

# Toroidal integer homology three-spheres have irreducible $SU(2)$ -representations

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## Abstract

We prove that if an integer homology three-sphere contains an embedded incompressible torus, then its fundamental group admits irreducible  $SU(2)$ -representations.

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## 1 | INTRODUCTION

The fundamental group is one of the most powerful invariants to distinguish closed three-manifolds. In fact, by Perelman's proof of Thurston's Geometrization conjecture [28–30], fundamental groups determine closed, orientable three-manifolds up to orientations of the prime factors and up to the indeterminacy arising from lens spaces. Prominently, the three-dimensional Poincaré conjecture, a special case of Geometrization, characterizes  $S^3$  as the unique closed, simply connected three-manifold. For a three-manifold with non-trivial fundamental group, it is then useful to quantify the non-triviality of the fundamental group. Since the

Geometrization theorem implies that three-manifolds have residually finite fundamental groups [18], this non-triviality can be measured by representations to finite groups. However, there is not a finite group  $G$  such that every three-manifold group has a non-trivial homomorphism to  $G$ . Therefore, a more uniform measurement of non-triviality can be found in the following conjecture.

**Conjecture 1** (Kirby Problem 3.105(A), [19]). *If  $Y$  is a closed, connected, three-manifold other than  $S^3$ , then  $\pi_1(Y)$  admits a non-trivial  $SU(2)$ -representation.*

Note that this conjecture is equivalent to the statement that the fundamental groups of all integer homology three-spheres other than  $S^3$  admit irreducible  $SU(2)$ -representations. Indeed, every three-manifold whose first homology group is non-zero admits non-trivial abelian representations to  $SU(2)$ . Moreover, lens spaces are examples of manifolds that admit non-trivial  $SU(2)$ -representations of their fundamental groups, but no irreducible ones. There are also three-manifolds with non-abelian fundamental group which do not admit irreducible representations [26]. However, for representations of perfect groups to  $SU(2)$ , non-triviality is equivalent to irreducibility.

For comparison, the third author showed in [36] that Conjecture 1 is true if one replaces  $SU(2)$  with  $SL_2(\mathbb{C})$ . The reader may also relate Conjecture 1 with characterizing the three-manifolds with simplest instanton or Heegaard Floer homologies. One side of the L-space conjecture predicts that every prime integer homology three-sphere other than  $S^3$  and the Poincaré homology three-sphere admits a co-orientable taut foliation. This fact, together with the gauge-theoretic methods used by Kronheimer–Mrowka in [21], would then imply Conjecture 1.

There are many families of integer homology three-spheres for which Conjecture 1 has been established, such as those which are Seifert fibred (although the methods go back to Fintushel–Stern [13], this can be found explicitly in [32, Theorem 2.1]), branched double covers of non-trivial knots with determinant 1 [8, Theorem 3.1] and [35, Corollary 9.2],  $1/n$ -surgeries on non-trivial knots in  $S^3$  [20], those that are filled by a Stein manifold which is not a homology ball [1], or for splittings of knots in  $S^3$  [36].

It follows again from Geometrization that there are three (non-disjoint) types of prime integer homology three spheres: Seifert fibred, hyperbolic, and toroidal ones. We remark that although some toroidal integer homology three spheres are Seifert fibred, they are never hyperbolic. The third author established that if all hyperbolic integer homology three spheres have irreducible  $SU(2)$ -representations, then Conjecture 1 holds in general. While we are unable to complete the remaining step in this program, we confirm the existence of  $SU(2)$ -representations for toroidal integer homology three spheres.

**Theorem 1.1.** *Let  $Y$  be a toroidal integer homology three spheres. Then  $\pi_1(Y)$  admits an irreducible  $SU(2)$ -representation.*

A proof of Theorem 1.1 could be obtained by showing that toroidal integer homology three spheres have non-trivial instanton Floer homology. Although we expect the latter to be true (see [19, Problem 3.106]), we do not prove it in this article. Our proof of Theorem 1.1 instead relies on holonomy perturbations in a manner similar to the proof of [36, Theorem 8.3]. If  $Y$  is a toroidal integer homology three-sphere, then  $Y$  can be viewed as a splice of knots  $K_i$  in integer homology three spheres  $Y_i$  for  $i = 1, 2$  (see, for example, [11, Proof of Corollary 6.2]). If some  $Y_i$  has an

irreducible  $SU(2)$ -representation, then there is a  $\pi_1$ -surjective map from  $Y$  to  $Y_i$  and we can pull back to an irreducible  $SU(2)$ -representation for  $Y$ . If not, then we will study the image of the space of representations of the knot exterior  $Y_i \setminus N(K_i)^\circ$  in the character variety for the boundary torus (that is, in the pillowcase). Here,  $N(K_i)$  denotes a closed tubular neighborhood of  $K_i$ , and  $N(K_i)^\circ$  denotes its interior. Similar to the case of non-trivial knots in  $S^3$ , if  $Y_i$  has no irreducible representations, we will show that the image in the pillowcase contains a suitably essential loop. The loops for the two exteriors will have a non-trivial intersection, and therefore, the spliced manifold  $Y$  will admit an irreducible  $SU(2)$ -representation.

Theorem 1.1 gives a simpler proof of [36, Theorem 9.4] since it avoids the use of a finiteness result of Boileau–Rubinstein–Wang.

**Corollary 1.2** (Theorem 9.4, [36]). *Every integer homology three-sphere other than  $S^3$  has an irreducible  $SL_2(\mathbb{C})$ -representation of its fundamental group.*

*Proof.* By the remarks above we have to consider three cases: Seifert fibred, hyperbolic, and toroidal integer homology three spheres. Let  $Y$  be an integer homology three-sphere other than  $S^3$ . If  $Y$  is hyperbolic, it admits an irreducible  $SL_2(\mathbb{C})$ -representation by lifting the holonomy representation to  $PSL_2(\mathbb{C})$  [9]. If  $Y$  is Seifert fibred, then  $\pi_1(Y)$  admits an irreducible  $SU(2)$ -representation by [32, Theorem 2.1]. If  $Y$  is toroidal, the result now follows from Theorem 1.1.  $\square$

In order to generalize the holonomy perturbation machinery developed by the third author from non-trivial knots in  $S^3$ , we will need to establish a non-vanishing result which may be of independent interest.

**Theorem 1.3.** *Let  $J$  be a knot in an integer homology three-sphere  $Y$  such that the exterior of  $J$  is irreducible and boundary-incompressible. Suppose that  $I_*(Y) = 0$ . Then,  $I_*^w(Y_0(J)) \neq 0$ .*

Here, and throughout this article,  $I_*$  denotes Floer's original version of instanton Floer homology and  $I_*^w$  denotes instanton Floer homology for an admissible  $SO(3)$ -bundle with second Stiefel–Whitney class  $w$ . (Note that  $Y_0(J)$  admits only one such bundle.)

The proof of Theorem 1.3 is a combination of (1) Kronheimer–Mrowka's non-vanishing result for instanton Floer homology of three-manifolds with a taut-sutured manifold hierarchy [22], (2) the surgery exact triangle in instanton Floer homology, and (3) Gordon's description of surgery on cable knots [17]. The argument is similar to Kronheimer–Mrowka's proof of *Property P* [21].

While Theorem 1.3 itself may not be particularly interesting, it does lead to the following corollary, whose analog in Heegaard Floer homology has been established by Ni [27, p.1144] and Conway and Tosun [7]. The proof of the corollary appears in Section 2 below.

**Corollary 1.4.** *Let  $Y \neq S^3$  be an integer homology three-sphere which bounds a Mazur manifold. Then,  $I_*(Y) \neq 0$ , and hence  $\pi_1(Y)$  admits an irreducible  $SU(2)$ -representation.*

Recall that Baldwin–Sivek prove that if an integer homology three-sphere  $Y$  bounds a Stein domain with non-trivial homology, then  $\pi_1(Y)$  admits an irreducible  $SU(2)$ -representation [1, Theorem 1.1]. In light of Conjecture 1, the following conjecture would be a natural extension of their work.

**Conjecture 2.** *If  $Y \neq S^3$  is an integer homology three-sphere which bounds a Stein integer homology ball, then  $\pi_1(Y)$  admits an irreducible  $SU(2)$ -representation.*

Since Stein domains admit handlebody decompositions with no three handles [12], Corollary 1.4 proves this conjecture for the boundaries of Stein integer homology balls with the simplest possible handle decompositions.

Theorem 1.1 also has two simple corollaries. The first one is obtained by considering branched covers over satellite knots in  $S^3$ . Remarkably, its proof requires no use of gauge theory, beyond our main result. Its proof appears in Section 5 below.

**Corollary 1.5.** *Let  $K$  be a prime, satellite knot in  $S^3$ . Conjecture 1 holds for any non-trivial cyclic branched cover of  $K$ .*

To obtain the second corollary, define a graph manifold integer homology three-sphere to be a closed, orientable three-manifold whose torus decomposition has no hyperbolic pieces.<sup>†</sup> As discussed above, the fundamental groups of Seifert integer homology three spheres other than  $S^3$  admit irreducible  $SU(2)$ -representations, and hence we obtain the following.

**Corollary 1.6.** *Let  $Y$  be a graph manifold integer homology three-sphere other than  $S^3$ . Then  $\pi_1(Y)$  admits an irreducible  $SU(2)$ -representation.*

A first alternate proof of this corollary can be obtained by noting that every integer homology three-sphere other than  $S^3$  which is a graph manifold can be realized as the branched double cover of a non-trivial (arborescent) knot in  $S^3$ , see [4]. A second alternate proof can be obtained by noting that every prime graph manifold integer homology three-sphere  $Y$  other than  $S^3$  or  $\Sigma(2, 3, 5)$  admits a co-orientable taut foliation by [3, Corollary 0.3]. This implies that  $I_*(Y) \neq 0$ , and this in turn implies that there exists an irreducible  $SU(2)$ -representation. On the other hand, the binary dodecahedral group is well known to admit two conjugacy classes of irreducible representations, completing the proof. Note that, unlike for Seifert integer homology three spheres, the Casson invariant of a non-trivial graph manifold can be zero. For example, the three-manifold  $Y$  obtained as the splice of two copies of the exterior of the right-handed trefoil has trivial Casson invariant [5, 16].

## Outline

In Section 2 we establish the main technical result Theorem 1.3 whose strategy also leads us to prove Corollary 1.4 about Mazur manifolds. In Section 3, we review the pillowcase construction and prove Theorem 1.1 in subsection 3.3, using a technical result about invariance under holonomy perturbations in instanton Floer homology reviewed in Section 4. The material in Section 4 is mostly known (or at least folklore knowledge) and can be found elsewhere, but the reader might appreciate our synthesis of the role of holonomy perturbations and our sketch of invariance in order to follow more easily through the proof of our main results. In Section 5, we prove Corollary 1.5.

<sup>†</sup> Some authors impose additional constraints, such as primeness or a non-trivial torus decomposition.

## 2 | INSTANTON FLOER HOMOLOGY OF 0-SURGERY

In this section we rely solely on formal properties of instanton Floer homology to prove Theorem 1.3 regarding the instanton Floer homology of 0-surgeries, and Corollary 1.4 regarding the instanton Floer homology of integer homology three spheres that bound Mazur manifolds. More concrete aspects of instanton Floer homology groups, in particular those regarding perturbations, appear in Section 4, but in this section we wish to place the focus on the usefulness of formal properties for purposes of computations.

We consider instanton Floer homology for admissible bundles, as introduced by Floer [15]. For integer homology three spheres, this is the trivial  $SU(2)$ -bundle over  $Y$ . For three-manifolds with positive first Betti number, this is an  $SO(3)$ -bundle  $P \rightarrow Y$  such that there is a surface  $\Sigma \subseteq Y$  on which the second Stiefel–Whitney class  $w := w_2(P)$  evaluates non-trivially, that is, such that  $\langle w_2(P), [\Sigma] \rangle \neq 0$ . The instanton Floer homology group is defined as a version of Morse homology of the Chern–Simons function on the space of connections on the admissible bundle [10, 15]. It is denoted by  $I_*(Y)$  for the trivial bundle on integer homology three spheres, and it is denoted by  $I_*^w(Y)$  for  $SO(3)$ -bundles  $P \rightarrow Y$  with  $w_2(P) = w$ . We remark here that for an integer homology three-sphere, the trivial connection is isolated and is the unique reducible connection (up to gauge equivalence). In the other cases, the admissibility condition ensures that there are no reducible flat connections on the bundle.

In the case of a knot  $K$  in an integer homology three-sphere  $Y$ , there is a unique admissible bundle on the 0-surgery  $Y_0(K)$ , because  $H^2(Y_0(K); \mathbb{Z}/2) \cong \mathbb{Z}/2$ . Therefore, the instanton Floer homology group  $I^w(Y_0(K))$  is defined without ambiguity.

**Proposition 2.1.** *Instanton Floer homology satisfies the following properties.*

- (1) For  $Y$  an integer homology three-sphere and any  $n \in \mathbb{Z}$ , the three-manifolds  $Y_{1/n}(K)$ ,  $Y_{1/(n+1)}(K)$ , and  $Y_0(K)$  fit into an exact triangle

$$\begin{array}{ccc} & I_*(Y_{1/n}(K)) & \\ & \nearrow & \searrow \\ I_*(Y_{1/(n+1)}(K)) & \longleftarrow & I_*^w(Y_0(K)) \end{array}$$

- (2) If  $M$  is an irreducible three-manifold with  $b_1(M) = 1$ , then  $I_*^w(M) \neq 0$ .  
 (3) For  $Y$  an integer homology three-sphere, if  $\pi_1(Y)$  admits no irreducible  $SU(2)$ -representations, then  $I_*(Y) = 0$ .  
 (4)  $I_*^w(S^2 \times S^1) = 0$ .

*Proof.* The surgery exact triangle in (2.1(1)) is originally due to Floer [15, Theorem 2.4] with details given in [6, Theorem 2.5]. The non-triviality result in (2.1(2)) is precisely [22, Theorem 7.21]. Next, (2.1(3)) follows from [14, Theorem 1], since if  $\pi_1(Y)$  admits no irreducible  $SU(2)$ -representations, then the generating set for the instanton Floer chain groups is empty. Finally, (2.1(4)) follows from (2.1(3)) and (2.1(1)), by considering the surgery exact triangle for surgery on the unknot in  $S^3$ . Alternatively, this follows from the definition of  $I_*^w$  (see Section 4), since  $\pi_1(S^2 \times S^1)$  admits no representations to  $SO(3)$  which do not lift to  $SU(2)$ .  $\square$

We will be particularly interested in integer homology three spheres whose fundamental groups do not admit irreducible  $SU(2)$ -representations. We therefore establish the following definition.

**Definition 2.2.** An integer homology three-sphere  $Y$  is  $SU(2)$ -cyclic if every  $SU(2)$ -representation of  $\pi_1(Y)$  is trivial.

Notice that Conjecture 1 states that  $S^3$  is the only  $SU(2)$ -cyclic integer homology three-sphere.

Having stated the above formal properties of instanton Floer homology, the proofs of Theorem 1.3 and Corollary 1.4 now follow easily.

### 2.1 | Non-vanishing of instanton Floer homology

In this subsection we illustrate the way the formal properties from Proposition 2.1 can be used to show that the instanton homology groups are non-zero in two cases: (1) three-manifolds obtained as 0-surgery along knots in  $SU(2)$ -cyclic integer homology three spheres whose exterior is irreducible and boundary incompressible, and (2) three-manifolds other than  $S^3$  obtained as the boundary of a Mazur manifold.

*Proof of Theorem 1.3.* We assume that  $I_*^w(Y_0(K))$  is trivial and argue by contradiction. By Proposition 2.1(1) the three-manifolds  $Y_{1/n}(K)$ ,  $Y_{1/(n+1)}(K)$ , and  $Y_0(K)$  fit together in an exact triangle

$$\begin{array}{ccc}
 & I_*(Y_{1/n}(K)) & \\
 & \nearrow & \searrow \\
 I_*(Y_{1/(n+1)}(K)) & \longleftarrow & I_*^w(Y_0(K)).
 \end{array}$$

The assumption  $I_*^w(Y_0(K)) = 0$  implies that there is an isomorphism

$$I_*(Y_{1/(n+1)}(K)) \cong I_*(Y_{1/n}(K)) \text{ for each } n \in \mathbb{Z}.$$

In particular, if  $n = 0$ , then  $I_*(Y_1(K)) \cong I_*(Y) = 0$  thus showing that for all  $n \in \mathbb{Z}$ ,

$$I_*(Y_{1/n}(K)) = 0. \tag{1}$$

Now, a result of Gordon [17, Lemma 7.2] shows that  $Y_{1/4}(K)$  is diffeomorphic to  $Y_1(K_{2,1})$ , where  $K_{2,1}$  is the (2,1)-cable of the knot  $K$  (see Figure 5 for an example of  $K_{2,1}$ ). This together with Equation 1 implies  $I_*(Y_1(K_{2,1})) = 0$ . An iteration of an exact triangle as in Proposition 2.1(1) for surgeries along  $K_{2,1}$  gives  $I_*^w(Y_0(K_{2,1})) = 0$ .

We now consider a decomposition of  $Y_0(K_{2,1})$  that includes the knot exterior of  $K$  in  $Y$ . Denote by  $C_{2,1}$  a closed curve that lies in the boundary of a ‘small’ solid torus  $S^1 \times \partial D_{1/2}^2 \subset S^1 \times D^2$ , and representing the class  $2[S^1] + [\partial D_{1/2}^2]$  in  $H_1(S^1 \times \partial D_{1/2}^2)$ . Notice that the 0-framing of  $K_{2,1}$  in  $Y$  induces the framing on  $C_{2,1}$  determined by the curve  $\lambda$  in  $\partial N(C_{2,1})$  that represents the class  $2[S^1]$  in  $H_1(S^1 \times \partial D^2)$  (see [17, p. 692]). Therefore, the manifold  $Y_0(K_{2,1})$  can be expressed as the union of the knot exterior  $Y \setminus N(K)$ , and the result of Dehn surgery on  $S^1 \times D^2$  along the curve  $C_{2,1}$

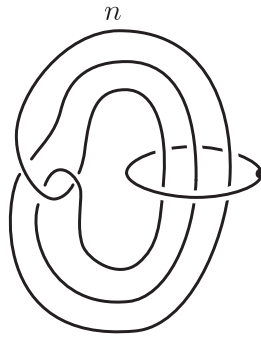


FIGURE 1 A Mazur manifold with one two-handle attached with framing given by  $n$  for some  $n \in \mathbb{N}$

with framing given by  $\lambda$ . By hypothesis the knot exterior  $Y \setminus N(K)$  is irreducible and boundary-incompressible, and by [17, Lemma 7.2] the 0-surgery along the curve  $C_{2,1}$  is a Seifert-fibred space with incompressible boundary. Hence  $Y_0(K_{2,1})$  is an irreducible closed three-manifold with first Betti number equal to 1 and with trivial instanton Floer homology. However, this contradicts Proposition 2.1(2), which says that  $I_*^w(Y_0(K_{2,1})) \neq 0$ .  $\square$

Next, consider integer homology three spheres that bound a Mazur manifold, that is, a four-manifold that admits a handle decomposition in terms of exactly one 0-handle, one 1-handle, and one 2-handle which algebraically runs exactly once over the 1-handle, such as in Figure 1. Then we have the following.

*Proof of Corollary 1.4.* If  $Y$  bounds a Mazur manifold, then there exists a knot  $J$  in  $Y$  such that  $Y_0(J) = S^2 \times S^1$ . Moreover, if  $I_*(Y) = 0$ , a combination of the surgery exact triangle from Proposition 2.1(1) and the computation  $I_*^w(S^2 \times S^1) = 0$  from Proposition 2.1(4) shows once again that  $I_*(Y_{1/4}(J)) = 0$ . The same argument used above in the proof of Theorem 1.3 then gives  $I_*(Y_0(J_{2,1})) = 0$ .

However, the exterior of a knot in  $S^2 \times S^1$  which generates homology is either irreducible and boundary-incompressible or a solid torus. To see this claim, suppose that the complement of a knot  $K$  in  $S^2 \times S^1$  is reducible. Since  $K$  is non-trivial in homology, it intersects every non-separating  $S^2$ , and so there are no  $S^2 \times S^1$  summands in the exterior of  $K$ . Hence, the exterior of  $K$  is the connected sum of a closed three-manifold  $N$  other than  $S^3$  and a three-manifold with torus boundary  $M$ . But we know that there is a filling of the exterior of  $K$  which is  $S^2 \times S^1$ , and, in particular, this is irreducible. Therefore, the separating 2-sphere must bound a ball on the side of  $M$  after the filling. But this implies that  $N$  is  $S^2 \times S^1$ , and this is a contradiction. On the other hand, if the exterior of  $K$  has compressible boundary and is not a solid torus, then that means that the exterior is the connected sum of a solid torus with a closed three-manifold. The exterior of  $K$  is then reducible, a contradiction we have already established. So the only possibility that remains is that the exterior is a solid torus.

The case where the exterior of a knot in  $S^2 \times S^1$  is a solid torus corresponds to  $Y = S^3$ , so by our assumption the exterior of  $J$  is irreducible and boundary-incompressible. As in the proof of Theorem 1.3 we get that  $Y_0(J_{2,1})$  is irreducible with  $b_1 = 1$ . But this contradicts Proposition 2.1(2).  $\square$

### 3 | THE PILLOWCASE ALTERNATIVE

In this section we recall the relevant background on  $SU(2)$ -character varieties and generalize work of the third author [36] to prove Theorem 1.1, our main result.

#### 3.1 | The pillowcase

Given a connected manifold  $M$ , we denote by

$$R(M) = \text{Hom}(\pi_1(M), SU(2))/SU(2)$$

the space of  $SU(2)$ -representations of its fundamental group, up to conjugation. We will write  $R(M)^*$  for the subset of irreducible representations. For example, the space  $R(T^2)$  is identified with the *pillowcase*, an orbifold homeomorphic to a two-dimensional sphere with four corner points. To see this, notice that since  $\pi_1(T^2) \cong \mathbb{Z}^2$  is abelian, the image of any representation  $\rho : \pi_1(T^2) \rightarrow SU(2)$  is contained in a maximal torus subgroup of  $SU(2)$ . Up to conjugation, this torus can be identified with the circle group consisting of matrices of the form  $\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$  for  $\theta \in [0, 2\pi]$ . Thus, if we denote the generators of  $\pi_1(T^2) \cong \mathbb{Z}^2$  by  $m$  and  $l$ , then, again after conjugation, a representation  $\rho \in R(T^2)$  is determined by

$$\rho(m) = \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix} \quad \text{and} \quad \rho(l) = \begin{bmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{bmatrix},$$

and hence we can associate to  $\rho$  a pair  $(\alpha, \beta) \in [0, 2\pi] \times [0, 2\pi]$ . However, conjugation of  $\rho$  by the element  $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  gives rise to the representation associated to the pair  $(2\pi - \alpha, 2\pi - \beta)$ . This is the only ambiguity, however, as can be seen using the fact that the trace of an element in  $SU(2)$  determines its conjugacy class. Therefore  $R(T^2)$  is isomorphic to the quotient of the fundamental domain  $[0, \pi] \times [0, 2\pi]$  by identifications on the boundary as indicated in Figure 2.

If we have a three-manifold  $M$  with torus boundary, then the inclusion  $i : T^2 \cong \partial M \hookrightarrow M$  induces a map  $i^* : R(M) \rightarrow R(T^2)$  by restricting a representation to the boundary. For instance, if  $K$  is a knot in a three-manifold  $Y$ , then the three-manifold  $Y(K) := Y \setminus N(K)^\circ$  obtained by removing the interior of a tubular neighborhood  $N(K)$  of  $K$  from  $Y$  is a three-manifold with boundary a two-dimensional torus. Figure 2 shows the image of  $R(S^3(K))$  when  $K$  is the right-handed trefoil in  $S^3$ , once in the pillowcase, and once in the fundamental domain  $[0, \pi] \times [0, 2\pi]$ . Here we use the convention that the first coordinate corresponds to  $\rho(m_K)$ , where  $m_K$  is a meridian to the knot  $K$ , and the second coordinate corresponds to  $\rho(l_K)$ , where  $l_K$  is a longitude of the knot  $K$ .

For a knot  $K$  in a three-manifold  $Y$  there is a well-defined notion of meridian  $m_K$ , and if the knot is nullhomologous, there is a well-defined notion of longitude  $l_K$ . In particular, this is the case for any knot  $K$  in an integer homology three-sphere  $Y$ . In what follows, we will use the notation  $R(K) := R(Y(K))$  if it is clear which integer homology three-sphere  $Y$  we have in mind, and we will stick to the above convention of the coordinates in  $R(T^2)$  corresponding to the meridian and longitude of  $K$ . With these conventions, all abelian representations in  $R(K)$  map under  $i^*$  to the thick red line  $\{\beta = 0 \pmod{2\pi\mathbb{Z}}\}$  ‘at the bottom’ of the pillowcase  $R(T^2)$ . Indeed,  $l_K$  is a product of commutators in the fundamental group of the knot complement, so an abelian



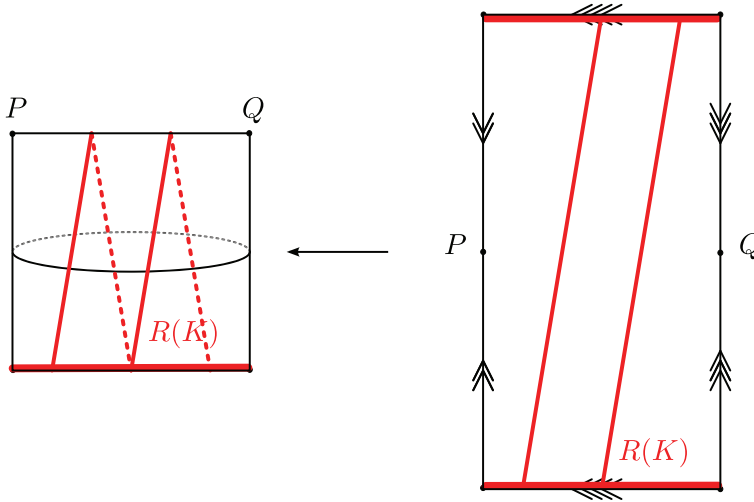


FIGURE 2 The gluing pattern for obtaining the pillowcase from a rectangle, and the image of the representation variety  $R(K)$  of the trefoil in the pillowcase

representation necessarily maps  $l_K$  to the identity. Furthermore, for any  $\alpha \in [0, \pi]$  we can find an abelian representation of  $R(K)$  whose restriction to  $R(T^2)$  corresponds to  $(\alpha, 0)$ .

If we cut the pillowcase open along the lines  $a_0 := \{\alpha = 0 \pmod{2\pi\mathbb{Z}}\}$  and  $a_\pi := \{\alpha = \pi \pmod{2\pi\mathbb{Z}}\}$ , we obtain a cylinder  $C = [0, \pi] \times \mathbb{R}/2\pi\mathbb{Z}$ . In the gluing pattern of Figure 2 this means that we do not perform the identifications along the four indicated vertical boundary lines.

Our main goal is to prove Theorem 3.5 below, which asserts that if  $K$  is a knot in an  $SU(2)$ -cyclic integer homology three-sphere whose 0-surgery has non-trivial instanton homology, then the image of  $R(K)$  in the pillowcase contains a homologically non-trivial embedded closed curve in the cylinder  $C$ . In order to derive Theorem 1.1 from this, we need a more refined statement, namely, that there is a homologically non-trivial embedded closed curve in  $i^*(R(K))$  that is disjoint from a neighborhood of the two lines  $a_0$  and  $a_\pi$ . Notice that for a knot in  $S^3$ , there are no representations with  $\rho(l_K) \neq \text{id}$  and  $\rho(m_K) = \pm \text{id}$ . This is because the fundamental group of a knot complement in  $S^3$  is normally generated by the meridian of the knot. In particular, there are no representations in  $i^*(R(K))$  that have coordinates  $(\alpha, \beta)$  with  $\beta \neq 0$ , and  $\alpha = 0$  or  $\alpha = \pi$ . In [36, Proposition 8.1], it is shown that the image of  $R(K)^*$ , the subset of irreducible representations in  $R(K)$ , in fact, stays outside a neighborhood of these two lines. We begin with a generalization of this fact.

**Lemma 3.1.** *Let  $K$  be a knot in an  $SU(2)$ -cyclic integer homology three-sphere  $Y$ . There is a neighborhood of the lines  $\{\alpha = 0 \pmod{2\pi\mathbb{Z}}\}$  and  $\{\alpha = \pi \pmod{2\pi\mathbb{Z}}\}$  in the pillowcase which is disjoint from the image of  $R(K)^*$ .*

*Proof.* Suppose by contradiction that the image of  $R(K)^*$  intersects every neighborhood of the lines  $\{\alpha = 0 \pmod{2\pi\mathbb{Z}}\}$  and  $\{\alpha = \pi \pmod{2\pi\mathbb{Z}}\}$ . If that was the case, then we could find a sequence of elements in  $R(K)^*$  whose image under  $i^*$  converges to a point on one of the two lines. By the compactness of  $R(K)$ , the limit is the image of a representation  $\rho : \pi_1(Y(K)) \rightarrow SU(2)$  sending every meridional curve  $\mu$  to  $\pm 1$ . We first claim that  $\rho$  must be a central representation (and hence reducible), and so its image under  $i^*$  can only be  $(0, 0)$  or  $(\pi, 0)$ .

First, if  $\rho(\mu) = 1$ , then  $\rho : \pi_1(Y(K)) \rightarrow SU(2)$  is really a representation of  $\pi_1(Y)$ . Since  $Y$  is assumed to be  $SU(2)$ -cyclic, then the representation is trivial and therefore  $\rho(\lambda) = 1$ . As a consequence, if the limit of elements in  $R(K)^*$  is an element of the line  $\{\alpha = 0 \pmod{2\pi\mathbb{Z}}\}$ , then it is the point  $(0,0)$  in the pillowcase. Next, consider the case that  $\rho(\mu) = -1$ . If the representation  $\rho$  is irreducible, then we obtain an irreducible representation  $\tilde{\rho} : \pi_1(Y) \rightarrow SO(3)$ . The obstruction to lifting an  $SO(3)$  representation into an  $SU(2)$ -representation is an element of  $H^2(Y; \mathbb{Z}/2)$ , and since  $Y$  is an integer homology three-sphere, the obstruction vanishes and  $\tilde{\rho}$  would lift to an irreducible representation to  $SU(2)$ , contradicting the fact that  $Y$  is  $SU(2)$ -cyclic. Therefore, a representation  $\rho : \pi_1(Y(K)) \rightarrow SU(2)$  satisfying  $\rho(\mu) = -1$  is reducible and hence abelian, and so factors through  $H_1(Y(K))$ . Because  $\lambda$  is trivial in  $H_1(Y(K))$ , we see that  $\rho$  is the central representation sending  $\mu$  to  $-1$  and  $\lambda$  to  $1$ , and this corresponds to the point  $(-\pi, 0)$  in the pillowcase. All of this shows that if a sequence of elements in  $i^*R(K)$  converges to a point on the lines  $\{\alpha = 0 \pmod{2\pi\mathbb{Z}}\}$  and  $\{\alpha = \pi \pmod{2\pi\mathbb{Z}}\}$ , then the limit point is a central representation. For notation, we will call these representations  $\rho_{\pm}$  for the sign of the image of  $\mu$ .

Now, it remains to show that the points  $(0,0)$  and  $(\pi, 0)$  cannot be limits of irreducible representations. We remark here that this fact does not require that  $Y$  is  $SU(2)$ -cyclic. Let  $\Gamma = \pi_1(Y(K))$ . A result of Weil [33] expanded in [25, Chapter 2] shows that  $T_{\rho}R(K)$  corresponds to  $H^1(\Gamma; \mathfrak{su}(2)_{\text{ad},\rho})$ . This group is identified with the first cohomology group (with twisted coefficients) of a  $K(\Gamma, 1)$ -space, or more generally, with the first (twisted) cohomology of any CW complex with fundamental group isomorphic to  $\Gamma$ . This shows that  $H^1(\Gamma; \mathfrak{su}(2)_{\text{ad},\rho}) = H^1(Y(K); \mathfrak{su}(2)_{\text{ad},\rho})$  and so  $T_{\rho}R(K) = H^1(Y(K); \mathfrak{su}(2)_{\text{ad},\rho})$ . Next, since each representation  $\rho_{\pm}$  is central, then  $\text{ad} \circ \rho_{\pm}$  is the trivial representation and so

$$H^1\left(Y(K); \mathfrak{su}(2)_{\text{ad},\rho_{\pm}}\right) = H^1\left(Y(K); \mathbb{R}^3\right) \cong \mathbb{R}^3.$$

This shows that the tangent space to  $R(K)$  at  $\rho_{\pm}$  is three-dimensional. Since we obtain three dimensions of freedom by abelian representations near  $\rho_{\pm}$  in  $R(K)$ , the entire tangent space to  $R(K)$  consists of tangent vectors to abelian representations and so there cannot be irreducible representations near  $\rho_{\pm}$ , completing the proof.  $\square$

### 3.2 | Essential curves in the pillowcase

In this section, we relate the instanton Floer homology of 0-surgery on a knot to the image of the character variety of the knot exterior in the pillowcase. This will be the key step in the proof of Theorem 1.1, found at the end of this subsection.

We next establish some notation, following Kronheimer–Mrowka in [20], which will be useful in the proof of our next theorem.

**Definition 3.2.** For a subset  $L \subseteq R(T^2)$ , we denote by  $R(K|L)$  the set of elements  $[\rho] \in R(K)$  such that  $[i^*\rho] \in L$ .

**Theorem 3.3.** *Let  $K$  be a knot in an integer homology three-sphere  $Y$ , and assume that the instanton Floer homology of the 0-surgery is non-vanishing,  $I_*^w(Y_0(K)) \neq 0$ . Then any topologically embedded path from  $P = (0, \pi)$  to  $Q = (\pi, \pi)$  in the associated pillowcase has an intersection point with the image of  $R(K)$ .*

Before proving the theorem, we point out that this generalizes [36, Theorem 7.1], from knots in  $S^3$  to knots in general integer homology three spheres. The main difference in the argument compared to [36, Theorem 7.1] is that here we make use of the non-trivial instanton Floer homology of the 0-surgery in an essential way, which is exploited through its connection with holonomy perturbations of the Chern–Simons functional. The arguments of the third author in [36] instead use holonomy perturbations of a moduli space which computes the Donaldson invariants of a closed 4-manifold containing the 0-surgery as a hypersurface. In that case, the non-vanishing result builds on the existence of a taut foliation on  $S_0^3(K)$  for a non-trivial knot  $K$ . In the case at hand, we do not know whether  $Y_0(K)$ , the 0-surgery on a knot  $K$  in the integer homology three-sphere  $Y$ , admits a taut foliation.

*Proof.* Suppose by contradiction that there is a continuous embedded path  $c$  from  $P$  to  $Q$  such that its image is disjoint from  $i^*(R(K)) \subseteq R(T^2)$ . (We will not distinguish between paths and their image for the remainder of this proof.) In other words,  $R(K|c)$  is empty. In particular, we may suppose that  $c$  is disjoint from the bottom line  $\{\beta = 0\}$  of the pillowcase  $R(T^2)$ , since any element of this line lies in the image of  $i^*$ . Since the image  $i^*(R(K))$  is compact, there is a neighborhood  $U \subseteq R(T^2)$  of the image of  $c$  in  $R(T^2)$  which is still disjoint from  $i^*(R(K))$ . Since  $R(K|c)$  is empty, for  $c'$  sufficiently close to  $c$ ,  $R(K|c')$  is empty as well.

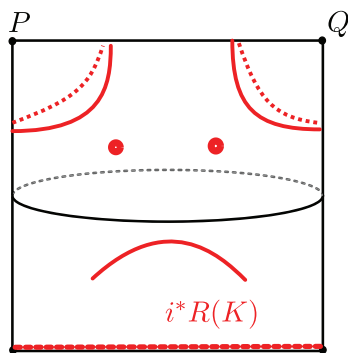
Associated to a three-manifold and admissible bundle, we consider two objects: the Chern–Simons functional and holonomy perturbations of the Chern–Simons functional. These are described in detail in Section 4, in particular Sections 4.2 and 4.3, but their definition is not needed for the proof. Given a three-manifold  $Z$  with admissible bundle represented by  $w$  and a holonomy perturbation  $\Psi$ , let  $R_\Psi^w(Z)$  denote the set of critical points of the Chern–Simons functional perturbed by  $\Psi$ . By Theorem 4.4 below (which is essentially a synthesis of [36, Theorem 4.2 and Proposition 5.3]), there exists a path  $c'$  arbitrarily close to  $c$  and a (holonomy) perturbation  $\Psi$  of the Chern–Simons functional such that  $R_\Psi^w(Y_0(K))$  is a double cover of  $R(K|c')$ . Therefore,  $R_\Psi^w(Y_0(K))$  is empty, so computing Morse homology with respect to this perturbation of the Chern–Simons functional produces a trivial group. However, Theorem 4.5 below asserts that computing Morse homology with respect to the particular perturbation  $\Psi$  produces a group isomorphic to  $I_*^w(Y_0(K))$ , which is non-zero by assumption. Therefore, we obtain a contradiction.  $\square$

*Remark 3.4.* Although [36, Proposition 5.3] is only stated for knots in  $S^3$ , the arguments used in its proof apply for a knot in an arbitrary  $SU(2)$ -cyclic integer homology three sphere.

If we combine the constraint that  $Y$  is  $SU(2)$ -cyclic with the assumption that  $I_*^w(Y_0(K))$  is non-trivial, then we obtain the following generalization of [36, Theorem 7.1], which will be the last step before the proof of our main theorem.

**Theorem 3.5** (Pillowcase alternative). *Suppose that  $Y$  is an  $SU(2)$ -cyclic integer homology three-sphere. Suppose that  $K$  is a knot in  $Y$  such that the 0-surgery  $Y_0(K)$  has non-trivial instanton Floer homology  $I_*^w(Y_0(K))$ , where  $w$  is the non-zero class in  $H^2(Y_0(K); \mathbb{Z}/2) \cong \mathbb{Z}/2$ . Then the image  $i^*(R(Y(K)))$  in the cut-open pillowcase  $C = [0, \pi] \times (\mathbb{R}/2\pi\mathbb{Z})$  contains a topologically embedded curve which is homologically non-trivial in  $H_1(C; \mathbb{Z}) \cong \mathbb{Z}$ .*

*Proof.* The hypothesis implies that the lines  $\{(0, \beta) \in R(T^2) \mid \beta \neq 0\}$  and  $\{(\pi, \beta) \in R(T^2) \mid \beta \neq 0\}$  have empty intersection with  $i^*(R(Y(K)))$  (Figure 3). The conclusion then follows from



**FIGURE 3** This is a hypothetical image of a representation variety  $i^*(R(K))$  of a knot  $K$  in an integer homology three-sphere  $Y$ . The homology three-sphere  $Y$  is assumed to satisfy  $I_*^w(Y_0(K)) \neq 0$  and assumed to not be  $SU(2)$ -cyclic. As a consequence,  $i^*(R(K))$  intersects every path joining  $P$  and  $Q$  as in Theorem 3.3, but it does not contain a curve which is homologically non-trivial in the cut-open pillowcase  $C = [0, \pi] \times (\mathbb{R}/2\pi\mathbb{Z})$ . This hypothetical example thus illustrates that the  $SU(2)$ -cyclic assumption is necessary in Theorem 3.5.

Theorem 3.3 together with the Alexander duality argument of [36, Lemma 7.3]. For Alexander duality to work we use the fact that  $i^*(R(Y(K)))$  is a semi-algebraic set of dimension 1, and hence a finite graph, see [2]. □

### 3.3 | Main result

In this subsection we prove that if an integer homology three-sphere contains an embedded incompressible torus, then the fundamental group of the homology three-sphere admits irreducible  $SU(2)$ -representations. To derive our result we first recall that we can realize a toroidal integer homology three-sphere as a splice, as in [11, Proof of Corollary 6.2]. We then study the image of the two knot exteriors in the pillowcase of the incompressible torus. With this in mind, we include the following definition.

**Definition 3.6.** Let  $K_1 \subset Y_1$  and  $K_2 \subset Y_2$  be oriented knots in oriented integer homology three spheres. For  $i = 1, 2$ , denote by  $\mu_i, \lambda_i \subset \partial N(K_i)$  a meridian and longitude for  $K_i$  in  $Y_i$ . Form a three-manifold  $Y$  as

$$(Y_1 \setminus N(K_1)^\circ) \cup_h (Y_2 \setminus N(K_2)^\circ),$$

where  $h : \partial N(K_1) \rightarrow \partial N(K_2)$  identifies  $\mu_1$  with  $\lambda_2$  and  $\lambda_1$  with  $\mu_2$ . The manifold  $Y$  is called the splice of  $Y_1$  and  $Y_2$  along knots  $K_1$  and  $K_2$ .

Let  $Y$  be an integer homology three sphere and let  $T$  be a two-dimensional torus embedded in  $Y$  in such manner that its normal bundle is trivial. A simple application of the Mayer–Vietoris sequence shows that  $Y \setminus N(T)^\circ$  has two connected components  $M_1, M_2$ , and that each component has the same homology groups as  $S^1$ . The ‘half lives, half dies’ principle shows that for each  $i = 1, 2$  there exists a basis  $(\alpha_i, \beta_i)$  for the peripheral subgroup of  $\partial M_i$  such that  $\beta_i$  is nullhomologous in  $M_i$ . Therefore, if  $Y_i$  denotes the union of  $M_i$  and a solid torus  $S^1 \times D^2$  in such a way that the curve

$\{1\} \times \partial D^2$  gets identified with  $\alpha_i$ , then  $Y_i$  is an integer homology three-sphere. Moreover, since  $T$  is incompressible in  $Y$ , then the core of the solid torus in  $Y_i$  is a non-trivial knot  $K_i$ . In other words, every toroidal integer homology three-sphere can be expressed as a splice of non-trivial knots  $K_1$  and  $K_2$  in integer homology three spheres  $Y_1$  and  $Y_2$ .

With all of this in place, we are ready to prove our main result.

*Proof of Theorem 1.1.* Realize  $Y$  as a splice  $(Y_1 \setminus N(K_1)^\circ) \cup_h (Y_2 \setminus N(K_2)^\circ)$ , with  $K_1, K_2$  non-trivial knots. Suppose first that  $Y_i \setminus N(K_i)^\circ$  is reducible, in other words, that  $Y_i \setminus N(K_i)^\circ = Q_i \# (Z_i \setminus N(J_i)^\circ)$  where  $Q_i, Z_i$  are integer homology three spheres and  $J_i \subset Z_i$  has irreducible and boundary-incompressible exterior. As a consequence of van Kampen’s theorem, there exists a surjection  $\pi_1(Y_i \setminus N(K_i)^\circ) \rightarrow \pi_1(Z_i \setminus N(J_i)^\circ)$ , and this surjection induces a  $\pi_1$ -surjection from  $Y$  to the splice of  $(Z_1, J_1)$  and  $(Z_2, J_2)$ . Thus, our proof reduces to the case when  $Y$  is the splice of two knots with irreducible and boundary-incompressible exteriors, which we assume from now on.

Next, by the Seifert–van Kampen theorem, the pieces of the decomposition fit into the following commutative diagram:

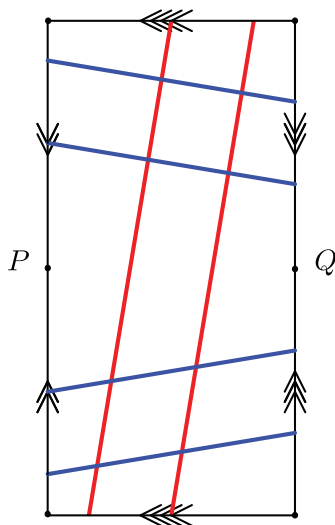
$$\begin{array}{ccc}
 & \pi_1(Y_1 \setminus N(K_1)^\circ) & \\
 \pi_1(T) & \swarrow & \searrow \\
 & & \pi_1(Y) \\
 & \searrow & \swarrow \\
 & \pi_1(Y_2 \setminus N(K_2)^\circ) &
 \end{array}$$

and since each  $Y_i \setminus N(K_i)^\circ$  is a homology circle, there exists a  $\pi_1$ -surjection from  $Y$  to each  $Y_i$ . Therefore, our proof reduces further to the case when both  $Y_1$  and  $Y_2$  are  $SU(2)$ -cyclic since an irreducible representation for  $Y_i$  gives rise to one for  $Y$ .

To recap, the previous two paragraphs allow us to assume that  $Y$  is the splice of  $(Y_1, K_1), (Y_2, K_2)$  with each  $Y_i$  an  $SU(2)$ -cyclic homology three sphere, and each  $K_i \subset Y_i$  a knot with irreducible and boundary-incompressible exterior. Then, as a consequence of Proposition 2.1(3) we have that each  $Y_i$  has trivial instanton Floer homology. Moreover, since each  $Y_i \setminus N(K_i)^\circ$  is irreducible and boundary-incompressible, Theorem 1.3 shows that the instanton Floer homology of 0-surgery on  $Y_i$  along  $K_i$  is non-zero. Therefore, the hypotheses of both Theorem 3.5 and Lemma 3.1 hold, and the proof now follows exactly as in [36, Proof of Theorem 8.3(i)] with [36, Theorem 7.1] and [36, Proposition 8.1(ii)] replaced by Theorem 3.5 and Lemma 3.1 respectively (Figure 4).  $\square$

## 4 | REVIEW OF INSTANTON FLOER HOMOLOGY AND HOLONOMY PERTURBATIONS

We start this section with a disclaimer: We do not claim to prove any original or new result in this section. However, we review instanton Floer homology and holonomy perturbations to the extent which is necessary in order to understand the proof of our main results above. For instance, Section 4.3 below contains a synthesis of the third author’s results about holonomy perturbations from [36] which we hope that the reader unfamiliar with this reference will appreciate. Section 4.5 contains a result about invariance under holonomy perturbations in the context of an admissible bundle with non-trivial second Stiefel–Whitney class, together with a sketch of proof. Again, this result is already contained in [14] and [10], but by looking up these references it may not be immediately clear whether these results apply verbatim in our situation.



**FIGURE 4** Let  $Y$  be the three-manifold obtained as the splice of two copies of the exterior of a right-handed trefoil, and let  $T$  be the incompressible torus given as the intersection of the two knot exteriors. The figure shows the image of each copy of  $R(T_{2,3})^*$  in the pillowcase. Note that any representation of the splice corresponding to an intersection of the red and blue curves is irreducible.

The proof of Theorem 3.3 relies on a non-vanishing result of an instanton Floer homology group  $I_{*,\Phi}^w(Y_0(K))$ , computed with suitable perturbation terms  $\Phi$  of the Chern–Simons function. We will review the construction of these perturbation terms below, which are built from the holonomy along families of circles, parametrized by embedded surfaces. The critical points of the complex underlying the homology group  $I_{*,\Phi}^w(Y_0(K))$  will have a clear interpretation in terms of intersections of the representation variety  $R(K)$  with certain deformations of the path given by the straight line  $\{\beta = \pi\}$  in the pillowcase, resulting as the representation variety of the boundary of the exterior of  $K$  in  $Y$  as before.

On the other hand, Theorem 1.3 yields a non-vanishing result for  $I_*^w(Y_0(K))$ , defined in the usual way, and in particular without the above class of perturbation terms. We can therefore complete the proof from the fact that the two instanton Floer homology groups,  $I_*^w(Y_0(K))$  and  $I_{*,\Phi}^w(Y_0(K))$ , are isomorphic, and we sketch the proof of this below.

*Remark 4.1.* In the construction of both  $I_*^w(Y_0(K))$  and  $I_{*,\Phi}^w(Y_0(K))$  there are typically perturbation terms involved for the sake of transversality. These can be chosen as small as one likes, in a suitable sense. We will omit these auxiliary perturbations from our notation. The perturbations labeled by the terms  $\Phi$ , however, will have a clear geometric purpose, and the discussion below will focus on these.

### 4.1 | The Chern–Simons function

For details on the holonomy perturbations we use we refer the reader to Donaldson’s book [10], Floer’s original article [15], and the third author’s article [36].

If we deal with an admissible  $SO(3)$ -bundle  $F \rightarrow Y$  over a three-manifold  $Y$  with second Stiefel–Whitney class  $w$ , we may suppose that it arises from an  $U(2)$ -bundle  $E \rightarrow Y$  as its adjoint bundle  $\mathfrak{su}(E)$ , see, for instance, [10, Section 5.6]. Then  $w = w_2(E) \equiv c_1(E) \pmod{2}$ . The space of  $SO(3)$ -connections on  $F$  is then naturally isomorphic to the space of  $U(2)$ -connections on  $E$  that induce a fixed connection  $\theta$  in the determinant line bundle  $\det(E)$ , which we will suppress from notation.

When dealing with functoriality properties, it is more accurate to consider  $w$  to be an embedded 1-manifold which is Poincaré dual to  $w_2(E) = w_2(F)$ , see [23].

We will fix a reference connection  $A_0$  on  $E$  and consider the Chern–Simons function

$$\text{CS} : \mathcal{A} \rightarrow \mathbb{R}$$

$$A \mapsto \int_Y \text{tr}(2a \wedge (F_{A_0})_0 + a \wedge d_{A_0} a + \frac{1}{3} a \wedge [a \wedge a]),$$

defined on the affine space  $\mathcal{A}$  of connections  $A$  in  $E$  which induce  $\theta$  in  $\det(E)$ , and where we have written  $A = A_0 + a$  with  $a \in \Omega^1(Y; \mathfrak{su}(E))$ . The term  $F_A$  denotes the curvature of a connection  $A$ , and  $(F_A)_0$  denotes its trace-free part, and  $d_A$  denotes the exterior derivative associated to a connection  $A$ . We denote by  $\mathcal{G}$  the group of bundle automorphisms of  $E$  which have determinant 1. The Chern–Simons function induces a circle-valued function  $\overline{\text{CS}} : \mathcal{B} \rightarrow \mathbb{R}/\mathbb{Z}$  on the space  $\mathcal{B} = \mathcal{A}/\mathcal{G}$  of connections modulo gauge equivalence, and the instanton Floer homology  $I_*^w(Y)$  is the Morse homology, in a suitable sense, of the Chern–Simons function  $\overline{\text{CS}}$ . To carry this out, one has to deal with a suitable grading on the critical points, which will only be a relative  $\mathbb{Z}/8$ -grading, with suitable compactness arguments (Uhlenbeck compactification and ‘energy running down the ends’), and with transversality arguments. In particular, one will in general add a convenient perturbation term to the Chern–Simons function to obtain the required transversality results. This is usually done by the use of holonomy perturbations that we discuss below. By a Sard–Smale-type condition, this term can be chosen as small as one wants, in the respective topologies one is working with. Therefore, we are suppressing these perturbations for the sake of transversality from our notation. One then needs to prove independence of the various choices involved, and in particular the Riemannian metric and the perturbation terms required for transversality.

One may also deal with orientations, but we do not need this in our situation, where  $\mathbb{Z}/2$ -coefficients in the Floer homology will be sufficient.

## 4.2 | Review of holonomy perturbations

To set up the perturbation of the Chern–Simons function we are using, we need to introduce some notation. Let  $\chi : SU(2) \rightarrow \mathbb{R}$  be a class function, that is, a smooth conjugation invariant function. Any element in  $SU(2)$  is conjugate to a diagonal element, and hence there is a  $2\pi$ -periodic even function  $g : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\chi \left( \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix} \right) = g(t) \quad (2)$$

for all  $t \in \mathbb{R}$ . Furthermore, let  $\Sigma$  be a compact surface with boundary, and let  $\mu$  be a real-valued two-form which has compact support in the interior of  $\Sigma$  and with  $\int_{\Sigma} \mu = 1$ . Let  $\iota : \Sigma \times S^1 \rightarrow Y$  be an embedding. Let  $N \subseteq Y$  be a codimension-zero submanifold containing the image of  $\iota$ , and

such that the bundle  $E$  is trivialized over  $N$  in such a way that the connection  $\theta$  in  $\det(E)$  induces the trivial product connection in the determinant line bundle of our trivialization of  $E$  over  $N$ . This means that connections in  $\mathcal{A}$  can be understood as  $SU(2)$ -connections in  $E$  when restricted to  $N$ .

Associated to this data, we can define a function

$$\Phi : \mathcal{A} \rightarrow \mathbb{R}$$

which is invariant under the action of the gauge group  $\mathcal{G}$ . For  $z \in \Sigma$ , we denote by  $\iota_z : S^1 \rightarrow Y$  the circle  $t \mapsto \iota(z, t)$ . A connection  $A \in \mathcal{A}$  provides an  $SU(2)$ -connection over the image of  $\iota$ . The holonomy  $\text{Hol}_{\iota_z}(A)$  of  $A$  around the loop  $\iota_z$  (with variable starting point) is a section of the bundle of automorphisms of  $E$  with determinant 1 over the loop. Since  $\chi$  is a class function,  $\chi(\text{Hol}_{\iota_z}(A))$  is well defined. We can therefore define

$$\Phi(A) = \int_{\Sigma} \chi(\text{Hol}_{\iota_z}(A)) \mu(z), \tag{3}$$

and this function is invariant under the action of the gauge group  $\mathcal{G}$ . It depends on the data  $(\iota, \chi, \mu)$  and a trivialization of the bundle over a codimension-zero submanifold  $N$ , but we will omit the latter from notation.

We will have to work with a finite sequence of such embeddings, all supported in a submanifold  $N$  of codimension zero over which the bundle  $E \rightarrow N$  is trivial. For some  $n \in \mathbb{N}$ , let  $\iota_k : S^1 \times \Sigma_k \rightarrow N \subseteq Y$  be a sequence of embeddings for  $k = 0, \dots, n - 1$  such that the interior of the image of  $\iota_k$  is disjoint from the interior of the image of  $\iota_l$  for  $k \neq l$ . We also suppose class functions  $\chi_k : SU(2) \rightarrow \mathbb{R}$  corresponding to even,  $2\pi$ -periodic functions  $g_k : \mathbb{R} \rightarrow \mathbb{R}$  as above to be chosen, for  $k = 0, \dots, n - 1$ , and we assume that  $\mu_k$  is a two form on  $\Sigma_k$  with support in the interior of  $\Sigma_k$  and integral 1. Just as in the case of (3), the data determine a finite sequence of functions

$$\Phi_k : \mathcal{A} \rightarrow \mathbb{R}, \quad k = 0, \dots, n - 1,$$

and we are interested in the Morse homology of the function

$$\overline{\text{CS}} + \Psi : \mathcal{B} \rightarrow \mathbb{R}/\mathbb{Z}, \quad \text{where } \Psi = \sum_{k=0}^{n-1} \Phi_k. \tag{4}$$

**Definition 4.2.** We denote by  $R_{\Psi}^w(Y)$  the space of critical points  $[A] \in \mathcal{A}/\mathcal{G}$  of the function  $\overline{\text{CS}} + \Psi : \mathcal{B} \rightarrow \mathbb{R}/\mathbb{Z}$ , where  $\Psi$  is specified by the holonomy perturbation data  $\{\iota_k, \chi_k\}$  as above.

If the holonomy perturbation data  $\Psi$  are chosen in a way such that  $R_{\Psi}^w(Y)$  does not contain equivalence classes of connections  $[A]$  such that  $A$  is reducible, then the construction for defining a Floer homology  $I_{\Psi}^w(Y)$  with generators given by critical points of the perturbed Chern–Simons function  $\overline{\text{CS}} + \Psi$ , and with differentials defined from negative gradient flow lines, goes through in the same way as in [10, 15]. This will require additional small perturbations in order to make the critical points non-degenerate and in order to obtain transversality for the moduli spaces of flow lines. In fact, we really have not done anything new compared to the constructions in these



references since the same perturbations already appear there for the sake of obtaining transversality of the moduli spaces involved in the construction. The only slight difference is that in Floer's work, the surfaces  $\Sigma_k$  appearing in the definition of the embeddings  $t_k$  are always chosen to be disks, whereas those used in the proof of Theorem 3.3 above, that is, in [36, Theorem 4.2 and Proposition 5.3], the surfaces  $\Sigma_k$  are all annuli.

More specifically, for completeness, we recall a bit more on the implementation of the holonomy perturbations used in the work of the third author as needed in the previous section for studying  $Y_0(K)$ . Suppose that we are given a smoothly embedded path  $c$  from  $P = (0, \pi)$  to  $Q = (\pi, \pi)$  avoiding  $(0, 0)$  and  $(\pi, 0)$ , and which is homotopic to the straight line segment  $c_0 := \{\beta = \pi\}$  from  $P$  to  $Q$  relative to the four corner points of the pillowcase  $P, Q, (0, 0)$  and  $(\pi, 0)$ . There is an isotopy  $\phi_t$  through area-preserving maps of the pillowcase  $R(T^2)$  such that  $\phi_1$  maps the straight line  $c_0$  to the path  $c$ , and such that  $\phi_t$  fixes the four corner points of the pillowcase. [36, Theorem 3.3] states that isotopies through area-preserving maps can be  $C^0$ -approximated by isotopies through finitely many *shearing* maps. For details on shearing maps we refer the reader to [36, Sections 2 and 3]. The essential relationship is outlined in the following subsection, which we include for the sake of clarity and completeness of our exposition.

### 4.3 | Review of holonomy perturbations and shearing maps

We denote by  $R(N)$  the space of flat  $SU(2)$ -connections in the trivial  $SU(2)$ -bundle over  $N = S^1 \times \Sigma$  up to gauge equivalence, where  $\Sigma = S^1 \times I = S^1 \times [0, 1]$  is an annulus. The two inclusion maps  $i_- : S^1 \times (S^1 \times \{0\}) \rightarrow N$  and  $i_+ : S^1 \times (S^1 \times \{1\}) \rightarrow N$  induce restriction maps  $r_-, r_+ : R(N) \rightarrow R(T^2)$  to the representation varieties of the two boundary tori, which are pillowcases. In this situation, we have that both  $r_-$  and  $r_+$  are homeomorphisms, and under the natural identification of these tori we have  $r_- = r_+$ .

Now if  $\chi$  is a class function as in Equation (2) above, then instead of the flatness equation  $F_A = 0$  for connections  $A$  on the trivial bundle over  $N$  one may consider the equation

$$F_A = \chi'(\text{Hol}_l(A)) \mu, \quad (5)$$

where the notation means the following: We denote by  $l_z = S^1 \times \{z\}$  the 'longitude' in  $N = S^1 \times \Sigma$  parametrized by a point  $z \in \Sigma$ , the holonomy of the connection  $A$  along such a loop is denoted by  $\text{Hol}_{l_z}(A)$ , the map  $\chi' : SU(2) \rightarrow \mathfrak{su}(2)$  is the trace dual of the derivative  $d\chi$  of  $\chi$ , and  $\mu$  is a 2-form in  $N$  which is the pull-back of a 2-form  $\mu_\Sigma$  via the projection  $S^1 \times \Sigma \rightarrow \Sigma$ , and  $\mu_\Sigma$  has compact support in the interior of  $\Sigma$  and satisfies  $\int_\Sigma \mu_\Sigma = 1$ . Then the equation we consider instead of the flatness equation  $F_A = 0$  is the equation  $(F_A)_{(w,z)} = \chi'(\text{Hol}_{l_z}(A)) \mu_{(w,z)}$ , for  $(w, z) \in N = S^1 \times \Sigma$ .

It can be proved that solutions  $A$  of this equation are reducible, and that furthermore the holonomy  $\text{Hol}_{l_z}(A)$  does not depend on the point  $z \in \Sigma$ . Both claims can be found in [6, Lemma 4], and are reproved in [36, Proposition 2.1] by the third author. This finally justifies our notation in Equation (5).

If we denote by  $R_\chi(N)$  the solutions of Equation (5) up to gauge equivalence, then we still have two restriction maps  $r_\pm : R_\chi(N) \rightarrow R(T^2)$ . However, in this situation we have the following relationship.

**Proposition 4.3.** *The two restriction maps  $r_{\pm}$  are homeomorphisms and fit into a commutative diagram*

$$\begin{array}{ccc}
 & R_{\chi}(N) & \\
 r_{-} \swarrow & & \searrow r_{+} \\
 R(T^2) & \xrightarrow{\phi_{\chi}} & R(T^2),
 \end{array} \tag{6}$$

where  $\phi$  is a shearing map that relates to  $\chi$  as follows.

If we write  $m_{-} = \{pt\} \times S^1 \times \{0\}$  and  $m_{+} = \{pt\} \times S^1 \times \{1\}$  for ‘meridians’ given by the boundaries of  $\Sigma$  in  $\{pt\} \times \Sigma$ , and if

$$\text{Hol}_{m_{\pm}}(A) = \begin{bmatrix} e^{i\beta_{\pm}} & 0 \\ 0 & e^{-i\beta_{\pm}} \end{bmatrix}, \text{ and } \text{Hol}_l(A) = \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix}, \tag{7}$$

which we may suppose up to gauge equivalence, then we have

$$\phi_{\chi} \begin{pmatrix} \alpha \\ \beta_{-} \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta_{-} + f(\alpha) \end{pmatrix}, \tag{8}$$

where  $f : \mathbb{R} \rightarrow \mathbb{R}$  is the derivative of the function  $g$  appearing in Equation (2). Here,  $(\alpha, \beta_{\pm})$  determine points in  $R(T^2)$  determined by  $\text{Hol}_{m_{\pm}}(A)$  and  $\text{Hol}_l(A)$  as in Equation (7) above.

Equation (6) is essentially proved in [6, Lemma 4], and a proof also appears in [36, Proposition 2.1].

Of course, one can iterate this construction: One may choose a finite collection of disjoint embeddings  $\iota_k : S^1 \times \Sigma$  into a closed three-manifold  $Y$ , and class functions  $\chi_k$ . The embeddings may be chosen to be ‘parallel’ in that the image of  $\iota_k$  corresponds to  $S^1 \times (S^1 \times [k, k + 1]) \subseteq S^1 \times (S^1 \times [0, n]) \subseteq Y$ , but the role of ‘meridian’ and ‘longitude’ may be chosen arbitrarily in an  $SL_2(\mathbb{Z})$  worth of possible choices. In this case the restriction maps to the two boundary components of  $S^1 \times (S^1 \times [0, n])$  in the diagram analogous to Equation (6) will be related by a composition of shearing maps.

### 4.4 | Holonomy perturbations and the pillowcase

The main application of holonomy perturbations we have in mind is stated as Theorem 4.4 below. To put it into context, note first that for a non-trivial bundle, the critical space of the Chern–Simons function  $R^w(Y_0(K))$  is a double cover of  $R(K|c_0)$ , where  $c_0$  is the straight line from  $(0, \pi)$  to  $(\pi, \pi)$  in the pillowcase, see [36, Proposition 5.1]. If we choose holonomy perturbations associated to some data  $\{\iota_k, \chi_k\}_{k=0}^{n-1}$  as above, where the image of  $\iota_k$  corresponds to  $S^1 \times (S^1 \times [k, k + 1]) \subseteq S^1 \times (S^1 \times [0, n]) \subseteq Y$  in a collar neighborhood of the Dehn filling torus in  $Y_0(K)$ , then repeated use of Proposition 4.3 above will imply that for the holonomy perturbation  $\Psi$  determined by the data  $\{\iota_k, \chi_k\}_{k=0}^{n-1}$ , the critical space of  $R^w_{\Psi}(Y_0(K))$  will correspond to  $R(K|c')$ , where  $c'$  is the image of  $c_0$  under a composition of shearing maps  $\phi_{n-1} \circ \dots \circ \phi_0$ , with ‘directions’

determined by the embeddings  $\iota_k$ . (In Equation (8) we are dealing with a shearing in direction  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ , but we can pick any direction in  $\mathbb{Z}^2$ .)

The main point of [36, Theorem 4.2] is that the area-preserving maps of the pillowcase obtained by composition of shearing maps is  $C^0$ -dense in the space of all area-preserving maps of the pillowcase, and this yields the following result.

**Theorem 4.4** (Theorem 4.2 and Proposition 5.3, [36]). *Let  $K$  be a knot in an  $SU(2)$ -cyclic integer homology three-sphere  $Y$ . Let  $c$  be an embedded path from  $(0, \pi)$  to  $(\pi, \pi)$  missing the other orbifold points of the pillowcase, and which is homotopic to the straight line segment  $c_0 := \{\beta = \pi\}$  from  $P$  to  $Q$  relative to the four corner points of the pillowcase  $P, Q, (0, 0)$  and  $(\pi, 0)$ . Then, there exists an embedded path  $c'$  arbitrarily close to  $c$  and a holonomy perturbation  $\Psi$  along disjoint embeddings of  $S^1 \times (S^1 \times I)$  parallel to the boundary of a neighborhood of  $K$  such that  $R_\Psi^w(Y_0(K))$  double-covers  $R(K|c')$ .*

(To see that we get a double-cover here we refer the reader to [36, Remark 1.2]).

We only stress the fact that we must assume that there are no reducible connections in  $R_\Psi^w(Y)$ , since the presence of such solutions will result in a failure of the transversality arguments involved in the discussion.

## 4.5 | Invariance of instanton Floer homology

The instanton Floer homology groups  $I^w(Y)$  and  $I_\Psi^w(Y)$ , the latter being defined under the additional assumption that  $R_\Psi^w(Y)$  does not contain reducible connections, depend on additional data that we have already suppressed from notation, notably the choice of a Riemannian metric on  $Y$  and holonomy perturbations just as defined above in order to achieve transversality. More explicitly, holonomy perturbations have already been implicit in the definition of instanton Floer homology unless the critical points of CS had been non-degenerate at the start and the moduli space defining the flow lines had been cut out transversally. In Floer's original work [15], and elaborated in more detail in Donaldson's book [10], invariance under the choice of Riemannian metric and the choice of holonomy perturbations follows from a more general concept, namely, the functoriality of instanton Floer homology under cobordisms. See also the discussion in [23, Section 3.8.]

**Theorem 4.5** (Invariance under holonomy perturbations). *Suppose that the space of critical points  $R_\Psi^w(Y)$  of the perturbed Chern–Simons function 4 appearing in Definition 4.2 above does not contain equivalence classes of reducible connections. Then the associated instanton Floer homology groups  $I_*^w(Y)$  and  $I_{*,\Psi}^w(Y)$  are isomorphic.*

*Sketch of Proof.* The proof of this statement is standard, so we will describe a chain map determining the isomorphism on homology and outline the ideas along which the result is proved.

Slightly more generally, suppose that we are dealing with a smooth map  $[0, 1] \rightarrow C^\infty(\mathcal{A}, \mathbb{R})$ ,  $s \mapsto \Gamma(s)$ . We may suppose that this map is constant near 0 and 1. The Floer differential counts flow lines of the Chern–Simons function, possibly suitably perturbed. Instead of doing this, we may also consider the downward gradient flow equation of the *time-dependent* function  $CS + \Gamma(s)$ , where we extend  $\Gamma(s)$  to a map  $(-\infty, \infty) \rightarrow C^\infty(\mathcal{A}, \mathbb{R})$  which is constant  $\Gamma(0)$  on  $(-\infty, 0]$  and

constant  $\Gamma(1)$  on  $[1, \infty)$ . If we are given critical points  $\rho_0$  of  $CS + \Gamma(0)$  and  $\rho_1$  of  $CS + \Gamma(1)$  of the same index, then we consider a zero-dimensional moduli space  $M_{\rho_0, \rho_1}$  of connections  $\mathbf{A} = \{A(t)\}_t$  on  $E \rightarrow \mathbb{R} \times Y$  of finite  $L^2$ -norm (inducing  $\theta$  on  $\det(E)$ , pulled back to  $\mathbb{R} \times Y$ ), such that the equation

$$\frac{dA}{dt} = -\text{grad}(CS + \Gamma(t))(A(t)) \tag{9}$$

holds on  $\mathbb{R} \times Y$ , where  $\text{grad}$  denotes the  $L^2$ -gradient, and  $\mathbf{A}$  limits to  $\rho_0$  and  $\rho_1$  in the limit  $t \rightarrow \pm\infty$ , respectively. Finally, we also require that the moduli space  $M_{\rho_0, \rho_1}$  is cut out transversally.

We require that the addition of the term  $-\text{grad}(\Gamma(t))(A(t))$  to the gradient flow Equation 9 for the Chern–Simons function does not alter the linearized deformation theory for  $\mathbf{A}$ , see, for instance, [10, Sections 3 and 4]. Furthermore, we have to require that the Uhlenbeck compactification goes through with the perturbation we have in mind. It is shown in [10, Section 5.5] that both hold for the function  $\Gamma$  built from holonomy perturbations as described in Equation 10. One essential feature is that the holonomy perturbation term appearing in the flow equation is uniformly bounded.

A suitable interpolation between the holonomy perturbation data  $\Gamma(0) = 0$  and  $\Gamma(1) = \Psi$  for  $\Psi$  as in Equation (4) is given, for instance, by the following formula. Suppose that  $\Psi$  is determined by data  $\{\iota_k, \chi_k\}_{k=0}^{n-1}$ . Then for  $t \in [\frac{k}{n}, \frac{k+1}{n}]$  we define

$$\Gamma(t) = \sum_{l=0}^{k-1} \Phi_l + \beta(t - k/n)\Phi_k \tag{10}$$

for any  $k \in \{0, \dots, n - 1\}$ . Here  $\beta : [0, \frac{1}{n}] \rightarrow [0, 1]$  is a smooth function which is 0 in a neighborhood of 0 and 1 in a neighborhood of  $\frac{1}{n}$ .

Now the moduli space  $M_{\rho_0, \rho_1}$  does not contain any reducibles, because if it did, then the limits  $\rho_0$  and  $\rho_1$  in  $R^w(Y)$  and  $R_\Psi^w(Y)$ , respectively, would also be reducible, and by our assumption and the setup for instanton Floer homology for admissible bundles, this does not occur.

One defines a linear map  $\zeta : C^w(Y) \rightarrow C_\Psi^w(Y)$  of the underlying chain complexes such that the ‘matrix entry’ corresponding to the elements  $\rho_0 \in C^w(Y)$  and  $\rho_1 \in C_\Psi^w(Y)$  is given by the signed count of the moduli space  $M_{\rho_0, \rho_1}$ , where the sign is determined in the usual way by the choice of a *homology orientation*. That  $\zeta$  is a chain map follows from analyzing the compactification of suitable 1-dimensional moduli spaces, making use of Uhlenbeck compactification — no bubbling can occur here due to the dimension of the moduli space — and the chain convergence discussed in [10, Section 5.1], together with suitable gluing results.

That different interpolations yield chain homotopic chain maps follows from studying the compactification of  $(-1)$ -dimensional moduli spaces over a 1-dimensional family, defining a chain homotopy equivalence between the two different interpolations.

That  $\zeta$  defines a chain homotopy equivalence follows from the functoriality property: One may consider a further path  $\Gamma' : [1, 2] \rightarrow C^\infty(\mathcal{A}, \mathbb{R})$  such that  $\Gamma'(1) = \Gamma(1)$ , similar as above. This defines a corresponding chain map  $\zeta' : C_\Psi^w \rightarrow C_{\Gamma'(2)}^w$ . On the other hand, one may concatenate the path  $\Gamma(t)$  and the path  $\Gamma'(t)$  and build a corresponding interpolation  $\Gamma'' : [0, 2] \rightarrow C^\infty(\mathcal{A}, \mathbb{R})$ , resulting in a chain map  $\zeta''$  as above. A neck stretching argument then shows that  $\zeta''$  and  $\zeta' \circ \zeta$  are chain homotopy equivalent, and hence induce the same maps on homology. In our situation we take  $\Gamma'(2)$  to be 0, meaning that this defines again the ‘unperturbed’ chain complex  $C^w(Y)$  (which, again, may contain some perturbations for the sake of regularity omitted in our notation). One

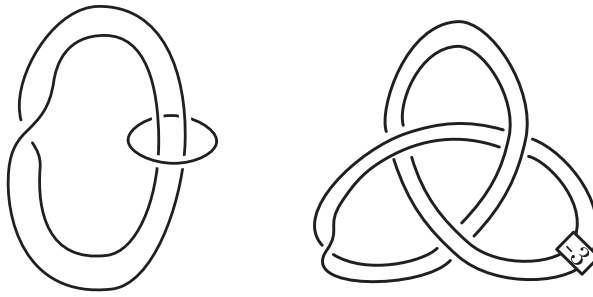


FIGURE 5 Left: The pattern representing the (2,1)-cable satellite operation. Right: The (2,1)-cable for the right-handed trefoil. The extra twisting appears as a consequence of the requirement that a longitude in  $S^1 \times D^2$  maps to the canonical longitude of the trefoil.

may finally interpolate between  $\Gamma''$  and the 0-term along a 1-dimensional family. Analyzing again the compactification of suitable  $(-1)$ -dimensional moduli spaces over a 1-dimensional family, one obtains a chain homotopy equivalence between  $\zeta''$  and the identity.  $\square$

*Remark 4.6.* There is some confusion about invariance under ‘small’ and ‘large’ holonomy perturbations in the field. If one is given holonomy perturbation data for which the underlying space of critical points and moduli spaces defining the differentials are already cut out transversally, then for small enough perturbations the same will still hold, and the resulting chain complexes will be isomorphic. This is due to the fact that the condition of being cut out transversally is an *open* condition, expressed as the surjectivity of the deformation operators involved in the linearized equation together with the Coulomb gauge fixing.

If, on the other hand, one is given a situation where the critical points and the unperturbed moduli spaces are not cut out transversally, then one needs to perturb, and even if these perturbations are chosen ‘small’, the resulting chain complexes will in general not be isomorphic but only chain homotopy equivalent. In this situation, the proof of invariance is really the same as proving the invariance under ‘large’ perturbations, and already present in [15] and [10].

## 5 | BRANCHED COVERS OF PRIME SATELLITE KNOTS

In this section, we prove Corollary 1.5, establishing the existence of a non-trivial  $SU(2)$  representation for cyclic branched covers of prime satellite knots. We begin with a definition of satellite knots.

**Definition 5.1.** Let  $P \subset S^1 \times D^2$  be an oriented knot in the solid torus. Consider an orientation-preserving embedding  $h : S^1 \times D^2 \rightarrow S^3$  whose image is a tubular neighborhood of a knot  $K$  so that  $S^1 \times \{*\in \partial D^2\}$  is mapped to the canonical longitude of  $K$ . The knot  $h(P)$  is called the *satellite knot with pattern  $P$  and companion  $K$* , and is denoted as  $P(K)$ . The *winding number* of the satellite is defined to be the algebraic intersection number of  $P$  with  $\{*\} \times D^2$ . See Figure 5 for an example.

**Corollary 1.5.** *Let  $K$  be a prime, satellite knot in  $S^3$  and let  $\Sigma(K)$  be any non-trivial cyclic cover of  $S^3$  branched over  $K$ . Then  $\pi_1(\Sigma(K))$  admits a non-trivial  $SU(2)$  representation.*

*Proof.* Let  $K$  be a prime satellite knot in  $S^3$ . If  $\Sigma(K)$  is not an integer homology sphere, then there is a non-trivial abelian representation. In the case when  $\Sigma(K)$  is an integer homology three sphere, then by Theorem 1.1 it suffices to show that  $\Sigma(K)$  is toroidal. Write  $K = P(J)$  and observe that if  $\Sigma(K)$  is the  $d$ -fold cover of  $S^3$  branched over  $P(J)$ , then there is a decomposition of  $\Sigma(P(J))$  as the union of  $\Sigma(S^1 \times D^2, P)$ , the  $d$ -fold cover of  $S^1 \times D^2$  branched over  $P$ , and a  $d$ -fold covering space of the knot complement  $S^3 \setminus N(J)$ . The isomorphism type of this latter covering space depends only on the greatest common divisor between  $d$  and the winding number of  $S^1 \times D^2$ , see, for example, [31] or [24, p. 220]. Since the exterior of  $J$  has incompressible boundary, the same is true of any cover. Therefore, we just need to show that  $\Sigma(D^2 \times S^1, P)$  has incompressible boundary. We claim the following. Let  $P$  be a non-trivial pattern knot in  $D^2 \times S^1$  which does not correspond to a connect-sum and which is not contained in an embedded  $B^3$ . Then for any cyclic branched cover over  $P$ ,  $\Sigma(D^2 \times S^1, P)$  has incompressible boundary. This claim is standard and proved in the lemma below for completeness.  $\square$

**Lemma 5.2.** *Let  $P$  be a non-trivial pattern knot in  $D^2 \times S^1$  which does not correspond to a connect-sum and which is not contained in an embedded  $B^3$ . Then for any cyclic branched cover over  $P$ , the manifold  $M = \Sigma(D^2 \times S^1, P)$  has incompressible boundary.*

*Proof.* Suppose that  $\gamma$  is an essential loop on  $\partial M$ , which is nullhomotopic in  $M$ . Let  $G$  denote the group of covering transformations of  $M$  and consider the action of  $G$  on the boundary. We first claim that  $\gamma$  can be isotoped on the boundary such that for each  $g \in G$ , either  $g(\gamma) \cap \gamma = \emptyset$  or  $g(\gamma) = \gamma$ . Of course, we only need to consider the elements of  $G$  that fix setwise the boundary component containing  $\gamma$ . Since the action of  $G$  on  $T^2$  is of finite order and free, it must be the standard covering transformation of a covering map from  $T^2$  to itself, which fixes all homology classes. In particular, the homology class of  $\gamma$  in  $\partial M$  is fixed by this action, and the claim easily follows. Now, because of this claim, and because the curve  $\gamma$  is disjoint from the lift of  $P$ , the equivariant Dehn's lemma [34] implies that there exists a disk  $D$  in  $M$  bounding  $\gamma$  such that for all  $g$ , either  $g(D) \cap D = \emptyset$  or  $g(D) = D$ , and furthermore,  $D$  is transverse to the lift of the branch set. Consider the (possibly disconnected) surface  $\Sigma = \bigcup_{g \in G} g(D)$ . Then,  $\Sigma/G$  is a collection of disks in  $D^2 \times S^1$  and  $\Sigma \rightarrow \Sigma/G$  is a branched cover (although some components of  $\Sigma$  may have trivial branch locus). Furthermore, each component of the boundary of  $\Sigma/G$  is an essential curve on the boundary of the solid torus. For homology reasons, it is necessarily a meridional curve on the solid torus and each component of  $\Sigma/G$  is a meridional disk. (The components cannot have any other topology, since a disk can only cover/branch cover another disk.) Now, if any component of  $\Sigma/G$  does not intersect  $P$ , then we can cut  $D^2 \times S^1$  along one of these disks, and see that  $P$  is contained in  $B^3$  and we have a contradiction. If some component of  $\Sigma/G$  does intersect  $P$ , it must intersect in exactly one point, since a disk cannot be a cyclic cover of a disk branched along more than one point. (Here we are using the fact that the branch points all correspond to intersections of  $P$  with the disk.) In other words,  $P$  is the pattern for a connect-sum, and again we have a contradiction. This proves the claim and completes the proof of the lemma.  $\square$

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