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Horocycle averages on closed manifolds and transfer operators

Alexander Adam and Viviane Baladi

We study the ergodic integrals of the horocycle flows h_ρ of C^r codimension one mixing Anosov flows. In dimension three, for any suitably bunched C^3 contact Anosov flow with orientable strong-stable distribution E_-, we show that $\left|\frac{1}{T}\int_0^T \varphi\circ h_\rho(x)\,\mathrm{d}\rho-\mu(\varphi)\right| \le \frac{C}{T^\epsilon}\|\varphi\|_{C^3}$ for some $\epsilon>0$, with μ the invariant measure of h_ρ. We thereby implement the toy model program of Giulietti–Liverani (2017) in the natural setting of geodesic flows in variable negative curvature, where nontrivial resonances exist.

1. Introduction

Anosov introduced a class of C^2 flows $g_\alpha\colon M\to M$, now bearing his name [3], on closed (i.e., compact and boundaryless) orientable manifolds[1] M of dimension $d\ge 3$. We focus on topologically mixing Anosov flows. A special class of mixing Anosov flows are those preserving a contact structure. Geodesic flows on the unit tangent bundle of a closed manifold with (possibly variable) negative sectional curvature are well-studied classes of contact Anosov flows.

Every Anosov flow g_α admits a strong stable foliation, tangent to a vector bundle denoted E_-. If this foliation is orientable and has dimension d_- equal to one, and if g_α is mixing, one associates with g_α another flow, the *horocycle* flow $h_\rho\colon M\to M$, such that for every $x\in M$ the trajectory $h_{\mathbb{R}}(x)$ is a strong stable leaf (defined up to speed reparametrisation). Horocycle flows were first introduced in the case of

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[1] All manifolds are implicitly endowed with a Riemannian metric.

geodesic Anosov flows; see [43, p. 84] or [37]. In a setting more general than ours (with $d_- \geq 1$), Bowen and Marcus [15] then proved that the horocycle flow is uniquely ergodic and minimal. Its invariant probability measure μ (related to, but distinct from, the measure of maximal entropy of g_α; see Remark 4.16) plays an important role below.

Since the horocycle flow is induced by the Anosov flow, there exists $\tau(\rho, \alpha, x) > 0$ such that

$$g_\alpha \circ h_\rho(x) = h_{\tau(\rho,\alpha,x)} \circ g_\alpha(x), \ \ \forall x \in M, \ \ \forall \alpha, \rho \in \mathbb{R}_+.$$

We call $\tau(\rho, \alpha, x)$ the *renormalisation time*. In the setting of unit speed geodesic flows on finite-volume surfaces of constant negative curvature, compact or not, renormalisation has been used effectively in the work of Flaminio and Forni [28] to study the *horocycle integrals*

$$\gamma_x(\varphi, T) := \int_0^T \varphi \circ h_\rho(x)\, \mathrm{d}\rho, \quad x \in M, \quad T > 0,$$

for $\varphi : M \to \mathbb{R}$ in Sobolev spaces of positive order. Flaminio and Forni found that the speed of convergence of $\gamma_x(\varphi, T)/T$ to $\mu(\varphi)$ as $T \to \infty$ is controlled by *invariant distributions* under the push-forward of the horocyclic vector field. These distributions are also eigendistributions under the push-forward of the geodesic vector field, and the eigenvalues give the powers of T appearing in the expansion of $\gamma_x(\varphi, T)/T - \mu(\varphi)$.

Their approach inspired Giulietti and Liverani [29] to study a toy model, replacing the Anosov flow with a hyperbolic diffeomorphism F, using the renormalisation dynamics as a key to study $\gamma_x(\varphi, T)$. Letting h_{top} be the topological entropy of F, they show analogously (for the corresponding invariant measure μ) that the speed of convergence to zero of $\gamma_x(\varphi, T)/T - \mu(\varphi)$ is controlled by eigenvalues in the annulus $1 < |z| < e^{h_{\mathrm{top}}}$ (and the corresponding eigendistributions) of a weighted transfer operator of F. Unfortunately, in the setting of [29], there are in fact no eigenvalues in the annulus $1 < |z| < e^{h_{\mathrm{top}}}$; see [6]. The approach of Giulietti and Liverani has been applied successfully in the meantime by Faure–Gouëzel–Lanneau [27] to the linear flow in the stable direction of a two-dimensional linear pseudo-Anosov map, and by Butterley–Simonelli [19] to parabolic flows on (3-dimensional) Heisenberg nilmanifolds which are renormalized by partially hyperbolic automorphisms (circle extensions of Anosov diffeomorphisms). In both these algebraic applications, nontrivial eigenvalues are present.

Giulietti and Liverani suggested in [29, Conjecture 2.12] that a similar expansion exists for more general (nonalgebraic) Anosov flows than in [28]: e.g., for the geodesic flow of a surface with variable negative curvature. More precisely, letting h_{top} be the topological entropy of the time-one map g_1, we expect that there exists

$\delta > 0$ such that, for smooth enough observables φ, the following expansion holds,[2] analogously to [28; 29]:

$$\gamma_x(\varphi, T) = T \int \varphi \, d\mu + \sum_{\delta < \operatorname{Re}\lambda < h_{\text{top}}} T^{\frac{\operatorname{Re}\lambda}{h_{\text{top}}}} \tilde{c}_\lambda(T, x) \mathcal{O}_\lambda(\varphi) + \mathcal{E}_{T,x}(\varphi). \tag{1}$$

Here, $\mathcal{E}_{T,x} = O(T^{\frac{\delta}{h_{\text{top}}}})$, uniformly in x; the $\mathcal{O}_\lambda$ are generalised eigendistributions associated to the eigenvalue λ for the adjoint (or dual) of the generator of a weighted transfer operator (see (3)) acting on an anisotropic Banach space; the real parameter δ is an upper bound on the essential spectral bound of the generator; and $\tilde{c}_\lambda(T, x) \in \mathbb{C}$ satisfies $\sup_{x,T} |\tilde{c}_\lambda(T, x)| |\log T|^{-J_\lambda} < \infty$, where $J_\lambda \geq 0$ is the size of the largest Jordan block of λ.

The main result of this work, Theorem 4.8, provides an asymptotic expansion (1) for C^r time reparametrisations of the unit speed horocycle flow of codimension one topologically mixing C^r Anosov flows, if $r > 2$ and the distribution E_- is C^{r-1}, under an essential spectral gap condition ($\lambda_{\min}^{s,t,p} < h_{\text{top}}$), and a weak Dolgopyat condition on the resolvent (Condition 3.12). As a consequence, we get power-law convergence of the ergodic averages (Corollary 4.9). In Proposition 4.10, we show that the conditions of Theorem 4.8 hold for C^3 contact Anosov flows in dimension three, with orientable strong stable bundle E_-, under the following bunching assumption: Recalling that $d_- = 1$, define

$$\lambda_+ = \lim_{\alpha \to \infty} \frac{\sup_x \log \|Dg_{-\alpha}(x)|_{E_-}\|}{\alpha}, \quad \lambda_- = -\lim_{\alpha \to \infty} \frac{\sup_x \log \|Dg_\alpha(x)|_{E_-}\|}{\alpha},$$

and $\hat{\varpi} := 2\frac{\lambda_-}{\lambda_+} \in (0, 2]$. The bunching condition is

$$\hat{\varpi} > \tfrac{8}{5}. \tag{2}$$

For constant negative curvature geodesic flows, we have $\hat{\varpi} = 2$. Assumption (2) thus holds for geodesic flows with variable strictly negative curvature close enough to a constant, but the reader is warned that it does not apply to generic three-dimensional contact Anosov flows.

For compact surfaces of constant negative curvature, Randol [44] proved that there exist eigenvalues of the Laplacian arbitrarily close to 1. This provides examples for which the expansion of Flaminio–Forni [28], and thus the expansion in Theorem 4.8, is not reduced to $T \int \varphi \, d\mu + \mathcal{E}_{T,x}(\varphi)$.

As in the work of Giulietti and Liverani [29], the key idea to study $\gamma_x(\varphi, T)$ is to introduce the weighted semigroup of transfer operators, with generator $X + V$,

[2] For Anosov flows, h_{top} is positive; see [3]. For geodesic flows on finite-volume negatively curved surfaces, $h_{\text{top}} = 1$.

defined by

$$\begin{aligned}\mathcal{L}_{\alpha,V} : W_p^{s,t,q}(M) \to W_p^{s,t,q}(M), \quad \mathcal{L}_{\alpha,V}\varphi = \phi_\alpha \cdot (\varphi \circ g_{-\alpha}),\\ \phi_\alpha(x) = e^{\int_0^\alpha V\circ g_\beta(x)\,\mathrm{d}\beta}, \quad \alpha \geq 0,\end{aligned} \tag{3}$$

where the potential is $V = -\partial_\alpha \partial_\rho \tau(0, 0, \cdot)$ (so that $\phi_\alpha = \partial_\rho \tau(0, -\alpha, \cdot)$), and where $W_p^{s,t,q}(M)$ is an anisotropic Banach space with regularity parameters $s < 0 < q \leq t < r - 1 + s$ and $p \in (1, \infty)$. In the case of the unit speed parametrisation of h_ρ, we shall see that $\phi_\alpha = \det \mathrm{D}g_{-\alpha}|_{E^-}$ is just the Jacobian along the strong stable distribution at a negative time $-\alpha$, and $V = \mathrm{div}\,(X|_{E_-})$.

The paper is organised as follows: The transfer operator $\mathcal{L}_{\alpha,V}$ is defined in Section 2A (for more general potentials). The new anisotropic Banach spaces $W_p^{s,t,q}(M)$ are constructed in Section 2C after introducing admissible cones for C^r Anosov flows in Section 2B (if $p = 2$ we get Hilbert spaces). These spaces are a flow analogue to the spaces constructed in [8] to study hyperbolic diffeomorphisms. Anisotropic Banach spaces are now a standard tool for hyperbolic dynamics (see [7; 10; 12; 30; 33; 41; 50], for example). Although we do not study here the dynamical determinant or zeta function associated to the transfer operator $\mathcal{L}_{\alpha,V}$, we believe that the spaces introduced in the present work are well suited for this purpose (see [5]). Guedes Bonthonneau and Lefeuvre very recently [14] applied a (microlocal) flow implementation of the spaces from [8] to study some dynamical and geometric problems.

In Section 3, we establish properties of the transfer operator semigroup, its generator $X + V$ and the resolvent $\mathcal{R}_z$ (see (9)). Most of these results do not require the contact assumption. Among those are norm bounds yielding a Lasota–York inequality for the resolvent (Theorem 3.8). Then, in Corollary 3.9, we obtain a strip in the spectrum of the generator containing at most countable eigenvalues of finite multiplicity. Proposition 3.13 puts the weak Dolgopyat condition, 3.12, in more standard form. These tools are used in Section 4 to show the main results, Theorem 4.8 and Proposition 4.10. (Our proofs highlight sufficient conditions for intrinsicness of resonances and portability of Dolgopyat bounds on the resolvent when navigating between different Banach spaces.) We close with some supporting results needed in the text: Appendix A contains elementary lemmas on integration by parts, adapted from [8], Appendix B recalls the fragmentation/reconstitution lemmas from [8], and Appendix C is devoted to interpolation and mollifiers.

We end this introduction with some remarks.

(1) The conjecture that the distributions $\mathcal{O}_v$ in the expansion (1) are fixed by the (adjoint) of the horocycle flow, which was the starting point in [28], remains open for general codimension one mixing Anosov flows (see [29, Remark 2.10]). For smooth contact Anosov flows with $d = 3$, invariance was proved by Faure–Guillarmou [35].

(2) The anisotropic Banach spaces $W^{s,t,q}_p(M)$ in this paper are based on those in [8]. We could also define spaces $\mathcal{B}^{s,t}(M)$ based on those in [9] (or [5, Chapter 5]). We expect that the following variational upper bound may be obtained for the essential spectrum of the semigroup $\mathcal{L}_{\alpha,V}$ on $\mathcal{B}^{s,t}(M)$:[3]

$$\lambda^{s,t}_{\min}(X,V) := \sup_{\tilde\mu}\Big\{h_{\tilde\mu}(g_1)+\chi_{\tilde\mu}\Big(\frac{\phi_1}{\det(Dg_1|_{E_+})}\Big)+\max\{t\chi_{\tilde\mu}(Dg_1|_{E_-}),\,|s|\chi_{\tilde\mu}(Dg_{-1}|_{E_+})\}\Big\}. \quad (4)$$

The above is in general better (even in the volume preserving case) than the bound $\lambda^{s,t,p}_{\min}(X,V)$ we obtain in Corollary 3.9 (see (54)). Since $\lambda^{0,0}_{\min}(X,V)=h_{\mathrm{top}}$, the essential spectral gap condition $\lambda^{s,t}_{\min}<h_{\mathrm{top}}$ would thus hold for $\mathcal{B}^{s,t}(M)$ for arbitrarily small $s<0$ and $t>0$ (so that the assumptions of Proposition 4.10 could be weakened accordingly, and s' could be taken arbitrarily close to 0 in (118)). However, the scale $\mathcal{B}^{s,t}(M)$ is more messy[4] to define, it is not an interpolation scale, it does not include a Hilbert space, and showing (4) would require a thermodynamic analysis of the sums over subcovers in the proof of Lemma 3.6. To keep the paper short, we restrict to the scale $W^{s,t,q}_p(M)$.

(3) The renormalisation time $\tau(\rho,\alpha,\cdot)$ inherits the smoothness of the invariant bundle E_-, which is only Hölder in general. We add the extra assumption that E_- is smooth enough and that an essential spectral gap holds in Theorem 4.8 (and Lemma 4.15), and we give settings where this is satisfied in Proposition 4.10. To work with anisotropic spaces with higher regularity (depending only on r), one could lift the dynamics to the Grassmannian [33; 29]. We have chosen to avoid the cumbersome corresponding technicalities for the sake of readability.

(4) Finally, we mention two directions of future research: First, the expansions of Flaminio and Forni [28] (or Faure–Tsujii [24; 26]; see also [20]) are not limited to finite sets of eigenvalues. Our methods do not currently allow to go beyond the smallest δ such that $\Sigma_\delta=\sigma(X+V)|_{W^{s,t,q}_p(M)}\cap\{\mathrm{Re}\,z>\delta\}$ is finite ($\delta=\frac{1}{2}$ for [28]). Second, even if the Dolgopyat condition 3.12 holds for some $\delta<0$, we cannot improve the remainder due to the term with $\|\varphi\|_0$ in Lemma 4.14. Although it is hoped that this term is spurious, an analogous error term is present in [28, Theorem 1.5] or [29, Theorem 2.8].

2. The transfer operators and the Banach spaces

2A. ***Transfer operators associated to a flow g_α and weight ϕ_α. The generator $X+V$.*** Throughout, M is a compact, boundaryless, connected, orientable, smooth

[3]Here $h_{\tilde\mu}$ is the entropy of an ergodic g_1 invariant probability measure and $\chi_{\tilde\mu}(A)$ is the largest Lyapunov exponent of A.

[4]See also the caveat in [5, Remark 5.18] regarding the lack of validity of (27).

manifold of dimension $d \geq 3$, and $r > 1$ is fixed, while $g_\alpha : M \to M$, $\alpha \in \mathbb{R}$, is a C^r *Anosov flow* on M. By definition, there is a Dg_α-invariant splitting of the tangent space $TM = E_- \oplus E_+ \oplus E_0$ of the tangent space such that for some $C_* \geq 1$ and $0 < \theta < 1$, we have

$$\|\mathrm{D}g_\alpha v\| \leq C_* \theta^\alpha \|v\|, \forall v \in E_-, \quad \|\mathrm{D}g_{-\alpha} v\| \leq C_* \theta^\alpha \|v\|, \forall v \in E_+, \quad \forall \alpha \geq 0, \tag{5}$$

while $E_0 = \langle X \rangle$, where the C^{r-1} vector field X is the generator of the flow defined by

$$X := \partial_\alpha g_{-\alpha}|_{\alpha=0}. \tag{6}$$

The (*strong*) *stable and unstable distributions* E_- and E_+ are Hölder. For $x \in M$, we split $T_x M$ as $E_{-,x} \oplus E_{+,x} \oplus E_{0,x}$. The cotangent space T^*M (the dual space of TM) is split analogously:

$$T^*M = E_-^* \oplus E_+^* \oplus E_0^*, \quad T_x^*M = E_{-,x}^* \oplus E_{+,x}^* \oplus E_{0,x}^*, \quad x \in M.$$

The splitting above is $(\mathrm{D}g_\alpha)^{\mathrm{tr}}$-invariant and, up to taking larger C_*, we have

$$\begin{cases} C_*^{-1}\|\xi\| \leq \|(\mathrm{D}g_{-\alpha})^{\mathrm{tr}}\xi\| \leq C_*\|\xi\|, & \forall \xi \in E_0^*, \ \forall \alpha \geq 0, \\ \|(\mathrm{D}g_{-\sigma\alpha})^{\mathrm{tr}}\xi\| \leq C_*\theta^\alpha\|\xi\|, & \forall \xi \in E_\sigma^*, \ \sigma = \pm, \ \forall \alpha \geq 0. \end{cases} \tag{7}$$

The dimensions of the spaces $E_{\sigma,x}$ do not depend on x, and we set $d_- := \dim E_- = \dim E_-^*$ and $d_+ := \dim E_+ = \dim E_+^* = d - 1 - d_-$. Fixing a *potential* $V \in C^{r-1}(M, \mathbb{R})$, we introduce the ϕ_α-weighted transfer operators

$$\mathcal{L}_{\alpha,V}(\varphi) := \phi_\alpha \cdot (\varphi \circ g_{-\alpha}), \quad \alpha \geq 0, \ \varphi \in C^{r-1}(M), \tag{8}$$

where

$$\phi_\alpha(x) := \exp\Big(\int_0^\alpha V \circ g_{-\beta}(x)\,\mathrm{d}\beta\Big), \quad \text{i.e., } V = \partial_\alpha \phi_\alpha|_{\alpha=0^+}.$$

Given an *integrability parameter*, $p \in (1, \infty)$, and suitable anisotropic *regularity parameters*, s, t, and q as in (21) below, we will construct Banach spaces $W_p^{s,t,q}(M)$, containing $C^{r-1}(M)$ as a dense subspace, on which the operators $\mathcal{L}_{\alpha,V}$ extend continuously to form a strongly continuous semigroup (Lemma 3.7). In particular, for all $\varphi \in C^{r-1}(M)$,

$$\partial_\alpha \mathcal{L}_{\alpha,V}\varphi|_{\alpha=0^+} = X\varphi + V\varphi \in W_p^{s,t,q}(M).$$

The generator of the semigroup is $X + V$, we denote by $\mathcal{R}_z$ its resolvent

$$\mathcal{R}_z\varphi = (z - V - X)^{-1}\varphi, \quad z \notin \sigma(X+V)|_{W_p^{s,t,q}}, \quad \varphi \in W_p^{s,t,q}(M), \tag{9}$$

where $\sigma(X+V)|_{\mathcal{B}}$ denotes the spectrum of the operator $X + V$ on $\mathcal{B}$. Theorem 3.8 will provide a Lasota–Yorke inequality for $\mathcal{R}_z$ for large $\operatorname{Re} z$. This gives a vertical strip in the complex plane in which $\sigma(X+V)|_{W_p^{s,t,q}(M)}$ contains only isolated eigenvalues of finite multiplicity (Corollary 3.9).

2B. *Cone ensembles. The atlas $\mathcal{A}$. Cone hyperbolicity. Admissible cones for g_α.* A *cone* is a nonempty convex set $\mathcal{C} \subset \mathbb{R}^d$ such that $\lambda\xi \in \mathcal{C}$ for all $\xi \in \mathcal{C}$ and $\lambda \in \mathbb{R}$. We say that a cone $\mathcal{C}$ is d'-dimensional if $d' \geq 1$ is the maximal dimension of a linear subset of $\mathcal{C}$. A cone $\mathcal{C}$ is *compactly included* in another cone $\mathcal{C}'$, denoted by $\mathcal{C} \Subset \mathcal{C}'$, if $\bar{\mathcal{C}} \subseteq \operatorname{int} \mathcal{C}' \cup \{0\}$. Two cones $\mathcal{C}$ and $\mathcal{C}'$ are *transversal* if $\mathcal{C} \cap \mathcal{C}' = \{0\}$.

We identify T^*M with $\mathbb{R}^d$, and for any $d' \geq 1$ we denote the norm of $\xi \in \mathbb{R}^{d'}$ by $|\xi| = \left(\sum_j \xi_j^2\right)^{1/2}$. For $\xi \in T_x^*M$, write $\xi = \xi^- + \xi^+ + \xi^0$, where $\xi^\sigma \in E^*_{\sigma,x}$ for $\sigma \in \{\pm, 0\}$. For $\gamma > 0$, define two transversal closed cone fields on[5] T^*M, of respective dimensions d_- and d_+, by

$$\begin{aligned} \mathcal{C}_-^\gamma(x) &:= \left\{\xi \mid \max\{|\xi^+|, |\xi^0|\} \leq \gamma|\xi^-|\right\}, \\ \mathcal{C}_+^\gamma(x) &:= \left\{\xi \mid \max\{|\xi^-|, |\xi^0|\} \leq \gamma|\xi^+|\right\}, \end{aligned} \tag{10}$$

and define a one-dimensional closed cone field on T^*M by

$$\mathcal{C}_0^\gamma(x) := \left\{\xi \in T_x^*M \mid \max\{|\xi^-|, |\xi^+|\} \leq \gamma|\xi^0|\right\}.$$

For $\sigma \in \{\pm, 0\}$ we have $E^*_{\sigma,x} \subset \mathcal{C}_\sigma^\gamma(x)$ and, if $\gamma' > \gamma$, then $\mathcal{C}_\sigma^\gamma(x) \Subset \mathcal{C}_\sigma^{\gamma'}(x)$. Moreover $T_x^*M \subset \bigcup_\sigma \mathcal{C}_\sigma^\gamma(x)$, if $\gamma \geq 1$ (any line through the origin must cross one side of the unit cube in $\mathbb{R}^3$), while $\mathcal{C}_\sigma^\gamma(x)$ and $\mathcal{C}_\tau^\gamma(x)$ are transversal if $\sigma \neq \tau$ and $\gamma < 1$. Last but not least, the lemma below is the key to construct admissible cones:[6]

Lemma 2.1. *Let $C_* \in [1, \infty)$ and $\theta \in (0, 1)$ be the constants from* (7). *Then for any $\gamma, \gamma' \in (0, 1)$ and all $\alpha > 0$ such that $C_*^2\theta^\alpha\gamma < \gamma'$, we have, recalling $\mathcal{C}_\gamma^-(x)$ and $\mathcal{C}_\gamma^+(x)$ from* (10),

$$(\mathrm{D}g_{-\alpha})^{\mathrm{tr}}\mathcal{C}_\gamma^-(x) \Subset \mathcal{C}_{\gamma'}^-(g_\alpha(x)), \quad (\mathrm{D}g_\alpha)^{\mathrm{tr}}\mathcal{C}_\gamma^+(x) \Subset \mathcal{C}_{\gamma'}^+(g_{-\alpha}(x)), \ \forall x \in M.$$

Proof. We show the first claim. The proof of the second claim is analogous. Let $\xi = \xi^- + \xi^+ + \xi^0 \in \mathcal{C}_\gamma^-(x)$. We estimate

$$\begin{aligned} \max\left\{|(\mathrm{D}g_{-\alpha})_x^{\mathrm{tr}}\xi^+|, |(\mathrm{D}g_{-\alpha})_x^{\mathrm{tr}}\xi^0|\right\} &\leq C_* \max\{|\xi^+|, |\xi^0|\} \\ &\leq C_*\gamma|\xi^-| \leq C_*^2\theta^\alpha\gamma|(\mathrm{D}g_{-\alpha})^{\mathrm{tr}}\xi^-|. \end{aligned}$$

Since $\mathcal{C}^-_{\gamma'-\epsilon}(g_\alpha(x)) \Subset \mathcal{C}^-_{\gamma'}(g_\alpha(x))$ for all $\epsilon \in (0, \gamma')$, we conclude. □

Next, we adapt to flows the cone ensembles for hyperbolic diffeomorphisms from [5; 8]:

[5] These cones have nonempty interior while [40, Proposition 17.4.4] uses "flat" cones included in $E_+^* \oplus E_-^*$.

[6] No such property holds for $\mathcal{C}_0^\gamma(x)$. The cones in (10) are strictly expanding and contracting, respectively, and this is not true for $\mathcal{C}_0^\gamma(x)$.

Definition 2.2 (cone ensembles Θ for flows; coverings $\widetilde{\Phi}$). A cone ensemble of $\mathbb{R}^d$, with $d = d_- + d_+ + 1$, $d_\pm \geq 1$, is a pair $\Theta = (\mathcal{C}, \Phi)$, where $\mathcal{C} = (\mathcal{C}_-, \mathcal{C}_+, \mathcal{C}_0)$ is a triplet of pairwise transversal closed cones with nonempty interiors, of respective dimensions d_-, d_+, and 1, while $\Phi = (\Phi_-, \Phi_+, \Phi_0)$, where each Φ_σ is a C^∞ map from the unit sphere $\mathbb{S}^d$ to $[0, 1]$, such that

$$\Phi_- + \Phi_+ + \Phi_0 \equiv 1 , \qquad \Phi_\sigma|_{\mathcal{C}_\sigma \cap \mathbb{S}^d} \equiv 1 , \quad \sigma \in \{\pm, 0\} .$$

In addition, we require that $\mathcal{C}_0 = \{\xi \mid |\xi| \leq \gamma_0 |\xi_d|\}$ for some finite γ_0.

For two cone ensembles Θ and Θ' of $\mathbb{R}^d$, we say that$\Theta' < \Theta$ if

$$\mathbb{R}^d \setminus (\mathcal{C}_+ \cup \mathcal{C}_0) \Subset \mathcal{C}'_- \quad \text{and} \quad \mathbb{R}^d \setminus \mathcal{C}_+ \Subset \mathcal{C}'_0 \cup \mathcal{C}'_- .^7$$

Finally, for a cone ensemble Θ, we say that a triplet $\widetilde{\Phi} = (\widetilde{\Phi}_+, \widetilde{\Phi}_-, \widetilde{\Phi}_0)$ is a covering of Θ if each $\widetilde{\Phi}_\sigma : \mathbb{S}^d \to [0, 1]$ is C^∞, with $\widetilde{\Phi}_\sigma|_{\operatorname{supp} \Phi_\sigma} \equiv 1$.

Definition 2.3 (cone hyperbolicity). Let $K \subset \mathbb{R}^d$ be compact with nonempty interior, and let $\Theta = (\mathcal{C}, \Phi)$, $\Theta' = (\mathcal{C}', \Phi')$ be cone ensembles. A diffeomorphism $F \colon K \to F(K)$ is called cone-hyperbolic from Θ' to Θ (on K) if

$$(\mathrm{D}_x F)^{\mathrm{tr}} \left(\mathbb{R}^d \setminus (\mathcal{C}'_+ \cup \mathcal{C}'_0)\right) \Subset \mathcal{C}_- , \quad (\mathrm{D}_x F)^{\mathrm{tr}} \left(\mathbb{R}^d \setminus \mathcal{C}'_+\right) \Subset \mathcal{C}_0 \cup \mathcal{C}_- ,^8 \quad \forall x \in K . \tag{11}$$

The conditions in (11) ensure that no parts of higher regularity in the anisotropic Banach spaces of Section 2C are mapped to parts of lower regularity (see (37) in the proof of (36) below).

We next introduce a crucial ingredient to construct the anisotropic Banach spaces.

Lemma 2.4 (admissible cone ensembles for $g_{-\alpha}$). *There exists an atlas $\mathcal{A}$, formed of a finite open cover $\{V_\omega \subseteq M \mid \omega \in \Omega\}$ of M and C^r local diffeomorphisms $\kappa_\omega \colon V_\omega \to \mathbb{R}^d$, such that, setting $K_\omega := \kappa_\omega(V_\omega)$, we have*

$$\min_{\omega \neq \omega'} d(K_\omega, K_{\omega'}) > 1 \quad \textit{and} \quad K_M := \bigcup_\omega \overline{K}_\omega \textit{ is compact,} \tag{12}$$

and, fixing coordinates $(x_1, \ldots, x_d) \in \mathbb{R}^d$ and recalling (6), the flow box condition

$$(\mathrm{D}\kappa_\omega)(X|_{V_\omega}) = \partial_{x_d}|_{\kappa_\omega(V_\omega)} \tag{13}$$

holds, and further, setting $V_{\alpha,\omega\omega'} := V_\omega \cap g_\alpha(V_{\omega'})$ for each $\alpha \in \mathbb{R}$ and $\omega, \omega' \in \Omega$ such that $V_\omega \cap g_\alpha(V_{\omega'}) \neq \varnothing$, and also defining $F_{-\alpha,\omega\omega'} : \kappa_\omega(V_{\alpha,\omega\omega'}) \to \kappa_{\omega'}(V_{\omega'})$ as

$$F_{-\alpha,\omega\omega'} := \kappa_{\omega'} \circ g_{-\alpha} \circ \kappa_\omega^{-1},$$

[7] For our purposes, the second condition could be replaced by the weaker condition

$$\mathbb{R}^d \setminus (\mathcal{C}_- \cup \mathcal{C}_+) \Subset \mathcal{C}'_0 \cup \mathcal{C}'_- .$$

[8] For our purposes, the second condition could be replaced by $(D_x F)^{\mathrm{tr}} (\mathbb{R}^d \setminus (\mathcal{C}'_- \cup \mathcal{C}'_+)) \Subset \mathcal{C}_0 \cup \mathcal{C}_-$.

there exists $\alpha_0 > 0$ and, for each ω, there exist cone ensembles Θ_ω such that the cone $D\kappa_\omega^{tr}(\mathcal{C}_{\sigma,\omega})$ in the cotangent space contains the normal subspace E_σ^* and is bounded away from E_τ^* for $\tau \neq \sigma$, and, for all $\alpha \geq \alpha_0$, the map $F := F_{-\alpha,\omega\omega'}$ is cone-hyperbolic from Θ_ω to $\Theta_{\omega'}$ on $K := \kappa_\omega(V_{\alpha,\omega\omega'})$.

Remark 2.5 [5, Remark 4.12]. For any $\Theta' < \Theta$ the identity is cone-hyperbolic from Θ to Θ'. If F is cone-hyperbolic from Θ' to Θ, then there exists $\widetilde{\Theta}' < \Theta'$ such that F is cone-hyperbolic from $\widetilde{\Theta}'$ to Θ, and there exists $\widetilde{\Theta} > \Theta$ such that F is cone-hyperbolic from Θ' to $\widetilde{\Theta}$. Thus, Lemma 2.4 implies that there exist cone ensembles $\Theta'_\omega < \Theta_\omega$ such that for all $\alpha \geq \alpha_0$ the map $F_{-\alpha,\omega\omega'}$ is cone-hyperbolic from Θ'_ω to $\Theta_{\omega'}$ on $\kappa_\omega(V_{\alpha,\omega\omega'})$. Finally, the proof of Lemma 2.4 provides an atlas $\mathcal{A}$, cone ensembles $\Theta'_\omega < \Theta_\omega$, and $\alpha_0 > 0$ such that $\kappa_\omega \circ \kappa_{\omega'}^{-1}$ is cone-hyperbolic from $\Theta_{\omega'}$ to Θ'_ω and $F_{-\alpha,\omega\omega'}$ is cone-hyperbolic from Θ'_ω to $\Theta_{\omega'}$ for all $\alpha \geq \alpha_0$. (Such pairs $\{\Theta_\omega\}$, $\{\Theta'_\omega\}$ are called *adapted* to $\mathcal{A}$ and g_α. They are used in Lemma C.2.)

Proof of Lemma 2.4. Let $cc(A)$ denote the convex closure of a set A.[9] By uniform continuity of the stable and unstable distributions, setting

$$\mathcal{C}^\gamma_{\sigma,\omega} := cc\Big(\bigcup_{x\in V_\omega} (\mathrm{D}\kappa_\omega^{-1})^{\mathrm{tr}}\mathcal{C}^\gamma_\sigma(x)\Big), \quad \sigma \in \{\pm\},\ \omega \in \Omega,\ \gamma \in (0,1),$$

we may choose (small) V_ω, κ_ω satisfying (12)–(13) and $\gamma^* \in (0,1)$, $\tilde{\gamma}^* \in (0,\gamma^*)$ such that

$$(\mathrm{D}_x\kappa_\omega)^{\mathrm{tr}}\mathcal{C}^\gamma_{-,\omega} \Subset \mathcal{C}^{\gamma^*}_-(x),\ (\mathrm{D}_x\kappa_\omega)^{\mathrm{tr}}\mathcal{C}^\gamma_{+,\omega} \Subset \mathcal{C}^{\gamma^*}_+(x),\ \forall\,\omega \in \Omega,\ x \in V_\omega, \tag{14}$$

and

$$(\mathrm{D}_{\kappa_\omega(x)}\kappa_\omega^{-1})^{\mathrm{tr}}\mathcal{C}^{\tilde{\gamma}^*}_-(x) \Subset \mathcal{C}^{\gamma,\omega}_-, \quad (\mathrm{D}_{\kappa_\omega(x)}\kappa_\omega^{-1})^{\mathrm{tr}}\mathcal{C}^{\tilde{\gamma}^*}_+(x) \Subset \mathcal{C}^{\gamma,\omega}_+,\ \ \forall\,\omega \in \Omega,\ x \in V_\omega. \tag{15}$$

For $C_* \geq 1$, $\theta < 1$ as in (7), and γ, $\tilde{\gamma}^*$, γ^* as above, let $\alpha_0 > 0$ be such that $C_*^2\theta^\alpha\gamma^* < \tilde{\gamma}^*$ for all $\alpha \geq \alpha_0$. By (14) and Lemma 2.1, we have, using the transversal closed cones $\mathcal{C}^\gamma_{\pm,\omega}$,

$$(\mathrm{D}g_\alpha)^{\mathrm{tr}}\,(\mathrm{D}_x\kappa_\omega)^{\mathrm{tr}}\mathcal{C}^\gamma_{+,\omega} \Subset (\mathrm{D}g_\alpha)^{\mathrm{tr}}\mathcal{C}^{\gamma^*}_+(x) \Subset \mathcal{C}^{\tilde{\gamma}^*}_+(g_{-\alpha}(x)), \quad \forall \alpha \geq \alpha_0,\ \forall x \in V_\omega.$$

We proceed similarly for $(\mathrm{D}g_{-\alpha})^{\mathrm{tr}}\,(\mathrm{D}_x\kappa_\omega)^{\mathrm{tr}}\mathcal{C}^\gamma_{-,\omega}$, using (15) and Lemma 2.1. Hence, for $\alpha \geq \alpha_0$ and ω, $\omega' \in \Omega$ such that $V_{\alpha,\omega\omega'} \neq \varnothing$, we have

$$(\mathrm{D}F_{-\alpha,\omega'\omega})^{\mathrm{tr}}\mathcal{C}^\gamma_{-,\omega} \Subset \mathcal{C}^\gamma_{-,\omega'}, \quad (\mathrm{D}F_{\alpha,\omega'\omega})^{\mathrm{tr}}\mathcal{C}^\gamma_{+,\omega} = (\mathrm{D}F^{-1}_{-\alpha,\omega'\omega})^{\mathrm{tr}}\mathcal{C}^\gamma_{+,\omega} \Subset \mathcal{C}^\gamma_{+,\omega'}. \tag{16}$$

Thus, there exist cone ensembles Θ_ω such that for any $\alpha \geq \alpha_0$ and ω, $\omega' \in \Omega$ with $V_{\alpha,\omega\omega'} \neq \varnothing$, using the first claim of (16), the first inclusion in (11) holds for $F = F_{-\alpha,\omega\omega'}$, $\Theta' = \Theta_{\omega'}$, $\Theta = \Theta_\omega$, while using the second claim of (16), the second inclusion in (11) holds. □

[9] Taking the convex closure may be useful for tiny $\gamma > 0$.

2C. ***The anisotropic Banach spaces.*** We shall make use of a Paley–Littlewood decomposition: Let $\chi_+\colon \mathbb{R}_+ \to [0,1]$ be a C^∞ map such that $\chi_+|_{[0,1]} \equiv 1$ and $\operatorname{supp}\chi_+ \subseteq [0,2]$. Set

$$\Psi_0(\xi) := \chi_+(|\xi|), \quad \Psi_n(\xi) := \chi_+(|2^{-n}\xi|) - \chi_+(|2^{1-n}\xi|),\ n \ge 1,\ \xi \in \mathbb{R}^d. \tag{17}$$

This defines a C^∞ partition of unity on $\mathbb{R}^d$ since $\sum_{n=0}^{\infty} \Psi_n(\xi) = 1$. We have $\Psi_n(\xi) = \Psi_1(2^{-n+1}\xi)$, and thus

$$\operatorname{supp}\Psi_n \subseteq \{\xi \in \mathbb{R}^d \mid 2^{n-1} \le |\xi| \le 2^{n+1}\},\ \forall n \ge 1.$$

Given a cone ensemble $\Theta = (\mathcal{C}, \Phi)$, we set

$$\Psi_{\sigma,0}(\xi) := \frac{\chi_+(\xi)}{3}, \quad \Psi_{\sigma,n}(\xi) := \Psi_n(\xi)\Phi_\sigma\left(\frac{\xi}{|\xi|}\right), \quad ,\sigma \in \{\pm, 0\},\ n \ge 1.$$

This also defines a C^∞ partition of unity on $\mathbb{R}^d$ since $\sum_{n=0}^{\infty}\sum_{\sigma\in\{\pm,0\}} \Psi_{n,\sigma}(\xi) = 1$.

Writing the inverse Fourier transform as $\mathbb{F}^{-1}v(x) := (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i\xi x} v(\xi)\,\mathrm{d}\xi$, where $\xi x := \langle \xi, x\rangle$ denotes the scalar product, we have, for $\sigma \in \{\pm, 0\}$, $n \ge 1$,

$$\|\mathbb{F}^{-1}\Psi_n\|_{L_1} = \|\mathbb{F}^{-1}\Psi_1\|_{L_1} < \infty \quad \text{and} \quad \|\mathbb{F}^{-1}\Psi_{\sigma,n}\|_{L_1} = \|\mathbb{F}^{-1}\Psi_{\sigma,1}\|_{L_1} < \infty,$$

Analogous estimates hold for $\mathbb{F}^{-1}\Psi_{\sigma,0}$ and $\mathbb{F}^{-1}\Psi_0$.

Using the convolution $v_1 * v_2(x) := \int_{\mathbb{R}^d} v_1(x-y)\, v_2(y)\,\mathrm{d}y$ of two distributions v_1, v_2, we associate to any Ψ with $\mathbb{F}^{-1}\Psi \in L_1(\mathbb{R}^d)$ a *pseudodifferential operator with symbol* Ψ via

$$\begin{aligned}\Psi^{\mathrm{Op}}v(x) &:= ((\mathbb{F}^{-1}\Psi) * v)(x) = \mathbb{F}^{-1}(\Psi\cdot\mathbb{F}v)(x)\\ &= (2\pi)^{-d}\int_K\int_{\mathbb{R}^d} e^{i(x-y)\xi}\Psi(\xi)v(y)\,\mathrm{d}\xi\,\mathrm{d}y.\end{aligned} \tag{18}$$

Young's inequality, $\|v_1 * v_2\|_{L_p} \le \|v_1\|_{L_1}\|v_2\|_{L_p}$ for all $p \in (1,\infty)$, gives

$$\|\Psi^{\mathrm{Op}}v\|_{L_p} \le \|\mathbb{F}^{-1}\Psi\|_{L_1}\|v\|_{L_p},\ \forall p \in (1,\infty). \tag{19}$$

The (Bochner) space $L_p(\mathbb{R}^d, \mathcal{H})$ associated to a Hilbert space $\mathcal{H}$ is defined by $\|v\|_{L_p(\mathbb{R}^d,\mathcal{H})} := \|\|v\|_{\mathcal{H}}\|_{L_p(\mathbb{R}^d)}$. The following is a variant of the Marcinkiewicz theorem, generalising (19):

Theorem 2.6 (see [47, Theorem 0.11.F], for example). *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, and let $L(\mathcal{H}_1, \mathcal{H}_2)$ be the space of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ endowed with the operator norm. If $\mathcal{Q}(\,\cdot\,) \in C^\infty(\mathbb{R}^d, L(\mathcal{H}_1, \mathcal{H}_2))$ satisfies*

$$\|\partial_\xi^\beta \mathcal{Q}(\xi)\|_{L(\mathcal{H}_1,\mathcal{H}_2)} \le C_\beta(1 + \|\xi\|^2)^{-|\beta|/2} \quad \textit{for each multi-index } \beta,$$

the operator $\mathcal{Q}^{\mathrm{Op}}$ *defined for compactly supported continuous* $a : \mathbb{R}^d \to \mathcal{H}_1$ *by*[10]

$$(\mathcal{Q}^{\mathrm{Op}}a)(x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i(x-y)\xi}\,\mathcal{Q}(\xi)a(y)\,\mathrm{d}y\,\mathrm{d}\xi \tag{20}$$

extends for each $1 < p < \infty$ *to a bounded operator* $L_p(\mathbb{R}^d, \mathcal{H}_1) \to L_p(\mathbb{R}^d, \mathcal{H}_2)$, *and*

$$\|Q^{\mathrm{Op}}\|_{L(L_p(\mathbb{R}^d,\mathcal{B}_1),L_p(\mathbb{R}^d,\mathcal{H}_2))} \le \|\mathbb{F}^{-1}Q\|_{L_1(\mathbb{R}^d,\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2))}\,.$$

We will mostly consider the three cases

$$\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C};\quad \mathcal{H}_1 = \mathcal{H}_2 = \ell_2(\mathbb{C});\quad \mathcal{H}_2 = \ell_2^c \text{ and } \mathcal{H}_1 = \ell_2^c \text{ or } \mathcal{H}_1 = \ell_2^{c'};$$

where ℓ_2^c, $\ell_2^{c'}$ are the Hilbert spaces associated, for fixed

$$-(r-1) < s < 0 < q \le t < r-1 \tag{21}$$

and $-(r-1) < s' < s$, $-(r-1) < q' \le q$, $-(r-1) < t' < t$, to

$$\|a\|_{\ell_2^c} := \Big(\sum_{\sigma,n} 4^{c(\sigma)n}|a_{\sigma,n}|^2\Big)^{\frac{1}{2}},\quad \|a\|_{\ell_2^{c'}} := \Big(\sum_{\sigma,n} 4^{c'(\sigma)n}|a_{\sigma,n}|^2\Big)^{\frac{1}{2}},$$

where we set

$$c(-) := s,\quad c(+) := t,\quad c(0) := q,\quad c'(-) := s',\quad c'(+) := t',\quad c'(0) := q'. \tag{22}$$

Set $C_0^{\tilde r}(K) := \{f : \mathbb{R}^d \to \mathbb{C} \mid f \text{ is } C^{\tilde r} \text{ with support in } K\}$ for $K \subseteq \mathbb{R}^d$ compact with nonempty interior and $\tilde r \in [0,\infty]$. We introduce the basic building block for our anisotropic spaces:

Definition 2.7 (local anisotropic norm and Banach space). Fix a cone ensemble Θ and[11]

$$p \in (1,\infty),\quad -(r-1) < s \le q \le t < r-1. \tag{23}$$

For a compactly supported C^∞ function $v : \mathbb{R}^d \to \mathbb{C}$, set

$$\|v\|_{W^{s,t,q}_{p,\Theta}} := \Big\|\Big(\sum_{n=0}^{\infty} 4^{ns}|\Psi^{\mathrm{Op}}_{-,n}v|^2 + 4^{nt}|\Psi^{\mathrm{Op}}_{+,n}v|^2 + 4^{nq}|\Psi^{\mathrm{Op}}_{0,n}v|^2\Big)^{\frac{1}{2}}\Big\|_{L_p(\mathbb{R}^d)}.$$

For $K \subset \mathbb{R}^d$ compact with nonempty interior, the Banach space $W^{s,t,q}_{p,\Theta}(K)$ is defined to be the completion of $C_0^\infty(K)$ under $\|\cdot\|_{W^{s,t,q}_{p,\Theta}}$.

We shall also use the auxiliary seminorm $\|v\|_{W^q_{p,\Theta}} := \Big\|\Big(\sum_{n=0}^{\infty} 4^{nq}|\Psi^{\mathrm{Op}}_{0,n}v|^2\Big)^{\frac{1}{2}}\Big\|_{L_p(\mathbb{R}^d)}.$

[10] With $\mathbb{F}^{-1}Q(x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix\xi}\,\mathcal{Q}(\xi)\,\mathrm{d}\xi$, for $x \in \mathbb{R}^d$, the notation (20) is compatible with (18).

[11] In the present work, we shall either have $s = q = t$ or $t-(r-1) < s < 0 < t < r-1$ with $q \le t$.

The definition of $\|v\|_{W^{s,t,q}_{p,\Theta}}$ is just like[12] [8, (2.4)], except that we have three cones instead of two. We record the following result for convenience; the continuous inclusion claim is obvious, while the compact inclusion claim — which is in fact not used in the present work — is proved exactly like [8, Proposition 5.1], using Arzelà–Ascoli:

Lemma 2.8 (continuous and compact embeddings, local spaces). *Let $K \subset \mathbb{R}^d$ be compact with nonempty interior. Let Θ be a cone ensemble and fix p, s, q, t as in* (23). *For any $s' \le s$, $q' \le q$, and $t' \le t$, the inclusion $W^{s,t,q}_{p,\Theta}(K) \subseteq W^{s',t',q'}_{p,\Theta}(K)$ is continuous. If $s' < s$, $t' < t$, and $q' < q$, the same inclusion is compact.*

Next, letting $\|v\|_{W^t_p} = \|(1+\Delta)^{t/2} v\|_{L_p}$ denote the classical isotropic Sobolev (Triebel–Lizorkin) norm, the arguments[13] of [8, Appendix A] give, for p, s, t, q as in (23), a constant $C \in (1,\infty)$ with[14]

$$\frac{1}{C^2}\|v\|_{W_p^{-r+1}} \le \frac{1}{C}\|v\|_{W^s_p} \le \|v\|_{W^{s,t,q}_{p,\Theta}} \le C\|v\|_{W^t_p} \le C^2\|v\|_{W^{r-1}_p}, \quad \forall v \in C^\infty_0(K). \tag{24}$$

It follows that $C^{r-1}_0(K) \subset W^{s,t,q}_{p,\Theta}(K)$ (as a dense subset).

We are finally ready to define our anisotropic space of distributions on M:

Definition 2.9 (anisotropic Banach space). Let

$$\mathcal{A} = \big\{\mathcal{V} = \{V_\omega\}_{\omega\in\Omega}, \{\kappa_\omega : V_\omega \to \mathbb{R}^d\}_{\omega\in\Omega}\big\},$$

$\alpha_0 > 0$, and cone ensembles $\{\Theta_\omega\}_{\omega\in\Omega}$ admissible for $\{g_{-\alpha}\}_{\alpha\ge\alpha_0}$ be as given by Lemma 2.4. Fix a C^r partition of unity $\{\vartheta_\omega\colon M \to [0,1]\}_{\omega\in\Omega}$, subordinate to $\mathcal{V}$, that is, with $\operatorname{supp}\vartheta_\omega \subset V_\omega$. For p, s, t, q as in (23), we put[15] for $\varphi \in C^\infty(M)$ (extending $\vartheta_\omega \circ \kappa_\omega^{-1}$ from $\kappa_\omega(V_\omega)$ to $\mathbb{R}^d$ by zero),

$$\|\varphi\|_{W^{s,t,q}_p} := \Big(\sum_{\omega\in\Omega}\int_0^{\alpha_0} \|(\vartheta_\omega \cdot \mathcal{L}_{\alpha,V}\varphi)\circ\kappa_\omega^{-1}\|^2_{W^{s,t,q}_{p,\Theta_\omega}}\,\mathrm{d}\alpha\Big)^{\frac{1}{2}}. \tag{25}$$

The Banach space $W^{s,t,q}_p(M)$ is defined as the completion of $C^\infty(M)$ under $\|\cdot\|_{W^{s,t,q}_p}$.

We show in Lemma C.1 that the scale $W^{s,t,q}_p(M)$ is an interpolation scale. (The proof also shows this for the scale $W^{s,t,q}_{p,\Theta}(K)$.) In Lemma C.2 we use mollifiers to approximate the identity.

[12] This is a "Sobolev"-type (Triebel–Lizorkin) norm. A "Besov–Hölder" version [8, (2.3)] should work too, in particular for $p = \infty$.

[13] See also [5, (4.17)–(4.18), footnote 15].

[14] Generalise [8, (A.2), (A.4), (A.5)] to three cones, noting that, if $s = t (= q)$, then Θ plays no role for the norm denoted by $W^{\Theta,s,t,(q,)p}_{\dagger\dagger}$ there, so that the condition $\Theta' > \Theta$ for [8, (A.5)] is immaterial.

[15] Integration with respect to α allows one to handle the times $\alpha \in [0, \alpha_0]$ where the flow is not sufficiently hyperbolic. (This is similar to [25, Definition 8.1] and, replacing the integral by a supremum, to [7, (3.6)]. See [18, Lemma 6] for a slightly different trick.) The identity (26) shows why the L_2 norm is natural.

Note that $W_p^{s,t,q}(M)$ depends on the atlas $\mathcal{A}$, the cone ensembles Θ_ω, and α_0. It follows from (24) and the bound $\sup_{\alpha\in[0,\alpha_0]} \|\mathcal{L}_{\alpha,V}\varphi\|_{C^{r-1}} \le C(\alpha_0)\|\varphi\|_{C^{r-1}}$ that $C^{r-1}(M) \subset W_p^{s,t,q}(M)$, as a dense subset. It is not hard to show that $W_p^{s,t,q}(M)$ is contained in the set of distributions of order $r-1+d/p$ on M. For $p=2$, the space $W_p^{s,t,q}$ is a Hilbert space since it satisfies the parallelogram law [11, Lemma 15.2]:

$$\|\varphi_1+\varphi_2\|^2_{W_2^{s,t,q}} + \|\varphi_1-\varphi_2\|^2_{W_2^{s,t,q}} = 2\|\varphi_1\|^2_{W_2^{s,t,q}} + 2\|\varphi_2\|^2_{W_2^{s,t,q}}. \tag{26}$$

Clearly, if $s'\le s$, $q'\le q$, and $t'\le t$, we have the continuous injection $W_p^{s,t,q}(M) \subseteq W_p^{s',t',q'}(M)$. Due to the integration over α in the definition (25) of the norm, the compact inclusion from Lemma 2.8 does not carry over[16] automatically to the space $W_p^{s,t,q}(M)$ (despite the fact that M is compact). We provide a direct proof of the following lemma instead:

Lemma 2.10 (compact embeddings). *Fix $p\in(1,\infty)$ and s,q,t as in* (23). *If $s'<s$, $q'<q$, and $t'<t$ satisfy $-r-1<s'\le q'\le t'$, the inclusion $W_p^{s,t,q}(M)\subset W_p^{s',t',q'}(M)$ is compact.*

Proof. It suffices to show that the inclusion $W_p^{s,t,q}(M)\subset W_p^{s',s',s'}(M)$ is compact. Indeed, for every $\epsilon>0$ there exists $C(\epsilon)<\infty$ such that, for any $v\in W^{s,t,q}_{p,\Theta}$,

$$\|v\|_{W^{s',t',q'}_{p,\Theta}} \le \epsilon\|v\|_{W^{s,t,q}_{p,\Theta}} + C(\epsilon)\|v\|_{W^{s',s',s'}_{p,\Theta}} \tag{27}$$

(This is easy to prove along the lines of [5, Remarks 2.22, 4.27]). Thus,

$$\|\varphi\|_{W_p^{s',t',q'}} \le \epsilon\|\varphi\|_{W_p^{s,t,q}} + C(\epsilon)\|\varphi\|_{W_p^{s',s',s'}}, \quad \forall\varphi\in W_p^{s,t,q}(M).$$

We now show that $W_p^{s,t,q}(M)\subset W_p^{s',s',s'}(M)$ is compact. For a C^r diffeomorphism $F:K\to F(K)$ and $f\in C_0^{r-1}(K)$, we introduce the local transfer operator

$$\mathcal{M}_{F,f}\colon C^{r-1}(F(K))\to C_0^{r-1}(K),\ \mathcal{M}_{F,f}(v)=f\cdot(v\circ F). \tag{28}$$

The key fact is that the operator $\mathcal{M}_{F,f}$ is bounded for the classical Sobolev norm W_p^s on $\mathbb{R}^d$ if $s\in(-r-1,r-1)$ (apply the results of [5, Chapter 2], for example), with norm depending only on $\|f\|_{C^{r-1}}$ and $\|F\|_{C^r}$. In particular, since $F_{\alpha,\omega\omega'} := \kappa_{\omega'}\circ g_\alpha\circ\kappa_\omega^{-1}$ and $f_{\alpha,\omega}:=(\vartheta_\omega\phi_{-\alpha})\circ\kappa_\omega^{-1}$ are C^r, respectively C^{r-1} (on K_ω) uniformly in $\alpha\in[0,\alpha_0]$, with $\phi_{-\alpha}\phi_\alpha\equiv 1$,[17] the decomposition

$$(\vartheta_\omega\varphi)\circ\kappa_\omega^{-1} = \big(\vartheta_\omega\phi_{-\alpha}\sum_{\omega'}(\vartheta_{\omega'}\phi_\alpha[\varphi\circ g_{-\alpha}])\circ\kappa_{\omega'}^{-1}\circ\kappa_{\omega'}\circ g_\alpha\big)\circ\kappa_\omega^{-1}$$

leads to, with $\mathcal{M}_{\alpha_0}=\sup_{\omega,\omega',\alpha\in[0,\alpha_0]}\|\mathcal{M}_{F_{\alpha,\omega\omega'},f_{\alpha,\omega}}\|_{W_p^s}<\infty$,

$$\|(\vartheta_\omega\varphi)\circ\kappa_\omega^{-1}\|_{W_p^s} \le \mathcal{M}_{\alpha_0}\sum_{\omega'}\|(\vartheta_{\omega'}[\mathcal{L}_\alpha\varphi])\circ\kappa_{\omega'}^{-1}\|_{W_p^s},\ \forall\varphi,\forall\omega,\forall\alpha\in[0,\alpha_0]. \tag{29}$$

[16] See [4] and [2], for instance, for relevant results in this context.

[17] It is useful here that $|\phi_\alpha|$ is bounded away from zero.

Now let φ_m be a sequence in the unit ball of $W_p^{s,t,q}(M)$. By definition (25) of the norm, for every m there exists $\alpha(m) \in [0, \alpha_0]$ such that

$$\left\|(\vartheta_{\omega'} \cdot \mathcal{L}_{\alpha(m),\phi_{\alpha(m)}} \varphi_m) \circ \kappa_{\omega'}^{-1}\right\|^2_{W^{s,s,s}_{p,\Theta_{\omega'}}} \le \|(\vartheta_{\omega'} \cdot \mathcal{L}_{\alpha(m),\phi_{\alpha(m)}} \varphi_m) \circ \kappa_{\omega'}^{-1}\|^2_{W^{s,t,q}_{p,\Theta_{\omega'}}} \le \frac{1}{\alpha_0}, \quad \forall \omega'.$$

Thus, using (29), and recalling C from (24),

$$\left\|(\vartheta_\omega \varphi_m) \circ \kappa_\omega^{-1}\right\|_{W_p^s} \le \mathcal{M}_{\alpha_0} \sum_{\omega'} \left\|(\vartheta_{\omega'} \cdot \mathcal{L}_{\alpha(m),\phi_{\alpha(m)}} \varphi_m) \circ \kappa_{\omega'}^{-1}\right\|^2_{W_p^s} \le M_{\alpha_0} \frac{C \cdot \#\Omega}{\alpha_0}, \qquad \forall \omega', \forall m. \tag{30}$$

Assume for a contradiction that there is $\epsilon > 0$ and, for any $k_0 \ge 1$, there are $k, \ell \ge k_0$ with

$$\|\varphi_k - \varphi_\ell\|_{W_p^{s',s',s'}} > \epsilon. \tag{31}$$

By definition, this implies that there exists $\omega \in \Omega$ with

$$\int_0^{\alpha_0} \left\|(\vartheta_\omega \cdot \mathcal{L}_{\alpha,V}(\varphi_k - \varphi_\ell)) \circ \kappa_\omega^{-1}\right\|^2_{W^{s',s',s'}_{p,\Theta_\omega}} \mathrm{d}\alpha > \frac{\epsilon^2}{\#\Omega}.$$

Now, using again the key fact, and setting $M'_{\alpha_0} = \sup_{\omega,\omega',\alpha\in[0,\alpha_0]} \|\mathcal{M}_{F_{-\alpha,\omega\omega'}, f_{-\alpha,\omega}}\|_{W_p^{s'}}$, we get

$$\begin{aligned}\int_0^{\alpha_0} &\left\|(\vartheta_\omega \cdot \mathcal{L}_{\alpha,V}(\varphi_k - \varphi_\ell)) \circ \kappa_\omega^{-1}\right\|^2_{W^{s',s',s'}_{p,\Theta_\omega}} \mathrm{d}\alpha \\ &\le C\alpha_0 \sup_{\alpha\in[0,\alpha_0]} \left\|(\vartheta_\omega \cdot \mathcal{L}_{\alpha,\phi_\alpha}(\varphi_k - \varphi_\ell)) \circ \kappa_\omega^{-1}\right\|^2_{W_p^{s'}} \\ &\le C\alpha_0 \cdot (M'_{\alpha_0})^2 \sum_{\omega'} \left\|(\vartheta_{\omega'}(\varphi_k - \varphi_\ell)) \circ \kappa_{\omega'}^{-1}\right\|^2_{W_p^{s'}}.\end{aligned} \tag{32}$$

Since (30) implies that $(\vartheta_{\omega'} \cdot (\varphi_k - \varphi_\ell)) \circ \kappa_{\omega'}^{-1}$ is a sequence in a bounded subset of $W_p^s(K_M)$ for each ω', and since the embedding $W_p^s(K_M) \subset W_p^{s'}(K_M)$ is compact, we find k_0 such that (32) is smaller than $\epsilon^2/\#\Omega$ for all $k \ge \ell \ge k_0$ and thus the desired contradiction with (31). □

3. Properties of the transfer operator, the generator, the resolvent

3A. ***Basic estimates on the local anisotropic space.*** The natural order $- < 0 < +$ on $\{-, +, 0\}$ is compatible with our choice $s = c(-) \le q = c(0) \le t = c(+)$ from (22). Inspired by [8], we introduce the following definition:

Definition 3.1 (arrow relation). For $K \subset \mathbb{R}^d$ compact with nonempty interior, let $F : K \to F(K)$ be a C^r cone hyperbolic diffeomorphism from Θ' to Θ on K. For a covering $\widetilde{\Phi}'$ of Θ', set

$$|F|_\tau := \sup_{\substack{x\in K\\ \eta\in\operatorname{supp}\widetilde{\Phi}'_\tau}} |\mathrm{D}F(x)^{\mathrm{tr}}\eta|, \quad |F^{-1}|_\sigma := \sup_{\substack{x\in F(K)\\ \xi\in\operatorname{supp}\Phi_\sigma}} |\mathrm{D}F^{-1}(x)^{\mathrm{tr}}\xi|. \tag{33}$$

Fix $s < 0 < q < t$. For $n, \ell \geq 0$, and $\sigma, \tau \in \{\pm, 0\}$, we say that $(\tau, \ell) \hookrightarrow_K (\sigma, n)$ if

$$\big(2^{sn} \leq |F|_+^t \text{ or } 2^{-q\ell} \leq |F^{-1}|_-^{|s|}\big) \text{ and } \sigma \leq \tau \text{ and } \frac{1}{2^4|F^{-1}|_\sigma} \leq 2^{n-\ell} \leq 2^4|F|_\tau, \tag{34}$$

and we say that $(\tau, \ell) \not\hookrightarrow_K (\sigma, n)$ otherwise.

Recalling (17), let $\widetilde{\Psi}_0, \widetilde{\Psi}_1 \in C^\infty$ be such that $\widetilde{\Psi}_0|_{\operatorname{supp}\Psi_0} \equiv 1$ and $\widetilde{\Psi}_1|_{\operatorname{supp}\Psi_1} \equiv 1$. Set $\widetilde{\Psi}_n(\xi) := \widetilde{\Psi}_1(2^{-n+1}\xi)$ for $n \geq 2$. With[18] (39), (40), and (41), the following lemma shows the usefulness of the arrow relation:

Lemma 3.2. *If F is a C^r cone hyperbolic diffeomorphism from Θ' to Θ on K, there exist a covering $\widetilde{\Phi}'$ of Θ' and a constant $C_1 = C_1(F, K) > 0$ such that, setting*

$$\widetilde{\Psi}'_{\sigma,0}(\xi) := \frac{\chi_+(\xi)}{3}, \quad \widetilde{\Psi}'_{\sigma,n}(\xi) := \widetilde{\Psi}_n(\xi)\widetilde{\Phi}'_\sigma\left(\frac{\xi}{|\xi|}\right), \quad \xi \in \mathbb{R}^d, \sigma \in \{\pm, 0\}, \ n \geq 1, \tag{35}$$

we have

$$\inf_{x\in K} d\big(\operatorname{supp}\Psi_{\sigma,n} - \mathrm{D}F(x)^{\mathrm{tr}}\operatorname{supp}\widetilde{\Psi}'_{\tau,\ell}\big) \geq C_1 2^{\max\{n,\ell\}}, \ \ \forall(\tau, \ell) \not\hookrightarrow_K (\sigma, n). \tag{36}$$

Proof. If $\sigma \leq \tau$, then (36) follows from (34) (without using cone-hyperbolicity). Indeed, if $n \geq \ell$,

$$\begin{aligned} d(\operatorname{supp}\Psi_{\sigma,n}, \mathrm{D}F(x)^{\mathrm{tr}}\operatorname{supp}\widetilde{\Psi}'_{\tau,\ell}) &\geq 2^{n-1} - 2^{\ell+2}|F|_\tau \\ &= 2^{n-1}(1 - 2^{\ell-n+3}|F|_\tau) > 2^{n-2}, \ \forall x \in K, \end{aligned}$$

while, if $n < \ell$, we have

$$d(\operatorname{supp}\Psi_{\sigma,n} - \mathrm{D}F(x)^{\mathrm{tr}}\operatorname{supp}\widetilde{\Psi}'_{\tau,\ell}\big) \geq 2^{\ell-1}(2^{n-\ell} - 2^3|F|_\tau) > 2^{\ell+2}|F|_\tau, \ \forall x \in K.$$

If $\sigma > \tau$, then either $\tau = 0$ and $\sigma = +$, or $\tau = -$ and $\sigma \in \{0, +\}$. In both cases, cone-hyperbolicity of F implies

$$\bigcup_{x\in K} (\operatorname{supp}\Psi_\sigma) \cap (\mathrm{D}F(x)^{\mathrm{tr}}\operatorname{supp}\Psi'_\tau) = \{0\}, \tag{37}$$

which is a trivial intersection of closed cones. Hence there exists a covering $\widetilde{\Phi}'$ such that $\bigcup_{x\in K}(\operatorname{supp}\Psi_\sigma) \cap \big(\mathrm{D}F(x)^{\mathrm{tr}}\operatorname{supp}\widetilde{\Psi}'_\tau\big) = \{0\}$, and (36) holds for suitable C_1. $\square$

For a C^r diffeomorphism $F : K \to F(K)$, cone-hyperbolic from Θ' to Θ and a covering $\widetilde{\Phi}'$ of Θ' satisfying (36) and $f \in C_0^{r-1}(K)$, recalling the weighted

[18] In our application below, $|F|_+ < 1$ while $|F^{-1}|_- > 1$.

composition operator $\mathcal{M}_{F,f}(v) = f \cdot (v \circ F)$ from (28), set, for $a = (a_{\tau,\ell})$ in $L_p(\mathbb{R}^d, \ell_2^c)$ (Lemmas 3.3 and 3.4 providing the necessary summability),

$$(Q^{\mathrm{Op}}_{\hookrightarrow_K} a)_{\sigma,n} := \Psi^{\mathrm{Op}}_{\sigma,n} \sum_{(\tau,\ell)\hookrightarrow_K(\sigma,n)} \mathcal{M}_{F,f}(a_{\tau,\ell}),$$
$$(Q^{\mathrm{Op}}_{\not\hookrightarrow_K} a)_{\sigma,n} := \Psi^{\mathrm{Op}}_{\sigma,n} \sum_{(\tau,\ell)\not\hookrightarrow_K(\sigma,n)} \mathcal{M}_{F,f}(\widetilde{\Psi}'^{\,\mathrm{Op}}_{\tau,\ell} a_{\tau,\ell}).$$

Then, taking $a_{\tau,\ell} := (\Psi')^{\mathrm{Op}}_{\tau,\ell} v$ for the ensemble Θ, we have

$$\|\mathcal{M}_{F,f} v\|_{W^{s,t,q}_{p,\Theta}} = \|Q^{\mathrm{Op}}_{\not\hookrightarrow_K} a + Q^{\mathrm{Op}}_{\hookrightarrow_K} a\|_{L_p(\mathbb{R}^d,\ell_2^c)}. \tag{38}$$

Lemma 3.3 describes the $\hookrightarrow$ term in the decomposition above. It will give the "contracting" factor $C_\pm$ in Lemma 3.5 for $\sigma = \pm$, while the term with C_0 in Lemma 3.5 for $\sigma = 0$ will become compact for the resolvent, see Lemmas 3.11 and 3.6).

Lemma 3.3 (the bounded term). *Fix $p \in (1,\infty)$ and s, q, t as in* (21). *There exists $C < \infty$ such that, for each compact $K \subset \mathbb{R}^d$ with nonempty interior, each C^r cone-hyperbolic*[19] *diffeomorphism $F : K \to F(K)$ from Θ' to Θ, each covering $\widetilde{\Phi}'$ of Θ', and each $f \in C_0^{r-1}(K)$*

$$\|Q^{\mathrm{Op}}_{\hookrightarrow_K} a\|_{L_p(\mathbb{R}^d,\ell_2^c)} \le C \max\{|F|^t_+, |F^{-1}|^{|s|}_-\} \sup_K \big|f\,|\det DF|^{-1/p}\big| \cdot \|a\|_{L_p(\mathbb{R}^d,\ell_2^c)}$$
$$+ C|F|^q_0 \sup_K \big|f\,|\det DF|^{-1/p}\big| \cdot \Big\|\Big(\sum_\ell 4^{q\ell}|a_{0,\ell}|^2\Big)^{\frac12}\Big\|_{L_p}.$$

Proof. Recall (22). There exists $C < \infty$, independent of F, such that

$$\sum_{n:(\tau,\ell)\hookrightarrow_K(\sigma,n)} 2^{c(\sigma)n-c(\tau)\ell} = \sum_{n:(\tau,\ell)\hookrightarrow_K(\sigma,n)} 2^{(c(\sigma)-c(\tau))n+c(\tau)(n-\ell)} \le \sum_{n:(\tau,\ell)\hookrightarrow_K(\sigma,n)} 2^{c(\tau)(n-\ell)}$$
$$\le C \max\{|F|^t_+, |F^{-1}|^{|s|}_-\},$$
$$\forall(\tau,\ell),\ \sigma,\ \text{with } (\sigma,\tau) \ne (0,0). \tag{39}$$

Similarly, up to taking a larger constant $C < \infty$, we have

$$\sum_{\ell:(\tau,\ell)\hookrightarrow_K(\sigma,n)} 2^{c(\sigma)n-c(\tau)\ell} \le C \max\{|F|^t_+, |F^{-1}|^{|s|}_-\},$$
$$\forall(\sigma,n),\ \tau,\ \text{with } (\sigma,\tau) \ne (0,0). \tag{40}$$

Theorem 2.6 applied to $\mathcal{H}_1 = \mathcal{H}_2 = \ell_2^c$ and $(Qb)_{\sigma,n}(\xi) = \Psi_{\sigma,n}(\xi) b_{\sigma,n}(\xi)$ gives D_1 such that

$$\|Q^{\mathrm{Op}}_{\hookrightarrow_K} a\|_{L_p(\mathbb{R}^d,\ell_2^c)} \le D_1 \Big\|\Big(\sum_{\sigma,n} 4^{c(\sigma)n}\Big|\sum_{(\tau,\ell)\hookrightarrow_K(\sigma,n)} \mathcal{M}_{F,f} a_{\tau,\ell}\Big|^2\Big)^{\frac12}\Big\|_{L_p}, \quad \forall f, F, a.$$

Set $\lambda_{F,s,t} = \max\{|F|^t_+, |F^{-1}|^{|s|}_-\}$. By Cauchy–Schwarz and (39)–(40), we find D_2

[19] Cone hyperbolicity is not really needed.

and D_3 such that

$$\begin{aligned}
3D_1 &\sum_{(\tau,\sigma)\neq(0,0)} \Big\| \Big(\sum_n \sum_{j:(\tau,j)\hookrightarrow_K(\sigma,n)} 2^{c(\sigma)n-c(\tau)j} \sum_{\ell:(\tau,\ell)\hookrightarrow_K(\sigma,n)} 2^{c(\sigma)n+c(\tau)\ell} |\mathcal{M}_{F,f} a_{\tau,\ell}|^2 \Big)^{\frac{1}{2}} \Big\|_{L_p} \\
&\le D_2 \sum_{(\tau,\sigma)\neq(0,0)} \Big\| \Big(\lambda_{F,s,t} \sum_n \sum_{\ell:(\tau,\ell)\hookrightarrow_K(\sigma,n)} 2^{c(\sigma)n+c(\tau)\ell} |\mathcal{M}_{F,f} a_{\tau,\ell}|^2 \Big)^{\frac{1}{2}} \Big\|_{L_p} \\
&= D_2 \sum_{(\tau,\sigma)\neq(0,0)} \Big\| \Big(\lambda_{F,s,t} \sum_\ell 4^{c(\tau)\ell} |\mathcal{M}_{F,f} a_{\tau,\ell}|^2 \sum_{n:(\tau,\ell)\hookrightarrow_K(\sigma,n)} 2^{c(\sigma)n-c(\tau)\ell} \Big)^{\frac{1}{2}} \Big\|_{L_p} \\
&\le D_3 \sup_K \big| f\, |\det DF|^{-1/p} \big| \sum_{\sigma,\tau} \Big\| \Big(\lambda^2_{F,s,t} \sum_\ell 4^{c(\tau)\ell} |a_{\tau,\ell}|^2 \Big)^{\frac{1}{2}} \Big\|_{L_p}, \qquad \forall f,\ F,\ a.
\end{aligned}$$

Finally, since there exists $C_{00} < \infty$, independent of F, such that

$$\begin{aligned}
\sum_{n:(0,\ell)\hookrightarrow_K(0,n)} 2^{c(0)(n-\ell)} &\le C_{00} |F|_0^q, \ \forall \ell, \quad \text{and} \\
\sum_{\ell:(0,\ell)\hookrightarrow_K(0,n)} 2^{c(0)(n-\ell)} &\le C_{00} |F|_0^q, \ \forall n,
\end{aligned} \tag{41}$$

we find D_4 such that

$$\begin{aligned}
&\Big\| \Big(\sum_n 4^{c(0)n} \Big| \sum_{(0,\ell)\hookrightarrow_K(0,n)} \mathcal{M}_{F,f} a_{\tau,\ell} \Big|^2 \Big)^{\frac{1}{2}} \Big\|_{L_p} \\
&\qquad \le D_4 \sup_K \big| f\, |\det DF|^{-1/p} |F|_0^q \big| \Big\| \Big(\sum_\ell 4^{q\ell} |a_{0,\ell}|^2 \Big)^{\frac{1}{2}} \Big\|_{L_p}, \ \forall F,\ f,\ a. \quad \square
\end{aligned}$$

We next bound the other term in the decomposition (38) of $\mathcal{M}_{F,f}$. For this, we need the following strengthening of condition (21):

$$t - (r-1) < s < 0 < q < t. \tag{42}$$

Lemma 3.4 (the compact term). *Fix $p \in (1,\infty)$, and fix s, q, t as in* (42). *Let $s' < s$, $q' \le q$ and $t' < t$ satisfy $t-(r-1) < s' < 0 < q' \le t'$. Let F be a C^r cone-hyperbolic diffeomorphism from Θ' to Θ on K, let $\widetilde{\Phi}'$ be given by Lemma 3.2, and let $f \in C_0^{r-1}(K)$. Then there exists $C(F, f) < \infty$ such that*

$$\| Q^{\mathrm{Op}}_{\not\hookrightarrow_K} a \|_{L_p(\mathbb{R}^d, \ell_2^c)} \le C(F, f) \cdot \big\| \|a\|_{\ell_2^{c'}} \big\|_{L_p}, \ \forall a.$$

Proof. We shall work along the lines of [8, pp. 144–147], using Lemma 3.2 and integration by parts. For $x \in \mathbb{R}^d$ and $(\tau,\ell) \not\hookrightarrow_K (\sigma,n)$, write

$$\begin{aligned}
(J^{\tau,\ell}_{\sigma,n} a)(x) &:= \frac{(2\pi)^{2d}}{2^{(n+\ell)d}} \Psi^{\mathrm{Op}}_{\sigma,n} \mathcal{M}_{F,f} (\widetilde{\Psi}'^{\mathrm{Op}}_{\tau,\ell} b_{\tau,\ell})(x) \\
&= \int_{y\in K} \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} e^{i2^n\tilde{\xi}(x-y)} e^{i2^\ell\tilde{\eta}(F(y)-w)} \Psi_{\sigma,1}(\tilde{\xi}) \widetilde{\Psi}'_{\tau,1}(\tilde{\eta}) f(y) a_{\tau,\ell}(w) \, \mathrm{d}w \, \mathrm{d}\tilde{\xi} \, \mathrm{d}\tilde{\eta} \, \mathrm{d}y,
\end{aligned}$$

where we used the change of variables $\tilde{\xi} = 2^{-n}\xi$ and $\tilde{\eta} = 2^{-\ell}\eta$. Integrating by parts

$r-1$ times (see Lemmas A.1–A.2) in y, and using (36), we rewrite $(J^{\tau,\ell}_{\sigma,n}a)(x)$ as

$$\int_{y\in K}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} e^{i2^n\tilde{\xi}(x-y)}e^{i2^\ell\tilde{\eta}(F(y)-w)}\Psi_{\sigma,1}(\tilde{\xi})\widetilde{\Psi}'_{\tau,1}(\tilde{\eta}) \times \frac{f_{r-1,n,\ell}(\tilde{\eta},\tilde{\xi},y)}{2^{\max\{n,\ell\}(r-1)}}a_{\tau,\ell}(w)\,\mathrm{d}w\,\mathrm{d}\tilde{\xi}\,\mathrm{d}\tilde{\eta}\,\mathrm{d}y,$$

where all partial derivatives of $f_{r-1,n,\ell}(\tilde{\eta},\tilde{\xi},y)$ with respect to $\tilde{\eta}$ and $\tilde{\xi}$ are bounded by a constant $C_2(F,f)$ uniformly in n, ℓ, and $(\tilde{\xi},\tilde{\eta},y)\in\operatorname{supp}\Psi_{\sigma,1}\times\operatorname{supp}\widetilde{\Psi}'_{\tau,1}\times K$. Define $b:\mathbb{R}^d\to[0,1]$ by

$$b(y):=\begin{cases}1 & \text{if } |y|\le 1,\\ |y|^{-d-1} & \text{if } |y|>1.\end{cases}$$

If $|x-y|2^n>1$ we integrate $(d+1)$-times by parts in $\tilde{\xi}$, and if $|w-F(y)|2^\ell>1$ we integrate $(d+1)$-times by parts in $\tilde{\eta}$. Hence, we arrive at the following formula for $(J^{\tau,\ell}_{\sigma,n}a)(x)$:

$$\int_{y\in K}\int_{\operatorname{supp}\Psi_{\sigma,1}\times\operatorname{supp}\widetilde{\Psi}'_{\tau,1}}\int_{\mathbb{R}^d}\frac{\tilde{f}_{r-1,n,\ell}(\tilde{\xi},\tilde{\eta},y)}{2^{\max\{n,\ell\}(r-1)}}b_n(x-y)b_\ell(w-F(y))a_{\tau,\ell}(w)\,\mathrm{d}w\,\mathrm{d}\tilde{\xi}\,\mathrm{d}\tilde{\eta}\,\mathrm{d}y,$$

where $b_m(w)=b(2^m w)$ $(m\ge 0)$, and $\tilde{f}_{r-1,n,\ell}(\tilde{\xi},\tilde{\eta},y)$ is uniformly bounded by $C'_2(F,f)$. Thus, there exists $C_3<\infty$ such that for all $x\in\mathbb{R}^d$

$$|(J^{\tau,\ell}_{\sigma,n}a)(x)|\le C_3\,C_2(F,f)\,2^{-\max\{n,\ell\}(r-1)}\big(b_n*(b_\ell\circ F)*|a_{\tau,\ell}|\big)(x),\quad \text{if } (\tau,\ell)\not\leftrightarrow_K(\sigma,n). \tag{43}$$

Since $r-1>t-s'>0$ and $c(\sigma)\le t$, $c'(\tau)\ge s'$, there exists $\epsilon>0$ such that for all σ and τ,

$$2^{(c(\sigma)+\epsilon)n-c'(\tau)\ell-\max\{n,\ell\}(r-1)}\le 2^{(t+\epsilon)n-s'\ell-\max\{n,\ell\}(r-1)}\le 2^{-\epsilon\ell},\quad \forall n\ge 1,\forall\ell\ge 1. \tag{44}$$

We can assume $n\cdot\ell\ne 0$ since if $n=0$ or $\ell=0$ then ξ or η is bounded. (By note 18, we have $n\cdot\ell\ne 0$ in our application.) Hence, using the triangle inequality and then (43), we find $C(\epsilon)<\infty$ such that, for all a,

$$\begin{aligned}\|Q^{\mathrm{Op}}_{\not\leftrightarrow_K}a\|_{L_p(\mathbb{R}^d,\ell^c_2)} &\le C(\epsilon)\sup_{\sigma,n}2^{(c(\sigma)+\epsilon)n}\Big\|\Psi^{\mathrm{Op}}_{\sigma,n}\sum_{(\tau,\ell)\not\leftrightarrow_K(\sigma,n)}\mathcal{M}_{F,f}(\widetilde{\Psi}'^{\mathrm{Op}}_{\tau,\ell}a_{\tau,\ell})\Big\|_{L_p(\mathbb{R}^d)}\\ &\le C(\epsilon)\sup_{\sigma,n}\sum_{(\tau,\ell)\not\leftrightarrow_K(\sigma,n)}2^{(c(\sigma)+\epsilon)n-c'(\tau)\ell}2^{c'(\tau)\ell}\|\Psi^{\mathrm{Op}}_{\sigma,n}\mathcal{M}_{F,f}(\widetilde{\Psi}'^{\mathrm{Op}}_{\tau,\ell}a_{\tau,\ell})\|_{L_p}\\ &= C(\epsilon)\frac{2^{d(n+\ell)}}{(2\pi)^{2d}}\sup_{\sigma,n}\sum_{(\tau,\ell)\not\leftrightarrow_K(\sigma,n)}2^{(c(\sigma)+\epsilon)n-c'(\tau)\ell}2^{c'(\tau)\ell}\|J^{\tau,\ell}_{\sigma,n}(a)\|_{L_p}.\end{aligned}$$

Applying Young's inequality (with $\|b_m\|_{L_1} = 2^{-dm}\|b\|_{L^1}$, for $m = \ell$ and n) and (44) then yields finite constants C_4, C_5, C_6 (depending on ϵ, F and f) such that, again for all a,

$$
\begin{aligned}
&\| Q^{\mathrm{Op}}_{\not\hookrightarrow_K} a \|_{L_p(\mathbb{R}^d,\ell_2^c)} \\
&\quad \le C_4 \sup_{\sigma,n} \sum_{\tau,\ell} 2^{(c(\sigma)+\epsilon)n - c'(\tau)\ell - \max\{n,\ell\}(r-1)} 2^{(n+\ell)d} 2^{c'(\tau)\ell} \| b_n * (b_\ell \circ F) * a_{\tau,\ell} \|_{L_p} \\
&\quad \le C_5 \sum_{\tau,\ell} 2^{-\ell\epsilon} 2^{c'(\tau)\ell} \| a_{\tau,\ell} \|_{L_p} \\
&\quad \le C_6 \sup_{\tau,\ell} 2^{c'(\tau)\ell} \| a_{\tau,\ell} \|_{L_p} \le C(F, f)\, \| a \|_{L_p(\mathbb{R}^d,\ell_2^{c'})} . \qquad \square
\end{aligned}
$$

We end this subsection with a bound on the transfer operator $\mathcal{M}_{F,f}$ from (28).

Lemma 3.5 (bounding the local transfer operator). *Let $K \subset \mathbb{R}^d$ be compact with nonempty interior. Fix $p \in (1,\infty)$ and fix s, q, t as in* (42). *Let $s' < s$, $q' \le q$ and $t' < t$ satisfy $t - (r-1) < s' < 0 < q' \le t'$. Then there exists $C < \infty$ such that for any cone-hyperbolic C^r-diffeomorphism F from Θ' to Θ on K, taking the covering $\widetilde{\Phi}'$ given by Lemma 3.2, we have, for any $f : \mathbb{R}^d \to \mathbb{C}$ in $C_0^{r-1}(K)$ and all $v \in W^{s,t,q}_{p,\Theta,F(K)}$,*

$$
\begin{aligned}
&\| \mathcal{M}_{F,f} v \|_{W^{s,t,q}_{p,\Theta}} \\
&\qquad \le C\Big(C(F,f) \| v \|_{W^{s',t',q'}_{p,\Theta',F(K)}} + C_0(F,f) \| v \|_{W^{q}_{p,\Theta',F(K)}} + C_\pm(F,f) \| v \|_{W^{s,t,q}_{p,\Theta',F(K)}} \Big),
\end{aligned}
$$

where $C(F, f)$ is from Lemma 3.4, and, recalling $|F|_+$, $|F|_0$, and $|F^{-1}|_-$ from (33),

$$
\begin{aligned}
C_0(F,f) &= \sup_K \left| \frac{f}{|\det \mathrm{D}F|^{1/p}} \right| \cdot \max\{1, |F|_0^q\}, \\
C_\pm(F,f) &= \sup_K \left| \frac{f}{|\det \mathrm{D}F|^{1/p}} \right| \cdot \max\{|F|_+^t, |F^{-1}|_-^{|s|}\} .
\end{aligned}
$$

Proof. Let $v \in W^{s,t,q}_{p,\Theta,F(K)} \subset W^{s',t',q'}_{p,\Theta,F(K)}$. For $\tau \in \{\pm, 0\}$, $\ell \ge 0$, set $a_{\tau,\ell} = (\Psi'_{\tau,\ell})^{\mathrm{Op}} v$. Recalling (22), we have $a = a(v) \in L_p(\mathbb{R}^d, \ell_2^c) \subseteq L_p(\mathbb{R}^d, \ell_2^{c'})$, and more precisely

$$
\| a \|_{L_p(\mathbb{R}^d,\ell_2^c)} = \| v \|_{W^{s,t,q}_{p,\Theta',F(K)}}, \quad \| a \|_{L_p(\mathbb{R}^d,\ell_2^{c'})} = \| v \|_{W^{s',t',q'}_{p,\Theta',F(K)}},
$$
$$
\Big\| \Big(\sum_\ell 4^{q\ell} \big| a_{0,\ell} \big|^2 \Big)^{\frac12} \Big\|_{L_p} = \| v \|_{W^q_{p,\Theta',F(K)}} .
$$

Letting $\hookrightarrow_K$ be as in Definition 3.1, the decomposition (38) gives

$$
\begin{aligned}
\| \mathcal{M}_{F,f} v \|_{W^{s,t,q}_{p,\Theta,K}} &= \| Q^{\mathrm{Op}}_{\not\hookrightarrow,K} a + Q^{\mathrm{Op}}_{\hookrightarrow,K} a \|_{L_p(\mathbb{R}^d,\ell_2^c)} \\
&\le \| Q^{\mathrm{Op}}_{\not\hookrightarrow,K} a \|_{L_p(\mathbb{R}^d,\ell_2^c)} + \| Q^{\mathrm{Op}}_{\hookrightarrow,K} a \|_{L_p(\mathbb{R}^d,\ell_2^c)} .
\end{aligned}
$$

We conclude using Lemmas 3.3 and 3.4. $\square$

3B. ***Lasota–Yorke type bounds on the transfer operator.*** For $s < 0 < t$ and $\alpha > 0$, set

$$\lambda^{(s,t,\alpha)}(x) := \max\left\{ \|(\mathrm{D}g_{-\alpha})^{\mathrm{tr}}|_{E^*_{+,x}}\|^t, \|(\mathrm{D}g_\alpha)^{\mathrm{tr}}|_{E^*_{-,g_{-\alpha}(x)}}\|^{-s} \right\}, \quad x \in M. \tag{45}$$

There exists $C' < \infty$ such that $\sup_x \lambda^{(s,t,\alpha)}(x) \le C' \theta^{\min\{t,|s|\}\alpha}$, by property (7).

Lemma 3.6 (bounding the transfer operator). *Fix $p \in (1,\infty)$ and fix s, q, t as in* (42). *Let $t-(r-1) < s' < s < 0 < q < t' < t$. There exist $A = A(X,V) < \infty$ and $C = C(X,V) < \infty$, such that, for all $\varphi \in W^{s,t,q}_p(M)$,*

$$\|\mathcal{L}_{\alpha,V}\varphi\|_{W^{s,t,q}_p} \le Ce^{A\alpha}\|\varphi\|_{W^{s',t',q}_p} + C\|\phi_\alpha|\det \mathrm{D}g_{-\alpha}|^{-1/p}\lambda^{(s,t,\alpha)}\|_{L_\infty}\|\varphi\|_{W^{s,t,q}_p}, \ \forall \alpha \ge 0.$$

This bound shows that $\mathcal{L}_{\alpha,V}$ is an operator semigroup on $W^{s,t,q}_p(M)$. As usual for flows, however, it is not a true Lasota–Yorke bound since $W^{s,t,q}_p(M)$ is not compactly embedded in $W^{s',t',q'}_p(M)$ if $q' = q$. However, using Lemma 3.11, we will prove in Theorem 3.8 that the resolvent $(z - V - X)^{-1}$ satisfies a Lasota–Yorke bound.

Proof. If $\alpha < \alpha_0$, using $\int_0^{\alpha_0} = \int_0^{\alpha_0-\alpha} + \int_{\alpha_0-\alpha}^{\alpha_0}$, we find

$$\|\mathcal{L}_{\alpha,V}\varphi\|_{W^{s,t,q}_p} \le \|\varphi\|_{W^{s,t,q}_p} + \|\mathcal{L}_{\alpha_0,\phi_{\alpha_0}}\varphi\|_{W^{s,t,q}_p}.$$

We may assume from now on that $\alpha \ge \alpha_0$. Recalling the charts $\kappa_\omega : V_\omega \to \mathbb{R}^d$, the partition of unity ϑ_ω, and the cone systems Θ_ω (from Lemma 2.4) above Definition 2.9, write, as before,

$$V_{\alpha,\omega\omega'} = V_\omega \cap g_\alpha(V_{\omega'}) \quad \text{and} \quad F_{-\alpha,\omega\omega'}(x) = \kappa_{\omega'} \circ g_{-\alpha} \circ \kappa_\omega^{-1}(x), \ x \in \kappa_\omega(V_{\alpha,\omega\omega'}),$$
$$\alpha \ge \alpha_0, \ \omega, \omega' \in \Omega.$$

Since $\alpha \ge \alpha_0$, each $F_{-\alpha,\omega\omega'}$ is cone-hyperbolic from $\Theta_{\omega'}$ to Θ_ω on $\kappa_\omega(V_{\alpha,\omega\omega'})$.

The intersection multiplicity of a family of sets is the maximal number of sets having nonempty intersection, while the intersection multiplicity of a family of functions is the intersection multiplicity of the family of the supports of the functions.

We claim that there exists an integer $\nu_d \ge 2$, depending only on d, such that the following holds: There exist $C = C_{\alpha_0} < \infty$ and, for each $\alpha \ge \alpha_0$, a finite refinement $\mathcal{W}_\alpha = \{W_{\alpha,\vec{\omega}}\}_{\vec{\omega}\in\Omega_\alpha}$ of $\mathcal{V}_\alpha = \{V_{\alpha,\omega\omega'}\}_{(\omega,\omega')\in\Omega^2}$, of intersection multiplicity at most ν_d, such that

$$\sup_W \left|\phi_\alpha \cdot |\det \mathrm{D}g_{-\alpha}|^{-1/p}\right| \le C \inf_W \left|\phi_\alpha \cdot |\det \mathrm{D}g_{-\alpha}|^{-1/p}\right|, \ \forall\, W \in \mathcal{W}_\alpha. \tag{46}$$

Indeed, since there exists $K_\alpha < \infty$ with $\sup_{\beta\in[0,\alpha]} d(g_{-\beta}(x), g_{-\beta}(y)) \le K_\alpha d(x,y)$, while $\log \phi_\alpha = \int_0^\alpha V(g_{-\beta}(x))\,\mathrm{d}\beta$, and (noting that $\alpha - \alpha_0[\alpha/\alpha_0] \in [0,\alpha_0)$)

$$\log|\det \mathrm{D}g_{-\alpha}|$$
$$= \log|\det \mathrm{D}g_{-(\alpha-\alpha_0[\alpha/\alpha_0])}| \circ g_{-([\alpha/\alpha_0]-1)\alpha_0} + \sum_{\ell=0}^{[\alpha/\alpha_0]-1} \log|\det \mathrm{D}g_{-\alpha_0}| \circ g_{-\ell\alpha_0},$$

where V and $|\det Dg_{-\alpha_0}|$, $|\det Dg_{-(\alpha-\alpha_0[\alpha/\alpha_0])}|$ are uniformly continuous on M (they are in fact γ-Hölder for $\gamma = \min\{r-1, 1\}$), there exists a finite refinement[20] $\widetilde{\mathcal{V}}_\alpha$ of $\mathcal{V}_\alpha$ such that (46) holds for all $W \in \widetilde{\mathcal{V}}_\alpha$. A finite refinement $\mathcal{W}_\alpha$ of $\widetilde{\mathcal{V}}_\alpha$ satisfying the claimed intersection multiplicity bound can then be obtained, e.g., by covering M with d-dimensional balls of radius the Lebesgue number of $\widetilde{\mathcal{V}}_\alpha$ centred on an appropriate lattice; see, e.g., [5, footnote 19, p. 46]. (Note that the cardinality of $\widetilde{\mathcal{V}}_\alpha$ or $\mathcal{W}_\alpha$ is immaterial in view of the use of the reconstitution Lemma B.2 below.) Finally, fix a C^r partition[21] of unity $\{\vartheta_{\alpha,\vec{\omega}}\}_{\vec{\omega}\in\Omega_\alpha}$ of M, subordinate to the cover $\mathcal{W}_\alpha$, with intersection multiplicity at most ν_d.

After these preliminaries, we perform the estimate. Let $\varphi \in W_p^{s,t,q}(M)$; by definition, we have

$$\|\mathcal{L}_{\alpha,V}\varphi\|^2_{W_p^{s,t,q}} \le \#\Omega \max_{\omega\in\Omega} \int_0^{\alpha_0} \left\|(\vartheta_\omega \cdot (\mathcal{L}_{\alpha',V} \circ \mathcal{L}_{\alpha,V}\varphi)) \circ \kappa_\omega^{-1}\right\|^2_{W^{s,t,q}_{p,\Theta_\omega}} \, d\alpha' .$$

Next, using $\mathcal{L}_{\alpha',V} \circ \mathcal{L}_{\alpha,V} = \mathcal{L}_{\alpha,V} \circ \mathcal{L}_{\alpha',V}$ and setting $\varphi_{\alpha'} = \mathcal{L}_{\alpha',V}(\varphi)$, we find

$$\begin{aligned}
&\left\|(\vartheta_\omega \cdot (\mathcal{L}_{\alpha',V} \circ \mathcal{L}_{\alpha,V}\varphi)) \circ \kappa_\omega^{-1}\right\|_{W^{s,t,q}_{p,\Theta_\omega}} \\
&\quad = \Big\| \sum_{\omega'\in\Omega} \sum_{\vec{\omega}\in\Omega_\alpha} (\vartheta_\omega \vartheta_{\alpha,\vec{\omega}} \cdot \phi_\alpha) \circ \kappa_\omega^{-1} \cdot (\vartheta_{\omega'} \cdot \varphi_{\alpha'}) \circ \kappa_{\omega'}^{-1} \circ F_{-\alpha,\omega\omega'} \Big\|_{W^{s,t,q}_{p,\Theta_\omega}} \\
&\quad \le C \nu_d^{(p-1)/p} \max_{\omega'\in\Omega} \Big(\sum_{\vec{\omega}\in\Omega_\alpha} \left\|(\vartheta_\omega \vartheta_{\alpha,\vec{\omega}} \cdot \phi_\alpha) \circ \kappa_\omega^{-1} \cdot (\vartheta_{\omega'} \cdot \varphi_{\alpha'}) \circ \kappa_{\omega'}^{-1} \circ F_{-\alpha,\omega\omega'}\right\|^p_{W^{s,t,q}_{p,\Theta_\omega}} \Big)^{\frac{1}{p}} \\
&\qquad + \widetilde{C}_{\vartheta_\alpha} \max_{\omega'\in\Omega} \sum_{\vec{\omega}\in\Omega_\alpha} \left\|(\vartheta_\omega \vartheta_{\alpha,\vec{\omega}} \cdot \phi_\alpha) \circ \kappa_\omega^{-1} \cdot (\vartheta_{\omega'} \cdot \varphi_{\alpha'}) \circ \kappa_{\omega'}^{-1} \circ F_{-\alpha,\omega\omega'}\right\|_{W^{s',t',q'}_{p,\Theta_\omega}} ,
\end{aligned} \tag{47}$$

using the fragmentation lemma, B.1. By Lemma 3.5 the term on the last line of (47) is bounded by

$$\widetilde{C}_{0,\alpha}(X, V) \max_{\omega'\in\Omega} \|(\vartheta_{\omega'}\varphi_{\alpha'}) \circ \kappa_{\omega'}^{-1}\|_{W^{s',t',q'}_{p,\Theta_{\omega'}}} , \tag{48}$$

for $\widetilde{C}_{0,\alpha}(X, V) < \infty$. Remark 2.5 gives systems $\Theta'_\omega < \Theta_\omega$ (independent of α) such that $F_{-\alpha,\omega\omega'}$ is cone-hyperbolic from $\Theta'_{\omega'}$ to Θ_ω on $\kappa_\omega(V_{\alpha,\omega\omega'})$. For $\alpha \ge \alpha_0$ and $\vec{\omega} \in \Omega_\alpha$, let $\tilde{\vartheta}_{\alpha,\vec{\omega}} : M \to [0, 1]$ be C^{r-1}, supported in $W_{\alpha,\vec{\omega}}$, and such that $\tilde{\vartheta}_{\alpha,\vec{\omega}}\vartheta_{\alpha,\vec{\omega}} = \vartheta_{\alpha,\vec{\omega}}$, and set

$$f_{\alpha,\vec{\omega}} = (\vartheta_\omega \tilde{\vartheta}_{\alpha,\vec{\omega}}\phi_\alpha) \circ \kappa_\omega^{-1}, \quad \bar{\vartheta}_{\alpha,\vec{\omega}} = (\vartheta_\omega \vartheta_{\alpha,\vec{\omega}}) \circ \kappa_\omega^{-1} \circ F^{-1}_{-\alpha,\omega,\omega'} . \tag{49}$$

Then, Lemma 3.5 gives $C_p < \infty$ and $\widehat{C}_{0,\alpha}(X, V) < \infty$ such that each term in the

[20]Using the bounded distortion argument for hyperbolic maps [40, Proposition 20.2.6], one can construct $\widetilde{\mathcal{V}}_\alpha$ by first iterating $[\alpha/\alpha_0]$ times the cover $\{V_\omega\}$ and then refining to guarantee that the diameter in the last coordinate in charts is $< \alpha^{-1/\gamma}$. The cardinality of such $\widetilde{\mathcal{V}}_\alpha$ grows like $C\alpha^{1/\gamma}C^\alpha$, for $C > 1$. This is not needed.

[21]We can ensure that $\alpha^{-1/\gamma}\|\vartheta_{\alpha,\vec{\omega}}\|_{C^{r-1}}$ is controlled by the largest expansion of $F_{-\alpha,\omega\omega'}$. This is not needed.

sum on the second-to-last line of (47) is bounded by

$$C_p C_\pm(F_{-\alpha,\omega\omega'}, f_{\alpha,\vec\omega}) \|\bar\vartheta_{\alpha,\vec\omega} \cdot ((\vartheta_{\omega'} \cdot \varphi_{\alpha'}) \circ \kappa_{\omega'}^{-1})\|^p_{W^{s,t,q}_{p,\Theta'_{\omega'}}} + \widehat{C}_{0,\alpha}(X, V) \|(\vartheta_{\omega'} \cdot \varphi_{\alpha'}) \circ \kappa_{\omega'}^{-1}\|^p_{W^{s',t',q}_{p,\Theta'_{\omega'}}}. \tag{50}$$

Due to the strict inequality between cone ensembles, the reconstitution Lemma B.2 bounds the p-th root of the sum of (50) over $\vec\omega$, uniformly in α. Recalling (48), taking the square, the maximum over ω, and integrating over α', we find $\overline{C} < \infty$ and $C'_\alpha = C'_\alpha(X, V) < \infty$ such that

$$\|\mathcal{L}_{\alpha,V}(\varphi)\|^2_{W^{s,t,q}_p} \le C'_\alpha \cdot \|\varphi\|^2_{W^{s',t',q}_p} + \overline{C} \max_{\omega,\omega',\vec\omega} (C_\pm(F_{-\alpha,\omega\omega'}, f_{\alpha,\vec\omega}))^2 \cdot \|\varphi\|^2_{W^{s,t,q}_p}, \quad \forall \alpha \ge \alpha_0. \tag{51}$$

We next estimate $C_\pm(F_{-\alpha,\omega\omega'}, f_{\alpha,\vec\omega})$. By construction of Θ_ω in the proof of Lemma 2.4, and since the covering $\widetilde{\Phi}'$ from Lemma 3.2 used in Lemma 3.5 can be taken such that $\operatorname{supp} \widetilde{\Phi}_\sigma$ is bounded away from E^*_τ (in charts) if $\tau \ne \sigma$, there exists $C < \infty$ such that, recalling (33), we have for all $\alpha \ge \alpha_0$, all ω, ω', and all $\vec\omega \in \Omega_\alpha$, setting $K_{\alpha,\vec\omega} = \kappa_\omega(\operatorname{supp} \tilde\vartheta_{\alpha,\vec\omega})$,

$$|F_{-\alpha,\omega\omega'}|_{+,K_{\alpha,\vec\omega}} \le C \sup_{x \in K_{\alpha,\vec\omega}} \left\|(\mathrm{D}g_{-\alpha})^{\mathrm{tr}}|_{E^*_{+,x}}\right\|,$$
$$|F^{-1}_{-\alpha,\omega\omega'}|_{-,K_{\alpha,\vec\omega}} \le C \sup_{x \in K_{\alpha,\vec\omega}} \left\|(\mathrm{D}g_{\alpha})^{\mathrm{tr}}|_{E^*_{-,g_{-\alpha}(x)}}\right\|.$$

Thus, using (46) and $\inf|\psi_1| \sup|\psi_2| \le \sup|\psi_1\psi_2|$ for continuous ψ_1, ψ_2, we find $C < \infty$ such that

$$\begin{aligned} C_\pm(F_{-\alpha,\omega\omega'}, f_{\alpha,\vec\omega}) &\le C \max_{W \in \mathcal{W}_\alpha} \left(\sup_W |\phi_\alpha \cdot |\det \mathrm{D}g_{-\alpha}|^{-1/p}| \cdot \sup_W |\lambda^{(s,t,\alpha)}|\right) \\ &\le \overline{C} \max_{W \in \mathcal{W}_\alpha} \left(\inf_W \left|\frac{\phi_\alpha}{|\det \mathrm{D}g_{-\alpha}|^{1/p}}\right| \cdot \sup_W |\lambda^{(s,t,\alpha)}|\right) \\ &\le \widehat{C} \sup_M \left|\frac{\phi_\alpha}{|\det \mathrm{D}g_{-\alpha}|^{1/p}} \lambda^{(s,t,\alpha)}\right|, \quad \forall \alpha \ge \alpha_0. \end{aligned}$$

In view of (51), we have proved the lemma. □

Strong continuity suffices (it is not necessary [16]) to show that $X + V$ is the generator of $\mathcal{L}_{\alpha,V}$:

Lemma 3.7 (strong continuity; domain of the generator $X + V$). *Let $p \in (1, \infty)$ and fix s, q, t as in (42). The family $\{\mathcal{L}_{\alpha,V}\}_{\alpha \ge 0}$ of bounded operators on $W^{s,t,q}_p(M)$ forms a strongly continuous semigroup. The generator of this semigroup is*

$$X + V : D(X + V) \to W^{s,t,q}_p(M),$$

which is closed on its (dense) domain $D(X+V)\subseteq W_p^{s,t,q}(M)$. *Moreover, if* $q<r-2$ *or if* ϕ_α *is* C^r *in the flow direction, setting* $\mathcal{D}_{r-1}:=C^{r-1}(M)$ *if* $q<r-2$, *and otherwise*

$$\mathcal{D}_{r-1}:=C^{r-1,r}(M)=\{\varphi\in C^{r-1}(M)\mid \varphi \text{ is } C^r \text{ in the flow direction}\},$$

$D(X+V)$ *contains*[22] $\mathcal{D}_{r-1}$ *as a dense subset for the graph norm* $\|\cdot\|_{W_p^{s,t,q}(M)}+\|(X+V)(\cdot)\|_{W_p^{s,t,q}(M)}$.

Proof. To establish strong continuity, it suffices to show $\lim_{\alpha\downarrow 0}\|\mathcal{L}_\alpha\varphi-\varphi\|_{W_p^{s,t,q}(M)}=0$ for all $\varphi\in W_p^{s,t,q}(M)$ ([22, Proposition I.1.3]). Lemmas 3.6 and 2.8 give $C<\infty$ such that $\|\mathcal{L}_{\alpha,V}\varphi\|_{W_p^{s,t,q}}\le C\|\varphi\|_{W_p^{s,t,q}}$ for all $\alpha\in[0,1]$. By density of $C^\infty(M)$, for every $\epsilon>0$ there is $\tilde\varphi=\tilde\varphi_\epsilon\in C^{r-1,r}(M)$ such that $\|\varphi-\tilde\varphi\|_{W_p^{s,t,q}}\le\epsilon$. Therefore,

$$\begin{aligned}\|\mathcal{L}_{\alpha,V}\varphi-\varphi\|_{W_p^{s,t,q}}&\le\|\mathcal{L}_{\alpha,V}(\varphi-\tilde\varphi)\|_{W_p^{s,t,q}}+\|\varphi-\tilde\varphi\|_{W_p^{s,t,q}}+\|\mathcal{L}_{\alpha,V}\tilde\varphi-\tilde\varphi\|_{W_p^{s,t,q}}\\&\le(C+1)\epsilon+\|\mathcal{L}_{\alpha,V}\tilde\varphi-\tilde\varphi\|_{W_p^{s,t,q}},\ \ \forall\epsilon>0,\forall\alpha\in[0,1].\end{aligned}\tag{52}$$

Since $\tilde\varphi\in C^{r-1,r}(M)$ (if $0<q<r-2$ then the argument can be adapted to $\tilde\varphi\in C^{r-1}$) we have $\partial_{\alpha'}\mathcal{L}_{\alpha',V}\tilde\varphi\in C^{r-1}(M)$. Thus, there exists $C(\tilde\varphi_\epsilon)<\infty$ such that

$$\begin{aligned}\|\mathcal{L}_{\alpha,V}\tilde\varphi-\tilde\varphi\|_{W_p^{s,t,q}}&=\Big\|\int_0^\alpha\partial_{\tilde\alpha}\mathcal{L}_{\tilde\alpha,V}\tilde\varphi\,\mathrm{d}\tilde\alpha\Big\|_{W_p^{s,t,q}}\\&\le\alpha\sup_{0\le\tilde\alpha\le\alpha}\|\partial_{\tilde\alpha}\mathcal{L}_{\tilde\alpha,V}\tilde\varphi\|_{W_p^{s,t,q}}\le C(\tilde\varphi_\epsilon)\alpha.\end{aligned}\tag{53}$$

Combining (52) and(53) yields $\lim_{\alpha\downarrow 0}\|\mathcal{L}_\alpha\varphi-\varphi\|_{W_p^{s,t,q}(M)}=0$. We also have $\lim_{\alpha\downarrow 0}\alpha^{-1}(\mathcal{L}_\alpha\varphi-\varphi)=X\tilde\varphi+V\tilde\varphi$ for $\tilde\varphi\in C^{r-1,r}(M)$. Strong continuity and Theorem II.1.4 of [22] then imply that $X+V$ is the generator of the semigroup, and that it is closed with domain dense in $W_p^{s,t,q}(M)$. Clearly, $\mathcal{L}_{\alpha,V}(C^{r-1}(M))\subseteq C^{r-1}(M)$, and, if ϕ_α is C^r in the flow direction, then $\mathcal{L}_{\alpha,V}(C^{r-1,r}(M))\subseteq C^{r-1,r}(M)$. The final claim thus follows from [22, Proposition II.1.7], since $\mathcal{L}_{\alpha,V}(\mathcal{D}_{r-1})\subset\mathcal{D}_{r-1}$ and $\lim_{\alpha\to 0}\alpha^{-1}(\mathcal{L}_{\alpha,V}\varphi-\varphi)$ exists for $\varphi\in\mathcal{D}_{r-1}$ in the two cases considered. □

3C. ***Lasota–Yorke bounds for the resolvent. Discrete spectrum of $X+V$.*** Recall $\lambda^{(s,t,\alpha)}$ from (45). We set

$$\lambda_{\min}=\lambda_{\min}^{s,t,p}(X,V):=\lim_{\alpha\to\infty}\frac{1}{\alpha}\log\big\|\phi_\alpha|\det Dg_{-\alpha}|^{-1/p}\lambda^{(s,t,\alpha)}\big\|_{L_\infty(M)};\tag{54}$$

the limit exists and is finite by superadditivity.

[22] As observed in [16], strong continuity implies that the $\|\cdot\|_{W_p^{s,t,q}(M)}$-completion $\mathcal{D}$ of

$$\mathcal{D}_0=\big\{\textstyle\int_0^\beta\mathcal{L}_{\alpha,V}\varphi\,\mathrm{d}\alpha\mid\varphi\in W_p^{s,t,q}(M),\ \beta>0\big\}$$

is a dense subset of $W_p^{s,t,q}(M)$, so that $\mathcal{D}=W_p^{s,t,q}(M)$. Clearly, $\mathcal{L}_{\alpha,V}(\mathcal{D}_0)\subset\mathcal{D}_0$ and $\mathcal{D}_0\subset D(X+V)$. Thus, $\mathcal{D}_0$ is a dense subset of $D(X+V)$ for the graph norm, without any conditions on q or ϕ_α.

Recalling $A(X, V)$ from Lemma 3.6, we may and shall replace $A(X, V)$ by $\max\{A(X, V), \lambda_{\min}\}$ from now on. The following theorem will furnish an essential spectral bound for $X + V$:

Theorem 3.8 (Lasota–Yorke inequality for the resolvent). *Let $p \in (1, \infty)$ and let s, q, t be as in* (42). *Let $s' < s$, $t' < t$ and $q' < q$ satisfy $q - 1 \le q'$ and $t - (r - 1) < s' < 0 < q' \le t'$. For any $\epsilon > 0$, there exists $C < \infty$ such that for all $\delta > 0$, all $z \in \mathbb{C}$ with* $\operatorname{Re} z > A(X, V) + \delta$, *all $n \in \mathbb{N}$, and all $\varphi \in W_p^{s,t,q}(M)$, recalling our notation $\mathcal{R}_z = (z - X - V)^{-1}$,*

$$\delta \,\|\mathcal{R}_z^{n+1}\varphi\|_{W_p^{s,t,q}} \le \frac{C(|z|+1)}{(\operatorname{Re} z - A(X, V))^n}\|\varphi\|_{W_p^{s',t',q'}} + \frac{C}{(\operatorname{Re} z - \epsilon - \lambda_{\min}^{s,t,p}(X, V))^n}\|\varphi\|_{W_p^{s,t,q}}\,.$$

Theorem 3.8 implies that the spectral radius of the resolvent $\mathcal{R}_z$ on $W_p^{s,t,q}(M)$ is bounded by $|\operatorname{Re} z - A(X, V)|^{-1}$ if $\operatorname{Re} z > A(X, V)$ (a very rough bound). In addition, we have:

Corollary 3.9 (essential spectral radius). *For all $z \in \mathbb{C}$ with* $\operatorname{Re} z > A(X, V)$, *the essential spectral radius of $\mathcal{R}_z$ on $W_p^{s,t,q}(M)$ is bounded by $|\operatorname{Re} z - \lambda_{\min}^{s,t,p}(X, V)|^{-1}$. Moreover, the set $\{\lambda \in \sigma(X+V)|_{W_p^{s,t,q}(M)} \mid \operatorname{Re}\lambda > \lambda_{\min}^{s,t,p}(X, V)\}$ consists of isolated eigenvalues of finite multiplicity.*

Proof. Since the inclusion $W_p^{s,t,q}(M) \subset W_p^{s',t',q'}(M)$ is compact by Lemma 2.10, the first claim follows from a result of Hennion [36, Corollaire 1] and Theorem 3.8. The second claim then follows from the spectral mapping theorem[23] for the resolvent [22, Theorem V.1.13]. □

If $\lambda_{\max}^{s,t,q,p}(X, V) > \lambda_{\min}^{s,t,p}(X, V)$, where

$$\lambda_{\max}^{s,t,q,p}(X, V) := \sup \operatorname{Re} \sigma(X + V)|_{W_p^{s,t,q}(M)}\,,$$

then the isolated eigenvalues furnished by Corollary 3.9 are called the *Ruelle–Pollicott resonances* of $X + V$ on $W_p^{s,t,q}(M)$. We will apply the following theorem to our scale $W_p^{s,t,q}(M)$ and, in Lemma 4.15, to the scale from [30]:

Theorem 3.10 (intrinsicness of Ruelle–Pollicott resonances). *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two Banach spaces of distributions on M on which $\{\mathcal{L}_{\alpha,V}\}$ is a strongly continuous semigroup with generator $X+V$. Assume that both $\mathcal{B}_1$ and $\mathcal{B}_2$ contain $C^{r-1}(M)$ as a dense subset and are continuously embedded in the dual of $C^{r-1}(M)$. If there exists $\lambda_{\min} > -\infty$ such that the sets $D_i = \{\lambda \in \sigma(X+V)|_{\mathcal{B}_i} \mid \operatorname{Re}\lambda > \lambda_{\min}\}$, $i = 1, 2$, consist of isolated eigenvalues of finite multiplicity, then $D_1 = D_2$, including multiplicities.*

[23]This spectral mapping theorem says that $\sigma(\mathcal{R}_z)\setminus\{0\} = \{(z-\lambda)^{-1} \mid \lambda \in \sigma(X+V)\}$ if $z \notin \sigma(X+V)$, and $(z - \lambda)^{-1}$ is an eigenvalue if and only if λ is an eigenvalue.

In particular, the corresponding generalised eigenvectors belong to $\mathcal{B}_1 \cap \mathcal{B}_2$ (in fact, to the intersection of the domains of $X + V$ on $\mathcal{B}_1$ and on $\mathcal{B}_2$).

Proof. If $r = \infty$, this is a special case of [32, Theorem 2.3], which refers to [23, Theorem 1.5]. If $r < \infty$[24] the proof of [23, Theorem 1.5] using meromorphic extensions of suitable resolvents applies, replacing $L_2(M)$ by the dual of $C^{r-1}(M)$ and using that $C^{r-1}(M)$ is a dense subset of both $\mathcal{B}_1$ and $\mathcal{B}_2$. □

Lemma 4.15 says that $\lambda_{\max}^{s,t,q,p}(X, V) = h_{\mathrm{top}}$ (for suitable s, t, q, p) for V as in Section 4.

The remainder of Section 3C is devoted to the proof of Theorem 3.8. Since the resolvent can be written as a Laplace transform (integrating along the flow), the proof will follow from the flow box condition (13), Lemma 3.6, and the next lemma:

Lemma 3.11 (integration along the flow). *Fix $p \in (1, \infty)$. There exists $C < \infty$ such that*

$$\Big\| \Big(\sum_{n=0}^{\infty} 4^{\tilde r n} |\Psi_{0,n}^{\mathrm{Op}} v|^2 \Big)^{\frac12} \Big\|_{L_p} \le C \Big\| \Big(\sum_{n=0}^{\infty} 4^{(\tilde r - 1)n} |\Psi_{0,n}^{\mathrm{Op}} \partial_{x_d} v|^2 \Big)^{\frac12} \Big\|_{L_p}, \ \forall \tilde r > 0.$$

Moreover, adapting the proof gives

$$\Big\| \Big(\sum_{n=0}^{\infty} 4^{(\tilde r - 1)n} |\Psi_{0,n}^{\mathrm{Op}} \partial_{x_d} v|^2 \Big)^{\frac12} \Big\|_{L_p} \le C \Big\| \Big(\sum_{n=0}^{\infty} 4^{\tilde r n} |\Psi_{0,n}^{\mathrm{Op}} v|^2 \Big)^{\frac12} \Big\|_{L_p}.$$

Proof. It is enough to consider the terms with $n > 0$. The starting point is

$$\Psi_{0,n}^{\mathrm{Op}}(\partial_{x_d} v) = (\mathbb{F}^{-1}\Psi_{0,n}) * (\partial_{x_d} v) = (\partial_{x_d}\mathbb{F}^{-1}\Psi_{0,n}) * v = 2^n (\mathcal{D}_d{}^{\mathrm{Op}} v)_n, \ \forall v \in L_p(\mathbb{R}^d),$$

where $(\mathcal{D}_d(\xi)b)_n := 2^{-n} i \xi_d \Psi_{0,n}(\xi) b$, for $n \in \mathbb{N}$, $\xi \in \mathbb{R}^d$, and $b \in \mathbb{C}$.

For a sequence a of complex numbers with $\|a\|_{\ell_2(\tilde r)} := \big(\sum_{n=1}^{\infty} 4^{\tilde r n} |a_n|^2\big)^{1/2} < \infty$, we put

$$(Q_d(\xi)a)_n := -i \frac{2^n}{\xi_d} \widetilde{\Psi}'_{0,n}(\xi) a_n, \quad \xi \in \mathbb{R}^d, \ n \in \mathbb{N},$$

where $\widetilde{\Psi}'_{0,n}$ is associated via (35) to a covering $\widetilde{\Psi}'$ of Θ with $\operatorname{supp} \widetilde{\Psi}'_0$ contained in a cone around ξ_d. Then $(Q_d{}^{\mathrm{Op}} \mathcal{D}_d{}^{\mathrm{Op}} v)_n = \Psi_{0,n}^{\mathrm{Op}} v$. Since there exists $\tilde\gamma_0 < \infty$ such that $2^{n-1} \le |\xi| \le \tilde\gamma_0 |\xi_d|$ for any $\xi \in \operatorname{supp} \widetilde{\Psi}'_{0,n}$, the map Q_d satisfies the decay condition in Theorem 2.6. Hence, taking $\mathcal{H}_1 = \mathcal{H}_2 = \{a \mid \|a\|_{\ell_2(\tilde r)} < \infty\}$, the map $Q_d{}^{\mathrm{Op}}$ is a bounded linear operator on $L_p(\mathbb{R}^d, \ell_2(\tilde r))$. This concludes the proof, since it gives $C < \infty$ such that $\| \|Q_d{}^{\mathrm{Op}} \mathcal{D}_d{}^{\mathrm{Op}} v\|_{\ell_2(\tilde r)} \|_{L_p} \le C \| \|\mathcal{D}_d{}^{\mathrm{Op}} v\|_{\ell_2(\tilde r)} \|_{L_p}$. □

Proof of Theorem 3.8. By Lemma 3.6, for $z \in \mathbb{C}$ such that $\operatorname{Re} z > A(X, V)$ (see [22, Corollary II.1.11]), we have

$$\mathcal{R}_z^n \varphi = \int_0^\infty \frac{\alpha^{n-1} e^{-z\alpha}}{(n-1)!} \mathcal{L}_{\alpha,V} \varphi \, d\alpha, \ \forall n \in \mathbb{N}, \tag{55}$$

[24] We expect that intrinsicness can also be proved by using dynamical determinants.

for all $\varphi \in W_p^{s,t,q}(M)$. Introducing the truncated iterated resolvent

$$\mathcal{R}^n_{tr,z}\varphi := \int_0^{\alpha_0} \frac{\alpha^{n-1}e^{-z\alpha}}{(n-1)!}\mathcal{L}_{\alpha,V}\varphi \, \mathrm{d}\alpha\,, \tag{56}$$

we claim that

$$\|\mathcal{R}^n_{tr,z}\varphi\|_{W_p^{s,t,q}} \le \frac{C}{(\operatorname{Re} z+\Delta)^n}\|\varphi\|_{W_p^{s,t,q}}\,, \\ \forall \Delta \ge 0,\ \forall \operatorname{Re} z > 0,\ \forall n > e\cdot\alpha_0\cdot(\operatorname{Re} z+\Delta)\,. \tag{57}$$

This bound holds because, using Lemma 3.6, $\sup_{\alpha\in[0,\alpha_0]} e^{-\operatorname{Re} z\alpha} \le 1$, and

$$\int_0^{\alpha_0} \frac{\alpha^{n-1}}{(n-1)!}\mathrm{d}\alpha = \frac{\alpha_0^n}{n!} \le \frac{1}{(\operatorname{Re} z+\Delta)^n}$$

if $n > e\cdot\alpha_0\cdot(\operatorname{Re} z+\Delta)$ (recall that $n! \ge n^n e^{-n}$), we find

$$\int_0^{\alpha_0} \frac{\alpha^{n-1}e^{-\operatorname{Re} z\alpha}}{(n-1)!}\|\mathcal{L}_{\alpha,V}\varphi\|_{W_p^{s,t,q}}\,\mathrm{d}\alpha \le C\frac{\alpha_0^n}{n!}\|\varphi\|_{W_p^{s,t,q}} \le \frac{C\|\varphi\|_{W_p^{s,t,q}}}{(\operatorname{Re} z+\Delta)^n}\,, \\ \forall \operatorname{Re} z > 0,\ \forall n > e\alpha_0(\operatorname{Re} z+\Delta)\,.$$

We can therefore focus on times $\alpha \ge \alpha_0$ in (55) and invoke Remark 2.5.

Lemma 3.6 gives $C_1 = C_1(\epsilon) < \infty$ such that for all $n \in \mathbb{N}$

$$\begin{aligned}\|\mathcal{R}_z^{n+1}\varphi\|_{W_p^{s,t,q}} &\le \int_0^{\infty} \frac{\alpha^{n-1}e^{-\operatorname{Re} z\alpha}}{(n-1)!}\|\mathcal{L}_{\alpha,V}\mathcal{R}_z\varphi\|_{W_p^{s,t,q}}\mathrm{d}\alpha \\ &\le \frac{C_1}{(\operatorname{Re} z - A(X,V))^n}\|\mathcal{R}_z\varphi\|_{W^{s',t',q}_{p,\Theta'}} + \frac{C_1}{(\operatorname{Re} z-\epsilon-\lambda_{\min})^n}\|\mathcal{R}_z\varphi\|_{W_p^{s,t,q}}\,,\end{aligned} \tag{58}$$

where, for $\Theta'_\omega < \Theta_\omega$ as in Remark 2.5, we replaced Θ by Θ' in the first term of the last line. Lemma 3.6 also gives $C_2 < \infty$ such that

$$\|\mathcal{R}_z\varphi\|_{W_p^{s,t,q}} \le \frac{C_2}{\operatorname{Re} z - A(X,V)}\|\varphi\|_{W_p^{s,t,q}}\,, \tag{59}$$

so that the last term in (58) is bounded as claimed. The starting point to bound the first term is the fact that the flow box condition (13) gives $(\mathrm{D}\kappa_\omega^{-1})(\partial_{x_d}|_{\kappa_\omega(V_\omega)}) = X|_{V_\omega}$, and hence

$$\partial_{x_d}((\vartheta_\omega\cdot\tilde\varphi)\circ\kappa_\omega^{-1}) = ((X\vartheta_\omega)\cdot\tilde\varphi + \vartheta_\omega\cdot(X\tilde\varphi))\circ\kappa_\omega^{-1}\,. \tag{60}$$

Using the triangle inequality (and $-\infty < q'$) to separate the contribution of $\Theta'_{\omega,0}$, and applying Lemma 3.11 with (60) (for $\tilde\varphi = \mathcal{L}_{\alpha',V}\mathcal{R}_z\varphi$) to bound this term, we find $C_3 < \infty$ such that

$$\begin{aligned}\big\|(\vartheta_\omega\cdot\mathcal{L}_{\alpha',V}\mathcal{R}_z\varphi)\circ\kappa_\omega^{-1}\big\|_{W^{s',t',q}_{p,\Theta'_\omega}(K_\omega)} &\le \big\|(\vartheta_\omega\cdot\mathcal{L}_{\alpha',V}\mathcal{R}_z\varphi)\circ\kappa_\omega^{-1}\big\|_{W^{s',t',q'}_{p,\Theta'_\omega}(K_\omega)} \\ &\quad + C_3\big\|((X\vartheta_\omega)\cdot\mathcal{L}_{\alpha',V}\mathcal{R}_z\varphi)\circ\kappa_\omega^{-1}\big\|_{W^{q-1}_{p,\Theta'_\omega}(K_\omega)} \\ &\quad + C_3\big\|(\vartheta_\omega\cdot X\mathcal{R}_z\mathcal{L}_{\alpha',V}\varphi)\circ\kappa_\omega^{-1}\big\|_{W^{q-1}_{p,\Theta'_\omega}(K_\omega)}\,,\end{aligned}$$

where we used for the last term that (55) implies $\mathcal{L}_{\alpha',V}\mathcal{R}_z\varphi = \mathcal{R}_z\mathcal{L}_{\alpha',V}\varphi$. Since $(X\vartheta_\omega)\circ\kappa_\omega^{-1} = \partial_{x_d}(\vartheta_\omega\circ\kappa_\omega^{-1}) \in C_0^{r-1}(\kappa_\omega(V_\omega))$ (using that ϑ_ω and κ_ω are C^r, with ϑ_ω is compactly supported in V_ω) and $q-1\le q'$, Lemma 3.5 for the identity map gives $C_4<\infty$ such that

$$\big\|((X\vartheta_\omega)\cdot\mathcal{L}_{\alpha',V}\mathcal{R}_z\varphi)\circ\kappa_\omega^{-1}\big\|_{W^{q-1}_{p,\Theta'_\omega}(K_\omega)} \le C_4\sup_{\omega\in\Omega}\big\|(\vartheta_\omega\cdot\mathcal{L}_{\alpha',V}\varphi)\circ\kappa_\omega^{-1}\big\|_{W^{s',t',q'}_{p,\Theta_\omega}(K_\omega)}.$$

Using $X\mathcal{R}_z\varphi = z\mathcal{R}_z\varphi - V\mathcal{R}_z\varphi - \varphi$, and, again, $\mathcal{R}_z\mathcal{L}_{\alpha',V}\varphi = \mathcal{L}_{\alpha',V}\mathcal{R}_z\varphi$, we find

$$\begin{aligned}\big\|(\vartheta_\omega\cdot X\mathcal{R}_z\mathcal{L}_{\alpha',V}\varphi)\circ\kappa_\omega^{-1}\big\|_{W^{q-1}_{p,\Theta'_\omega}(K_\omega)} &\le |z|\,\big\|(\vartheta_\omega\cdot\mathcal{L}_{\alpha',V}\mathcal{R}_z\varphi)\circ\kappa_\omega^{-1}\big\|_{W^{s',t',q-1}_{p,\Theta'_\omega}(K_\omega)}\\ &\quad+\big\|(\vartheta_\omega\cdot V\mathcal{L}_{\alpha',V}\mathcal{R}_z\varphi)\circ\kappa_\omega^{-1}\big\|_{W^{s',t',q-1}_{p,\Theta'_\omega}(K_\omega)}\\ &\quad+\big\|(\vartheta_\omega\cdot\mathcal{L}_{\alpha',V}\varphi)\circ\kappa_\omega^{-1}\big\|_{W^{s',t',q-1}_{p,\Theta'_\omega}(K_\omega)}. \qquad (61)\end{aligned}$$

Since $V\in C^{r-1}(M)$, we may bound the first term in (61) with Lemma 3.5 for the identity map. Using the definition of $\|\mathcal{R}_z\varphi\|_{W_p^{s',t',q}}$ as an integral of local norms over $\alpha'\in[0,\alpha_0]$, this bounds the first term of (58) as claimed (we use (59) for the terms with $\mathcal{R}_z\varphi$). □

3D. ***Dolgopyat bounds for the resolvent of weighted transfer operators.*** In the contact Anosov case and for the potential V introduced in the next section, the spectrum of $X+V$ has already been studied [30], on a different Banach space. We will use in the proof of Lemma 4.15 that the discrete spectra of $X+V$ on our Banach spaces and the spaces of [30] coincide in a big enough half-plane ("intrinsicness"), but it is not clear how to exploit this to obtain the bounds on the resolvent needed in Section 4. Indeed, in the self-adjoint case, there exist good bounds on the iterated resolvent $\mathcal{R}_z^n$ in terms of the distance between z and the spectrum. However, even when $W_p^{s,t,q}(M)$ is a Hilbert space, the operator $X+V$ is not self-adjoint a priori, so the existence of a spectral gap for $X+V$ does *not* imply good bounds on the resolvent in general (see [20; 26; 49] for special cases where such bounds are known). For this reason, we introduce the following condition:

Condition 3.12 (weak Dolgopyat bounds on the resolvent). There exist $p\in(1,\infty)$, s, q, t as in (42), constants

$$s''\in\mathbb{R},\quad \delta'\in(\lambda_{\min}^{s,t,p},\lambda_{\max}^{s,t,q,p}),$$

and constants $a_0>0$, $b_0'>1$, $c_1<1<C_1$, such that, for all $a\ge a_0$ and γ' in the range

$$aC_1<\gamma'<\frac{c_1}{\log(1+(\lambda_{\max}^{s,t,q,p}-\delta')/a)}, \qquad (62)$$

we have

$$\|\mathcal{R}^n_{a+ib+\lambda_{\max}^{s,t,q,p}}\varphi\|_{W_p^{s''}} \le C_1|a+(\lambda_{\max}^{s,t,q,p}-\delta')|^{-n}\|\varphi\|_{C^1},\quad \forall |b|\ge b_0',$$

where $n=\lceil\gamma'\log|b|\rceil$.

Using the mollification ideas introduced by Liverani in [7, §5, §7] and [10, §9], we will show that Condition 3.12 implies norm estimates on the resolvent (see [16, Remark 2.6]):

Proposition 3.13 (strong Dolgopyat bounds on the resolvent). *If there exists $C_0 < \infty$ with*

$$\|\mathcal{L}_{\alpha,V}\|_{W_p^{s',t',q'}} \le C_0 e^{\lambda_{\max}^{s,t,q,p}\alpha}, \ \forall \alpha \ge 0, \tag{63}$$

for $t-(r-1) < s' < 0 < q' < t'$, with $t-t' = q-q' = s-s' > 0$, for some s, q, t and $p > \max\{d/t, d/(r-1+s)\}$ such that Condition 3.12 holds for c_1, C_1, then there exist $\delta \in (\lambda_{\min}^{s,t,p}, \lambda_{\max}^{s,t,q,p})$, $a > 0$, $b_0 > 1$, $C < \infty$, and γ satisfying the unprimed version of (62), *all such that*

$$\|\mathcal{R}^n_{a+ib+\lambda_{\max}}\|_{W_p^{s,t,q}} \le C|a + (\lambda_{\max}^{s,t,q,p} - \delta)|^{-n}, \ \forall |b| \ge b_0, \tag{64}$$

where $n = \lceil \gamma \log |b| \rceil$.

Before proving Proposition 3.13, we make further remarks. Bounds (64) on suitable anisotropic Banach spaces are used in many places in the literature, starting with Liverani's breakthrough paper [41] (see, for example, [7; 10; 30]). This has been axiomatized by Butterley in [16]: together with a weak Lipschitz control on $\alpha \mapsto \mathcal{L}_{\alpha,V}$, the bounds (63) and (64) imply the *spectral gap* property

$$\sigma(X+V)|_{W_p^{s,t,q}(M)} \cap \{\operatorname{Re}\lambda > \delta\} \text{ is a finite set.}[25]$$

The above spectral gap can be used to get exponential decay of correlations, but the implication in the other direction is not known in general. (Dolgopyat [21] obtained exponential decay of correlations for Gibbs measures with arbitrary Hölder potentials for geodesic flows on surfaces of strictly negative curvature or, more generally, C^5 Anosov flows such that E_- and E_+ are C^1 and not jointly integrable, using symbolic dynamics. His ideas led to results of Liverani on the SRB measure of contact Anosov flows [41]. See [34] and [51], and references therein, for recent sufficient conditions ensuring exponential mixing for Gibbs measures and Anosov flows.)

The bounds (64) have been established [41; 49; 7; 30] for the generator X associated to contact Anosov flows and the potential $V = 0$, replacing our spaces $W_p^{s,t,q}$ by other anisotropic Banach spaces. For the potential V used in Section 4, Dolgopyat bounds are shown in [30, §7] (see also the argument sketched by Faure and Guillarmou before [35, Proposition 3.4]). We expect that (64) or Condition 3.12

[25]Beware that this does not imply a spectral gap (quasicompactness) for the time-one transfer operator: we do not expect $\mathcal{L}_{\alpha,V}$ to be eventually norm continuous [22, Theorem §II.5.3], so a priori we only have $\sigma(\mathcal{L}_{\alpha,V}) \subset \exp(\alpha\sigma(X+V))$ for $\alpha \ge 0$ (equality holds for eigenvalues and residual spectrum); see [22, §V.2.b]. A spectral gap for the time-one transfer operator is only known in special cases [26; 49].

can be proved directly in our setting. For our purposes it is sufficient instead to refer to [30, §7] in Section 4 to establish Condition 3.12, and then invoke Proposition 3.13. We thereby illustrate how to build bridges between results for different anisotropic spaces (once the essential radius is controlled, exact growth is obtained, and, for the Dolgopyat estimate, mollification bounds are known).

Proof of Proposition 3.13. Let $\{\Theta'_\omega\}$ and $\{\Theta_\omega\}$ form an adapted pair for $\mathcal{A}$ and g_α in the sense of Remark 2.5. Denote by $\|\varphi\|_{W_p^{s',t',q'}(\Theta')}$ the norm constructed with Θ'_ω instead of Θ_ω. We start with three trivial but useful observations: First, for any $\beta > 0$, $\delta_2 > 0$, and $\delta_1 \geq 0$, we have for all $|b| \geq 1$ and $a > \delta_1$ that

$$|a-\delta_1|^{-\lceil\gamma'\log|b|\rceil}|b|^{-\beta} \leq |a+\delta_2|^{-\lceil\gamma'\log|b|\rceil}, \quad \forall\gamma' \in \left(0, \frac{\beta}{\log\left(1+\frac{\delta_2}{a}\right)-\log\left(1-\frac{\delta_1}{a}\right)}\right). \tag{65}$$

(If $\delta_1 = 0$ and $\beta > 0$, taking $\delta_2 > 0$ small enough, we can choose γ' arbitrarily large in (65).)

Second, for any $\beta' > 0$ and $\delta_3 > \delta_2 > 0$, we have, for all $|b| \geq 1$ and $a > 0$,

$$|a+\delta_3|^{-\lceil\gamma'\log|b|\rceil}|b|^{\beta'} \leq |a+\delta_2|^{-\lceil\gamma'\log|b|\rceil}, \ \forall\gamma' > \frac{\beta'}{\log\left(1+\frac{\delta_3}{a}\right)-\log\left(1+\frac{\delta_2}{a}\right)}. \tag{66}$$

Third, for any $0 < \delta_0 < \delta_2$ and $\delta_1 \geq 0$, we have, for all $a > \delta_1$ and $m_1, m_2 \in \mathbb{N}$,

$$|a+\delta_2|^{-m_1}|a-\delta_1|^{-m_2} \leq |a+\delta_0|^{-m_1-m_2} \quad \text{if} \quad \frac{m_1}{m_2} \geq \frac{\log\left(1+\frac{\delta_0}{a}\right)-\log\left(1-\frac{\delta_1}{a}\right)}{\log\left(1+\frac{\delta_2}{a}\right)-\log\left(1+\frac{\delta_0}{a}\right)}. \tag{67}$$

(If $\delta_1 = 0$, for fixed $\delta_2 > 0$, taking $\delta_0 > 0$ small enough, we can choose m/n arbitrarily small.)

Set $\lambda_{\max} = \lambda_{\max}^{s,t,q,p}$. To deduce (64) from Condition 3.12, we use the Lasota–Yorke estimate: We may assume that $s'' < \min\{-d-1, s\}$. Then, for any $\tilde{\delta}_2 \in (0, \lambda_{\max} - \lambda_{\min}^{s,t,p})$, Theorem 3.8 with (27) give C, $C(s'', m)$, and $A(X, V) \geq \lambda_{\max}$, such that for all $\varphi \in W_p^{s,t,q}(M)$ and all $m, n \in \mathbb{N}$,

$$\begin{aligned}\|\mathcal{R}^{m+1+n}_{a+ib+\lambda_{\max}}\varphi\|_{W_p^{s,t,q}} \leq{}& \frac{2C(s'', m)|b|}{(a+\lambda_{\max}-A(X, V))^m}\|\mathcal{R}^n_{a+ib+\lambda_{\max}}\varphi\|_{W_p^{s'',s'',s''}} \\ &+ \frac{C}{(a+\tilde{\delta}_2)^m}\|\mathcal{R}^n_{a+ib+\lambda_{\max}}\varphi\|_{W_p^{s,t,q}}, \\ &\forall|b| \geq 1, \forall a > A(X,V)-\lambda_{\max}+1.\end{aligned} \tag{68}$$

Then, we proceed as in [7, §5, §7] or [10, §9]: First, since the condition (63) gives $\|\mathcal{R}^n_{a+ib+\lambda_{\max}}\varphi\|_{W_p^{s,t,q}} \leq Ca^{-n}\|\varphi\|_{W_p^{s,t,q}}$, for any $\epsilon_0 > 0$ (to be fixed in the next paragraph and by (75)) there exist m and n with $m \leq \epsilon_0 n$, such that the last term on the right-hand side of (68) satisfies the required condition: indeed, apply (67) for

$m_1 = m$, $m_2 = n$, $\delta_1 = 0$, $\delta_2 = \tilde{\delta}_2$, taking $\delta_0 > 0$ small enough such that $m \le \epsilon_0 n$ is allowed, and choose $\delta \ge \min(\epsilon_1, \lambda_{\max} - \delta_0)$.

For the first term on the right-hand side of (68), it is enough to bound

$$\frac{2C(s'', m)|b|}{(a + \lambda_{\max} - A(X, V))^m} \|\mathcal{R}_z^n \varphi - \mathcal{R}_{tr,z}^n \varphi\|_{W_p^{s'',s'',s''}}.$$

Indeed, it is not hard to see that there exists $\bar{C}(s'') \ge 1$ such that $C(s'', m) \le C(s'')^m$. Let $\delta_1(s'') > A(X, V) - \lambda_{\max} > 0$ be such that

$$\frac{2C(s'', m)}{(a - A(X, V) - \lambda_{\max})^m} \le \frac{2C(s'')^m}{(a - A(X, V) - \lambda_{\max})^m} \le \frac{C}{(a - \delta_1(s''))^m}, \qquad \forall m \ge 1, \quad \forall a > \delta_1(s'').$$

Then, the contribution of $\|\mathcal{R}_{tr,z}^n \varphi\|_{W_p^{s'',s'',s''}} \le \|\mathcal{R}_{tr,z}^n \varphi\|_{W_p^{s,t,q}}$ is controlled by (57), applying (67) for $m_1 = n$, $m_2 = m$, $\delta_1 = \delta_1(s'') > 0$, and $\delta_2 = \Delta > \delta_0 > 0$, for large enough Δ, taking ϵ_0 small enough that

$$n \ge m \frac{\log(1 + \delta_0/a) - \log(1 - \delta_1(s'')/a)}{\log(1 + \Delta/a) - \log(1 + \delta_0/a)}, \tag{69}$$

and taking b_0 large enough to ensure $n = \lceil \gamma' \log|b| \rceil > e\alpha_0(a + \lambda_{\max} + \Delta)$ if $|b| \ge b_0$, for $\gamma' \ge C_1 a$ determined below. (Again, choose $\delta \ge \min(\epsilon_1, \lambda_{\max} - \delta_0)$.)

Set $\mathcal{R}_{*,z}^n := \mathcal{R}_z^n \varphi - \mathcal{R}_{tr,z}^n$. Fixing s', q', t' with $q' - q = t' - t = s' - s < 0$, and $t - (r - 1) < s' < 0 < q' < t'$, we decompose, for any $s'' \le s'$,

$$|b| \|\mathcal{R}_{*,z}^n \varphi\|_{W_p^{s''}} \le |b| \|\mathcal{R}_{*,z}^n (\mathbb{M}_\epsilon \varphi)\|_{W_p^{s''}} + C|b| \|\mathcal{R}_{*,z}^n (\varphi - \mathbb{M}_\epsilon \varphi)\|_{W_p^{s',t',q'}}, \tag{70}$$

where $\mathbb{M}_\epsilon$ is the mollification operator in charts defined by (116), for $\epsilon = |b|^{-\kappa}$, with $\kappa > 1$ to be chosen later. Let $\{\Theta'_\omega\}$ form an adapted pair with $\{\Theta_\omega\}$. By (63) we have

$$\|\mathcal{R}_{*,a+ib+\lambda_{\max}}^n (\varphi - \mathbb{M}_\epsilon \varphi)\|_{W_p^{s',t',q'}} \le \frac{C}{a^n} \|\varphi - \mathbb{M}_\epsilon \varphi\|_{W_p^{s',t',q'}(\Theta')}. \tag{71}$$

Then the mollification estimate Lemma C.2 gives

$$\frac{C}{a^n} \|\varphi - \mathbb{M}_\epsilon \varphi\|_{W_p^{s',t',q'}(\Theta')} \le \frac{C}{a^n} \epsilon^{s-s'} \|\varphi\|_{W_p^{s,t,q}} \le \frac{\bar{C}}{a^n} |b|^{-\kappa(s-s')} \|\varphi\|_{W_p^{s,t,q}}, \tag{72}$$

If $\kappa > 1/(s - s')$, applying (65) with $\beta = \kappa(s - s') - 1 > 0$ and $a > \delta_1 = 0$, $\delta_2 = \lambda_{\max} - \delta$, the bounds (71) and (72) take care of the second term in the right-hand side of (70), assuming

$$\gamma' < \frac{\kappa(s - s') - 1}{\log\bigl(1 + (\lambda_{\max} - \delta)/a\bigr)}. \tag{73}$$

Note that this inequality is compatible with $\gamma' > aC_1$ if κ is large enough.

Fix $\eta_0 \in (0, \min\{t, r-1+s\})$, small. By the Sobolev embeddings for $W_p^{1+\eta_0} = F_{p,2}^{1+\eta_0}$ and $B_{\infty,\infty}^{1+}$ [45, Theorem 2.2.3(i)] in dimension d, we have

$$\|\tilde{\varphi}\|_{C^1} \leq \widehat{C}\|\tilde{\varphi}\|_{W_p^{1+\eta_0}}, \quad \text{if } p > \frac{d}{\eta_0}.$$

Thus, Condition 3.12 bounds the first term in the right-hand side of (70) by

$$\frac{C_1|b|}{|a+\lambda_{\max}-\delta'|^n}\|\mathbb{M}_\epsilon\varphi\|_{C^1} \leq \frac{\bar{C}|b|}{|a+\lambda_{\max}-\delta'|^n}\|\mathbb{M}_\epsilon\varphi\|_{W_p^{1+\eta_0}}.$$

Since the charts in $\mathcal{A}$ are C^r, the classical isotropic mollification estimate of [7, Lemma 5.3] (replacing X_0 by M and 2 by r there) becomes: For each $p \in (1, \infty)$ and all $-r+1 < s \leq s' < r+s \leq r$, there exists $C_\#$ such that for all small enough $\epsilon > 0$ and every $\varphi \in W_p^s(M)$, we have

$$\|\mathbb{M}_\epsilon(\varphi)\|_{W_p^{s'}(M)} \leq C_\#\epsilon^{s-s'}\|\varphi\|_{W_p^s(M)}.$$

Therefore, since $-r+1 < s < 0 < 1+\eta_0 < r+s$, taking $s' = 1+\eta$ and recalling (24), we obtain

$$\frac{\bar{C}|b|}{|a+\lambda_{\max}-\delta'|^n}\|\mathbb{M}_\epsilon\varphi\|_{W_p^{1+\eta_0}} \leq \frac{\bar{C}|b|\,\epsilon^{s-1-\eta_0}}{|a+\lambda_{\max}-\delta'|^n}\|\varphi\|_{W_p^{s,s,s}} = \frac{\bar{C}\,|b|^{1+\kappa r}}{|a+\lambda_{\max}-\delta'|^n}\|\varphi\|_{W_p^{s,s,s}}.$$

We need to multiply the above by $C(a-\delta_1(s''))^{-m}$. For this, we use that

$$\gamma' > \frac{1+\kappa r}{\log(1+(\lambda_{\max}-\delta')/a)-\log(1+\delta_2/a)} \tag{74}$$

is compatible with the upper bound (73) on γ', so long as we take $\delta_2 \in (0, \lambda_{\max}-\delta')$ on the right-hand side of (66) small enough, for $\beta' = 1+\kappa r$ and $\delta_3 = \lambda_{\max}-\delta'$.

We conclude the proof of the proposition by applying (67) for $a > \delta_1 = \delta_1(s'')$, δ_2 as in the previous paragraph, $m_1 = n$, $m_2 = m$, for $\delta_0 \in (0, \delta_2)$, and $\epsilon_0 \in (0, 1)$ such that

$$\frac{\log(1+\delta_0/a)-\log(1-\delta_1(s'')/a)}{\log(1+\delta_2/a)-\log(1+\delta_0/a)} < \frac{1}{\epsilon_0}. \tag{75}$$

Indeed, taking $\delta > \epsilon_1$ closer to $\lambda_{\max}$ if necessary to ensure $\delta \leq \lambda_{\max}-\delta_0$, and for $\gamma' > aC_1$ satisfying (73)–(74), take $\gamma > 0$ such that (using $m \leq \epsilon_0 n$)

$$\gamma\lceil\log|b|\rceil = m+n \leq (\epsilon_0+1)\gamma'\lceil\log|b|\rceil.$$

Thanks to (73), we can ensure that $\gamma < 1/\log\left(1+\frac{\lambda_{\max}-\delta}{a}\right)$, up to taking $\delta < \lambda_{\max}$ closer to $\lambda_{\max}$, $\square$

Remark C.3 explains why we are not able to carry out successfully the bounds in the previous proof by using mollifiers through isotropic spaces as in [7, Lemma 5.4, (7.5)–(7.6)].

4. Asymptotics of horocycle integrals

In this section, we assume throughout that $r \geq 2$, the C^r Anosov flow g_α on M is topologically mixing with stable dimension $d_- = 1$, and that the strong-stable distribution E_- is orientable.

4A. ***Horocycle flow h_ρ. Horocycle integral $\gamma_x(\varphi, T)$. Renormalisation time $\tau(\rho, \alpha, x)$.*** We shall focus on stable horocycle flows. Analogous results exist for unstable horocycle flows.

Definition 4.1 (stable horocycle flow). A (stable) horocycle flow for a topologically mixing C^r Anosov flow g_α on M with $d_- = 1$ and E_- orientable is a C^0 flow h_ρ on M such that $\partial_\rho h_\rho \in E_- \setminus \{0\}$ for all $\rho \in \mathbb{R}$.

Remark 4.2 (unit speed parametrisation). The stable manifolds of the flow g_α are the submanifolds tangent to the bundle E_- (this bundle is in general only Hölder, and existence is ensured by the stable manifold theorem; see [40, Theorem 17.4.3], for example). We can parametrise stable manifolds by the arc-length induced by the Riemannian metric on M. Since we assumed that E_- is orientable, this defines uniquely a horocycle flow with $|\partial_\rho h_\rho| \equiv 1$, called the unit speed horocycle flow. All other horocycle flows are obtained by time reparametrisations. Topological mixing of g_α implies that each stable manifold is dense in M [43, p. 84] so any horocycle flow is minimal.

Our main object of interest is the following (stable) horocycle integral:

Definition 4.3 (horocycle integral). The horocycle integral of the horocycle flow h_ρ for the observable $\varphi \in C^0(M)$ at $x \in M$ is defined by

$$\gamma_x(\varphi, T) = \int_0^T \varphi \circ h_\rho(x)\, \mathrm{d}\rho\,. \tag{76}$$

Writing $\mu(\varphi) = \int \varphi \mathrm{d}\mu$, where μ is the unique[26] h_ρ-invariant probability measure, we have

$$\gamma_x(\varphi, T) = T \cdot \mu(\varphi) + \mathcal{E}_{T,x}(\varphi)\,, \quad \lim_{T\to\infty} \frac{\mathcal{E}_{T,x}(\varphi)}{T} = 0\,, \quad \forall x \in M,\ \forall \varphi \in C^0(M)\,. \tag{77}$$

Our main result, Theorem 4.8 in Section 4B, gives a more precise asymptotic expansion, involving the spectrum and eigendistributions of a suitably weighted transfer operator $\mathcal{L}_{\alpha,V}$. A crucial ingredient in our analysis is the renormalisation time (first introduced by Marcus [43, p. 83] to study ergodic properties of the horocycle flow):

[26] See [15] for a proof of unique ergodicity. If g_α preserves a smooth measure see also [42]. See also Remark 4.16.

Definition 4.4 (pointwise renormalisation time). A map $\tau\colon \mathbb{R}^2 \times M \to \mathbb{R}$ that satisfies

$$g_\alpha \circ h_\rho(x) = h_{\tau(\rho,\alpha,x)} \circ g_\alpha(x), \qquad \forall \rho, \alpha \in \mathbb{R}, \forall x \in M, \tag{78}$$

is called a (pointwise) renormalisation time for the stable horocycle flow h_ρ.

For the unit speed horocycle flow of the geodesic flow on a compact surface of constant negative curvature, the renormalisation time is $\tau(\rho, \alpha, x) = \rho \exp(-\alpha h_{\mathrm{top}})$. More generally:

Lemma 4.5 (properties of $\tau(\rho, \alpha, x)$). *There exists a unique solution $\tau(\rho, \alpha, x)$ to* (78). *In addition $\tau(\rho, \alpha, x)$ is differentiable in ρ, and we have*[27]

$$\tau(\rho, \alpha, x) = \gamma_x(\partial_\rho \tau(0, \alpha, \cdot\,), \rho), \qquad \forall x \in M,\ \forall \alpha \in \mathbb{R},\ \forall \rho \in \mathbb{R}, \tag{79}$$

$$\partial_\rho \tau(0, \alpha, x) = \det \mathrm{D}g_\alpha|_{E_-}(x) \cdot \frac{(\partial_\rho h_0(x))^*(\partial_\rho h_0(x))}{(\partial_\rho h_0 \circ g_\alpha(x))^*(\partial_\rho h_0 \circ g_\alpha(x))}, \qquad \forall x \in M,\ \forall \alpha \in \mathbb{R}. \tag{80}$$

In particular, $\partial_\rho \tau(0, \alpha, x) > 0$, $\tau(0, \alpha, x) = 0$, $\tau(\rho, 0, x) = \rho$. *Moreover, there exists $C < \infty$ with*

$$\frac{1}{C} \le \frac{\tau(\rho, -\alpha, x)}{\rho} e^{-h_{\mathrm{top}}\alpha} \le C,\ \forall x \in M,\ \forall \rho \in \mathbb{R} \text{ with } |\rho| \ge 1,\ \forall \alpha \ge 0, \tag{81}$$

$$\frac{1}{C} \le \frac{\rho}{\tau(\rho, \alpha, x)} e^{-h_{\mathrm{top}}\alpha} \le C,\ \forall x \in M,\ \forall \rho \in \mathbb{R},\ \forall \alpha \ge 0 \text{ with } |\tau(\rho, \alpha, x)| \ge 1. \tag{82}$$

These bounds will come from [30, Appendix C]. That $\lim_{\rho\to\infty} \frac{\tau(\rho, \alpha, x)}{\rho} = e^{-\alpha h_{\mathrm{top}}}$ for all $\alpha \ge 0$ follows from [43]; see the proof of Lemma 4.6.

The key fact behind our main result (Theorem 4.8) is the following consequence[28] of (78), known as *renormalisation*:

$$\int_0^T \varphi \circ h_\rho(x)\, \mathrm{d}\rho = \int_0^{\tau(T,\alpha,x)} (\mathcal{L}_{\alpha,V}\varphi) \circ h_\rho \circ g_\alpha(x)\, \mathrm{d}\rho, \tag{83}$$

where the transfer operator $\mathcal{L}_{\alpha,V}$ is defined by (8), choosing

$$V \equiv -\partial_\alpha \partial_\rho \tau(0, 0, \cdot\,), \quad \text{i.e., } \phi_\alpha = \partial_\rho \tau(0, -\alpha, \cdot\,), \tag{84}$$

and assuming that ϕ_α is C^{r-1}. The underlying idea will be to take $\alpha = O(\log T)$ so that $\tau(T, \alpha, x) = O(1)$, and then exploit the information on the spectrum of the semigroup $\mathcal{L}_{\alpha,V}$ obtained in the previous section.

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[27] In (80), we denote by $(\partial_\rho h_0)^* \in E_-^*$ the canonical dual of $\partial_\rho h_0 := \partial_\rho h_\rho|_{\rho=0}$.

[28] The proof of (83) is a simplification of that of Sublemma 4.13 below.

Since the derivative of the Jacobian is the divergence and $\partial_\alpha g_{-\alpha}|_{\alpha=0} = X$, we find for the unit speed horocycle flow that (80) implies

$$V = \operatorname{div}(X|_{E_-}) \text{ and } \phi_\alpha = \det \mathrm{D}g_{-\alpha}|_{E^-}.$$

Hence, if E_- is C^{r-1} then $\phi_\alpha \in C^{r-1}(M)$. More generally, if E_- is C^{r-1}, for any C^r time reparametrisation of the unit speed horocycle flow, the weight ϕ_α is C^{r-1} by (80). (Compare [29, Remark 2.4].) In order to fit in the Banach norm setting of Sections 2 and 3, we will need ϕ_α to be C^{r-1} for $r \geq 2$, and we will have to introduce the horocycle integrals (92) localised by smooth cutoff functions (following [29], see the proof of Lemma 4.14), replacing thus (83) by the more involved version of "renormalisation" in Sublemma 4.13.

Before proving Lemma 4.5, we state and prove a consequence of (83) and classical results:

Lemma 4.6 (the invariant measure μ as an eigenvector). *If ϕ_α from* (84) *is C^{r-1}, we have*

$$\mu(\mathcal{L}_{\alpha,V}\varphi) = e^{h_{\mathrm{top}}\alpha}\mu(\varphi), \quad \forall \alpha \geq 0,\ \forall \varphi \in C^0(M). \tag{85}$$

Remark 4.7 (spectrum of $X + V$ on $L^1(\mu)$). Lemma 4.6 gives $\mu(|\mathcal{L}_{\alpha,V}\varphi|) \leq \mu(\mathcal{L}_{\alpha,V}|\varphi|) = \mu(|\varphi|)$ for all $\alpha \geq 0$ and any $\varphi \in C^0(M)$. Therefore, since μ is a Radon measure, for each $\alpha \geq 0$, the operator $\mathcal{L}_{\alpha,V}$ is bounded on the Banach space $L^1(\mu)$, with norm equal to $e^{\alpha h_{\mathrm{top}}}$. Hence, using [22, Corollary II.1.11] and (55), the spectral radius of $\mathcal{R}_z = (z - (X+V))^{-1}$ on $L^1(\mu)$ is bounded by $|\mathrm{Re}\, z - h_{\mathrm{top}}|$ if $\mathrm{Re}\, z > h_{\mathrm{top}}$. The spectrum of $X+V$ on $L^1(\mu)$ thus lies in the half-plane $\mathrm{Re}\, z \leq h_{\mathrm{top}}$.

Proof of Lemma 4.6. Unique ergodicity (77) (twice), renormalisation (83), and a result of Marcus [43, Lemma 3.1, p. 84] give, for all $\alpha \geq 0$ and $\varphi \in C^0(M)$,

$$\begin{aligned}\mu(\varphi) = \lim_{T\to\infty} \frac{1}{T}\gamma_x(\varphi, T) &= \lim_{T\to\infty} \frac{\tau(T,\alpha,x)}{T}\frac{1}{\tau(T,\alpha,x)}\gamma_{g_{\alpha(x)}}(\mathcal{L}_{\alpha,V}\varphi, \tau(T,\alpha,x)) \\ &= e^{-\alpha h_{\mathrm{top}}}\mu(\mathcal{L}_{\alpha,V}\varphi).\end{aligned}$$

□

Proof of Lemma 4.5. Since stable leaves are dense and the flow h_ρ is nonsingular, this flow does not admit any periodic orbits. For $x \in M$ and $\rho, \alpha \in \mathbb{R}$, set $h_{\alpha,\rho}(x) := g_\alpha \circ h_\rho \circ g_{-\alpha}(x)$. Then $\partial_\rho h_{\alpha,\rho} \in E_{-,x} \setminus \{0\}$. Hence $h_{\alpha,\rho}(x)$ parametrises the same stable manifold as $h_\rho(x)$. If there were two different pointwise times τ, there would exist $\rho_1 < \rho_2 \in \mathbb{R}$ such that $h_{\alpha,\rho}(x) = h_{\rho_1}(x) = h_{\rho_2}(x)$, and this would contradict the absence of periodic orbits. Thus, $\tau(\rho,\alpha,x)$ is uniquely defined and differentiable in ρ. We deduce from (78) that

$$\begin{aligned}h_{\tau(\rho,\alpha_1+\alpha_2,\cdot)}(g_{\alpha_1+\alpha_2}(\,\cdot\,)) = g_{\alpha_1+\alpha_2}(h_\rho(\,\cdot\,)) &= g_{\alpha_1}(h_{\tau(\rho,\alpha_2,\cdot)}(g_{\alpha_2}(\,\cdot\,))) \\ &= h_{\tau(\tau(\rho,\alpha_2,\cdot),\alpha_1,g_{\alpha_2}(x))}(g_{\alpha_1+\alpha_2}(\,\cdot\,))\end{aligned}$$

and

$$h_{\tau(\rho_1+\rho_2,\alpha,x)}(g_\alpha(x)) = g_\alpha(h_{\rho_1+\rho_2}(x)) = g_\alpha(h_{\rho_1}\circ g_{-\alpha}\circ g_\alpha(h_{\rho_2}\circ g_{-\alpha}\circ g_\alpha(x)) \\ = h_{\tau(\rho_1,\alpha,h_{\rho_2}(x))}(h_{\tau(\rho_2,\alpha,x)}(g_\alpha(x))\,.$$

This implies that, for all $\alpha_1, \alpha_2, \rho_1, \rho_2 \in \mathbb{R}$,

$$\begin{aligned} \tau(\rho, \alpha_1+\alpha_2, \cdot\,) &= \tau(\tau(\rho,\alpha_2,\cdot\,), \alpha_1, g_{\alpha_2}(\,\cdot\,))\,, \\ \tau(\rho_1+\rho_2, \cdot\,, x) &= \tau(\rho_1, \cdot\,, h_{\rho_2}(x)) + \tau(\rho_2, \cdot\,, x)\,. \end{aligned} \tag{86}$$

Then, using $\tau(0,\alpha,x)=0$, and differentiating the identities in (86) at $\rho=0$ and $\rho_1=0$, we find

$$\begin{aligned} \partial_\rho\tau(\rho,\alpha,x) &= \partial_\rho\tau(0,\alpha,h_\rho(x))\,, \quad \forall\alpha, \\ \partial_\rho\tau(0,\alpha_1,g_{\alpha_2}(x))\partial_\rho\tau(0,\alpha_2,x) &= \partial_\rho\tau(0,\alpha_1+\alpha_2,x)\,,\ \forall\alpha_1,\alpha_2\,. \end{aligned} \tag{87}$$

Next (79) follows from the definition (76) of γ_x and the first claim of (87).

To show (80), we take derivatives on both sides of (78) with respect to ρ:

$$\mathrm{D}g_\alpha\partial_\rho h_\rho(x) = \partial_\rho\tau(\rho,\alpha,x)\cdot(\partial_\rho h_0)\circ h_{\tau(\rho,\alpha,x)}\circ g_\alpha(x)\,. \tag{88}$$

We have

$$\begin{aligned} (\partial_\rho h_0\circ g_\alpha)^*(\mathrm{D}g_\alpha\partial_\rho h_0) &= (\partial_\rho h_0\circ g_\alpha)^*((g_\alpha)_*\partial_\rho h_0) = (g_\alpha)^*(\partial_\rho h_0\circ g_\alpha)^*(\partial_\rho h_0) \\ &= \det(\mathrm{D}g_\alpha|_{E_-})^*(\partial_\rho h_0)^*(\partial_\rho h_0) \\ &= \det\mathrm{D}g_\alpha|_{E_-}(\partial_\rho h_0)^*(\partial_\rho h_0)\,. \end{aligned} \tag{89}$$

Setting $\rho=0$ in (88), we obtain (80), using (89) and the nonsingularity of the horocycle flow. That $\partial_\rho\tau(0,\alpha,x)>0$ follows from (80), for instance.

Next, since h_ρ is nonsingular, the stable manifold $W_y := h_{[0,1]}(y)$ has length bounded from above and below uniformly in $y \in M$. Using (80) and[29] [30, Lemma C.3, Remark C.4] (recalling that g_α is transitive), we find $C_3, C_4, C_5 < \infty$ such that

$$\begin{aligned} \tau(\rho,-\alpha,x) &\le C_3\int_0^\rho \det\mathrm{D}g_{-\alpha|E_-}\circ h_\rho(x)\,\mathrm{d}\rho \le C_4\rho\sup_{y\in M}\int\det\mathrm{D}g_{-\alpha|E_-}\mathrm{d}W_y \\ &\le C_5\rho\sup_{y\in M}\mathrm{vol}\,(g_{-\alpha}(W_y)) \le C_6\rho e^{h_{\mathrm{top}}\alpha}\,, \quad \forall\rho\ge 1\,,\ \alpha\ge 0\,,\ x\in M\,. \end{aligned}$$

A lower bound for $\tau(\rho,-\alpha,x)$ is obtained analogously, using [30, Lemma C.1]. This shows (81) for $\rho \ge 1$. We get (81) for all $\rho \le -1$ since (79) implies $\tau(-\rho,\alpha,x) = -\tau(\rho,\alpha,h_{-\rho}(x))$.

Finally, (82) follows from (80) and the following consequence of the first claim of (86):

$$\rho = \tau(\tau(\rho,\alpha,x),-\alpha,g_\alpha(x)) = \tau(|\tau(\rho,\alpha,x)|,-\alpha,g_\alpha(x))\,. \qquad \square$$

[29]There is a typo in [30, Section C] and the set W there should actually be unstable.

4B. ***Main result: asymptotic expansion for the horocycle integral.*** To state our main result, Theorem 4.8, we need some notation. Denote by $(X+V)'$ the dual of $X+V$ (acting on the dual of $W_p^{s,t,q}(M)$). Recall that $\sigma((X+V)')|_{(W_p^{s,t,q}(M))'} = \sigma(X+V)|_{W_p^{s,t,q}(M)}$ (by [22, Section II.2.5], strong continuity of the dual semigroup is not needed for this). Therefore, by Corollary 3.9, each $\lambda \in \sigma(X+V)|_{W_p^{s,t,q}(M)}$ with $\operatorname{Re}\lambda > \lambda_{\min}^{s,t,p}$ is an eigenvalue of finite geometric multiplicity n_λ and finite algebraic multiplicities $m_{\lambda,i}$, $1 \le i \le n_\lambda$, of $(X+V)'$, with generalised eigenstates $\mathcal{O}_{(\lambda,i,j)}$ in the domain of $(X+V)'$, for $1 \le j \le m_{\lambda,i}$, with

$$((X+V)'-\lambda)^j \mathcal{O}_{(\lambda,i,j)} = 0, \quad ((X+V)'-\lambda)^{j-1}\mathcal{O}_{(\lambda,i,j)} \neq 0, \ 1 \le j \le m_{\lambda,i}. \tag{90}$$

We may now state our main theorem:

Theorem 4.8 (an expansion for horocycle integrals). *Let $r > 2$ and let h_ρ be a C^r reparametrisation of the unit speed horocycle flow of a topologically mixing C^r Anosov flow g_α, such that $d_- = 1$, with E_- orientable and*[30] *C^{r-1}. Assume that there exist $t-(r-1) < s < 0 < q < t < r-2$ and $p \in (d/\min\{t, r-1+s\}, \infty)$ with*

$$\lambda_{\min}^{s,t,p} < h_{\text{top}}, \ \textit{and Condition 3.12 holds for some } \delta \textit{ with } \lambda_{\min}^{s,t,p} < \delta < h_{\text{top}}.$$

Then $\Sigma_\delta := \sigma(X+V)|_{W_p^{s,t,q}(M)} \cap \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > \max\{0,\delta\}\}$ is finite, and there exists $T_0 > 1$ such that for each (λ, i, j) with $\lambda \in \Sigma_\delta$, $1 \le i \le n_\lambda$, and $1 \le j \le m_{\lambda,i}$, there are[31] *functions*

$$c_{(\lambda,i,j)} : (T_0,\infty) \times M \to \mathbb{C}, \ \textit{with} \sup_{T > T_0,\, x \in M} |c_{(\lambda,i,j)}(T,x)| < \infty,$$

and for any $\tilde{\delta} > \max\{0,\delta\}$ there exists $C_{\tilde{\delta}} < \infty$ such that for all $\varphi \in C^r(M)$ and all $T \ge T_0$

$$\int_0^T \varphi \circ h_\rho(x)\,\mathrm{d}\rho = T\mu(\varphi) + \mathcal{E}_{T,x,\Sigma_\delta}(\varphi) + \sum_{\substack{\lambda \in \Sigma_\delta \\ \lambda \neq h_{\text{top}}}} \sum_{\substack{1 \le i \le n_\lambda \\ 1 \le j \le m_{\lambda,i}}} T^{\frac{\lambda}{h_{\text{top}}}} (\log T)^{j-1} c_{(\lambda,i,j)}(T,x) \cdot \mathcal{O}_{(\lambda,i,j)}(\varphi),$$

where

$$\sup_{x \in M} |\mathcal{E}_{T,x,\Sigma_\delta}(\varphi)| \le C_{\tilde{\delta}}(T^{\tilde{\delta}/h_{\text{top}}}\|\varphi\|_{C^r} + \|\varphi\|_{C^0}). \tag{91}$$

The proof of Theorem 4.8 is given at the end of Section 4D. We record an immediate corollary:

[30] Hence $\partial_\rho \tau(0,-\alpha,\cdot) \in C^{r-1}(M)$ for all $\alpha \ge 0$.

[31] Note that (109) gives a formula for $c_{(\lambda,i,j)}$ using the generalised eigenvector of (λ, i, j). We do not show $\inf_{T > T_0,\, x} |c_{(\lambda,i,j)}(T,x)| > 0$.

Corollary 4.9 (power law convergence). *Under the assumptions of Theorem 4.8, there exist $\epsilon \in (0, \min\{1, 1-\delta/h_{\mathrm{top}}\})$ and $C_\epsilon < \infty$ such that, for all $\varphi \in C^r(M)$,*

$$\left|\frac{1}{T}\int_0^T \varphi \circ h_\rho(x)\,\mathrm{d}\rho - \mu(\varphi)\right| \le \frac{C_\epsilon}{T^\epsilon}\|\varphi\|_{C^r} + \frac{C_\epsilon}{T}\|\varphi\|_{C^0}, \ \forall T > 0.$$

A *contact form* is a 1-form $\upsilon \in T^*M$ such that $\upsilon \wedge \left(\bigwedge_{n=1}^{\frac{d-1}{2}} \mathrm{d}\upsilon\right)$ vanishes nowhere, where d denotes the exterior derivative. (A contact form can only exist if d is odd.) A flow g_α on M is a *contact flow* if there exists a C^1 contact form υ which is preserved by the pullback of g_α. Geodesic flows on (the unit tangent bundle of) negatively curved compact manifolds are examples of Anosov contact flows. Contact Anosov flows are topologically mixing [39, Theorem 3.6].

Proposition 4.10. *Let g_α be a C^3 contact Anosov flow on a compact manifold M of dimension $d = 3$. Assume the strong-stable distribution E_- is orientable. Then for any $\epsilon_1 > 0$, we may choose $r \in (2, 3)$ and $t - (r-1) < s < 0 < t < r-2$ such that E_- is C^{r-1} and such that, for any C^r reparametrisation of the unit speed horocycle flow, we have $\lambda_{\min}^{s,t,p} < \epsilon_1$ for all $p \in (1, \infty)$. In addition, if the flow satisfies the bunching condition* (2), *there exists $p_0 > 1$ such that Condition 3.12 holds for some $\delta' \in [\max\{\lambda_{\min}^{s,t,p}, 0\}, h_{\mathrm{top}})$ if $p > p_0$.*

The proposition above is proved in Section 4E. Its assumptions hold if g_α is the geodesic flow on a C^3 surface of strictly negative curvature (E_- is $C^{2-\tilde\eta}$ for any $\tilde\eta \in (0, 1)$ by [38, Theorem 3.1]; for the orientability of E_-, see [30, Lemma B.1]) with $\hat\varpi$ satisfying (2). In particular, they hold if g_α is the geodesic flow on a C^3 compact surface of constant negative curvature, where $\hat\varpi = 2$.

We compare our main theorem and Proposition 4.10 with the results of Flaminio and Forni [28]: Let M be the unit tangent bundle of a compact surface of constant negative curvature, and let g_α be its unit speed geodesic flow. Then the canonical volume form vol on M is the measure of maximal entropy for g_α. The unit speed horocycle flow leaves vol invariant as well, so that $\mu = \mathrm{vol}$. Also, the vector fields X and $V = h_{\mathrm{top}}$ are constant, $r = \infty$, and $h_{\mathrm{top}} = 1$ because $\tau(\rho, \alpha, x) = \rho\exp(-\alpha)$. In this setting [28, Theorem 1.5] the noninteger obstructions to convergence, corresponding to our eigenvalues λ, are connected to the nonzero eigenvalues σ of the Laplacian via $\lambda = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \sigma}$. The eigenvalue $h_{\mathrm{top}} = 1$ is simple, there are no other eigenvalues of real part equal to 1, all eigenvalues with $\mathrm{Re}\,\lambda > \frac{1}{2}$ are semisimple, and there are only finitely many eigenvalues with $\mathrm{Re}\,\lambda > \frac{1}{2}$. Moreover, since $r = \infty$, for any $p \in (1, \infty)$ (including $p = 2$) the parameters $-s$, t can be taken large enough to ensure $\lambda_{\min}^{s,t,p} = \lambda_{\min}^{s,t} < 0$. Since Condition 3.12 holds for some $\delta > \frac{1}{2}$ (see Proposition 4.10), we find $c_{(\lambda,i,1)}$ and $\mathcal{E}_{T,x,\Sigma_\delta}$ as in Theorem 4.8

such that

$$\int_0^T \varphi \circ h_\rho(x)\,\mathrm{d}\rho = T\,\mathrm{vol}(\varphi) + \sum_{\lambda\in\Sigma_\delta\setminus\{1\}} \sum_{i=1}^{n_\lambda} T^\lambda c_{(\lambda,i,1)}(T,x)\mathcal{O}_{(\lambda,i,1)}(\varphi) + \mathcal{E}_{T,x,\Sigma_\delta}(\varphi)\,.$$

4C. ***Localised horocycle integrals, properties of the renormalisation time τ.*** In view of the smooth cutoff decomposition of $\gamma_x(\,\cdot\,,T)$ in Lemma 4.14 below, we introduce localised horocycle integrals as follows. For any bounded compactly supported $w:\mathbb{R}\to\mathbb{R}$, let

$$\gamma_{w,x}(\varphi) := \int_{\mathbb{R}} w(\rho)\cdot(\varphi\circ h_\rho(x))\,\mathrm{d}\rho\,,\;\; x\in M\,,\;\; \varphi\in C^0(M)\,. \tag{92}$$

To show Theorem 4.8, it will be useful[32] to view $\gamma_{w,x}$ as an element of the dual of $W_p^{s,t,q}(M)$:

Lemma 4.11. *There exists $\bar{C}<\infty$, depending on* $\max_{\alpha\in[0,\alpha_0]}\|\phi_{-\alpha}\circ g_{-\alpha}\|_{C^{r-1}}$, *the partition of unity ϑ_ω, and the charts κ_ω* (Definition 2.9), *such that for any $w\in C_0^{r-1}(\mathbb{R})$ and $\varphi\in W_p^{s,t,q'}(M)$*

$$\sup_{x\in M}|\gamma_{w,x}(\varphi)| \le \bar{C}|\mathrm{supp}\, w|\cdot\|w\|_{C^{|s|}}\cdot\|\varphi\|_{W_p^{s,t,q'}}\,,$$
$$\forall p\in(1,\infty)\,,\;\forall t-(r-1)<s<0<q'\le t\,.$$

Before proving it, we show an easy consequence of Lemma 4.11: The unique h_ρ-invariant measure μ belongs to the dual space of $W_p^{s,t,q}(M)$.

Corollary 4.12. *Let $p\in(1,\infty)$ and let s,q,t be as in* (42). *If ϕ_α from* (84) *is C^{r-1} then $\mu\in(W_p^{s,t,q}(M))'$. Also, by Lemma 4.6, we have*

$$\lambda_{\max}^{s,t,q,p}\ge h_{\mathrm{top}}\in\sigma(X+V)|_{W_p^{s,t,q}(M)}\,.$$

If $\lambda_{\min}^{s,t,q}<\lambda_{\max}^{s,t,q,p}$, the statement of this corollary could alternatively be obtained from [30, §4]; see the proof of Lemma 4.15.

Proof. Fix $\epsilon>0$ small (much smaller than the diameter of M). For $x\in M$ denote by $C_{x,\epsilon}^{r-1}(M)$ the set of $\varphi\in C^{r-1}(M)$ which vanish in an ϵ neighbourhood of x. Then there exist $\delta(\epsilon)>0$ and $C(\epsilon)$ such that for any $T>1$ with $d(h_T(x),x)<\delta$ there exists $w^{T,\epsilon}\in C_0^r(\mathbb{R},[0,1])$ with $|\mathrm{supp}\,(w^{T,\epsilon})|\le T+2$ and $\|w^{T,\epsilon}\|_{C^r}\le C(\epsilon)$, such that

$$1_{[0,T]}(\rho)\varphi(h_\rho(x)) = w^{T,\epsilon}(\rho)\varphi(h_\rho(x))\,,\quad \forall\varphi\in C_{x,\epsilon}^{r-1}(M)\,,\;\forall\rho\in\mathbb{R}.$$

For any $x\in M$, since $h_\rho(x)$ is dense, there is a sequence $T_n=T_n(x,\epsilon)$ such that $d(h_{T_n}(x),x)<n^{-1}$ and $T_n\to\infty$. By unique ergodicity (77) and Lemma 4.11, we

[32]The corresponding result for the anisotropic norms of [29] is slightly more intuitive.

have

$$
\begin{aligned}
|\mu(\varphi)| \le \lim_{n\to\infty}\left|\frac{1}{T_n}\gamma_x(\varphi, T_n)\right| &= \lim_{n\to\infty}\left|\frac{1}{T_n}\gamma_{w^{T_n,\epsilon},x}(\varphi)\right| \\
&\le 2C(\epsilon)\bar{C}\|\varphi\|_{W_p^{s,t,q}}, \quad \forall\varphi \in C^{r-1}_{x,\epsilon}(M). \qquad (93)
\end{aligned}
$$

Next, using a C^∞ function $\psi = \psi_{x,y,\epsilon} : M \to [0, 1]$, vanishing in an ϵ neighbourhood of x and identically 1 in an ϵ neighbourhood of some $y \ne x$, we can write any $\varphi \in C^{r-1}(M)$ as $\psi\varphi + (1-\psi)\varphi$, where $\psi\varphi \in C^{r-1}_{x,\epsilon}(M)$ and $(1-\psi)\varphi \in C^{r-1}_{y,\epsilon}(M)$. Applying (93) at x and y gives

$$
|\mu(\varphi)| \le 2C(\epsilon)\bar{C}\big(\|\psi\varphi\|_{\widetilde{W}_p^{s,t,q}} + \|(1-\psi)\varphi\|_{\widetilde{W}_p^{s,t,q}}\big), \ \ \forall\varphi \in C^{r-1}(M),
$$

where $\widetilde{W}_p^{s,t,q}$ is defined like $W_p^{s,t,q}$, but using systems of cones $\widetilde{\Theta}_\omega$ (see Remark 2.5) ensuring that $\|\psi\varphi\|_{\widetilde{W}_p^{s,t,q}} \le C\|\psi\|_{C^r}\|\varphi\|_{W_p^{s,t,q}}$ for some $C < \infty$ and all φ, ψ. We conclude by density of C^{r-1} functions in $W_p^{s,t,q}(M)$. □

Proof of Lemma 4.11. Let δ_* denote the Dirac distribution, fix $w \in C_0^{r-1}(\mathbb{R})$, and set

$$
\begin{aligned}
w_{x,\phi,\omega_1,\omega_2,\alpha}(z) &:= (\vartheta_{\omega_2}\cdot\phi_{-\alpha}\circ g_{-\alpha})\circ\kappa_{\omega_1}^{-1}(z)\int_{-\infty}^{\infty} w(\rho)\delta_*(z-\kappa_{\omega_1}\circ g_\alpha\circ h_\rho(x))\,\mathrm{d}\rho, \\
\varphi_{\omega,\alpha}(z) &:= (\vartheta_\omega\cdot\mathcal{L}_{\alpha,V}\varphi)\circ\kappa_\omega^{-1}(z),
\end{aligned}
$$

for $z \in \mathbb{R}^d$, $x \in M$, $\alpha \ge 0$, $\omega, \omega_1, \omega_2 \in \Omega$, $\varphi \in C^{r-1}(M)$. Since h_ρ has no periodic orbits and M is compact, $w_{x,\phi,\omega_1,\omega_2,\alpha}$ is a bounded function supported in the interior of a subset J of the (one-dimensional) stable leaf at $g_\alpha(x)$ (using (78)) in charts. In addition, there exists $\bar{C}_0$ such that

$$
|J| \le \bar{C}_0\,|\mathrm{supp}\,w| \quad \text{and} \quad \sup_{\substack{\alpha\in[0,\alpha_0] \\ x\in M,\,\omega_i\in\Omega}} \|w_{x,\phi,\omega_1,\omega_2,\alpha}\|_{L_\infty} \le \bar{C}_0\|w\|_{L_\infty}.
$$

Next, since $\phi_{-\alpha}\circ g_{-\alpha} = 1/\phi_\alpha$, we have, exchanging the integrals with respect to z and ρ (so that $z = \kappa_{\omega_1}\circ g_\alpha\circ h_\rho(x)$)

$$
\gamma_{w,x}(\varphi) = \sum_{\omega_1,\omega_2\in\Omega}\int_{\mathbb{R}^d} w_{x,\phi,\omega_1,\omega_2,\alpha}(z)\cdot\varphi_{\omega_1,\alpha}(z)\,\mathrm{d}z, \ \ \forall\alpha \ge 0.
$$

Recalling $\widetilde{\Psi}'_{\sigma,n}$ from (35), we find, using Plancherel's theorem for the inner product of two functions, $\widetilde{\Psi}'_{\sigma,n}\Psi_{\sigma,n} = \Psi_{\sigma,n}$, Cauchy–Schwarz for the sum in σ and n, Hölder's inequality, a constant $C < \infty$ such[33] that, for all $p \in (1,\infty)$ and all $t-(r-1) < s < 0 < q' \le t$,

[33]The bounds below can be viewed as yet another avatar of integration by parts.

$$\begin{aligned}
&\alpha_0\,|\gamma_{w,x}(\varphi)| \\
&\quad = \int_0^{\alpha_0} \Big| \sum_{\omega_1,\omega_2} \int_{\mathbb{R}^d} w_{x,\phi,\omega_1,\omega_2,\alpha}(z)\,\varphi_{\omega_1,\alpha}(z)\,\mathrm{d}z \Big| \,\mathrm{d}\alpha \\
&\quad \le \int_0^{\alpha_0} \sum_{\omega_1,\omega_2} \Big| \int_{\mathbb{R}^d} \sum_{\sigma,n} 2^{-c(\sigma)n}\, \widetilde{\Psi}'^{\mathrm{Op}}_{\sigma,n}(w_{x,\phi,\omega_1,\omega_2,\alpha})(z) 2^{c(\sigma)n}\, \Psi^{\mathrm{Op}}_{\sigma,n}(\varphi_{\omega_1,\alpha})(z)\,\mathrm{d}z \Big| \,\mathrm{d}\alpha \\
&\quad \le C \sup_{\alpha\in[0,\alpha_0]} \sum_{\omega_1,\omega_2} \Big\| \Big(\sum_{\sigma,n} 4^{-c(\sigma)n} |\widetilde{\Psi}'^{\mathrm{Op}}_{\sigma,n}(w_{x,\phi,\omega_1,\omega_2,\alpha})|^2 \Big)^{\frac{1}{2}} \Big\|_{L_{1-\frac{1}{p}}} \|\varphi\|_{W_p^{s,t,q'}}\,, \qquad (94)
\end{aligned}$$

using the definition (25) of the norm in (94). To conclude, it suffices to find $C_0 < \infty$ such that

$$\max_{\substack{\sigma\in\{+,0\}\\ \omega_1,\omega_2\in\Omega}} \sup_{x\in M} \|\widetilde{\Psi}'^{\mathrm{Op}}_{\sigma,n}(w_{x,\phi,\omega_1,\omega_2,\alpha})\|_{L_{1-1/p}} \le C_0\,|\operatorname{supp} w|\,\|w\|_{L_\infty}\,, \qquad \forall 0 \le \alpha \le \alpha_0\,,\ \forall n \in \mathbb{N}, \quad (95)$$

and (since $c(+) > 0$ and $c(0) > 0$, it is enough to consider $\sigma = -$)

$$\max_{\omega_1,\omega_2\in\Omega} \sup_{x\in M} \big\| \widetilde{\Psi}'^{\mathrm{Op}}_{-,n}(w_{x,\phi,\omega_1,\omega_2,\alpha}) \big\|_{L_{1-1/p}} \le \frac{C_0}{2^{(r-1)n}}\,|\operatorname{supp} w|\,\|w\|_{C^{|s|}}\,, \qquad \forall 0 \le \alpha \le \alpha_0\,, \forall n \in \mathbb{N}. \quad (96)$$

Now Young's inequality for $\big\| \mathbb{F}^{-1}(\widetilde{\Psi}'_{\sigma,n}) * w_{x,\phi,\omega_1,\omega_2,\alpha} \big\|_{L_{1-1/p}}$ gives (95) for $C_0 = \overline{C}_0 \max \operatorname{diam} V_\omega$.

Finally, we show (96). There are $\widetilde{C}_0$ (depending on $\max_{\alpha\in[0,\alpha_0]} \|\phi_{-\alpha} \circ g_{-\alpha}\|_{C^{r-1}}$, ϑ_ω, and κ_ω), a subset $\tilde{J} \subset \mathbb{R}$, and a C^{r-1} diffeomorphism $\tilde{y} : \tilde{J} \to J \subset \mathbb{R}^d$ with[34] $|\tilde{J}| \le C|J| \le \widetilde{C}_0 |\operatorname{supp} w|$, with $\max\{\|\tilde{y}\|_{C^{r-1}}, \|\tilde{y}^{-1}\|_{C^{r-1}}\} \le \widetilde{C}_0$, and

$$\begin{aligned}
\widetilde{\Psi}'^{\mathrm{Op}}_{-,n} w_{x,\phi,\omega_1,\omega_2,\alpha}(z) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \widetilde{\Psi}'_{-,n}(\xi) e^{i\xi(z-\tilde{z})} w_{x,\phi,\omega_1,\omega_2,\alpha}(\tilde{z})\,\mathrm{d}\xi\,\mathrm{d}\tilde{z} \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d}\int_{\tilde{J}} \widetilde{\Psi}'_{-,n}(\xi) e^{i\xi(z-\tilde{y}(y))} w_{x,\phi,\omega_1,\omega_2,\alpha}(\tilde{y}(y))\,\mathrm{d}\xi\,\mathrm{d}y,
\end{aligned}$$

with $w_{x,\phi,\omega_1,\omega_2,\alpha} \circ \tilde{y}$ a C^{r-1} function supported in the interior of $\tilde{J}$ such that

$$\sup_{\substack{\alpha\in[0,\alpha_0]\\ x\in M,\,\omega_i\in\Omega}} \|w_{x,\phi,\omega_1,\omega_2,\alpha} \circ \tilde{y}\|_{C^{\tilde{r}}(\tilde{J})} \le \widetilde{C}_0 \|w\|_{C^{\tilde{r}}}\,,\ \forall \tilde{r} \le r-1\,.$$

Note that J lies in a stable cone in charts. Thus, there exists $C_1 > 0$ such that $|\partial_y(\xi\,\tilde{y}(y))| \ge C_1 2^n$ for any ξ in the support of $\widetilde{\Psi}'_{-,n}$ (which lies inside E^*_- in charts). Finally, integrating $\lfloor |r-1| \rfloor$ times by parts with respect to y, following by a regularised integration by parts if $|r-1|$ is not an integer (Lemmas A.1 and A.2), and ending with Young's inequality, we get (96). □

[34] As a warmup, the reader is invited to think of the case when J is a subset of a coordinate axis in $\mathbb{R}^d$.

The next two lemmas use the following version of the renormalisation equation (83) for the localised horocycle integral (92).

Sublemma 4.13 (renormalisation and smooth localisation). *Fix $x \in M$ and $\varphi \in C^0$. Then*

$$\gamma_{w,x}(\varphi) = \int_{\mathbb{R}} w(\tau(\rho, -\alpha, g_\alpha(x))) \cdot \mathcal{L}_{\alpha,V}\varphi(h_\rho(g_\alpha(x)))\,\mathrm{d}\rho\,,\ \forall \alpha \geq 0\,.$$

Proof. By definition and our choice $\phi_\alpha = \partial_\rho \tau(0, -\alpha, \cdot\,)$,

$$\int_{\mathbb{R}} w(\tau(\rho, -\alpha, g_\alpha(x))) \cdot \mathcal{L}_{\alpha,V}\varphi(h_\rho(g_\alpha(x)))\,\mathrm{d}\rho \\ = \gamma_{w(\tau(\,\cdot\,,-\alpha,g_\alpha(x)),g_\alpha(x)}(\mathcal{L}_{\alpha,\partial_\rho\tau(0,-\alpha,\cdot)}\varphi)\,.$$

Thus, the sublemma follows from (78) and the first claims of (86) and (87), since

$$\begin{aligned}\gamma_{w,x}(\varphi) &= \int_{-\infty}^{\infty} w(\rho) \cdot \varphi \circ g_{-\alpha} \circ h_{\tau(\rho,\alpha,x)} \circ g_\alpha(x)\,\mathrm{d}\rho \\ &= \int_{-\infty}^{\infty} w(\tau(\rho, -\alpha, g_\alpha(x))) \cdot \varphi \circ g_{-\alpha} \circ h_\rho \circ g_\alpha(x) \cdot \partial_\rho \tau(\rho, -\alpha, g_\alpha(x))\,\mathrm{d}\rho \\ &= \int_{-\infty}^{\infty} w(\tau(\rho, -\alpha, g_\alpha(x))) \cdot (\partial_\rho \tau(0, -\alpha, \cdot\,) \cdot \varphi \circ g_{-\alpha}) \circ h_\rho \circ g_\alpha(x)\,\mathrm{d}\rho \\ &= \gamma_{w\circ\tau(\,\cdot\,,-\alpha,g_\alpha(x)),g_\alpha(x)}(\partial_\rho \tau(0, -\alpha, \cdot\,) \cdot \varphi \circ g_{-\alpha})\,, \qquad \forall \alpha \geq 0. \qquad \square\end{aligned}$$

Taking $w = w_T$ to be the characteristic function $w_T = 1_{[0,T]}$, we have $\gamma_{w_T,x}(\varphi) = \gamma_x(\varphi, T)$, and by choosing $\alpha = O(\log T)$ we can ensure (in view of Lemma 4.5) that the support of $w_T \circ \tau(\,\cdot\,, -\alpha, g_\alpha(x))$ has size $O(1)$, uniformly in x. In order to apply Lemma 4.11, some regularity of w is required: we thus need a more clever choice of localisation function $w_T = w_{T,x}$. We state the corresponding result, similar to [28, Lemma 5.16], [29, Lemma 3.19].

Lemma 4.14 (bounds for localised horocycle integrals). *Let $C > 1$ be as in* (81)–(82) *and fix $\bar{C} > \max\{C, 4\}$. If ϕ_α from* (84) *is C^{r-1}, then for every $T > C\bar{C}$ and $x \in M$ there exists a compactly supported C^r function $w = w_{T,x} : \mathbb{R} \to [0, 1]$ with*

$$|\gamma_x(\varphi, T) - \gamma_{w_{T,x},x}(\varphi)| \leq 2C\bar{C}\|\varphi\|_{C^0}\,,\ \forall \varphi \in C^0(M)\,. \tag{97}$$

Moreover, for $p \in (1, \infty)$ and $t - (r-1) < s < 0 < q' \leq q \leq t$, there exists $\widetilde{C} < \infty$ such that, if $\tilde{\varphi} \in W_p^{s,t,q}(M)$ satisfies

$$\|\mathcal{L}_{\alpha,V}\tilde{\varphi}\|_{W_p^{s,t,q'}} \leq \exp(\alpha a)\max\{1, |\alpha|^{j-1}\}C_{\tilde{\varphi}}\,,\ \forall \alpha \geq 0\,, \tag{98}$$

for some[35] $a > 0$, $j \geq 1$, *and $C_{\tilde{\varphi}} < \infty$, then, setting* $C(a) = \dfrac{1}{1-(C/\bar{C})^{a/h_{\mathrm{top}}}}$, *we have*

[35]The proof gives (99), replacing $\widetilde{C}C(a) \cdot T^{a/h_{\mathrm{top}}}(\log T)^{j-1}$ by $\widetilde{C} \cdot (\log T)^j$ if $a = 0$ and by $\widetilde{C} \cdot (\log T)^{j-1}$ if $a < 0$.

$$\sup_{x\in M} |\gamma_{w_{T,x},x}(\tilde\varphi)| \le \widetilde{C}\, C(a) T^{a/h_{\mathrm{top}}} (\log T)^{j-1} C_{\tilde\varphi}, \quad \forall T > C\bar{C}. \tag{99}$$

Proof. For $x \in M$ and $T > C\bar{C}$, inductively define sequences $\alpha_k^{\pm} = \alpha_k^{\pm}(x, T) \in \mathbb{R}$, $k \ge 1$, by

$$\begin{aligned}
&\bar{C} = \tau(T, \alpha_1^+, x),\\
&1 = \tau(\tau(\bar{C}, -\alpha_k^+, g_{\alpha_k^+}(x)), \alpha_{k-1}^+, x) = \tau(\bar{C}, \alpha_{k-1}^+ - \alpha_k^+, g_{\alpha_k^+}(x)),\\
&\alpha_1^- = \alpha_1^+,\\
&-1 = \tau(\tau(-\bar{C}, -\alpha_k^-, g_{\alpha_k^-}(h_T(x))), \alpha_{k-1}^-, h_T(x)) = \tau(-\bar{C}, \alpha_{k-1}^+ - \alpha_k^+, g_{\alpha_k^-}(h_T(x))),
\end{aligned}$$

where we used the first claim of (86). In the special case when $\tau(\rho, \alpha, x) = \rho e^{-\alpha h_{\mathrm{top}}}$ we find

$$\alpha_1^- = \alpha_1^+ = \frac{\log(T/\bar{C})}{h_{\mathrm{top}}} > 0, \qquad \alpha_k^+ - \alpha_{k-1}^+ = \alpha_k^- - \alpha_{k-1}^- = \frac{\log(1/\bar{C})}{h_{\mathrm{top}}} < 0, \;\; k \ge 2.$$

More generally, since $\tau(T, 0, x) = T$ and $\tau(T, \alpha, x)$ is continuous in α, the bounds (81)–(82) give $0 < \log(T/(\bar{C}C)) \le h_{\mathrm{top}} \cdot \alpha_1^{\pm} \le \log(TC/\bar{C})$. It is also easy to check that $\alpha_k^+ < \alpha_{k-1}^+$ and $\alpha_k^- < \alpha_{k-1}^-$ for all $k \ge 2$, and that (81) gives

$$\{e^{h_{\mathrm{top}}(\alpha_k^- - \alpha_{k-1}^-)}, e^{h_{\mathrm{top}}(\alpha_k^+ - \alpha_{k-1}^+)}\} \in \left[\frac{1}{C\bar{C}}, \frac{C}{\bar{C}}\right], \;\; \forall k \ge 2, \; \forall x \in M, \forall T > C\bar{C}. \tag{100}$$

Thus, we have

$$T/(C\bar{C})^k \le e^{h_{\mathrm{top}}\alpha_k^{\pm}} \le TC^k/\bar{C}^k, \quad \forall k \ge 1. \tag{101}$$

Fixing a C^∞ function $\chi : \mathbb{R} \to [0, 1]$ such that $\chi|_{[1,\infty)} \equiv 1$ and $\chi|_{(-\infty,0]} \equiv 0$, we put

$$w_1(\rho) = \chi(\tau(\rho, \alpha_1^+, x)) \cdot \chi(-\tau(\rho - T, \alpha_1^-, h_T(x))),$$

and, for $k \ge 2$ (note that w_1 and the $w_k^{\pm}$ depend on x and T),

$$\begin{aligned}
w_k^+(\rho) &= \chi(\tau(\rho, \alpha_k^+, x)) - \chi(\tau(\rho, \alpha_{k-1}^+, x)),\\
w_k^-(\rho) &= \chi(-\tau(\rho - T, \alpha_k^-, h_T(x))) - \chi(-\tau(\rho - T, \alpha_{k-1}^-, h_T(x))).
\end{aligned}$$

Then, for any $n_{\pm} \ge 1$, we have

$$\begin{aligned}
w_1(\rho) + \sum_{k=2}^{n_+} w_k^+(\rho) + \sum_{k=2}^{n_-} w_k^-(\rho)&\\
= w_1(\rho) + \chi(\tau(\rho, \alpha_n^+, x)) - \chi(\tau(\rho, \alpha_1^+, x))&\\
+\chi(-\tau(\rho - T, \alpha_n^-, h_T(x)) - \chi(-\tau(\rho - T, \alpha_1^-, h_T(x))),&
\end{aligned} \tag{102}$$

so that (82) and (101) imply $w_1 + \sum_{k=1}^{\infty}(w_k^+ + w_k^-) = 1|_{(0,T)}$. Define $N_\pm \geq 1$ by $\min\{\alpha^+_{N_+}, \alpha^-_{N_-}\} \geq 0$ and $\max\{\alpha^+_{N_++1}, \alpha^-_{N_-+1}\} \leq 0$; it follows from (101) that $N_\pm \in [1, \log(T)/\log(\bar{C}/C)]$. Then put

$$w_{T,x}(\rho) = w_1(\rho) + \sum_{k=2}^{N_+} w_k^+(\rho) + \sum_{k=2}^{N_-} w_k^-(\rho).$$

Setting $n_\pm = N_\pm$ in (102), the definitions of χ and $\alpha^\pm_{N+1}$ then give

$$\operatorname{supp}(1|_{[0,T]} - w_{T,x}) \subset \\ \big[0, \tau(\bar{C}, -\alpha^+_{N_++1}, g_{\alpha^+_{N_++1}}(x))\big] \cup \big[T + \tau(-\bar{C}, -\alpha^-_{N_-+1}, g_{\alpha^-_{N_-+1}}(h_T(x)), T)\big]. \quad (103)$$

Using $\max\{\alpha^+_{N_++1}, \alpha^-_{N_-+1}\} \leq 0$, the claim (97) now follows from (82).

Next, for $\tilde{\varphi} \in W_p^{s,t,q}(M)$, using $\gamma_{v,x}(\tilde{\varphi}) = \gamma_{v\circ(\cdot+T),h_T(x)}(\tilde{\varphi})$, Sublemma 4.13 gives[36]

$$\gamma_{w_{T,x},x}(\tilde{\varphi}) = \\ \gamma_{\tilde{w}_1, g_{\alpha_1^+}(x)}(\mathcal{L}_{\alpha_1^+,V}\tilde{\varphi}) + \sum_{k=2}^{N_+} \gamma_{\tilde{w}_k^+, g_{\alpha_k^+}(x)}(\mathcal{L}_{\alpha_k^+,V}\tilde{\varphi}) + \sum_{k=2}^{N_-} \gamma_{\tilde{w}_k^-, g_{\alpha_k^-}}(h_T(x))(\mathcal{L}_{\alpha_k^-,V}\tilde{\varphi}), \quad (104)$$

where, recalling (86), we put $\tilde{w}_1 = w_1(\tau(\,\cdot\,, -\alpha_1^+, g_{\alpha_1^+}(x))) = \chi \cdot \chi(\bar{C} - \cdot)$, and

$$\tilde{w}_k^+(\rho) = w_k^+(\tau(\rho, -\alpha_k^+, g_{\alpha_k^+}(x))),$$
$$\tilde{w}_k^-(\rho) = w_k^-(T + \tau(\,\cdot\,, -\alpha_k^-, g_{\alpha_k^-}(h_T(x))),$$

for $k \geq 2$. Since $\tilde{w}_k^+(\rho) = \chi(\rho) - \chi(\tau(\rho, \alpha^+_{k-1} - \alpha_k^+, g_{\alpha_k^+}(x)))$ and also $\tilde{w}_k^-(\rho) = \chi(-\rho) - \chi(-\tau(\rho, \alpha^-_{k-1} - \alpha_k^-, g_{\alpha_k^-}(h_T(x))))$, we find

$$\operatorname{supp}\tilde{w}_1 \subseteq [0, C\bar{C}] \text{ and } \operatorname{supp}\tilde{w}_k^+ \cup -\operatorname{supp}\tilde{w}_k^- \subseteq [0, C\bar{C}], \quad \forall k \in \mathbb{N}. \quad (105)$$

Since $\phi_\alpha \in C^{r-1}$, (79)–(80) imply $\sup_{\alpha \geq 0, x \in M} \|\partial_\rho \tau(\,\cdot\,, \alpha, x)\|_{C^{r-1}} < \infty$, so (100) gives

$$\sup_{x \in M} \max\{\|\tilde{w}_1\|_{C^r}, \sup_{k \geq 2} \sup_{T > \bar{C}C} \|\tilde{w}_k^\pm\|_{C^r}\} < \infty. \quad (106)$$

Thus, if (98) holds for $\tilde{\varphi}$, applying Lemma 4.11, and (101) to (104), we find $\hat{C} < \infty$ such that

$$\sup_x |\gamma_{w_{T,x},x}(\tilde{\varphi})| \leq \hat{C} C_{\tilde{\varphi}} T^{\frac{a}{h_{\mathrm{top}}}} (\log T)^{j-1} \sum_{k=1}^{\max\{N_-,N_+\}} (C/\bar{C})^{k\frac{a}{h_{\mathrm{top}}}}, \quad (107)$$

for all $T > C\bar{C}$. Since $4 < C < \bar{C}$, summing $\sum_{k=1}^{\infty}(C/\bar{C})^{k\frac{a}{h_{\mathrm{top}}}}$ gives (99). □

[36] This is analogous to the decomposition in [29, Lemma 3.1]. We use more explicit smoothing functions.

4D. ***Exact bounds*** **($\sup_{\alpha\geq 0} \|e^{-\alpha h_{\mathrm{top}}}\mathcal{L}_{\alpha,V}\|_{W_p^{s,t,q}} < \infty$).** ***Proof of Theorem 4.8.*** We saw in Lemma 4.6 that μ, the unique h_ρ-invariant probability, is a fixed point of $e^{-h_{\mathrm{top}}\alpha}\mathcal{L}'_{\alpha,V}$ acting on Radon measures, in Remark 4.7 that

$$\sup_{\alpha\geq 0} \|e^{-h_{\mathrm{top}}\alpha}\mathcal{L}_{\alpha,V}\|_{L^1(\mu)} \leq 1,$$

and in Corollary 4.12 that $\mu \in (W_p^{s,t,q}(M))'$, so that $\lambda_{\max}^{s,t,q,p} \geq h_{\mathrm{top}}$. If $\lambda_{\min}^{s,t,p} < h_{\mathrm{top}}$, we get more.[37]

Lemma 4.15 (peripheral spectrum and exact growth). *If E_- is C^{r-1} and $\lambda_{\min}^{s,t,p} < h_{\mathrm{top}}$ for some $p \in (1,\infty)$ and s, q, t as in* (42), *then for all $0 < q \leq t$ we have $\lambda_{\max}^{s,t,q,p} = h_{\mathrm{top}}$. Moreover, h_{top} is a simple eigenvalue and the only element of $\{\lambda \in \sigma(X+V)|_{W_p^{s,t,q}(M)},\ \mathrm{Re}\,\lambda = h_{\mathrm{top}}\}$. In particular, there are no maximal Jordan blocks, and*[38] $\sup_{\alpha\geq 0} \|e^{-\alpha h_{\mathrm{top}}}\mathcal{L}_{\alpha,V}\|_{W_p^{s,t,q}} < \infty$.

For the potential V associated to the SRB measure, where $\lambda_{\max}^{s,t,q,p} = 0$, the results above are well known, see [17, Lemma 5.1] and [18] (the claims there are for other Banach spaces, but intrinsicness can be applied as in our proof of Lemma 4.15).

Remark 4.16 (MME and bypassing unique ergodicity). Exploiting the results of [30] as in the proof of Lemma 4.15, it can be shown (without using Corollary 4.12), that the unique fixed point of $e^{-h_{\mathrm{top}}\alpha}\mathcal{L}'_{\alpha,V}$ in the dual of $W_p^{s,t,q}(M)$ is a Radon measure μ, and letting $\nu \in W_p^{s,t,q}(M)$ be the unique fixed point of $e^{-h_{\mathrm{top}}\alpha}\mathcal{L}_{\alpha,V}$, that the distribution formally defined by $\mu_*(\varphi) = \mu(\varphi\nu)$ is a Radon measure, and it is the unique measure of maximal entropy (MME) of g_α, which is (exponentially) mixing. (See, e.g., [33] for the discrete-time analogue.) In fact, unique ergodicity of h_ρ — that is, (77) — could be obtained from the information on the peripheral spectrum of $\mathcal{L}_{\alpha,V}$, bypassing the results of Bowen and Marcus from [15]. To keep the paper short, we refer to [15].

Before showing Lemma 4.15 we state and prove consequences of the exact growth.

Corollary 4.17 (exact growth for the resolvent). *Assume that E_- is C^{r-1} and $\lambda_{\min}^{s,t,p} < h_{\mathrm{top}}$ for $p \in (1,\infty)$ and s, q, t as in* (42). *Fix $0 < q \leq t$. There exists $C < \infty$ such that*

$$\|\mathcal{R}_z^n\varphi\|_{W_p^{s,t,q}} \leq \frac{C}{(\mathrm{Re}\,z - h_{\mathrm{top}})^n}\|\varphi\|_{W_p^{s,t,q}}, \quad \forall\,\mathrm{Re}\,z > h_{\mathrm{top}},\ \forall n \geq 1.$$

[37] In fact, only the exact growth claim is needed from Lemma 4.15: the rest of the information about the peripheral spectrum could be obtained by an ad hoc argument based on (85), the identity in the proof of Lemma 4.6, (99), and Sublemma 4.13. See [1, Lemma 5.18(v) and last claim of Lemma 5.14].

[38] This exact growth estimate is a key ingredient, e.g., for [16, Assumption 1] used in the proof of Theorem 4.8. See, e.g., the inverse Laplace transform in [16, Lemma 4.3].

Moreover, recalling (56), *there exist* $C < \infty$ *and a system* $\Theta = \{\Theta'_\omega\}$ *with* $\Theta'_\omega < \Theta_\omega$ *such that*

$$\|\mathcal{R}_z^n \varphi - \mathcal{R}_{tr,z}^n \varphi\|_{W_p^{s,t,q}} \leq \frac{C}{(\operatorname{Re} z - h_{\mathrm{top}})^n} \|\varphi\|_{W_{p,\Theta'}^{s,t,q}}, \quad \forall \operatorname{Re} z > h_{\mathrm{top}}, \ \forall n \geq 1 .$$

Proof. The first bound follows from exact growth ($\sup_{\alpha \geq 0} \|e^{-\alpha h_{\mathrm{top}}} \mathcal{L}_{\alpha,V}\|_{W_p^{s,t,q}} < \infty$), simplifying (58). The second claim follows from exact growth, using Remark 2.5 (for $\alpha \geq \alpha_0$) with (55). □

Proof of Lemma 4.15. By Corollary 3.9, since $\lambda_{\min}^{s,t,p} < h_{\mathrm{top}}$, we claim that we can exploit Theorem 3.10 about intrinsicness to transfer[39] the results of Giulietti, Liverani, and Pollicott in [30] to our spaces.

Indeed, first recall that a C^{r-1} one-form is a C^{r-1} section of the cotangent bundle T^*M, or equivalently, a C^{r-1} map from the tangent space TM to $\mathbb{R}$ whose restriction to each fibre T_xM is a linear functional on T_xM. Using that the Anosov flow g_α is topologically mixing (and E_- is orientable), they showed in [30, (4.5), Lemma 4.7, Proposition 4.9, for $\ell = d_- = 1$] that h_{top} (denoted σ_{d_s} there, with $d_-s = d_-$) is a simple eigenvalue and the only element λ of the spectrum with $\operatorname{Re}\lambda \geq h_{\mathrm{top}}$ for the generator $Y^{(d_-)}$ of the pullback semigroup $\mathcal{L}_\alpha^{(d_-)}$ of $g_{-\alpha}$ [30, (2.9)] acting on the closure $\widetilde{\mathcal{B}}^{1,|s|,d_-}$ of C^{r-1} one-forms on M vanishing in the flow direction, for an anisotropic Banach norm (see [30, Definition 3.6 and (4.6)] for $\ell = d_- = 1$, $p = 1$, and $q = t$, and note that this is equivalent to letting the pullback semigroup act on the Grassmannian of line bundles in TM as in [33; 29]). A key step for this is the fact that, setting $\tilde{\lambda}_{\min}^{1,|s|} := h_{\mathrm{top}} + \min\{1, |s|\} \log \theta < h_{\mathrm{top}}$, the intersection of the spectrum of $Y^{(d_-)}$ on $\widetilde{\mathcal{B}}^{1,|s|,d_-}$ with the half-plane $\operatorname{Re}\lambda > \tilde{\lambda}_{\min}^{1,|s|}$ contains only isolated eigenvalues of finite multiplicity (this is shown by establishing the corresponding result for the resolvent $\mathcal{R}_z^{(d_-)}$ [30, Definition 4.4, Lemma 4.8]).

Next, recall that, by our assumptions, $r \geq 2$ and E_- is C^{r-1} (so that E_-^* is C^{r-1} too). The (closed) subspace $\overline{\mathcal{B}}^{1,|s|}$ of $\widetilde{\mathcal{B}}^{1,|s|,d_-}$ obtained by taking the closure (for the norm of $\widetilde{\mathcal{B}}^{1,|s|,d_-}$) of the space $\Omega_{E_-}^{r-1}$ of those C^{r-1} one-forms taking values in E_-^*, is invariant under $\mathcal{L}_\alpha^{(d_-)}$. Using the natural bijection $\varphi \mapsto (\varphi(\,\cdot\,), E_-^*(\,\cdot\,))$ from $C^{r-1}(M)$ to $\Omega_{E_-}^{r-1}$, we see that the restriction of $\mathcal{L}_\alpha^{(d_-)}$ to $\Omega_{E_-}^{r-1}$ coincides with our operator $\mathcal{L}_{\alpha,V}$ on $C^{r-1}(M)$. It is well-known[40] that restricting a bounded operator $\mathcal{R}$ to a closed invariant subspace $\overline{\mathcal{B}} \subset \widetilde{\mathcal{B}}$ can fill up the holes (a hole in a compact set of $\mathbb{C}$ is a bounded connected component of its complement) in the original spectrum, but the spectrum of the restriction does not intersect the unbounded connected component of $\sigma(\mathcal{R}|_{\widetilde{\mathcal{B}}})$ (see [46, Corollary 4.1]). Hence, the intersection of the

[39] An independent proof should exist. To exclude maximal Jordan blocks, a geometric argument is needed see [30, §4.3]. Maybe (81)–(82) can help; see [1, Lemma 5.17].

[40] For example, the spectrum of the two-sided shift restricted to sequences vanishing on one side is the whole disc, while the original spectrum is the circle.

spectrum of $Y^{(d_-)}|_{\overline{\mathcal{B}}^{1,|s|}}$ with the half-plane $\operatorname{Re}\lambda > \tilde{\lambda}^{1,|s|}_{\min}$ still contains only isolated eigenvalues of finite multiplicity. Some of the eigenvalues of $Y^{(d_-)}$ on $\widetilde{\mathcal{B}}^{1,|s|,d_-}$ can disappear for the restricted operator, but we already established in Corollary 4.12 that h_{top} is an eigenvalue in our space. Finally, since $\max\{\lambda^{s,t,p}_{\min}, \tilde{\lambda}^{1,|s|}_{\min}\} < h_{\text{top}}$, we can apply Theorem 3.10, using also that C^{r-1} functions are dense in $W^{s,t,q}_p$ and in $\overline{\mathcal{B}}^{1,|s|}$, and that $W^{s,t,q}_p$ and $\overline{\mathcal{B}}^{1,|s|}$ are both continuously embedded into the dual of $C^{|s|+\max\{1,t\}}$ [30, Lemma 3.10]. □

Proof of Theorem 4.8. The starting point of the proof is

$$\gamma_x(1,T)\mu(\varphi) = \int_0^T \mu(\varphi)d\rho = T\mu(\varphi);$$

this is trivial for the unit speed horocycle flow, an easy computation otherwise. Thus, we may and shall assume $\mu(\varphi) = 0$, replacing φ by $\varphi - \mu(\varphi)$ (constants belong to our Banach space).

Fix $0 < q \le t$. By Lemma 4.15, $\lambda^{s,t,p}_{\max} = h_{\text{top}}$, it is a simple eigenvalue and the only maximal eigenvalue of $X+V$ on $W^{s,t,q}_p(M)$. Hence, $\mathcal{O}_{h_{\text{top}}} := \mathcal{O}_{h_{\text{top}},1,1} = \mu$, so that $\mathcal{O}_{h_{\text{top}}}(\varphi) = 0$.

For a general Ruelle–Pollicott resonance $\lambda \in \sigma(X+V)|_{W^{s,t,q}_p(M)}$ with $\operatorname{Re}\lambda > \lambda^{s,t,p}_{\min}$, recalling the notation introduced above (90), we denote by $\mathcal{D}_{(\lambda,i,j)} \in D(X+V)$, for $1 \le i \le n_\lambda$ and $1 \le j \le m_{\lambda,i}$, its generalised eigenstates, i.e.,

$$(X+V-\lambda)^j \mathcal{D}_{(\lambda,i,j)} = 0, \quad (X+V-\lambda)^{j-1}\mathcal{D}_{(\lambda,i,j)} \neq 0.$$

We write $\mathcal{D}_{h_{\text{top}}} := \mathcal{D}_{h_{\text{top}},1,1}$. There is a curve Γ_λ around λ with

$$\frac{1}{2i\pi}\oint_{\Gamma_\lambda} (z-(X+V))^{-1}\varphi \,\mathrm{d}z = \sum_{i=1}^{n_\lambda} \Pi_{\lambda,i}\varphi,$$

where $\Pi_{\lambda,i}$ is a projector of rank $m_{\lambda,i}$, with

$$\Pi_{\lambda,i} = \sum_{j=1}^{m_{\lambda,i}} \mathcal{D}_{(\lambda,i,j)} \otimes \mathcal{O}_{(\lambda,i,j)}, \ 1 \le i \le n_\lambda, \quad \mathcal{O}_{(\lambda,i,j)} \in D((X+V)'),$$

where

$$\mathcal{O}_{(\lambda_1,i_1,j_1)}(\mathcal{D}_{(\lambda_2,i_2,j_2)}) = \begin{cases} 1 & \text{if } (\lambda_1,i_1,j_1) = (\lambda_2,i_2,j_2), \\ 0 & \text{otherwise.} \end{cases}$$

In addition, there are finite-rank nilpotent operators $\mathcal{N}_{\lambda,i}$, for $1 \le i \le n_\lambda$, such that

$$\Pi_{\lambda_1,i_1}\Pi_{\lambda_2,i_2} \equiv 0 \text{ and } \mathcal{N}_{\lambda_1,i_1}\mathcal{N}_{\lambda_2,i_2} \equiv 0 \ \text{ if } \lambda_1 \neq \lambda_2 \text{ or } i_1 \neq i_2,$$

$$\mathcal{N}^{m_{\lambda,i}-1}_{\lambda,i} \equiv 0, \qquad \Pi_{\lambda_1,i_1}\mathcal{N}_{\lambda_2,i_2} = \mathcal{N}_{\lambda_2,i_2}\Pi_{\lambda_1,i_1} = \begin{cases} \mathcal{N}_{\lambda_2,i_2} & \text{if } \lambda_1 = \lambda_2 \text{ and } i_1 = i_2, \\ 0 & \text{if } \lambda_1 \neq \lambda_2 \text{ or } i_1 \neq i_2, \end{cases}$$

and, using the surjection from eigenvalues of $X+V$ to those of the semigroup [22, V (2.3)],

$$\mathcal{L}_{\alpha,V}\Pi_{\lambda,i} = \exp(\alpha\lambda)\exp(\alpha\mathcal{N}_{\lambda,i})\Pi_{\lambda,i}\,,\ \forall\alpha \geq 0\,.$$

(See also [16], for example.) Therefore, for each (λ, i, j) there exists $C_{i,j} < \infty$ such that

$$\|\mathcal{L}_{\alpha,V}\mathcal{D}_{(\lambda,i,j)}\|_{W_p^{s,t,q}} \leq C_{i,j}\exp(\alpha\operatorname{Re}\lambda)\max\{1, |\alpha|^{j-1}\}\|\mathcal{D}_{(\lambda,i,j)}\|_{W_p^{s,t,q}}\,,\ \forall\alpha\in\mathbb{R}. \quad (108)$$

In other words, $\mathcal{D}_{(\lambda,i,j)}$ satisfies (98) for all $\alpha\in\mathbb{R}$ if $a = \operatorname{Re}\lambda > \lambda_{\min}^{s,t,p}$. Assume that $\delta > 0$, let

$$\lambda \in \Sigma_\delta = \sigma(X+V)|_{W_p^{s,t,q}(M)} \cap \{z\in\mathbb{C} \mid \operatorname{Re} z \geq \delta\}$$

and fix $x\in M$.

Let $T \geq T_0 = \bar{C}C > 1$ and $w_{T,x}\in C_0^{r-1}$ be given by Lemma 4.14, and define

$$c_{(\lambda,i,j)}(T,x) := T^{-\frac{\lambda}{h_{\mathrm{top}}}}(\log T)^{1-j}\gamma_{w_{T,x},x}(\mathcal{D}_{(\lambda,i,j)}) \in\mathbb{C},$$
$$1\leq i\leq n_\lambda,\ 1\leq j\leq m_{\lambda,i}\,. \quad (109)$$

Then (99) from Lemma 4.14 implies that $\sup_{x,T\geq T_0}|c_{(\lambda,i,j)}(T,x)| < \infty$. With the decomposition

$$\begin{aligned}\gamma_{w_{T,x},x}(\Pi_{\lambda,i}\varphi) &= \sum_{j=1}^{m_{\lambda,i}}\mathcal{O}_{(\lambda,i,j)}(\varphi)\gamma_{w_{T,x},x}(\mathcal{D}_{(\lambda,i,j)})\\ &= \sum_{j=1}^{m_{\lambda,i}} c_{(\lambda,i,j)}T^{\frac{\lambda}{h_{\mathrm{top}}}}(\log T)^{j-1}\mathcal{O}_{(\lambda,i,j)}(\varphi)\end{aligned}$$

and using Lemma 4.15, we find for any finite subset $\Lambda_\delta\subset\Sigma_\delta$ that

$$\begin{aligned}\gamma_x(\varphi,T) &= \gamma_{w_{T,x},x}(\mathcal{D}_{h_{\mathrm{top}}})\mu(\varphi)\\ &\quad + \sum_{\substack{\lambda\in\Lambda_\delta\\ \lambda\neq h_{\mathrm{top}}}}\sum_{i=1}^{n_\lambda}\sum_{j=1}^{m_{\lambda,i}} c_{(\lambda,i,j)}T^{\frac{\lambda}{h_{\mathrm{top}}}}(\log T)^{j-1}\mathcal{O}_{(\lambda,i,j)}(\varphi) + \mathcal{E}_{T,x,\Lambda_\delta}(\varphi)\,,\end{aligned}$$

where $\gamma_{w_{T,x},x}(\mathcal{D}_{h_{\mathrm{top}}})\mu(\varphi) = 0$ and

$$\mathcal{E}_{T,x,\Lambda_\delta}(\varphi) = \gamma_{w_{T,x},x}\Big(\varphi - \sum_{\lambda\in\Lambda_\delta}\sum_{i=1}^{n_\lambda}\Pi_{\lambda,i}\varphi\Big) + \gamma_x(\varphi,T) - \gamma_{w_{T,x},x}(\varphi)\,.$$

To conclude, we show that finiteness of Σ_δ and the claimed bound on $\mathcal{E}_{T,x,\Sigma_\delta}$ follow from Condition 3.12. We first check that Assumptions 1, 2, and 3A from [16] hold for the semigroup $e^{-h_{\mathrm{top}}\alpha}\mathcal{L}_{\alpha,V}$ on $W_p^{s,t,q}(M)$ (with generator $X+V-h_{\mathrm{top}}$ and resolvent $\mathcal{R}_{z+h_{\mathrm{top}}}$): Note that $\mathcal{R}_{h_{\mathrm{top}}} = (h_{\mathrm{top}} - (X+V))^{-1}$ is bounded on the codimension one subset $W(h_{\rm top})$ of $W_p^{s,t,q}$ formed of those $\tilde\varphi$ such that $\mu(\tilde\varphi)=0$. Therefore, the norm on $W(h_{\rm top})$ defined by

$$\|\tilde\varphi\|_{\rm weak} = \frac{\|(h_{\rm top}-(X+V))^{-1}(\tilde\varphi)\|_{W_p^{s,t,q}}}{\|\mathcal{R}_{h_{\rm top}}\|}$$

satisfies $\|\tilde\varphi\|_{\rm weak} \le \|\tilde\varphi\|_{W_p^{s,t,q}}$. The identity

$$\tilde\varphi - e^{-h_{\rm top}\alpha}\mathcal{L}_{\alpha,V}\tilde\varphi = (h_{\rm top}-(X+V))\int_0^\alpha e^{-h_{\rm top}\tilde\alpha}\mathcal{L}_{\tilde\alpha,V}\tilde\varphi \mathrm{d}\tilde\alpha$$

thus implies Assumption 1 in [16], i.e.,

$$\sup_{\alpha>0}\frac{1}{\alpha}\|\mathrm{id}-e^{-h_{\rm top}\alpha}\mathcal{L}_{\alpha,V}\|_{W_p^{s,t,q}(M)\to\text{weak}} < \infty,\ \forall\tilde\varphi\in W(h_{\rm top}). \tag{110}$$

(Indeed, it is enough to consider $\alpha\in(0,1]$ in (110) due to the exact growth.) Since $h_{\rm top}-\lambda_{\min}^{s,t,p}>0$, the essential spectral radius of $\mathcal{R}_{z+h_{\rm top}}$ is not larger than $|\mathrm{Re}\, z + h_{\rm top} - \lambda_{\min}^{s,t,p}|^{-1}$ by Corollary 3.9, giving Assumption 2 in [16]. Finally, since $p > d/\min\{t, r-1+s\}$, Proposition 3.13 and Corollary 4.17 imply that Condition 3.12 gives (64), i.e., Assumption 3A from [16] for $\mathcal{R}_{z+h_{\rm top}}$.

Thus [16, Theorem 1] gives $\#\Sigma_\delta<\infty$ and furnishes, for $\tilde\delta>\delta$, a constant $C_B(\tilde\delta)<\infty$ with

$$\left\|e^{-h_{\rm top}\alpha}\mathcal{L}_{\alpha,V}\Big(\psi-\sum_{\lambda\in\Sigma_\delta}\sum_{i=1}^{n_\lambda}\Pi_{\lambda,i}\psi\Big)\right\|_{\rm weak} \le C_B e^{\tilde\delta\alpha}\|(h_{\rm top}-(X+V))\psi\|_{W_p^{s,t,q}},\quad \forall\alpha\ge 0, \tag{111}$$

for all $\psi\in W(h_{\rm top})$.

Finally, Lemma 4.14 gives the bound (91) for $\sup_x|\mathcal{E}_{T,x,\Sigma_\delta}(\varphi)|$: First note that (97) implies that $\sup_{x,T}|\gamma_x(\varphi,T)-\gamma_{w_{T,x},x}(\varphi)|\le C\|\varphi\|_{C^0}$. Then, introducing $\psi_j=(h_{\rm top}-(X+V))^j\varphi$, for $j=1,2$, we have $\max_{j=1,2}\|\psi_j\|_{W_p^{s,t,q}}\le C\|\varphi\|_{C^r}$, because $V\in C^{r-1}$, and $t<r-2$. Then (99), with $\tilde\varphi=\psi_1\in W_p^{s,t,q}(M)$ and

$$C_{\tilde\varphi} = C_B\|(h_{\rm top}-(X+V))\tilde\varphi\|_{W_p^{s,t,q}},$$

gives $T_0<\infty$ such that

$$\sup_x\left|\gamma_{w_{T,x},x}\Big(\varphi-\sum_{\lambda\in\Sigma_\delta}\sum_{i=1}^{n_\lambda}\Pi_{\lambda,i}\varphi\Big)\right| \le \widetilde{C}C(\tilde\delta)C_B(\tilde\delta)T^{\tilde\delta/h_{\rm top}}\big\|(h_{\rm top}-(X+V))\psi_1\big\|_{W_p^{s,t,q}},\quad \forall T\ge T_0. \qquad\square$$

4E. *Proof of Proposition 4.10.* We assumed the flow fixes a C^1 contact 1-form $\upsilon\in T^*M$. In particular, υ is annihilated on E_++E_- and the volume in $\bigwedge^3 T^*M$ is preserved by the flow. Then, since $d=3$, we already mentioned that [38, Theorem 3.1] gives that E_- is $C^{2-\tilde\eta}$ for any $\tilde\eta\in(0,1)$. Taking $r=3-\tilde\eta$, we find $\partial_\rho\tau(0,-\alpha,\cdot)\in C^{r-1}$ for any C^r reparametrisation of the unit speed horocycle flow.

It follows from (80) and Lemma 3.5 that the transfer operators associated to C^r reparametrisations are conjugate to each other and thus have the same spectrum (using Remark 2.5). For the unit speed parametrisation we have $\phi_\alpha = \partial_\rho \tau(0, -\alpha, 0) = \det \mathrm{D}g_{-\alpha}|_{E_-}$. We claim that

$$\lambda^{s,t,p}_{\min}(X, V) = \lim_{\alpha\to\infty} \frac{1}{\alpha} \log \|\phi_\alpha \left|\det(\mathrm{D}g_\alpha)^{\mathrm{tr}}|_{E^*_-}\right|^{\min\{t,-s\}} \|_{L_\infty(M)}. \tag{112}$$

Indeed, $d_- = d_+ = 1$, and since the flow g_α preserves volume, i.e., $|\det \mathrm{D}g_\alpha| \equiv 1$, we find

$$\left|\det(\mathrm{D}g_{-\alpha})^{\mathrm{tr}}|_{E^*_+}\right| = \left|\det(\mathrm{D}_{g_{-\alpha}}g_\alpha)^{\mathrm{tr}}|_{E^*_0}\right| \left|\det(\mathrm{D}_{g_{-\alpha}}g_\alpha)^{\mathrm{tr}}|_{E^*_-}\right|.$$

Using (45) and the upper and lower bounds on $\left|\det(\mathrm{D}_{g_{-\alpha}}g_\alpha)^{\mathrm{tr}}|_{E^*_0}\right|$, we get (112).

Then, taking $-s = t = \frac{r-1}{2} - \frac{\tilde\eta}{2} = 1 - \tilde\eta$, we have $t - r + 1 \le s < 0 < t < r - 2$, and formula (112) together with (80) give $\lambda^{s,t,p}_{\min} < \epsilon_1$, if $\tilde\eta > 0$ is small enough.

It remains to discuss Condition 3.12. (This condition is stable under reparametrizations of the horocycle flows, using (55) and the conjugacy mentioned in the previous paragraph.) Since we assumed (2), the second[41] claim of [30, Proposition 7.5] holds for $\ell = d_-$. Therefore, since our operator $\mathcal{R}_z$ coincides with the operator denoted $\mathcal{R}^{(d_s)}(z)$ in [30] restricted to one-forms that take their images in E^*_- (as in the proof of Lemma 4.15), the second claim of [30, Proposition 7.5], combined with [30, Lemma 7.4], [30, (7.1)] (which holds due to the exact growth bounds) and (65) (for $\delta_1 = 0$, $\delta_2 > 0$ and $\beta = 3\gamma_0$), gives $\eta \in (0, 1)$, $a_0 \ge 1$, $b'_0 \ge 1$, $\delta_2 \in (0, h_{\mathrm{top}})$, $C < \infty$, $\gamma_0 \in (0, 1)$, $C_1 > 1$, and, for any $a \ge a_0$ and $\gamma' \ge aC_1$ such that[42] $\gamma' < 3\gamma_0/\log(1 + \delta_2/a)$, we have

$$\|\mathcal{R}^{\lceil\gamma'\log|b|\rceil}_{a+ib+h_{\mathrm{top}}}\varphi\|_{\bar{\mathcal{B}}^{0,1+\eta}} \le C|a + \delta_2|^{-\lceil\gamma'\log|b|\rceil}\|\varphi\|_{\bar{\mathcal{B}}^{1,\eta}}, \ \ \forall |b| \ge b'_0, \tag{113}$$

for the anisotropic Banach spaces $\bar{\mathcal{B}}^{j,\eta}$, $j = 0, 1$, in the scale from the proof of Lemma 4.15. (The statement of [30, Proposition 7.5] is for $a \in [a_0, 2a_0]$ and $\gamma' \ge C_1$; the proof gives (113).)

By [30, Lemma 3.10] the space $\bar{\mathcal{B}}^{1,\eta}$ lies in the dual of $C^{1+\eta}(M)$. Thus, if $s'' < -1 - \eta - 3 - \frac{3}{p}$, we have $\|\tilde\varphi\|_{W^{s'',s'',s''}_p} \le C\|\tilde\varphi\|_{\bar{\mathcal{B}}^{0,1+\eta}}$. (Use Sobolev embeddings [45, Theorem 2.2.3(i)] for the dual $B^{-1-\eta}_{1,1}$ of $b^{1+\eta}_{\infty,\infty} \supset C^{1+\eta}(M)$ [45, Definition 2.1.3.1(ii), Remark 2.1.5.1] and $W^{s''}_p = F^{s''}_{p,2}$ in dimension $d = 3$.) The last bound of [30, Remark 3.8] gives $\|\tilde\varphi\|_{\bar{\mathcal{B}}^{1,\eta}} \le C\|\tilde\varphi\|_{C^1}$, ending the proof.

[41]The proof of [30, Proposition 7.5] has a gap since a factor $e^{z\tau_W\circ H_{\beta,i,W}}$ is missing from [30, (7.14)]. However, the statement is correct [31, Theorem 1 and its proof], replacing the condition $\min\{1, \hat\varpi\} > \frac{2}{3}$ in [30] by: "$\lambda_+ - \lambda_- < \vartheta_0\lambda_-$ with $\vartheta_0 \in \left(0, \frac{1}{4}\right)$ and $\varpi \ge 1$," which hold since we assumed (2).

[42]We may take $\delta_2 > 0$ small enough in (65) to ensure $aC_1 < 3\gamma_0/\log\left(1 + \delta_2/a\right)$.

Appendix A: Integration by parts

Lemma A.1 (integration by parts; cf. text after Remark 3.3 in [8]). *Let $f : \mathbb{R}^d \to \mathbb{R}$ be C^1 and compactly supported. For any C^2 function $G : \mathbb{R}^d \to \mathbb{R}$ such that $\inf_{\mathrm{supp}\, f} \sum_{j=1}^d (\partial_j G)^2 > 0$,*

$$\int_{\mathbb{R}^d} e^{iG(z)} f(z)\,\mathrm{d}z = i \int_{\mathbb{R}^d} e^{iG(z)} \sum_{k=1}^d \partial_k \frac{(\partial_k G(z)) f(z)}{\sum_{j=1}^d (\partial_j G(z))^2} \mathrm{d}z .$$

For $z \in \mathbb{R}^d$ we write $\nabla_z G = (\partial_j G(z))_{j=1,\dots,d}$ for the gradient and $\nabla_z^{\mathrm{tr}} G = \sum_{j=1}^d \partial_j G(z)$ for the divergence of $G : \mathbb{R}^d \to \mathbb{R}$. Let $\nu : \mathbb{R}^d \to \mathbb{R}_+$ be C^∞, supported in the unit ball, and such that $\int_{\mathbb{R}^d} \nu(x)\,\mathrm{d}x = 1$. Then we have:

Lemma A.2 (regularised integration by parts [8, (3.4)]). *Fix $1 < r < 2$. Let $f : \mathbb{R}^d \to \mathbb{C}$ be a compactly supported C^{r-1}-map, let $G : \mathbb{R}^d \to \mathbb{R}$ be C^r and such that $|\nabla_z G(z)|^2 = \sum_{j=1}^d (\partial_j G)^2 > 0$ on $\mathrm{supp}\, f$. Set*[43]

$$h(z) := i \frac{\nabla_z G(z) f(z)}{|\nabla_z G(z)|^2}, \quad h_\delta := \delta^{-d} \cdot h * \nu\Big(\frac{\cdot}{\delta}\Big),\ \ \delta > 0 .$$

Then, for every $L \geq 1$,

$$\int_{\mathbb{R}^d} e^{iLG(z)} f(z)\,\mathrm{d}z = \frac{1}{L} \int_{\mathbb{R}^d} e^{iLG(z)} \nabla_z^{\mathrm{tr}} h_\delta(z)\,\mathrm{d}z - i \int_{\mathbb{R}^d} e^{iLG(z)} \nabla_z^{\mathrm{tr}} G(z) \big(h(z) - h_\delta(z)\big)\,\mathrm{d}z .$$

Appendix B: Fragmentation and reconstitution

A finite set of C^r functions $\vartheta_j : \mathbb{R}^d \to [0, 1]$ such that $\sum_j \vartheta_j(x) \leq 1$ for all $x \in \mathbb{R}^d$ is called a C^r subpartition of unity. The fragmentation and reconstitution lemmas of [8] and [5] extend straightforwardly to our anisotropic spaces. The first lemma is a variant of [8, Lemma 7.1]:

Lemma B.1 (fragmentation). *Let $1 < p < \infty$ and let s, q, t as in (21) be real numbers, and let $K \subset \mathbb{R}^d$ be compact. For any s', t', $q' \in \mathbb{Z}$, there exists $C < \infty$ such that, for any C^r subpartition of unity $\{\vartheta_j\}_{j=1,\dots J}$ of K with intersection multiplicity ν, there exists $\tilde{C}_\vartheta < \infty$ such that (in the applications, we take $s' < s$, $t' < t$, and $q' \leq q$)*

$$\Big\| \sum_{j=1}^J \vartheta_j v \Big\|_{W^{s,t,q}_{p,\Theta}} \leq C \nu^{(p-1)/p} \Big(\sum_j \|\vartheta_j v\|^p_{W^{s,t,q}_{p,\Theta}} \Big)^{1/p} + \tilde{C}_\vartheta \sum_{j=1}^J \|\vartheta_j v\|_{W^{s',t',q'}_{p,\Theta}} . \quad (114)$$

The last lemma, a variant of [8, Proposition 7.2] (see also [5, Lemma 4.29]), is useful to group partitions of unity associated with a fixed cone system:

[43] In particular, there exists $\bar{C} \geq 1$ such that $\|\nabla_z h_\delta\|_{L_\infty} \leq \bar{C} \|h\|_{C^{r-1}} \delta^{r-1}$ and $\|h - h_\delta\|_{L_\infty} \leq \bar{C} \|h\|_{C^{r-1}} \delta^{r-1}$.

Lemma B.2 (reconstitution). *Let $1 < p < \infty$, let s, q, t as in* (21) *be real, and let $K \subset \mathbb{R}^d$ be compact. If $\Theta' < \Theta$ then for any $s', q', t' \in \mathbb{Z}$, there exists $C < \infty$ such that, for any C^r subpartition of unity $\{\vartheta_j\}_{j=1,\dots J}$ of K with intersection multiplicity ν, there exists $C'_\vartheta < \infty$ such that* (*in the applications, we take $s' < s$, $t' < t$, and $q' \le q$*)

$$\left(\sum_{j=1}^{J} \|\vartheta_j v\|^p_{W^{s,t,q}_{p,\Theta'}}\right)^{\frac{1}{p}} \le C\nu^{1/p}\|v\|_{W^{s,t,q}_{p,\Theta}} + C'_\vartheta \|v\|_{W^{s',t',q'}_{p,\Theta}}. \tag{115}$$

Appendix C: Interpolation, mollification, and approximations of the identity

Let $[\mathcal{B}_1, \mathcal{B}_2]_u$, for $u \in (0, 1)$, denote the complex (Calderón) interpolation of an interpolation pair of Banach spaces $\mathcal{B}_1$, $\mathcal{B}_2$. The Banach spaces in this paper are complex interpolation spaces:

Lemma C.1 (interpolation). *Setting $w(x_1, x_2) = (1 - w)x_1 + wx_2$ for $w \in (0, 1)$, we have for any $p \in (1, \infty)$, all $t_j - (r - 1) < s_j < 0 < q_j \le t_j$, $j = 1, 2$, and all $w \in (0, 1)$ that*

$$[W^{s_1,t_1,q_1}_p(M), W^{s_2,t_2,q_2}_p(M)]_w = W^{s,t,q}_p(M),$$
$$s = w(s_1, s_2), \quad t = w(t_1, t_2), \quad q = w(q_1, q_2).$$

Proof. The norms on $\mathbb{R}$ given by $\|x\|_{n,u} = 2^{nw}|x|$, $n \in \mathbb{N}$, form a complex interpolation scale with respect to $w \in \mathbb{R}$. The lemma thus follows from using [48, Theorems 1.18.1, 1.18.4] to show that the local norms $\|\cdot\|_{W^{s,t,q}_{\Theta_\omega,p}}$ have the desired interpolation property, and then applying [48, Theorem 1.18.4] to the function $\alpha \mapsto \|(\vartheta_\omega \mathcal{L}_{\alpha,V}\varphi) \circ \kappa_\omega^{-1}\|_{W^{s,t,q}_{\Theta_\omega,p}}$ on $[0, \alpha_0]$. □

Recall the finite atlas $\mathcal{A}$, indexed by $\omega \in \Omega$, and the pair $\{\Theta_\omega\}$, $\{\Theta'_\omega\}$ of adapted cone systems from Lemma 2.4 and Remark 2.5. Let $\{U_\omega\}$ be an open cover of M with $\bar U_\omega \subset V_\omega$. Let ν be as in Lemma A.2 and set $\nu_\epsilon(x) = \epsilon^{-d}\nu(x/\epsilon)$. Fix C^∞ functions $\bar\vartheta_\omega : M \to [0, 1]$, with $\bar\vartheta_\omega$ supported in U_ω, such that $\sum_\omega \bar\vartheta_\omega(x) = 1$ for all $x \in M$. Finally, let $\epsilon > 0$ be such that the ϵ-neigbourhood of $\kappa_\omega(U_\omega)$ is contained in $\kappa_\omega(V_\omega)$ for each ω. As in [7, (5.4)], define a mollifier operator $\mathbb{M}_\epsilon$, by setting, for any distribution φ of order at most r on M,

$$\begin{aligned}(\mathbb{M}_\epsilon(\varphi))_\omega(u) &= \int_{\mathbb{R}^d} \nu_\epsilon(u - v)\psi(\kappa_\omega^{-1}(v))\,\mathrm{d}v = [\nu_\epsilon * (\psi \circ \kappa_\omega^{-1})](u), \\ &\qquad\qquad \omega \in \Omega,\ u \in \kappa_\omega(U_\omega), \\ \mathbb{M}_\epsilon(\varphi) &= \sum_{\omega\in\Omega} \bar\vartheta_\omega \cdot (((\mathbb{M}_\epsilon(\varphi))_\omega \circ \kappa_\omega).\end{aligned} \tag{116}$$

Since $\{\Theta_\omega\}$ and $\{\Theta'_\omega\}$ are adapted to $\mathcal{A}$ and g_α, the fact that $\Theta'_\omega < \Theta_\omega$ in the next lemma is not a problem:

Lemma C.2 (approximation of the identity). *For any $p \in (1, \infty)$, all $s', t', q' \in \mathbb{R}$ and all $\eta > 0$ such that $-(r - 1) + t' + \eta < s' < -\eta < 0 < q' < t'$, there exists*

$C<\infty$ such that, letting $W_p^{s',t',q'}(\Theta')$ be the space constructed with Θ',

$$\|\mathbb{M}_\epsilon\varphi-\varphi\|_{W_p^{s',t',q'}(\Theta')}\le C\epsilon^\eta\|\varphi\|_{W_p^{s'+\eta,t'+\eta,q'+\eta}}\,,\qquad \forall\varphi,\ \forall\epsilon>0\,.$$

Proof. Minkowski-type integral bounds hold for the local norms $W^{s,t,q}_{p,\Theta'_\omega}$: There exists $C_M<\infty$ such that for any $\psi:\mathbb{R}^d\to\mathbb{R}_+$ and any family $\varphi_y\in W^{s,t,q}_{p,\Theta'_\omega}$, uniformly bounded in y,

$$\Big\|\int_{\mathbb{R}^d}\psi(y)\varphi_y(\,\cdot\,)\,dy\Big\|_{W^{s,t,q}_{p,\Theta'_\omega}}\le C_M\|\psi\|_{L^1(\mathbb{R}^d)}\sup_y\|\varphi_y\|_{W^{s,t,q}_{p,\Theta'_\omega}}\,. \tag{117}$$

(See [7, Remark 5.1]. We already established interpolation for our spaces.) So we proceed as in the proof of [7, Lemma 5.4]. The changes of charts $\kappa_{\omega'}\circ\kappa_\omega^{-1}$ (one chart is for the mollifier and the other for the norm) are cone-hyperbolic from Θ_ω to $\Theta'_{\omega'}$ by construction. □

Remark C.3. If we attempted to show Dolgopyat bounds using mollifiers through isotropic spaces as in [7, Lemma 5.4, (7.5)–(7.6)], we would face a factor

$$\|\mathcal{R}^n_{a+ib+h_{\mathrm{top}}}\|_{W_p^{s',s',s'}}\le\frac{C}{(a+s'\log\Theta)^n} \tag{118}$$

instead of Ca^{-n} in (72). After applying (65) with $\beta=\kappa(s-s')-1>0$, we would end up with an upper bound

$$\gamma'<\frac{\kappa(s-s')-1}{\log\big(1+(\lambda_{\max}-\delta)/a\big)-\log\big(1-|s'|(\log\Theta)/a\big)}.$$

In our main application, Proposition 4.10, we need to take s' close to -1 to guarantee $\lambda_{\min}<\lambda_{\max}$. The upper bound would then conflict with (74). This is why (proving (4) would give another solution to this problem) we used mollification through anisotropic spaces as[44] in [30, Lemma D.2], taking advantage of the exact growth from Lemma 4.15. Our norms are different from those of [30]: Their drawback is that we need to go through charts twice to prove Lemma C.2 (we have Remark 2.5 to save us). Their strength is that we can use the interpolation[45] Lemma C.1 and Minkowski inequalities to prove Lemma C.2.

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[44] For another use of approximations of identity with anisotropic norms see [13, (2.77) proof of Corollary 2.3, sentence after (3.27) in the proof of Lemma 3.5].

[45] Interpolation was not available in [7] due to the presence of a supremum in the norm there.

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ALEXANDER ADAM:

a.adam.sci@protonmail.com
Aachen, Germany

VIVIANE BALADI:

baladi@lpsm.paris
Sorbonne Université and Université Paris Cité, CNRS, Laboratoire de Probabilités, Statistique et Modélisation, F-75005 Paris, France