



On Formality and Combinatorial Formality for Hyperplane Arrangements

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Abstract

A hyperplane arrangement is called formal provided all linear dependencies among the defining forms of the hyperplanes are generated by ones corresponding to intersections of codimension two. The significance of this notion stems from the fact that complex arrangements with aspherical complements are formal. The aim of this note is twofold. While work of Yuzvinsky shows that formality is not combinatorial, in our first main theorem we prove that the combinatorial property of factoredness of arrangements does entail formality. Our second main theorem shows that formality is hereditary, i.e., is passed to restrictions. This is rather counter-intuitive, as in contrast the known sufficient conditions for formality, i.e., asphericity, freeness and factoredness (owed to our first theorem), are not hereditary themselves. We also demonstrate that the stronger property of k -formality, due to Brandt and Terao, is not hereditary.

Keywords Hyperplane arrangements · Formality · k -Formality · Combinatorial formality · Free arrangements · Factored arrangements · $K(\pi, 1)$ -Arrangements

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1 Introduction

In the study of hyperplane arrangements it is of central concern to connect algebraic or topological properties (over \mathbb{C}) with combinatorial invariants encoded by their intersection lattices. Yet another important theme is to investigate the behavior of a given notion under the standard arrangement constructions of restriction and localization. Following this philosophy, the aim of our note is to relate the combinatorial notion of factoredness with the general concept of formality on the one hand and on the other to complete the picture about the behavior of formality with respect to restriction and localization, complementing previous results in the literature.

A hyperplane arrangement \mathcal{A} is *formal* provided all linear dependencies among the defining forms of the hyperplanes in \mathcal{A} are generated by dependencies corresponding to intersections of codimension 2 of the members of \mathcal{A} (see Sect. 2.3). Over the complex numbers this property is implied by $K(\pi, 1)$ -arrangements (see Theorem 2.4).

A property for arrangements is said to be *combinatorial* if it only depends on the intersection lattice of the underlying arrangement. Yuzvinsky [25, Ex. 2.2] demonstrated that formality is not combinatorial, answering a question raised by Falk and Randell [14] in the negative. Yuzvinsky's insight motivates the following notion. Suppose \mathcal{A} is a formal arrangement. We say \mathcal{A} is *combinatorially formal* if every arrangement with an intersection lattice isomorphic to the one of \mathcal{A} is also formal. In view of Yuzvinsky's result it is therefore startling that even this strong form of formality is afforded by the combinatorial property of *factoredness* (see Sect. 2.8). This is the content of our first main result.

Theorem 1.1 *If \mathcal{A} is factored, then it is combinatorially formal.*

In [14], the authors raised the question whether *freeness* (see Sect. 2.7) also implies formality. This was settled affirmatively by Yuzvinsky in [25, Cor. 2.5].

While both, freeness and asphericity, do imply formality, it is not known whether the former are combinatorial or not (cf. [15, Prob. 3.8]). For the concept of freeness this is a longstanding conjecture due to Terao (cf. [19, Conj. 4.138]).

A property for arrangements is said to be *hereditary* if it is inherited by every restriction of the underlying arrangement. Freeness, asphericity, and factoredness are known to *not* be hereditary in general (cf. [11], [2, Thm. 1.1], and Example 2.16, respectively). It is thus rather surprising and counter-intuitive that formality on the other hand, which is entailed by each of these properties, *is* hereditary. This is the content of our second main theorem.

Theorem 1.2 *Formality is hereditary.*

The notion of formality was further refined to the concept of *k-formality* by Brandt and Terao [5, Def. 4.4] (see Sect. 5). In the setting of [5], formality coincides with 2-formality. Higher formality is stronger than formality, cf. [5, Ex. 5.1]. In [5, Thm. 4.15], Brandt and Terao extended Yuzvinsky's result [25, Cor. 2.5] by showing that free arrangements are *k*-formal for all *k*. Extending [5, Thm. 4.15] further, in [10, Thm. 1.1], DiPasquale employed homological methods based on those developed in [5] to characterize freeness of a multiplicity of an arrangement and showed as a consequence that

the latter implies k -formality for all k , [10, Cor. 4.10]. We emphasize that Theorem 1.2 does not extend to the stronger notion of k -formality, see Example 5.1.

A property for arrangements is said to be *local* if it is passed to every localization of the ambient arrangement. Freeness, asphericity, and factoredness are all local (cf. [19, Thm. 4.37], [20, Lem. 1.1], and the proof of [24, Cor. 2.11], respectively). In contrast, this fails for formal arrangements of rank at least 4 (cf. the example following [25, Def. 2.3]). However, as an application of Theorem 1.2, we in turn show that localizations at modular flats of formal arrangements are formal again, see Corollary 5.2.

Moreover, it is also known that if X is a modular flat of \mathcal{A} of corank 1 and the localization \mathcal{A}_X is free, $K(\pi, 1)$, or factored, then so is \mathcal{A} itself (cf. [22, Rem. 2.8], [23], and [22, Rem. 2.25], respectively). In Corollary 5.3, we show that this property also holds for formality.

Following Yuzvinsky [25, Def. 2.3], we say that \mathcal{A} is *locally formal* provided that each localization of \mathcal{A} is formal. Since freeness, asphericity, and factoredness are all local and each affords formality, each of these properties implies local formality. Since formality is not passed to localizations in general, as noted above, the class of formal arrangements properly encompasses the classes of free, factored and $K(\pi, 1)$ -arrangements. The following is immediate from Theorem 1.2.

Corollary 1.3 *Local formality is hereditary.*

The paper is organized as follows. In Sects. 2.1 and 2.2, we recall some standard terminology and introduce further notation on hyperplane arrangements and record some basic facts on modular elements in the lattice of intersections of an arrangement. The notions of formality and k -formality are recapitulated from [5] in Sect. 2.3. This is followed by a brief discussion of the concept of line closure of a subset of an arrangement. Here we review Falk’s criterion that the presence of an lc-basis entails combinatorial formality (Proposition 2.3). This in turn is the key ingredient in our proof of Theorem 1.1. Short recollections on the fundamental notions of $K(\pi, 1)$ -arrangements (Sect. 2.5) and simplicial arrangements (Sect. 2.6) follow. In Sects. 2.7 and 2.8, the concepts of free and factored arrangements are recalled. Theorems 1.1 and 1.2 are proved in Sects. 3 and 4, respectively.

In our final section, we present some complements to our main developments. Here we demonstrate in Example 5.1 that Theorem 1.2 does not extend to higher formality. In addition, we present a family of combinatorially formal arrangements \mathcal{K}_n of rank $n \geq 3$ which fail to be free, $K(\pi, 1)$, and factored (Example 5.4). These are natural subarrangements of the Coxeter arrangement of type B_n .

For general information about arrangements we refer the reader to [19].

2 Recollections and Preliminaries

2.1 Hyperplane Arrangements

Let \mathbb{K} be a field and let $V = \mathbb{K}^\ell$ be an ℓ -dimensional \mathbb{K} -vector space. A *hyperplane arrangement* $\mathcal{A} = (\mathcal{A}, V)$ in V is a finite collection of hyperplanes in V each con-

taining the origin of V . We also use the term ℓ -arrangement for \mathcal{A} . We denote the empty arrangement in V by \mathcal{A}_\emptyset .

The lattice $L(\mathcal{A})$ of \mathcal{A} is the set of subspaces of V of the form $H_1 \cap \dots \cap H_i$ where $\{H_1, \dots, H_i\}$ is a subset of \mathcal{A} . For $X \in L(\mathcal{A})$, we have two associated arrangements, firstly $\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subseteq H\} \subseteq \mathcal{A}$, the *localization of \mathcal{A} at X* , and secondly, the *restriction of \mathcal{A} to X* , (\mathcal{A}^X, X) , where $\mathcal{A}^X := \{X \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_X\}$. Note that V belongs to $L(\mathcal{A})$ as the intersection of the empty collection of hyperplanes and $\mathcal{A}^V = \mathcal{A}$. The lattice $L(\mathcal{A})$ is a partially ordered set by reverse inclusion: $X \leq Y$ provided $Y \subseteq X$ for $X, Y \in L(\mathcal{A})$.

Throughout, we only consider arrangements \mathcal{A} such that $0 \in H$ for each H in \mathcal{A} . These are called *central*. In that case the *center* $T(\mathcal{A}) := \bigcap_{H \in \mathcal{A}} H$ of \mathcal{A} is the unique maximal element in $L(\mathcal{A})$ with respect to the partial order. A *rank function* on $L(\mathcal{A})$ is given by $r(X) := \text{codim}_V(X)$. The *rank of \mathcal{A}* is defined as $r(\mathcal{A}) := r(T(\mathcal{A}))$.

For $1 \leq k \leq r(\mathcal{A})$ we write $L_k(\mathcal{A})$ for the collection of all $X \in L(\mathcal{A})$ of rank k .

2.2 Modular Elements in $L(\mathcal{A})$

We say that $X \in L(\mathcal{A})$ is *modular* provided $X+Y \in L(\mathcal{A})$ for every $Y \in L(\mathcal{A})$, [19, Cor. 2.26]. We require the following characterization of modular members of $L(\mathcal{A})$ of rank $r-1$, see the proof of [4, Thm. 4.3].

Lemma 2.1 *Let \mathcal{A} be an arrangement of rank r . Suppose that $X \in L(\mathcal{A})$ is of rank $r-1$. Then X is modular if and only if for any two distinct $H_1, H_2 \in \mathcal{A} \setminus \mathcal{A}_X$ there is an $H_3 \in \mathcal{A}_X$ such that $r(H_1 \cap H_2 \cap H_3) = 2$, i.e., H_1, H_2, H_3 are linearly dependent.*

Next, we record a refined geometric version of a standard fact about modular elements in a geometric lattice, cf. [1, Prop. 2.42] or [19, Lem. 2.27].

Lemma 2.2 *Let \mathcal{A} be an arrangement of rank r . Suppose that $X \in L(\mathcal{A})$ is modular of rank q . Then there is a complementary intersection $Y \in L(\mathcal{A})$ of rank $r-q$ (i.e., $X \cap Y = T(\mathcal{A})$) such that the arrangements $(\mathcal{A}_X/X, V/X)$ and $(\mathcal{A}^Y/T(\mathcal{A}), Y/T(\mathcal{A}))$ are linearly isomorphic.*

Proof Let $X = H_1 \cap \dots \cap H_q$. Then there are $H_{q+1}, \dots, H_r \in \mathcal{A}$ such that $T(\mathcal{A}) = H_1 \cap \dots \cap H_r$, i.e., $Y = H_{q+1} \cap \dots \cap H_r$ is complementary to X . Due to the modularity of X for all $K = H' \cap Y \in \mathcal{A}^Y$, we have $H = X + K \in \mathcal{A}_X$, $H' \cap Y = H \cap Y = K$, and furthermore, this correspondence is bijective. Consequently, the quotient map $Y/T(\mathcal{A}) \rightarrow V/X, v+T(\mathcal{A}) \mapsto v+X$, which is an isomorphism by the choice of Y , maps $\mathcal{A}^Y/T(\mathcal{A})$ to \mathcal{A}_X/X . \square

2.3 Formality and k -Formality

The notion of formality is due to Falk and Randell [14]. Here we give the equivalent definition by Brandt and Terao [5]. Let $(e_H \mid H \in \mathcal{A})$ be the basis of a \mathbb{K} -vector space indexed by the hyperplanes in \mathcal{A} . For each $H \in \mathcal{A}$, choose a linear form $\alpha_H \in V^*$ such that $\ker \alpha_H = H$. Consider the map

$$\bigoplus \mathbb{K}e_H \rightarrow V^* \quad \text{defined by} \quad e_H \mapsto \alpha_H$$

and let $F(\mathcal{A})$ be the kernel of this map. Note that \mathcal{A} is linearly isomorphic to the arrangement $\{F(\mathcal{A})^\perp \cap \ker x_H \mid H \in \mathcal{A}\}$ in $F(\mathcal{A})^\perp$, where $(x_H \mid H \in \mathcal{A})$ is the dual basis to $(e_H \mid H \in \mathcal{A})$. We say an element $x \in F(\mathcal{A})$ has length p if x has exactly p non-zero entries. Let $F_2(\mathcal{A})$ be the subspace of $F(\mathcal{A})$ generated by all $x \in F(\mathcal{A})$ of length at most 3. Then, \mathcal{A} is said to be formal if $F_2(\mathcal{A}) = F(\mathcal{A})$. Note that if $r(\mathcal{A}) = 2$, then $F_2(\mathcal{A}) = F(\mathcal{A})$, so each rank 2 arrangement is formal. As indicated in the introduction, a localization of a formal arrangement need not be formal again, see the example after [25, Def. 2.3].

The following generalization of formality is due to Brandt and Terao [5]. If $X \in L(\mathcal{A})$, then there is a natural inclusion $F_2(\mathcal{A}_X) \hookrightarrow F_2(\mathcal{A})$. If $x \in F(\mathcal{A})$ is of length 3, then there is an associated $X \in L_2(\mathcal{A})$ with $x \in F_2(\mathcal{A}_X) \subset F(\mathcal{A})$. Thus, \mathcal{A} is formal if and only if the map

$$\pi_2: \bigoplus_{X \in L_2(\mathcal{A})} F(\mathcal{A}_X) \longrightarrow F(\mathcal{A})$$

is a surjection. The definition of k -formality in [5] is recursive and does not require the choice of linear forms as before. To emphasize the use of this alternative definition, we follow the notation from [5], which differs from the one above.

Let $R_0(\mathcal{A}) = T(\mathcal{A})^*$ be the dual space of the center of \mathcal{A} . For each $X \in L(\mathcal{A})$, we have a surjective map $i_0(X): R_0(\mathcal{A}_X) \rightarrow R_0(\mathcal{A})$ defined by restricting the domain to $T(\mathcal{A})$. Then, for $1 \leq k \leq r(\mathcal{A})$, define $R_k(\mathcal{A})$ recursively as the kernel of the map

$$\pi_{k-1}(\mathcal{A}): \bigoplus_{X \in L_{k-1}} R_{k-1}(\mathcal{A}_X) \longrightarrow R_{k-1}(\mathcal{A}),$$

where the maps π_k are defined as follows. (Note that $\bigoplus_{X \in L_0} R_0(\mathcal{A}_X) = R_0(\Phi_\ell) = V^*$ and $\pi_0: V^* \rightarrow R_0(\mathcal{A}) = T(\mathcal{A})^*$ is just restriction.) For each $k \geq 1$ and each $Y \in L(\mathcal{A})$ with $\text{rk}(Y) \geq k$, there is a map

$$i_k(Y) := i_k(\mathcal{A}, Y): R_k(\mathcal{A}_Y) \rightarrow R_k(\mathcal{A}),$$

defined recursively via the commutative diagram

$$\begin{array}{ccccc} R_k(\mathcal{A}_Y) & \longrightarrow & \bigoplus_{\substack{X \in L_{k-1} \\ X \leq Y}} R_{k-1}((\mathcal{A}_Y)_X) & \xrightarrow{\pi_{k-1}(\mathcal{A}_Y)} & R_{k-1}(\mathcal{A}_Y) \\ \downarrow i_k(Y) & & \downarrow j_{k-1}(Y) & & \downarrow i_{k-1}(Y) \\ R_k(\mathcal{A}) & \longrightarrow & \bigoplus_{X \in L_{k-1}} R_{k-1}(\mathcal{A}_X) & \xrightarrow{\pi_{k-1}(\mathcal{A})} & R_{k-1}(\mathcal{A}) \end{array}$$

In the diagram, $R_k((\mathcal{A}_Y)_X) = R_k(\mathcal{A}_X)$ since $X \leq Y$, so $j_{k-1}(Y)$ is defined as the direct sum of identity maps and zero maps. Then $i_k(Y)$ is defined through the universal property of the kernel $R_k(\mathcal{A})$. With this define π_k as the sum of the $i_k(X)$ for $X \in L_k(\mathcal{A})$.

If \mathcal{A} is formal, we say it is *2-formal*. For $k \geq 3$, we say \mathcal{A} is *k-formal* if \mathcal{A} is $(k - 1)$ -formal and the map π_k is surjective. Note that an arrangement of rank r which is $(r - 1)$ -formal is trivially k -formal for all $k \geq r$.

2.4 Line Closure and Combinatorial Formality

We want to establish further combinatorial properties of \mathcal{A} that imply formality. The following definitions are due to Falk [12] for matroids, but they can easily be applied to arrangements as well. Let $\mathcal{B} \subset \mathcal{A}$ be a subset of hyperplanes. We say \mathcal{B} is *closed* if $\mathcal{B} = \mathcal{A}_Y$ for $Y = \bigcap_{H \in \mathcal{B}} H$. We call \mathcal{B} *line-closed* if for every pair $H, H' \in \mathcal{B}$ of hyperplanes, we have $\mathcal{A}_{H \cap H'} \subset \mathcal{B}$. The *line-closure* $\text{lc}(\mathcal{B})$ of \mathcal{B} is defined as the intersection of all line-closed subsets of \mathcal{A} containing \mathcal{B} . The arrangement \mathcal{A} is called *line-closed* if every line-closed subset of \mathcal{A} is closed. With these notions, we have the following criterion for combinatorial formality, see [12, Cor. 3.8]:

Proposition 2.3 *Let \mathcal{A} be an arrangement of rank r . Suppose $\mathcal{B} \subseteq \mathcal{A}$ consists of r hyperplanes such that $r(\mathcal{B}) = r$ and $\text{lc}(\mathcal{B}) = \mathcal{A}$. Then \mathcal{A} is combinatorially formal.*

We call such a subset $\mathcal{B} \subseteq \mathcal{A}$ as in Proposition 2.3 an *lc-basis* of \mathcal{A} . Note that the converse of Proposition 2.3 is false [12], i.e., a combinatorially formal \mathcal{A} need not admit an lc-basis.

Furthermore, the existence of an lc-basis does not imply higher formality. Brandt and Terao give an example of an arrangement that is 2-formal but not 3-formal, see [5, Ex. 5.1]. Here an lc-basis is easily calculated, e.g. take $\{H_1, H_2, H_4, H_5\}$ in *loc. cit.*

Finally, we note that the presence of an lc-basis does not descend to localizations, i.e., if $\mathcal{B} \subseteq \mathcal{A}$ is as in Proposition 2.3 and $X \in L(\mathcal{A})$, then it need not be the case that \mathcal{B}_X is an lc-basis of \mathcal{A}_X , see Example 5.1.

2.5 $K(\pi, 1)$ -Arrangements

A complex ℓ -arrangement \mathcal{A} is called *aspherical*, or a $K(\pi, 1)$ -arrangement (or that \mathcal{A} is $K(\pi, 1)$ for short), provided the complement $M(\mathcal{A})$ of the union of the hyperplanes in \mathcal{A} in \mathbb{C}^ℓ is aspherical, i.e., is a $K(\pi, 1)$ -space. That is, the universal covering space of $M(\mathcal{A})$ is contractible and the fundamental group $\pi_1(M(\mathcal{A}))$ of $M(\mathcal{A})$ is isomorphic to the group π . This is an important topological property, for the cohomology ring $H^*(X, \mathbb{Z})$ of a $K(\pi, 1)$ -space X coincides with the group cohomology $H^*(\pi, \mathbb{Z})$ of π . The crucial point here is that the intersections of codimension 2 determine the fundamental group $\pi_1(M(\mathcal{A}))$ of $M(\mathcal{A})$.

By Deligne's seminal result [9], complexified simplicial arrangements are $K(\pi, 1)$. Likewise for complex supersolvable arrangements, cf. [13] and [23] (cf. [19, Prop. 5.12, Thm. 5.113]). As restrictions of simplicial (resp. supersolvable) arrangements are again simplicial (resp. supersolvable), the $K(\pi, 1)$ -property of these kinds of arrangements is inherited by their restrictions. However, we emphasize that in general, a restriction of a $K(\pi, 1)$ -arrangement need not be $K(\pi, 1)$ again, see [2] for examples of this kind.

The following theorem due to Falk and Randell [14, Thm. 4.2] establishes formality as a necessary condition for asphericity.

Theorem 2.4 *If \mathcal{A} is a $K(\pi, 1)$ -arrangement, then it is formal.*

Thanks to [7], simpliciality is a combinatorial property. Thus, as simplicial arrangements are $K(\pi, 1)$, it follows that all such are combinatorially formal, see also Sect. 2.6. It is not known whether this is true for aspherical arrangements in general.

Remark 2.5 Thanks to an observation by Oka, if the complex arrangement \mathcal{A} is $K(\pi, 1)$, then so is every localization \mathcal{A}_X for $X \in L(\mathcal{A})$, e.g., see [20, Lem. 1.1].

The following is an immediate consequence of Terao’s work [23] (see also [19, Sect. 5.5]).

Lemma 2.6 *Let \mathcal{A} be a complex arrangement of rank r . Suppose that $X \in L(\mathcal{A})$ is modular of rank $r - 1$. If \mathcal{A}_X is $K(\pi, 1)$, then so is \mathcal{A} .*

2.6 Simplicial Arrangements

Let \mathcal{A} be a real arrangement in $V \cong \mathbb{R}^\ell$. Then, the connected components of the complement $V \setminus \bigcup_{H \in \mathcal{A}} H$ are called chambers and are denoted by $\mathcal{C}(\mathcal{A})$. The arrangement \mathcal{A} is called *simplicial* if all $C \in \mathcal{C}(\mathcal{A})$ are (open) simplicial cones. It was already mentioned above that simplicial arrangements are aspherical and therefore formal. Furthermore, simpliciality is actually a combinatorial property of the intersection lattice, as was observed in [7]. Consequently, the formality of (real) simplicial arrangements is combinatorial.

In Proposition 2.8 below, we demonstrate that a simplicial arrangement naturally satisfies the stronger property of having an lc-basis. Explicitly, owing to Proposition 2.8, the walls of each chamber of a simplicial arrangement yield such a special basis.

For a chamber $C \in \mathcal{C}(\mathcal{A})$, we write $\mathcal{W}^C := \{H \in \mathcal{A} \mid \langle \overline{C} \cap H \rangle = H\}$ for the walls of C , where $\langle A \rangle$ denotes the linear span of a subset $A \subseteq V$. Two chambers $C, D \in \mathcal{C}(\mathcal{A})$ are *adjacent* provided $\langle \overline{C} \cap \overline{D} \rangle \in \mathcal{A}$.

The following lemma, which is a special case of [8, Lem. 3.3], supplies the essential argument for the proof of Proposition 2.8 below.

Lemma 2.7 *Let $C, D \in \mathcal{C}(\mathcal{A})$ be two adjacent and simplicial chambers with a common wall $H \in \mathcal{W}^C \cap \mathcal{W}^D$, and let $H' \in \mathcal{W}^D \setminus \{H\}$. Then, $H' \in \mathcal{W}^C$ if and only if $|\mathcal{A}_{H \cap H'}| = 2$.*

We require a bit more notation, cf. [8, Sect. 2.2]. A sequence $(C_0, C_1, \dots, C_{n-1}, C_n)$ of distinct chambers in $\mathcal{C}(\mathcal{A})$ is called a *gallery* if for all $1 \leq i \leq n$ the chambers C_i and C_{i-1} are adjacent. The set of all galleries is denoted by $\mathcal{G}(\mathcal{A})$. The *length* $l(G)$ of a gallery $G \in \mathcal{G}(\mathcal{A})$ is one less than the number of chambers in G . For a gallery $G = (C_0, \dots, C_n)$, we denote by $b(G) = C_0$ the first chamber and by $t(G) = C_n$ the last chamber in G . For two chambers $C, D \in \mathcal{C}(\mathcal{A})$, we set $\text{dist}(C, D) := \min \{l(G) \mid G \in \mathcal{G}(\mathcal{A}), b(G) = C, t(G) = D\}$, and for $H \in \mathcal{A}$ we set $\text{dist}(H, C) := \min \{l(G) \mid b(G) = C, t(G) = D, H \in \mathcal{W}^D\}$.

Proposition 2.8 *Let \mathcal{A} be a simplicial arrangement. Then for each $C \in \mathcal{C}(\mathcal{A})$ we have $\text{lc}(\mathcal{W}^C) = \mathcal{A}$.*

Proof Let $H \in \mathcal{A}$. We argue by induction on $\text{dist}(H, C)$. Firstly, we have $\text{dist}(H, C) = 0$ if and only if $H \in \mathcal{W}^C$. Hence, $H \in \text{lc}(\mathcal{W}^C)$ in this case. Now suppose $\text{dist}(H, C) > 0$. For the induction step, let $D \in \mathcal{C}(\mathcal{A})$ be such that $\text{dist}(H, C) = \text{dist}(C, D)$ and $H \in \mathcal{W}^D$. There is a chamber $E \in \mathcal{C}(\mathcal{A})$ adjacent to D with $\text{dist}(C, E) < \text{dist}(C, D)$. By the induction hypothesis, we see that $\mathcal{W}^E \subseteq \text{lc}(\mathcal{W}^C)$. Now, set $H' := \langle \overline{D} \cap \overline{E} \rangle$. By considering the distances, we apparently have $H \in \mathcal{W}^D \setminus \mathcal{W}^E$ and consequently, by Lemma 2.7, there is another $H'' \in \mathcal{W}^E \setminus \{H'\}$ such that $H \in \mathcal{A}_{H' \cap H''}$, i.e., $H \in \text{lc}(\mathcal{W}^E) \subseteq \text{lc}(\mathcal{W}^C)$, which concludes the induction. \square

In view of Proposition 2.3, Proposition 2.8 again shows that simplicial arrangements are combinatorially formal.

Remark 2.9 Proposition 2.8 implies that an arrangement \mathcal{A} over some field \mathbb{K} with $L(\mathcal{A})$ isomorphic to the intersection lattice of a real simplicial arrangement is combinatorially formal. For instance, if \mathbb{K} is a finite field, the combinatorial formality of \mathcal{A} could previously not be deduced by the line of implications “real simplicial $\Rightarrow K(\pi, 1) \Rightarrow$ formal” due to Deligne [9] and Falk and Randell [14].

2.7 Free Hyperplane Arrangements

Let $S = S(V^*)$ be the symmetric algebra of the dual space V^* of V . Let $\text{Der}(S)$ be the S -module of \mathbb{K} -derivations of S . Since S is graded, $\text{Der}(S)$ is a graded S -module.

Let \mathcal{A} be an arrangement in V . For $H \in \mathcal{A}$, we fix $\alpha_H \in V^*$ with $H = \ker \alpha_H$. The defining polynomial $Q(\mathcal{A})$ of \mathcal{A} is given by $Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H \in S$. The module of \mathcal{A} -derivations is defined by

$$D(\mathcal{A}) := \{\theta \in \text{Der}(S) \mid \theta(Q(\mathcal{A})) \in Q(\mathcal{A})S\}.$$

We say that \mathcal{A} is free if $D(\mathcal{A})$ is a free S -module, cf. [19, Sect. 4].

Remark 2.10 Note that the class of free arrangements is closed with respect to taking localizations, cf. [19, Thm. 4.37]. For $X \in L(\mathcal{A})$ modular of rank $r - 1$, also the converse holds, e.g. see [22, Rem. 2.8].

In [25, Cor. 2.5], Yuzvinsky showed that freeness entails formality. This was extended to k -formality by Brandt and Terao in [5, Thm. 4.15]:

Theorem 2.11 *If \mathcal{A} is free, then it is k -formal for each k .*

Note that the converse of Theorem 2.11 is false, e.g. see Example 5.1.

Following DiPasquale [10, Def. 4.9], we say that \mathcal{A} is totally formal provided \mathcal{A} is locally k -formal for all k . Thanks to [19, Thm. 4.37] and Theorem 2.11, if \mathcal{A} is free, it is totally formal. Using [10, Cor. 4.10], it is easy to see that this property of \mathcal{A} is passed to the restriction \mathcal{A}^H of \mathcal{A} to a hyperplane $H \in \mathcal{A}$. For, owing to [26,

Thm. 11], the Ziegler multiplicity is free on \mathcal{A}^H and so [10, Cor. 4.10] shows that \mathcal{A}^H is totally formal. Note that Corollary 1.3 is valid without the freeness requirement on \mathcal{A} and gives local formality for any restriction \mathcal{A}^H .

2.8 Factored Arrangements

The notion of a *factored* arrangement is due to Terao [24]. It generalizes the concept of a supersolvable arrangement, see [18, Thm. 5.3] and [19, Prop. 2.67, Thm. 3.81]. Factoredness is a combinatorial property and provides a general combinatorial framework to deduce tensor factorizations of the underlying Orlik–Solomon algebra, see also [19, Sect. 3.3]. We recall the relevant notions from [24] (cf. [19, Sect. 2.3]):

Definition 2.12 Let $\pi = (\pi_1, \dots, \pi_s)$ be a partition of \mathcal{A} .

- (a) π is called *independent*, provided for any choice $H_i \in \pi_i$ for $1 \leq i \leq s$, the resulting s hyperplanes are linearly independent, i.e., $r(H_1 \cap \dots \cap H_s) = s$.
- (b) Let $X \in L(\mathcal{A})$. The *induced partition* π_X of \mathcal{A}_X is given by the non-empty blocks of the form $\pi_i \cap \mathcal{A}_X$.
- (c) π is a *factorization* of \mathcal{A} provided
 - π is independent, and
 - for each $X \in L(\mathcal{A}) \setminus \{V\}$, the induced partition π_X admits a block which is a singleton.

If \mathcal{A} admits a factorization, then we also say that \mathcal{A} is *factored*.

We record some consequences of the main results from [24] (cf. [19, Sect. 3.3]). Here $A(\mathcal{A})$ denotes the *Orlik–Solomon algebra* of \mathcal{A} .

Corollary 2.13 Let $\pi = (\pi_1, \dots, \pi_s)$ be a factorization of \mathcal{A} . Then the following hold:

- (i) $s = r = r(\mathcal{A})$ and

$$\text{Poin}(A(\mathcal{A}), t) = \prod_{i=1}^r (1 + |\pi_i|t);$$

- (ii) the multiset $\{|\pi_1|, \dots, |\pi_r|\}$ only depends on \mathcal{A} ;
- (iii) for any $X \in L(\mathcal{A})$, we have

$$r(X) = |\{i \mid \pi_i \cap \mathcal{A}_X \neq \emptyset\}|.$$

Remark 2.14 If \mathcal{A} is non-empty and π is a factorization of \mathcal{A} , then the non-empty parts of the induced partition π_X form a factorization of \mathcal{A}_X for each $X \in L(\mathcal{A}) \setminus \{V\}$; cf. the proof of [24, Cor. 2.11]. For X in $L(\mathcal{A})$ modular of rank $r - 1$, also the converse holds, e.g. see [22, Rem. 2.25].

We record the following relation between factored and aspheric arrangements, due to Paris, [21].

Theorem 2.15 If \mathcal{A} is a complexified, factored arrangement in \mathbb{C}^3 , then \mathcal{A} is $K(\pi, 1)$.

Falk and Randell posed the question whether Theorem 2.15 does hold in arbitrary dimensions [15, Probl. 3.12]. Theorem 1.1 is strong evidence for that.

We close this section with an example illustrating that factoredness is not hereditary, using Theorem 2.15.

Example 2.16 Let \mathcal{A} be the *ideal arrangement* in the Weyl arrangement of type D_4 where the associated ideal in the set of positive roots of the root system of type D_4 consists of all roots of height at least 4. Owing to [22, Thm. 1.31 (ii)], \mathcal{A} is (inductively) factored. In [2, Ex. 3.2], a restriction \mathcal{A}^H of \mathcal{A} is exhibited which is not aspherical. It then follows from Theorem 2.15 that \mathcal{A}^H is not factored.

3 Proof of Theorem 1.1

Let $\pi = (\pi_1, \dots, \pi_\ell)$ be a factorization of \mathcal{A} . We call a subset $\{H_1, \dots, H_\ell\}$ of \mathcal{A} a *section* of π if $H_i \in \pi_i$ for each $i = 1, \dots, \ell$. Note that because π is a factorization of \mathcal{A} , every section of π consists of linearly independent hyperplanes. The following lemma paves the way for our proof of Theorem 1.1.

Lemma 3.1 *Let π be a factorization of \mathcal{A} , S be a section of π and $H \in \mathcal{A}$. Then there exists a section T of π such that $S \cup \{H\} \subset \text{lc}(T)$.*

Proof Since π is a factorization of \mathcal{A} , if we pick two hyperplanes H, H' in a block of π , then the induced partition $\pi_{H \cap H'}$ contains a singleton block. If $H \neq H'$ then $|\mathcal{A}_{H \cap H'}| \geq 3$ and if $H = H'$ then $\pi_{H \cap H'} = (\{H\})$; in either case the singleton block of $\pi_{H \cap H'}$ is uniquely determined by H, H' . Define $s(H, H')$ to be the hyperplane in this singleton block. Note that $s(H, H') = H$ if and only if $H = H'$.

Let $\pi = (\pi_1, \dots, \pi_\ell)$, and let $\iota: \mathcal{A} \rightarrow \{1, \dots, \ell\}$ be the indicator function of each block, i.e., $H \in \pi_{\iota(H)}$ for each $H \in \mathcal{A}$. Pick an arbitrary section S of π and an arbitrary hyperplane $H \in \mathcal{A}$ and define the sequences

$$\begin{aligned} \mathbf{H}: \mathbb{N}_0 &\rightarrow \mathcal{A}, \quad i \mapsto H_i, \\ \bar{\cdot}: \mathbb{N}_0 &\rightarrow \{1, \dots, \ell\}, \quad \bar{i} := \iota(H_i), \quad \text{and} \\ \mathbf{S}: \mathbb{N}_0 &\rightarrow \{\text{sections of } \pi\}, \quad i \mapsto S_i \end{aligned}$$

inductively as follows: As starting parameters choose $H_0 := H$, and $S_0 := S$. In the i th step writing $S_i = \{K_{i,1}, \dots, K_{i,\ell}\}$ such that $K_{i,j} \in \pi_j$ for $j = 1, \dots, \ell$, define

$$H_{i+1} := s(H_i, K_{i,\bar{i}}) \quad \text{and} \quad K_{i+1,j} := \begin{cases} H_i & \text{if } j = \bar{i} \\ K_{i,j} & \text{else} \end{cases} \quad \text{for } j = 1, \dots, \ell.$$

This means H_i is swapped against $K_{i,\bar{i}}$ in the new section S_{i+1} and H_{i+1} becomes the singleton hyperplane associated to H_i and $K_{i,\bar{i}}$. Note, both $H_i, K_{i,\bar{i}}$ belong to $\pi_{\bar{i}}$ and so, by construction, S_{i+1} is again a section of π for every $i \in \mathbb{N}_0$.

The following property is easy to see. For all integers p, q with $0 \leq p < q$ and for every $K \in \{H_p, K_{p,1}, \dots, K_{p,\ell}\}$, either $K \in S_q$ or there is a $j \in \{p, \dots, q-1\}$

such that $K = K_{j,\bar{j}}$. Since \mathcal{A} is a finite set, we can choose the smallest two integers $p < q$ such that $H_p = H_q$. We argue by (reverse) induction that $K_{j,\bar{j}} \in \text{lc}(S_q)$ for every $j = 0, \dots, q$.

For $j = q$ there is nothing to prove. Assume that $0 \leq j < q$ and the hypothesis is true for k with $j < k \leq q$. If $H_j = K_{j,\bar{j}}$ or $H_{j+1} = K_{j,\bar{j}}$, then the sequence becomes stationary and we have $K_{j,\bar{j}} = K_{q,\bar{q}} \in S_q$.

Otherwise both H_j and H_{j+1} are either in S_q or of the form $K_{k,\bar{k}}$ for some $j < k \leq q$. Due to the induction hypothesis, we have $H_j, H_{j+1} \in \text{lc}(S_q)$. Since $H_{j+1} = s(H_j, K_{j,\bar{j}})$, we have $K_{j,\bar{j}} \in \mathcal{A}_{H_j \cap H_{j+1}} \subset \text{lc}(S_q)$, which completes the proof. \square

Corollary 3.2 *If π is a factorization of \mathcal{A} , then there exists a section S of π such that $\text{lc}(S) = \mathcal{A}$.*

Proof Let S be a section of π such that $\text{lc}(S)$ is of maximal cardinality amongst all sections of π . For $H \in \mathcal{A}$, with Lemma 3.1, there is a section T such that $\{H\} \cup S \subset \text{lc}(T)$. Thus, $\{H\} \cup \text{lc}(S) \subset \text{lc}(T)$, and because S was chosen maximal, we get $\text{lc}(S) = \text{lc}(T)$ and finally $H \in \text{lc}(S)$. \square

Armed with Proposition 2.3 and Corollary 3.2, we can now address Theorem 1.1.

Proof of Theorem 1.1 Let π be a factorization of \mathcal{A} . With Corollary 3.2 there is a section S of π such that $\text{lc}(S) = \mathcal{A}$. Since π is independent, we have $r(\bigcap S) = \ell$. Thus, by Proposition 2.3, \mathcal{A} is combinatorially formal. \square

We show the versatility of Theorem 1.1 by improving a result from [16]. In [16, Thm. 3.5], the authors prove an addition-deletion theorem for factored arrangements. For, let π be a partition of \mathcal{A} and let $H_0 \in \pi_1$ be fixed. Consider the induced partitions π' of \mathcal{A}' by removing H_0 and π'' of \mathcal{A}'' by intersecting the parts π_2, \dots, π_ℓ with H_0 . To ensure that π'' is a factorization, in addition, Hoge and Röhrle require the bijectivity of the restriction map $\rho: \mathcal{A} \setminus \pi_1 \rightarrow \mathcal{A}''$ defined by $\rho(H) = H \cap H_0$.

Theorem 3.3 *Let \mathcal{A} be a non-empty arrangement of rank ℓ and let $\pi = (\pi_1, \dots, \pi_\ell)$ be a partition of \mathcal{A} . For $H_0 \in \pi_1$, consider the arrangement triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$. Two of the following statements imply the third:*

- (i) π is a factorization of \mathcal{A} ;
- (ii) π' is a factorization of \mathcal{A}' ;
- (iii) $\rho: \mathcal{A} \setminus \pi_1 \rightarrow \mathcal{A}''$ is bijective and π'' is a factorization of \mathcal{A}'' .

Using Theorem 1.1, we can prove a stronger criterion for the deletion part of the theorem. It turns out that the bijectivity of $\rho: \mathcal{A} \setminus \pi_1 \rightarrow \mathcal{A}''$ is a strong enough requirement on its own.

Proposition 3.4 *Let $\pi = (\pi_1, \dots, \pi_\ell)$ be a factorization of \mathcal{A} . Let $H_0 \in \pi_1$. Suppose that the restriction map*

$$\rho: \mathcal{A} \setminus \pi_1 \rightarrow \mathcal{A}'', \quad H \mapsto H \cap H_0,$$

is bijective. Then π' is a factorization for \mathcal{A}' and π'' is a factorization for \mathcal{A}'' .

Proof Since ρ is bijective, π'' is an independent partition of \mathcal{A}'' . Let $Y \in L(\mathcal{A}'')$. Note that Y can also be considered as an element of $L(\mathcal{A})$. Since π is a factorization, there is a singleton part in π_Y . If $\pi_i \cap \mathcal{A}_Y$ is a singleton for some $i > 1$, then the bijectivity of ρ implies that $\pi'' \cap \mathcal{A}_Y''$ is a singleton as well. Therefore, π'' is not a factorization of \mathcal{A}'' if and only if for some $Y \in L(\mathcal{A}'')$, the only singleton part in π_Y is $\{H_0\} = \pi_1 \cap \mathcal{A}_Y$. Suppose that this is the case. Then, if $H \in \mathcal{A}_Y \setminus \{H_0\}$, the bijectivity of ρ implies that $\mathcal{A}_{H \cap H_0} = \{H, H_0\}$, so there are no linear dependencies of rank 2 in \mathcal{A}_Y involving H_0 . With Remark 2.14, \mathcal{A}_Y is factored, so by Theorem 1.1 it is formal. The formality of \mathcal{A}_Y implies that H_0 is a separator inside \mathcal{A}_Y , i.e., $r(\mathcal{A}_Y \setminus \{H_0\}) = r(\mathcal{A}_Y) - 1$. Because of this there exists $X \in L(\mathcal{A})$ with $\mathcal{A}_X = \mathcal{A}_Y \setminus \{H_0\}$. Again, by Remark 2.14, \mathcal{A}_X is factored, so π_X has a singleton part. Our assumption implies $\pi_1 \cap \mathcal{A}_X = \emptyset$ and $|\pi_i \cap \mathcal{A}_X| = |\pi_i \cap \mathcal{A}_Y| > 1$ for $2 \leq i \leq \ell$, a contradiction. It follows that π'' is a factorization for \mathcal{A}'' . By the deletion part of Theorem 3.3, π' is also a factorization for \mathcal{A}' . □

Remark 3.5 (a). The converse of Theorem 1.1 is false. In [17], Möller constructs a real 3-arrangement which is combinatorially formal whose characteristic polynomial does not factor over the integers, thus in particular, it is not factored, cf. Corollary 2.13 (i). In addition, Möller’s example also fails to be $K(\pi, 1)$, as it satisfies the “simple triangle” condition of Falk and Randell, [14, Cor. 3.3]. Note that this example is actually k -formal for all k .

(b). In view of Theorem 2.4, Theorem 1.1 is strong support for an affirmative answer to a question raised by Falk and Randell [15, Probl. 3.12], whether complexified factored arrangements are always $K(\pi, 1)$, extending Theorem 2.15.

4 Proof of Theorem 1.2

Let V be an ℓ -dimensional vector space over some field \mathbb{K} . For a subspace $X \subseteq V$ we denote by $\text{Ann}_{V^*}(X) = \{f \in V^* \mid f|_X \equiv 0\}$ the subspace of V^* of all forms which vanish on X .

Recall from Sect. 2.3 that for an arrangement \mathcal{A} in V , $F(\mathcal{A})$ is defined as the kernel of the map $\pi_1 : \mathbb{K}(\mathcal{A}) := \bigoplus_{H \in \mathcal{A}} \text{Ann}_{V^*}(H) \rightarrow V^*$, $(\alpha_H \mid H \in \mathcal{A}) \mapsto \sum_{H \in \mathcal{A}} \alpha_H$. For a subarrangement $\mathcal{B} \subseteq \mathcal{A}$ we clearly have a natural inclusion $F(\mathcal{B}) \hookrightarrow F(\mathcal{A})$ (induced by the natural inclusion $\mathbb{K}(\mathcal{B}) \hookrightarrow \mathbb{K}(\mathcal{A})$). If $\mathcal{B} = \mathcal{A}_X$ we denote this inclusion by i_X . Recall that an arrangement \mathcal{A} is formal if and only if the map

$$\begin{aligned} \pi_2 := \sum_{X \in L_2(\mathcal{A})} i_X : \bigoplus_{X \in L_2(\mathcal{A})} F(\mathcal{A}_X) &\rightarrow F(\mathcal{A}), \\ (a_X \mid X \in L_2(\mathcal{A})) &\mapsto \sum_{X \in L_2(\mathcal{A})} i_X(a_X), \end{aligned}$$

is surjective.

In the following, we fix an arrangement \mathcal{A} and an intersection $Z \in L(\mathcal{A})$. The subsequent propositions yield Theorem 1.2 step by step.

Proposition 4.1 *There exists a surjective map $\psi_1: \mathbb{K}(\mathcal{A}) \rightarrow \mathbb{K}(\mathcal{A}^Z)$ such that the diagram*

$$\begin{CD} \mathbb{K}(\mathcal{A}) @>\pi_1>> V^* \\ @V\psi_1VV @VV\text{res}|_ZV \\ \mathbb{K}(\mathcal{A}^Z) @>\pi_1^Z>> Z^* \end{CD} \tag{4.2}$$

commutes, where π_1^Z denotes the corresponding map for \mathcal{A}^Z . Furthermore, for $Y \geq Z$ the induced diagram

$$\begin{CD} \mathbb{K}(\mathcal{A}_Y) @<< \hookrightarrow \mathbb{K}(\mathcal{A}) \\ @V\psi_{1,Y}VV @VV\psi_1V \\ \mathbb{K}(\mathcal{A}_Y^Z) @<< \hookrightarrow \mathbb{K}(\mathcal{A}^Z) \end{CD} \tag{4.3}$$

also commutes.

Proof For $K \in \mathcal{A}^Z$ set

$$\begin{aligned} \psi_K: \bigoplus_{H \in \mathcal{A}_K \setminus \mathcal{A}_Z} \text{Ann}_{V^*}(H) &\rightarrow \text{Ann}_{Z^*}(K), \\ (\alpha_H \mid H \in \mathcal{A}_K \setminus \mathcal{A}_Z) &\mapsto \sum_{H \in \mathcal{A}_K \setminus \mathcal{A}_Z} \alpha_H|_Z, \end{aligned}$$

which is obviously surjective. Note that $\mathcal{A} \setminus \mathcal{A}_Z = \bigsqcup_{K \in \mathcal{A}^Z} (\mathcal{A}_K \setminus \mathcal{A}_Z)$. Then

$$\begin{CD} \mathbb{K}(\mathcal{A}) @>\psi_1 := (\bigoplus \psi_K) \circ \text{proj}>> \mathbb{K}(\mathcal{A}^Z) \\ @V\text{proj}VV @AA\bigoplus \psi_KA \\ \mathbb{K}(\mathcal{A} \setminus \mathcal{A}_Z) @>> \end{CD}$$

does the job (since for $H \in \mathcal{A}_Z$ and $\alpha_H \in \text{Ann}_{V^*}(H)$ we have $\alpha_H|_Z = 0$). The commutativity of the diagram (4.3) is obvious. □

Proposition 4.2 *The canonical map $\tilde{\psi}_1: F(\mathcal{A}) \rightarrow F(\mathcal{A}^Z)$ between the kernels induced by the diagram (4.2) is surjective.*

Proof By Proposition 4.1 we have the following diagram with exact rows:

$$\begin{CD} 0 @>> F(\mathcal{A}) @>> \mathbb{K}(\mathcal{A}) @>\pi_1>> V^* @>> 0 \\ @. @V\tilde{\psi}_1VV @VV\psi_1V @VV\text{res}|_ZV \\ 0 @>> F(\mathcal{A}^Z) @>> \mathbb{K}(\mathcal{A}^Z) @>\pi_1^Z>> Z^* @>> 0 \end{CD}$$

If we look at the map induced by π_1 between the kernels of ψ_1 and $\text{res}|_Z$, we see that under this map $\mathbb{K}(\mathcal{A}_Z) \subseteq \ker(\psi_1)$ maps onto $\text{Ann}_{V^*}(Z) = \ker(\text{res}|_Z)$. Since ψ_1 is surjective, by Proposition 4.1, the snake lemma yields $\text{coker}(\tilde{\psi}_1) = 0$. □

Proposition 4.3 For the map $\tilde{\psi}_1$ we have

$$\tilde{\psi}_1(\text{im}(\pi_2)) \subseteq \text{im}(\pi_2^Z).$$

In particular, the induced map $\text{coker}(\pi_2) \rightarrow \text{coker}(\pi_2^Z)$ is surjective.

Proof Since, by definition, we have $\text{im}(\pi_2) = \sum_{X \in L_2(\mathcal{A})} i_X(F(\mathcal{A}_X))$, it suffices to show that $\tilde{\psi}_1(i_X(F(\mathcal{A}_X))) \subseteq \text{im}(\pi_2^Z)$ for each $X \in L_2(\mathcal{A})$. For $Y \geq X$, we have $i_X(F(\mathcal{A}_X)) \subseteq i_Y(F(\mathcal{A}_Y))$. If $X \in L_2(\mathcal{A})$ then there is a $Y \in L_2(\mathcal{A}^Z)$ with $Y \geq X$. Since the diagram (4.3) commutes, the same holds for the induced maps on the kernels, i.e., $\tilde{\psi}_1 \circ i_Y = i_Y^Z \circ \tilde{\psi}_{1,Y}$, and we get

$$\tilde{\psi}_1(i_X(F(\mathcal{A}_X))) \subseteq \tilde{\psi}_1(i_Y(F(\mathcal{A}_Y))) \subseteq i_Y^Z(F(\mathcal{A}_Y^Z)) \subseteq \text{im}(\pi_2^Z),$$

which concludes the proof. □

Corollary 4.4 (Theorem 1.2) Let \mathcal{A} be a formal arrangement. Then for any $Z \in L(\mathcal{A})$ the restriction \mathcal{A}^Z is formal, too.

5 Complements and Examples

In this section, we provide some complements and applications to the main developments.

Thanks to Theorem 1.2, formality is inherited by restrictions. In view of the results by DiPasquale [10] and due to a lack of examples of non-free k -formal arrangements in dimensions at least 5, it is not clear whether Theorem 1.2 extends to higher formality (i.e., k -formality for $k \geq 3$). The fact that this is not the case is demonstrated by our next example. This was found by means of an implementation of a modified “greedy search” algorithm based on ideas presented by Cuntz in [6].

Example 5.1 Consider the 5-arrangement \mathcal{A} consisting of 11 hyperplanes in \mathbb{K}^5 with defining polynomial

$$Q(\mathcal{A}) = x_1x_2x_3x_4x_5(x_1 + x_3)(x_1 + x_3 + x_5)(x_1 + x_2 + x_4)(x_1 + x_2 + x_4 + x_5) \cdot (x_1 + x_2 + x_3 + x_5)(x_1 + x_2 + x_3 + x_4 + x_5).$$

We note that the underlying matroid of \mathcal{A} is *regular*, i.e., is realizable over any field \mathbb{K} . We proceed to show that \mathcal{A} is 4-formal (and hence k -formal for all k). To see that \mathcal{A} is 2-formal, first observe that there are six relations of rank 2:

$$\begin{aligned} \alpha_1 + \alpha_3 - \alpha_6 &= 0, & \alpha_2 + \alpha_7 - \alpha_{10} &= 0, \\ \alpha_3 + \alpha_9 - \alpha_{11} &= 0, & \alpha_4 + \alpha_{10} - \alpha_{11} &= 0, \\ \alpha_5 + \alpha_6 - \alpha_7 &= 0, & \alpha_5 + \alpha_8 - \alpha_9 &= 0, \end{aligned}$$

where α_i is the i th linear form in the order as it appears in $Q(\mathcal{A})$ above. Thus, the map $\pi_2: \bigoplus_{X \in L_2} F_2(\mathcal{A}_X) \rightarrow F(\mathcal{A})$ is defined by the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix},$$

which is of rank 6. Since $\dim F(\mathcal{A}) = |\mathcal{A}| - r(\mathcal{A}) = 11 - 5 = 6$, it follows that π_2 is surjective, so \mathcal{A} is 2-formal. We also see that $\ker(\pi_2) = R_3(\mathcal{A}) = \{0\}$ and therefore $R_3(\mathcal{A}_X) = \{0\}$ for any $X \in L_3(\mathcal{A})$ as well. Thus, $\pi_3: \bigoplus R_3(\mathcal{A}_X) \rightarrow R_3(\mathcal{A})$ is surjective and its kernel is $R_4(\mathcal{A}) = \{0\}$, so the map $\pi_4: \bigoplus R_4(\mathcal{A}_X) \rightarrow R_4(\mathcal{A})$ is surjective and \mathcal{A} is 4-formal.

The five coordinate hyperplanes $\ker(x_1), \ker(x_2), \dots, \ker(x_5)$ form an lc-basis of \mathcal{A} and thus the arrangement is combinatorially formal by Proposition 2.3. Yet, the restriction \mathcal{A}^H to $H = \ker(x_2)$ in \mathbb{K}^4 , which has defining polynomial

$$Q(\mathcal{A}^H) = x_1x_2x_3x_4(x_1 + x_3)(x_1 + x_3 + x_4)(x_1 + x_2)(x_1 + x_2 + x_4) \cdot (x_1 + x_2 + x_3 + x_4),$$

fails to be 3-formal. To see this, again consider the rank 2 relations of \mathcal{A}^H :

$$\begin{aligned} \alpha_1 + \alpha_2 - \alpha_7 &= 0, & \alpha_1 + \alpha_3 - \alpha_5 &= 0, \\ \alpha_2 + \alpha_6 - \alpha_9 &= 0, & \alpha_3 + \alpha_8 - \alpha_9 &= 0, \\ \alpha_4 + \alpha_5 - \alpha_6 &= 0, & \alpha_4 + \alpha_7 - \alpha_8 &= 0, \end{aligned}$$

where α_i is the i th linear form in the order as it appears in $Q(\mathcal{A}^H)$ above. The matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}$$

is of rank 5, so its kernel $R_3(\mathcal{A}^H)$ is of dimension 1. Consider the following table of the non-trivial rank 3 intersections in $L(\mathcal{A}^H)$ and the non-trivial rank 2 flats they are contained in. For the set $\{H_{i_1}, \dots, H_{i_k}\}$ we write $i_1i_2 \dots i_k$ as a shorthand.

From the table it is obvious that the rank 2 relations in every rank 3 localization of \mathcal{A}^H are linearly independent, so $R_3((\mathcal{A}^H)_X) = \{0\}$ for every $X \in L_3(\mathcal{A}^H)$. We conclude that the map $\pi_3: \bigoplus R_3((\mathcal{A}^H)_X) \rightarrow R_3(\mathcal{A}^H)$ is not surjective and therefore \mathcal{A}^H is not 3-formal.

Moreover, for $X = \ker(x_1) \cap \ker(x_2) \cap \ker(x_4)$, the rank 3 localization \mathcal{A}_X is obviously not formal. Consequently, \mathcal{A}_X fails to be factored, $K(\pi, 1)$ (for $\mathbb{K} = \mathbb{C}$), and free, owing to Theorems 1.1, 2.4, and 2.11, respectively, and thus also \mathcal{A} fails to be factored, $K(\pi, 1)$ (for $\mathbb{K} = \mathbb{C}$), and free, thanks to Remarks 2.14, 2.5, and 2.10, respectively.

Finally, the argument in the last paragraph also shows that the given lc-basis of \mathcal{A} does not descend to one for \mathcal{A}_X .

Non-trivial $X \in L_3(\mathcal{A}^H)$	Non-trivial $Y \in L_2(\mathcal{A}^H)$ with $Y < X$
12357	125, 134
12478	124, 345
12679	124, 235
13456	124, 345
13589	123, 245
23689	135, 245
24569	145, 234
34789	145, 234
45678	123, 145

Owing to [25] or by the example above, formality is not inherited by arbitrary localizations. The following consequence of Theorem 1.2 shows however that it is passed to special localizations.

Corollary 5.2 *Suppose $X \in L(\mathcal{A})$ is modular. If \mathcal{A} is formal, then so is \mathcal{A}_X .*

Proof Since $X \in L(\mathcal{A})$ is modular, it follows from Lemma 2.2 that $\mathcal{A}_X/X \cong \mathcal{A}^Y/T(\mathcal{A})$ for a $Y \in L(\mathcal{A})$ complementary to X . As \mathcal{A} is formal, so is the restriction \mathcal{A}^Y , thanks to Theorem 1.2. □

The following consequence of Lemma 2.1 gives a converse to Corollary 5.2 in corank 1.

Corollary 5.3 *If $X \in L(\mathcal{A})$ is modular of corank 1, then \mathcal{A}_X is formal if and only if \mathcal{A} is formal.*

In the following example, we present a family of real arrangements \mathcal{K}_n of rank $n \geq 3$, none of which is free, $K(\pi, 1)$, or factored. Nevertheless, each is combinatorially formal. So that in particular, here formality is not a consequence of any of the other sufficient properties.

Example 5.4 Let \mathcal{B}_n be the reflection arrangement of the hyperoctahedral group of type B_n and let \mathcal{A}_{n-1} be the braid arrangement of type A_{n-1} , a subarrangement of \mathcal{B}_n . Let

$$\mathcal{K}_n := \mathcal{B}_n \setminus \mathcal{A}_{n-1}$$

be the complement of \mathcal{A}_{n-1} in \mathcal{B}_n . Thus \mathcal{K}_n has defining polynomial

$$Q(\mathcal{K}_n) = \prod_{i=1}^n x_i \prod_{1 \leq i < j \leq n} (x_i + x_j).$$

It was shown in [3, Ex. 2.7] that \mathcal{K}_n is not $K(\pi, 1)$ and not free for $n \geq 3$.

We next show by induction on n that \mathcal{K}_n is also not factored either for $n \geq 3$. Owing to Theorem 2.15, \mathcal{K}_3 is not factored, as it is not $K(\pi, 1)$. Now suppose that $n > 3$ and that \mathcal{K}_{n-1} is not factored. Let $X := \bigcap_{i=1}^{n-1} \ker(x_i)$. Then one readily checks that

$$(\mathcal{K}_n)_X \cong \mathcal{K}_{n-1}.$$

It follows from our induction hypothesis and Remark 2.14 that also \mathcal{K}_n fails to be factored.

Finally, we observe that \mathcal{K}_n admits an lc-basis. For, let $\mathcal{B} \subset \mathcal{K}_n$ be the Boolean subarrangement, i.e., $Q(\mathcal{B}) = \prod_{i=1}^n x_i$. Then one easily checks that $\text{lc}(\mathcal{B}) = \mathcal{K}_n$. It thus follows from Proposition 2.3 that \mathcal{K}_n is combinatorially formal; formality for \mathcal{K}_3 was already observed by Falk and Randell in [14, (3.12)].

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