CORRIGENDUM TO: SPIN^h AND FURTHER GENERALISATIONS OF SPIN

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Abstract. We qualify a claim made in [AM21], regarding the dimensions in which all orientable manifolds admit spin^h structures, with a compactness assumption, and comment on when this assumption can be removed.

In [AM21], the present authors made the following statement ([AM21, Theorem 1.3], [AM21, Corollary 3.10]): *Every orientable manifold of dimension* ≤ 7 *is spin*^{h}.

Here and throughout, we take all manifolds to be smooth. The argument for manifolds of dimension 6 and 7 invoked Cohen's immersion theorem [Coh85] in order to obtain a codimension 4 immersion in Euclidean space, followed by an application of [AM21, Proposition 3.9] that such an immersion guarantees the existence of a spin^h structure. However, in order to apply [Coh85], one should qualify the statement by assuming the manifolds in question are compact.

In the present corrigendum, we note that the following holds, using only (other) results of [AM21] and results preceding [Coh85]:

Theorem. *The following hold:*

(1) *Every (not necessarily compact) orientable manifold of dimension* ≤ 5 *is spin*^{h}.

(2) *Compact orientable manifolds of dimension* 6 *and* 7 *are spin*^{h}.

(3) *A non-compact orientable manifold* M *of dimension* 6 *or* 7 *is spin*^{h} *if and only if* $W_5(TM) = 0$.

(4) *A non-compact orientable manifold* M *of dimension* 6 *or* 7 *with no elements of order exactly four in* $H^5(M;\mathbb{Z})$ *is spin*^{h}.

By "no elements of order exactly four" we mean that any $x \in H^5(M;\mathbb{Z})$ satisfying $4x = 0$ also satisfies $2x = 0$.

Proof. Part (1) is already proved in [AM21, p.5] without appealing to [Coh85] as a corollary of [AM21, Corollary 2.6], which states that the primary obstruction to the existence of a spin^h structure on an oriented manifold is the fifth integral Stiefel–Whitney class W_5. This is an integral class of order two and hence vanishes on all orientable manifolds of dimension ≤ 5. Here and throughout, we use the fact that the top cohomology (with any coefficients) of a non-compact manifold vanishes, e.g. see [Wh61, Lemma 2.1]; see also [Br62, Theorem 2] that every (not necessarily compact) manifold with boundary admits a collar neighborhood of its boundary, and hence it deformation retracts onto its interior.

That compact orientable six–manifolds immerse in $\mathbb{R}^{10}$ is proved in [Hir61, Corollary 9], and hence they admit spin^h structures by [AM21, Proposition 3.9]. Note that the statement of [Hir61, Corollary 9] does not include compactness, though it is clear from the proof that the manifold is assumed to be closed; the general compact case then follows by taking the double if the boundary is non-empty.

For compact orientable seven–manifolds M, we use the result listed in the second table of [AtDu72, p.25] (see also the footnote (1) in loc. cit.), that the only obstruction to a compact seven–manifold admitting three linearly independent vector fields is the integral Bockstein of w_4, i.e. W_5. Again, the result is stated for closed manifolds, and the compact-with-boundary case follows by considering the double. (Note, for orientable seven–manifolds, W_5 is a priori the single obstruction to finding three linearly independent sections over the five–skeleton.) This class vanishes by [Mas62, Theorem 3]. Hence the tangent bundle of M splits off a trivial rank three bundle, giving an orientable rank four bundle with the same w_2 as TM, and we again apply [AM21, Proposition 3.9]. This proves part (2).

Now let M be a non-compact orientable six–manifold. Since $H^6(M;\mathbb{Z}) = 0$, there are no secondary or higher obstructions to admitting a spin^h structure beyond W_5; this establishes part (3) for six–manifolds. Choose an increasing exhaustion $\{M_i\}$ by compact manifolds with boundary. For an abelian group A and integer $k > 1$, we have the short exact Milnor sequence [Sw17, Proposition 7.66]

$$0 \to \varprojlim{}^1 H^{k-1}(M_i; A) \to H^k(M; A) \to \varprojlim H^k(M_i; A) \to 0.$$

From the long exact sequence in cohomology associated to the short exact coefficient sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\bmod 2} \mathbb{Z}/2 \to 0$$

we have the following commutative diagram:

$$\begin{tikzcd}
0 & 0 & 0 \\
\varprojlim H^5(M_i;\mathbb{Z}) \arrow[u] \arrow[r] & \varprojlim H^5(M_i;\mathbb{Z}) \arrow[u] \arrow[r] & \varprojlim H^5(M_i;\mathbb{Z}/2) \arrow[u] \\
H^5(M;\mathbb{Z}) \arrow[u] \arrow[r, "\cdot 2"] & H^5(M;\mathbb{Z}) \arrow[u] \arrow[r, "\bmod 2"] & H^5(M;\mathbb{Z}/2) \arrow[u] \\
\varprojlim{}^1 H^4(M;\mathbb{Z}) \arrow[u] \arrow[r] & \varprojlim{}^1 H^4(M;\mathbb{Z}) \arrow[u] \arrow[r] & \varprojlim{}^1 H^4(M;\mathbb{Z}/2) \arrow[u] \\
0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

For each M_i, we have $W_5(M_i) = 0$ by taking the double and applying [Mas62, Theorem 2] (or crossing the double with a circle and applying [Mas62, Theorem 3] again). Generally for an orientable manifold, the mod 2 reduction of W_5 is w_5. From here and by naturality, $w_5(M) \in H^5(M;\mathbb{Z}/2)$ maps to the zero element in $\varprojlim H^5(M_i;\mathbb{Z}/2)$. By [MiSt74, Lemma 10.3], the term $\varprojlim{}^1 H^4(M;\mathbb{Z}/2)$ vanishes. Therefore $w_5(M)$ must be zero as well. Now, $W_5(M) \in H^5(M;\mathbb{Z})$ is an element of order two which maps to $w_5(M) = 0$ by mod 2 reduction. Therefore it is in the image of the map $H^5(M;\mathbb{Z}) \xrightarrow{\cdot 2} H^5(M;\mathbb{Z})$. Since by assumption there are no elements of order exactly four in $H^5(M;\mathbb{Z})$, it follows that $W_5(M)$ must be the zero class. This establishes part (4) for six–manifolds.

Now let M be an orientable non-compact seven–manifold. We will show that the secondary obstruction to the existence of a spin^h structure vanishes, establishing parts (3) and (4). Take an exhaustion $\{M_i\}$ by compact seven–manifolds with boundary. If $W_5(M) = 0$, which will for instance be true given the torsion condition on $H^5(M;\mathbb{Z})$ by the argument above, we can choose a lift of the classifying map of the tangent bundle $M \to BSO(7)$ to E_1, the second stage of the relative Postnikov tower of $BSO(4) \to BSO(7)$,

$$\begin{tikzcd}
 & BSO(4) \arrow[d] & \\
 & \vdots \arrow[d] & \\
 & E_1 \arrow[d] & K(\mathbb{Z},4) \arrow[l] \\
M \arrow[r] \arrow[ur] & BSO(7) \arrow[r, "W_5"] & K(\mathbb{Z},5)
\end{tikzcd}$$

Restricting to the M_i gives a compatible system of lifts to E_1. We consider now the secondary obstruction $\mathfrak{o}(M)$ to admitting three linearly independent vector fields. This is a class in $H^6(M;\pi_5(V(3,7)))$, where $V(3,7)$ is the Stiefel manifold of 3–frames in $\mathbb{R}^7$. We have the following exact sequence of homotopy groups:

$$\pi_6(BSO(7)) \to \pi_5(V(3,7)) \to \pi_5(BSO(4)) \to \pi_5(BSO(7)).$$

The natural map $BSO(7) \to BSO$ is an isomorphism on $\pi_{\leq 6}$, and hence we have $\pi_5(BSO(7)) = \pi_6(BSO(7)) = 0$. Furthermore, $\pi_5(BSO(4)) \cong \pi_4(SO(4)) \cong \pi_4(S^3 \times S^3) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Since we are fixing the lifts $M \to E_1$ and $M_i \to E_1$, and they are compatible, the secondary obstruction to lifting further to E_2 is natural, i.e. $\mathfrak{o}(M_i)$ is the restriction of $\mathfrak{o}(M)$.

$$\begin{tikzcd}
 & & E_2 \arrow[d] & K(\mathbb{Z}/2 \oplus \mathbb{Z}/2,5) \arrow[l] \\
 & & E_1 \arrow[d] \arrow[r] & K(\mathbb{Z}/2 \oplus \mathbb{Z}/2,6) \\
M \arrow[urr] \arrow[rr] & & BSO(7) & \\
 & M_i \arrow[ul, hook] \arrow[uur] \arrow[ur] & &
\end{tikzcd}$$

Let us now argue that $\mathfrak{o}(M_i) = 0$. We will use [Du74, Theorem 1.1], which gives us that for *any* choice of lift to E_1 on a *closed* orientable seven–manifold, the secondary obstruction vanishes. In order to apply this to M_i, we consider the double DM_i. We will argue that the lift $M_i \xrightarrow{f_i} E_1$ (obtained by restricting $M \xrightarrow{f} E_1$) extends to a lift $DM_i \to E_1$. Then, by applying loc. cit., we will have $\mathfrak{o}(DM_i) = 0$ and hence $\mathfrak{o}(M_i) = 0$.

$$\begin{tikzcd}
 & & E_1 \arrow[d] & K(\mathbb{Z},4) \arrow[l] \\
M_i \arrow[urr, "f_i"] \arrow[rr] \arrow[dr, hook, "j"'] & & BSO(7) & \\
 & DM_i \arrow[ur] \arrow[uur, dashed] & &
\end{tikzcd}$$

First, choose any lift $DM_i \xrightarrow{G} E_1$ of $DM_i \to BSO(7)$; this exists since W_5 vanishes on any closed orientable seven–manifold. Now, f_i and the restriction of G to M_i differ by the action of an element

x in $[M_i, K(\mathbb{Z},4)] = H^4(M_i;\mathbb{Z})$ (this group acts simply transitively on the homotopy classes of lifts to E_1). Let us denote this by $[f_i] = x \cdot [G|_{M_i}]$.

Now observe that x is the restriction of a class $X \in H^4(DM_i;\mathbb{Z})$. Namely, consider the Mayer–Vietoris sequence for the double:

$$\cdots \to H^4(DM_i;\mathbb{Z}) \to H^4(M_i;\mathbb{Z}) \oplus H^4(M_i;\mathbb{Z}) \to H^4(\partial M_i;\mathbb{Z}) \to \cdots$$

The element (x,x) maps to zero, and hence $x = j^*X$ for some $X \in H^4(DM_i;\mathbb{Z})$.

Therefore, if we consider the (class of the) lift $X \cdot [G]$ on DM_i instead of $[G]$, by naturality we have that its restriction to M_i is $x \cdot [G|_{M_i}]$, i.e. $[f_i]$.

Now we have that $\mathfrak{o}(M_i) = 0$ for all i. Consider the short exact sequence

$$0 \to \varprojlim{}^1 H^3(M_i;\mathbb{Z}/2 \oplus \mathbb{Z}/2) \to H^4(M;\mathbb{Z}/2 \oplus \mathbb{Z}/2) \to \varprojlim H^4(M;\mathbb{Z}/2 \oplus \mathbb{Z}/2) \to 0.$$

Since $H^*(-;\mathbb{Z}/2 \oplus \mathbb{Z}/2)$ is naturally isomorphic to $H^*(-;\mathbb{Z}/2) \oplus H^*(-;\mathbb{Z}/2)$, the $\varprojlim^1$ term vanishes. Further, since $\mathfrak{o}(M)$ maps to $(\mathfrak{o}(M_i))_i$, which is the zero element, by injectivity we have that $\mathfrak{o}(M) = 0$. Since M has the homotopy type of a six–complex, the secondary obstruction is also the final obstruction to admitting three linearly independent vector fields, and we conclude that M admits a spin^h structure. □

Likewise, the statement in the paragraph preceding [AM21, Remark 3.5], that every orientable n–manifold is spin$^{n-\alpha(n)}$ (where $\alpha(n)$ is the number of one's in the binary expansion of n) should be qualified with a compactness assumption. Removing the compactness assumption, we can of course appeal to Whitney's immersion theorem to conclude that every orientable n–manifold is spin^{n-1}. Similarly, compactness should be assumed where appropriate in [AM21, Section 5].

Remark. (1) We record also that there are some inaccuracies in the table of homotopy groups of Stiefel manifolds in [EDM93, p.1747]. Namely, taking $n = 4$ and $m = 3, 8s-1, 8s+3$, one sees from the long exact sequence in homotopy groups, as above, that $\pi_5(V(m, m+4))$ – in the notation of loc. cit. this is $\pi_5(V(m+4, m))$ – is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, contrary to what is listed therein.

(2) The primary obstruction to a spin^c structure on an orientable manifold is W_3, and compact orientable four–manifolds are spin^c. An analogous argument to the above then shows that non-compact orientable four–manifolds with no elements of order exactly four in $H^3(-;\mathbb{Z})$ are spin^c. The four–torsion assumption can be removed in this case [TV], and it is not clear whether one should expect this in the theorem above.

Acknowledgements. The authors would like to thank Gustavo Granja and Peter Teichner for helpful conversations, together with the Instituto Superior Técnico in Lisbon and the Max-Planck-Institut für Mathematik in Bonn for their generous hospitality.

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