(z_1, z_2) \in M_i \mid z_1 = 0\}. \tag{72}$$

This is an embedded 2-sphere in M, and in particular it is a closed orientable submanifold of M. We give R_i the orientation for which the standard coordinate vector fields $(\partial/\partial x_1\, , \partial/\partial x_2\,)$ for M_i form a positively oriented frame for R_i. Then the fundamental classes $[R_1], \ldots, [R_{n-1}]$ are elements of $H_2(M,\mathbb{R})$, and we claim that they form a basis.

To this end, let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth function with $\operatorname{supp}\psi \subseteq (0,1)$ and $\int_0^1 \psi(t)\,dt \neq 0$. For each i, we define a 2-form ω_i on M_i by $\omega_i = \psi(|z_1| - |z_2|^2)\, d\big(|z_1| - |z_2|^2\big) \wedge d(\arg z_1)$. Since ω_i is supported in M_i, we can extend it by zero to all of M. Clearly, $\int_{R_i} \omega_j = 0$ whenever $i \neq j$, and

$$\int_{R_i} \omega_i = \int_0^{2\pi} \int_0^1 \psi(t)\, dt\, d\theta \neq 0. \tag{73}$$

Thus, the fundamental classes $[R_i]$ are linearly independent, and since $\dim(H_2(M,\mathbb{R})) = n-1$, this shows that they form a basis.

It remains to compute the intersection form in terms of this basis. For $|i-j| > 1$, the submanifolds R_i and R_j are disjoint, and thus the intersection product $[R_i] \cdot [R_j]$ vanishes. Two adjacent submanifolds R_i and R_{i+1} intersect in exactly one point, and they do so transversely, and from this is follows that $[R_i] \cdot [R_{i+1}] = \pm 1$. Flipping orientations of the submanifolds R_i appropriately, we ensure that $[R_i] \cdot [R_{i+1}] = 1$ for all i, noting that flipping orientation has no effect on the self-intersection numbers $[R_i] \cdot [R_i]$

To calculate the self-intersection numbers $[R_i] \cdot [R_i]$, we have to perturb R_i into another submanifold representative of the same homology class, which intersects R_i transversely. We first consider the case where $d_i \geq 0$. In this case, we take a smooth function $f : \mathbb{C} \to \mathbb{C}$ such that $f(z) = z^{d_i} - 3^{-d_i}$ when $|z| \le 2/3$, and $f(z) = (z/|z|)^{d_i}$ when $|z| \geq 1$, and such that $f(z) \neq 0$ when $2/3 \le |z| \le 1$. Defining a submanifold by

$$R_i' = \{(z_1, z_2) \in M_i \mid z_2 = f(z_1)\} \cup \{(z_1, z_2) \in M_{i+1} \mid z_1 = 1\}, \tag{74}$$

it is clear that R_i' and R_i represent the same homology class. Since f has exactly d_i roots, and since it is holomorphic with non-vanishing derivative near these roots, it follows that R_i and R_i' intersect transversely in d_i points, and that they intersect positively at each such point (with respect to the orientation of M). In other words, $[R_i] \cdot [R_i] = d_i$. The case where $d_i < 0$ is exactly the same, except that we let $f(z) = \overline{z}^{-d_i} - 3^{d_i}$ for $|z| \le 2/3$. In that case, the function f is antiholomorphic (instead of holomorphic) in that region, so that R_i and R_i' intersect negatively at each of the roots. □

4.1 Hitchin–Thorpe Inequality

In order to state the results in this section, we shall first need the definition of hyper-Kähler.

Definition 4. A Riemannian manifold (M, g) is said to be *hyper-Kähler* if it admits three almost complex structures I, J and K such that

- (M, g) is a Kähler manifold with respect to I, J and K separately.
- $I^2 = J^2 = K^2 = -1$.

In its original form, the Hitchin–Thorpe inequality is a statement about compact Einstein 4-manifolds. For such manifolds M, the inequality states that

$$2\chi(M) \geq 3|\tau(M)|, \tag{75}$$

and it further states that equality occurs if and only if the universal cover of M is hyper-Kähler. As a special case, this inequality applies in the case where M is Ricci-flat. In general, the inequality does not hold without modification in the non-compact case. For Ricci-flat ALE or ALF manifolds there are however variants of the inequality which involve extra boundary terms. In the case of ALE manifolds, for instance, we have the following variant of the inequality, whose proof can be found in [20, Theorem 4.2].

Theorem 4 (Hitchin–Thorpe Inequality for Ricci-Flat ALE Manifolds)**.** *Let (M, g) be an oriented Ricci-flat ALE manifold with group Γ. Then*

$$2\left(\chi(M) - \frac{1}{|\Gamma|}\right) \geq 3|\tau(M) + \eta_S(S^3/\Gamma)|. \tag{76}$$

Equality holds if and only if the universal cover of M is hyper-Kähler.

The number $\eta_S(S^3/\Gamma)$ occuring in Theorem 4 is the so-called *eta-invariant of the signature operator of the space form S^3/Γ*, which is a spectral invariant of this space form. Although we will not describe the eta-invariant in detail, we mention that the eta-invariant is an orientation-preservingly isometric invariant of the space form S^3/Γ, and that reversing the orientation of this space form flips the sign of the eta-invariant. We also cite the following formula for the eta-invariants of lens spaces, whose proof can be found in [16, Theorem 4].

Theorem 5. *For $p, q \in \mathbb{Z}$ with $0 \leq q < p$ and $\gcd(p, q) = 1$, the eta-invariant of the signature operator of $L(p, q)$ is*

$$\eta_S(L(p,q)) = \frac{1}{3p}(p-1)(3pq - 2p - q + 3) - \frac{2}{p}\sum_{k=1}^{q-1}\left\lfloor \frac{kp}{q} \right\rfloor^2. \tag{77}$$

Now let M be a toric ALE (with group Γ) instanton, with rod structure $(v_0, \ldots, v_n)$, where we assume that $\det\begin{pmatrix} v_{i-1} & v_i \end{pmatrix} = 1$. Then Theorem 4 applies, provided that we orient M consistently with the asymptotic diffeomorphism. By assumption, M is simply connected, which means that the statement about its universal cover applies directly to M. As we saw in the proof of Theorem 3, the boundary at infinity of M is a lens space $L(p, q')$, where $p = |\det\begin{pmatrix} v_0 & v_n \end{pmatrix}|$ and $q' = |\det\begin{pmatrix} v_1 & v_n \end{pmatrix}|$. By [1, Theorem 7.28], h-cobordant lens spaces are isometric, and since the eta-invariant is an isometric invariant (up to sign), it follows that $\eta_S(S^3/\Gamma) = \pm\eta_S(L(p, q'))$. Letting q be the unique integer in the interval $0 \leq q < p$ satisfying $q \equiv q' \pmod{p}$, the well-known fact that $L(p, q') = L(p, q)$, means that up to sign, $\eta_S(S^3/\Gamma)$ is given by the right hand side of (77). It is well known (and can be seen directly by considering the construction in the proof of Theorem 1) that the Euler characteristic is given by $\chi(M) = n$. Finally, the group Γ is isomorphic to the fundamental group of S^3/Γ, and since S^3/Γ is homotopy equivalent to $L(p, q)$, it follows that $|\Gamma| = p$. This shows that we can express all of the quantities occuring in Theorem 4 directly in terms of the rod structure. Thus Theorem 4 can be interpreted as a necessary condition that rod structures of toric ALE instantons have to satisfy.

As mentioned before, there is also a variant of the Hitchin–Thorpe inequality for ALF manifolds. For our purposes we will only need to consider the case of ALF-A_k. In this case we have the following result, whose proof can be found in [16, Theorem 4].

Theorem 6 (Hitchin–Thorpe Inequality for Ricci-Flat ALF-A_k Manifolds)**.** *Let (M, g) be an oriented Ricci-flat manifold which is ALF-A_k for some integer k. Then*

$$2\chi(M) \geq 3\left|\tau(M) - \frac{e}{3} + \operatorname{sgn}(e)\right|, \tag{78}$$

where $e = -k - 1$. Equality holds if and only if the universal cover of M is hyper-Kähler.

4.2 Rod Structures with Three Turning Points

Consider a rod structure (v_0, v_1, v_2, v_3) with three turning points, satisfying $v_0 = (0, 1)$ and $v_1 = (-1, 0)$ and $\det\begin{pmatrix} v_{i-1} & v_i \end{pmatrix} = 1$; any such rod structure can be written as in Figure 1, for some $a, b \in \mathbb{Z}$. For a toric gravitational instanton (M, g) with this rod structure we have $\chi(M) = 3$, and the signature $\tau(M)$ is just the difference between the number of positive and negative roots, respectively, of the polynomial $\lambda^2 - (a + b)\lambda + ab - 1$. Assume now, in addition, that (M, g) is ALE. As we shall see, this additional assumption places restriction the possible pairs (a, b). An obvious restriction is of course that $p = |1 - ab|$ is non-zero: otherwise, the boundary at infinity would be $S^2 \times S^1$, which is incompatible with ALE geometry. This line of reasoning rules out the cases $(a, b) = \pm(1, 1)$, but it cannot rule out any of the other cases where $p \neq 0$, since in these cases the boundary at infinity has topology $L(p, q)$, where $q \equiv |b| \pmod p$, $0 \leq q < p$, which is compatible with ALE.

Theorem 4 implies that either

$$2\left(3 - \frac{1}{p}\right) \geq 3|\tau(M) + \eta_S(L(p, q))| \tag{79}$$

or

$$2\left(3 - \frac{1}{p}\right) \geq 3|\tau(M) - \eta_S(L(p, q))|, \tag{80}$$

where $\eta_S(L(p, q))$ is given by (77). The inequalities (79) and (80) are just statements about a and b, and for any specific values of a and b, it is straightforward to check whether they hold or not. In other words, if for some $a, b \in \mathbb{Z}$, neither of (79) and (80) holds, there cannot exist any toric ALE instantons with this rod structure. Even if one of (79) and (80) holds, although we cannot rule the rod structure from belonging to a toric ALE instanton, we can still rule out the case where this manifold or the orientation-reversed space is hyper-Kähler, provided that exact equality holds in neither of the inequalities. Similarly, even if either of (79) and (80) holds, but if neither hold with strict inequality, we can rule out the case where the manifold (or its orientation-reversed space) is *not* hyper-Kähler. Finally, we can also rule out the case where one of a, b is zero. Indeed, in this case M would have to be AE, since $p = 1$, and therefore homeomorphic to $\mathbb{R}^4$, contradicting the fact that $\chi(M) = 3 \neq 1 = \chi(\mathbb{R}^4)$. Putting the conditions together, we get the following theorem.

Theorem 7. *Let M be a toric ALE instanton with the rod structure in Figure 1, for some $a, b \in \mathbb{Z}$, let $p = |1 - ab|$, $q \equiv |b| \pmod p$, $0 \leq q < p$, and define*

$$\Delta_\pm = 2\left(3 - \frac{1}{p}\right) - 3|\tau(M) \pm \eta_S(L(p, q))|, \tag{81}$$

where $\tau(M)$ is the difference between the number of positive and negative roots, respectively, of the polynomial $\lambda^2 - (a + b)\lambda + ab - 1$, and $\eta_S(L(p, q))$ is given by (77). Then the following holds.

(a) $p > 1$.

(b) *At least one of* $\Delta_\pm$ *is non-negative.*

(c) *If* M *is hyper-Kähler, at least one of* $\Delta_\pm$ *is zero. Otherwise, at least one of them is positive.*

Theorem 7 can be interpreted as an algorithm to systematically rule out rod structures for toric ALE instantons. Figure 2 shows the result of running this algorithm, checking all a, b with $\max(|a|, |b|) \leq 17$, with the plotted points corresponding to pairs (a, b) which the algorithm does *not* rule out. The black points correspond to pairs (a, b) where one of the inequalities hold, but where equality holds in neither (ruling out the case of hyper-Kähler)), and for the red points, we have equality in one of (79) and (80) but strict inequality in neither (permitting the case of hyper-Kähler). Interestingly enough, there are no pairs (a, b) where both of the inequalities hold, one of them with equality, and the other with strict inequality.

We claim that if $\max(|a|, |b|) \geq 13$ and $a, b \neq 0$, then the inequalities (79) and (80) can never hold; this implies that Figure 2 covers all the cases. To see this, first note that both inequalities imply

$$6 \geq 2\left(3 - \frac{1}{p}\right) \geq 3(|\eta_S(L(p, q))| - |\tau(M)|) \geq 3(|\eta_S(L(p, q))| - 2). \tag{82}$$

On the other hand,

$$\eta_S(L(p, q)) \geq \frac{1}{3p}(p-1)(3pq - 2p - q + 3) - \frac{2}{p}\sum_{k=1}^{q-1}\left(\frac{kp}{q}\right)^2 = f(q), \tag{83}$$

where

$$f(t) = \frac{1}{3p}(p-1)(3pt - 2p - t + 3) - \frac{p(t-1)(2t-1)}{3t}. \tag{84}$$

A quick computation shows that

$$\dot{f}(t) = \frac{1}{3}\left(p + \frac{1}{p} + \frac{p}{t^2} - 4\right) \tag{85}$$

for $t > 0$, and since $p = |1 - ab| \geq |a||b| - 1 \geq 12$ we have $\dot{f}(t) > 0$. Thus f is increasing, and since

$$f(1) = \frac{p+1}{3} - \frac{2}{3p} > 4 \tag{86}$$

and $q \geq 1$, we $\eta_S(L(p, q)) > 4$. This contradicts (82), and therefore rules out this case.

Consider now the same question for ALF: Given $a, b \in \mathbb{Z}$, does there exist a toric ALF instanton (M, g) with the rod structure in Figure 2? Such a manifold M is necessarily ALF-A_k, where $|k+1| = |\det\begin{pmatrix} v_0 & v_3 \end{pmatrix}| =: p$. Thus, the boundary at infinity is $S^2 \times S^1$ if $p = 0$, and is $L(p, 1)$ if $p > 0$. In the latter case, since the boundary at infinity is also $L(p, q)$, we get the condition $q \equiv \pm 1 \pmod p$.

Theorem 6 implies that either

$$6 \geq 3\left|\tau(M) - \frac{p}{3} + 1\right| \tag{87}$$

or

$$6 \geq 3\left|\tau(M) + \frac{p}{3} - 1\right|, \tag{88}$$

where $p = |1 - ab|$. Of course, the remarks about hyper-Kähler apply in this case, too. When equality holds in one of (87) and (88) and strict inequality does *not* hold in the other, any manifold with that rod structure must be hyper-Kähler. By the classification of hyper-Kähler ALF-A_k gravitational instantons (see [19]), it must then be isometric to a multi-Taub–NUT space. Since $\chi(M) = 3$, and since the only multi-Taub–NUT space with Euler characteristic 3 is the triple-Taub–NUT space, whose boundary at infinity is $L(3, 1)$, we must have $p = 3$ and $q \not\equiv 0$ (mod 3), a condition which allows us to further rule out pairs (a, b). Summarizing, we have the following theorem.

Theorem 8. *Let M be a toric ALF instanton with the rod structure in Figure 1, for some $a, b \in \mathbb{Z}$, let $p = |1 - ab|$, $q \equiv |b| \pmod p$, $0 \le q < p$, and define*

$$\Delta_\pm = 6 - 3\Big|\tau(M) \pm \Big(-\frac{p}{3} + \operatorname{sgn} p\Big)\Big|, \tag{89}$$

where $\tau(M)$ is the difference between the number of positive and negative roots, respectively, of the polynomial $\lambda^2 - (a + b)\lambda + ab - 1$.

(a) *If $p > 0$, then $q = 1$ or $q = p - 1$.*

(b) *At least one of $\Delta_\pm$ is non-negative.*

(c) *If M is hyper-Kähler, then $p = 3$, $q \neq 0$, and at least one of $\Delta_\pm$ is zero.*

(d) *Otherwise, at least one of $\Delta_\pm$ is positive.*

As in the ALE case, we interpret this as an algorithm, and run the algorithm for $\max(|a|, |b|) \le 17$, giving the result shown in Figure 2.[10] Unlike for ALE, the figure does not show every remaining possibility. Indeed, if a or b is zero, then $p = 1$ and $\tau(M) = 0$, so that (87) and (88) both hold with strict inequality. On the other hand, we claim that these are the only possibilities for $\max(|a|, |b|) \ge 17$. Assuming for a contradiction that $a, b \neq 0$ and $\max(|a|, |b|) \ge 17$, then similarly to in ALE case, the existence of such an (M, g) implies (by Theorem 6) that

$$6 \ge 3\Big|\tau(M) \pm \Big(-\frac{e}{3} + \operatorname{sgn} e\Big)\Big|, \tag{90}$$

(with $|e| = |1 - ab|$) holding at least for one of the choices of $+$ and $-$. In any case,

$$6 \ge 3\Big(\Big|\frac{e}{3}\Big| - |\operatorname{sgn} e| - |\tau(M)|\Big) \ge |e| - 9 \ge |a||b| - 10 \ge 17, \tag{91}$$

a contradiction.

Remark 6. One can also look at rod structure with four turning points. As in the case with three turning points we can assume that $v_0 = (0, 1), v_1 = (-1, 0)$ and $\det\begin{pmatrix} v_{i-1} & v_i \end{pmatrix} = 1$. In general, such a rod structure will then have the form shown in Figure 4, with three integer parameters a, b, c. Restricting attention to the AF case, we are left with two families of rod structures, shown in Figure 5, along with four exceptional rod structures, shown in Figure 6. For AF rod structures with four turning points, it then turns out that the Hitchin–Thorpe inequality is always satisfied with strict inequality. Thus, we cannot rule out any of these rod structures.

Since the inequalities are all strict, we can at least conclude that a simply connected AF toric gravitational instanton with four turning points cannot be hyper-Kähler. However, this is already known: by the classification of ALF-A_k hyper-Kähler gravitational instantons, any AF hyper-Kähler gravitational instanton must be a product of $\mathbb{R}^3$ with a circle, and in particular is not simply connected.

In other words, the Hitchin–Thorpe inequality gives no information at all in the case of AF rod structures with four turning points.

[10] The color convention is the same as for ALE. There are no pairs (a, b) where both of the inequalities hold, one of them with equality, and the other with strict inequality, in this case either.

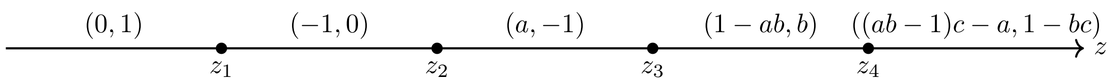

Figure 4: A general rod structure with four turning points.

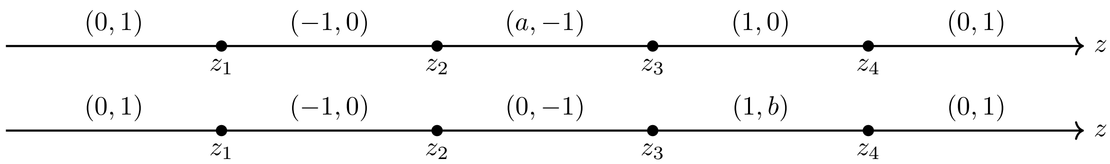

Figure 5: Two families of AF rod structures with four turning points.

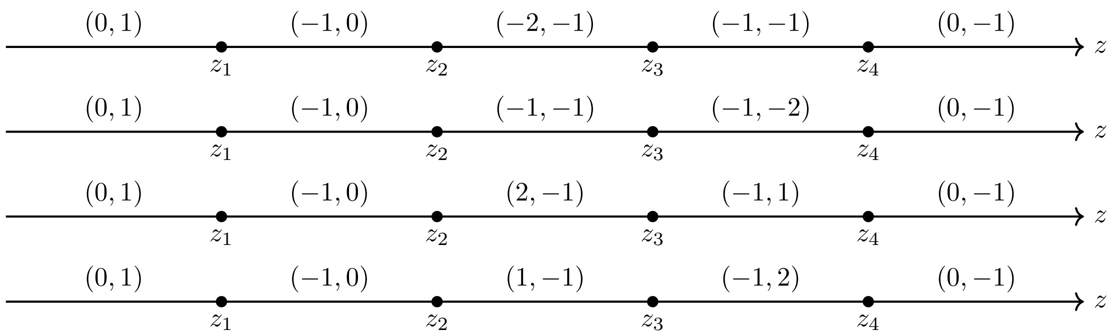

Figure 6: Four exceptional AF rod structures with four turning points.

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