Max-Planck-Institut für Mathematik Bonn

On spectral properties of the Schreier graphs of the
Thompson group F

by

Artem Dudko
Rostislav Grigorchuk

On spectral properties of the Schreier graphs of the Thompson group F

by
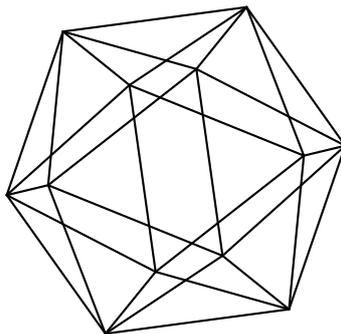
Artem Dudko
Rostislav Grigorchuk

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

IMPAN
Warsaw
Poland

Texas A&M University
College Station, TX
USA

On spectral properties of the Schreier graphs of the Thompson group F

Artem Dudko

IMPAN, Warsaw, Poland

adudko@impan.pl

Rostislav Grigorchuk

Texas A&M University, College Station, TX, USA

grigorch@math.tamu.edu

0 Introduction

The Thompson's group F is one of the most famous and most important groups related to many areas of mathematics (see the survey [3]). The question about amenability of this group remains open for more than 50 years despite many attempts to solve it. By a remarkable Kesten's criterion of amenability a group G is amenable if and only if 1 belongs to the spectrum of the Markov operator associated to a symmetric random walk on G. This criterion also holds for graphs of uniformly bounded degree (see [4]). Amenability of the Thompson's group F (which is equivalent to amenability of the Cayley graph of F) would imply that every Schreier graph of F is amenable. This motivates studying spectral properties of such graphs.

This article is devoted to studying spectral properties of the family of Schreier graphs associated to the action of F on the interval $[0, 1]$ (see [3]). A Schreier graph of a group G is determined by a triple (G, H, S), where H is a subgroup of G and S is a system of generators for G. These graphs naturally arise in association with an action of G on a set X. Given $x \in X$ one can introduce the Schreier graph $\Gamma_x = \Gamma(G, G_x, S)$, where G_x is the stabilizer of x.

An interesting and important case is when a subgroup $H < G$ is maximal and of infinite index or when H is weakly maximal (i.e. H has infinite index in G and is maximal among the groups with this property). Study of spectral, combinatorial and dynamical properties of Schreier graphs associated with weakly maximal subgroups of branch groups of intermediate growth constructed by the second author got a lot of attention and led to many unexpected results and connections. Among them are

relations to holomorphic dynamics ([1], [17],[5]) and to random Shrödinger operators ([12]). In [9] this approach was used to show that there exists an uncountable family of isospectral but not quasi-isometric Cayley graphs.

The new method introduced in [1] for studying spectra of graphs using the relation to the spectra of operators associated with the Koopman representation for groups acting on rooted trees was generalized by the authors to the case of actions (G, X, μ) where μ is a quasi-invariant measure [7].

Around 2010 the second author suggested to D. Savchuk to inspect Schreier graphs Γ_x, $x \in [0, 1]$, of Thompson group F associated with natural action of F on the interval $[0, 1]$ by piecewise-linear transformations. It was discovered by Savchuk that the stabilizers F_x of points $x \in (0, 1)$ are maximal subgroups and the Schreier graphs Γ_x are amenable. He described their shape. The graph for the case $x = \frac{1}{2}$ is shown on Figure 1. This graph (which we denote by Υ) was used by Kaimanovich to show that
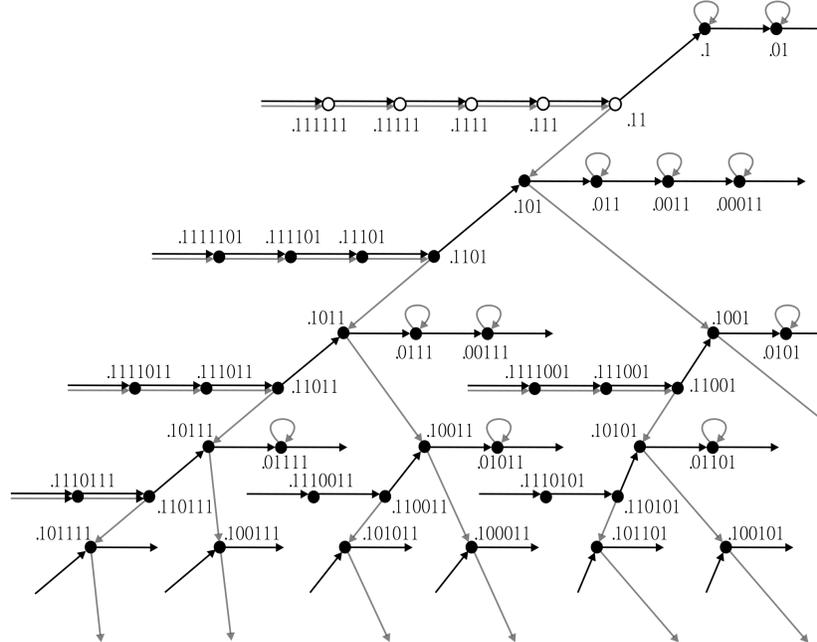
Figure 1: The Schreier graph Υ of the action of F on the orbit of $\frac{1}{2}$. The black colored edges correspond to the action of the generator a and the grey colored edges correspond to the action of the generator b of F.

F has no Liouville property [19], i.e. there are non-constant harmonic functions on F. The graph Υ is one of the main objects of investigation in this article.

Due to the amenability of Schreier graphs Γ_x, $x \in [0, 1]$, the spectra of Markov operators contain 1, but it is still interesting to know what exactly is the spectrum

2

and what can be said about spectral measures μ_v associated with delta-functions δ_v, where v is a vertex of the graph. The graphs Γ_x associated with F are tree-like graphs. Spectra of tree-like graphs have been studied for many years. Usually, they are unions of finitely many intervals with the spectrum of Lebesgue type (i.e. the spectral measure is absolutely continuous with respect to the Lebesgue measure). For instance, in the case of the Cayley graph of a free group F_m the spectrum of the Markov operator associated to the simple random walk is the interval $I_m = [-\frac{\sqrt{2m-1}}{m}, \frac{\sqrt{2m-1}}{m}]$. The spectral measure associated with the delta function δ_e, where e is the identity element of F_m, is equal to $d\mu(x) = \frac{\sqrt{2m-1-m^2x^2}}{\pi(1-x^2)}dx$ on I_m. In the case of the random walk on the set of integers with probability $1 > p > 0$ of going one step to the right and probability $q = 1 - p$ of going one step to the left the spectrum is $[-\sqrt{4pq}, \sqrt{4pq}]$ and spectral measure is equal to $d\mu(x) = \frac{dx}{\pi\sqrt{4pq-x^2}}$.

In our study we inspect spectra not only of Markov operators but more generally of so called Laplace type operators associated to weighted Schreier graphs Γ_x. Here weight can take negative or even complex values. Given an action of a group G on a set X, a point $x \in X$ and an element $m = \sum_{i=1}^{n} \alpha_i g_i \in \mathbb{C}[G]$ of the group algebra, where $n \in \mathbb{N}, \alpha_i \in \mathbb{C}, g_i \in G$, we consider the associated Laplace type operator on $l^2(Gx)$ denoted by $H_{m,x}$ (see Section 1 and formula (1)). Our main results concern the case of the action of the Thompson group F on $[0,1]$ and $m = \alpha(a+a^{-1}) + \beta(b+b^{-1}) \in \mathbb{R}[F]$, where a, b are the standard generators of F (see Subsection 1.2 for details) and $\alpha, \beta \in \mathbb{R}$. One of our main results is a description of the Kesten spectral measure of $H_{m,1/2}$ with respect to the vertex $1/2$ (Theorem 1). Another result is a description of the spectrum of $H_{m,x}$ for the case of real α, β of the same sign (Theorem 2).

For $\alpha = \beta = 1/4$ the operator $H_{m,x}$ coincides with the Markov operator of simple random walk on Γ_x. Closely related to the Markov operator M is the Laplace operator $\Delta = \mathrm{Id} - M$, where Id is the identity operator. Given a vertex v of the Schreier graph Γ_x denote by η_v the spectral measure of Δ with respect to the delta function $\delta_v \in l^2(\Gamma_x)$. Let $N_v(s) = \eta_v((-\infty, s])$ be the distribution function of η_v. These functions play important role in studying Laplace operators. In the case of amenable group the behavior of $N_v(s)$ near zero is of special interest (see [2], [23]). If $N_v(s) \sim s^\gamma$ near zero then γ is called Novikov-Shubin invariant. This parameter is also related to the so-called Lifshitz tails (see e.g. [22]). For Schreier graphs much less is known. One of our results states that for the Schreier graph Υ (see Figure 1) one has $N_{1/2}(s) \sim s^{3/2}$ near zero (Proposition 4).

For the Cayley graph of the free group $F_m, m \geqslant 2$, or d-regular rooted tree $T_d, d \geqslant 2$, probabilities $P_{v_0,v_0}^{(n)}$ of returning to the initial vertex v_0 after n steps decay exponentially as follows, for instance, from direct computations or Kesten's criterium of amenability. For $\mathbb{Z}_+^d$ the decay of these probabilities is of the power type $n^{-\frac{d}{2}}$. When the graph is

a combination of parts of non-amenable graphs and amenable graphs, the behavior of return probabilities is unclear in general. We show that for the Schreier graph Υ of the action of Thomspon group F on the orbit of $1/2$ these probabilites decay as $n^{-\frac{3}{2}}$ (Theorem 3).

Finally, we address the question about the behavior of spectrum for covering of graphs. In [9] the authors proved a theorem which they called "Hulanicki type theorem for graphs" (due to its relation to the famous Hulanicki criterion of amenability in terms of weak inclusion of the trivial representation into a regular representation). Namely, they showed that if $\Gamma_1 \to \Gamma_2$ is a covering between two weighted graphs of uniformly bounded degree and with uniformly bounded weights then the inclusion of spectra $\sigma(\Delta_2) \subset \sigma(\Delta_1)$ of the corresponding weighted Laplace operators holds in two cases: 1) Γ_1 has subexponential growth; 2) Γ_1 is amenable and Γ_2 is finite. In Appendix we show that in the second case the inclusion may fail if Γ_2 is not finite.

The paper is organized as follows. In Section 1 we recall the main definitions and formulate the main results. In Section 2 we present some graph operations and describe how they affect the generating series of return probabilities. In particular, we calculate the generating series corresponding to the graph $\Upsilon = \Gamma_{1/2}$ (see Figure 1). In Section 3 we calculate spectral measures associated to Υ using Cauchy-Stieltjes transform and prove Theorem 1. In Section 4 we study the generating series associated to simple random walk on Υ and prove Theorem 3 and Proposition 4. In Section 5 we investigate the dependence of Kesten spectral measure on the vertex of the graph and prove Theorem 2. Finally, in Section 7 we present an example of graph covering related to Hulanicki Type Theorem for Graphs from [9] that answers a question raised there.

Acknowledgements

1   Preliminaries and the main results

1.1   Laplace type operators and spectral measures.

Let G be a finitely generated group with a symmetric (i.e. $S = S^{-1}$) generating set $S = \{g_1, \ldots, g_n\}$. Assume that G acts on a set X. Given a point $x \in X$ we consider

the undirected Schreier graph Γ_x whose vertex set is the orbit Gx and the edge set is $\{\{y, g_i y\} : y \in Gx, 1 \leqslant i \leqslant n\}$. Given a linear combination of elements from S

$$m = \sum_{i=1}^{n} \alpha_i g_i \in \mathbb{C}[G]$$

we introduce a Laplace type operator $H_{m,x}$ on $l^2(Gx)$ by

$$(H_{m,x}f)(y) = \sum_{i=1}^{n} \alpha_i f(g_i^{-1}y) \quad \forall y \in Gx, \tag{1}$$

where $f \in l^2(Gx)$. In particular, if $\alpha_i = \frac{1}{n}$ for all i then $H_{m,x}$ is the Markov operator associated to the simple random walk on Γ_x.

Recall that any self-adjoint operator A on a separable Hilbert space $\mathcal{H}$ admits a spectral decomposition

$$A = \int\limits_{[-\|A\|, \|A\|]} \lambda E(\lambda),$$

where $E(\lambda)$ is a projection-valued spectral measure defined on Borel subsets of $[-\|A\|, \|A\|]$ and taking values on projections of $\mathcal{H}$. Given a vector $\xi \in \mathcal{H}$ one can define a real valued spectral measure μ on $[-\|A\|, \|A\|]$ (depending on ξ) by

$$\mu(B) = (E(B)\xi, \xi), \tag{2}$$

where B is a Borel subset of $[-\|A\|, \|A\|]$ and $(\cdot, \cdot)$ is the scalar product on $\mathcal{H}$.

In this paper we study the spectral measures $\mu_{m,x}$ of $H_{m,x}$ (in the case when these operators are self-adjoint) associated to the functions $\delta_x \in l^2(\Gamma_x)$. Such measures were introduced by Kesten in [20, 21] for the case of random walks on groups. Sometimes they are called Kesten spectral measures. In general, computing such spectral measures is a complicated task. One of results in this direction concerns the group of intermediate growth constructed by the second author in [15]. The spectral measures for the action of this group on the boundary of the binary rooted tree were computed in [16]. This result was generalized to the case of spinal groups [14].

1.2 Thompson group F.

It is known that F is generated by two elements a, b where

$$a(x) = \begin{cases} \frac{x}{2} & \text{if } x \in [0, \frac{1}{2}), \\ x - \frac{1}{4} & \text{if } x \in [\frac{1}{2}, \frac{3}{4}), \\ 2x - 1 & \text{if } x \in [\frac{3}{4}, 1]; \end{cases} \quad b(x) = \begin{cases} x & \text{if } x \in [0, \frac{1}{2}), \\ \frac{1}{2}a(2x - 1) + \frac{1}{2} & \text{if } x \in [\frac{1}{2}, 1]. \end{cases} \tag{3}$$

5

For a point $x \in (0,1)$ one can introduce a Schreier graph Γ_x associated to the action of F on the orbit Fx. These graphs were described by Savchuk in [24, 25]. We compute spectral measures of certain Laplace type operators associated to the graph $\Upsilon = \Gamma_{1/2}$. As a corollary we obtain the description of spectra of these Laplace type operators for Γ_x for all $x \in (0,1)$.
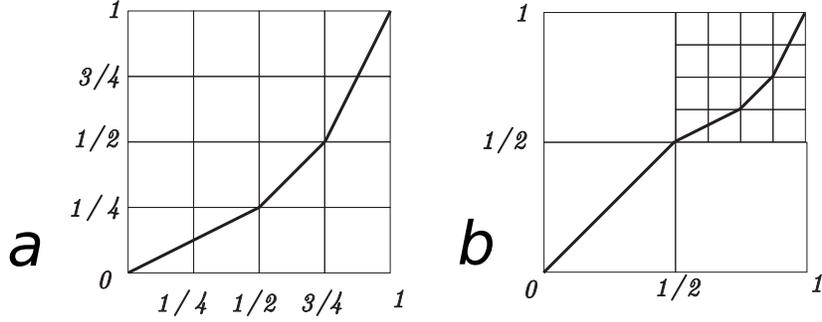
Figure 2: The graphs of the generators a, b of F

1.3   Main results.

Our main results concern spectra and spectral measures of Laplace type operators $H_{m,x}$ on $l^2(\Gamma_x)$, where

$$m = \alpha(a + a^{-1}) + \beta(b + b^{-1}) \in \mathbb{R}[F]. \tag{4}$$

Fix $\alpha, \beta \in \mathbb{R} \setminus \{0\}$. Introduce functions P_*, Q_* on $\mathbb{R}$ by the formulas

$$P_*(z) = \tfrac{z}{2} \pm \tfrac{1}{2}\sqrt{z^2 - 4(\alpha + \beta)^2}, \quad Q_*(z) = \tfrac{z}{2} + \beta \pm \tfrac{1}{2}\sqrt{z^2 - 4\beta z + 4(\beta^2 - \alpha^2)}. \tag{5}$$

Here if the expression under the root sign is nonnegative we choose the sign equal to the sign of z in the formula for $P_*(z)$ and choose the sign equal to the sign of $z - 2\beta$ in the formula for $Q_*(z)$. If the expression under the root sign is negative we choose the root with a positive imaginary part.

Consider the equation

$$\beta^4 v^3 + (P_*(z) + Q_*(z))\beta^2 v^2 + (P_*(z)Q_*(z) + (\beta^2 - \alpha^2))v + P_*(z) = 0. \tag{6}$$

We will show that for any $z \in \mathbb{R}$ it has at most one solution V with positive imaginary part. Recall that the support of a measure μ on $\mathbb{R}$ (denoted by $\operatorname{supp}(\mu)$) is the set of points $s \in \mathbb{R}$ such that for every open neighborhood U of s one has $\mu(U) > 0$. Below we list the main results of the paper.

6

Theorem 1. *The spectral measure $\mu_{m,1/2}$ of the operator $H_{m,1/2}$ on $l^2(\Gamma_{1/2})$ is absolutely continuous with respect to the Lebesgue measure λ on $\mathbb{R}$. The support of $\mu_{m,1/2}$ is the closure of the set of points z for which (6) has a solution $V = V(z)$ with positive imaginary part. In particular, $\mathrm{supp}(\mu_{m,1/2})$ is a union of at most finitely many closed intervals. For $z \in \mathrm{supp}(\mu_{m,1/2})$ one has:*

$$\frac{\mathrm{d}\mu_{m,1/2}(z)}{\mathrm{d}\lambda(z)} = \mathrm{Im}\frac{V(z)}{\pi(\beta V(z) + 1)^2}. \tag{7}$$

The Radon-Nikodym derivative above is continuous everywhere and is analytic everywhere except finitely many points z.

In this paper by $\sigma(A)$ we denote the spectrum of an operator A.

Theorem 2. *If α, β are of the same sign then for every $x \in (0, 1)$ the spectrum of $H_{m,x}$ is an interval and one has*

$$\sigma(H_{m,x}) = \mathrm{supp}(\mu_{m,x}) = [-2|\alpha + \beta|, 2|\alpha + \beta|]. \tag{8}$$

In addition, we compute the asymptotics of the return probabilities for the simple random walk on $\Upsilon = \Gamma_{\frac{1}{2}}$. As usual, let $O(s)$ stand for a function defined near zero (or infinity) such that $\limsup |O(s)/s|$ is finite when $s \to 0$ (respectively, when $s \to \infty$).

Theorem 3. *Consider the simple random walk on Υ. Let p_n be the probability starting at $\frac{1}{2}$ to return after n steps to $\frac{1}{2}$. Then there exist constants A, B such that one has:*

$$p_{2n} = An^{-\frac{3}{2}} + O(n^{-2}), \quad p_{2n+1} = Bn^{-\frac{3}{2}} + O(n^{-2}), \quad when \ n \to \infty.$$

The formula (8) shows that in the case of simple random walk (i.e. when $m = \frac{1}{4}(a + b + a^{-1} + b^{-1})$) one has $\sigma(H_{m,1/2}) = \mathrm{supp}(\mu_{m,1/2}) = [-1, 1]$. The asymptotic behavior of the spectral distribution $N_{1/2}(s)$ of the Laplace operator $\mathrm{Id} - H_{m,1/2}$ with respect to the vector $\delta_{1/2}$ is given in the text statement.

Proposition 4. *For some constant $C > 0$ one has: $N_{1/2}(s) = Cs^{\frac{3}{2}} + O(s^2)$ near zero, where $s > 0$.*

As Savchuk showed in [24], Schreir graphs Γ_x of the action of F on $[0, 1]$ are amenable. Observe also that the action of F on $[0, 1]$ is hyperfinite, i.e. the partition on orbits is an increasing union of finite Borel equivalence relations. Indeed, as it was noted in [13], the orbit equivalence relation of the action of F on $[0, 1]$ is isomorphic (modulo countable set of points) to the tail equivalence relation on $\{0, 1\}^{\mathbb{N}}$ with the standard Borel structure. The latter is hyperfinite by [6].

Given a representation π of a group G in a Hilbert space and an element $f \in \mathbb{C}[G]$ set

$$\pi(f) = \sum_{g \in \mathrm{supp}(f)} f(g)\pi(g).$$

As a corollary of Theorem 2 using Theorem 1 from [7] we obtain:

Corollary 5. *Let κ be the Koopman representation of F on $L^2([0,1], \lambda)$. Let m be as above with $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ of the same sign. Then $\sigma(\kappa(m)) = [-2|\alpha + \beta|, 2|\alpha + \beta|]$.*

As an interesting example we show:

Proposition 6. *For $\alpha = -1, \beta = 1$ one has:*

$$\mathrm{supp}(\mu_{m,1/2}) = [z_1, z_2] \cup [0, 4],$$

where $z_1 \approx -2.766, z_2 \approx -0.062$ are the real roots of the polynomial $z^4 - 2z^2 + 16z + 1$.

2 Weighted graphs and generating functions.

The aim of this section is to establish the framework for obtaining information about the moments of the spectral measures of the Laplace type operators $H_{m,x}$ described above.

2.1 General setting.

Let a countable group G act on a set X. Fix an element $m \in \mathbb{C}[G]$. We will view m as a function from G to $\mathbb{C}$ and denote by $\mathrm{supp}(m)$ the set of elements $g \in G$ such that $m(g) \neq 0$. Consider a symmetric ($S = S^{-1}$) generating set $S = \{g_1, g_2, \ldots g_n\}$ that contains $\mathrm{supp}(m)$. Given a point $x \in X$ we consider the corresponding Schreier graph Γ_x. To study the spectrum of the operator $H_{m,x}$ on $l^2(\Gamma_x)$ it is convenient to use weighted directed graphs.

Definition 7. For any undirected graph Γ we call a directed edge an edge together with a choice of its initial and terminal vertices. By weighted (directed) graph we mean a graph with an assignment of a weight $\omega_e \in \mathbb{C}$ for every directed edge e.

Note that every edge of a graph Γ gives rise to two directed edges. Thus, every edge of a graph Γ has two weights assigned to it which might or might not coincide. This also concerns the loops. Per our definition, weighted graphs are always directed.

Definition 8. Given $m \in \mathbb{C}[G]$ and any Schreier graph Γ of G we define by Γ_m the weighted graph with weight $m(g)$ assigned to each directed edge (v, gv), $v \in \Gamma, g \in S$.

Given any weighted graph Γ of bounded degree with uniformly bounded weight (i.e. there exists $D > 0$ such that $|\omega_e| \leqslant D$ for each directed edge e of Γ) one can introduce an associated Laplace type operator H_Γ as follows. For a vertex v of Γ let E_v denote the set of directed edges ending at v. For a directed edge e let i_e stand for its initial vertex. Set

$$(H_\Gamma f)(v) = \sum_{e \in E_v} \omega_e f(i_e) \ \ \forall v \in \Gamma. \tag{9}$$

Observe that uniform boundedness of the degree and of the weight implies that the operator H_Γ is bounded. Moreover, in the case of real symmetric weight the operator H_Γ is self-adjoint. In the case of the weighted graph associated to $m \in \mathbb{C}[G]$ the above Laplace type operator coincides with the one defined in (1).

Further, each path γ in Γ with a chosen starting and terminal vertices can be written as a chain of directed edges $\gamma = (e_1, \ldots, e_k)$. Set

$$\omega_\gamma = \prod_{e \in \gamma} \omega_e.$$

For a vertex v of Γ denote by $\mathcal{P}_{\Gamma,v}$ the set of all paths (including trivial) starting and ending at v. Denote by $\mathcal{F}_{\Gamma,v}$ the set of non-trivial paths from $\mathcal{P}_{\Gamma,v}$ which do not visit v except at starting and ending vertices (the paths of first-return). Let $|\gamma|$ stand for the length (total number of edges) of the path γ. For $n \geqslant 0$ set

$$\mathcal{P}_{\Gamma,v}^{(n)} = \{\gamma \in \mathcal{P}_{\Gamma,v} : |\gamma| = n\}, \ \ p_{\Gamma,v}^{(n)} = \sum_{\gamma \in \mathcal{P}_{\Gamma,v}^{(n)}} \omega_\gamma. \tag{10}$$

Similarly define $\mathcal{F}_{\Gamma,v}^{(n)}$ and $f_{\Gamma,v}^{(n)}$:

$$\mathcal{F}_{\Gamma,v}^{(n)} = \{\gamma \in \mathcal{F}_{\Gamma,v} : |\gamma| = n\}, \ \ f_{\Gamma,v}^{(n)} = \sum_{\gamma \in \mathcal{F}_{\Gamma,v}^{(n)}} \omega_\gamma. \tag{11}$$

Note that in the case when ω_e are transition probabilities for a random walk on Γ the value $p_{\Gamma,v}^{(n)}$ (correspondingly $f_{\Gamma,v}^{(n)}$) is the probability that starting at v the random walk after n steps returns to v (correspondingly, first time returns to v). For any weighted graph Γ one has

$$p_{\Gamma,v}^{(n)} = (H_\Gamma^n \delta_v, \delta_v) \tag{12}$$

where $\delta_v \in l^2(\Gamma)$ is the Kroneker delta-function at the vertex v. In particular, if H_Γ is self-adjoint then $p_{\Gamma,v}^{(n)}$ are the moments of the spectral measure of H_Γ corresponding to δ_v.

9

Now, introduce the generating functions

$$P_{\Gamma,v} = P_{\Gamma,v}(t) = \sum_{\gamma \in \mathcal{P}_{\Gamma,v}} \omega_\gamma t^{|\gamma|}, \quad F_{\Gamma,v} = F_{\Gamma,v}(t) = \sum_{\gamma \in \mathcal{F}_{\Gamma,v}} \omega_\gamma t^{|\gamma|}. \tag{13}$$

In the case when $\Gamma = \Gamma_m$ is a Schreier graph and m is a symmetric probability distribution on the set of generators the generating function $P_{\Gamma,v}$ is the Green's function of the corresponding random walk on Γ. The following relations are well known:

Lemma 9. *One has*

$$P_{\Gamma,v} = \frac{1}{1 - F_{\Gamma,v}}, \quad F_{\Gamma,v} = 1 - \frac{1}{P_{\Gamma,v}}.$$

Remark 10. In the case of a graph of a bounded degree with uniformly bounded weight for any t such that $|t| < \|H_\Gamma\|^{-1}$ the series $P_{\Gamma,v}(t)$ and $F_{\Gamma,v}(t)$ converge and moreover

$$P_{\Gamma,v}(t) = ((I - tH_\Gamma)^{-1}\delta_v, \delta_v).$$

Further, for a graph Γ (possibly, weighted) and a vertex v of Γ denote by Γ_v the corresponding rooted graph. For a rooted graph Γ_v denote by $\delta + \Gamma_v$ the rooted graph obtained by adding a new vertex δ (serving as a new root) to Γ_v connected by a single edge to v. In case of a weighted graph we keep the weights on the edges of Γ and equip the new edge joining δ and w with new weights $\omega_{(\delta,w)}, \omega_{(w,\delta)} \in \mathbb{C}$. Thus, for a weighted graph Γ the result of the operation $"\delta + "$ depends on two complex numbers $\omega_{(\delta,w)}$ and $\omega_{(w,\delta)}$. We do not incorporate these numbers in the notation of the operation to simplify the notations.

For two graphs Γ and Δ (possibly, weighted) and two vertices $v \in \Gamma, w \in \Delta$ denote by $\Gamma_v \cup \Delta_w$ the rooted graph obtained by taking disjoint copies of Γ and Δ and identifying v and w. If Γ and Δ are weighted graphs the graph $\Gamma_v \cup \Delta_w$ is a weighted graph with the weights inherited from Γ and Δ. Note that the operation $" \cup "$ is commutative and associative.

To describe the Schreier graphs of Thompson group it is convenient to introduce an additional operation $\star$:

$$\Gamma_v \star \Delta_w = \Gamma_v \cup (\delta + \Delta_w). \tag{14}$$

It is not hard to see on simple examples that the operation $"\star"$ is neither commutative nor associative. Note that, by construction, the results of operations $"\delta + "$, $" \cup "$ and $"\star"$ are rooted graphs (weighted if the graphs involved in the operations are weighted). These operations are illustrated on Figure 3.

Lemma 11. *One has*

$$F_{\Gamma_v \cup \Delta_u, v} = F_{\Gamma,v} + F_{\Delta,u}, \quad F_{\delta + \Gamma_v, \delta} = \omega_{(\delta,v)}\omega_{(v,\delta)}t^2 P_{\Gamma,v}, \quad F_{\Gamma_v \star \Delta_u, v} = F_{\Gamma,v} + \omega_{(v,u)}\omega_{(u,v)}t^2 P_{\Delta,u}$$

10
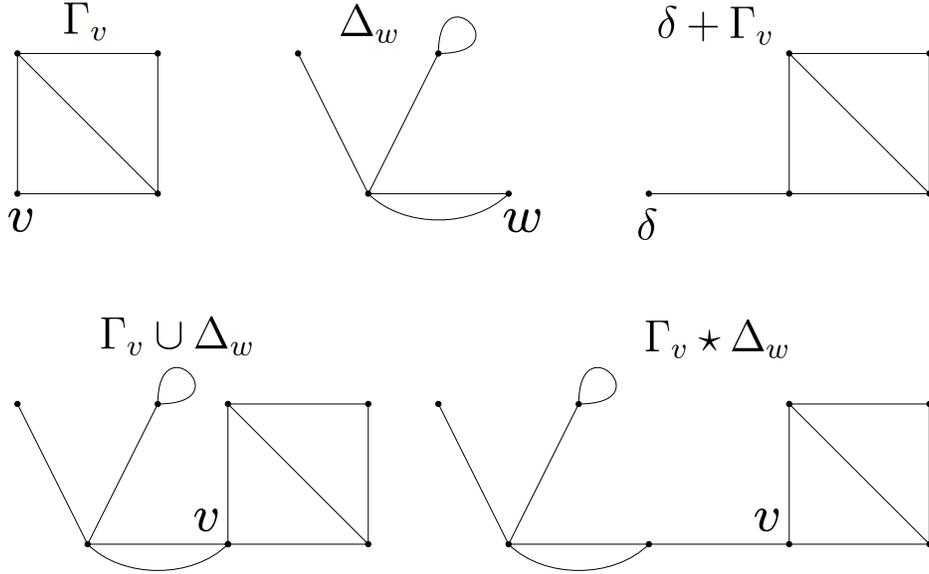
Figure 3: Illustration to operations on graphs.

Proof. Since Γ and Δ as subgraphs of $\Gamma_v \cup \Delta_u$ intersect only at one vertex v any non-trivial first return path $\gamma \in \Gamma_v \cup \Delta_u$ at the vertex v either belongs to Γ or belongs to Δ, and the two possibilities are mutually exclusive. Thus, $\mathcal{F}_{\Gamma_v \cup \Delta_u, v}$ can be written as a disjoint union of $\mathcal{F}_{\Gamma, v}$ and $\mathcal{F}_{\Delta, u}$. By definition of the first-return series (13) this implies the first formula of Lemma 11.

To prove the second formula of Lemma 11 observe that any first return path γ at the vertex δ in $\delta + \Gamma_v$ starts from the edge (δ, v), follows a path γ_1 in Γ (starting and ending at v), and finishes by the edge (v, δ).

The third formula of Lemma 11 follows from (14) and the combination of the first two. $\qquad\square$

2.2 Weighted Schreier graphs of the action of F on $[0, 1]$.

In this subsection we give a description of the building blocks of Schreier graphs Γ_x of the action of F on $[0, 1]$ and obtain a description of $\Upsilon = \Gamma_{1/2}$ (see Figure 1) in terms of the operation $\star$ and simpler graphs which are subgraphs of Υ. For all graphs we construct below we assume the property that $\omega_{(v,w)} = \omega_{(w,v)}$ for any edge $e = (v, w)$.

Introduce directed graphs Φ and Ψ with vertex sets labeled by positive and negative integers correspondingly such that:
• in Φ every two consecutive integers are connected by one edge, every integer $i > 1$ has a loop attached to it and there are no other edges;

11

• in Ψ every two consecutive integers are connected by two edges and there are no other edges.

Fix two nonzero real numbers α, β. We make Φ and Ψ weighted graphs by:
• marking all directed "horizontal" (i.e. joining consecutive integers) edges in Φ by α and all directed loops by β;
• for every i marking one edge of Ψ between i and $i+1$ by α (in both directions) and another by β (in both directions).

The shapes of graphs Φ and Ψ are shown at Figure 4.

Lemma 12. *There exists a unique connected weighted rooted graph Δ_v with countably many vertices such that*
$$\Delta_v = (\Phi_1 \star \Delta_v) \star (\Psi_{-1} \star \Delta_v) \tag{15}$$
and, in addition, $\omega_{(1,v)} = \omega_{(-1,v)} = \beta$, $\omega_{(-1,1)} = \alpha$.

The graph Δ_v from (15) is illustrated at Figure 5.

Proof. Using the equation (15) one can construct Δ_v as a union of increasing sequence of weighted rooted graphs Γ_n defined inductively. We start with a graph Γ_0 consisting of a single vertex v which will be the root of Δ_v. On step n we will add some vertices and weighted edges to already constructed graph Γ_n. Namely, for each leaf w (vertex adjacent to one edge) of Γ_n we do the following:

- attach a new copy of Φ to w so that the vertex 1 coincide with w;

- connect w by an edge with weight β to a new vertex w_1;

- connect w by an edge with weight α to a vertex -1 of a new copy of Ψ;

- connect -1 by an edge with weight β to a new vertex w_2.

Denote the obtained graph by Γ_{n+1}. Note that for each leaf w of Γ_n the vertices w_1, w_2 are leaves of Γ_{n+1}.

Figure 4: Graphs Ψ and Φ. Every black edge has weight α and every grey edge has weight β.
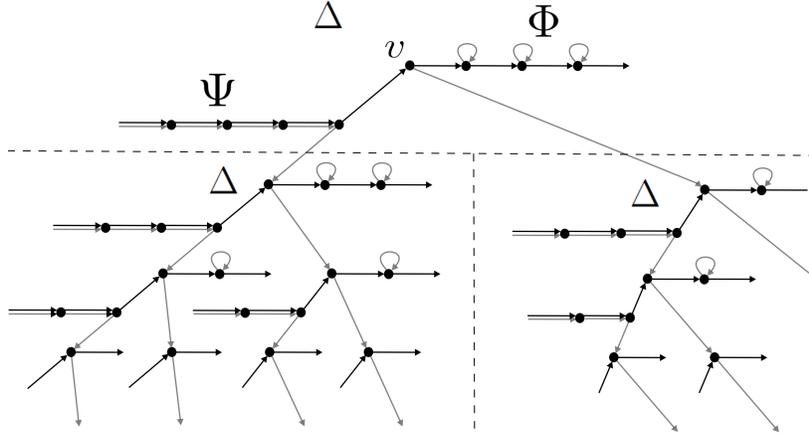
Figure 5: Illustration to formula (15). Every black edge has weight α and every grey edge has weight β.

It is straightforward to verify that $\Gamma_\infty = \bigcup\limits_{n=1}^\infty \Gamma_n$ satisfies (15) and any connected weighted rooted graph Δ_v with countably many vertices satisfying (15) is isomorphic to Γ_∞. $\square$

As before let a, b be the generators of the Thompson group F and $\alpha, \beta \in \mathbb{R} \setminus \{0\}$. Let $m = m_{\alpha,\beta} = \alpha(a + a^{-1}) + \beta(b + b^{-1}) \in \mathbb{R}[F]$. Recall that one of the main objects of investigation of this article is the graph $\Upsilon = \Gamma_{1/2,m}$, which is the weighted Schreier graph of the action of F on $[0, 1]$ at point $\frac{1}{2}$ with the weight defined by $m_{\alpha,\beta}$ (see Figure 1). Using the description of Υ given in [24] (Proposition 1), and [25] (Proposition 2.1), we obtain:

Corollary 13. *Let Δ_v be the graph from Lemma 12. Then*

$$\Upsilon = \tilde{\Phi}_1 \star (\Psi_{-1} \star \Delta_v) \tag{16}$$

with $\omega_{(-1,v)} = \beta$ and $\omega_{(-1,1)} = \alpha$, where $\tilde{\Phi}_1$ is obtained from Φ_1 by adding a loop at the vertex 1 with the weight β.

2.3 Computing the generating functions.

To simplify notations set

$$q(t) = F_{\Phi,1} - 1, \quad p(t) = F_{\Psi,-1}(t) - 1, \quad x(t) = P_{\Delta,v}(t) \text{ (see (13))}.$$

We will drop the variable t where it does not create confusion. For instance, we will write p, q, x instead of $p(t), q(t), x(t)$. Note that the functions $p(t), q(t)$ are different

from the functions $P_*(z), Q_*(z)$ defined in (5) (see also the formula (26)). In particular, the variables z and t are related by $z = t^{-1}$.

By Remark 10, $q(t), p(t)$ and $x(t)$ are convergent for t near the origin, and thus they are elements of the ring $\mathbb{C}\{t\}$ of germs of complex analytic functions near $0 \in \mathbb{C}$.

Lemma 14. *In the space $\mathbb{C}\{t\}$ of germs at the origin one has:*

$$q = -\tfrac{1}{2} - \beta t - \tfrac{1}{2}\sqrt{(2\beta t - 1)^2 - 4\alpha^2 t^2}, \quad p = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - 4(\alpha + \beta)^2 t^2}, \quad (17)$$

where the branch of the radical is chosen so that $\sqrt{1} = 1$.

Proof. Observe that Φ_1 is isomorphic as rooted graph to $\delta + \tilde{\Phi}_1$, where δ is the graph consisting of one vertex without edges. Recall that $\tilde{\Phi}_1$ is obtained from Φ_1 by adding a loop at 1 with weight β. The weight of the edge connecting $\tilde{\Phi}$ to δ is equal to α. In the graph $\tilde{\Phi}$ any 1-first-return path is either a 1-first-return path of Φ or is equal to the loop at 1 in one of the two possible directions. It follows that

$$F_{\tilde{\Phi},1} = 2\beta t + F_{\Phi,1}. \quad (18)$$

Using Lemmas 11 and 9 we obtain:

$$F_{\Phi,1} = \alpha^2 t^2 P_{\tilde{\Phi},1} = \frac{\alpha^2 t^2}{1 - F_{\tilde{\Phi},1}} = \frac{\alpha^2 t^2}{1 - 2\beta t - F_{\Phi,1}}$$

from which taking into account that $F_{\Phi,1}(0) = 0$ the formula for $q = F_{\Phi,1} - 1$ follows.

To prove the second formula note that the weighted graph Ψ is isomorphic to the graph obtained by connecting the vertex -1 of a copy of Ψ by two edges with weights α and β to the one-vertex graph δ. It follows that

$$F_{\Psi,-1} = (\alpha + \beta)^2 t^2 P_{\Psi,-1} = \frac{(\alpha + \beta)^2 t^2}{1 - F_{\Psi,-1}}.$$

Solving the latter equation for $F_{\Psi,-1}$ we prove the formula for $p = F_{\Psi,-1} - 1$. $\square$

Observe that $x(0) = 1$ and $p(0) = -1$ and thus the formulas

$$\frac{1}{x(t)} \quad \text{and} \quad \frac{\alpha^2 t^2}{p(t) + \beta^2 t^2 x(t)}$$

define germs of analytic functions near the origin (i.e. elements of $\mathbb{C}\{t\}$).

Proposition 15. *In the ring $\mathbb{C}\{t\}$ one has:*

$$q + \frac{1}{x} + \beta^2 t^2 x - \frac{\alpha^2 t^2}{p + \beta^2 t^2 x} = 0. \quad (19)$$

Equivalently,

$$\beta^4 t^4 x^3 + (p + q)\beta^2 t^2 x^2 + (pq + (\beta^2 - \alpha^2)t^2)x + p = 0. \quad (20)$$

Proof. Using Lemmas 11 and 12 we obtain:

$$F_{\Delta,v} = F_{\Phi_1 \star \Delta_v, 1} + \alpha^2 t^2 P_{\Psi_{-1} \star \Delta_v, -1} = F_{\Phi,1} + \beta^2 t^2 P_{\Delta,v} + \frac{\alpha^2 t^2}{1 - F_{\Psi_{-1} \star \Delta_v, -1}},$$

$$F_{\Psi_{-1} \star \Delta_v, -1} = F_{\Psi, -1} + \beta^2 t^2 P_{\Delta,v}.$$

Setting $x = P_{\Delta,v}$ using Lemma 9 we arrive at:

$$\frac{x-1}{x} = q + 1 + \beta^2 t^2 x - \frac{\alpha^2 t^2}{p + \beta^2 t^2 x},$$

which is equivalent to the equation (19). $\qquad\square$

Corollary 16. *The following holds:*

$$F_{\Upsilon, 1/2} = 1 + 2\beta t - \beta^2 t^2 x - \tfrac{1}{x}.$$

Proof. By Corollary 13 and Lemmas 9 and 11 one has:

$$F_{\Upsilon, 1/2} = F_{\tilde{\Phi}, 1} + \frac{\alpha^2 t^2}{1 - F_{\Psi_{-1} \star \Delta_v, -1}},$$

$$F_{\Psi_{-1} \star \Delta_v, -1} = F_{\Psi, -1} + \beta^2 t^2 P_{\Delta,v}.$$

Combining these equations using (18) and (19) we obtain the desired. $\qquad\square$

3 Proofs of Theorem 1 and Proposition 6.

3.1 Cauchy-Stieltjes transform.

Let μ be a finite measure on a segment $I = [s_1, s_2] \subset \mathbb{R}$. Its Cauchy-Stieltjes transform is defined by

$$S_\mu(z) = \int_I \frac{\mathrm{d}\mu(y)}{y - z}, \quad z \in \mathbb{C} \setminus \mathrm{supp}(\mu).$$

It is well-known that the function $S_\mu(z)$ is complex analytic on $\mathbb{C} \setminus \mathrm{supp}(\mu)$. Since for $y \in \mathbb{R}$ and $z \in \mathbb{C} \setminus \mathbb{R}$ the sign of $\mathrm{Im}\frac{1}{y-z}$ coincides with the sign of $\mathrm{Im}z$ the definition of Cauchy-Stieltjes transform implies that

$$\mathrm{Im}S_\mu(z) > 0 \text{ if } \mathrm{Im}z > 0, \quad \mathrm{Im}S_\mu(z) < 0 \text{ if } \mathrm{Im}z < 0. \tag{21}$$

Introduce the moments and the generating function of the measure μ by:

$$m_k = \int_I y^k \mathrm{d}\mu(y), \quad k \geqslant 0, \quad R_\mu(t) = \sum_{k \geqslant 0} m_k t^k.$$

Observe that
$$S_\mu(z) = -z^{-1}R_\mu(z^{-1}) \ \text{if} \ |z| > \max\{|s_1|, |s_2|\}. \tag{22}$$

Recall that λ stands for the Lebesgue measure on $\mathbb{R}$. For a finite measure μ on $\mathbb{R}$ denote by G_μ its distribution function:

$$G_\mu(s) = \mu((-\infty, s]), \ \ s \in \mathbb{R}. \tag{23}$$

Our computation of spectral measures relies on the following result from [26]:

Theorem 17. *[Silverstein-Choi] Suppose that for some $s \in I$ the limit $S_+(s) = \lim\limits_{z \to s, \text{Im} z > 0} \text{Im} S_\mu(z)$ exists. Then the distribution function G_μ is differentiable at s and $G'_\mu(s) = \frac{1}{\pi}S_+(s)$.*

Theorem 17 implies the following:

Corollary 18. *Let $s \in \mathbb{R}$. Then $s \notin \text{supp}(\mu)$ if and only if there exists an open segment J containing s such that*
$$\lim_{z \to z_0, \text{Im} z > 0} \text{Im} S_\mu(z) = 0$$
for every $z_0 \in J$.

Proof. Indeed, if such segment J exists, then by Theorem 17 G_μ is constant on J, therefore $\mu(J) = 0$ and $s \notin \text{supp}(\mu)$. On the other hand, if $s \notin \text{supp}(\mu)$ then there exists an open segment J containing s which does not intersect $\text{supp}(\mu)$. By definition of Cauchy-Stieltjes transform $S_\mu(z)$ is well-defined, analytic, and has real values on J. It follows that $S_+(z) = 0$ on J. $\square$

Though the domain of definition of $S_\mu(z)$ is $\mathbb{C} \setminus \text{supp}(\mu)$ it is possible in some cases to continue analytically S_μ to neighborhoods of points from $\text{supp}(\mu)$. To study these cases let us introduce an auxiliary

Definition 19. We will say that S_μ has an isolated singularity at a point $z_0 \in \text{supp}(\mu)$ if it admits an analytic continuation from $\mathbb{H}_+ = \{z \in \mathbb{C} : \text{Re}(z) > 0\}$ to a punctured neighborhood of z_0. In other words, there exists an open neighborhood $U \subset \mathbb{C}$ of z_0 and an analytic function ϕ on $U \setminus \{z_0\}$ such that $\phi(z) = S_\mu(z)$ for all $z \in U \cap \mathbb{H}_+$.

If z_0 is a pole of ϕ we will say that S_μ has a pole at z_0.

If $\phi(z) \in \mathbb{R}$ for all $z \in (U \cap \mathbb{R}) \setminus \{z_0\}$ we will say that S_μ has a real isolated singularity at z_0.

Remark 20. Since S_μ is real on the real line one has $S_\mu(\bar{z}) = \overline{(S_\mu(z))}$ for all $z \notin I$. Moreover, (21) implies that all zeros of S_μ on $\mathbb{R} \setminus \text{supp}(\mu)$ are simple and all real isolated singularities of S_μ on $\text{supp}(\mu)$ are simple poles.

16

Now, let Γ be a weighted graph of bounded degree with a uniformly bounded weight, let v be a vertex of Γ, and let $\mu_{\Gamma,v}$ be the correpsonding spectral measure of H_Γ (see (9)). Then the nth moment m_n of $\mu_{\Gamma,v}$ is equal to $p_{\Gamma,v}^{(n)}$ for every $n \in \mathbb{Z}_+$ (see (10) and (12)). Therefore, $R_\mu(t) = P_{\Gamma,v}(t)$. In particular, we have:

$$S_{\mu_{\Gamma,v}}(z) = -z^{-1}P_{\Gamma,v}(z^{-1}) \ \text{ if } \ |z| > \|H_\Gamma\|. \tag{24}$$

3.2 The spectral measure $\mu_{\Delta,v}$ of H_Δ.

Formulas (17) define analytic functions $p(t), q(t)$ near zero which admit analytic continuations to the lower half-plane $\mathbb{H}_- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$. We then extend these functions by continuity onto the real line. We denote the obtained functions on $\overline{\mathbb{H}}_-$ by $P(t)$ and $Q(t)$ correspondingly.

Recall that $x(t)$ (see Subsection 2.3) is a convergent power series in t near 0. From (22) we deduce that x admits an analytic continuation to $\mathbb{H}_-$. Denote the obtained function on $\mathbb{H}_-$ by $X(t)$. Using (20) and Cardano formulas we obtain that $X(t)$ extends to a continuous function on $\overline{\mathbb{H}}_-$. Moreover,

$$\beta^4 t^4 X^3(t) + (P(t)+Q(t))\beta^2 t^2 X^2(t) + (P(t)Q(t)+(\beta^2-\alpha^2)t^2)X(t) + P(t) = 0 \tag{25}$$

for all $t \in \overline{\mathbb{H}}_-$.

Further, let

$$P_*(z) = -zP(z^{-1}), \quad Q_*(z) = -zQ(z^{-1}), \quad V(z) = -z^{-1}X(z^{-1}) \tag{26}$$

for $z \in \overline{\mathbb{H}}_+ \setminus \{0\}$.

Remark 21. By (22) one has

$$V(z) = S_{\mu_\Delta}(z) \ \text{ on } \ \mathbb{H}_+,$$

where $\mu_\Delta = \mu_{\Delta,v}$ is the spectral measure of H_Δ corresponding to the vector δ_v. In particular, $\operatorname{Im}V(z) > 0$ for $z \in \mathbb{H}_+$.

Observe that for any $z \in \overline{\mathbb{H}}_+ \setminus \{0\}$ one has:

$$P_*(z) = \tfrac{z}{2} \pm \tfrac{1}{2}\sqrt{z^2 - 4(\alpha+\beta)^2}, \quad Q_*(z) = \tfrac{z}{2} + \beta \pm \tfrac{1}{2}\sqrt{z^2 - 4\beta z + 4(\beta^2-\alpha^2)}. \tag{27}$$

It is not hard to see that P_* and Q_* extend by continuity to 0.

From (25) we obtain that for all z from $\overline{\mathbb{H}}_+ \setminus \{0\}$ the value $V(z)$ is a solution of the following equation:

$$\beta^4 v^3 + (P_*(z)+Q_*(z))\beta^2 v^2 + (P_*(z)Q_*(z)+(\beta^2-\alpha^2))v + P_*(z) = 0. \tag{28}$$

17

If β is nonzero then using Cardano formulas we obtain that $V(z)$ extends by continuity to 0. If $\beta = 0$ and $\alpha \neq 0$ then

$$P_*(z) = Q_*(z) = \tfrac{z}{2} \pm \tfrac{1}{2}\sqrt{z^2 - 4\alpha^2}, \ \text{ and so } \ V(z) = \frac{P_*(z)}{\alpha^2 - P_*^2(z)}$$

also extends by continuity to 0. Using Theorem 17 we obtain:

Corollary 22. *The spectral measure μ_Δ is absolutely continuous with respect to the Lebesgue measure λ. Moreover,*

$$\frac{\mathrm{d}\mu_\Delta(z)}{\mathrm{d}\lambda(z)} = \tfrac{1}{\pi}\mathrm{Im}V(z) \ \text{ for } \ z \in \mathbb{R}.$$

Lemma 23. *For $z \in \overline{\mathbb{H}_+}$ unless $z = 2(\alpha + \beta) = 0$ one has*

$$P_*(z) \neq 0, \ V(z) \neq 0, \ P_*(z) + \beta^2 V(z) \neq 0$$

and, moreover, $V(z)$ is a solution of the equation

$$Q_*(z) + \frac{1}{v} + \beta^2 v - \frac{\alpha^2}{P_*(z) + \beta^2 v} = 0. \tag{29}$$

Proof. Assume $z \in \overline{\mathbb{H}_+}$ and either $z \neq 0$ or $z \neq 2(\alpha + \beta)$. Then $P_*(z) \neq 0$ by (17), (26) and (27). Consequently, $V(z) \neq 0$ by (28). Now, since $\alpha^2 V(z) \neq 0$ from (28) we derive that $P_*(z) + \beta^2 V(z) \neq 0$. Formula (29) follows from (28). $\square$

3.3 The spectral measure $\mu_{m,1/2}$ of H_m.

As before assume that $\alpha, \beta \in \mathbb{R} \setminus \{0\}$. To simplify the notations set $y = P_{\Upsilon,1/2}$, where as before Υ is the Schreier graph of the action of the Thompson group F on the orbit of $1/2$. Then $y(t)$ is a convergent series near the origin. Set $W(z) = -z^{-1}y(z^{-1})$. By (24) one has $W(z) = S_{\mu_m}(z)$, where $\mu_m = \mu_{m,1/2}$ is the spectral measure of the operator H_m corresponding to the vector $\delta_{\frac{1}{2}}$.

Proposition 24. *$W(z)$ extends to a continuous function on $\overline{\mathbb{H}_+}$ analytic on $\mathbb{H}_+$. One has:*

$$W(z) = \frac{V(z)}{(\beta V(z) + 1)^2}. \tag{30}$$

For $z \in \mathbb{R}$ $\mathrm{Im}W(z) > 0$ if and only if $\mathrm{Im}V(z) > 0$.

Proof. From Lemma 9 and Corollary 16 we obtain:

$$y = P_{\Upsilon,1/2} = \frac{1}{1 - F_{\Upsilon,1/2}} = \frac{1}{\beta^2 t^2 x + \frac{1}{x} - 2\beta t}.$$

Substituting $W(z) = -z^{-1}y(z^{-1})$ and $V(z) = -z^{-1}x(z^{-1})$ in the latter formula we obtain formula (30).

If $\beta V(z_0) + 1 = 0$ for some $z_0 \in \mathbb{R}$ then $W(z)$ has a pole of order at least 2 at z_0. This contradicts to Remark 20 and shows that $\beta V(z) + 1 \neq 0$ for all $z \in \mathbb{R}$.

Further, from (30) we obtain that $W(z)$ is real whenever $V(z)$ is real. From (29) we obtain:

$$\frac{1}{W(z)} = -Q_*(z) + 2\beta + \frac{\alpha^2}{P_*(z) + \beta^2 V(z)}.$$

Since $\operatorname{Im} P_*(z) \geqslant 0$ and $\operatorname{Im} Q_*(z) \geqslant 0$ for $z \in \overline{\mathbb{H}}_+$ we obtain that $\operatorname{Im} W(z) > 0$ whenever $\operatorname{Im} V(z) > 0$. This finishes the proof. $\qquad\square$

3.4   Spectrum of H_Δ

Lemma 25. *Assume that one of the following two conditions hold:*

1) $z \in \mathbb{H}_+$;

2) $z \in \mathbb{R}$ *such that* $|\alpha + \beta| > \frac{|z|}{2}$ *or* $|\alpha| > |\frac{z}{2} - \beta|$,

then the equation (28) *has no real solutions.*

Proof. From Lemma 14 and (26) we deduce that $\operatorname{Im} P_*(z) > 0$ and $\operatorname{Im} Q_*(z) > 0$ on $\mathbb{H}_+$. In addition, for $z \in \mathbb{R}$ one has

- if $|\alpha + \beta| > \frac{|z|}{2}$ then $\operatorname{Im} P_*(z) > 0$;

- if $|\alpha| > |\frac{z}{2} - \beta|$ then $\operatorname{Im} Q_*(z) > 0$.

Let $v \in \mathbb{R}$ be a solution of (28). If z satisfies at least one of the conditions of Proposition 25 then $v \neq 0$ and $P_*(z) + \beta^2 v \neq 0$ (see the proof of Lemma 23). Therefore (29) is true and so

$$\operatorname{Im}\left(Q_*(z) - \frac{\alpha^2}{P_*(z) + \beta^2 v}\right) = 0.$$

This contradicts to the fact that $\operatorname{Im} P_*(z) \geqslant 0$, $\operatorname{Im} Q_*(z) \geqslant 0$ and at least one of the inequalities is strict. $\qquad\square$

Recall that Δ is the subgraph of the Schreier graph of the action of F on the orbit of $1/2$ defined in Lemma 12 and illustrated on Figure 5.

19

Proposition 26. *If $\alpha, \beta \neq 0$ are of the same sign then the support of the spectral measure $\mu_\Delta = \mu_{\Delta,v}$ of the Laplace type operator H_Δ (see (9)) is $[-2|\alpha + \beta|, 2|\alpha + \beta|]$ and coincides with the spectrum of H_Δ.*

Proof. Without loss of generality assume that $\alpha, \beta > 0$. Lemma 25, Corollary 18 and Remark 21 imply that $\sigma(H_\Delta) \supset [-2(\alpha + \beta), 2(\alpha + \beta)]$. On the other hand, since for every vertex of Δ the sum of the weights of adjacent edges is at most $2(\alpha + \beta)$ we have $\|H_\Delta\| \leqslant 2(\alpha + \beta)$. It follows that $\sigma(H_\Delta) \subset [-2(\alpha + \beta), 2(\alpha + \beta)]$. $\qquad\square$

Proposition 27. *For any $\alpha, \beta \in \mathbb{R}$ at least one of which is nonzero and any $z \in \overline{\mathbb{H}}_+$ the equation (28) has at most one solution $v = v(z)$ (counting multiplicity) such that $\mathrm{Im}(v) > 0$. For $z \in \mathbb{H}_+$ such solution is exactly one and coincides with $V(z)$. If $z \in \mathbb{R}$ and such solution exists then it coincides with $V(z)$.*

Proof. If $\beta = 0, \alpha \neq 0$ then the equation (28) has only one solution. Let $\beta \neq 0$. Without loss of generality we may assume that $\beta = 1$. Let z_0 be such that (28) for $z = z_0$ has two solutions (counting multiplicity) with positive imaginary part. If $z_0 \in \mathbb{R}$ using either Cardano's formulas or Implicit Function Theorem we obtain that one can find near z_0 a point $z_1 \in \mathbb{H}_+$ satisfying the same condition. Therefore, without loss of generality we may assume that $z_0 \in \mathbb{H}_+$. Since the coefficients of (28) depend continuously on α Cardano's formulas together with Lemma 25 imply that for $\alpha = 0$ and $z = z_0$ (28) also have two solutions with positive imaginary part. Finally, same arguments show that for $z = 0$ there are also two solutions with positive imaginary part. But for $\alpha = 0, \beta = 1$ one has $P_*(0) = i, Q_*(0) = 0$ and the equation (28) for $z = 0$ has the following form:

$$v^3 + iv^2 + v + i = 0.$$

The latter equation has one simple solution $v = i$ with positive imaginary part. This contradiction together with Remark 21 finishes the proof. $\qquad\square$

Recall that a cubic equation with real coefficients has imaginary roots if and only its discriminant is negative. Denote by $D(z)$ the discriminant of the equation (28):

$$D(z) = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2, \qquad (31)$$

where $a = \beta^4, b = \beta^2(P_*(z) + Q_*(z)), c = P_*(z)Q_*(z) + \beta^2 - \alpha^2, d = P_*(z)$. Using Corollary 18 and Remark 21, Lemma 25 and Proposition 27 we obtain that

$$\mathrm{supp}(\mu_\Delta) = \overline{\{z \in \mathbb{R} : |\alpha + \beta| > \tfrac{|z|}{2} \text{ or } |\alpha| > |\tfrac{z}{2} - \beta| \text{ or } D(z) > 0\}}. \qquad (32)$$

Let us prove an auxiliary lemma.

Lemma 28. *Let $U \subset C$ be a domain, $p_1(z), \ldots, p_n(z)$ be complex polynomials and $s_1(z), \ldots s_n(z)$ be analytic functions on U such that $s_j^2(z) = p_j(z)$, $j = 1, \ldots, n$, for all $z \in U$. Let R be a polynomial in $n + 1$ variable. Then the equation*

$$R(z, s_1(z), \ldots, s_n(z)) = 0 \tag{33}$$

either holds for all $z \in U$ or has only finitely many solutions on U.

Proof. The equation (33) can be written in the form

$$s_n(z) R_1(z, s_1(z), \ldots, s_{n-1}(z)) = R_2(z, s_1(z), \ldots, s_{n-1}(z)),$$

where R_1, R_2 are polynomials of n variables. Thus, if z is a solution of (33) we have:

$$p_n^2(z) R_1^2(z, s_1(z), \ldots, s_{n-1}(z)) - R_2^2(z, s_1(z), \ldots, s_{n-1}(z)) = 0.$$

From the latter one can complete the proof by induction on n. $\qquad\square$

Let I be one of the open intervals (finite or infinite) on which the points $-2(\alpha + \beta), 2(\alpha + \beta), 2(\beta - \alpha)$ divide $\mathbb{R}$. Let U be an open simply-connected set containing I but not containing any of the points $-2(\alpha + \beta), 2(\alpha + \beta), 2(\beta - \alpha)$. From Lemma 28 we obtain that $D(z) = 0$ either a) has only finitely many solutions on U or b) is true for all $z \in U$. In case a) set $S_{\alpha,\beta} = \{z \in \mathbb{R} : D(z) = 0\}$. By analyticity and continuity arguments the case b) would imply that $D(z) = 0$ on $\overline{\mathbb{H}_+}$. In this case we set $S_{\alpha,\beta} = \varnothing$.

Using Theorem 17 we deduce the following:

Proposition 29. *For any $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ the set $\mathrm{supp}(\mu_\Delta)$ is a union of at most finite number of closed intervals with the end-points belonging to $S_{\alpha,\beta} \cup \{-2(\alpha + \beta), 2(\alpha + \beta), 2(\beta - \alpha)\}$.*

3.5   Proof of Theorem 1.

By (22) one has $S_{\mu_m}(z) = W(z)$ on $\mathbb{H}_+$. Using Theorem 17, Proposition 24 and Corollary 18 we obtain that

$$\mathrm{supp}(\mu_m) = \overline{\{z \in \mathbb{R} : \mathrm{Im} W(z) > 0\}} = \overline{\{z \in \mathbb{R} : \mathrm{Im} V(z) > 0\}} = \mathrm{supp}(\mu_\Delta). \tag{34}$$

Moreover, μ_m is absolutely continuous with respect to the Lebesgue measure λ on $\mathbb{R}$ and (7) is true. Finally, from Proposition 29 we obtain that $\mathrm{supp}(\mu_m)$ is a union of a finite number of closed intervals.

21

3.6 Proof of Proposition 6.

Let $\alpha = -1, \beta = 1$. Lemma 25 and Theorem 17 imply that $\mathrm{supp}(\mu_\Delta) \supset [0, 4]$. Observe that for $z \notin [0, 4]$ the coefficients of the cubic equation (28) are real. Clearly, $\mathrm{supp}(\mu_\Delta) \subset [-4, 4]$. For $z \in [-4, 0)$ the discriminant $D(z)$ of (25) (see (31)) is of the form:

$$D(z) = -4z - 14z^2 + 2z^4 + \sqrt{z(z-4)}(6z - 4z^2 - 2z^3).$$

Algebraic transformations show that the set of zeros of $D(z)$ on $[-4, 0)$ is a subset of the set of roots of $z^4 - 2z^2 + 16z + 1$. The latter equation has two real roots z_1, z_2 defined in the formulation of Proposition 6. One can check that z_1, z_2 are zeros of $D(z)$. Moreover on $(-4, 0) \setminus [z_1, z_2]$ the function $D(z)$ is positive and so (28) has three real solutions. On (z_1, z_2) the function $D(z)$ is negative and so (28) has one real and two complex conjugate solutions with nonzero imaginary part. Using Proposition 27 we obtain that $\mathrm{supp}(\mu_\Delta) = [z_1, z_2] \cup [0, 4]$. Finally, using (34) we obtain Proposition 6.

4 On simple random walk on Υ: proofs of Theorem 3 and Proposition 4.

Let $\alpha = \beta = 1/4$ so that the operator $H_{m,1/2}$ is the Markov operator associated to the simple random walk on $\Upsilon = \Gamma_{1/2}$. The moment generating function for the operator $H_{m,1/2}$ is $P_{\Upsilon,1/2}$. Recall that by Lemma 9 and Corollary 16 we have:

$$P_{\Upsilon,1/2} = \frac{1}{1 - F_{\Upsilon,1/2}}, \quad F_{\Upsilon,1/2} = 1 + \tfrac{t}{2} - \tfrac{1}{16}t^2x^2 - \tfrac{1}{x}, \tag{35}$$

where $x(t)$ is a solution of the following equation (see Proposition 15 and Lemma 14):

$$\tfrac{1}{256}t^4x^3 + \tfrac{1}{16}(p + q)t^2x^2 + pqx + p = 0. \tag{36}$$

By (17) one has

$$q(t) = -\tfrac{1}{2} - \tfrac{t}{4} - \tfrac{1}{2}\sqrt{1 - t}, \quad p(t) = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - t^2}.$$

To study the **behavior of $x(t)$ near** $t = 1$ we substitute $t = 1 - s^2$. We obtain for $t \leqslant 1$ and $s \geqslant 0$:

$$q(t) = -\frac{3}{4} + \frac{1}{4}s^2 - \frac{1}{2}s = -\frac{3}{4} - \frac{1}{2}s + O(s^2), \quad p(t) = -\frac{1}{2} - \frac{1}{2}s\sqrt{2 - s^2} = -\frac{1}{2} - \frac{\sqrt{2}}{2}s + O(s^3).$$

Let $u(s) = x(1 - s^2)$. The equation (36) can be rewritten as:

$$F(u, s) := a_0(s)u^3 + a_1(s)u^2 + a_2(s)u + a_3(s) = 0,$$

$$a_0(s) = \frac{1}{256} + O(s^2), \quad a_1(s) = -\frac{5}{64} - \frac{(\sqrt{2} + 1)s}{32} + O(s^2),$$

$$a_2(s) = \frac{3}{8} + \frac{(2 + 3\sqrt{2})s}{8} + O(s^2), \quad a_3(s) = -\frac{1}{2} - \frac{\sqrt{2}s}{2} + O(s^2),$$

where $a_i(s)$ are analytic near $s = 0$. The solutions of the cubic equation $F(u, 0) = 0$ are:

$$u_1 = 4(2 + \sqrt{2}), \quad u_2 = 4, \quad u_3 = 4(2 - \sqrt{2}).$$

One has:

$$\frac{\partial F}{\partial u}\Big|_{s=0} = \frac{3}{256}u^2 - \frac{5}{32}u + \frac{3}{8}.$$

The latter expression is nonzero at $u = u_j$ for $j = 1, 2, 3$. We also calculate:

$$\frac{\partial F}{\partial s}\Big|_{s=0} = -\frac{1 + \sqrt{2}}{32}u^2 + \frac{2 + 3\sqrt{2}}{8}u - \frac{\sqrt{2}}{2}.$$

By Implicit Function Theorem we have that for each $j = 1, 2, 3$ the equation $F(u, s) = 0$ has an analytic solution $U_j(s)$ near $s = 0$ such that $U_j(0) = u_j$. Moreover,

$$U_j'(0) = -\frac{\partial F}{\partial s}(u_j, 0)\Big/\frac{\partial F}{\partial u}(u_j, 0) = \frac{\frac{1 + \sqrt{2}}{32}u_j^2 - \frac{2 + 3\sqrt{2}}{8}u_j + \frac{\sqrt{2}}{2}}{\frac{3}{256}u_j^2 - \frac{5}{32}u_j + \frac{3}{8}}.$$

Substituting the values of u_j in the above formula we obtain:

$$U_1'(0) \approx 9.657, \quad U_2'(0) \approx 19.314, \quad U_3'(0) \approx -9.657.$$

Recall that the coefficients of $x(t)$ are the return probabilities of a random walk, and so are nonnegative. It follows that $x'(t) \geq 0$ for $0 \leq t < 1$. We obtain that $u'(s) = -2sx'(t) \leq 0$ for $1 > s > 0$. Therefore, $u(s) = U_3(s)$. Thus, $x(1 - s^2) = u(s) = C_1 + C_2 s + O(s^2)$, where $C_1 = 4(2 - \sqrt{2}), C_2 \approx -9.657$. We arrive at $x(t) = C_1 + C_2\sqrt{1 - t} + O(1 - t)$ for t near 1.

Similarly, to study **the behavior of** $x(t)$ **near** $t = -1$ we substitute $t = s^2 - 1$, $s \geq 0$. We have

$$p(t) = -\frac{1}{2} - \frac{1}{2}s\sqrt{2 - s^2} = -\frac{1}{2} - \frac{\sqrt{2}}{2}s + O(s^2),$$

$$q(t) = -\frac{1}{2} - \frac{s^2 - 1}{4} - \frac{1}{2}\sqrt{2 - s^2} = -\frac{1}{4} - \frac{\sqrt{2}}{2} + O(s^2).$$

23

Now, let $u(s) = x(s^2 - 1)$. The equation (36) becomes

$$F(u, s) := b_0(s)u^3 + b_1(s)u^2 + b_2(s)u + b_3(s) = 0, \quad \text{where}$$

$$b_0(s) = \frac{1}{256} + O(s^2), \quad b_1(s) = -\frac{3 + 2\sqrt{2}}{64} - \frac{\sqrt{2}}{32}s + O(s^2), \qquad (37)$$

$$b_2(s) = \frac{1 + 2\sqrt{2}}{8} + \frac{4 + \sqrt{2}}{8}s + O(s^2), \quad b_3(s) = -\frac{1}{2} - \frac{\sqrt{2}}{2}s + O(s^2),$$

and $b_1(s), b_2(s), b_3(s)$ are analytic near $s = 0$.

The equation (37) for $s = 0$ has the following form:

$$\frac{1}{256}u^3 - \frac{3 + 2\sqrt{2}}{64}u^2 + \frac{1 + 2\sqrt{2}}{8}u - \frac{1}{2} = 0. \qquad (38)$$

The roots are:

$$u_1 = 6 + 2\sqrt{2} - 2\sqrt{11 + 2\sqrt{2}} \approx 1.3911, \quad u_2 = 4\sqrt{2} \approx 5.657,$$

$$u_3 = 6 + 2\sqrt{2} + 2\sqrt{11 + 2\sqrt{2}} \approx 16.267. \qquad (39)$$

One has:

$$\frac{\partial F}{\partial u}\Big|_{s=0} = \frac{3}{256}u^2 - \frac{3 + 2\sqrt{2}}{32}u + \frac{1 + 2\sqrt{2}}{8}.$$

The latter expression is nonzero at u_1, u_2, u_3. We have:

$$\frac{\partial F}{\partial s}\Big|_{s=0} = -\frac{\sqrt{2}}{32}u^2 + \frac{4 + \sqrt{2}}{8}u - \frac{\sqrt{2}}{2}.$$

By Implicit Function Theorem we have that for each $j = 1, 2, 3$ the equation $F(u, s) = 0$ has an analytic solution $U_j(s)$ near $s = 0$ such that $U_j(0) = u_j$. We have $x(s^2 - 1) = U_j(s)$ for some j for $s \geqslant 0$ sufficiently close to zero. Since the Taylor coefficients of $x(t)$ are nonnegative, we have $u_j = x(-1) < x(1) = 4(2 - \sqrt{2}) \approx 2.343$. Comparing the latter with (39) we obtain that $j = 1$ and $x(s^2 - 1) = U_1(s)$. Further,

$$U_1'(0) = -\frac{\partial F}{\partial s}(u_1, 0) \Big/ \frac{\partial F}{\partial u}(u_1, 0) = \frac{\frac{\sqrt{2}}{32}u_1^2 - \frac{4 + \sqrt{2}}{8}u_1 + \frac{\sqrt{2}}{2}}{\frac{3}{256}u_1^2 - \frac{3 + 2\sqrt{2}}{32}u_1 + \frac{1 + 2\sqrt{2}}{8}} \approx -0.6.$$

In particular, $U_1'(0) \approx -0.6$. We obtain that $x(t) = C_3 + C_4\sqrt{1 + t} + O(1 + t)$ for t near -1 with $C_3 = 6 + 2\sqrt{2} - 2\sqrt{11 + 2\sqrt{2}} \approx 1.3911, C_4 \approx -0.6$.

Now, using formulas (35) we obtain, that $P_{\Upsilon,1/2}(t)$ has near ± 1 the asymptotical behavior similar to the asymptotical behavior of $x(t)$:

$$P_{\Upsilon,1/2}(t) = \widetilde{C}_1 + \widetilde{C}_2\sqrt{1 - t} + O(1 - t) \quad \text{near} \quad t = 1,$$

$$P_{\Upsilon,1/2}(t) = \widetilde{C}_3 + \widetilde{C}_4\sqrt{1 + t} + O(1 + t) \quad \text{near} \quad t = -1 \qquad (40)$$

24

for some constants $\widetilde{C}_j \in \mathbb{R}$, $1 \leqslant j \leqslant 4$. Further, let $P_{\Upsilon,1/2}(t) = \sum\limits_{n=0}^{\infty} p_n t^n$. The function

$$G(s) = \tfrac{1}{2}(P_{\Upsilon,1/2}(\sqrt{s}) + P_{\Upsilon,1/2}(-\sqrt{s})) = \sum_{n=0}^{\infty} p_{2n} s^n$$

is analytic in a region of the form $D_\epsilon = \{s \in \mathbb{C} : |s| < 1+\epsilon\} \setminus [1, 1+\epsilon)$ for some $\epsilon > 0$ and has the following asymptotical behavior near $s = 1$:

$$G(s) = \tfrac{1}{2}(\widetilde{C}_1 + \widetilde{C}_3) + \tfrac{1}{2}(\widetilde{C}_2 + \widetilde{C}_4)\sqrt{1-s} + O(1-s).$$

Using results of Flajolet-Odlyzko [10] (see also Theorem of Section 16.8 in [27]) we obtain that $p_{2n} = -\tfrac{1}{2}(\widetilde{C}_2 + \widetilde{C}_4)n^{-3/2} + O(n^{-2})$ when $n \to \infty$. Similarly, we obtain that $p_{2n+1} = -\tfrac{1}{2}(\widetilde{C}_2 - \widetilde{C}_4)n^{-3/2} + O(n^{-2})$ when $n \to \infty$. This finishes the proof of Theorem 3.

From the asymptotics (40) of $P_{\Upsilon,1/2}(t)$ near $t = 1$ and (24) we obtain:

$$S_{\mu_{\Upsilon,1/2}}(z) = -\widetilde{C}_1 - \widetilde{C}_2\sqrt{z-1} + O(z-1)$$

near $z = 1$. Notice that (40) and the fact that $P_{\Upsilon,1/2}(t)$ is increasing on $[0,1)$ implies that $\widetilde{C}_2 < 0$. Using Theorem 17 we get

$$\frac{\mathrm{d}\mu_{\Upsilon,1/2}(z)}{\mathrm{d}\lambda(z)} = -\tfrac{1}{\pi}\widetilde{C}_2\sqrt{1-z} + O(1-z)$$

for $z < 1$ near $z = 1$. Integrating we arrive at:

$$\mu_{\Upsilon,1/2}(1-z, 1] = -\tfrac{2}{3\pi}\widetilde{C}_2(1-z)^{\frac{3}{2}} + O(1-z)^2,$$

which finishes the proof of Proposition 4.

5 Inclusion of spectra and dependence of spectral measure μ_v on v.

Here we study the relation between the spectrum of the Laplace type operator and the supports of the spectral measures for a certain class of graphs including Schreier graphs of the action of F on $[0,1]$.

Observe that for the simple random walk on a Cayley graph of a finitely generated countable group spectral measures corresponding to different vertices coincide and their support is equal to the spectrum of the Markov operator (see [21]). However,

25

for Laplace type operators on general graphs (or even on Schreier graphs) this is not always true. Below we provide two examples.

Let Γ be a finite weighted graph with real symmetric weight. Consider the corresponding Laplace-type operator $L = H_\Gamma$. One can find the spectral measure $\mu_v = \mu_{\Gamma,v}$ corresponding to a vertex v as follows. Since L is self-adjoint, it has a spectral decomposition. Let $\lambda_i, \xi_i, i = 1, \ldots, |\Gamma|$, be the eigenvalues and the eigenvectors of L, where $|\Gamma|$ is the number of vertices of Γ. We normalize ξ_i to have $\|\xi_i\| = 1$, $i = 1, \ldots, |\Gamma|$. Let

$$\delta_v = \sum \alpha_i \xi_i$$

be the decomposition of δ_v over the basis $\{\xi_i\}$ of $l^2(\Gamma)$. Then the spectral measure μ_v is supported on the set $\{\lambda_i : 1 \leqslant i \leqslant n, \alpha_i \neq 0\}$ and

$$\mu_v(\{\lambda_i\}) = |\alpha_i|^2 = |(\delta_v, \xi_i)|^2 \ \text{ for } \ 1 \leqslant i \leqslant n. \tag{41}$$

Example 30. Let Γ be the undirected graph with the vertex set $V = \{-2, -1, 0, 1, 2\}$ and the edge set $E = \{(-2, -1), (-1, 0), (0, 1), (1, 2)\}$. Assign weight 1 to each edge. Then the corresponding Laplace-type operator can be written as a 5×5 matrix with 1 at each entry directly above or below the diagonal and 0 every where else. The eigenvalues and the eigenvectors are:

$$\lambda_1 = 0, \ \xi_1 = \tfrac{1}{\sqrt{3}}(1, 0, -1, 0, 1), \ \lambda_2 = 1, \ \xi_2 = \tfrac{1}{\sqrt{3}}(-1, -1, 0, 1, 1),$$
$$\lambda_3 = -1, \ \xi_3 = \tfrac{1}{2}(-1, 1, 0, -1, 1), \ \lambda_4 = \sqrt{3}, \ \xi_4 = \tfrac{1}{2\sqrt{3}}(1, \sqrt{3}, 2, \sqrt{3}, 1),$$
$$\lambda_5 = -\sqrt{3}, \ \xi_5 = \tfrac{1}{2\sqrt{3}}(1, -\sqrt{3}, 2, -\sqrt{3}, 1).$$

From (41) we obtain

$$\mathrm{supp}(\mu_{-2}) = \mathrm{supp}(\mu_2) = \{\lambda_i : 1 \leqslant i \leqslant 5\},$$
$$\mathrm{supp}(\mu_{-1}) = \mathrm{supp}(\mu_1) = \{\lambda_i : 2 \leqslant i \leqslant 5\}, \ \mathrm{supp}(\mu_0) = \{\lambda_1, \lambda_4, \lambda_5\}.$$

Remark 31. As Pierre de la Harpe pointed to the authors, the graph with three vertices $V = \{-1, 0, 1\}$ and the edge set $E = \{\{-1, 0\}, \{0, 1\}\}$ also has this property, that for some vertex (namely, the vertex 0) the support of the spectral measure is a strict subset of the spectrum of the Markov operator.

Example 32. Consider the Schreier graph Γ of the action of the Hanoi Towers group $H^{(3)}$ on the second level $V_2 \simeq \{0, 1, 2\}^2$ of the ternary rooted tree T_3 (see [11]). This Schreier graph is shown on Figure 6. Let M be the Markov operator corresponding to the simple random walk on Γ. One can show that $-2/3$ is a simple eigenvalue of M with the eigenvector given by:

$$\alpha_{00} = \alpha_{11} = \alpha_{22} = 0, \ \ \alpha_{10} = \alpha_{02} = \alpha_{21} = 1, \ \ \alpha_{20} = \alpha_{12} = \alpha_{01} = -1.$$
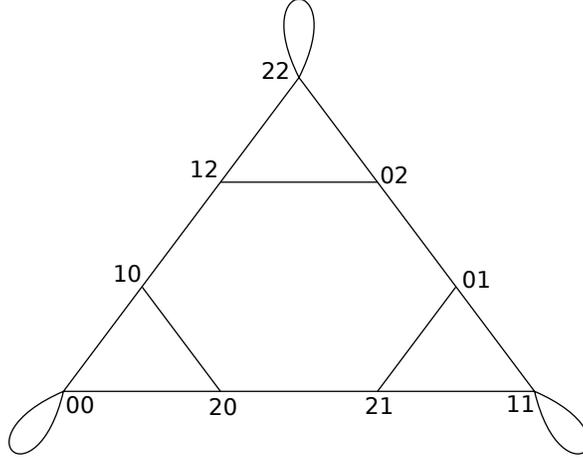
Figure 6: The Schreier graph of the action of $H^{(3)}$ on V_2.

This eigenvector is orthogonal to δ_{00} but not orthogonal to δ_{10}. Thus, by (41) $-2/3$ is contained in the support of μ_{10} but not in the support of μ_{00}.

Definition 33. Let us call a connected graph Γ *tree-like* if for every pair of adjacent vertices $v \neq w$ the graph obtained by removing from Γ all the edges joining v and w is disconnected.

Note that a connected graph is tree-like if and only if after removing all loops and replacing multiple edges by single edges it becomes a tree. It is not hard to see that the Schreier graphs of the action of F on $[0, 1]$ are tree-like.

Lemma 34. *Let Γ be a weighted graph of bounded degree with real symmetric uniformly bounded weight. Let $a \in \mathbb{R}$ and let v be a vertex of Γ. Then a is an isolated atom of $\mu_{\Gamma,v}$ if and only if a is a real pole of $S_{\mu_{\Gamma,v}}$ (per Definition 19). Moreover, if the above equivalent conditions hold then a is an eigenvalue of H_Γ.*

Proof. Assume that a is an isolated atom of $\mu := \mu_{\Gamma,v}$. Using the formula for S_μ:

$$S_\mu(z) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu(y)}{y - z}$$

(see Section 3.1 for details) we immediately obtain that a is a real pole of S_μ. Let $E_{\{a\}}$ be the spectral orthogonal projection of H_Γ corresponding to the one-point set $\{a\}$. Then $(E_{\{a\}}\delta_v, \delta_v) = \mu(\{a\}) \neq 0$ and so $E_{\{a\}}\delta_v \neq 0$. One has $H_\Gamma E_{\{a\}}\delta_v = a E_{\{a\}}\delta_v$. Thus, a is an eigenvalue of H_Γ.

Now assume that a is a real pole of S_μ. By Corollary 18 and Remark 20, $a \in \mathrm{supp}(\mu)$ and there is an open neighborhood $U \subset \mathbb{R}$ of a such that $\mu(U \setminus \{a\}) = 0$. Thus, a is an isolated atom of μ. $\square$

27

Note that, as example 30 shows, it is possible even for a tree-like graph that H_Γ has an eigenvalue a which does not belong to $\mathrm{supp}(\mu_{\Gamma,v})$ for some vertex v.

The next statement shows that for the graphs under considerations supports of the measures $\mu_{\Gamma,v}$ for different vertices v may differ only by isolated points representing eigenvalues. Let $A \triangle B$ stand for the symmetric difference of sets A, B.

Proposition 35. *Let Γ be a connected tree-like weighted graph of bounded degree with real symmetric uniformly bounded weights and v, w be two vertices of Γ. Then $\mathrm{supp}(\mu_{\Gamma,v}) \triangle \mathrm{supp}(\mu_{\Gamma,w})$ is a subset of the set of eigenvalues of H_Γ.*

Proof. Clearly, it is sufficient to prove the statement under the assumption that v, w are two adjacent vertices of Γ. In addition, if v and w are connected by more than one edge we will replace these edges by one edge with the weight $\alpha_{(v,w)}$ equal to the sum of the weights between v and w. The latter does not change either operator H_Γ or measures $\mu_{\Gamma,v}$, $\mu_{\Gamma,w}$. Thus, we reduce the general case to the case with v and w connected by exactly one edge.

Let Γ^v and Γ^w be the connected components of the graph obtained by removing from Γ the edge connecting v and w. Recall that for a graph Ω and a vertex u of Ω the notation Ω_u means the corresponding rooted graph with the root at u. One has:

$$\Gamma_v = \Gamma_v^v \cup (\delta + \Gamma_w^w), \ \ \Gamma_w = (\delta + \Gamma_v^v) \cup \Gamma_w^w.$$

Using Lemma 11 we obtain:

$$F_{\Gamma,v}(t) = F_{\Gamma^v,v}(t) + \alpha_{(v,w)}^2 t^2 P_{\Gamma^w,w}(t), \ \ F_{\Gamma,w}(t) = F_{\Gamma^w,w}(t) + \alpha_{(v,w)}^2 t^2 P_{\Gamma^v,v}(t). \quad (42)$$

Using Lemma 9 and (24) setting $z = t^{-1}$ we arrive at the following equations:

$$\frac{1}{S_{\mu_{\Gamma,v}}(z)} = \frac{1}{S_{\mu_{\Gamma^v,v}}(z)} - \alpha_{(v,w)}^2 S_{\mu_{\Gamma^w,w}}(z), \ \ \frac{1}{S_{\mu_{\Gamma,w}}(z)} = \frac{1}{S_{\mu_{\Gamma^w,w}}(z)} - \alpha_{(v,w)}^2 S_{\mu_{\Gamma^v,v}}(z), \quad (43)$$

whenever all terms of the corresponding equation are defined. Let $I \subset \mathbb{R}$ be an open segment which does not intersect $\mathrm{supp}(\mu_{\Gamma,v})$. By definition of Cauchy-Stieltjes transform (see Subsection 3.1) the function $S_{\mu_{\Gamma,v}}(z)$ is real analytic on I (i.e. is analytic and has real values on I). In particular, the set $Z = \{z \in I : S_{\mu_{\Gamma,v}}(z) = 0\}$ does not have any limit points in I.

Further, for every $z_0 \in I \setminus Z$ one has

$$\lim_{z \to z_0} \mathrm{Im} \frac{1}{S_{\mu_{\Gamma,v}}(z)} = 0.$$

Taking into account (21) from the first equation of (43) we obtain:

$$\lim_{z \to z_0, \mathrm{Im} z > 0} \mathrm{Im} S_{\mu_{\Gamma^w,w}}(z) = 0.$$

28

By Corollary 18, the latter implies that $I \setminus Z$ does not intersect $\text{supp}(\mu_{\Gamma^w,w})$. It follows that $S_{\mu_{\Gamma^w,w}}(z)$ is real analytic on $I \setminus Z$. Therefore, the function $\frac{1}{S_{\mu_{\Gamma,v}}(z)} + \alpha^2_{(v,w)} S_{\mu_{\Gamma^w,w}}(z)$ is real analytic on $I \setminus Z$. By Remark 20 each of the points $z \in Z$ is either a removable singularity or a simple pole of $S_{\mu_{\Gamma^w,w}}$. We conclude that the set

$$Z_1 = \left\{ z \in I : \frac{1}{S_{\mu_{\Gamma,v}}(z)} + \alpha^2_{(v,w)} S_{\mu_{\Gamma^w,w}}(z) = 0, \text{ or } S_{\mu_{\Gamma,v}}(z) = 0, \text{ or } S_{\mu_{\Gamma^w,w}}(z) = 0 \right\}$$

does not have any limit points in I.

Now, using the first equation of (43) we obtain that $S_{\mu_{\Gamma^v,v}}(z)$ is nonzero and real analytic on $I \setminus Z_1$. Using the second equation of (43) we obtain that $\frac{1}{S_{\mu_{\Gamma,w}}(z)}$ is real analytic on $I \setminus Z_1$, and so $S_{\mu_{\Gamma,w}}(z)$ is real analytic on $I \setminus Z_1$ except possibly a set of poles without a limit point in I. By Remark 20 each of the points $z \in Z_1$ is either a point of analyticity or a simple pole of $S_{\mu_{\Gamma,w}}$. It follows that the set Z_2 of poles of $S_{\mu_{\Gamma,w}}$ in I does not have limit points in I. From Corollary 18 we obtain that $\text{supp}(\mu_{\Gamma,w}) \cap (I \setminus Z_2) = \varnothing$. Using Lemma 34 we derive that Z_2 is a subset of the set of eigenvalues of H_Γ.

Thus, we obtain that $\text{supp}(\mu_{\Gamma,w}) \setminus \text{supp}(\mu_{\Gamma,v})$ is a subset of the set of eigenvalues of H_Γ. Observing that the choice of the vertices v and w is symmetric, we finish the proof. $\square$

The following folklore statement is a consequence of the application of Spectral Theorem to self-adjoint Laplace type operators.

Proposition 36. *Let Γ be a weighted graph of bounded degree with real symmetric uniformly bounded weights. Then*

$$\sigma(H_\Gamma) = \overline{\bigcup_{v \in \Gamma} \text{supp}(\mu_v)}.$$

Proof. Let $\lambda \in \sigma(H_\Gamma)$. Then for any open interval I containing λ the corresponding spectral orthogonal projection E_I of H_Γ is nonzero, and so there exists a vertex v such that $\mu_v(I) = (E_I \delta_v, \delta_v) \neq 0$. This means that $I \cap \text{supp}(\mu_v) \neq \varnothing$. We obtain that $\lambda \in \overline{\bigcup_{v \in \Gamma} \text{supp}(\mu_v)}$.

On the other hand, if $\lambda \in \overline{\bigcup_{v \in \Gamma} \text{supp}(\mu_v)}$ then for any open interval I containing λ there exists a vertex v of Γ such that $\mu_v(I) \neq 0$, and so $E_I \neq 0$. This implies that $\lambda \in \sigma(H_\Gamma)$. $\square$

Recall that Υ is the Schreier graph of the action of the Thompson group F on the orbit of $1/2$ (see Figure 1) and Δ is the subgraph of Υ defined in Section 2.2 (see Figure 5).

29

Proposition 37. *Let $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ be of the same sign. Then the operators H_Υ and H_Δ have no eigenvalues.*

Proof. Let us show that H_Υ has no eigenvalues. Dividing all weights of H_Υ by α we reduce the general case to the case $\alpha = 1$. Assume that H_Υ has an eigenvalue λ. Note that $\lambda \in \mathbb{R}$ since H_Υ is self-adjoint. Let $f \in l^2(\Upsilon)$ be an eigenvector corresponding to λ. In particular, f is not identically equal to zero. Let u be a vertex of Υ such that $f(u) \neq 0$. The vertex u has a copy of at least one of the graphs Φ and Ψ attached to it (see Figure 1). In fact, it is not hard to see that both cases occur for vertices u with $f(u) \neq 0$.

Case 1. Assume that a copy of Ψ is attached to u. We label the vertices of this copy of Ψ by negative integers as in Subsection 2.2. Then the values of f satisfy the equation:

$$(1 + \beta)f(-j) - \lambda f(-j-1) + (1 + \beta)f(-j-2) = 0, \quad j \in \mathbb{N}.$$

The roots of the characteristic equation of this recursion are

$$t_{1,2} = \frac{\lambda \pm \sqrt{\lambda^2 - 4(1+\beta)^2}}{2(1+\beta)}.$$

It follows that

$$f(-j) = c_1 t_1^j + c_2 t_2^j, \quad j \in \mathbb{N},$$

where c_1, c_2 are constants. Note that $t_1 t_2 = 1$. Moreover, if $|\lambda| \leqslant 2(1+\beta)$ then $|t_1| = |t_2| = 1$. If $|\lambda| > 2(1+\beta)$ then only one of the roots

$$t_1 := \frac{\lambda - \text{sign}(\lambda)\sqrt{\lambda^2 - 4(1+\beta)^2}}{2(1+\beta)}$$

has absolute value less than 1. Since $f \in l^2(\Upsilon)$ we obtain that $|\lambda| > 2(1+\beta)$ and $f(j) = c_1 t_1^j$.

Further, the vertex $u = 1$ has three neighbors: the vertex labeled by 2 and two other vertices (possibly, coinciding) which we denote by u_1 and u_2 correspondingly. We assume that the edge connecting u and u_1 has weight 1 and the edge connecting u and u_2 has weight β. One has: $f(u_1) + \beta f(u_2) + (1+\beta)f(2) = \lambda f(1)$. Taking into account that $|\lambda| > 2(1+\beta)$ and $|f(2)| < |f(1)|$ we obtain that

$$\min\{|f(u_1)|, |f(u_2)|\} > |f(1)|.$$

Case 2. Assume that a copy of Φ is attached to u. We label the vertices of this copy of Φ by positive integers as in Subsection 2.2. Then the values of f satisfy the equation:

$$f(j) + (2\beta - \lambda)f(j+1) + f(j+2) = 0, \quad j \in \mathbb{N}.$$

30

Solving the recurrent relation we obtain

$$f(j) = c_1 t_1^j + c_2 t_2^j, \quad t_{1,2} = \frac{\lambda - 2\beta \pm \sqrt{(\lambda - 2\beta)^2 - 4}}{2}, \quad j \in \mathbb{N},$$

where c_1, c_2 are constants. Observe that if $|\lambda - 2\beta| \leqslant 2$ then $|t_1| = |t_2| = 1$. If $|\lambda - 2\beta| > 2$ then only one of the roots

$$t_1 := \frac{\lambda - 2\beta - \mathrm{sign}(\lambda - 2\beta)\sqrt{(\lambda - 2\beta)^2 - 4}}{2}$$

has absolute value less than 1. Since $f \in l^2(\Upsilon)$ we obtain that $|\lambda - 2\beta| > 2$ and $f(j) = c_1 t_1^j$.

Further, let the neighbors of u outside the copy of Φ be u_1, u_2, u_3 (some of them might coincide), where the edge between u and u_3 has weight 1. Then

$$\beta f(u_1) + \beta f(u_2) + f(u_3) + f(2) = \lambda f(1).$$

Taking into account that $|\lambda| > 2(1 + \beta)$ (see case 1), $\beta > 0$ and $|f(2)| < |f(1)|$ we obtain that $\min\{|f(u_1)|, |f(u_2)|, |f(u_3)|\} > |f(1)|$.

Thus, for every vertex u such that $f(u) \neq 0$ there is an adjacent vertex w such that $|f(w)| > |f(u)|$. This implies that $f \notin l^2(\Upsilon)$. This contradiction shows that H_Υ has no eigenvalues.

The case of operator H_Δ can be treated similarly. The only difference from the above arguments is that in case 2 it is possible that $u = 1$ have only two neighbors u_1, u_2 outside Φ (precisely when $u = v$, see Figure 5). Then $\beta f(u_1) + f(u_2) + f(2) = \lambda f(1)$ and we conclude that $\min\{|f(u_1)|, |f(u_2)|\} > |f(1)|$. $\qquad\square$

For a graph Γ and a vertex v of Γ let $B_n(v)$ stand for the ball of radius n around v in Γ. That is, $B_n(v) = (V_n(v), E_n(v))$, where $V_n(v)$ is the set of all vertices which can be reached by a path from v of length at most n in Γ, $E_n(v)$ is the set of all edges from Γ with both endpoints in $V_n(v)$.

Definition 38. For two uniformly bounded weighted graphs Γ_1, Γ_2 we will say that Γ_1 is locally contained in Γ_2 if for every vertex $v \in \Gamma_1$ and every $n \in \mathbb{N}$ there exists $w \in \Gamma_2$ such that $B_n(v) \subset \Gamma_1$ is isomorphic as a weighted graph to $B_n(w) \subset \Gamma_2$.

Let us recall a useful Lemma from [8].

Lemma 39. *Let A be any bounded nonzero linear operator on a Hilbert space and $R \geqslant 2\|A\|$. Then the following assertions are equivalent:*

1) $\lambda \in \sigma(A)$,

2) $1 \in \sigma(\mathrm{I} - \frac{1}{R^2}(A - \lambda\mathrm{I})(A - \lambda\mathrm{I})^*) \cup \sigma(\mathrm{I} - \frac{1}{R^2}(A - \lambda\mathrm{I})^*(A - \lambda\mathrm{I}))$,

where I *is the identity operator.*

The following statement is a straightforward generalization of Proposition 11 from [7] (which also appears in [8] as Proposition 1).

Proposition 40. *Let Γ_1, Γ_2 be two graphs of bounded degree with uniformly bounded weights such that Γ_1 is locally contained in Γ_2. Then $\sigma(H_{\Gamma_1}) \subset \sigma(H_{\Gamma_2})$.*

Proof. Let $\lambda \in \sigma(H_{\Gamma_1})$. Take $R > 2\max\{\|H_{\Gamma_1}\|, \|H_{\Gamma_2}\|\}$. Using Lemma 39 we obtain that

$$1 \in \sigma(\mathrm{I} - \tfrac{1}{R^2}(H_{\Gamma_1} - \lambda\mathrm{I})(H_{\Gamma_1} - \lambda\mathrm{I})^*) \cup \sigma(\mathrm{I} - \tfrac{1}{R^2}(H_{\Gamma_1} - \lambda\mathrm{I})^*(H_{\Gamma_1} - \lambda\mathrm{I})).$$

Assume, for instance, that $1 \in \sigma(\mathrm{I} - \frac{1}{R^2}(H_{\Gamma_1} - \lambda\mathrm{I})(H_{\Gamma_1} - \lambda\mathrm{I})^*)$ (the second case can be treated similarly). Denote by V_i, $i = 1, 2$, the vertex set of Γ_i. Using the operations on graphs introduced in [8] we define for $i = 1, 2$ the graph $\widetilde{\Gamma}_i$ with the vertex set V_i such that

$$H_{\widetilde{\Gamma}_i} = \mathrm{I} - \tfrac{1}{R^2}(H_{\Gamma_i} - \lambda\mathrm{I})(H_{\Gamma_i} - \lambda\mathrm{I})^*, \quad i = 1, 2.$$

Namely, we set

$$\widetilde{\Gamma}_i = 1 - \tfrac{1}{R^2}(\Gamma_i - \lambda) \circ (\Gamma_i - \lambda)^*, \quad i = 1, 2.$$

We refer the reader to Lemma 4 in [8] for details.

The graph $\widetilde{\Gamma}_1$ is locally contained in the graph $\widetilde{\Gamma}_2$. The operators $H_{\widetilde{\Gamma}_i}, i = 1, 2$ are positive of norm at most 1. In fact, $\|H_{\widetilde{\Gamma}_1}\| = 1$ since $1 \in H_{\widetilde{\Gamma}_1}$ by our assumptions. Let $\epsilon > 0$. Since

$$\sup_{\xi: \|\xi\|=1} (H_{\widetilde{\Gamma}_1}\xi, \xi) = 1$$

we can find $v \in V_1$, $l \in \mathbb{N}$ and a vector $\eta \in l^2(V_1)$ supported on $B_l(v) \subset \widetilde{\Gamma}_1$ such that $(H_{\widetilde{\Gamma}_1}\eta, \eta) > 1 - \epsilon$. Let $w \in V_2$ be such that $B_{l+1}(w) \subset \widetilde{\Gamma}_2$ is isomorphic (as a rooted weighted graph) to $B_{l+1}(v)$. Let $\eta' \in l^2(B_l(w)) \subset l^2(V_2)$ be a copy of η via this isomorphism. We get:

$$(H_{\widetilde{\Gamma}_2}\eta', \eta') = (H_{\widetilde{\Gamma}_1}\eta, \eta) > 1 - \epsilon.$$

Since $\epsilon > 0$ is arbitrary it follows that $\|H_{\widetilde{\Gamma}_2}\| \geqslant 1$. By construction, $\|H_{\widetilde{\Gamma}_2}\| \leqslant 1$ and therefore $\|H_{\widetilde{\Gamma}_2}\| = 1$. By positivity of $H_{\widetilde{\Gamma}_2}$ we have $1 \in \sigma(H_{\widetilde{\Gamma}_2})$. Using Lemma 39 we obtain that $\lambda \in \sigma(H_{\Gamma_2})$ which finishes the proof. $\square$

6   Proof of Theorem 2.

Recall that in Theorem 2 we assume that $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ are of the same sign and $x \in (0,1)$. Let $\Gamma = \Gamma_x$ be the corresponding weighted Schreier graph of the action of F and $\mu_x = \mu_{m,x}$ be the spectral measure of Γ corresponding to the vector δ_x. We need to show that

$$\operatorname{supp}(\mu_x) = \sigma(H_\Gamma) = [-2|\alpha + \beta|, 2|\alpha + \beta|].$$

From the description of Schreier graphs of the action of F on $[0,1]$ (see [25], Theorem 3.3) it follows that Γ contains a copy of Δ. Thus, for some vertex $y \in \Gamma$ one has

$$\Gamma_y = \Delta_v \cup \widetilde{\Gamma}_y,$$

where Γ_y is the graph obtained from Γ_x by moving the root to y and $\widetilde{\Gamma}_y$ is some subgraph of Γ_y. We have:

$$F_{\Gamma,y}(t) = F_{\Delta,v}(t) + F_{\widetilde{\Gamma},y}(t).$$

From Lemma 9 and (24) we obtain:

$$\frac{1}{S_{\mu_{\Gamma,y}}(z)} = z + \frac{1}{S_{\mu_{\Delta,v}}(z)} + \frac{1}{S_{\mu_{\widetilde{\Gamma},y}}(z)}. \tag{44}$$

Further, let $I \subset (-2|\alpha + \beta|, 2|\alpha + \beta|)$ be an open segment. Let us show that $I \cap \operatorname{supp}(\mu_{\Gamma,y}) \neq \varnothing$. Assume the contrary. Then $S_{\mu_{\Gamma,y}}(z)$ is real analytic (i.e. has real values and is analytic) on I. Clearly, $S_{\mu_{\Gamma,y}}(z)$ is not identically equal to zero. Therefore, there exists an open segment $J \subset I$ such that $S_{\mu_{\Gamma,y}}(z) \neq 0$ on J. It follows that

$$\lim_{z \to z_0, \operatorname{Im} z > 0} \operatorname{Im} \frac{1}{S_{\mu_{\Gamma,y}}(z)} = 0$$

for all $z_0 \in J$. On the other hand, from Lemma 25, Remark 21 and (21) we have for every $z_0 \in (-2|\alpha + \beta|, 2|\alpha + \beta|)$:

$$\liminf_{z \to z_0, \operatorname{Im} z > 0} \operatorname{Im} \frac{1}{S_{\mu_{\Delta,v}}(z)} < 0, \quad \liminf_{z \to z_0, \operatorname{Im} z > 0} \operatorname{Im} \frac{1}{S_{\mu_{\widetilde{\Gamma},y}}(z)} \leqslant 0,$$

where the limit is allowed to be $-\infty$. We obtain a contradiction to (44) from which we conclude that $I \cap \operatorname{supp}(\mu_{\Gamma,y}) \neq \varnothing$. Since $I \subset (-2|\alpha + \beta|, 2|\alpha + \beta|)$ is an arbitrary open segment and $\operatorname{supp}(\mu_{\Gamma,y})$ is closed we obtain that

$$[-2|\alpha + \beta|, 2|\alpha + \beta|] \subset \operatorname{supp}(\mu_{\Gamma,y}) \subset \sigma(H_\Gamma).$$

On the other hand, since for every vertex of Γ the sum of the weights of the edges adjacent to it is $2(\alpha + \beta)$ we have $\|H_\Gamma\| \leqslant 2|\alpha + \beta|$ and so $\operatorname{supp}(\mu_{\Gamma,y}) \subset \sigma(H_\Gamma) \subset [-2|\alpha + \beta|, 2|\alpha + \beta|]$. This finishes the proof.

7 Addendum: on Hulanicki type theorem for graphs.

In [9] the authors showed the following:

Theorem 41 (Weak Hulanicki Theorem for Graphs). *Let $\widetilde{\Gamma}$ be a connected graph of bounded degree with uniformly bounded weight. Assume that $\widetilde{\Gamma}$ covers a weighted graph Γ such that either*

a) *$\widetilde{\Gamma}$ is amenable and Γ is finite or*

b) *$\widetilde{\Gamma}$ has subexponential growth.*

Let $\widetilde{H}, H$ be the Laplace type operators associated to $\widetilde{\Gamma}, \Gamma$. Then $\sigma(H) \subset \sigma(\widetilde{H})$.

They raised a question wether the part $a)$ can be proven without the assumption of finiteness of Γ. Here we answer this question by presenting a pair of amenable graphs $\widetilde{\Gamma}$ and Γ such that $\widetilde{\Gamma}$ covers Γ but the spectrum of Γ is not contained in the spectrum of $\widetilde{\Gamma}$. See figure 1 for graphic representations of these graphs. This example shows that Theorem 41 will not be true if dropping the condition of finiteness of Γ in part $a)$.

Denote by $\Gamma_{\mathbb{N}}$ the graph with the vertex set identified with $\mathbb{N}$ and two vertices i, k connected by an edge if and only if $|i - k| = 1$. Let T_4 be the infinite 4-regular tree (each vertex has degree 4). Denote by Γ the graph obtained by adding two loops at vertex 1 of $\Gamma_{\mathbb{N}}$. Denote by $\widetilde{\Gamma}$ the graph obtained by attaching a copy of $\Gamma_{\mathbb{N}}$ at every vertex of T_4. Then the map $\phi : \widetilde{\Gamma} \to \Gamma$ sending each vertex of T_4 to the vertex 1 of Γ and each copy of $\Gamma_{\mathbb{N}}$ from $\widetilde{\Gamma}$ identically onto the copy of $\Gamma_{\mathbb{N}}$ in Γ defines a covering of graphs. We make each of the graphs described above a weighted graph by putting weight 1 on each edge. Observe that the graphs $\widetilde{\Gamma}$ and Γ are amenable. The main result of this section is

Theorem 42. *One has*

$$\sigma(H_\Gamma) = [-2, 2] \cup \tfrac{17}{4}, \quad \sigma(H_{\widetilde{\Gamma}}) = \left[-\frac{13\sqrt{3}}{6}, \frac{13\sqrt{3}}{6} \right].$$

In particular, $\sigma(H_\Gamma)$ is not a subset of $\sigma(H_{\widetilde{\Gamma}})$.

Note that in the paper [18], Lemma 5, the authors show that for a uniform random walk on an infinite graph given that the Markov operator has spectral radius 1 the eigenvalue 1 cannot be isolated. In the proof of Lemma 5 it is important that the sum of the weights of edges adjacent to a vertex does not depend on the vertex. In our example the graph Γ does not satisfy this condition. That is why it is possible for the Laplace type operator H_Γ to have the largest by absolute value eigenvalue $\frac{17}{4}$ isolated.
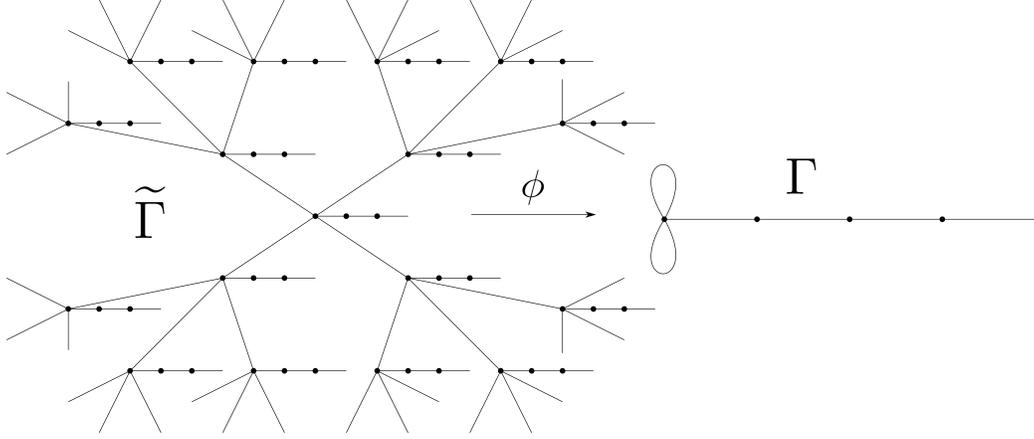
Figure 7: The graphs Γ and $\widetilde{\Gamma}$.

Lemma 43. *One has*

$$F_{\Gamma_{\mathbb{N},1}}(t) = \tfrac{1}{2}(1 - \sqrt{1 - 4t^2}), \quad F_{\Gamma,1}(t) = 4t + F_{\Gamma_{\mathbb{N},1}}(t).$$

Proof. The weighted rooted graph $\Gamma_{\mathbb{N},1}$ satisfies the equation

$$\Gamma_{\mathbb{N},1} = \delta + \Gamma_{\mathbb{N},1} \quad \text{(see Subsection 2.1).}$$

Using Lemmas 9 and 11 we obtain that the function $f(t) = F_{\Gamma_{\mathbb{N},1}}(t)$ satisfies the equation

$$f(t) = \frac{t^2}{1 - f(t)}, \quad f(0) = 0,$$

solving which we arrive at

$$f(t) = \tfrac{1}{2}(1 - \sqrt{1 - 4t^2}).$$

Using Lemma 11 we also get $F_{\Gamma,1}(t) = 4t + F_{\Gamma_{\mathbb{N},1}}(t)$. $\qquad\square$

Lemma 44. *Let Θ be a weighted graph, v a vertex of Θ and $\Omega = \Theta_v \cup \Gamma_{\mathbb{N},1}$. Assume that H_Ω has a real eigenvalue λ and f is a corresponding eigenvector. Then $|\lambda| > 2$ and the values of f on $\Gamma_{\mathbb{N}} \subset \Omega$ are of the form*

$$f(j) = c \left(\frac{\lambda - \operatorname{sign}(\lambda)\sqrt{\lambda^2 - 4}}{2} \right)^j, \quad j \in \mathbb{N}, \tag{45}$$

where c is a constant and $\operatorname{sign}(\lambda)$ stands for the sign of λ.

35

Proof. An eigenvector f of H_Ω with an eigenvalue λ has values on $\Gamma_{\mathbb{N}}$ satisfying:

$$f(j+2) - \lambda f(j+1) + f(j) = 0, \quad j \in \mathbb{N}. \tag{46}$$

The characteristic polynomial of this recursive equation $t^2 - \lambda t + 1 = 0$ has roots

$$t_{1,2} = \frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2}.$$

The general solution of (46) is

$$f(j) = c_1 t_1^j + c_2 t_2^j,$$

where c_1, c_2 are constants. Observe that for $\lambda \in [-2, 2]$ we have $|t_1| = |t_2| = 1$. Since f is a vector from $l^2(\Omega)$, $f(j) \to 0$, we obtain in this casse that $c_1 = c_2 = 0$. If $\lambda \notin [-2, 2]$ then only one of the roots $t_1 := \frac{\lambda - \text{sign}(\lambda)\sqrt{\lambda^2 - 4}}{2}$ has absolute value less than one. It follows that $c_2 = 0$ which finishes the proof of Lemma 44. $\square$

Lemma 45. *One has:*

$$\sigma(H_{\Gamma_{\mathbb{N}}}) = [-2, 2], \quad \sigma(H_\Gamma) = [-2, 2] \cup \tfrac{17}{4}.$$

Note that the first part $\sigma(H_{\Gamma_{\mathbb{N}}}) = [-2, 2]$ is well known. For the readers convenience we present a proof here.

Proof. Observe that
$$F_{\Gamma_{\mathbb{N}}, 1}(t) = \tfrac{1}{2}(1 - \sqrt{1 - 4t^2})$$

defines a function analytic on $\mathbb{H}_+ = \{z \in \mathbb{C} : \text{Im} z > 0\}$ and continuous on $\overline{\mathbb{H}_+}$ having real values for $t \in [-1/2, 1/2]$ and non-real values for $t \in \mathbb{R} \setminus [-1/2, 1/2]$. The same property holds for

$$P_{\Gamma_{\mathbb{N}}, 1}(t) = \frac{1}{1 - F_{\Gamma_{\mathbb{N}}, 1}(t)},$$

since $\tfrac{1}{2}(1 - \sqrt{1 - 4t^2}) \neq 1$ for $t \in \mathbb{R}$. Using (22) and Corollary 18 we obtain that $\text{supp}(\mu_1) = [-2, 2]$, where μ_1 is the spectral measure of $H_{\Gamma_{\mathbb{N}}}$ at vertex 1.

Further, assume that $\Gamma_{\mathbb{N}}$ has an eigenvalue λ. By Lemma 44 $|\lambda| > 2$ and the corresponding eigenvector f is of the form (45). Since 1 has only one edge attached to it (leading to 2) we get $\lambda f(1) = f(2)$ from which we get that the value of c from (45) is zero and $f \equiv 0$. This contradiction shows that $H_{\Gamma_{\mathbb{N}}}$ does not have eigenvectors. Using Proposition 35 we obtain that

$$\text{supp}(\mu_j) = \text{supp}(\mu_1) = [-2, 2] \text{ for every } j \in \mathbb{N}.$$

36

Finally, Proposition 36 implies that $\sigma(H_{\Gamma_\mathbb{N}}) = [-2, 2]$.

In case of the graph Γ there are two differences. First, the equation $F_{\Gamma,1}(t) = 1$ has a unique real solution $t = \frac{4}{17}$. Indeed,

$$4t + \tfrac{1}{2}(1 - \sqrt{1 - 4t^2}) = 1 \Leftrightarrow 8t - 1 = \sqrt{1 - 4t^2} \Rightarrow 68t^2 - 16t = 0 \Rightarrow t \in \{0, \tfrac{4}{17}\}.$$

From two solutions $t = 0$ and $t = \frac{4}{17}$ only the second satisfies the initial equation. It follows that $P_{\Gamma,1}(t) = 1/(F_{\Gamma,1}(t) - 1)$ has a pole at $\frac{4}{17}$, is real analytic on $[-1/2, 1/2] \setminus \{\frac{4}{17}\}$ and is non-real on $\mathbb{R} \setminus [-1/2, 1/2]$. Using (22) and Corollary 18 we obtain that $\mathrm{supp}(\mu_1) = [-2, 2] \cup \{\frac{17}{4}\}$, where μ_1 now is the spectral measure of H_Γ at vertex 1.

Second, for an eigenvector f of H_Γ corresponding to an eigenvalue λ we have $\lambda f(1) = 4f(1) + f(2)$. Together with (45) this leads to:

$$\lambda = 4 + \frac{\lambda - \mathrm{sign}(\lambda)\sqrt{\lambda^2 - 4}}{2}.$$

Setting $\lambda = t^{-1}$ we arrive at the equation considered above and conclude that H_Γ has a unique eigenvalue $\lambda = \frac{17}{4}$. The corresponding eigenvector can be given by $f(j) = \frac{1}{4^j}$. Using Propositions 35 and 36 we obtain that $\sigma(H_\Gamma) = [-2, 2] \cup \{\frac{17}{4}\}$. $\qquad\square$
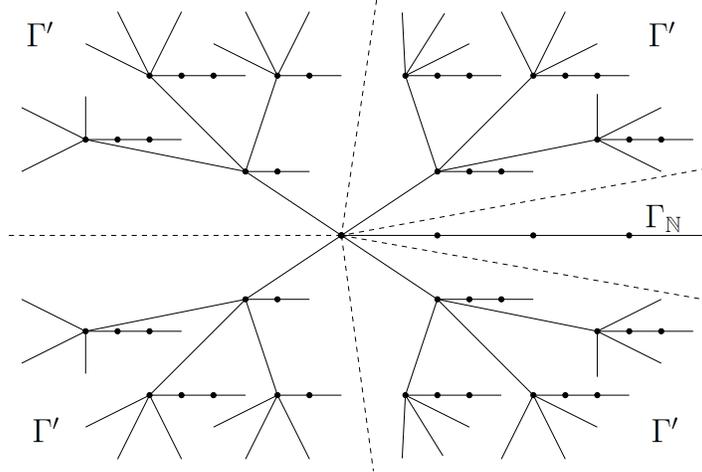
Figure 8: Illustration to formula (47).

Fix a vertex $v \in T_4 \subset \widetilde{\Gamma}$. The graph $\widetilde{\Gamma}$ can be obtained as a union of four isomorphic copies of some graph Γ' and a copy of $\Gamma_\mathbb{N}$ intersecting at v (see Figure 8):

$$\widetilde{\Gamma}_v = \Gamma'_v \cup \Gamma'_v \cup \Gamma'_v \cup \Gamma'_v \cup \Gamma_{\mathbb{N},1}. \tag{47}$$

Observe that Γ'_v satisfies the equation:

$$\Gamma'_v = \delta + (\Gamma'_v \cup \Gamma'_v \cup \Gamma'_v \cup \Gamma_{\mathbb{N},1}). \tag{48}$$

37

Proposition 46. *One has*

$$F_{\widetilde{\Gamma},v}(t) = \tfrac{1}{6}\left(5 - \sqrt{1 - 4t^2} - 2\sqrt{2 - 52t^2 + 2\sqrt{1 - 4t^2}}\right).\qquad(49)$$

Proof. Using Lemmas 9 and 11 we obtain from (48)

$$F_{\Gamma',v} = \frac{t^2}{1 - 3F_{\Gamma',v} - F_{\Gamma_{\mathbb{N}},1}}.$$

Thus, $g = g(t) := F_{\Gamma',v}(t)$ satisfies the quadratic equation:

$$3g^2 - (1 - F_{\Gamma_{\mathbb{N}},1})g + t^2 = 0,$$

solving which taking into account that $g(0) = 0$ we obtain:

$$g = \tfrac{1}{6}\left(1 - F_{\Gamma_{\mathbb{N}},1} - \sqrt{(1 - F_{\Gamma_{\mathbb{N}},1})^2 - 12t^2}\right) =$$

$$\tfrac{1}{12}\left(1 + \sqrt{1 - 4t^2} - \sqrt{2 - 52t^2 + 2\sqrt{1 - 4t^2}}\right).$$

From Lemma 11 and (47) we obtain $F_{\widetilde{\Gamma},v} = 4F_{\Gamma',v} + F_{\mathbb{N},1}$ from which Proposition 46 follows. $\qquad\square$

Proposition 47. *The operator $H_{\widetilde{\Gamma}}$ has no eigenvalues.*

Proof. Assume that λ is an eigenvalue of $H_{\widetilde{\Gamma}}$. Let f be the corresponding eigenvector. It is not hard to see that there exists a vertex $w \in T_4 \subset \widetilde{\Gamma}$ such that $f(w) \neq 0$. For simplicity, identify the vertices of the copy of $\Gamma_{\mathbb{N}}$ containing w with positive integers. In particular, w is identified with 1. Let w_1, w_2, w_3, w_4 be the neighbors of w in T_4. By Lemma 44,

$$f(2) = \gamma f(w), \quad \text{where} \quad \gamma = \frac{\lambda - \text{sign}(\lambda)\sqrt{\lambda^2 - 4}}{2}.$$

On the other hand, $f(w_1) + f(w_2) + f(w_3) + f(w_4) + f(2) = \lambda f(w)$. It follows that

$$f(w_1) + f(w_2) + f(w_3) + f(w_4) = (\lambda - \gamma)f(w).\qquad(50)$$

The equation (50) holds for any $w \in T_4$ such that $f(w) \neq 0$.

Now, let $w \in T_4$ be such that $f(w) = 0$. Then by Lemma 44 the values of f on the vertices of the corresponding copy of $\Gamma_{\mathbb{N}}$ are equal to zero. Therefore, $f(w_1) + f(w_2) + f(w_3) + f(w_4) = \lambda f(w) = 0 = (\lambda - \gamma)f(w)$. Thus, (50) holds for all $w \in T_4$. This implies that $\lambda - \gamma$ is an eigenvalue of H_{T_4}. However, it is known that the Markov operator of the simple random walk on a d-regular tree does not have eigenvalues. This contradiction finishes the proof of Proposition 47. $\qquad\square$

Proposition 48. *One has $\sigma(H_{\widetilde{\Gamma}}) = [-\frac{2\sqrt{3}}{13}, \frac{2\sqrt{3}}{13}]$.*

Proof. Consider the equation:

$$2 - 52t^2 + 2\sqrt{1-4t^2} = 0 \;\Rightarrow\; 1 - 4t^2 = (1-26t^2)^2 \;\Rightarrow\; 676t^4 - 48t^2 = 0 \;\Rightarrow\; t \in \left\{ 0, \pm\frac{2\sqrt{3}}{13} \right\}.$$

Using the latter we derive that $2 - 52t^2 + 2\sqrt{1-4t^2}$ has non-negative real values on $\left[-\frac{2\sqrt{3}}{13}, \frac{2\sqrt{3}}{13} \right]$ and either negative or non-real values outside this interval. From formulae (49) we obtain that $F_{\widetilde{\Gamma},v}(t)$ has real values on $\left[-\frac{2\sqrt{3}}{13}, \frac{2\sqrt{3}}{13} \right]$ and non-real values outside this interval. The function $P_{\widetilde{\Gamma},v}(t) = 1/(1 - F_{\widetilde{\Gamma},v}(t))$ satisfies the same property, since $H_{\widetilde{\Gamma}}$ has no eigenvalues by Proposition 47 and so $P_{\widetilde{\Gamma},v}(t)$ has no poles by Lemma 34. Using Corollary 18 and (24) we arrive at

$$\operatorname{supp}(\mu_v) = \left[-\frac{13\sqrt{3}}{6}, \frac{13\sqrt{3}}{6} \right].$$

Using Propositions 35 and 47 we obtain that $\operatorname{supp}(\mu_w) = \operatorname{supp}(\mu_v)$ for every vertex $w \in \widetilde{\Gamma}$. Finally, using Proposition 36 we derive that $\sigma(H_{\widetilde{\Gamma}}) = \left[-\frac{13\sqrt{3}}{6}, \frac{13\sqrt{3}}{6} \right]$. $\qquad\square$

References

[1] L. BARTHOLDI AND R. I. GRIGORCHUK, *On the spectrum of Hecke type operators related to some fractal groups*, Tr. Mat. Inst. im. V.A. Steklova, Ross. Akad. Nauk, 231 (2000), pp. 5–45.

[2] A. BENDIKOV, C. PITTET, AND R. SAUER, *Specral distribution and l^2-isoperimetric profile of laplace operators on groups*, Math. Ann., 354 (2012), pp. 43–71.

[3] J. W. CANNON, J. W. FLOYD, AND W. R. PARRY, *Introductory notes on Richard Thompson's groups*, Enseign. Math. (2), 42 (1996), pp. 215–256.

[4] T. CECCHERINI-SILBERSTEIN, R. GRIGORCHUK, AND P. DE LA HARPE, *Amenability and paradoxical decompositions for pseudogroups and discrete metric spaces*, Tr. Mat. Inst. im. V.A. Steklova, Ross. Akad. Nauk, 224 (1999), pp. 68–11.

[5] N.-B. DANG, R. GRIGORCHUK, AND M. LYUBICH, *Self-similar groups and holomorphic dynamics: Renormalization, integrability, and spectrum*, ArXiv e-prints, (2020).

[6] R. Dougherty, S. Jackson, and A. S. Kechris, *The structure of hyperfinite Borel equivalence relations*, Trans. of AMS, 341 (1994), pp. 193–225.

[7] A. Dudko and R. Grigorchuk, *On spectra of Koopman, groupoid and quasi-regular representations*, J. of Mod. Dyn., 11 (2017), pp. 99–123.

[8] ——, *On spectra of representations and graphs. Erratum*, ArXiv e-prints, (2017).

[9] ——, *On the question "Can one hear the shape of a group?" and Hulanicki type theorem for graphs*, Israel J. Math., 237 (2020), pp. 53–74.

[10] P. Flajolet and A. Odlyzko, *Singularity analysis of generating functions*, Siam J. Disc. Math., 3 (1990), pp. 216–240.

[11] R. Grigorchuk and Z. Šunić, *Schreier spectrum of the Hanoi Towers group on three pegs*, in Analysis on graphs and its applications, vol. 77 of Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, 2008, pp. 183–198.

[12] R. Grigorchuk, D. Lenz, and T. Smirnova-Nagnibeda, *Spectra of Schreier graphs of Grigorchuk's group and Schroedinger operators with aperiodic order*, Math. Ann., 370 (2018), pp. 1607–1637.

[13] R. Grigorchuk, V. Nekrashevych, and V. Sushchanskii, *Automata, dynamical systems and infinite groups*, Proc. Steklov Inst. Math., 231 (2000), pp. 134–214.

[14] R. Grigorchuk, A. Pérez, and T. Smirnova-Nagnibeda, *On spectra and spectral measures of Schreier and Cayley graphs*, ArXiv e-prints, (2020).

[15] R. I. Grigorchuk, *Burnside's problem on periodic groups*, Funkts. Anal. Prilozh., 14 (1) (1980), pp. 53–54.

[16] R. I. Grigorchuk and Y. S. Krylyuk, *The spectral measure of the Markov operator related to 3-generated 2-group of intermediate growth and its Jacobi parameters*, Algebra and Discrete Mathematics, 13 (2012), pp. 237–272.

[17] R. I. Grigorchuk and V. Nekrashevych, *Self-similar groups, operator algebras and Schur complement*, J. Modern Dyn., 1 (2007), pp. 323–370.

[18] R. I. Grigorchuk and A. Żuk, *On the asymptotic spectrum of random walks on infinite families of graphs*, Random walks and discrete potential theory (Cortona, 1997), (1999), pp. 188–204.

[19] V. Kaimanovich, *Thompsons group F is not Liouville*, in Groups, Graphs and Random Walks, T. Ceccherini-Silberstein, M. Salvatori, and E. Sava-Huss, eds., London Mathematical Society Lecture Notes Series, Cambridge University Press, 2017, pp. 300–342.

[20] H. Kesten, *Full Banach mean values on countable groups*, Math. Scand., 7 (1959), pp. 146–156.

[21] ——, *Symmetric random walks on groups*, Trans. Amer. Math. Soc., 92 (1959), pp. 336–354.

[22] W. Kirsch and B. Metzger, *The integrated density of states for random Schrödinger operators*, vol. 76, Part 2 of Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, 2007, p. 649–696.

[23] W. Lück, L^2–*invariants and their applications to geometry, group theory and spectral theory*, Springer, Berlin, 2001, p. 859–871.

[24] D. Savchuk, *Some graphs related to Thompson's group F*, in Combinatorial and geometric group theory, Trends Math., Birkhäuser Basel, 2010, pp. 279–296.

[25] ——, *Schreier graphs of actions of Thompson's group F on the unit interval and on the cantor set*, Geom. Dedicata, 175 (2015), pp. 355–372.

[26] J. W. Silverstein and S. I. Choi, *Analysis of the limiting spectral distribution of large dimensional random matrices*, J. Multivariate Anal., 54 (1995), pp. 295–309.

[27] Woess, *Random walks on infinite graphs and groups: a survey on selected topics*, Bull. London Math. Soc., 26 (1994), pp. 1–60.