# ON FORMALITY AND COMBINATORIAL FORMALITY FOR HYPERPLANE ARRANGEMENTS 

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#### Abstract

A hyperplane arrangement is called formal provided all linear dependencies among the defining forms of the hyperplanes are generated by ones corresponding to intersections of codimension two. The significance of this notion stems from the fact that complex arrangements with aspherical complements are formal.

The aim of this note is twofold. While work of Yuzvinsky shows that formality is not combinatorial, in our first main theorem we prove that the combinatorial property of factoredness of arrangements does entail formality.

Our second main theorem shows that formality is hereditary, i.e. is passed to restrictions. This is rather counter-intuitive, as in contrast the known sufficient conditions for formality, i.e. asphericity, freeness and factoredness (owed to our first theorem), are not hereditary themselves. We also demonstrate that the stronger property of $k$-formality, due to Brandt and Terao, is not hereditary.


## 1. Introduction

In the study of hyperplane arrangements it is of central concern to connect algebraic or topological properties (over $\mathbb{C}$ ) with combinatorial invariants encoded by their intersection lattices. Yet another important theme is to investigate the behavior of a given notion under the standard arrangement constructions of restriction and localization. Following this philosophy, the aim of our note is to relate the combinatorial notion of factoredness with the general concept of formality on the one hand and on the other to complete the picture about the behavior of formality with respect to restriction and localization, complementing previous results in the literature.

A hyperplane arrangement $\mathscr{A}$ is formal provided all linear dependencies among the defining forms of the hyperplanes in $\mathscr{A}$ are generated by dependencies corresponding to intersections of codimension 2 of the members of $\mathscr{A}$ (see §2.3). Over the complex numbers this property is implied by $K(\pi, 1)$-arrangements (see Theorem 2.4).

A property for arrangements is said to be combinatorial if it only depends on the intersection lattice of the underlying arrangement. Yuzvinsky [Y93, Ex. 2.2] demonstrated that formality is not combinatorial, answering a question raised by Falk and Randell [FR87] in the negative. Yuzvinsky's insight motivates the following notion. Suppose $\mathscr{A}$ is a formal arrangement. We say $\mathscr{A}$ is combinatorially formal if every arrangement with an intersection lattice isomorphic to the one of $\mathscr{A}$ is also formal. In view of Yuzvinsky's result it is therefore startling that

[^0]even this strong form of formality is afforded by the combinatorial property of factoredness (see §2.8). This is the content of our first main result.

Theorem 1.1. If $\mathscr{A}$ is factored, then it is combinatorially formal.
In [FR87], the authors raised the question whether freeness (see §2.7) also implies formality. This was settled affirmatively by Yuzvinsky in [Y93, Cor. 2.5].
While both, freeness and asphericity, do imply formality, it is not known whether the former are combinatorial or not (cf. [FR00, Prob. 3.8]). For the concept of freeness this is a longstanding conjecture due to Terao (cf. [OT92, Conj. 4.138]).

A property for arrangements is said to be hereditary if it is inherited by every restriction of the underlying arrangement. Freeness, asphericity, and factoredness are known to not be hereditary in general (cf. [ER91], [AMR18, Thm. 1.1], and Example 2.15, respectively). It is thus rather surprising and counter-intuitive that formality on the other hand, which is entailed by each of these properties, is hereditary. This is the content of our second main theorem.

## Theorem 1.2. Formality is hereditary.

The notion of formality was further refined to the concept of $k$-formality by Brandt and Terao [BT94, Def. 4.4] (see §5). In the setting of [BT94], formality coincides with 2-formality. Higher formality is stronger than formality, cf. [BT94, Ex. 5.1]. In [BT94, Thm. 4.15], Brandt and Terao extended Yuzvinsky's result [Y93, Cor. 2.5] by showing that free arrangements are $k$-formal for all $k$. Extending [BT94, Thm. 4.15] further, in [DiP18, Thm. 1.1], DiPasquale employed homological methods based on those developed in [BT94] to characterize freeness of a multiplicity of an arrangement and showed as a consequence that the latter implies $k$-formality for all $k$, [DiP18, Cor. 4.10]. We emphasize that Theorem 1.2 does not extend to the stronger notion of $k$-formality, see Example 5.1.

A property for arrangements is said to be local if it is passed to every localization of the ambient arrangement. Freeness, asphericity, and factoredness are all local (cf. [OT92, Thm. 4.37], [P93, Lem. 1.1], and the proof of [T92, Cor. 2.11], respectively). In contrast, this fails for formal arrangements of rank at least 4 (cf. the example following [Y93, Def. 2.3]). However, as an application of Theorem 1.2, we in turn show that localizations at modular flats of formal arrangements are formal again, see Corollary 5.2.

Moreover, it is also known that if $X$ is a modular flat of $\mathscr{A}$ of corank 1 and the localization $\mathscr{A}_{X}$ is free, $K(\pi, 1)$, or factored, then so is $\mathscr{A}$ itself (cf. [R17, Rem. 2.8], [T86], and [R17, Rem. 2.25], respectively). In Corollary 5.3, we show that this property also holds for formality.

Following Yuzvinsky [Y93, Def. 2.3], we say that $\mathscr{A}$ is locally formal provided that each localization of $\mathscr{A}$ is formal. Since freeness, asphericity, and factoredness are all local and each affords formality, each of these properties implies local formality. Since formality is not passed to localizations in general, as noted above, the class of formal arrangements properly encompasses the classes of free, factored and $K(\pi, 1)$-arrangements. The following is immediate from Theorem 1.2.

Corollary 1.3. Local formality is hereditary.
The paper is organized as follows. In $\S 2.1$ and $\S 2.2$, we recall some standard terminology and introduce further notation on hyperplane arrangements and record some basic facts on modular elements in the lattice of intersections of an arrangement. The notions of formality and $k$-formality are recapitulated from [BT94] in §2.3. This is followed by a brief discussion of the concept of line closure of a subset of an arrangement. Here we review Falk's criterion that the presence of an lc-basis entails combinatorial formality (Proposition 2.3). This in turn is the key ingredient in our proof of Theorem 1.1. Short recollections on the fundamental notions of $K(\pi, 1)$-arrangements (§2.5) and simplicial arrangements (§2.6) follow. In §2.7 and $\S 2.8$, the concepts of free and factored arrangements are recalled.

Theorems 1.1 and 1.2 are proved in $\S 3$ and $\S 4$, respectively.
In our final section, we present some complements to our main developments. Here we demonstrate in Example 5.1 that Theorem 1.2 does not extend to higher formality.

In addition, we present a family of combinatorially formal arrangements $\mathscr{K}_{n}$ of rank $n \geq 3$ which fail to be free, $K(\pi, 1)$, and factored (Example 5.4). These are natural subarrangements of the Coxeter arrangement of type $B_{n}$.
For general information about arrangements we refer the reader to [OT92].

## 2. Recollections and Preliminaries

2.1. Hyperplane arrangements. Let $\mathbb{K}$ be a field and let $V=\mathbb{K}^{\ell}$ be an $\ell$-dimensional $\mathbb{K}$ vector space. A hyperplane arrangement $\mathscr{A}=(\mathscr{A}, V)$ in $V$ is a finite collection of hyperplanes in $V$ each containing the origin of $V$. We also use the term $\ell$-arrangement for $\mathscr{A}$. We denote the empty arrangement in $V$ by $\Phi_{\ell}$.
The lattice $L(\mathscr{A})$ of $\mathscr{A}$ is the set of subspaces of $V$ of the form $H_{1} \cap \cdots \cap H_{i}$ where $\left\{H_{1}, \ldots, H_{i}\right\}$ is a subset of $\mathscr{A}$. For $X \in L(\mathscr{A})$, we have two associated arrangements, firstly $\mathscr{A}_{X}:=\{H \in$ $\mathscr{A} \mid X \subseteq H\} \subseteq \mathscr{A}$, the localization of $\mathscr{A}$ at $X$, and secondly, the restriction of $\mathscr{A}$ to $X$, $\left(\mathscr{A}^{X}, X\right)$, where $\mathscr{A}^{X}:=\left\{X \cap H \mid H \in \mathscr{A} \backslash \mathscr{A}_{X}\right\}$. Note that $V$ belongs to $L(\mathscr{A})$ as the intersection of the empty collection of hyperplanes and $\mathscr{A}^{V}=\mathscr{A}$. The lattice $L(\mathscr{A})$ is a partially ordered set by reverse inclusion: $X \leq Y$ provided $Y \subseteq X$ for $X, Y \in L(\mathscr{A})$.
Throughout, we only consider arrangements $\mathscr{A}$ such that $0 \in H$ for each $H$ in $\mathscr{A}$. These are called central. In that case the center $T(\mathscr{A}):=\cap_{H \in \mathscr{A}} H$ of $\mathscr{A}$ is the unique maximal element in $L(\mathscr{A})$ with respect to the partial order. A rank function on $L(\mathscr{A})$ is given by $r(X):=\operatorname{codim}_{V}(X)$. The rank of $\mathscr{A}$ is defined as $r(\mathscr{A}):=r(T(\mathscr{A}))$.
For $1 \leq k \leq r(\mathscr{A})$ we write $L_{k}(\mathscr{A})$ for the collection of all $X \in L(\mathscr{A})$ of rank $k$.
2.2. Modular elements in $L(\mathscr{A})$. We say that $X \in L(\mathscr{A})$ is modular provided $X+Y \in$ $L(\mathscr{A})$ for every $Y \in L(\mathscr{A})$, [OT92, Cor. 2.26]. We require the following characterization of modular members of $L(\mathscr{A})$ of rank $r-1$, see the proof of [BEZ90, Thm. 4.3].

Lemma 2.1. Let $\mathscr{A}$ be an arrangement of rank $r$. Suppose that $X \in L(\mathscr{A})$ is of rank $r-1$. Then $X$ is modular if and only if for any two distinct $H_{1}, H_{2} \in \mathscr{A} \backslash \mathscr{A}_{X}$ there is a $H_{3} \in \mathscr{A}_{X}$ so that $r\left(H_{1} \cap H_{2} \cap H_{3}\right)=2$, i.e. $H_{1}, H_{2}, H_{3}$ are linearly dependent.

Next, we record a refined geometric version of a standard fact about modular elements in a geometric lattice, cf. [A79, Prop. 2.42] or [OT92, Lem. 2.27].

Lemma 2.2. Let $\mathscr{A}$ be an arrangement of rank r. Suppose that $X \in L(\mathscr{A})$ is modular of rank $q$. Then there is a complementary intersection $Y \in L(\mathscr{A})$ of rank $r-q$ (i.e. $X \cap Y=$ $T(\mathscr{A}))$ such that the arrangements $\left(\mathscr{A}_{X} / X, V / X\right)$ and $\left(\mathscr{A}^{Y} / T(\mathscr{A}), Y / T(\mathscr{A})\right)$ are linearly isomorphic.

Proof. Let $X=H_{1} \cap \ldots \cap H_{q}$. Then there are $H_{q+1}, \ldots, H_{r} \in \mathscr{A}$ such that $T(\mathscr{A})=$ $H_{1} \cap \ldots \cap H_{r}$, i.e. $Y=H_{q+1} \cap \ldots \cap H_{r}$ is complementary to $X$. Due to the modularity of $X$ for all $K=H^{\prime} \cap Y \in \mathscr{A}^{Y}$, we have $H=X+K \in \mathscr{A}_{X}, H^{\prime} \cap Y=H \cap Y=K$, and furthermore, this correspondence is bijective. Consequently, the quotient map $Y / T(\mathscr{A}) \rightarrow$ $V / X, v+T(\mathscr{A}) \mapsto v+X$, which is an isomorphism by the choice of $Y$, maps $\mathscr{A}^{Y} / T(\mathscr{A})$ to $\mathscr{A}_{X} / X$.
2.3. Formality and $k$-formality. The notion of formality is due to Falk and Randell [FR87]. Here we give the equivalent definition by Brandt and Terao [BT94]. Let $\left(e_{H} \mid\right.$ $H \in \mathscr{A}$ ) be the basis of a $\mathbb{K}$-vector space indexed by the hyperplanes in $\mathscr{A}$. For each $H \in \mathscr{A}$, choose a linear form $\alpha_{H} \in V^{*}$ such that ker $\alpha_{H}=H$. Consider the map

$$
\bigoplus \mathbb{K} e_{H} \rightarrow V^{*} \quad \text { defined by } \quad e_{H} \mapsto \alpha_{H}
$$

and let $F(\mathscr{A})$ be the kernel of this map. Note that $\mathscr{A}$ is linearly isomorphic to the arrangement $\left\{F(\mathscr{A})^{\perp} \cap \operatorname{ker} x_{H} \mid H \in \mathscr{A}\right\}$ in $F(\mathscr{A})^{\perp}$, where $\left(x_{H} \mid H \in \mathscr{A}\right)$ is the dual basis to $\left(e_{H} \mid H \in \mathscr{A}\right)$. We say an element $x \in F(\mathscr{A})$ has length $p$ if $x$ has exactly $p$ nonzero entries. Let $F_{2}(\mathscr{A})$ be the subspace of $F(\mathscr{A})$ generated by all $x \in F(\mathscr{A})$ of length at most 3 . Then, $\mathscr{A}$ is said to be formal if $F_{2}(\mathscr{A})=F(\mathscr{A})$. Note that if $r(\mathscr{A})=2$, then $F_{2}(\mathscr{A})=F(\mathscr{A})$, so each rank 2 arrangement is formal. As indicated in the introduction, a localization of a formal arrangement need not be formal again, see the example after [Y93, Def. 2.3].

The following generalization of formality is due to Brandt and Terao [BT94]. If $X \in L(\mathscr{A})$, then there is a natural inclusion $F_{2}\left(\mathscr{A}_{X}\right) \hookrightarrow F_{2}(\mathscr{A})$. If $x \in F(\mathscr{A})$ is of length 3 , then there is an associated $X \in L_{2}(\mathscr{A})$ with $x \in F_{2}\left(\mathscr{A}_{X}\right) \subset F(\mathscr{A})$. Thus, $\mathscr{A}$ is formal if and only if the map

$$
\pi_{2}: \bigoplus_{X \in L_{2}(\mathscr{A})} F\left(\mathscr{A}_{X}\right) \longrightarrow F(\mathscr{A})
$$

is a surjection. The definition of $k$-formality in [BT94] is recursive and does not require the choice of linear forms as before. To emphasize the use of this alternative definition, we follow the notation from [BT94], which differs from the one above.

Let $R_{0}(\mathscr{A})=T(\mathscr{A})^{*}$ be the dual space of the center of $\mathscr{A}$. For each $X \in L(\mathscr{A})$, we have a surjective map $i_{0}(X): R_{0}\left(\mathscr{A}_{X}\right) \rightarrow R_{0}(\mathscr{A})$ defined by restricting the domain to $T(\mathscr{A})$. Then, for $1 \leq k \leq r(\mathscr{A})$, define $R_{k}(\mathscr{A})$ recursively as the kernel of the map

$$
\pi_{k-1}(\mathscr{A}): \bigoplus_{X \in L_{k-1}} R_{k-1}\left(\mathscr{A}_{X}\right) \longrightarrow R_{k-1}(\mathscr{A})
$$

where the maps $\pi_{k}$ are defined as follows. (Note that $\bigoplus_{X \in L_{0}} R_{0}\left(\mathscr{A}_{X}\right)=R_{0}\left(\Phi_{\ell}\right)=V^{*}$ and $\pi_{0}: V^{*} \rightarrow R_{0}(\mathscr{A})=T(\mathscr{A})^{*}$ is just restriction.) For each $k \geq 1$ and each $Y \in L(\mathscr{A})$ with $\operatorname{rk}(Y) \geq k$, there is a map

$$
i_{k}(Y):=i_{k}(\mathscr{A}, Y): R_{k}\left(\mathscr{A}_{Y}\right) \rightarrow R_{k}(\mathscr{A}),
$$

defined recursively via the commutative diagram

$$
\begin{array}{rc}
R_{k}\left(\mathscr{A}_{Y}\right) & \longrightarrow \bigoplus_{\substack{X \in L_{k-1} \\
X \leq Y}} R_{k-1}\left(\left(\mathscr{A}_{Y}\right)_{X}\right) \xrightarrow{\pi_{k-1}(\mathscr{A} Y)} \\
\downarrow_{k-1}\left(\mathscr{A}_{Y}\right) \\
\downarrow_{k}(Y) & \downarrow^{j_{k-1}(Y)} \\
R_{k}(\mathscr{A}) & \longrightarrow \downarrow_{X \in L_{k-1}} R_{k-1}\left(\mathscr{A}_{X}\right) \\
\bigoplus_{k-1}(Y) \\
\pi_{k-1}(\mathscr{A}) & R_{k-1}(\mathscr{A})
\end{array}
$$

In the diagram, $R_{k}\left(\left(\mathscr{A}_{Y}\right)_{X}\right)=R_{k}\left(\mathscr{A}_{X}\right)$ since $X \leq Y$, so $j_{k-1}(Y)$ is defined as the direct sum of identity maps and zero maps. Then $i_{k}(Y)$ is defined through the universal property of the kernel $R_{k}(\mathscr{A})$. With this define $\pi_{k}$ as the sum of the $i_{k}(X)$ for $X \in L_{k}(\mathscr{A})$.

If $\mathscr{A}$ is formal, we say it is 2 -formal. For $k \geq 3$, we say $\mathscr{A}$ is $k$-formal if $\mathscr{A}$ is $(k-1)$-formal and the map $\pi_{k}$ is surjective. Note that an arrangement of rank $r$ which is $(r-1)$-formal is trivially $k$-formal for all $k \geq r$.
2.4. Line closure and combinatorial formality. We want to establish further combinatorial properties of $\mathscr{A}$ that imply formality. The following definitions are due to Falk [F02] for matroids, but they can easily be applied to arrangements as well. Let $\mathscr{B} \subset \mathscr{A}$ be a subset of hyperplanes. We say $\mathscr{B}$ is closed if $\mathscr{B}=\mathscr{A}_{Y}$ for $Y=\bigcap_{H \in \mathscr{B}} H$. We call $\mathscr{B}$ line-closed if for every pair $H, H^{\prime} \in \mathscr{B}$ of hyperplanes, we have $\mathscr{A}_{H \cap H^{\prime}} \subset \mathscr{B}$. The line-closure $\operatorname{lc}(\mathscr{B})$ of $\mathscr{B}$ is defined as the intersection of all line-closed subsets of $\mathscr{A}$ containing $\mathscr{B}$. The arrangement $\mathscr{A}$ is called line-closed if every line-closed subset of $\mathscr{A}$ is closed. With these notions, we have the following criterion for combinatorial formality, see [F02, Cor. 3.8]:

Proposition 2.3. Let $\mathscr{A}$ be an arrangement of rank $r$. Suppose $\mathscr{B} \subseteq \mathscr{A}$ consists of $r$ hyperplanes such that $r(\mathscr{B})=r$ and $\operatorname{lc}(\mathscr{B})=\mathscr{A}$. Then $\mathscr{A}$ is combinatorially formal.

We call such a subset $\mathscr{B} \subseteq \mathscr{A}$ as in Proposition 2.3 an lc-basis of $\mathscr{A}$. Note that the converse of Proposition 2.3 is false [F02], i.e. a combinatorially formal $\mathscr{A}$ need not admit an lc-basis.

Furthermore, the existence of an lc-basis does not imply higher formality. Brandt and Terao give an example of an arrangement that is 2-formal but not 3-formal, see [BT94, Ex. 5.1]. Here an lc-basis is easily calculated, e.g. take $\left\{H_{1}, H_{2}, H_{4}, H_{5}\right\}$ in [BT94, Ex. 5.1].

Finally, we note that the presence of an lc-basis does not descend to localizations, i.e. if $\mathscr{B} \subseteq \mathscr{A}$ is as in Proposition 2.3 and $X \in L(\mathscr{A})$, then it need not be the case that $\mathscr{B}_{X}$ is an lc-basis of $\mathscr{A}_{X}$, see Example 5.1.
2.5. $K(\pi, 1)$-arrangements. A complex $\ell$-arrangement $\mathscr{A}$ is called aspherical, or a $K(\pi, 1)$ arrangement (or that $\mathscr{A}$ is $K(\pi, 1)$ for short), provided the complement $M(\mathscr{A})$ of the union of the hyperplanes in $\mathscr{A}$ in $\mathbb{C}^{\ell}$ is aspherical, i.e. is a $K(\pi, 1)$-space. That is, the universal covering space of $M(\mathscr{A})$ is contractible and the fundamental group $\pi_{1}(M(\mathscr{A}))$ of $M(\mathscr{A})$ is isomorphic to the group $\pi$. This is an important topological property, for the cohomology ring $H^{*}(X, \mathbb{Z})$ of a $K(\pi, 1)$-space $X$ coincides with the group cohomology $H^{*}(\pi, \mathbb{Z})$ of $\pi$. The crucial point here is that the intersections of codimension 2 determine the fundamental group $\pi_{1}(M(\mathscr{A}))$ of $M(\mathscr{A})$.
By Deligne's seminal result [De72], complexified simplicial arrangements are $K(\pi, 1)$. Likewise for complex supersolvable arrangements, cf. [FR85] and [T86] (cf. [OT92, Prop. 5.12, Thm. 5.113]). As restrictions of simplicial (resp. supersolvable) arrangements are again simplicial (resp. supersolvable), the $K(\pi, 1)$-property of these kinds of arrangements is inherited by their restrictions. However, we emphasize that in general, a restriction of a $K(\pi, 1)$ arrangement need not be $K(\pi, 1)$ again, see [AMR18] for examples of this kind.

The following theorem due to Falk and Randell [FR87, Thm. 4.2] establishes formality as a necessary condition for asphericity.

Theorem 2.4. If $\mathscr{A}$ is a $K(\pi, 1)$-arrangement, then it is formal.

Thanks to [CG15], simpliciality is a combinatorial property. Thus, as simplicial arrangements are $K(\pi, 1)$, it follows that all such are combinatorially formal, see also $\S 2.6$. It is not known whether this is true for aspherical arrangements in general.

Remark 2.5. Thanks to an observation by Oka, if the complex arrangement $\mathscr{A}$ is $K(\pi, 1)$, then so is every localization $\mathscr{A}_{X}$ for $X \in L(\mathscr{A})$, e.g., see [P93, Lem. 1.1].

The following is an immediate consequence of Terao's work [T86] (see also [OT92, §5.5]).
Lemma 2.6. Let $\mathscr{A}$ be a complex arrangement of rank $r$. Suppose that $X \in L(\mathscr{A})$ is modular of rank r-1. If $\mathscr{A}_{X}$ is $K(\pi, 1)$, then so is $\mathscr{A}$.
2.6. Simplicial arrangements. Let $\mathscr{A}$ be a real arrangement in $V \cong \mathbb{R}^{\ell}$. Then, the connected components of the complement $V \backslash \bigcup_{H \in \mathscr{A}} H$ are called chambers and are denoted by $\mathscr{C}(\mathscr{A})$. The arrangement $\mathscr{A}$ is called simplicial if all $C \in \mathscr{C}(\mathscr{A})$ are (open) simplicial cones. It was already mentioned above that simplicial arrangements are aspherical and therefore formal. Furthermore, simpliciality is actually a combinatorial property of the intersection lattice, as was observed in [CG15]. Consequently, the formality of simplicial arrangements is combinatorial.

In Proposition 2.8 below, we demonstrate that a simplicial arrangement naturally satisfies the stronger property of having an lc-basis. Explicitly, owing to Proposition 2.8, the walls of each chamber of a simplicial arrangement yield such a special basis.

For a chamber $C \in \mathscr{C}(\mathscr{A})$, we write $\mathscr{W}^{C}:=\{H \in \mathscr{A} \mid\langle\bar{C} \cap H\rangle=H\}$ for the walls of $C$. Two chambers $C, D \in \mathscr{C}(\mathscr{A})$ are adjacent if $\langle\bar{C} \cap \bar{D}\rangle \in \mathscr{A}$.

The following lemma, which is a special case of [CM19, Lem. 3.3], provides the essential argument for the proof of Proposition 2.8 below.

Lemma 2.7. Let $C, D \in \mathscr{C}(\mathscr{A})$ be two adjacent and simplicial chambers with a common wall $H \in \mathscr{W}^{C} \cap \mathscr{W}^{D}$, and let $H^{\prime} \in \mathscr{W}^{D} \backslash\{H\}$. Then, $H^{\prime} \in \mathscr{W}^{C}$ if and only if $\left|\mathscr{A}_{H \cap H^{\prime}}\right|=2$.

We require a bit more notation, cf. [CM19, Sec. 2.2]. A sequence $\left(C_{0}, C_{1}, \ldots, C_{n-1}, C_{n}\right)$ of distinct chambers in $\mathscr{C}(\mathscr{A})$ is called a gallery if for all $1 \leq i \leq n$ the chambers $C_{i}$ and $C_{i-1}$ are adjacent. The set of all galleries is denoted by $\mathcal{G}(\mathscr{A})$. The length $l(G)$ of a gallery $G \in \mathcal{G}(\mathscr{A})$ is one less than the number of chambers in $G$. For a gallery $G=\left(C_{0}, \ldots, C_{n}\right)$, we denote by $b(G)=C_{0}$ the first chamber and by $t(G)=C_{n}$ the last chamber in $G$. For two chambers $C, D \in \mathscr{C}(\mathscr{A})$, we set $\operatorname{dist}(C, D):=\min \{l(G) \mid G \in \mathcal{G}(\mathscr{A}), b(G)=C$ and $t(G)=D\}$, and for $H \in \mathscr{A}$, we set $\operatorname{dist}(H, C):=\min \{l(G) \mid b(G)=C, t(G)=D$ and $\left.H \in \mathscr{W}^{D}\right\}$.

Proposition 2.8. Let $\mathscr{A}$ be a simplicial arrangement. Then, for each $C \in \mathscr{C}(\mathscr{A})$ we have $\operatorname{lc}\left(\mathscr{W}^{C}\right)=\mathscr{A}$.

Proof. Let $H \in \mathscr{A}$. We argue by induction on $\operatorname{dist}(H, C)$.
Firstly, we have $\operatorname{dist}(H, C)=0$ if and only if $H \in \mathscr{W}^{C}$. Hence, $H \in \operatorname{lc}\left(\mathscr{W}^{C}\right)$ in this case.
Now suppose $\operatorname{dist}(H, C)>0$. For the induction step, let $D \in \mathscr{C}(\mathscr{A})$ be such that $\operatorname{dist}(H, C)=$ $\operatorname{dist}(C, D)$ and $H \in \mathscr{W}^{D}$. There is a chamber $E \in \mathscr{C}(\mathscr{A})$ adjacent to $D$ with $\operatorname{dist}(C, E)<$ $\operatorname{dist}(C, D)$. By the induction hypothesis, we see that $\mathscr{W}^{E} \subseteq \operatorname{lc}\left(\mathscr{W}^{C}\right)$. Now, set $H^{\prime}:=\langle\bar{D} \cap \bar{E}\rangle$. By considering the distances, we apparently have $H \in \mathscr{W}^{D} \backslash \mathscr{W}^{E}$ and consequently, by Lemma 2.7, there is another $H^{\prime \prime} \in \mathscr{W}^{E} \backslash\left\{H^{\prime}\right\}$ such that $H \in \mathscr{A}_{H^{\prime} \cap H^{\prime \prime}}$, i.e. $H \in \operatorname{lc}\left(\mathscr{W}^{E}\right) \subseteq \operatorname{lc}\left(\mathscr{W}^{C}\right)$, which concludes the induction.

In view of Proposition 2.3, Proposition 2.8 again shows that simplicial arrangements are combinatorially formal.
2.7. Free hyperplane arrangements. Let $S=S\left(V^{*}\right)$ be the symmetric algebra of the dual space $V^{*}$ of $V$. Let $\operatorname{Der}(S)$ be the $S$-module of $\mathbb{K}$-derivations of $S$. Since $S$ is graded, $\operatorname{Der}(S)$ is a graded $S$-module.

Let $\mathscr{A}$ be an arrangement in $V$. For $H \in \mathscr{A}$, we fix $\alpha_{H} \in V^{*}$ with $H=\operatorname{ker} \alpha_{H}$. The defining polynomial $Q(\mathscr{A})$ of $\mathscr{A}$ is given by $Q(\mathscr{A}):=\prod_{H \in \mathscr{A}} \alpha_{H} \in S$. The module of $\mathscr{A}$-derivations is defined by

$$
D(\mathscr{A}):=\{\theta \in \operatorname{Der}(S) \mid \theta(Q(\mathscr{A})) \in Q(\mathscr{A}) S\} .
$$

We say that $\mathscr{A}$ is free if $D(\mathscr{A})$ is a free $S$-module, cf. [OT92, §4].
Remark 2.9. Note that the class of free arrangements is closed with respect to taking localizations, cf. [OT92, Thm. 4.37]. For $X \in L(\mathscr{A})$ modular of rank $r-1$, also the converse holds, e.g. see [R17, Rem. 2.8].

In [Y93, Cor. 2.5], Yuzvinsky showed that freeness entails formality. This was extended to $k$-formality by Brandt and Terao in [BT94, Thm. 4.15]:

Theorem 2.10. If $\mathscr{A}$ is free, then it is $k$-formal for each $k$.
Note that the converse of Theorem 2.10 is false, e.g. see Example 5.1.
Following DiPasquale [DiP18, Def. 4.9], we say that $\mathscr{A}$ is totally formal provided $\mathscr{A}$ is locally $k$-formal for all $k$. Thanks to [OT92, Thm. 4.37] and Theorem 2.10 , if $\mathscr{A}$ is free, it is totally formal. Using [DiP18, Cor. 4.10], it is easy to see that this property of $\mathscr{A}$ is passed to the restriction $\mathscr{A}^{H}$ of $\mathscr{A}$ to a hyperplane $H$. For, owing to [Z89, Thm. 11], the Ziegler multiplicity is free on $\mathscr{A}^{H}$ and so [DiP18, Cor. 4.10] shows that $\mathscr{A}^{H}$ is totally formal. Note that Corollary 1.3 is valid without the freeness requirement on $\mathscr{A}$ and gives local formality for any restriction $\mathscr{A}^{H}$.
2.8. Factored arrangements. The notion of a factored arrangement is due to Terao [T92]. It generalizes the concept of a supersolvable arrangement, see [OST84, Thm. 5.3] and [OT92, Prop. 2.67, Thm. 3.81]. factoredness is a combinatorial property and provides a general combinatorial framework to deduce tensor factorizations of the underlying Orlik-Solomon algebra, see also [OT92, §3.3]. We recall the relevant notions from [T92] (cf. [OT92, §2.3]):

Definition 2.11. Let $\pi=\left(\pi_{1}, \ldots, \pi_{s}\right)$ be a partition of $\mathscr{A}$.
(a) $\pi$ is called independent, provided for any choice $H_{i} \in \pi_{i}$ for $1 \leq i \leq s$, the resulting $s$ hyperplanes are linearly independent, i.e. $r\left(H_{1} \cap \ldots \cap H_{s}\right)=s$.
(b) Let $X \in L(\mathscr{A})$. The induced partition $\pi_{X}$ of $\mathscr{A}_{X}$ is given by the non-empty blocks of the form $\pi_{i} \cap \mathscr{A}_{X}$.
(c) $\pi$ is a factorization of $\mathscr{A}$ provided
(i) $\pi$ is independent, and
(ii) for each $X \in L(\mathscr{A}) \backslash\{V\}$, the induced partition $\pi_{X}$ admits a block which is a singleton.

If $\mathscr{A}$ admits a factorization, then we also say that $\mathscr{A}$ is factored.

We record some consequences of the main results from [T92] (cf. [OT92, §3.3]). Here $A(\mathscr{A})$ denotes the Orlik-Solomon algebra of $\mathscr{A}$.

Corollary 2.12. Let $\pi=\left(\pi_{1}, \ldots, \pi_{s}\right)$ be a factorization of $\mathscr{A}$. Then the following hold:
(i) $s=r=r(\mathscr{A})$ and

$$
\operatorname{Poin}(A(\mathscr{A}), t)=\prod_{i=1}^{r}\left(1+\left|\pi_{i}\right| t\right)
$$

(ii) the multiset $\left\{\left|\pi_{1}\right|, \ldots,\left|\pi_{r}\right|\right\}$ only depends on $\mathscr{A}$;
(iii) for any $X \in L(\mathscr{A})$, we have

$$
r(X)=\mid\left\{\underset{8}{\left|\left\{i \mid \pi_{i} \cap \mathscr{A}_{X} \neq \varnothing\right\}\right| .}\right.
$$

Remark 2.13. If $\mathscr{A}$ is non-empty and $\pi$ is a factorization of $\mathscr{A}$, then the non-empty parts of the induced partition $\pi_{X}$ form a factorization of $\mathscr{A}_{X}$ for each $X \in L(\mathscr{A}) \backslash\{V\}$; cf. the proof of [T92, Cor. 2.11].

For $X$ in $L(\mathscr{A})$ modular of rank $r-1$, also the converse holds, e.g. see [R17, Rem. 2.25].

We record the following relation between factored and aspheric arrangements, due to Paris, [P95].

Theorem 2.14. If $\mathscr{A}$ is a complexified, factored arrangement in $\mathbb{C}^{3}$, then $\mathscr{A}$ is $K(\pi, 1)$.

Falk and Randell posed the question whether Theorem 2.14 does hold in arbitrary dimensions [FR00, Probl. 3.12]. Theorem 1.1 is strong evidence for that.

We close this section with an example illustrating that factoredness is not hereditary, using Theorem 2.14.

Example 2.15. Let $\mathscr{A}$ be the ideal arrangement in the Weyl arrangement of type $D_{4}$ where the associated ideal in the set of positive roots of the root system of type $D_{4}$ consists of all roots of height at least 4. Owing to [R17, Thm. 1.31(ii)], $\mathscr{A}$ is (inductively) factored. In [AMR18, Ex. 3.2], a restriction $\mathscr{A}^{H}$ of $\mathscr{A}$ is exhibited which is not aspherical. It then follows from Theorem 2.14 that $\mathscr{A}^{H}$ is not factored.

## 3. Proof of Theorem 1.1

Let $\pi=\left(\pi_{1}, \ldots, \pi_{\ell}\right)$ be a factorization of $\mathscr{A}$. We call a subset $\left\{H_{1}, \ldots, H_{\ell}\right\}$ of $\mathscr{A}$ a section of $\pi$ if $H_{i} \in \pi_{i}$ for each $i=1, \ldots, \ell$. Note that because $\pi$ is a factorization of $\mathscr{A}$, every section of $\pi$ consists of linearly independent hyperplanes. The following lemma paves the way for our proof of Theorem 1.1.

Lemma 3.1. Let $\pi$ be a factorization of $\mathscr{A}, S$ be a section of $\pi$ and $H \in \mathscr{A}$. Then there exists a section $T$ of $\pi$ such that $S \cup\{H\} \subset \operatorname{lc}(T)$.

Proof. Since $\pi$ is a factorization of $\mathscr{A}$, if we pick two hyperplanes $H, H^{\prime}$ in a block of $\pi$, then the induced partition $\pi_{H \cap H^{\prime}}$ contains a singleton block. Define $s\left(H, H^{\prime}\right)$ to be the hyperplane in this singleton block. Note that $s\left(H, H^{\prime}\right)=H$ if and only if $H=H^{\prime}$.

Let $\pi=\left(\pi_{1}, \ldots, \pi_{\ell}\right)$, and let $\iota: \mathscr{A} \rightarrow\{1, \ldots, \ell\}$ be the indicator function of each block, i.e. $H \in \pi_{\iota(H)}$ for each $H \in \mathscr{A}$. Pick an arbitrary section $S$ of $\pi$ and an arbitrary hyperplane $H \in \mathscr{A}$ and define the sequences

$$
\begin{aligned}
\mathrm{H} & : \mathbb{N}_{0} \rightarrow \mathscr{A}, i \mapsto \mathrm{H}_{i}, \\
- & : \mathbb{N}_{0} \rightarrow\{1, \ldots, \ell\}, \bar{i}:=\iota\left(\mathrm{H}_{i}\right), \text { and } \\
\mathrm{S} & : \mathbb{N}_{0} \rightarrow\{\text { sections of } \pi\}, i \mapsto \mathrm{~S}_{i}
\end{aligned}
$$

inductively as follows: As starting parameters choose $\mathrm{H}_{0}:=H$, and $\mathrm{S}_{0}:=S$. In the $i$-th step writing $\mathrm{S}_{i}=\left\{K_{i, 1}, \ldots, K_{i, \ell}\right\}$ such that $K_{i, j} \in \pi_{j}$ for $j=1, \ldots, \ell$, define

$$
\mathrm{H}_{i+1}:=s\left(\mathrm{H}_{i}, K_{i, \bar{i}}\right) \text { and } K_{i+1, j}:=\left\{\begin{array}{ll}
\mathrm{H}_{i} & \text { if } j=\bar{i} \\
K_{i, j} & \text { else }
\end{array} \text { for } j=1, \ldots, \ell .\right.
$$

This means $\mathrm{H}_{i}$ is swapped against $K_{i, \bar{i}}$ in the new section $\mathrm{S}_{i+1}$ and $\mathrm{H}_{i+1}$ becomes the singleton hyperplane associated to $\mathrm{H}_{i}$ and $K_{i, \bar{i}}$. Note, both $\mathrm{H}_{i}, K_{i, \bar{i}}$ belong to $\pi_{i}$ and so, by construction, $\mathrm{S}_{i+1}$ is again a section of $\pi$ for every $i \in \mathbb{N}_{0}$.

The following property is easy to see. For all integers $p, q$ with $0 \leq p<q$ and for every $K \in\left\{\mathrm{H}_{p}, K_{p, 1}, \ldots, K_{p, \ell}\right\}$, either $K \in \mathrm{~S}_{q}$ or there is a $j \in\{p, \ldots, q-1\}$ such that $K=K_{j, \bar{j}}$. Since $\mathscr{A}$ is a finite set, we can choose the smallest two integers $p<q$ such that $\mathrm{H}_{p}=\mathrm{H}_{q}$. We argue by (reverse) induction that $K_{j, \bar{j}} \in \operatorname{lc}\left(\mathrm{~S}_{q}\right)$ for every $j=0, \ldots, q$.

For $j=q$ there is nothing to prove. Assume that $0 \leq j<q$ and the hypothesis is true for $k$ with $j<k \leq q$. If $\mathrm{H}_{j}=K_{j, \bar{j}}$ or $\mathrm{H}_{j+1}=K_{j, \bar{j}}$, then the sequence becomes stationary and we have $K_{j, \bar{j}}=K_{q, \bar{q}} \in \mathrm{~S}_{q}$.
Otherwise both $\mathrm{H}_{j}$ and $\mathrm{H}_{j+1}$ are either in $\mathrm{S}_{q}$ or of the form $K_{k, \bar{k}}$ for some $j<k \leq q$. Due to the induction hypothesis, we have $\mathrm{H}_{j}, \mathrm{H}_{j+1} \in \operatorname{lc}\left(\mathrm{~S}_{q}\right)$. Since $\mathrm{H}_{j+1}=s\left(\mathrm{H}_{j}, K_{j, \bar{j}}\right)$, we have $K_{j, \bar{j}} \in \mathscr{A}_{\mathrm{H}_{j} \cap \mathrm{H}_{j+1}} \subset \operatorname{lc}\left(\mathrm{~S}_{q}\right)$, which completes the proof.

Corollary 3.2. If $\pi$ is a factorization of $\mathscr{A}$, then there exists a section $S$ of $\pi$ such that $\operatorname{lc}(S)=\mathscr{A}$.

Proof. Let $S$ be a section of $\pi$ such that $\operatorname{lc}(S)$ is of maximal cardinality amongst all sections of $\pi$. For $H \in \mathscr{A}$, with Lemma 3.1, there is a section $T$ such that $\{H\} \cup S \subset \operatorname{lc}(T)$. Thus, $\{H\} \cup \operatorname{lc}(S) \subset \operatorname{lc}(T)$, and because $S$ was chosen maximal, we get $\operatorname{lc}(S)=\operatorname{lc}(T)$ and finally $H \in \operatorname{lc}(S)$.

Armed with Proposition 2.3 and Corollary 3.2, we can now address Theorem 1.1.
Proof of Theorem 1.1. Let $\pi$ be a factorization of $\mathscr{A}$. With Corollary 3.2 there is a section $S$ of $\pi$ such that $\operatorname{lc}(S)=\mathscr{A}$. Since $\pi$ is independent, we have $r(\cap S)=\ell$. Thus, by Proposition $2.3, \mathscr{A}$ is combinatorially formal.

We show the versatility of Theorem 1.1 by improving a result from [HR16]. In [HR16, Thm. 3.5], the authors prove an addition-deletion theorem for factored arrangements. For, let $\pi$ be a partition of $\mathscr{A}$ and let $H_{0} \in \pi_{1}$ be fixed. Consider the induced partitions $\pi^{\prime}$ of $\mathscr{A}^{\prime}$ by removing $H_{0}$ and $\pi^{\prime \prime}$ of $\mathscr{A}^{\prime \prime}$ by intersecting the parts $\pi_{2}, \ldots, \pi_{\ell}$ with $H_{0}$. To ensure that $\pi^{\prime \prime}$ is a factorization, in addition, Hoge and Röhrle require the bijectivity of the restriction $\operatorname{map} \rho: \mathscr{A} \backslash \pi_{1} \rightarrow \mathscr{A}^{\prime \prime}$ defined by $\rho(H)=H \cap H_{0}$.

Theorem 3.3. Let $\mathscr{A}$ be a nonempty arrangement of rank $\ell$ and let $\pi=\left(\pi_{1}, \ldots, \pi_{\ell}\right)$ be a partition of $\mathscr{A}$. For $H_{0} \in \pi_{1}$, consider the arrangement triple $\left(\mathscr{A}, \mathscr{A}^{\prime}, \mathscr{A}^{\prime \prime}\right)$. Two of the following statements imply the third:
(i) $\pi$ is a factorization of $\mathscr{A}$;
(ii) $\pi^{\prime}$ is a factorization of $\mathscr{A}^{\prime}$;
(iii) $\rho: \mathscr{A} \backslash \pi_{1} \rightarrow \mathscr{A}^{\prime \prime}$ is bijective and $\pi^{\prime \prime}$ is a factorization of $\mathscr{A}^{\prime \prime}$.

Using Theorem 1.1, we can prove a stronger criterion for the deletion part of the theorem. It turns out that the bijectivity of $\rho: \mathscr{A} \backslash \pi_{1} \rightarrow \mathscr{A}^{\prime \prime}$ is a strong enough requirement on its own.

Proposition 3.4. Let $\pi=\left(\pi_{1}, \ldots, \pi_{\ell}\right)$ be a factorization of $\mathscr{A}$. Let $H_{0} \in \pi_{1}$. Suppose that the restriction map

$$
\rho: \mathscr{A} \backslash \pi_{1} \rightarrow \mathscr{A}^{\prime \prime}, H \mapsto H \cap H_{0}
$$

is bijective. Then $\pi^{\prime}$ is a factorization for $\mathscr{A}^{\prime}$ and $\pi^{\prime \prime}$ is a factorization for $\mathscr{A}^{\prime \prime}$.
Proof. Since $\rho$ is bijective, $\pi^{\prime \prime}$ is an independent partition of $\mathscr{A}^{\prime \prime}$. Let $Y \in L\left(\mathscr{A}^{\prime \prime}\right)$. Note that $Y$ can also be considered as an element of $L(\mathscr{A})$. Since $\pi$ is a factorization, there is a singleton part in $\pi_{Y}$. If $\pi_{i} \cap \mathscr{A}_{Y}$ is a singleton for some $i>1$, then the bijectivity of $\rho$ implies that $\pi_{i}^{\prime \prime} \cap \mathscr{A}_{Y}^{\prime \prime}$ is a singleton as well. Therefore, $\pi^{\prime \prime}$ is not a factorization of $\mathscr{A}^{\prime \prime}$ if and only if for some $Y \in L\left(\mathscr{A}^{\prime \prime}\right)$, the only singleton part in $\pi_{Y}$ is $\left\{H_{0}\right\}=\pi_{1} \cap \mathscr{A}_{Y}$. Suppose that this is the case. Then, if $H \in \mathscr{A}_{Y} \backslash\left\{H_{0}\right\}$, the bijectivity of $\rho$ implies that $\mathscr{A}_{H \cap H_{0}}=\left\{H, H_{0}\right\}$, so there are no linear dependencies of rank 2 in $\mathscr{A}_{Y}$ involving $H_{0}$. With Remark $2.13 \mathscr{A}_{Y}$ is factored, so by Theorem 1.1 it is formal. The formality of $\mathscr{A}_{Y}$ implies that $H_{0}$ is a separator inside $\mathscr{A}_{Y}$, i.e. $r\left(\mathscr{A}_{Y} \backslash\left\{H_{0}\right\}\right)=r\left(\mathscr{A}_{Y}\right)$ - 1. Because of this there exists $X \in L(\mathscr{A})$ with $\mathscr{A}_{X}=\mathscr{A}_{Y} \backslash\left\{H_{0}\right\}$. Again, by Remark 2.13, $\mathscr{A}_{X}$ is factored, so $\pi_{X}$ has a singleton part. Our assumption implies $\pi_{1} \cap \mathscr{A}_{X}=\emptyset$ and $\left|\pi_{i} \cap \mathscr{A}_{X}\right|=\left|\pi_{i} \cap \mathscr{A}_{Y}\right|>1$ for $2 \leq i \leq \ell$, a contradiction. It follows that $\pi^{\prime \prime}$ is a factorization for $\mathscr{A}^{\prime \prime}$. By the deletion part of Theorem 3.3, $\pi^{\prime}$ is also a factorization for $\mathscr{A}^{\prime}$.

Remarks 3.5. (a). The converse of Theorem 1.1 is false. In [M21], Möller constructs a real 3-arrangement which is combinatorially formal whose characteristic polynomial does not factor over the integers, thus in particular, it is not factored, cf. Corollary 2.12(i). In addition, Möller's example also fails to be $K(\pi, 1)$, as it satisfies the "simple triangle" condition of Falk and Randell, [FR87, Cor. 3.3]. Note that this example is actually $k$-formal for all $k$.
(b). In view of Theorem 2.4, Theorem 1.1 is strong support for an affirmative answer to a question raised by Falk and Randell [FR00, Probl. 3.12], whether complexified factored arrangements are always $K(\pi, 1)$, extending Theorem 2.14.

## 4. Proof of Theorem 1.2

Let $V$ be an $\ell$-dimensional vector space over some field $\mathbb{K}$. For a subspace $X \subseteq V$ we denote by $\operatorname{Ann}_{V^{*}}(X)=\left\{f \in V^{*}|f|_{X} \equiv 0\right\}$ the subspace of $V^{*}$ of all forms which vanish on $X$.

Recall from $\S 2.3$ that for an arrangement $\mathscr{A}$ in $V, F(\mathscr{A})$ is defined as the kernel of the map $\pi_{1}: \mathbb{K}(\mathscr{A}):=\bigoplus_{H \in \mathscr{A}} \operatorname{Ann}_{V^{*}}(H) \rightarrow V^{*},\left(\alpha_{H} \mid H \in \mathscr{A}\right) \mapsto \sum_{H \in \mathscr{A}} \alpha_{H}$. For a subarrangement $\mathscr{B} \subseteq \mathscr{A}$ we clearly have a natural inclusion $F(\mathscr{B}) \hookrightarrow F(\mathscr{A})$ (induced by the natural inclusion $\mathbb{K}(\mathscr{B}) \hookrightarrow \mathbb{K}(\mathscr{A}))$. If $\mathscr{B}=\mathscr{A}_{X}$ we denote this inclusion by $i_{X}$.

Recall, that an arrangement $\mathscr{A}$ is formal if and only if the map

$$
\begin{aligned}
\pi_{2}:=\sum_{X \in L_{2}(\mathscr{A})} i_{X}: \bigoplus_{X \in L_{2}(\mathscr{A})} F\left(\mathscr{A}_{X}\right) & \rightarrow F(\mathscr{A}), \\
\left(a_{X} \mid X \in L_{2}(\mathscr{A})\right) & \mapsto \sum_{X \in L_{2}(\mathscr{A})} i_{X}\left(a_{X}\right)
\end{aligned}
$$

is surjective.
In the following, we fix an arrangement $\mathscr{A}$ and an intersection $Z \in L(\mathscr{A})$. The subsequent propositions yield Theorem 1.2 step by step.

Proposition 4.1. There exists a surjective map $\psi_{1}: \mathbb{K}(\mathscr{A}) \rightarrow \mathbb{K}\left(\mathscr{A}^{Z}\right)$ such that the diagram

commutes, where $\pi_{1}^{Z}$ denotes the corresponding map for $\mathscr{A}^{Z}$.
Furthermore, for $Y \geq Z$ the induced diagram

also commutes.

Proof. For $K \in \mathscr{A}^{Z}$ set

$$
\begin{aligned}
\psi_{K} & : \bigoplus_{H \in \mathscr{A}_{K} \backslash \mathscr{A}_{Z}} \operatorname{Ann}_{V^{*}}(H) \rightarrow \operatorname{Ann}_{Z^{*}}(K), \\
& \left.\left(\alpha_{H} \mid H \in \mathscr{A}_{K} \backslash \mathscr{A}_{Z}\right) \mapsto \sum_{H \in \mathscr{A}_{K} \backslash \mathscr{A}_{Z}} \alpha_{H}\right|_{Z^{*}},
\end{aligned}
$$

which is obviously surjective. Note that $\mathscr{A} \backslash \mathscr{A}_{Z}=\bigsqcup_{K \in \mathscr{A}^{Z}}\left(\mathscr{A}_{K} \backslash \mathscr{A}_{Z}\right)$. Then

does the job (since for $H \in \mathscr{A}_{Z}$ and $\alpha_{H} \in \operatorname{Ann}_{V^{*}}(H)$ we have $\left.\alpha_{H}\right|_{Z^{*}}=0$ ).
The commutativity of the diagram (4.3) is obvious.

Proposition 4.4. The canonical map $\widetilde{\psi}_{1}: F(\mathscr{A}) \rightarrow F\left(\mathscr{A}^{Z}\right)$ between the kernels induced by the diagram (4.2) is surjective.

Proof. By Proposition 4.1 we have the following diagram with exact rows:


If we look at the map induced by $\pi_{1}$ between the kernels of $\psi_{1}$ and res $\left.\right|_{Z^{*}}$, we see that under this map $\mathbb{K}\left(\mathscr{A}_{Z}\right) \subseteq \operatorname{ker}\left(\psi_{1}\right)$ maps onto $\operatorname{Ann}_{V^{*}}(Z)=\operatorname{ker}\left(\right.$ res $\left.\left.\right|_{Z^{*}}\right)$. Since $\psi_{1}$ is surjective, by Proposition 4.1, the snake lemma yields: $\operatorname{coker}\left(\widetilde{\psi}_{1}\right)=0$.

Proposition 4.5. For the map $\widetilde{\psi}_{1}$ we have:

$$
\widetilde{\psi}_{1}\left(\operatorname{im}\left(\pi_{2}\right)\right) \subseteq \operatorname{im}\left(\pi_{2}^{Z}\right)
$$

In particular, the induced map $\operatorname{coker}\left(\pi_{2}\right) \rightarrow \operatorname{coker}\left(\pi_{2}^{Z}\right)$ is surjective.
Proof. Since, by definition, we have $\operatorname{im}\left(\pi_{2}\right)=\sum_{X \in L_{2}(\mathscr{A})} i_{X}\left(F\left(\mathscr{A}_{X}\right)\right)$, it suffices to show that $\widetilde{\psi}_{1}\left(i_{X}\left(F\left(\mathscr{A}_{X}\right)\right)\right) \subseteq \operatorname{im}\left(\pi_{2}^{Z}\right)$ for each $X \in L_{2}(\mathscr{A})$.
For $Y \geq X$, we have $i_{X}\left(F\left(\mathscr{A}_{X}\right)\right) \subseteq i_{Y}\left(F\left(\mathscr{A}_{Y}\right)\right)$. If $X \in L_{2}(\mathscr{A})$ then there is a $Y \in L_{2}\left(\mathscr{A}^{Z}\right)$ with $Y \geq X$. Since the diagram (4.3) commutes, the same holds for the induced maps on the kernels, i.e. $\widetilde{\psi}_{1} \circ i_{Y}=i_{Y}^{Z} \circ \widetilde{\psi}_{1, Y}$, and we get

$$
\widetilde{\psi}_{1}\left(i_{X}\left(F\left(\mathscr{A}_{X}\right)\right)\right) \subseteq \widetilde{\psi}_{1}\left(i_{Y}\left(F\left(\mathscr{A}_{Y}\right)\right)\right) \subseteq i_{Y}^{Z}\left(F\left(\mathscr{A}_{Y}^{Z}\right)\right) \subseteq \operatorname{im}\left(\pi_{2}^{Z}\right)
$$

which concludes the proof.
Corollary 4.6 (Theorem 1.2). Let $\mathscr{A}$ be a formal arrangement. Then for any $Z \in L(\mathscr{A})$ the restriction $\mathscr{A}^{Z}$ is formal, too.

## 5. Complements and examples

In this section, we provide some complements and applications to the main developments.
Thanks to Theorem 1.2, formality is inherited by restrictions. In view of the results by DiPasquale [DiP18] and due to a lack of examples of non-free $k$-formal arrangements in dimensions at least 5, it is not clear whether Theorem 1.2 extends to higher formality (i.e. $k$ formality for $k \geq 3$ ). The fact that this is not the case is demonstrated by our next example. This was found by means of an implementation of a modified "greedy search" algorithm based on ideas presented by Cuntz in [C21].

Example 5.1. Consider the 5-arrangement $\mathscr{A}$ consisting of 11 hyperplanes in $\mathbb{K}^{5}$ with defining polynomial

$$
Q(\mathscr{A})=x_{1} x_{2} x_{3} x_{4} x_{5}\left(x_{1}+x_{3}\right)\left(x_{1}+x_{3}+x_{5}\right)\left(x_{1}+x_{2}+x_{4}\right)\left(x_{1}+x_{2}+x_{4}+x_{5}\right) .
$$

$$
\left(x_{1}+x_{2}+x_{3}+x_{5}\right)\left(x_{1}+x_{2}+x_{3}+x_{4}+x_{5}\right) .
$$

We note that the underlying matroid of $\mathscr{A}$ is regular, i.e. is realizable over any field $\mathbb{K}$.
We proceed to show that $\mathscr{A}$ is 4 -formal (and hence $k$-formal for all $k$ ). To see that $\mathscr{A}$ is 2 -formal, first observe that there are 6 relations of rank 2 :

$$
\begin{array}{r}
\alpha_{1}+\alpha_{3}-\alpha_{6}=0, \\
\alpha_{2}+\alpha_{7}-\alpha_{10}=0, \\
\alpha_{3}+\alpha_{9}-\alpha_{11}=0, \\
\alpha_{4}+\alpha_{10}-\alpha_{11}=0, \\
\alpha_{5}+\alpha_{6}-\alpha_{7}=0, \\
\alpha_{5}+\alpha_{8}-\alpha_{9}=0,
\end{array}
$$

where $\alpha_{i}$ is the $i$-th linear form in the order as it appears in $Q(\mathscr{A})$ above. Thus, the map $\pi_{2}: \bigoplus_{X \in L_{2}} F_{2}\left(\mathscr{A}_{X}\right) \rightarrow F(\mathscr{A})$ is defined by the matrix

$$
\left(\begin{array}{rrrrrrrrrrr}
1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0
\end{array}\right),
$$

which is of rank 6. Since $\operatorname{dim} F(\mathscr{A})=|\mathscr{A}|-r(\mathscr{A})=11-5=6$, it follows that $\pi_{2}$ is surjective, so $\mathscr{A}$ is 2 -formal. We also see that $\operatorname{ker} \pi_{2}=R_{3}(\mathscr{A})=\{0\}$ and therefore $R_{3}\left(\mathscr{A}_{X}\right)=\{0\}$ for any $X \in L_{3}(\mathscr{A})$ as well. Thus, $\pi_{3}: \bigoplus R_{3}\left(\mathscr{A}_{X}\right) \rightarrow R_{3}(\mathscr{A})$ is surjective and its kernel is $R_{4}(\mathscr{A})=\{0\}$, so the map $\pi_{4}: \bigoplus R_{4}\left(\mathscr{A}_{X}\right) \rightarrow R_{4}(\mathscr{A})$ is surjective and $\mathscr{A}$ is 4-formal.

The 5 coordinate hyperplanes $\operatorname{ker}\left(x_{1}\right), \operatorname{ker}\left(x_{2}\right), \ldots, \operatorname{ker}\left(x_{5}\right)$ form an lc-basis of $\mathscr{A}$ and thus the arrangement is combinatorially formal by Proposition 2.3.

Yet, the restriction $\mathscr{A}^{H}$ to $H=\operatorname{ker}\left(x_{2}\right)$ in $\mathbb{K}^{4}$, which has defining polynomial

$$
Q\left(\mathscr{A}^{H}\right)=x_{1} x_{2} x_{3} x_{4}\left(x_{1}+x_{3}\right)\left(x_{1}+x_{3}+x_{4}\right)\left(x_{1}+x_{2}\right)\left(x_{1}+x_{2}+x_{4}\right)\left(x_{1}+x_{2}+x_{3}+x_{4}\right),
$$

fails to be 3 -formal. To see this, again consider the rank 2 relations of $\mathscr{A}^{H}$ :

$$
\begin{aligned}
& \alpha_{1}+\alpha_{2}-\alpha_{7}=0, \\
& \alpha_{1}+\alpha_{3}-\alpha_{5}=0, \\
& \alpha_{2}+\alpha_{6}-\alpha_{9}=0, \\
& \alpha_{3}+\alpha_{8}-\alpha_{9}=0, \\
& \alpha_{4}+\alpha_{5}-\alpha_{6}=0, \\
& \alpha_{4}+\alpha_{7}-\alpha_{8}=0,
\end{aligned}
$$

where $\alpha_{i}$ is the $i$-th linear form in the order as it appears in $Q\left(\mathscr{A}^{H}\right)$ above. The matrix

$$
\left(\begin{array}{rrrrrrrrr}
1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0
\end{array}\right)
$$

is of rank 5 , so its kernel $R_{3}\left(\mathscr{A}^{H}\right)$ is of dimension 1 . Consider the following table of the nontrivial rank 3 intersections in $L\left(\mathscr{A}^{H}\right)$ and the nontrivial rank 2 flats they are contained in. For the set $\left\{H_{i_{1}}, \ldots, H_{i_{k}}\right\}$ we write $i_{1} i_{2} \cdots i_{k}$ as a shorthand.

| nontrivial $X \in L_{3}\left(\mathscr{A}^{H}\right)$ | nontrivial $Y \in L_{2}\left(\mathscr{A}^{H}\right)$ with $Y<X$ |
| :--- | :--- |
| 12357 | 125,134 |
| 12478 | 124,345 |
| 12679 | 124,235 |
| 13456 | 124,345 |
| 13589 | 123,245 |
| 23689 | 135,245 |
| 24569 | 145,234 |
| 34789 | 145,234 |
| 45678 | 123,145 |

From the table it is obvious that the rank 2 relations in every rank 3 localization of $\mathscr{A}^{H}$ are linearly independent, so $R_{3}\left(\left(\mathscr{A}^{H}\right)_{X}\right)=\{0\}$ for every $X \in L_{3}\left(\mathscr{A}^{H}\right)$. We conclude that the map $\pi_{3}: \bigoplus R_{3}\left(\left(\mathscr{A}^{H}\right)_{X}\right) \rightarrow R_{3}\left(\mathscr{A}^{H}\right)$ is not surjective and therefore $\mathscr{A}^{H}$ is not 3 -formal.

Moreover, for $X=\operatorname{ker}\left(x_{1}\right) \cap \operatorname{ker}\left(x_{2}\right) \cap \operatorname{ker}\left(x_{4}\right)$, the rank 3 localization $\mathscr{A}_{X}$ is obviously not formal. Consequently, $\mathscr{A}_{X}$ fails to be factored, $K(\pi, 1)$ (for $\mathbb{K}=\mathbb{C}$ ), and free, owing to Theorems 1.1, 2.4, and 2.10, respectively, and thus also $\mathscr{A}$ fails to be factored, $K(\pi, 1)$ (for $\mathbb{K}=\mathbb{C}$ ), and free, thanks to Remarks 2.13, 2.5, and 2.9, respectively.

Finally, the argument in the last paragraph also shows that the given lc-basis of $\mathscr{A}$ does not descend to one for $\mathscr{A}_{X}$.

Owing to [Y93] or by the example above, formality is not inherited by arbitrary localizations. The following consequence of Theorem 1.2 shows however that it is passed to special localizations.

Corollary 5.2. Suppose $X \in L(\mathscr{A})$ is modular. If $\mathscr{A}$ is formal, then so is $\mathscr{A}_{X}$.

Proof. Since $X \in L(\mathscr{A})$ is modular, it follows from Lemma 2.2 that $\mathscr{A}_{X} / X \cong \mathscr{A}^{Y} / T(\mathscr{A})$ for a $Y \in L(\mathscr{A})$ complementary to $X$. As $\mathscr{A}$ is formal, so is the restriction $\mathscr{A}^{Y}$, thanks to Theorem 1.2.

The following consequence of Lemma 2.1 gives a converse to Corollary 5.2 in corank 1 .
Corollary 5.3. If $X \in L(\mathscr{A})$ is modular of corank 1 , then $\mathscr{A}_{X}$ is formal if and only if $\mathscr{A}$ is formal.

In the following example, we present a family of real arrangements $\mathscr{K}_{n}$ of rank $n \geq 3$, none of which is free, $K(\pi, 1)$, or factored. Nevertheless, each is combinatorially formal. So that in particular, here formality is not a consequence of any of the other sufficient properties.

Example 5.4. Let $\mathscr{B}_{n}$ be the reflection arrangement of the hyperoctahedral group of type $B_{n}$ and let $\mathscr{A}_{n-1}$ be the braid arrangement of type $A_{n-1}$, a subarrangement of $\mathscr{B}_{n}$. Let

$$
\mathscr{K}_{n}:=\mathscr{B}_{n} \backslash \mathscr{A}_{n-1}
$$

be the complement of $\mathscr{A}_{n-1}$ in $\mathscr{B}_{n}$. Thus $\mathscr{K}_{n}$ has defining polynomial

$$
Q\left(\mathscr{K}_{n}\right)=\prod_{i=1}^{n} x_{i} \prod_{1 \leq i<j \leq n}\left(x_{i}+x_{j}\right)
$$

It was shown in [AR19, Ex. 2.7] that $\mathscr{K}_{n}$ is not $K(\pi, 1)$ and not free for $n \geq 3$.
We next show by induction on $n$ that $\mathscr{K}_{n}$ is also not factored either for $n \geq 3$. Owing to Theorem 2.14, $\mathscr{K}_{3}$ is not factored, as it is not $K(\pi, 1)$. Now suppose that $n>3$ and that $\mathscr{K}_{n-1}$ is not factored. Let $X:=\cap_{i=1}^{n-1} \operatorname{ker} x_{i}$. Then one readily checks that

$$
\left(\mathscr{K}_{n}\right)_{X} \cong \mathscr{K}_{n-1}
$$

It follows from our induction hypothesis and Remark 2.13 that also $\mathscr{K}_{n}$ fails to be factored.
Finally, we observe that $\mathscr{K}_{n}$ admits an lc-basis. For, let $\mathscr{B} \subset \mathscr{K}_{n}$ be the Boolean subarrangement, i.e. $Q(\mathscr{B})=\prod_{i=1}^{n} x_{i}$. Then one easily checks that $\operatorname{lc}(\mathscr{B})=\mathscr{K}_{n}$. It thus follows from Proposition 2.3 that $\mathscr{K}_{n}$ is combinatorially formal; formality for $\mathscr{K}_{3}$ was already observed by Falk and Randell in [FR87, (3.12)].

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