J. August *et al.* (2024) "Categories for Grassmannian Cluster Algebras of Infinite Rank,"
International Mathematics Research Notices, Vol. 2024, No. 2, pp. 1166–1210
Advance Access Publication February 27, 2023
https://doi.org/10.1093/imrn/rnad004

Categories for Grassmannian Cluster Algebras of Infinite Rank

Jenny August[1,*], Man-Wai Cheung[2], Eleonore Faber[3], Sira Gratz[1], and Sibylle Schroll[4,5]

[1]Department of Mathematics, Aarhus University, Ny Munkegade 118, 8000 Aarhus C, Denmark, [2]School of Mathematics, Kavli IPMU (WPI), UTIAS, The University of Tokyo, Japan, [3]School of Mathematics, University of Leeds, Leeds, LS2 9JT, UK, [4]Department of Mathematics, Universität zu Köln, Weyertal 86-90, 50931 Köln, Germany, and [5]Institutt for matematiske fag, NTNU, N-7491 Trondheim, Norway

[*]*Correspondence to be sent to: e-mail: jennyaugust@math.au.dk*

We construct Grassmannian categories of infinite rank, providing an infinite analogue of the Grassmannian cluster categories introduced by Jensen, King, and Su. Each Grassmannian category of infinite rank is given as the category of graded maximal Cohen–Macaulay modules over a certain hypersurface singularity. We show that generically free modules of rank 1 in a Grassmannian category of infinite rank are in bijection with the Plücker coordinates in an appropriate Grassmannian cluster algebra of infinite rank. Moreover, this bijection is structure preserving, as it relates rigidity in the category to compatibility of Plücker coordinates. Along the way, we develop a combinatorial formula to compute the dimension of the Ext^1-spaces between any two generically free modules of rank 1 in the Grassmannian category of infinite rank.

1 Introduction

Grassmannians are objects of great combinatorial and geometric beauty, which arise in myriad contexts. Their coordinate rings serve as a classical example in the theory of cluster algebras, whose genesis by Fomin and Zelevinsky [9] was initially motivated

Received February 2, 2022; Revised December 19, 2022; Accepted January 4, 2023
Communicated by Prof. Lauren Williams

by total positivity in Lie theory as propagated by Lusztig; see for example [19]. The combinatorial structures of Grassmannians, in relation to total positivity, were first studied by Postnikov [22]. Employing these combinatorial tools, Scott [23] showed that coordinate rings of Grassmannians indeed carry a natural cluster algebra structure, which led to these objects becoming a staple in the study of cluster algebras.

Jensen *et al.* [18] introduce an additive categorification of the Grassmannian cluster algebra $\mathbb{C}[\mathrm{Gr}(k,n)]$ of finite rank via G-equivariant maximal Cohen–Macaulay modules over the plane curve singularities $R_{(k,n)} = \mathbb{C}[x,y]/(x^k - y^{n-k})$, where G is the cyclic group of order n acting on $R_{(k,n)}$ in a natural way (cf. Section 2.3). They show that rank 1 modules in $\mathrm{MCM}_G R_{(k,n)}$ are in one-to-one correspondence with Plücker coordinates in $\mathbb{C}[\mathrm{Gr}(k,n)]$ and that this bijection preserves structure: rigidity of subcategories of rank 1 modules is translated to compatibility of the corresponding Plücker coordinates (i.e., pairwise noncrossing of k-subsets; cf. Section 2.1.1). An interesting aspect of this relation is that it affords a formal connection between two famous examples of a priori unrelated ADE classifications, providing a bridge between skew-symmetric cluster algebras of finite type and simple plane curve singularities. More precisely, it relates Grassmannian cluster algebras $\mathbb{C}[\mathrm{Gr}(k,n)]$ of finite type to the simple plane curve singularities $x^k = y^{n-k}$, which occur precisely in the cases $k = 2$ or $n-2$ and $n \geq 4$ (type A_{n-3}), $k = 3$ or $n-3$ and $n = 6$ (type D_4), $k = 3$ or $n-3$ and $n = 7$ (type E_6), and $k = 3$ or $n-3$ and $n = 8$ (type E_8). Here, the type indicates both cluster algebra type and singularity type, respectively.

We extend the theory to the infinite rank setting, that is, we let n go to infinity. For a fixed $k \geq 2$, a natural object to consider on the cluster algebra side is the ring

$$\mathcal{A}_k = \frac{\mathbb{C}[p_I \mid I \subseteq \mathbb{Z}, |I| = k]}{\langle \text{Plücker relations} \rangle};$$

cf. Section 2.1.2 for details. This is a cluster algebra of infinite rank in the sense of [11] and can be viewed as a colimit of Grassmannian cluster algebras $\mathbb{C}[\mathrm{Gr}(k,n)]$ in the category of rooted cluster algebras (introduced by Assem *et al.* [1]), as discussed in depth in [13]. In fact, the ring $\mathcal{A}_k$ can be interpreted as the homogeneous coordinate ring of an infinite version of the Grassmannian under a generalised Plücker embedding. In the case $k = 2$, this is the space of 2D subspaces of a profinite-dimensional (topological) vector space under the Plücker embedding constructed by Groechenig in the appendix to [11]. This point of view naturally extends to $k \geq 3$.

We construct an analogue of the Jensen, King, and Su Grassmannian cluster categories in this infinite setting: for a fixed $k \geq 2$, the Grassmannian category of infinite rank is defined to be the category of finitely generated $\mathbb{Z}$-graded maximal Cohen–Macaulay modules over the ring $R_k = \mathbb{C}[x,y]/(x^k)$, where x is in degree 1 and y is in degree -1. From the point of view of the singularities, this is the natural category to consider—the singularity $x^k = 0$ is the limit of the singularities $x^k = y^{n-k}$ as n goes to infinity, and the cyclic group actions yield a circle action in the limit, giving rise to the $\mathbb{Z}$-grading.

We find that this gives categorical companions embodying the combinatorics of the infinite Grassmannians. For instance, we naturally rediscover the combinatorial description of Plücker coordinates through certain indecomposable objects of the Grassmannian category.

Theorem A. (Theorem 3.9) There is a one-to-one correspondence between Plücker coordinates in $\mathcal{A}_k$ and generically free modules of rank 1 in $\mathrm{MCM}_{\mathbb{Z}}R_k$.

In order to prove this, we show that every generically free module of rank 1 in $\mathrm{MCM}_{\mathbb{Z}}R_k$ arises as a syzygy of a finite dimensional module in $\mathrm{gr}R_k$. This allows us to reduce to the problem of classifying cofinite homogeneous ideals; we solve this problem explicitly by naturally constructing a Plücker coordinate from any such ideal.

Crucially, the correspondence from Theorem A is structure preserving, in the sense that it connects the concept of rigidity in $\mathrm{MCM}_{\mathbb{Z}}R_k$ with the concept of compatibility of Plücker coordinates and that of noncrossing of k-subsets.

Theorem B. (Theorem 4.1) Let I and J be generically free modules of rank 1 in $\mathcal{C} = \mathrm{MCM}_{\mathbb{Z}}R_k$ corresponding, under the bijection from Theorem A, to Plücker coordinates p_I and p_J, respectively. Then $\mathrm{Ext}^1_{\mathcal{C}}(I,J) = 0$ if and only if p_I and p_J are compatible.

Theorem B is a direct consequence of a general formula for the dimension of the Ext^1-space between any two given generically free modules of rank 1 that we provide in this paper. To prove this formula, we employ the combinatorial tool of staircase paths in a $(k \times k)$-grid to extract the dimension of the Ext^1-space between two such modules from the crossing pattern of the associated k-subsets $\underline{\ell}$ and $\underline{m}$ of the corresponding Plücker coordinates; cf. Section 4.1. A pair of staircase paths uniquely represents the crossing patterns of $\underline{\ell}$ and $\underline{m}$ and yields two significant numbers: the number $\alpha(\underline{\ell}, \underline{m})$ of diagonals strictly above one of the paths, and the number $\beta(\underline{\ell}, \underline{m})$ of diagonals strictly below the other; for precise details, see Definition 4.6.

Theorem C. (Theorem 4.10) Let I and J be two generically free modules of rank 1 in $\mathcal{C} = \mathrm{MCM}_{\mathbb{Z}} R_k$ corresponding, under the bijection from Theorem A, to the Plücker coordinates p_I and p_J, respectively, associated with the k-subsets $\underline{\ell}$ and $\underline{m}$. Then

$$\dim_{\mathbb{C}}(\mathrm{Ext}^1_{\mathcal{C}}(I,J)) = \alpha(\underline{\ell},\underline{m}) + \beta(\underline{\ell},\underline{m}) - k - |\underline{\ell} \cap \underline{m}|.$$

It is a direct consequence of this formula that Ext^1 on generically free modules of rank 1 is symmetric in its argument. This is not a coincidence: we provide an argument to show that the full subcategory of generically free maximal Cohen–Macaulay modules is stably 2-Calabi–Yau, using a result by Iyama and Takahashi [17]. We are grateful to Osamu Iyama and Michael Wemyss for suggesting this should be the case.

These connections provide a convincing argument for the study of Grassmannian categories of infinite rank as the appropriate categorical analogue to Grassmannian cluster algebras of infinite rank. As a further illustration, let us consider the $k = 2$ case. In the case of finite rank, the types $k = 2$ and $n \geq 4$ form the simplest family of Grassmannian cluster categories. In particular, the corresponding singularities are of finite Cohen–Macaulay type (i.e., these Grassmannian cluster categories have finitely many indecomposable objects) and exhibit Dynkin type A cluster combinatorics. In the limit, as n goes to ∞, this mild behaviour survives: the ring $R_2 = \mathbb{C}[x,y]/(x^2)$ has countable Cohen–Macaulay type, and indecomposable objects in the category $\mathrm{MCM}_{\mathbb{Z}}\mathbb{C}[x,y]/(x^2)$ can be classified via two-element subsets of $\mathbb{Z} \cup \{\infty\}$ (or, to use a geometric Dynkin type A_∞ model, by arcs in an ∞-gon with one marked accumulation point). Furthermore, this particular Grassmannian category of infinite rank has cluster tilting subcategories, which we classify in work in progress [2], recovering the classification for the one-accumulation point case by Paquette and Yıldırım [21] from a different perspective.

The infinite rank $k = 2$ case has been studied extensively in recent years from different perspectives, starting with the pioneering work by Holm and Jørgensen [14]. They study the finite derived category $D^f_{dg}(\mathbb{C}[y])$, where $\mathbb{C}[y]$ is viewed as a differential graded algebra with trivial differential, and y in cohomological degree -1, which they show exhibits cluster combinatorics of type A_∞. In fact, the stable category of the subcategory of $\mathrm{MCM}_{\mathbb{Z}}\mathbb{C}[x,y]/(x^2)$ generated by generically free modules of rank 1 is equivalent to $D^f_{dg}(\mathbb{C}[y])$. A different viewpoint on this category is given by a special case of the combinatorial construction of discrete cluster categories of type A_∞ by Igusa and Todorov [16]. Recent work by Paquette and Yıldırım [21] constructs a completion of the discrete cluster categories of type A_∞. We note that in the one-accumulation point case, this completion coincides with the stable category of our Grassmannian category of infinite rank $\mathrm{MCM}_{\mathbb{Z}}\mathbb{C}[x,y]/(x^2)$.

While the story is a satisfyingly conclusive one for the $k = 2$ case, we note that the $k \geq 3$ case, which we treat in this paper simultaneously, is a different matter entirely: already in the finite rank setting (bar a handful of exceptions), we are in wild Cohen–Macaulay type. As we let n go to ∞, this wildness, unsurprisingly, survives, and a classification of indecomposable objects in the Grassmannian categories of infinite rank for $k \geq 3$ is out of reach. It is striking that it is still possible to classify all generically free rank 1 Cohen–Macaulay modules via the combinatorially accessible tools from Theorem A and to draw a natural connection to Grassmannian combinatorics.

2 Preliminaries

2.1 Grassmannian cluster algebras

Grassmannian cluster categories are an additive categorification of Grassmannian cluster algebras, of which this section provides an overview.

2.1.1 *The finite rank case*

Coordinate rings of flag varieties provide an interesting source of cluster algebras. An important example thereof is the Grassmannian $\mathrm{Gr}(k,n)$ of k-subspaces of $\mathbb{C}^n$, viewed as a projective variety via the Plücker embedding. It was shown by Scott [23] that its homogeneous coordinate ring $\mathbb{C}[\mathrm{Gr}(k,n)]$ carries a natural cluster algebra structure, with Plücker coordinates providing a subset of cluster variables and exchange relations coming from Plücker relations.

Consider the Grassmannian $\mathrm{Gr}(k,n)$ of k-dimensional subspaces in $\mathbb{C}^n$ as a projective variety via the Plücker embedding. Its homogeneous coordinate ring is the ring

$$\mathcal{A}_{(k,n)} = \mathbb{C}[x_I \mid I \subseteq \{1,\dots,n\}, |I| = k]/\mathcal{I}_P$$

where $\mathcal{I}_P$ is the ideal generated by the Plücker relations, which are described as follows: for any two subsets $J, J' \subseteq \{1,\dots,n\}$ with $|J| = k+1$ and $|J'| = k-1$, with $J = \{j_0,\dots,j_k\}$ and $j_0 < \dots < j_k$, we get a Plücker relation

$$\sum_{l=0}^{k} (-1)^l x_{J'\cup\{j_l\}} x_{J\setminus\{j_l\}}. \tag{2.1}$$

We call a subset $I \subseteq \{1, \dots, n\}$, with $|I| = k$ a *k-subset*. The variables x_I labelled by k-subsets are called *Plücker coordinates*.

Given two k-subsets I, J, we say that I and J *cross*, if there exist $i_1, i_2 \in I \setminus J$ and $j_1, j_2 \in J \setminus I$ with

$$i_1 < j_1 < i_2 < j_2 \text{ or } j_1 < i_1 < j_2 < i_2,$$

and I and J are *noncrossing* if they do not cross. Two Plücker coordinates x_I and x_J are *compatible*, if the k-subsets I and J are noncrossing.

Scott [23] has shown that $\mathcal{A}_{(k,n)}$ has the structure of a cluster algebra, where the Plücker coordinates form a subset of the cluster variables and where maximal sets of mutually compatible Plücker coordinates provide examples of clusters in $\mathcal{A}_{(k,n)}$.

2.1.2 *Colimits*

A natural way of extending the cluster combinatorics of $\mathcal{A}_{(k,n)}$ to an infinite setting is by considering the ring

$$\mathcal{A}_k = \mathbb{C}[x_I \mid I \subseteq \mathbb{Z}, |I| = k] \big/ \mathcal{I}_P,$$

where $\mathcal{I}_P$ is the ideal generated by relations of the form (2.1). Note that here, the labelling k-subsets are subsets of $\mathbb{Z}$ of size k.

Indeed, the ring $\mathcal{A}_k$ can be endowed with the structure of an infinite rank cluster algebra in the sense of [11] in uncountably infinitely many ways—it requires us to choose some initial cluster, given, for example, by a maximal set of compatible Plücker coordinates. It was shown in [13] that these cluster algebras of infinite rank can be interpreted as colimits of cluster algebras of finite rank in the category of rooted cluster algebras. Indeed, for a fixed k, we can write it as a colimit of the cluster algebras $\mathcal{A}_{(k,n)}$ with appropriate fixed initial seeds, as illustrated in [12].

For $k = 2$, it was shown in the appendix to [11] that the ring $\mathcal{A}_k$ can be interpreted as the homogeneous coordinate ring of an infinite version of the Grassmannian under a generalisation of the Plücker embedding—this infinite version can be described as the space of 2D subspaces of a profinite-dimensional (topological) vector space (equivalently, 2D quotients of a countably infinite-dimensional vector space). This construction naturally extends to $k \geq 3$.

2.2 Maximal Cohen–Macaulay modules

Let R be a commutative ring. A finitely generated module M is *maximal Cohen–Macaulay (=MCM)* over R if $\operatorname{depth}(M_\mathfrak{p}) = \dim(R_\mathfrak{p})$ in $R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$. Note that if R is local, then this property can simply be stated as $\operatorname{depth}(M) = \dim(R)$. If R is a Gorenstein commutative ring (e.g., a hypersurface), a module M is maximal Cohen–Macaulay if and only if $\operatorname{Ext}^i_R(M, R) = 0$ for $i \neq 0$; see [5]. Note that MCM-modules are precisely the Gorenstein projectives in case R is Gorenstein.

2.3 Grassmannian cluster categories of finite rank

Jensen *et al.* [18] introduce an additive categorification of the natural cluster algebra structure on $\mathcal{A}_{(k,n)}$. In particular, their *Grassmannian cluster categories* are Frobenius categories, with projective-injectives corresponding to the consecutive Plücker coordinates, that is, the Plücker coordinates labelled by k-subsets of the form $\{i, i+1, \dots, i+k-1\}$, where we calculate modulo n. This extends the cluster structure of classical cluster categories with Grassmannian combinatorics to include the coefficients of the cluster algebra $\mathcal{A}_{(k,n)}$. We briefly recall their construction here. The combinatorics of these categories has been extensively studied by Baur *et al.* [3].

Let $k \in \mathbb{Z}_{\geq 2}$ and $n \geq k+2$. Consider the ring $S = \mathbb{C}[x, y]$. The group of n-th roots of unity

$$\mu_n = \{\zeta \in \mathbb{C} \mid \zeta^n = 1\}$$

acts on S via

$$x \mapsto \zeta x; \ y \mapsto \zeta^{-1} y.$$

Taking the quotient by the μ_n semi-invariant function $x^k - y^{n-k}$ yields the ring

$$R_{(k,n)} = S/(x^k - y^{n-k}).$$

The *Grassmannian cluster category* is the category

$$\operatorname{MCM}_{\mu_n} R_{(k,n)}$$

of μ_n-equivariant maximal Cohen–Macaulay $R_{(k,n)}$-modules. Its rank 1 modules are in one-to-one correspondence with the Plücker coordinates of $\mathcal{A}_{(k,n)}$ and under this correspondence, vanishing Ext^1 between two rank 1 modules corresponds to the corresponding Plücker coordinates being compatible; cf. [18, Section 5]. In fact, the Grassmannian cluster category $\operatorname{MCM}_{\mu_n} R_{(k,n)}$ is stably equivalent to the category $\operatorname{Sub} Q_k$ studied by

Geiß *et al.* [10]. Therefore, it has cluster tilting subcategories, and, under the above correspondence, maximal sets of compatible Plücker coordinates provide cluster tilting subcategories.

3 Grassmannian Categories of Infinite Rank

In this section, we introduce the construction of infinite rank versions of Grassmannian cluster categories.

3.1 Construction

We fix $k \in \mathbb{Z}_{\geq 2}$. We generalise the construction of Grassmannian cluster categories to the infinite case, by letting n go to infinity. Consider the action of the multiplicative group $\mathbb{G}_m$ (playing the role taken by μ_n in the finite case) on $S = \mathbb{C}[x, y]$ via

$$x \mapsto \zeta x; \ y \mapsto \zeta^{-1} y.$$

Now, as a semi-invariant function, we take x^k. We may think of this as the infinite version of the function $x^k - y^{n-k}$ as n goes to infinity; topologically, the neighbourhood (y^{n-k}) tends to 0 as n goes to infinity. This yields the Gorenstein ring

$$R_k := S/(x^k),$$

which, when we have fixed a choice of k, we will often simply denote by R. The category we are interested in is the category

$$\mathrm{MCM}_{\mathbb{G}_m} R_k$$

of $\mathbb{G}_m$-equivariant maximal Cohen–Macaulay R_k-modules. The character group of $\mathbb{G}_m$ is the group of integers $\mathbb{Z}$. This yields an equivalence of categories

$$\mathrm{mod}_{\mathbb{G}_m} R_k \cong \mathrm{gr} R_k$$

between the category $\mathrm{mod}_{\mathbb{G}_m} R_k$ of finitely generated $\mathbb{G}_m$-equivariant R_k-modules and the category of finitely generated $\mathbb{Z}$-graded R_k-modules $\mathrm{gr} R_k$, where R_k is viewed as a $\mathbb{Z}$-graded ring with x in degree 1, and y in degree -1. This induces an equivalence of categories

$$\mathrm{MCM}_{\mathbb{G}_m} R_k \cong \mathrm{MCM}_{\mathbb{Z}} R_k,$$

where $\mathrm{MCM}_{\mathbb{Z}} R_k$ is the category of graded Cohen–Macaulay R_k-modules, with the grading given above. Note that the objects in our category are graded MCM-modules over R_k and the morphisms are graded morphisms of degree 0. This means that for any morphism $f : M \to N$ of modules in $\mathrm{MCM}_{\mathbb{Z}} R_k$, one has $f(M_i) \subseteq N_i$, where M_i is the i-th graded piece of M. We call $\mathrm{MCM}_{\mathbb{Z}} R_k$ the *Grassmannian category of type* (k, ∞), or just a *Grassmannian category of infinite rank*, if k is clear from context.

3.2 Generically free modules of rank 1

Fix $k \geq 2$, and set $R = \mathbb{C}[x, y]/(x^k)$ as above. Define $\mathcal{F}$ to be the graded total ring of fractions of R, that is, the ring R localised at all homogeneous nonzero divisors:

$$\mathcal{F} = R_y = \mathbb{C}[x, y^{\pm}]/(x^k).$$

We consider $\mathcal{F}$ as a graded ring, with the grading induced by the grading of R.

Definition 3.1. A module $M \in \mathrm{gr}R$ is *generically free of rank n* if $M \otimes_R \mathcal{F}$ is a graded free $\mathcal{F}$-module of rank n.

Note that $\mathcal{F} \cong R_{(x)}$, where (x) is the graded minimal prime ideal of R.

Lemma 3.2. Every generically free module M of rank n in $\mathrm{gr}R$ has a maximal free submodule P of rank n such that M/P is finite dimensional.

Proof. Take P to be a maximal free submodule of M. This exists, since R is Noetherian, and M is finitely generated. First, we see that M/P is also generically free, by tensoring the short exact sequence

$$0 \to P \to M \to M/P \to 0$$

with $\mathcal{F}$, which yields the short exact sequence

$$0 \to \mathcal{F}^m \to \mathcal{F}^n \to M/P \otimes \mathcal{F} \cong (M/P)_y \to 0,$$

for some $m, n \geq 0$. This sequence splits, since $\mathcal{F}$ is graded self-injective, which can be seen using Baer's criterion. Therefore, $(M/P)_y$ is graded free. Next, we show that in fact $M/P \otimes \mathcal{F} \cong (M/P)_y = 0$, which implies that M/P is finite dimensional and P is of rank $m = n$. Indeed, if we have $(M/P)_y = 0$, then M/P is annihilated by some power of y and hence is isomorphic to some quotient of some power of R, say $(R/(y^l))^m = (\mathbb{C}[x, y]/(x^k, y^l))^m$, which is finite dimensional.

To show $(M/P)_y = 0$, assume as a contradiction that we have $(M/P)_y \neq 0$. Then there exists a free submodule of M/P: pick a nonzero divisor $0 \neq \frac{z}{y^l} \in (M/P)_y$. Since $(M/P)_y$ is free, we have that $x^i \frac{z}{y^l} \neq 0$ for $0 \leq i < k$. It follows that $x^i z \neq 0$ for all $0 \leq i < k$, so the annihilator of $z \in M/P$ vanishes and z generates a rank 1 free submodule P' of M/P. We get a diagram

$$\begin{tikzcd}
P \arrow[r] \arrow[d, equal] & Q \arrow[r, two heads] \arrow[d, tail] & P' \arrow[d, tail] \\
P \arrow[r] & M \arrow[r, two heads] & M/P
\end{tikzcd}$$

where the right-hand square is a pull-back. The top sequence splits, and we get that $Q \cong P \oplus P'$ is a free submodule of M, contradicting the maximality of P. ■

In the following, we denote the graded Hom by grHom, and graded Ext^1 by grExt. Throughout, $M(j)$ denotes the graded shift of M, that is, $M(j)_i = M_{i+j}$.

Lemma 3.3. If M is a generically free module of rank n in $\mathrm{MCM}_{\mathbb{Z}}R$, its dual $M^* = \mathrm{grHom}_R(M, R)$ is also a generically free module of rank n in $\mathrm{MCM}_{\mathbb{Z}}R$.

Proof. By [5, Lemma 4.2.2 (iii)], the dual M^* of the MCM M is again MCM. Furthermore, we have

$$M^* \otimes \mathcal{F} = \mathrm{grHom}_R(M, R) \otimes \mathcal{F} \cong \mathrm{grHom}_{\mathcal{F}}(M \otimes \mathcal{F}, \mathcal{F}) \cong \mathrm{grHom}_{\mathcal{F}}(\mathcal{F}^n, \mathcal{F}) \cong \mathcal{F}^n,$$

■

where the 1st equivalence follows from [20, Thm. 7.11].

Proposition 3.4. Every generically free module M in $\mathrm{MCM}_{\mathbb{Z}}R$ is a syzygy of a finite dimensional module in gr$R$. More precisely, we have a short exact sequence of the form

$$0 \to M \to \bigoplus_{i=1}^{m} R(-n_i) \to N \to 0,$$

where m is the rank of M and N is finite dimensional.

Proof. Assume M is a generically free module of rank m in $\mathrm{MCM}_{\mathbb{Z}}R$. Note that $m > 0$, since M is MCM. If M is free, we are done. So assume that M is not free. By Lemma 3.3, the dual M^* is also generically free, and by Lemma 3.2, it has a maximal free submodule P of rank m such that M^*/P is finite dimensional. So we have $P \cong \bigoplus_{i=1}^{m} R(n_i)$ and $n_i \in \mathbb{Z}$. This yields a short exact sequence

$$0 \to \bigoplus_{i=1}^{m} R(n_i) \to M^* \to M^*/P \to 0$$

and applying the graded $\mathrm{grHom}(-, R)$ yields the short exact sequence

$$0 \to M \to \bigoplus_{i=1}^{m} R(-n_i) \to \mathrm{grExt}(M^*/P, R) \to 0, \tag{3.1}$$

since $(M^*/P)^* = \mathrm{grHom}(M^*/P, R) = 0$ (as M^*/P is finite dimensional and thus annihilated by a power of y) and $\mathrm{grExt}(M^*, R) = 0$ (as M^* is MCM). Note that $M^{**} \cong M$, as MCM modules over a Gorenstein ring are reflexive by [5, Lemma 4.2.2 (iii)]. Again, since M^*/P is annihilated by some power of y, it is a graded $R/(y^i)$-module for some $i \in \mathbb{N}$. By [24, Corollary 3.3.7], $\mathrm{grExt}(M^*/P, R)$ is a graded $R/(y^i)$-module as well. Furthermore, since both M^*/P and R are finitely generated graded R-modules, so is $\mathrm{grExt}(M^*/P, R)$. To summarise, $\mathrm{grExt}(M^*/P, R)$ is a finitely generated graded R-module, which is annihilated by y^i, and hence it is finite dimensional. Thus, (3.1) is the desired sequence. ∎

Proposition 3.5. A graded module I in $\mathrm{MCM}_{\mathbb{Z}}R$ is generically free of rank 1 if and only if I is isomorphic to a graded ideal containing a power of y.

Proof. If I is a graded ideal of R containing a power of y, then there is an exact sequence

$$0 \to I \to R(n) \to M \to 0,$$

where $n \in \mathbb{Z}$ and M is finite dimensional. Note that tensoring with $\mathcal{F}$ is precisely localisation at y and thus is exact. Thus, we obtain the short exact sequence

$$0 \to I \otimes_R \mathcal{F} \to \mathcal{F}(n) \to M \otimes_R \mathcal{F} \to 0.$$

If $M \otimes_R \mathcal{F} \cong M_y \neq 0$, then no power of y acts trivially on M, and we have an infinite descending chain of ideals $M \supset yM \supset y^2M \supset \ldots$, contradicting M being finite dimensional. Therefore, the last term in the sequence vanishes and so I is generically free of rank 1.

Now assume that I is a generically free module of rank 1 in $\mathrm{MCM}_{\mathbb{Z}}R$. By Proposition 3.4, we have a short exact sequence

$$0 \to I \to R(-i) \to N \to 0$$

for some $i \in \mathbb{Z}$ and finite dimensional N. Therefore, $I \cong J(-i)$ for some ideal J, and I is cofinite and hence contains a power of y. ■

3.3 Bijection with Plücker coordinates

We can easily describe the graded ideals containing a power of y, thanks to the following.

Lemma 3.6. Every homogeneous ideal $I \subseteq R$ can be generated by monomials.

Proof. Let $f_m = a_0x^m + a_1x^{m+1}y + \cdots + a_{k-m-1}x^{k-1}y^{k-1-m}$ be a homogeneous polynomial contained in I, and notice that we must have $m < k$ since $x^k = 0$. Note also that m may be negative, in which case we assume that all $a_p = 0$ for $p < -m$.

Let p the smallest index such that $a_p \neq 0$. We will show by induction that $x^{k-i}y^{k-m-i} \in I$ for $i = 1, \ldots, k-m-p$, or in other words, the ideal generated by f_m is the same as the ideal generated by $x^{m+p}y^p$, and thus I is generated by monomials.

For the $i = 1$ case, multiply f_m by x^sy^s where s satisfies $m + p + s = k - 1$ and notice that $s \geq 0$ by the assumption that $a_p \neq 0$. Thus,

$$x^sy^sf_m = a_px^{k-1}y^{k-1-m} + x^k(\ldots) = a_px^{k-1}y^{k-1-m}$$

belongs to I and hence $x^{k-1}y^{k-1-m} \in I$.

Now, for the inductive step, assume $1 < i \leq k-m-p$ and $x^{k-j}y^{k-m-j} \in I$ for all $1 \leq j \leq i-1$. Multiply f_m by x^sy^s where s satisfies $m+p+s = k-i$, and notice that $s \geq 0$ by the assumption $i \leq k-m-p$. Thus,

$$\begin{aligned} x^sy^sf_m &= a_px^{k-i}y^{k-i-m} + a_{p+1}x^{k-i+1}y^{k-i-m+1} + \cdots + a_{p+i-1}x^{k-1}y^{k-1-m} + x^k(\ldots) \\ &= a_px^{k-i}y^{k-i-m} + a_{p+1}x^{k-i+1}y^{k-i-m+1} + \cdots + a_{p+i-1}x^{k-1}y^{k-1-m} \end{aligned}$$

also belongs to I. However, by the inductive hypothesis,

$$a_{p+1}x^{k-i+1}y^{k-i-m+1} + \cdots + a_{p+i-1}x^{k-1}y^{k-1-m}$$

belongs to I and therefore so does $x^{k-i}y^{k-i-m}$ as required. ■

Combining Proposition 3.5 and Lemma 3.6, we are able to prove the following.

Theorem 3.7. Let I be in $\mathrm{MCM}_{\mathbb{Z}}R$. Then I is a generically free module of rank 1 if and only if

$$I \cong (x^{k-1}, x^{k-2}y^{i_1}, x^{k-3}y^{i_2}, \ldots, xy^{i_{k-2}}, y^{i_{k-1}})(i_k)$$

for some $0 \leq i_1 \leq i_2 \leq \ldots \leq i_{k-1}$ and some $i_k \in \mathbb{Z}$.

Proof. Recall from Proposition 3.5 that a graded module I in $\mathrm{MCM}_{\mathbb{Z}}R$ is generically free of rank 1 if and only if I is isomorphic to a graded ideal containing a power of y. The "if" direction follows immediately. So now suppose that I is a graded module in $\mathrm{MCM}_{\mathbb{Z}}R$ that is generically free of rank 1. By Proposition 3.5, I is isomorphic to a homogeneous ideal of R containing a power of y, and by Lemma 3.6, this ideal must be of the form

$$(x^{k-1}y^{i_0}, x^{k-2}y^{i_1}, x^{k-3}y^{i_2}, \ldots, xy^{i_{k-2}}, y^{i_{k-1}})(i_k), \tag{3.2}$$

where $0 \leq i_0 \leq i_1 \leq \cdots \leq i_{k-1}$ and $i_k \in \mathbb{Z}$. However, notice that since y is a nonzero divisor, then as graded R-modules, the ideal in (3.2), and hence also I, is isomorphic to

$$(x^{k-1}, x^{k-2}y^{i_1-i_0}, x^{k-3}y^{i_2-i_0}, \ldots, xy^{i_{k-2}-i_0}, y^{i_{k-1}-i_0})(i_k + i_0)$$

as required. ■

We can depict the generically free module of rank 1 in $\mathrm{MCM}_{\mathbb{Z}}R$ from Theorem 3.7 as follows.

$\deg_I$:	$-i_k - i_{k-1}$	$\cdots$	$-i_k - i_{k-2} + 1$	$\cdots$	$-i_k - i_1 + k - 1$	$-i_k - i_1 + k - 2$	$\cdots$	$-i_k + k - 2$	$-i_k + k - 1$
$\cdots$			$\cdots$	$\cdots$	$x^{k-1}y^{i_1+2}$	$x^{k-1}y^{i_1+1}$	$\cdots$	$x^{k-1}y$	x^{k-1}
$\cdots$			$\cdots$	$\cdots$	$x^{k-2}y^{i_1+1}$	$x^{k-2}y^{i_1}$			
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$				
$\cdots$	$xy^{i_{k-1}+1}$	$\cdots$	$xy^{i_{k-2}}$						
$\cdots$	$y^{i_{k-1}}$								

Remark 3.8. Note that Theorem 3.7 includes the case

$$(x^{k-1}, x^{k-2}y^0, x^{k-3}y^0, \dots, xy^0, y^0)(i_k) \cong R(i_k) \cong (y^j)(i_k - j),$$

for any $j \geq 0$, where the latter isomorphism holds as y is a nonzero divisor.

For an ideal

$$I = (x^{k-1}, x^{k-2}y^{i_1}, x^{k-3}y^{i_2}, \dots, xy^{i_{k-2}}, y^{i_{k-1}})(i_k)$$

with $i_1 \leq i_2 < \dots \leq i_{k-1}$, and a homogeneous element $f \in I$, we write $\deg_I(f) = \deg(f) - i_k$. Associated with the ideal I is the k-subset

$$\begin{aligned}\underline{\ell}(I) &= \big(\deg_I(y^{i_{k-1}}), \deg_I(xy^{i_{k-2}}), \dots, \deg_I(x^{k-2}y^{i_1}), \deg_I(x^{k-1})\big)\\ &= \big(-i_k - i_{k-1}, -i_k - i_{k-2} + 1, \dots, -i_k - i_1 + k - 2, -i_k + k - 1\big),\end{aligned}$$

which we will view as a strictly increasing tuple throughout. We now consider again the cluster algebra of infinite rank

$$\mathcal{A}_k = \mathbb{C}[p_{\underline{\ell}} \mid \underline{\ell} \subseteq \mathbb{Z}, |\underline{\ell}| = k]/\mathcal{I}_P,$$

where we have relabelled the Plücker coordinates by $p_{\underline{\ell}}$ and where $\mathcal{I}_P$ is the ideal generated by the Plücker relations (2.1). For the next result, we adapt the general setup: to a k-subset $\underline{\ell} := (\ell_1, \dots, \ell_k)$, as always viewed as a tuple that is strictly increasing, we associate the following graded ideal:

$$I(\underline{\ell}) := (x^{k-1}, x^{k-2}y^{i_1}, x^{k-3}y^{i_2}, \dots, xy^{i_{k-2}}, y^{i_{k-1}})(i_k),$$

where $i_k = k - 1 - \ell_k$ and $i_{k-p} = \ell_k - \ell_p - (k - p)$.

Theorem 3.9. The generically free modules of rank 1 in $\mathrm{MCM}_{\mathbb{Z}}R$ are in bijection with the Plücker coordinates in $\mathcal{A}_k$. This bijection is given by the inverse maps

$$\begin{aligned}\{\text{generically free modules of rank 1 in } \mathrm{MCM}_{\mathbb{Z}}R\} &\rightarrow \{\text{Plücker coordinates of} \mathcal{A}_k\}\\ I &\mapsto p_{\underline{\ell}(I)}\\ I(\underline{\ell}) &\leftarrow\!\shortmid p_{\underline{\ell}}.\end{aligned}$$

Proof. This follows immediately from Theorem 3.7. ∎

3.4 The subcategory of generically free maximal Cohen–Macaulay modules

We denote by $\mathrm{MCM}^0_{\mathbb{Z}}R$ the full subcategory of $\mathrm{MCM}_{\mathbb{Z}}R$ consisting of generically free maximal Cohen–Macaulay modules. In particular, it contains the generically free modules of rank 1 that correspond to the Plücker coordinates of $\mathcal{A}_k$ by Theorem 3.9.

Note that generically free modules are closed under extensions and that they form an admissible subcategory of $\mathrm{MCM}_{\mathbb{Z}}R$, so $\mathrm{MCM}^0_{\mathbb{Z}}R$ is again a Frobenius category (see, e.g., [7]). Thus, the stable category $\underline{\mathrm{MCM}}^0_{\mathbb{Z}}R$ is a triangulated category and the goal of this section is to show that this category is 2-Calabi–Yau by applying results from Iyama and Takahashi [17].

Lemma 3.10. The Gorenstein parameter of $R = \mathbb{C}[x,y]/(x^k)$, with x in degree 1 and y in degree -1, is k.

Note that this agrees with the formula for the computation of the Gorenstein parameter provided in [15, Example 4.8f] or in [4, Examples 3.6.15]. Since our ring is nontrivial in both negative and positive degrees, we provide a direct computation for the peace of mind of the reader.

Proof. Let α denote the Gorenstein parameter of R. Since R has Krull-dimension 1, we have $\mathrm{grExt}^1_R(\mathbb{C},R) \cong \mathbb{C}(-\alpha)$, where $\mathrm{grExt}^j_R(A,B) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^j(A,B(i))$ for graded R-modules A and B.

To compute α, denote as before by $\mathcal{F}$ the graded total ring of fractions R_y, and consider the sequence

$$R \to \mathcal{F} \to \mathcal{F}/R,$$

where the 1st map is localisation at y. We first verify this is an injective resolution of R: indeed, since y is a nonzero divisor, the 1st map is injective. Furthermore, by Baer's criterion, $\mathcal{F}$ is injective over $\mathcal{F}$. Since $\mathcal{F}$ is flat over R, restriction of scalars sends injectives to injectives. It follows that $\mathcal{F}$ is injective over R. Finally, since R has injective dimension 1 as a graded module over itself, the cokernel $\mathcal{F}/R$ must be injective as well.

Now apply $\mathrm{grHom}_R(\mathbb{C},-)$ to this resolution. Note that the socle of $\mathcal{F}/R$ is generated by $y^{-1}x^{k-1}$ (up to multiplication by a scalar, this is the only element in $\mathcal{F}/R$ that gets annihilated by both x and y), which lives in degree k. Since $\mathbb{C}$ must map into the socle of $\mathcal{F}/R$, it follows that $\mathrm{grHom}(\mathbb{C}, \mathcal{F}/R) \cong \mathbb{C}(-k)$. We know that $\mathrm{grExt}^1_R(\mathbb{C}, R)$ is one dimensional, so we must have

$$\mathrm{grExt}^1_R(\mathbb{C}, R) \cong \mathrm{grHom}_R(\mathbb{C}, \mathcal{F}/R) \cong \mathbb{C}(-k).$$

The claim follows. ■

Lemma 3.11. Denote by Σ the suspension in the stable category $\underline{\mathrm{MCM}}_{\mathbb{Z}}R$. Then $\Sigma^2 \cong (k)$.

Proof. The well-known equivalence between the stable category of maximal Cohen–Macaulay modules $\underline{\mathrm{MCM}}(R)$ and the category of reduced matrix factorisations $\underline{\mathrm{MF}}(S, x^k)$ (see [8, 6.1,6.3] or [25, Theorem 7.4]) also holds in the graded case (cf. [6, Remark 1.8]). Thus, we have an exact equivalence

$$\underline{\mathrm{MCM}}_{\mathbb{Z}}R \cong \underline{\mathrm{MF}}_{\mathbb{Z}}(S, x^k),$$

where $S = \mathbb{C}[x, y]$ with x in degree 1 and y in degree -1 and $\underline{\mathrm{MF}}_{\mathbb{Z}}(S, x^k)$ denotes the homotopy category of graded matrix factorisations of x^k over S. Indeed, if (d_0, d_1) is a matrix factorisation of x^k, that is, $d_1 d_0$ is multiplication by x^k, and thus a degree k map. Suspension on matrix factorisations is given by the shift, when viewing them as (twisted) 2-periodic objects, so double suspension is just the degree shift by k and so Σ^2 acts as (k) on objects and morphisms. ■

Proposition 3.12. The category $\mathrm{MCM}^0_{\mathbb{Z}}R$ of generically free maximal Cohen–Macaulay modules is stably 2-Calabi–Yau.

Proof. By [17, Cor. 3.5], $\underline{\mathrm{MCM}}^0_{\mathbb{Z}}R$ has Serre functor $S = (\alpha)$ where α denotes the Gorenstein parameter of R. By Lemma 3.10, we have $\alpha = k$, and by Lemma 3.11, it follows that

$$S = (k) \cong \Sigma^2.$$

■

4 Compatibility

In this section, we fix $k \geq 2$ and continue to write R for the $\mathbb{Z}$-graded ring $\mathbb{C}[x, y]/(x^k)$ with x in degree 1 and y in degree -1. We set $\mathcal{C} := \mathrm{MCM}_{\mathbb{Z}}R$, and furthermore, we denote the Hom and Ext^1 bifunctors in $\mathcal{C}$ by $\mathrm{Hom}(-,-)$ and $\mathrm{Ext}^1(-,-)$, respectively. We will show that for two generically free MCM modules I and J of rank 1, we have $\mathrm{Ext}^1(I, J) = 0$, if and only if the Plücker coordinates corresponding to I and J are compatible; cf. Section 2.1.1.

The key intermediate result of this section is to compute the dimension of the Ext^1-spaces between generically free modules of rank 1 in $\mathrm{MCM}_{\mathbb{Z}}R$. A formula for this is provided in Theorem 4.10, and as a consequence, we deduce our following main result.

Theorem 4.1. Let I, J be two generically free rank 1 modules in $\mathrm{MCM}_{\mathbb{Z}}R$ and $p_{\underline{\ell}(I)}$ and $p_{\underline{\ell}(J)}$ the corresponding Plücker coordinates. Then $\mathrm{Ext}^1(I,J) = 0$ if and only if $p_{\underline{\ell}(I)}$ and $p_{\underline{\ell}(J)}$ are compatible.

To prove Theorem 4.1, we show that the k-subsets $\underline{\ell} := \underline{\ell}(I)$ and $\underline{m} := \underline{\ell}(J)$ are noncrossing.

This section is structured as follows. In Subsection 4.1, we develop a combinatorial tool to record the crossing pattern of $\underline{\ell}$ and $\underline{m}$. In Section 4.2, we provide a general formula to calculate the dimension of the Ext^1-space between any two generically free modules of rank 1, using the tool from Section 4.1. Section 4.3 provides a concrete example in the case $k = 3$. Finally, in Section 4.4, we prove Theorem 4.1, using reduction to the setting where $\underline{\ell}$ and $\underline{m}$ are disjoint sets.

4.1 Combinatorial tool

Given two k-subsets $\underline{\ell}$ and $\underline{m}$, we now introduce a combinatorial tool that will help us to calculate the dimension of $\mathrm{Ext}^1(I(\underline{\ell}), I(\underline{m}))$, as well as determining whether or not the subsets are crossing.

Definition 4.2. Let $A(\underline{\ell},\underline{m})$ (respectively $B(\underline{\ell},\underline{m})$) be a $(k \times k)$ grid where the vertex $A(\underline{\ell},\underline{m})_{i,j}$ is filled if $\ell_i \leq m_j$ (respectively $B(\underline{\ell},\underline{m})_{i,j}$ is filled if $\ell_i < m_j$) and is empty otherwise.

Example 4.3. Take $k = 4$, and consider the subsets $\underline{\ell}$ and $\underline{m}$ with

$$m_1 < \ell_1 < \ell_2 = m_2 < m_3 < \ell_3 < m_4 < \ell_4.$$

$$A(\underline{\ell},\underline{m}) = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \circ & \bullet & \bullet & \bullet \\ 2 & \circ & \bullet & \bullet & \bullet \\ 3 & \circ & \circ & \circ & \bullet \\ 4 & \circ & \circ & \circ & \circ \end{array} \qquad B(\underline{\ell},\underline{m}) = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \circ & \bullet & \bullet & \bullet \\ 2 & \circ & \circ & \bullet & \bullet \\ 3 & \circ & \circ & \circ & \bullet \\ 4 & \circ & \circ & \circ & \circ \end{array}$$

Lemma 4.4. If $\underline{\ell}$ and $\underline{m}$ are disjoint k-subsets, then $A(\underline{\ell},\underline{m}) = B(\underline{\ell},\underline{m})$.

Proof. Clear from Definition 4.2. ■

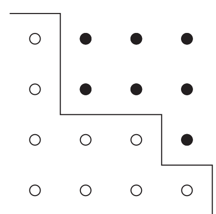
Lemma 4.5. If $A(\underline{\ell},\underline{m})_{i,j}$ is filled, then so is $A(\underline{\ell},\underline{m})_{p,q}$ for all $p \le i, q \ge j$. Further, if $A(\underline{\ell},\underline{m})_{i,j}$ is empty, then so is $A(\underline{\ell},\underline{m})_{p,q}$ for all $p \ge i, q \le j$.

Proof. Suppose $A(\underline{\ell},\underline{m})_{i,j}$ is filled, and hence by definition, $\ell_i \le m_j$. Since the k-subsets $\underline{\ell}$ and $\underline{m}$ are strictly increasing, if $p \le i$, then $\ell_p \le \ell_i$ and similarly, if $q \ge j$, then $m_j \le m_q$. Thus, for all pairs (p,q) with $p \le i, q \ge j$,

$$\ell_p \le \ell_i \le m_j \le m_q,$$

and hence $A(\underline{\ell},\underline{m})_{p,q}$ is also filled. The 2nd statement is proved similarly. ■

In other words, Lemma 4.5 tells us that there is a staircase path obtained by separating the shaded and empty regions of $A(\underline{\ell},\underline{m})$. In Example 4.3, we obtain the following path:



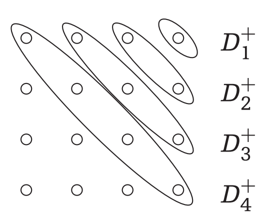
Note that there is a completely analogous statement of Lemma 4.5 for $B(\underline{\ell},\underline{m})$ and thus we also get a staircase path separating the shaded and unshaded regions there. We use these staircase paths to define two nonnegative integers associated with the pair $\underline{\ell}$ and $\underline{m}$ of k-subsets. For $1 \le p \le k$, we define the sets

$$D_p^+ = \{(i,j) \mid j - i = k - p\}$$

to be the *upper diagonals of a* $(k \times k)$*-grid* and

$$D_p^- = \{(i,j) \mid i - j = k - p\}$$

to be the *lower diagonals of a* $(k \times k)$*-grid*. Note that $D_k^+ = D_k^-$. Below is a picture of (4×4)-grid with the upper diagonals circled.



Definition 4.6. With the above notation, we introduce the following.

1. Let $\alpha(\underline{\ell}, \underline{m})$ be the number of upper diagonals that lie completely above the staircase path in $A(\underline{\ell}, \underline{m})$, that is,

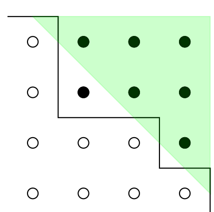
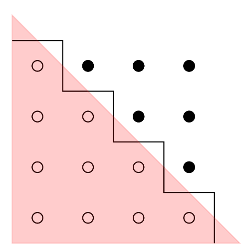
$$\alpha(\underline{\ell}, \underline{m}) := \begin{cases} \max_{1 \leq p \leq k}\{p \mid \forall (i,j) \in D_p^+,\ \ell_i \leq m_j\} & \text{if it exists} \\ 0 & \text{otherwise.} \end{cases}$$

2. Let $\beta(\underline{\ell}, \underline{m})$ be the number of lower diagonals that lie completely below the staircase path in $B(\underline{\ell}, \underline{m})$, that is,

$$\beta(\underline{\ell}, \underline{m}) := \begin{cases} \max_{1 \leq p \leq k}\{p \mid \forall (i,j) \in D_p^-,\ \ell_i \geq m_j\} & \text{if it exists} \\ 0 & \text{otherwise.} \end{cases}$$

When the choice of $\underline{\ell}$ and $\underline{m}$ is clear, we will often shorten $\alpha(\underline{\ell}, \underline{m})$ to α and $\beta(\underline{\ell}, \underline{m})$ to simply β.

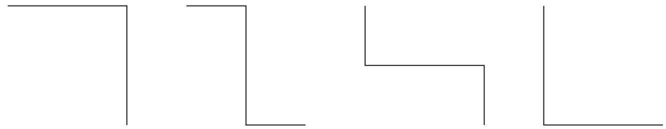
Example 4.7. Returning to Example 4.3, we see that $\alpha(\underline{\ell}, \underline{m}) = 3$ and $\beta(\underline{\ell}, \underline{m}) = 4$.

$A(\underline{\ell}, \underline{m}) =$  $B(\underline{\ell}, \underline{m}) =$ 

Lemma 4.8. If $\underline{\ell}$ and $\underline{m}$ are k-subsets, then $\alpha(\underline{\ell}, \underline{m}) = \beta(\underline{m}, \underline{\ell})$.

Proof. By definition, D_p^+ lies completely above the staircase path in $A(\underline{\ell}, \underline{m})$ if, for all (i,j) such that $j - i = k - p$, we have $l_i \leq m_j$. Similarly, D_p^- lies completely below the staircase path in $B(\underline{m}, \underline{\ell})$ if, for all (j,i) with $j - i = k - p$ we have $m_j \geq l_i$. Since these conditions are the same, the number $\alpha(\underline{\ell}, \underline{m})$ of upper diagonals that lie completely above the staircase path in $A(\underline{\ell}, \underline{m})$ is the same as the number $\beta(\underline{m}, \underline{\ell})$ of lower diagonals that lie completely below the staircase path in $B(\underline{m}, \underline{\ell})$. ∎

Lemma 4.9. If $\underline{\ell}$ and $\underline{m}$ are disjoint k-subsets, then they are noncrossing if and only if the staircase path consists of a single step, that is, looks like one of the following:

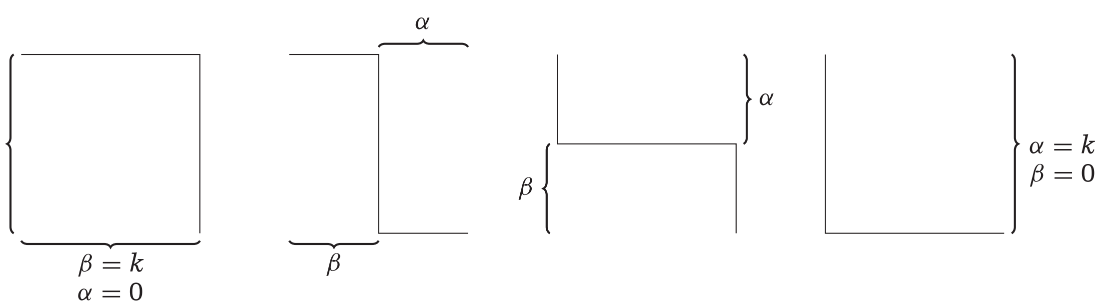


Moreover, this holds if and only if $\alpha(\underline{\ell},\underline{m}) + \beta(\underline{\ell},\underline{m}) = k$.

Proof. Since $\underline{\ell}$ and $\underline{m}$ are disjoint, $A(\underline{\ell},\underline{m}) = B(\underline{\ell},\underline{m})$ by Lemma 4.4 and so there is only one staircase path associated with the pair. It is straightforward to check that if the path is one of the four given, then $\underline{\ell}$ and $\underline{m}$ are noncrossing. On the other hand, if the staircase path for $\underline{\ell}$ and $\underline{m}$ is not one of the four given, the path must have one of the following local configurations:

In the former case, since $\underline{\ell}$ and $\underline{m}$ are disjoint, we have $m_j < l_{i-1} < m_{j+1} < l_i$ and thus $\underline{\ell}$ and $\underline{m}$ are crossing. The latter case follows similarly.

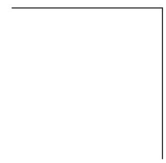
For the 2nd statement, first assume that the step path in $A(\underline{\ell},\underline{m})$ (and hence also in $B(\underline{\ell},\underline{m})$ since $\underline{\ell}$ and $\underline{m}$ are disjoint) is one of the four given cases. In each case, it is easy to read off α and β:



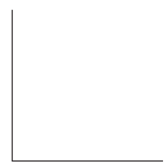
It is clear that in all cases $\alpha + \beta = k$.

Now assume that $\alpha + \beta = k$. Since $\underline{\ell}$ and $\underline{m}$ are disjoint, we have $A(\underline{\ell},\underline{m}) = B(\underline{\ell},\underline{m})$ and so both α and β can be determined by looking solely at $A(\underline{\ell},\underline{m})$.

Start by considering the case when $\beta = k$ and $\alpha = 0$. Since $\alpha = 0$, we know that $A(\underline{\ell},\underline{m})_{1,k}$ lies below the staircase, and thus by Lemma 4.5, $A(\underline{\ell},\underline{m})_{p,q}$ lies below the staircase for all $1 \leq p, q \leq k$. Then the staircase must be



Similarly, considering the case when $\alpha = k$ and $\beta = 0$, this implies $A(\underline{\ell}, \underline{m})_{k,1}$ lies above the staircase otherwise β would be at least one. By Lemma 4.5, this implies further that $A(\underline{\ell}, \underline{m})_{p,q}$ lies above the staircase for all $1 \leq p, q \leq k$, and thus the staircase must be



Now assume that $1 < \beta < k$. Since β is chosen to be maximal, we know that there must exist a pair $(i, j) \in D^-_{\beta+1}$ such that $A(\underline{\ell}, \underline{m})_{i,j}$ is above the staircase (i.e., one of the diamond vertices in Figure 1 must be above the staircase or β would be at least one larger). Note that such an (i, j) has the form $(i, i-(k-\beta)+1) = (i, i-\alpha+1)$ where $i \in \{\alpha, \ldots, k\}$. Similarly, since $1 < \alpha < k$ is also chosen to be maximal, there exists a pair $(p, \beta+p-1) \in D^+_{\alpha+1}$ with $p \in \{1, \ldots, \alpha+1\}$ such that $A(\underline{\ell}, \underline{m})_{p,q}$ lies below the staircase (i.e., one of the triangle vertices in Figure 1).

Suppose $A(\underline{\ell}, \underline{m})_{i,i-\alpha+1}$ lies above the staircase for some $i \in \{\alpha+1, \ldots, k-1\}$ (i.e., one of the inner vertices on the diagonal $D^-_{\beta+1}$). Then, for all $p \in \{1, \ldots, \alpha+1\}$, we have $p \leq i$ and

$$i - \alpha + 1 \leq (k-1) - \alpha + 1 = k - \alpha = \beta \leq \beta + p - 1,$$

and hence, by Lemma 4.5, all the points $A(\underline{\ell}, \underline{m})_{p,\beta+p-1}$ with $p \in \{1, \ldots, \alpha+1\}$ (all the triangle vertices in Figure 1) lie above the staircase. Or equivalently, the entire diagonal $D^+_{\alpha+1}$ lies above the staircase, contradicting the maximality of α. Therefore, we must have either $A(\underline{\ell}, \underline{m})_{\alpha,1}$ or $A(\underline{\ell}, \underline{m})_{k,\beta+1}$ lying above the staircase.

If $A(\underline{\ell}, \underline{m})_{k,\beta+1}$ lies above the staircase, then, similar to above, all the points $A(\underline{\ell}, \underline{m})_{p,\beta+p-1}$ with $p \in \{2, \ldots, \alpha+1\}$ lie above the staircase, and thus $A(\underline{\ell}, \underline{m})_{1,\beta}$ must lie below the staircase, so as not to contradict the maximality of α. Thus, we have $A(\underline{\ell}, \underline{m})_{k,\beta+1}$ above the staircase and $A(\underline{\ell}, \underline{m})_{1,\beta}$ below the staircase and so the staircase

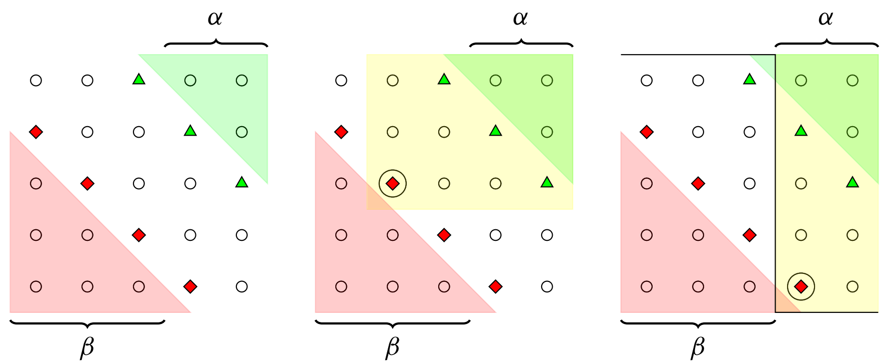
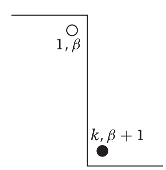
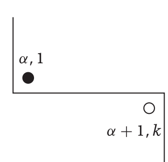


Fig. 1. With $k = 5$, $\beta = 3$, and $\alpha = 2$, one of the diamond vertices must lie above the staircase (not drawn), and one of the triangle vertices must lie below. If one of the inner diamond vertices lies above the staircase, the 2nd picture shows all the triangle vertices must also lie above. The last picture shows choosing one of the outer diamond vertices leaves one triangle vertex that may lie below.

must be



If $A(\underline{\ell}, \underline{m})_{\alpha,1}$ lies above the staircase, then all the points $A(\underline{\ell}, \underline{m})_{p,\beta+p-1}$ with $p \in \{1, \ldots, \alpha\}$ must lie above the staircase, and thus $A(\underline{\ell}, \underline{m})_{\alpha+1,k}$ must lie below the staircase, so as not to contradict the maximality of α. Hence, we have $A(\underline{\ell}, \underline{m})_{\alpha,1}$ above the staircase and $A(\underline{\ell}, \underline{m})_{\alpha+1,k}$ below the staircase and so the staircase must be



■

4.2 Dimension formula

Now we may use the combinatorial tool developed Section 4.1 to provide a formula for calculating the dimension of $\mathrm{Ext}^1(I(\underline{\ell}), I(\underline{m}))$.

Theorem 4.10. Given two k-subsets $\underline{\ell}$ and $\underline{m}$,

$$\dim_{\mathbb{C}}(\mathrm{Ext}^1(I(\underline{\ell}), I(\underline{m}))) = \alpha(\underline{\ell}, \underline{m}) + \beta(\underline{\ell}, \underline{m}) - k - |\underline{\ell} \cap \underline{m}|.$$

Remark 4.11. By Proposition 3.12, the subcategory $\mathrm{MCM}^0_{\mathbb{Z}} R$ of $\mathcal{C}$ consisting of generically free modules is stably 2-Calabi–Yau, and thus for any two generically free modules M and N in $\mathcal{C}$, it immediately follows that

$$\dim_{\mathbb{C}}(\mathrm{Ext}^1(M, N)) = \dim_{\mathbb{C}}(\mathrm{Ext}^1(N, M)).$$

It is easy to see that by combining Lemma 4.8 and Theorem 4.10, our combinatorial tool allows us to verify this symmetry directly for the generically free modules of rank 1.

To prove Theorem 4.10, we start by fixing the following notation:

- $I := I(\underline{\ell}) = (x^{k-1}, x^{k-2}y^{i_1}, x^{k-3}y^{i_2}, \dots, xy^{i_{k-2}}, y^{i_{k-1}})(i_k)$ where $i_k = k-1-\ell_k$ and $i_{k-p} = \ell_k - \ell_p - (k-p)$;
- $J := I(\underline{m}) = (x^{k-1}, x^{k-2}y^{j_1}, x^{k-3}y^{j_2}, \dots, xy^{j_{k-2}}, y^{j_{k-1}})(j_k)$ where $j_k = k-1-m_k$ and $j_{k-p} = m_k - m_p - (k-p)$;
- $\mathbf{J} := J(\deg_I(x^{k-1})) \oplus J(\deg_I(x^{k-2}y^{i_1})) \oplus \cdots \oplus J(\deg_I(y^{i_{k-1}}))$. This means that an element of $\mathbf{J}$ is a vector with m-th component in the ideal J shifted by $\deg_I(x^{k-m}y^{i_{m-1}})$.

Our 1st observation is that we may assume that $i_k = 0$, or equivalently $\ell_k = k-1$. Indeed, if this does not hold, we may shift both I and J by $-i_k$ to get to this setting, which will not affect the Ext calculation as we have only shifted the grading. Moreover, this corresponds to shifting both $\underline{\ell}$ and $\underline{m}$ by i_k and so it does not change $A(\underline{\ell}, \underline{m})$ or $B(\underline{\ell}, \underline{m})$ in any way. For future use, also note that

$$m_p = -j_k - j_{k-p} + p - 1 \quad \text{and} \quad \ell_p = -i_{k-p} + p - 1, \tag{4.1}$$

where, for ease of notation, we set $i_0 = 0 = j_0$.

4.2.1 *Matrix factorisations*

In the ring $R = \mathbb{C}[x, y]/(x^k)$, a matrix factorisation for the ideal

$$I = (x^{k-1}, x^{k-2}y^{i_1}, x^{k-3}y^{i_2}, \dots, xy^{i_{k-2}}, y^{i_{k-1}}),$$

where $0 \leq i_1 \leq i_2 \leq \ldots \leq i_{k-1}$ is given as

$$R^k \xrightarrow{M} R^k \xrightarrow{N} R^k \to I \to 0,$$

where M, N are the $k \times k$ upper triangular matrices:

$$M = \begin{pmatrix} x^{k-1} & x^{k-2}y^{i_1} & x^{k-3}y^{i_2} & \ldots & xy^{i_{k-2}} & y^{i_{k-1}} \\ 0 & x^{k-1} & x^{k-2}y^{i_2-i_1} & \ldots & x^2y^{i_{k-2}-i_1} & xy^{i_{k-1}-i_1} \\ 0 & 0 & x^{k-1} & \ldots & x^3y^{i_{k-2}-i_2} & x^2y^{i_{k-1}-i_2} \\ & & & \ddots & \vdots & \vdots \\ & & & & x^{k-1} & x^{k-2}y^{i_{k-1}-i_{k-2}} \\ & & & & 0 & x^{k-1} \end{pmatrix}$$

and

$$N = \begin{pmatrix} x & -y^{i_1} & 0 & 0 & & & \\ 0 & x & -y^{i_2-i_1} & 0 & & & \\ 0 & 0 & x & -y^{i_3-i_2} & & & \\ & & & \ddots & \ddots & & \\ & & & & x & -y^{i_{k-2}-i_{k-3}} & 0 \\ & & & & 0 & x & -y^{i_{k-1}-i_{k-2}} \\ & & & & 0 & 0 & x \end{pmatrix}.$$

In particular, a graded projective presentation of I is

$$\begin{array}{c} R(-\deg_I(x^{k-1})-k) \oplus R(-\deg_I(x^{k-2}y^{i_1})-k) \oplus \cdots \oplus R(-\deg_I(y^{i_{k-1}})-k) \\ \downarrow M \\ R(-\deg_I(x^{k-1})-1) \oplus R(-\deg_I(x^{k-2}y^{i_1})-1) \oplus \cdots \oplus R(-\deg_I(y^{i_{k-1}})-1) \\ \downarrow N \\ R(-\deg_I(x^{k-1})) \oplus R(-\deg_I(x^{k-2}y^{i_1})) \oplus \cdots \oplus R(-\deg_I(y^{i_{k-1}})) \\ \downarrow \\ I \\ \downarrow \\ 0 \end{array}$$

We remark that the matrix factorisations are not reduced if some of the i_js are equal.

4.2.2 *Strategy*

To calculate $\mathrm{Ext}^1(I,J)$, take the graded projective presentation of I above, and apply the graded $\mathrm{Hom}^{\mathbb{Z}}(-,J) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}(-,J(n))$. Since $\mathrm{Hom}^{\mathbb{Z}}(R(a),J) \cong J(-a)$, this gives

$$\mathbf{J} \xrightarrow{N^T} \mathbf{J}(1) \xrightarrow{M^T} \mathbf{J}(k)$$

and to obtain $\mathrm{Ext}^1(I,J)$, we calculate $(\ker(M^T)/\mathrm{im}(N^T))_0$ or equivalently $\ker(M^T)_0/\mathrm{im}(N^T)_0$. In fact, we will only be interested in the dimension of the Ext group that we can calculate as

$$\dim_{\mathbb{C}}(\mathrm{Ext}^1(I,J)) = \dim_{\mathbb{C}}(\ker(M^T)_0) - \dim_{\mathbb{C}}(\mathrm{im}(N^T)_0).$$

Since the maps are graded and each of the degree zero parts are finite-dimensional $\mathbb{C}$-vector spaces, we may use the standard rank-nullity theorem to say

$$\dim_{\mathbb{C}}(\ker(M^T)_0) = \dim_{\mathbb{C}}(\mathbf{J}(1)_0) - \dim_{\mathbb{C}}(\mathrm{im}(M^T)_0)$$

and

$$\dim_{\mathbb{C}}(\mathrm{im}(N^T)_0) = \dim_{\mathbb{C}}(\mathbf{J}_0) - \dim_{\mathbb{C}}(\ker(N^T)_0).$$

So our strategy to prove Theorem 4.10 is to determine the complex dimensions of $\mathbf{J}_0, \mathbf{J}(1)_0, \ker(N^T)_0$ and $\mathrm{im}(M^T)_0$, and then to combine them to determine $\dim_{\mathbb{C}}(\mathrm{Ext}^1(I,J))$.

4.2.3 *Calculating dimensions*

Lemma 4.12. A degree zero element of $\mathbf{J}$ has the following form:

$$\underline{a} = \begin{pmatrix} a_{11}x^{k-1}y^{-j_k} & + & a_{12}x^{k-2}y^{-j_k-1} & + & \dots & + & a_{1k}y^{-j_k+1-k} \\ a_{21}x^{k-1}y^{i_1-j_k+1} & + & a_{22}x^{k-2}y^{i_1-j_k} & + & \dots & + & a_{2k}y^{i_1-j_k+2-k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{k1}x^{k-1}y^{i_{k-1}-j_k+k-1} & + & a_{k2}x^{k-2}y^{i_{k-1}-j_k+k-2} & + & \dots & + & a_{kk}y^{i_{k-1}-j_k} \end{pmatrix}$$

where a_{pq} is the coefficient of $x^{k-q}y^{p-q+i_{p-1}-j_k}$ and $a_{pq} \in \mathbb{C}$ can be nonzero if and only if $\ell_{k+1-p} \leq m_{k+1-q}$.

Proof. Recall that $J(n)_0 = J_n$, and thus, remembering that J is an ideal shifted by j_k, monomials $x^a y^b$ lie in $J(n)_0$ precisely when $a - b = n + j_k$. In particular, if $a = k - q$ for some $q = 1, \ldots, k$, then

$$b = k - q - n - j_k.$$

Therefore, there is a $\mathbb{C}$-basis for $J(n)_0$ that consists of the subset of

$$x^{k-1}y^{k-1-n-j_k}, x^{k-2}y^{k-2-n-j_k}, \ldots, xy^{1-n-j_k}, y^{-n-j_k},$$

which lie in J. In particular, for each $p = 1, \ldots, k$, a degree zero element of $J(\deg(x^{k-p}y^{i_{p-1}}))$ is

$$\sum_{q=1}^{k} a_{pq}x^{k-q}y^{p-q+i_{p-1}-j_k},$$

where a_{pq} can be nonzero only if $x^{k-q}y^{p-q+i_{p-1}-j_k} \in J$, or equivalently,

$$\begin{aligned} p - q + i_{p-1} - j_k \geq j_{q-1} &\iff -j_{q-1} - j_k - q \geq -i_{p-1} - p \\ &\iff -j_{q-1} - j_k + (k+1-q) - 1 \geq -i_{p-1} + (k+1-p) - 1 \\ &\iff m_{k+1-q} \geq \ell_{k+1-p}. \qquad \text{(using (4.1))} \end{aligned}$$

■

Example 4.13. Take $k = 3$, and consider $\underline{\ell} = (-2, 0, 2)$ and $\underline{m} = (-1, 2, 3)$. These correspond to ideals

$$I = (x^2, xy, y^2) \quad \text{and} \quad J = (x^2, x, y^2)(-1).$$

In this case, a degree zero element of $\mathbf{J} = J(2) \oplus J(0) \oplus J(-2)$ is

$$\begin{pmatrix} a_{11}x^2y & + & a_{12}x & & \\ a_{21}x^2y^3 & + & a_{22}xy^2 & & \\ a_{31}x^2y^5 & + & a_{32}xy^4 & + & a_{33}y^3 \end{pmatrix} \tag{4.2}$$

where $a_{ij} \in \mathbb{C}$. Notice that a_{13} and a_{23} do not appear since $y^{-1}, y \notin J$. Compare this with $A(\underline{\ell}, \underline{m})$, and its image after rotating by a half turn:

$$A(\underline{\ell}, \underline{m}) = \begin{array}{ccc} \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet \\ \circ & \bullet & \bullet \end{array} \qquad \text{“rotated” } A(\underline{\ell}, \underline{m}) = \begin{array}{ccc} \bullet & \bullet & \circ \\ \bullet & \bullet & \circ \\ \bullet & \bullet & \bullet \end{array}$$

After rotation, the shape formed by the staircase path is precisely the same as that of the possibly nonzero coefficients in (4.2). This follows since a_{ij} can be nonzero if and only if $\ell_{k+1-i} \leq m_{k+1-j}$ that by definition is if and only if $A(\underline{\ell}, \underline{m})_{k+1-i,k+1-j}$ is shaded.

Lemma 4.14. With the setup above, $\dim_{\mathbb{C}}(\mathbf{J}_0) = |\{(i,j) \mid 1 \leq i,j \leq k$ and $A(\underline{\ell}, \underline{m})_{i,j}$ is shaded$\}|$.

Proof. It is clear from Lemma 4.12 that

$$\dim_{\mathbb{C}}(\mathbf{J}_0) = |\{(i,j) \mid 1 \leq i,j \leq k \text{ and } a_{ij} \text{ can be nonzero}\}|.$$

Moreover, we know that a_{ij} can be nonzero precisely when $\ell_{k+1-i} \leq m_{k+1-j}$ that, by definition, is if and only if $A(\underline{\ell}, \underline{m})_{k+1-i,k+1-j}$ is shaded. Thus,

$$\dim_{\mathbb{C}}(\mathbf{J}_0) = |\{(i,j) \mid 1 \leq i,j \leq k \text{ and } A(\underline{\ell}, \underline{m})_{k+1-i,k+1-j} \text{ is shaded}\}|.$$

The map $(i,j) \mapsto (k+1-i, k+1-j)$ precisely describes the rotation of the $(k \times k)$-grid as seen in Example 4.13. Since this gives a bijection from $\{1, \ldots, k\} \times \{1, \ldots, k\}$ to itself, the right-hand side is the same as $|\{(i,j) \mid 1 \leq i,j \leq k$ and $A(\underline{\ell}, \underline{m})_{i,j}$ is shaded$\}|$ completing the proof. ∎

Lemma 4.15. A degree zero element of $\mathbf{J}(1)$ has the following form:

$$\underline{b} = \begin{pmatrix} b_{11}x^{k-1}y^{-j_k-1} & + & b_{12}x^{k-2}y^{-j_k-2} & + & \ldots & + & b_{1k}y^{j_k-k} \\ b_{21}x^{k-1}y^{i_1-j_k} & + & b_{22}x^{k-2}y^{i_1-j_k-1} & + & \ldots & + & b_{2k}y^{i_1-j_k+1-k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ b_{k1}x^{k-1}y^{i_{k-1}-j_k+k-2} & + & b_{k2}x^{k-2}y^{i_{k-1}-j_k+k-3} & + & \ldots & + & b_{kk}y^{i_{k-1}-j_k-1} \end{pmatrix}$$

where b_{pq} is the coefficient of $x^{k-q}y^{p-q+i_{p-1}-j_k-1}$ and $b_{pq} \in \mathbb{C}$ can be nonzero if and only if $\ell_{k+1-p} < m_{k+1-q}$.

Proof. This proof is completely analogous to the proof of Lemma 4.12. For each $p = 1, \dots, k$, a degree zero element of $J(\deg(x^{k-p}y^{i_{p-1}}) + 1)$ is

$$\sum_{q=1}^{k} b_{pq} x^{k-q} y^{p-q+i_{p-1}-j_k-1}$$

(the y-index drops by one from Lemma 4.12 since we have shifted by one) where b_{pq} can be nonzero only if $x^{k-q}y^{p-q+i_{p-1}-j_k-1} \in J$, or equivalently,

$$\begin{aligned} p - q + i_{p-1} - j_k - 1 \geq j_{q-1} &\iff p - q + i_{p-1} - j_k > j_{q-1} \\ &\iff -j_{q-1} - j_k - q > -i_{p-1} - p \\ &\iff -j_{q-1} - j_k + (k+1-q) - 1 > -i_{p-1} + (k+1-p) - 1 \\ &\iff m_{k+1-q} > \ell_{k+1-p}. \qquad \text{(using (4.1))} \end{aligned}$$

■

Lemma 4.16. With the setup above, $\dim_{\mathbb{C}}(\mathbf{J}(1)_0) = |\{(i,j) \mid 1 \leq i,j \leq k$ and $B(\underline{\ell}, \underline{m})_{i,j}$ is shaded$\}|$.

Proof. Completely analogous to Lemma 4.14, but now using that, by definition, $m_{k+1-q} > \ell_{k+1-p}$ if and only if $B(\underline{\ell}, \underline{m})_{k+1-i,k+1-j}$ is shaded. ■

Corollary 4.17. With the setup above, $\dim_{\mathbb{C}}(\mathbf{J}_0) - \dim_{\mathbb{C}}(\mathbf{J}(1)_0) = |\underline{\ell} \cap \underline{m}|$.

Proof. Using Lemmas 4.14 and 4.16,

$$\begin{aligned} \dim_{\mathbb{C}}(\mathbf{J}_0) - \dim_{\mathbb{C}}(\mathbf{J}(1)_0) = |\{(i,j) \mid 1 \leq i,j \leq k \text{ and } A(\underline{\ell}, \underline{m})_{i,j} \text{ is shaded}\}| \\ - |\{(i,j) \mid 1 \leq i,j \leq k \text{ and } B(\underline{\ell}, \underline{m})_{i,j} \text{ is shaded}\}|. \end{aligned}$$

Since $\ell_i < m_j$ implies $\ell_i \leq m_j$, it is clear that if $B(\underline{\ell}, \underline{m})_{i,j}$ is shaded, then so is $A(\underline{\ell}, \underline{m})_{i,j}$, and hence the right-hand side is simply

$$\begin{aligned} |\{(i,j) \mid 1 \leq i,j \leq k \text{ and } A(\underline{\ell}, \underline{m})_{i,j} \text{ is shaded and } B(\underline{\ell}, \underline{m})_{i,j} \text{ is empty}\}| \\ = |\{(i,j) \mid 1 \leq i,j \leq k \text{ and } \ell_i \leq m_j \text{ and } \ell_i \geq m_j\}| \\ = |\{(i,j) \mid 1 \leq i,j \leq k \text{ and } \ell_i = m_j\}|. \end{aligned}$$

For each such pair (i,j), it is clear there is a corresponding element of $\underline{\ell} \cap \underline{m}$ and since $\underline{\ell}$ and $\underline{m}$ are strictly increasing sequences, each element of $\underline{\ell} \cap \underline{m}$ corresponds to a unique such pair (i,j). Thus,

$$|\{(i,j) \mid 1 \leq i,j \leq k \text{ and } \ell_i = m_j\}| = |\underline{\ell} \cap \underline{m}|$$

and so $\dim_{\mathbb{C}}(\mathbf{J}_0) - \dim_{\mathbb{C}}(\mathbf{J}(1)_0) = |\underline{\ell} \cap \underline{m}|$ as required. ■

Lemma 4.18. With the setup above, $\dim_{\mathbb{C}}(\ker(N^T)_0) = \alpha(\underline{\ell}, \underline{m})$.

A calculation for $\dim_{\mathbb{C}}(\ker(N^T)_0)$ when $k = 3$ is given in Example 4.21.

Proof. By Lemma 4.12, we know the form of a generic element $\underline{a}$ of $\mathbf{J}$ and applying N^T gives a vector $N^T(\underline{a})$ with 1st term

$$\sum_{q=1}^{k-1} a_{1q+1} x^{k-q} y^{-j_k-q},$$

and for $2 \leq p \leq k$, its p-th term is

$$\left(\sum_{q=1}^{k-1} (a_{pq+1} - a_{p-1q}) x^{k-q} y^{-j_k+i_{p-1}+p-q}\right) - a_{p-1k} y^{-j_k+i_{p-1}+p-q}.$$

In particular, $\underline{a} \in \ker(N^T)$ if and only if the coefficient of each monomial in each of these expressions is zero, that is,

1. $a_{1q} = 0$ for all $q = 2, \ldots, k$;
2. $a_{pk} = 0$ for all $p = 1, \ldots, k-1$;
3. $a_{pq+1} = a_{p-1q}$ for all $p = 2, \ldots, k, q = 1, \ldots, k-1$ or equivalently, $a_{p+1q+1} = a_{pq}$ for all $1 \leq p, q \leq k-1$.

Note that (3) holds if and only if, in the matrix of coefficients

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-1} & a_{k-1k} \\ a_{k1} & a_{k2} & \cdots & a_{kk-1} & a_{kk} \end{pmatrix} \tag{4.3}$$

the value along each of the diagonals is constant. If we further impose conditions (1) and (2), this shows that each of the diagonals above the main diagonal must be zero. Thus, $\underline{a} \in \ker(N^T)$ if and only if the a_{ij} above the main diagonal are zero, and the a_{ij} on each lower diagonal are constant. Hence, we see that for an element of $\ker(N^T)_0$, there are at most k-free choices—one for each of the lower diagonals. However, for a given diagonal D_p^- (where here, we are abusing notation by using $D_p^{\pm}$ to denote diagonals in matrices, as well as in $(k \times k)$ grids), we may only choose something nonzero if all the a_{ij} along D_p^- may be nonzero. Thus, $\dim_{\mathbb{C}}(\ker(N^T)_0)$ is precisely the number of lower diagonals in the matrix (4.3) along which all a_{ij} may be nonzero.

Recall from Lemma 4.12 that a_{ij} can be nonzero if and only if $\ell_{k+1-i} \leq m_{k+1-j}$ if and only if the entry $A(\underline{\ell}, \underline{m})_{k+1-i,k+1-j}$ is shaded. Thus, a lower diagonal in the matrix (4.3), say D_p^-, can be all nonzero if and only if the upper diagonal D_p^+ in $A(\underline{\ell}, \underline{m})$ is completely shaded. Hence, the number of lower diagonals in (4.3) along which all entries may be nonzero is the same as the number of upper diagonals in $A(\underline{\ell}, \underline{m})$ that are completely above the staircase path that, by definition, is $\alpha(\underline{\ell}, \underline{m})$. ∎

Lemma 4.19. With the setup above, $\dim_{\mathbb{C}}(\mathrm{im}(M^T)_0) = k - \beta(\underline{\ell}, \underline{m})$.

Another computation for $\dim_{\mathbb{C}}(\mathrm{im}(M^T)_0)$ in the $k = 3$ case will be shown in Example 4.22.

Proof. Recall that $(M^T)_{p,q} = 0$ if $p < q$ and $(M^T)_{p,q} = x^{k-1-p+q} y^{i_{p-1}-i_{q-1}}$ if $p \geq q$. Applying M^T to a generic element $\underline{b}$ of $\mathbf{J}(1)_0$ (cf. Lemma 4.15) is therefore

$$\begin{aligned}
(M^T(\underline{b}))_p &= \sum_{q=1}^{k} (M^T)_{p,q} (\underline{b})_q \\
&= \sum_{q=1}^{k} (M^T)_{p,q} \left(\sum_{r=1}^{k} b_{qr} x^{k-r} y^{i_{q-1}-j_k-1+q-r} \right) \\
&= \sum_{q=1}^{p} x^{k-1-p+q} y^{i_{p-1}-i_{q-1}} \left(\sum_{r=1}^{k} b_{qr} x^{k-r} y^{i_{q-1}-j_k-1+q-r} \right) \\
&= \sum_{q=1}^{p} \sum_{r=1}^{k} b_{qr} x^{2k-1-p+q-r} y^{i_{p-1}-j_k-1+q-r}.
\end{aligned}$$

Recall that $x^k = 0$ and so for a term $x^{2k-1-p+q-r}$ to be nonzero, it must be that

$$2k - 1 - p + q - r \leq k - 1 \iff r \geq k - p + q.$$

Thus, we may write

$$(M^T(\underline{b}))_p = \sum_{q=1}^{p} \sum_{r=k-p+q}^{k} b_{qr} x^{2k-1-p+q-r} y^{i_{p-1}-j_k-1+q-r}. \tag{4.4}$$

Now set $s = k + q - r$. Then, since $r \leq k$, we have $s = k + q - r \geq k + q - k = q$. Moreover, since $r \geq k - p + q$, we have $s = k + q - r \leq k + q - (k - p + q) = p$. Thus, we may reindex (4.4) to get

$$\begin{aligned}
(M^T(\underline{b}))_p &= \sum_{q=1}^{p} \sum_{s=q}^{p} b_{qk+q-s} x^{k-1-p+s} y^{i_{p-1}-j_k-1-k+s} \\
&= \sum_{s=1}^{p} \sum_{q=1}^{s} b_{qk+q-s} x^{k-1-p+s} y^{i_{p-1}-j_k-1-k+s} \\
&= \sum_{s=1}^{p} x^{k-1-p+s} y^{i_{p-1}-j_k-1-k+s} \left(\sum_{q=1}^{s} b_{qk+q-s} \right).
\end{aligned}$$

Notice that for each $1 \leq s \leq k$, the complex number $\gamma_s := \sum_{q=1}^{s} b_{q,k+q-s}$ appears as a coefficient in the terms $(M^T(\underline{b}))_p$ for $p = s, \ldots, k$, and the γ_s are mutually independent, as none of the b_{ij} appear as a summand in more than one γ_s. In particular, we may write $M^T(\underline{b})$ as

$$\gamma_1 \begin{pmatrix} x^{k-1} y^{-j_k-k} \\ x^{k-2} y^{i_1-j_k-k} \\ \vdots \\ x y^{i_{k-2}-j_k-k} \\ y^{i_{k-1}-j_k-k} \end{pmatrix} + \gamma_2 \begin{pmatrix} 0 \\ x^{k-1} y^{i_1-j_k-(k-1)} \\ \vdots \\ x^2 y^{i_{k-2}-j_k-(k-1)} \\ x y^{i_{k-1}-j_k-(k-1)} \end{pmatrix} + \cdots + \gamma_k \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ x^{k-1} y^{i_{k-1}-j_k-1} \end{pmatrix},$$

and thus the dimension of $\mathrm{im}(M^T)_0$ is the number of these vectors whose corresponding coefficient γ_s may be nonzero. But γ_s may be nonzero if and only if at least one of the b_{qk+q-s} for $q = 1, \ldots, s$ may be nonzero and so the dimension of $\mathrm{im}(M^T)_0$ is the number of upper diagonals in the coefficient matrix

$$\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1k-1} & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k-1} & b_{2k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{k-11} & b_{k-12} & \cdots & b_{k-1k-1} & b_{k-1k} \\ b_{k1} & b_{k2} & \cdots & b_{kk-1} & b_{kk} \end{pmatrix} \tag{4.5}$$

where at least one coefficient along that diagonal can be nonzero. Equivalently, the dimension $\dim_{\mathbb{C}}(\mathrm{im}(M^T)_0)$ is precisely k minus the number of upper diagonals where all the coefficients must be zero. Recall that b_{ij} can be nonzero if and only if $\ell_{k+1-i} < m_{k+1-j}$ if and only if $B(\underline{\ell}, \underline{m})_{k+1-i,k+1-j}$ is shaded. Thus, the upper diagonal D_p^+ in the matrix (4.5) has to be all zero if and only if the lower diagonal D_p^- in $B(\underline{\ell}, \underline{m})_{i,j}$ is completely unshaded. Hence, the number of upper diagonals in (4.5) along which all entries have to be zero is the same as the number of lower diagonals in $B(\underline{\ell}, \underline{m})$ that are completely below the staircase path that, by definition, is $\beta(\underline{\ell}, \underline{m})$. ■

We are now ready to prove Theorem 4.10.

Proof of Theorem 4.10. As explained in Section 4.2.2, $\dim_{\mathbb{C}}(\mathrm{Ext}^1(I(\underline{\ell}), I(\underline{m})))$ is calculated as

$$\dim_{\mathbb{C}}(\ker(M^T)_0) - \dim_{\mathbb{C}}(\mathrm{im}(N^T)_0),$$

which by rank-nullity is equal to

$$(\dim_{\mathbb{C}}(\mathbf{J}(1)_0) - \dim_{\mathbb{C}}((M^T)_0)) - (\dim_{\mathbb{C}}(\mathbf{J}_0) - \dim_{\mathbb{C}}(\ker(N^T)_0)).$$

Lemma 4.17 shows

$$\dim_{\mathbb{C}}(\mathbf{J}(1)_0) - \dim_{\mathbb{C}}(\mathbf{J}_0) = -|\underline{\ell} \cap \underline{m}|,$$

and Lemmas 4.18 and Lemma 4.19 respectively show that

$$\dim_{\mathbb{C}}(\ker(N^T)_0) = \alpha(\underline{\ell}, \underline{m}) \quad \text{and} \quad \dim_{\mathbb{C}}(\mathrm{im}(M^T)_0) = k - \beta(\underline{\ell}, \underline{m}),$$

so combining all of these gives

$$\dim_{\mathbb{C}}(\mathrm{Ext}^1(I(\underline{\ell}), I(\underline{m}))) = \beta(\underline{\ell}, \underline{m}) + \alpha(\underline{\ell}, \underline{m}) - k - |\underline{\ell} \cap \underline{m}|.$$

■

This gives all we need to prove our main result in the special case when $\underline{\ell}$ and $\underline{m}$ are disjoint.

Corollary 4.20. Given two disjoint k-subsets $\underline{\ell}$ and $\underline{m}$, $\dim_{\mathbb{C}}(\mathrm{Ext}^1(I(\underline{\ell}), I(\underline{m}))) = 0$ if and only if $\underline{\ell}$ and $\underline{m}$ are noncrossing.

Proof. Since $\underline{\ell}$ and $\underline{m}$ are disjoint, $|\underline{\ell} \cap \underline{m}| = 0$ and so by Theorem 4.10

$$\dim_{\mathbb{C}}(\mathrm{Ext}^1(I(\underline{\ell}), I(\underline{m}))) = \beta(\underline{\ell}, \underline{m}) + \alpha(\underline{\ell}, \underline{m}) - k.$$

Thus, $\mathrm{Ext}^1(I(\underline{\ell}), I(\underline{m})) = 0$ if and only if $\beta(\underline{\ell}, \underline{m}) + \alpha(\underline{\ell}, \underline{m}) = k$ which holds if and only if $\underline{\ell}$ and $\underline{m}$ are noncrossing by Lemma 4.9. ■

4.3 The $k = 3$ case

In this subsection, we will illustrate the calculations in Lemma 4.18 and Lemma 4.19 in the $k = 3$ case.

The following example will demonstrate Lemma 4.18, showing that $\dim_{\mathbb{C}}(\ker(N^T)_0) = \alpha(\underline{\ell}, \underline{m})$.

Example 4.21 ($k = 3$ example). Applying N^T to a generic element $\underline{a}$ of $\mathbf{J} = J(2) \oplus J(1 - i_1) \oplus J(-i_2)$ gives

$$N^T(\underline{a}) = \begin{pmatrix} a_{12}x^2y^{-j_3-1} + a_{13}xy^{-j_3-2} \\ (a_{22} - a_{11})x^2y^{i_1-j_3} + (a_{23} - a_{12})xy^{i_1-j_3-1} - a_{13}y^{i_1-j_3-2} \\ (a_{32} - a_{21})x^2y^{i_2-j_3+1} + (a_{33} - a_{22})xy^{i_2-j_3} - a_{23}y^{i_2-j_3-1} \end{pmatrix}.$$

In particular, $\underline{a}$ lies in $\ker(N^T)_0$ if and only if

$$a_{11} = a_{22} = a_{33}, \; a_{32} = a_{21}, \text{ and } a_{12} = a_{13} = a_{23} = 0.$$

Equivalently, in the matrix of coefficients

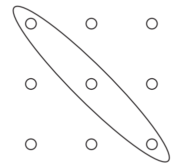
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

all the entries above the main diagonal must be zero, and those connected by a line must all be equal. Thus, the dimension of $\ker(N^T)_0$ is at most three, with possible basis vectors corresponding to each of these lines:

$$\begin{pmatrix} x^2y^{-j_3} \\ xy^{i_1-j_3} \\ y^{i_2-j_3} \end{pmatrix} \in \ker(N^T)_0 \iff a_{11}, a_{22}, a_{33} \text{ can all be nonzero}$$

$$\iff \ell_1 \le m_1, \ell_2 \le m_2, \ell_3 \le m_3,$$

which by definition is if and only if, in $A(\underline{\ell},\underline{m})$, all those vertices in diagonal D_3^+, circled below,

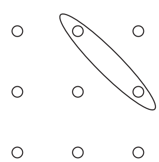


are shaded, or equivalently, this diagonal lies completely above the corresponding staircase path. Similarly,

$$\begin{pmatrix} 0 \\ x^2y^{i_1-j_3+1} \\ xy^{i_2-j_3+1} \end{pmatrix} \in \ker(N^T)_0 \iff a_{21}, a_{32} \text{ can both be nonzero}$$

$$\iff \ell_2 \le m_3, \ell_1 \le m_2,$$

which by definition is if and only if, in $A(\underline{\ell},\underline{m})$, all those vertices in diagonal D_2^+, circled below,

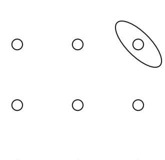


are shaded, or equivalently, this diagonal lies completely above the corresponding staircase path. And finally,

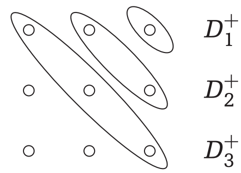
$$\begin{pmatrix} 0 \\ 0 \\ x^2y^{i_2-j_3+2} \end{pmatrix} \in \ker(N^T)_0 \iff a_{31} \text{ can be nonzero}$$

$$\iff \ell_1 \le m_3,$$

which by definition is if and only if, in $A(\underline{\ell},\underline{m})$, all those vertices in the circled diagonal D_1^+



are shaded, or equivalently, this diagonal lies completely above the corresponding staircase path. In other words, the dimension of $\ker(N^T)_0$ is precisely the number of upper diagonals



which lie completely above the staircase path in $A(\underline{\ell}, \underline{m})$, which by definition is $\alpha(\underline{\ell}, \underline{m})$.

This example will show the statement $\dim_{\mathbb{C}}(\mathrm{im}(M^T)_0) = k - \beta(\underline{\ell}, \underline{m})$ in Lemma 4.19 in the $k = 3$ case.

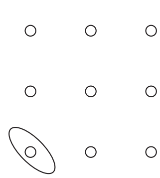
Example 4.22 ($k = 3$ example). Applying M^T to a generic element $\underline{b}$ of $\mathbf{J}(1) = J(3) \oplus J(2 - i_1) \oplus J(1 - i_2)$ gives

$$\begin{aligned} M^T(\underline{b}) &= \begin{pmatrix} b_{13}x^2y^{-j_3-3} \\ (b_{12}+b_{23})x^2y^{i_1-j_3-2} + b_{13}xy^{i_1-j_3-3} \\ (b_{11}+b_{22}+b_{33})x^2y^{i_2-j_3-1} + (b_{12}+b_{23})xy^{i_2-j_3-2} - b_{13}y^{i_2-j_3-3} \end{pmatrix} \\ &= b_{13}\begin{pmatrix} x^2y^{-j_3-3} \\ xy^{i_1-j_3-3} \\ y^{i_2-j_3-3} \end{pmatrix} + (b_{12}+b_{23})\begin{pmatrix} 0 \\ x^2y^{i_1-j_3-2} \\ xy^{i_2-j_3-2} \end{pmatrix} + (b_{11}+b_{22}+b_{33})\begin{pmatrix} 0 \\ 0 \\ x^2y^{i_2-j_3-1} \end{pmatrix}. \end{aligned}$$

From this, we see that the dimension of $\mathrm{im}(M^T)_0$ is at most three, with possible basis vectors:

$$\begin{aligned} \begin{pmatrix} x^2y^{-j_3-3} \\ xy^{i_1-j_3-3} \\ y^{i_2-j_3-3} \end{pmatrix} \in \mathrm{im}(M^T)_0 &\iff b_{13} \text{ can be nonzero} \\ &\iff \ell_3 < m_1, \end{aligned}$$

which by definition is if and only if, in $B(\underline{\ell}, \underline{m})$, at least one of the vertices in the diagonal D_1^-, circled below,

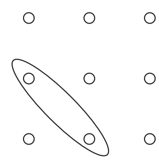


lies above the corresponding staircase path. Similarly,

$$\begin{pmatrix} 0 \\ x^2 y^{i_1 - j_3 - 2} \\ x y^{i_2 - j_3 - 2} \end{pmatrix} \in \operatorname{im}(M^T)_0 \iff \text{at least one of } b_{12}, b_{23} \text{ can be nonzero}$$

$$\iff \ell_3 < m_2 \text{ or } \ell_2 < m_1,$$

which by definition is if and only if, in $B(\underline{\ell}, \underline{m})$, at least one of the vertices in the diagonal D_2^-, circled below,

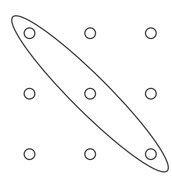


lies above the corresponding staircase path. And finally,

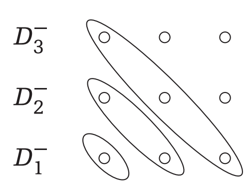
$$\begin{pmatrix} 0 \\ 0 \\ x^2 y^{i_2 - j_3 - 1} \end{pmatrix} \in \operatorname{im}(M^T)_0 \iff \text{at least one of } b_{11}, b_{22}, b_{33} \text{ can be nonzero}$$

$$\iff \ell_3 < m_3 \text{ or } \ell_2 < m_2 \text{ or } \ell_1 < m_1,$$

which by definition is if and only if, in $B(\underline{\ell}, \underline{m})$, at least one of the vertices in the diagonal D_3^-, circled below,



lies above the corresponding staircase path. In other words, the dimension of $\operatorname{im}(M^T)_0$ is the number of the circled diagonals in



which lie partially above the staircase path in $B(\underline{\ell}, \underline{m})$. Or equivalently, the dimension of $\operatorname{im}(M^T)_0$ is 3 minus the number of lower diagonals that lie completely below the staircase path in $B(\underline{\ell},\underline{m})$, which by definition is $\beta(\underline{\ell},\underline{m})$. Hence, $\dim_{\mathbb{C}}(\mathrm{im}(M^T)_0) = 3 - \beta(\underline{\ell},\underline{m})$.

Example 4.23. Returning to Example 4.13, we compute $\dim_{\mathbb{C}}(\mathrm{Ext}^1(I,J))$ for the graded ideals $I = (x^2, xy, y^2)$ and $J = (x^2, x, y^2)(-1)$ of $R = \mathbb{C}[x,y]/(x^3)$. Recall that these ideals correspond to the 3-subsets $\underline{\ell} = (-2, 0, 2)$ and $\underline{m} = (-1, 2, 3)$, and we may compute that $\alpha(\underline{\ell},\underline{m}) = 3$, $\beta(\underline{\ell},\underline{m}) = 2$ and $|\underline{\ell} \cap \underline{m}| = 1$. Using Theorem 4.10, this shows that

$$\dim_{\mathbb{C}}(\mathrm{Ext}^1(I,J)) = 3 + 2 - 3 - 1 = 1.$$

We see here that $\dim_{\mathbb{C}}(\mathrm{Ext}^1(I,J)) \neq 0$ that coincides with the fact that there is a crossing

$$\ell_1 < m_1 < \ell_2 < m_3.$$

4.4 Reduction to disjoint case

Return now to the general setting of $k \geq 2$. The dimension formula for Ext^1 given in Theorem 4.10 allowed us to directly prove Theorem 4.1 in the case where $\underline{\ell}$ and $\underline{m}$ are disjoint. In this final section, we complete the proof of Theorem 4.1 by showing that when $\underline{\ell}$ and $\underline{m}$ are not disjoint, we may reduce the problem to a setting where they are.

Suppose that $\underline{\ell}$ and $\underline{m}$ are k-subsets such that $\underline{\ell} \cap \underline{m}$ is non-empty. In particular, suppose that we have a pair (i,j) such that $\ell_i = m_j$. Note that this corresponds to a difference between $A(\underline{\ell},\underline{m})$ and $B(\underline{\ell},\underline{m})$; $A(\underline{\ell},\underline{m})_{i,j}$ will be shaded but $B(\underline{\ell},\underline{m})_{i,j}$ will not be. We may form two new $(k-1)$-subsets $\underline{\widetilde{\ell}}$ and $\underline{\widetilde{m}}$ by deleting $\ell_i = m_j$ from $\underline{\ell}$ and $\underline{m}$, respectively:

$$\widetilde{\ell}_p = \begin{cases} \ell_p & \text{if } 1 < p < i \\ \ell_{p+1} & \text{if } i \leq p \leq k-1 \end{cases} \quad \text{and} \quad \widetilde{m}_q = \begin{cases} m_q & \text{if } 1 < q < j \\ m_{q+1} & \text{if } j \leq q \leq k-1. \end{cases}$$

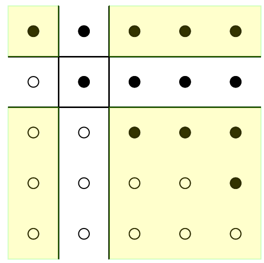
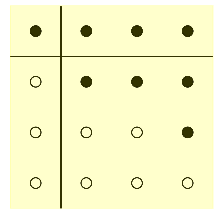
Example 4.24. ($k = 5$) Taking $\underline{\ell}$ and $\underline{m}$ with

$$\ell_1 < m_1 < \ell_2 = m_2 < \ell_3 < m_3 < m_4 < \ell_4 < m_5 < \ell_5$$

and removing $\ell_2 = m_2$ gives $\underline{\widetilde{\ell}}$ and $\underline{\widetilde{m}}$ satisfying

$$\widetilde{\ell}_1 < \widetilde{m}_1 < \widetilde{\ell}_2 < \widetilde{m}_2 < \widetilde{m}_3 < \widetilde{\ell}_3 < \widetilde{m}_4 < \widetilde{\ell}_4,$$

which have $A(\underline{\ell},\underline{m})$ and $A(\underline{\widetilde{\ell}},\underline{\widetilde{m}})$ as follows:

Lemma 4.25. Given k-subsets $\underline{\ell}$ and $\underline{m}$ as above, $A(\underline{\widetilde{\ell}},\underline{\widetilde{m}})$ is obtained from $A(\underline{\ell},\underline{m})$ be deleting row i and column j. Analogously, $B(\underline{\widetilde{\ell}},\underline{\widetilde{m}})$ is obtained from $B(\underline{\ell},\underline{m})$ be deleting row i and column j.

Proof. By deleting row i and column j in $A(\underline{\ell},\underline{m})$, we split $A(\underline{\ell},\underline{m})$ into (up to) four regions:

- $A(\underline{\ell},\underline{m})_{p,q}$ where $1 \leq p < i$ and $1 \leq q < j$;
- $A(\underline{\ell},\underline{m})_{p,q}$ where $1 \leq p < i$ and $j < q \leq k$;
- $A(\underline{\ell},\underline{m})_{p,q}$ where $i < p \leq k$ and $1 \leq q < j$;
- $A(\underline{\ell},\underline{m})_{p,q}$ where $i < p \leq k$ and $j < q \leq k$.

In the 1st case, we wish to identify $A(\underline{\ell},\underline{m})_{p,q}$ with $A(\underline{\widetilde{\ell}},\underline{\widetilde{m}})_{p,q}$. In this region, we have $\ell_p = \widetilde{\ell}_p$ and $m_q = \widetilde{m}_q$ and hence

$$\ell_p \leq m_q \iff \widetilde{\ell}_p \leq \widetilde{m}_q$$

or in other words, $A(\underline{\ell},\underline{m})_{p,q}$ is filled if and only if $A(\underline{\widetilde{\ell}},\underline{\widetilde{m}})_{p,q}$ is filled, as required. In the 2nd region, we wish to identify $A(\underline{\ell},\underline{m})_{p,q}$ with $A(\underline{\widetilde{\ell}},\underline{\widetilde{m}})_{p,q-1}$. In this region, we have we have $\ell_p = \widetilde{\ell}_p$ but since $q-1 \geq j$, we also have $m_q = \widetilde{m}_{q-1}$. Thus,

$$\ell_p \leq m_q \iff \widetilde{\ell}_p \leq \widetilde{m}_{q-1},$$

or in other words, $A(\underline{\ell},\underline{m})_{p,q}$ is filled if and only if $A(\underline{\widetilde{\ell}},\underline{\widetilde{m}})_{p,q-1}$ is filled, as required. In the 3rd region, we wish to identify $A(\underline{\ell},\underline{m})_{p,q}$ with $A(\underline{\widetilde{\ell}},\underline{\widetilde{m}})_{p-1,q}$. In this region, we have we have $m_q = \widetilde{m}_q$ but since $p-1 \geq i$, we also have $\ell_p = \widetilde{\ell}_{p-1}$. Thus,

$$\ell_p \leq m_q \iff \widetilde{\ell}_{p-1} \leq \widetilde{m}_q,$$

or in other words, $A(\underline{\ell},\underline{m})_{p,q}$ is filled if and only if $A(\underline{\widetilde{\ell}},\underline{\widetilde{m}})_{p-1,q}$ is filled, as required. In the final region, we wish to identify $A(\underline{\ell},\underline{m})_{p,q}$ with $A(\underline{\widetilde{\ell}},\underline{\widetilde{m}})_{p-1,q-1}$. In this region, we have we have $\ell_p = \widetilde{\ell}_{p-1}$ and $m_q = \widetilde{m}_{q-1}$. Thus,

$$\ell_p \leq m_q \iff \widetilde{\ell}_{p-1} \leq \widetilde{m}_{q-1}$$

or in other words, $A(\underline{\ell},\underline{m})_{p,q}$ is filled if and only if $A(\underline{\widetilde{\ell}},\underline{\widetilde{m}})_{p-1,q-1}$ is filled, as required. The proof for $B(\underline{\ell},\underline{m})$ is exactly the same with all the inequalities changed to strict inequalities. ■

Lemma 4.26. Suppose $\underline{\ell}$ and $\underline{m}$ are k-subsets with $\ell_i = m_j$. Then $j - i \leq k - \alpha(\underline{\ell},\underline{m})$ and $i - j \leq k - \beta(\underline{\ell},\underline{m})$.

Proof. Suppose for a contradiction that $j-i > k-\alpha(\underline{\ell},\underline{m})$. Since $\alpha(\underline{\ell},\underline{m}) \leq k$, this implies $j - i > 0$, and hence we have $j > 1$. Then,

$$\begin{aligned} j-i > k-\alpha(\underline{\ell},\underline{m}) &\iff (j-i)-1 > (k-\alpha(\underline{\ell},\underline{m}))-1 \\ &\iff (j-1)-i \geq k-\alpha(\underline{\ell},\underline{m}). \end{aligned}$$

This shows $(i, j-1)$ lies on the diagonal D_p^+ for some $p \leq \alpha(\underline{\ell},\underline{m})$, and so, by definition of $\alpha(\underline{\ell},\underline{m})$, the final line implies $A(\underline{\ell},\underline{m})_{i,j-1}$ is shaded. This implies $\ell_i \leq m_{j-1}$ that further implies $m_j = \ell_i \leq m_{j-1}$, which is a contradiction. The proof for $i - j \leq k - \beta(\underline{\ell},\underline{m})$ is similar. ■

Lemma 4.27. With the setup above $\alpha(\underline{\widetilde{\ell}},\underline{\widetilde{m}}) \geq \alpha(\underline{\ell},\underline{m}) - 1$.

Proof. First, note that since $\ell_i = m_j$, there is at least one vertex above the staircase path in $A(\underline{\ell},\underline{m})$ and hence $\alpha(\underline{\ell},\underline{m}) > 0$. Also, by the definition of $\alpha(\underline{\ell},\underline{m})$, we know that for all $s \leq \alpha(\underline{\ell},\underline{m})$ the diagonal D_s^+ is completely shaded, or equivalently,

$$\text{for all pairs } (p,q) \text{ with } 1 \leq p,q \leq k \text{ and } q-p \geq k-\alpha(\underline{\ell},\underline{m}), \text{ we have } \ell_p \leq m_q. \tag{4.6}$$

We will prove that in $A(\underline{\widetilde{\ell}},\underline{\widetilde{m}})$, the diagonal $D^+_{\alpha(\underline{\ell},\underline{m})-1}$ lies completely above the staircase path from which the result follows. Take a pair (p,q) on this diagonal, that is, with $1 \leq p,q \leq k-1$ and $q-p = k-\alpha(\underline{\ell},\underline{m}) = (k-1)-(\alpha(\underline{\ell},\underline{m})-1)$. Using (4.6), we see that

$$\ell_p \leq m_q \quad \text{and} \quad \ell_{p+1} \leq m_{q+1}, \tag{4.7}$$

where the latter holds since $1 \le p+1, q+1 \le k$ and $(q+1)-(p+1) = q-p \ge k-\alpha(\underline{\ell}, \underline{m})$. Now, the pair (p,q) must lie in one of four regions:

1. $1 \le p < i$ and $1 \le q < j$;
2. $i \le p \le k-1$ and $1 \le q < j$;
3. $1 \le p < i$ and $j \le q \le k-1$;
4. $i \le p \le k-1$ and $j \le q \le k-1$.

If (p,q) lies in the 1st region, then $\ell_p = \widetilde{\ell}_p$ and $m_q = \widetilde{m}_q$. Then, using (4.7),

$$\widetilde{\ell}_p = \ell_p \le m_q = \widetilde{m}_q.$$

If (p,q) lies in the 2nd region, then, using Lemma 4.26,

$$q-p < j-i \le k-\alpha(\underline{\ell}, \underline{m}),$$

and thus no such (p,q) lies on the diagonal $D_{\alpha(\underline{\ell},\underline{m})-1}$ in $A(\underline{\widetilde{\ell}}, \underline{\widetilde{m}})$. If (p,q) lies in the 3rd region, then $\ell_p = \widetilde{\ell}_p$ and $m_{q+1} = \widetilde{m}_q$. Then, using (4.7) and that $m_q < m_{q+1}$ shows that

$$\widetilde{\ell}_p = \ell_p \le m_q < m_{q+1} = \widetilde{m}_q.$$

If (p,q) lies in the 4th region, then $\ell_{p+1} = \widetilde{\ell}_p$ and $m_{q+1} = \widetilde{m}_q$. Then, using (4.7),

$$\widetilde{\ell}_p = \ell_{p+1} \le m_{q+1} = \widetilde{m}_q.$$

Thus, we have shown that in $A(\underline{\widetilde{\ell}}, \underline{\widetilde{m}})$, all (p,q) on the diagonal $D^+_{\alpha(\underline{\ell},\underline{m})-1}$ lie above the staircase path as required. ■

Lemma 4.28. With the setup above $\alpha(\underline{\widetilde{\ell}}, \underline{\widetilde{m}}) \le \alpha(\underline{\ell}, \underline{m}) - 1$.

Proof. Suppose that $\alpha(\underline{\ell}, \underline{m}) = k$. Since $\underline{\widetilde{\ell}}$ and $\underline{\widetilde{m}}$ are $(k-1)$-subsets, by definition, $\alpha(\underline{\widetilde{\ell}}, \underline{\widetilde{m}}) \le k-1$ and hence $\alpha(\underline{\widetilde{\ell}}, \underline{\widetilde{m}}) \le \alpha(\underline{\ell}, \underline{m}) - 1$ is trivial in this case. Now suppose $\alpha(\underline{\ell}, \underline{m}) < k$. Since $\alpha(\underline{\ell}, \underline{m})$ is maximal, there exists $(p,q) \in D^+_{\alpha(\underline{\ell},\underline{m})+1}$ (i.e., $1 \le p, q \le k$ and $q-p = k-\alpha(\underline{\ell}, \underline{m}) - 1$) such that $A(\underline{\ell}, \underline{m})_{p,q}$ lies below the staircase path or equivalently, such that $\ell_p > m_q$. Since $A(\underline{\ell}, \underline{m})_{i,j}$ lies above the staircase (as $\ell_i = m_j$), Lemma 4.5 shows that $A(\underline{\ell}, \underline{m})_{s,t}$ lies above the path whenever we have both $s \le i$ and $t \ge j$. Thus, we must have $p > i$ or $q < j$. Suppose $p > i$, or equivalently $p \geq i+1$. Then,

$$\begin{aligned}
q &= p + k - \alpha(\underline{\ell}, \underline{m}) - 1 \\
&\geq (i+1) + (k - \alpha(\underline{\ell}, \underline{m})) - 1 && \text{(since } p \geq i+1\text{)} \\
&\geq (i+1) + (j-i) - 1 && \text{(by Lemma 4.26)} \\
&= j.
\end{aligned}$$

So we have $p-1 \geq i$ and $q \geq j$. Thus, $\widetilde{\ell}_{p-1} = \ell_p$ and either

- $q = j$, and then $q-1 < j$ and so $\widetilde{m}_{q-1} = m_{q-1} \leq m_q$;
- $q > j$, and then $\widetilde{m}_{q-1} = m_q$.

Hence, we have $\widetilde{\ell}_{p-1} = \ell_p$ and $\widetilde{m}_{q-1} \leq m_q$, and so,

$$\widetilde{\ell}_{p-1} = \ell_p > m_q \geq \widetilde{m}_{q-1},$$

where the strict inequality holds as $A(\underline{\ell}, \underline{m})_{p,q}$ lies below the staircase path. In particular, the pair $(p-1, q-1)$ satisfies $(q-1)-(p-1) = q-p = k - \alpha(\underline{\ell}, \underline{m}) - 1$ and $A(\underline{\widetilde{\ell}}, \underline{\widetilde{m}})_{p-1,q-1}$ lies below the staircase. If $q < j$, or equivalently $q \leq j-1$, then,

$$\begin{aligned}
p &= q - k + \alpha(\underline{\ell}, \underline{m}) + 1 \\
&\leq (j-1) - (k - \alpha(\underline{\ell}, \underline{m})) + 1 && \text{(since } q \leq j-1\text{)} \\
&\leq (j-1) - (j-i) + 1 && \text{(by Lemma 4.26)} \\
&= i.
\end{aligned}$$

So we have $q < j$ and $p \leq i$. Thus, $\widetilde{m}_q = m_q$ and either

- $p = i$, and then $\widetilde{\ell}_p = \ell_{p+1} > \ell_p$;
- $p < i$, and then $\widetilde{\ell}_p = \ell_p$.

Hence, we have $\widetilde{\ell}_p \geq \ell_p$ and $\widetilde{m}_q = m_q$, and so,

$$\widetilde{\ell}_p \geq \ell_p > m_q = \widetilde{m}_q,$$

where the strict inequality holds as $A(\underline{\ell}, \underline{m})_{p,q}$ lies below the staircase path. In particular, the pair (p, q) satisfy $q - p = k - \alpha(\underline{\ell}, \underline{m}) - 1$ and $A(\underline{\widetilde{\ell}}, \underline{\widetilde{m}})_{p,q}$ lies below the staircase.

Moreover, if $p = k$, then since we know $q < j \leq k$, we have $k - \alpha(\underline{\ell}, \underline{m}) - 1 = q - p < 0$ contradicting $\alpha(\underline{\ell}, \underline{m}) < k$. Thus, we must have $1 \leq p, q \leq k - 1$.

Thus, we have shown that there exists $1 \leq p, q \leq k-1$ such that $q - p = (k-1) - \alpha(\underline{\ell}, \underline{m})$ and $A(\underline{\widetilde{\ell}}, \underline{\widetilde{m}})_{p,q}$ lies below the staircase. Equivalently, we have shown that in $A(\underline{\widetilde{\ell}}, \underline{\widetilde{m}})$, the diagonal $D^+_{\alpha(\underline{\ell}, \underline{m})}$ does not lie completely above the staircase path and thus $\alpha(\underline{\widetilde{\ell}}, \underline{\widetilde{m}}) \leq \alpha(\underline{\ell}, \underline{m}) - 1$. ■

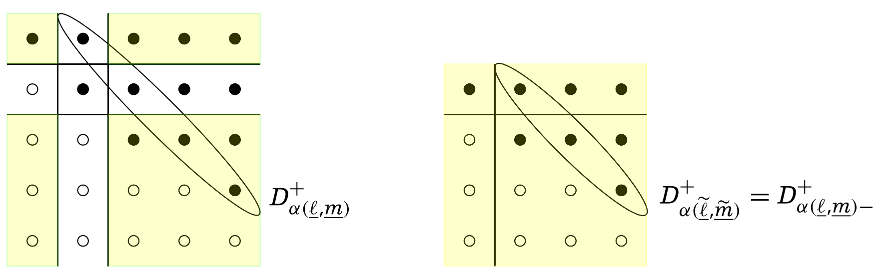
Corollary 4.29. With the setup above $\alpha(\underline{\widetilde{\ell}}, \underline{\widetilde{m}}) = \alpha(\underline{\ell}, \underline{m}) - 1$.

Proof. Combine Lemmas 4.27 and Lemma 4.28. ■

Corollary 4.30. With the setup above $\beta(\underline{\widetilde{\ell}}, \underline{\widetilde{m}}) = \beta(\underline{\ell}, \underline{m}) - 1$.

Proof. Combine Corollary 4.29 and Lemma 4.8. ■

Example 4.31. Continuing Example 4.24, we see that $\alpha(\underline{\ell}, \underline{m}) = 4$ and $\alpha(\underline{\widetilde{\ell}}, \underline{\widetilde{m}}) = 3$:



Corollary 4.32. A pair of k-subsets $\underline{\ell}$ and $\underline{m}$ are noncrossing if and only if

$$\alpha(\underline{\ell}, \underline{m}) + \beta(\underline{\ell}, \underline{m}) - |\underline{\ell} \cap \underline{m}| = k.$$

Proof. Suppose that $\underline{\ell}$ and $\underline{m}$ are disjoint. Then the result is precisely Lemma 4.9 and we are done.

If $\underline{\ell}$ and $\underline{m}$ are not disjoint, suppose that $\ell_i = m_j$. Then, consider the $k-1$-subsets $\underline{\widetilde{\ell}} := \underline{\ell} \setminus \{\ell_i\}$ and $\underline{\widetilde{m}} := \underline{m} \setminus \{m_j\}$. Then,

- $\underline{\widetilde{\ell}}$ and $\underline{\widetilde{m}}$ are noncrossing if and only if $\underline{\ell}$ and $\underline{m}$ are noncrossing;
- $|\underline{\widetilde{\ell}} \cap \underline{\widetilde{m}}| = |\underline{\ell} \cap \underline{m}| - 1$;
- $\alpha(\underline{\widetilde{\ell}}, \underline{\widetilde{m}}) = \alpha(\underline{\ell}, \underline{m}) - 1$ using Corollary 4.29;
- $\beta(\underline{\widetilde{\ell}}, \underline{\widetilde{m}}) = \beta(\underline{\ell}, \underline{m}) - 1$ using Corollary 4.30;

If $t := |\underline{\ell} \cap \underline{m}| = 1$, then $\widetilde{\underline{\ell}}$ and $\widetilde{\underline{m}}$ are disjoint. If not, repeat the process by removing another equality, and continue until you end up with disjoint $(k-t)$-subsets $\widetilde{\underline{\ell}}$ and $\widetilde{\underline{m}}$ such that

- $\widetilde{\underline{\ell}}$ and $\widetilde{\underline{m}}$ are noncrossing if and only if $\underline{\ell}$ and $\underline{m}$ are noncrossing;
- $\alpha(\widetilde{\underline{\ell}}, \widetilde{\underline{m}}) = \alpha(\underline{\ell}, \underline{m}) - t$;
- $\beta(\widetilde{\underline{\ell}}, \widetilde{\underline{m}}) = \beta(\underline{\ell}, \underline{m}) - t$;

Then,

$$\begin{aligned}
\alpha(\underline{\ell}, \underline{m}) + \beta(\underline{\ell}, \underline{m}) - |\underline{\ell} \cap \underline{m}| = k &\iff \alpha(\underline{\ell}, \underline{m}) + \beta(\underline{\ell}, \underline{m}) - t = k \\
&\iff (\alpha(\underline{\ell}, \underline{m}) - t) + (\beta(\underline{\ell}, \underline{m}) - t) = k - t \qquad \text{(rearrange)} \\
&\iff \alpha(\widetilde{\underline{\ell}}, \widetilde{\underline{m}}) + \beta(\widetilde{\underline{\ell}}, \widetilde{\underline{m}}) = k - t \\
&\iff \widetilde{\underline{\ell}} \text{ and } \widetilde{\underline{m}} \text{ are noncrossing} \qquad \text{(by Lemma 4.9)} \\
&\iff \underline{\ell} \text{ and } \underline{m} \text{ are noncrossing.}
\end{aligned}$$

■

We are now ready to prove Theorem 4.1, keeping the notation from the beginning of Section 4.1, where $\underline{\ell} = \underline{\ell}(I)$ and $\underline{m} = \underline{\ell}(J)$.

Proof of Theorem 4.1 Recall that we wish to show $\dim_{\mathbb{C}}(\mathrm{Ext}^1(I, J)) = 0$ if and only if $\underline{\ell}$ and $\underline{m}$ are noncrossing. But this follows directly from Theorem 4.10 and Corollary 4.32. ■

Funding

This work was supported by the Max Planck Institute for Mathematics; the Danish National Research Foundation [DNRF156 to J.A.] the NSF [DMS-1854512 to M.C.]; the AMS Simons Travel Grants; a Marie Skłodowska-Curie fellowship at the University of Leeds [789580 to E.F.]; and the EPSRC [EP/W007509/1 to E.F. and EP/P016294/1 to S.S.].

Acknowledgments

This project started from the WINART2 (Women in Noncommutative Algebra and Representation Theory) workshop, and the authors would like to thank the organisers for this wonderful opportunity. They also thank the London Mathematical Society (WS-1718-03), the University of Leeds, the US National Science Foundation (MS 1900575), the Association for Women in Mathematics (DMS-1500481), and the Alfred P. Sloan foundation for supporting the workshop.

Further thanks go to Alastair King for his keen interest in the project and comments on the 1st draft.

References

[1] Assem, I., G. Dupont, and R. Schiffler. "On a category of cluster algebras." *J. Pure Appl. Algebra* 218, no. 3 (2014): 553–82. https://doi.org/10.1016/j.jpaa.2013.07.005.

[2] August, J., M.-W. Cheung, E. Faber, S. Gratz, and S. Schroll. "Cluster structures for the A_∞ singularity." (2022): preprint arXiv:2205.15344.

[3] Baur, K., A. D. King, and B. R. Marsh. "Dimer models and cluster categories of Grassmannians." *Proc. London Math. Soc. (3)* 113, no. 2 (2016): 213–60. https://doi.org/10.1112/plms/pdw029.

[4] Bruns, W. and J. Herzog. *Cohen–Macaulay Rings*, vol. 39. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 1993.

[5] Buchweitz, R.-O. *Maximal Cohen–Macaulay Modules and Tate Cohomology*, vol. 262. Math. Surveys Monogr.Providence, RI: American Mathematical Society, 2021. With appendices and an introduction by Luchezar L. Avramov, Benjamin Briggs, Srikanth B. Iyengar, and Janina C. Letz.

[6] Buchweitz, R.-O., G.-M. Greuel, and F.-O. Schreyer. "Cohen–Macaulay modules on hypersurface singularities. II." *Invent. Math.* 88, no. 1 (1987): 165–82. https://doi.org/10.1007/BF01405096.

[7] Chen, X.-W. "Three results on Frobenius categories." *Math. Z.* 270, no. 1–2 (2012): 43–58. https://doi.org/10.1007/s00209-010-0785-3.

[8] Eisenbud, D. "Homological algebra on a complete intersection, with an application to group representations." *Trans. Amer. Math. Soc.* 260, no. 1 (1980): 35–64. https://doi.org/10.1090/S0002-9947-1980-0570778-7.

[9] Fomin, S. and A. Zelevinsky. "Cluster algebras. I. Foundations." *J. Amer. Math. Soc.* 15, no. 2 (2002): 497–529. https://doi.org/10.1090/S0894-0347-01-00385-X.

[10] Geiss, C., B. Leclerc, and J. Schröer. "Partial flag varieties and preprojective algebras." *Ann. Inst. Fourier (Grenoble)* 58, no. 3 (2008): 825–76. https://doi.org/10.5802/aif.2371.

[11] Grabowski, J. E. and S. Gratz. "Cluster algebras of infinite rank." *J. Lond. Math. Soc.* 89 (2014): 337–63. With an appendix by Michael Groechenig. https://doi.org/10.1112/jlms/jdt064.

[12] Grabowski, J. E. and S. Gratz. "Graded quantum cluster algebras of infinite rank as colimits." *J. Pure Appl. Algebra* 222, no. 11 (2018): 3395–413. https://doi.org/10.1016/j.jpaa.2017.12.014.

[13] Gratz, S. "Cluster algebras of infinite rank as colimits." *Math. Z.* 281, no. 3–4 (2015): 1137–69. https://doi.org/10.1007/s00209-015-1524-6.

[14] Holm, T. and P. Jørgensen. "On a cluster category of infinite Dynkin type, and the relation to triangulations of the infinity-gon." *Math. Z.* 270, no. 1–2 (2012): 277–95. https://doi.org/10.1007/s00209-010-0797-z.

[15] Huneke, C. "Tight Closure, Parameter Ideals, and Geometry." In *Six Lectures on Commutative Algebra*. Mod. Birkhäuser Class. 187–239. Basel: Birkhäuser, 2010.

[16] Igusa, K. and G. Todorov. "Cluster categories coming from cyclic posets." *Comm. Algebra* 43, no. 10 (2015): 4367–402. https://doi.org/10.1080/00927872.2014.946138.

[17] Iyama, O. and R. Takahashi. “Tilting and cluster tilting for quotient singularities.” *Math. Ann.* 356, no. 3 (2013): 1065–105. https://doi.org/10.1007/s00208-012-0842-9.

[18] Jensen, B. T., A. D. King, and X. Su. “A categorification of Grassmannian cluster algebras.” *Proc. Lond. Math. Soc. (3)* 113, no. 2 (2016): 185–212.

[19] Lusztig, G. “A survey of total positivity.” *Milan J. Math.* 76 (2008): 125–34. https://doi.org/10.1007/s00032-008-0083-2.

[20] Matsumura, H. *Commutative Ring Theory*, vol. 8, 2nd ed. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 1989. Translated from the Japanese by M. Reid.

[21] Paquette, C. and E. Yildirim. “Completions of discrete cluster categories of type $\mathbb{A}$.” *Trans. London Math. Soc.* 8, no. 1 (2021): 35–64. https://doi.org/10.1112/tlm3.12025.

[22] Postnikov, A. “Total positivity, Grassmannians, and networks.” (2006): preprint arXiv:0609764.

[23] Scott, J. S. “Grassmannians and cluster algebras.” *Proc. London Math. Soc. (3)* 92, no. 2 (2006): 345–80.

[24] Weibel, C. A. *An Introduction to Homological Algebra*, vol. 38. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 1994.

[25] Yoshino, Y. *Cohen–Macaulay Modules over Cohen–Macaulay Rings*, vol. 146. Lond. Math. Soc. Lecture Note Ser.Cambridge: Cambridge University Press, 1990.