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An intersection-theoretic proof of the Harer–Zagier formula

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Abstract

We provide an intersection-theoretic formula for the Euler characteristic of the moduli space of smooth curves. This formula reads purely in terms of Hodge integrals, and, as a corollary, the standard calculus of tautological classes gives a new short proof of the Harer–Zagier formula. Our result is based on the Gauss–Bonnet formula, and on the observation that a certain parametrisation of the Ω-class – the Chern class of the universal rth root of the twisted log canonical bundle – provides the Chern class of the log tangent bundle to the moduli space of smooth curves. These Ω-classes have been recently employed in a great variety of enumerative problems. We produce a list of their properties, proving new ones, collecting the properties already in the literature or only known to the experts, and extending some of them.

1. Introduction and results

Let $\mathcal{M}_{g,n}$ be the moduli space of smooth curves of genus g with n labelled and distinct marked points. Moduli spaces of curves are a topic of great interest both within pure algebraic geometry and, arguably even more charmingly, in the relation between algebraic geometry and different branches of mathematics and physics: string theory, mirror symmetry, Gromov–Witten theory, random matrix models, integrable systems and integrable hierarchies, as well as recent methods such as topological recursion in the sense of Eynard and Orantin.

The (orbifold) Euler characteristic $\chi_{g,n}$ of $\mathcal{M}_{g,n}$ represents one of the most fundamental invariants of these spaces and is computed by the famous Harer–Zagier formula.

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Theorem 1.1 (Harer–Zagier formula, [HZ86]). *The Euler characteristic of $\mathcal{M}_{g,n}$ is given by*

$$\chi_{g,n} = \begin{cases} (-1)^{n-3}(n-3)!\,, & g=0\,,\ n\geqslant 3\,,\\ (-1)^n\dfrac{(n-1)!}{12}\,, & g=1\,,\ n\geqslant 1\,,\\ (-1)^n(2g-3+n)!\dfrac{B_{2g}}{2g(2g-2)!}\,, & g\geqslant 2\,,\ n\geqslant 0\,, \end{cases}$$

where the B_{2g} are Bernoulli numbers.

This formula has been proved with quite different methods: via the combinatorics of polygons/graphs on surfaces and random matrix models, via representation theory, via topological recursion, via semi-infinite wedge formalism; see for instance [Lew19, MPS21] and references therein.

We provide the first expression of $\chi_{g,n}$ in terms of intersection theory of explicit tautological classes on the moduli spaces of stable curves, which is the main result of the paper.

Theorem 1.2. *The Euler characteristic $\chi_{g,n}$ is given by the following Hodge integrals:*

$$\chi_{g,n} = (-1)^{3g-3+n}\sum_{\ell\geqslant 0}\frac{1}{\ell!}\sum_{i=0}^{g}\int_{\overline{\mathcal{M}}_{g,n+\ell}}\frac{\lambda_i}{\prod_{j=1}^{\ell}(1+\psi_{n+j})}\psi_{n+1}^2\cdots\psi_{n+\ell}^2\,. \tag{1.1}$$

The product in the integrand is 1 when $\ell=0$. Beginning with this formula, a short manipulation of linear Hodge integrals provides the Harer–Zagier formula.

Corollary 1.3. *The Harer–Zagier formula holds true.*

The techniques used by Harer and Zagier in [HZ86] required $n>0$, and they deduced the $n=0$ case via an exact sequence. The formula (1.1) applies equally for all $n\geqslant 0$.

1.1 Strategy of the proof

In fact, Theorem 1.2 relies on an application of the Gauss–Bonnet theorem: the Euler characteristic of $\mathcal{M}_{g,n}$ is the integral over $\overline{\mathcal{M}}_{g,n}$ of the Chern class of the logarithmic tangent bundle. Such a Chern class is a specific instance of the Ω-classes that we now describe.

Let $\Omega(r,s;a)$ be the total Chern class of the direct image of the universal rth root of the twisted log canonical bundle $\omega_{\log}^{\otimes s}\big(-\sum_i a_i p_i\big)$ (see Section 2 for a precise definition). A formula for these Ω-classes in terms of ψ-classes, κ-classes, and boundary divisors was given by Bini [Bin03] for $r=1$ and by Chiodo [Chi08] for any r. The Ω-class $\Omega(r,s;a)$ is shown to form a semisimple cohomological field theory of rank r in [LPSZ17]. In the case $r=1$, the class $\Omega(1,s;0)$ can be simply expressed using the Givental action (for an overview on cohomological field theories and the definition of the Givental group action, we refer to [Pan19]). We give here their form explicitly.

Lemma 1.4. *For $V=\mathbb{Q}$, the vector space underlying the Frobenius algebra with metric $\eta(1,1)=1$, consider the R-matrix and translation given by*

$$R(u) = \exp\left(\sum_{m\geqslant 1}\frac{B_{m+1}}{m(m+1)}u^m\right),$$

$$T(u) = \begin{cases} u\left[1 - \prod_{j=1}^{s-1}(1+ju)\right] = -\sum_{\ell \geqslant 1} \mathsf{s}(s, s-\ell)u^{\ell+1} & \text{if } s > 1\,, \\ 0 & \text{if } s = 0 \text{ or } 1\,, \\ u\left[1 - \prod_{j=1}^{-s}\frac{1}{(1-ju)}\right] = -\sum_{\ell \geqslant 1} \mathsf{S}(\ell - s, -s)u^{\ell+1} & \text{if } s < 0\,. \end{cases}$$

Here $\mathsf{s}(n,m)$ and $\mathsf{S}(n,m)$ are Stirling numbers of the first and second kind, respectively, and the B_{m+1} are Bernoulli numbers. The class $\Omega(1,s;0)$ is then obtained by applying the unit-preserving R-matrix and the translation T on the trivial cohomological field theory $1 \in H^0\big(\overline{\mathcal{M}}_{g,n}\big)$:

$$\Omega(1,s;0) = TR.1\,,$$

where all classes are evaluated on the generator $1 \in V$. Equivalently, $\Omega(1,s;0)$ is obtained by applying the translation T on the Hodge cohomological field theory $\Lambda(-1) = \sum_{i=1}^{g}(-1)^i\lambda_i \in H^\bullet\big(\overline{\mathcal{M}}_{g,n}\big)$:

$$\Omega(1,s;0) = T\Lambda(-1)\,.$$

The class $\Omega(1,-1;0)$ is the Chern class of the log tangent bundle to $\mathcal{M}_{g,n}$. Thus, an application of the Gauss–Bonnet theorem leads to the following formula.

Proposition 1.5. *The Euler characteristic $\chi_{g,n}$ is given by the following integral over the moduli space of stable curves:*

$$\chi_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} \Omega(1,-1;0)\,. \tag{1.2}$$

In particular, notice that for $s = -1$, the translation for $\Omega(1,s;0)$ reads

$$T(u) = -\big(u^2 + u^3 + u^4 + \cdots\big).$$

Expanding the Ω-class in this form provides the statement of Theorem 1.2.

1.2 A parallel with Masur–Veech volumes via Ω-classes

We would like here to make a brief parallel with a slightly different story. Masur–Veech volumes $\pi^{-(6g-6+2n)}\mathsf{MV}_{g,n} \in \mathbb{Q}$ of the principal stratum of the moduli space of quadratic differentials have been shown to have a cohomological representation, given by the Segre class – as opposed to the Chern class – of the logarithmic tangent bundle (to make the comparison cleaner, we ignore the normalisation constant $2^{2g+1}(4g-4+n)!/(6g-7+2n)!$ due to the labelling of simple poles and zeros of the quadratic differentials, and the normalisation of the Masur–Veech measure).

Theorem 1.6 ([CMS19]). *The Masur–Veech volume $\mathsf{MV}_{g,n}$ is given by the following Ω-integral:*

$$\frac{\mathsf{MV}_{g,n}}{\pi^{6g-6+2n}} = (-1)^{3g-3+n}\int_{\overline{\mathcal{M}}_{g,n}} \Omega(1,2;0) = \int_{\overline{\mathcal{M}}_{g,n}} \big(\Omega(1,-1;0)\big)^{-1}\,. \tag{1.3}$$

The integrand is a Segre class instead of a Chern class, which produces a shift in the parameters of the Ω-class (see Remark 4.3 for a more general relation). In a similar fashion, by expanding the Chern characters of the Ω-class, one can express the Masur–Veech volumes $\mathsf{MV}_{g,n}$ as an explicit finite linear combination of Hodge integrals. In other words, $\chi_{g,n}$ and $\mathsf{MV}_{g,n}$ are obtained up to prefactors by integrating the Hodge cohomological field theory $\Lambda(-1)$, on which one applies the translation T for $s_\chi = -1$ in the first case, and its dual translation T for $s_{\mathsf{MV}} = 1 - s_\chi = 2$ in the second case.

In the same way, equation (1.3) is a dual statement to equation (1.2); the following statement corresponds to our result for the Euler characteristic in Theorem 1.2.

Theorem 1.7 ([CMS19]). *The Masur–Veech volume* $\mathsf{MV}_{g,n}$ *is given by the following Hodge integrals:*

$$\frac{\mathsf{MV}_{g,n}}{\pi^{6g-6+2n}} = \sum_{\ell \geqslant 0} \frac{1}{\ell!} \sum_{i=0}^{g} \int_{\overline{\mathcal{M}}_{g,n+\ell}} \lambda_i \psi_{n+1}^2 \cdots \psi_{n+\ell}^2 \, .$$

In fact, this result had tremendous consequences for the study of Masur–Veech volumes of quadratic differentials. First, the statement settled two conjectures on the large n behaviour of the Masur–Veech volumes and their relative Siegel–Veech constants elaborated in [ABC$^+$22]. Then, in two different works, Kazarian and Yang–Zagier produced fast recursive methods to compute these Masur–Veech volumes by exploiting, respectively, KP-type and ILW-type integrability properties of such linear Hodge integrals.

To conclude the parallelism, we point out that both enumerative problems $\mathsf{MV}_{g,n}$ and $\chi_{g,n}$ are generated by topological recursion in the sense of Eynard and Orantin [ABC$^+$22, DN11].

1.3 Properties of the Ω-classes

Lately, Ω-classes have been employed in a great variety of enumerative problems. We produce a list of their properties by collecting those in the literature and the ones known to experts; moreover, we extend some of these properties and prove new ones. The main new result is a pullback property that holds when the cohomological field theories given by the Ω-classes do not have a flat unit. We employ this property to prove new vanishings of the integrals of these classes.

Outline of the paper

In Section 2 we provide the necessary background on Ω-classes. In Section 3 we provide the proofs of the statements presented in the introduction. In Section 4 we establish a list of properties of the Ω-classes and prove them and also apply some of these properties to prove the vanishing of integrals of Ω-classes. Finally, in Section 5 we mention a curious coincidence observed by Pandharipande.

2. Background on Ω-classes

In [Mum83], Mumford derived a formula for the Chern character of the Hodge bundle on the moduli space of curves $\overline{\mathcal{M}}_{g,n}$ in terms of tautological classes and Bernoulli numbers. Among various applications, such a class appears in the celebrated ELSV formula [ELSV01], named after its four authors Ekedahl, Lando, Shapiro, Vainshtein, that is an equality between a simple Hurwitz number and an integral over the moduli space of stable curves.

A generalisation of Mumford's formula was found by Chiodo in [Chi08]. The moduli space $\overline{\mathcal{M}}_{g,n}$ is replaced by the proper moduli stack $\overline{\mathcal{M}}_{g;a}^{r,s}$ of rth roots of the line bundle

$$\omega_{\log}^{\otimes s}\left(-\sum_{i=1}^{n} a_i p_i\right),$$

where $\omega_{\log} = \omega\big(\sum_i p_i\big)$ is the log canonical bundle, r and s are integers with r positive, and

$a_1, \ldots, a_n$ are integers satisfying the modular constraint

$$a_1 + a_2 + \cdots + a_n \equiv (2g - 2 + n)s \pmod{r}\,.$$

This condition guarantees the existence of a line bundle whose rth tensor power is isomorphic to $\omega_{\log}^{\otimes s}\big(-\sum_i a_i p_i\big)$. The value $a_i \pmod{r}$ is called the *type* of the marked point p_i. Let $\pi\colon \overline{\mathcal{C}}^{r,s}_{g;a} \to \overline{\mathcal{M}}^{r,s}_{g;a}$ be the universal curve and $\mathcal{L} \to \overline{\mathcal{C}}^{r,s}_{g;a}$ the universal rth root. The lower strata of $\overline{\mathcal{M}}^{r,s}_{g;a}$ consist of moduli spaces $\overline{\mathcal{M}}^{r,s}_{g';a'}$ of lower dimension, and in particular a node of a stable curve has type a and $r-a$, again $\pmod{r}$, at the two branches. In complete analogy with the case of moduli spaces of stable curves, one can define ψ-classes and κ-classes. There is, moreover, a natural forgetful morphism

$$\epsilon\colon \overline{\mathcal{M}}^{r,s}_{g;a} \longrightarrow \overline{\mathcal{M}}_{g,n}\,,$$

which forgets the line bundle, otherwise known as the spin structure. It can be turned into an unramified covering (in the orbifold sense) of degree r^{2g-1} by slightly modifying the structure of $\overline{\mathcal{M}}_{g,n}$, introducing an extra $\mathbb{Z}/r\mathbb{Z}$-stabiliser for each node of each stable curve (see [JPPZ17]).

Let $B_m(x)$ denote the mth Bernoulli polynomial, that is, the polynomial defined by the generating series

$$\frac{te^{tx}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(x)\frac{t^m}{m!}\,.$$

The evaluations $B_m(0) = (-1)^m B_m(1) = B_m$ recover the usual Bernoulli numbers. Chiodo's formula provides an explicit formula for the Chern characters of the derived pushforward of the universal rth root $\mathrm{ch}_m(r,s;a) = \mathrm{ch}_m(R^\bullet \pi_* \mathcal{L})$.

Theorem 2.1 ([Chi08]). *The Chern characters* $\mathrm{ch}_m(r,s;a)$ *of the derived pushforward of the universal* r*th root have the following explicit expression in terms of* ψ*-classes,* κ*-classes, and boundary divisors:*

$$\mathrm{ch}_m(r,s;a) = \frac{B_{m+1}(s/r)}{(m+1)!}\kappa_m - \sum_{i=1}^{n} \frac{B_{m+1}(a_i/r)}{(m+1)!}\psi_i^m + \frac{r}{2}\sum_{a=0}^{r-1} \frac{B_{m+1}(a/r)}{(m+1)!} j_{a,*}\frac{(\psi')^m - (-\psi'')^m}{\psi' + \psi''}\,. \tag{2.1}$$

Here j_a *is the boundary morphism that represents the boundary divisor such that the two branches of the corresponding node are of type* a *and* $r-a$*, and* ψ' *and* ψ'' *are the* ψ*-classes at the two branches of the node.*

We can then consider the pushforward to the moduli space of stable curves of the family of Chern classes

$$\Omega^{[x]}_{g,n}(r,s;a) = \epsilon_* \exp\left(\sum_{m=1}^{\infty} (-x)^m (m-1)!\mathrm{ch}_m(r,s;a)\right) \in H^{\mathrm{even}}(\overline{\mathcal{M}}_{g,n})\,.$$

We will omit the degree variable x when $x = 1$ and the indices (g,n) whenever they are clear from the context. Notice that one recovers Mumford's formula for the Hodge class when $r = s = 1$ and $a = (1, \ldots, 1)$. For $r = 1$, general s, and $a = (s, \ldots, s)$, we get the generalised Hodge classes considered by Bini in [Bin03]. For any $r \in \mathbb{Z}_+$ and $s \in \mathbb{Z}$, we refer to these classes as Ω-classes, or an Ω-CohFT when referring to them as a collection or when exploiting some of their features as a cohomological field theory (CohFT), based on the following statement.

THEOREM 2.2 ([LPSZ17], Ω-classes form a CohFT). *Let r be a positive integer, and let $V = \langle v_1, \dots, v_r\rangle_{\mathbb{Q}}$ be a vector space. For any $s \in \mathbb{Z}$, the collection of maps*

$$\Omega_{g,n}(r,s;\bullet)\colon V^{\otimes n} \longrightarrow H^{\mathrm{even}}(\overline{\mathcal{M}}_{g,n})$$

associating with a vector $(v_{a_1}, \dots, v_{a_n})$ the cohomology class $\Omega_{g,n}(r,s;a_1,\dots,a_n)$, and extended by multilinearity, forms a cohomological field theory with associated V-metric η defined by

$$\eta(v_a, v_b) = \frac{1}{r}\delta_{a+b\equiv 0 \pmod r}$$

and whose unit is flat whenever s belongs to the set $\{\,0,\dots,r\,\}$.

Notice that the above result is restricted to $1 \leqslant a_1, \dots, a_n \leqslant r$. A relation among Ω-classes with a_i possibly outside this range will be given in Section 4.

2.1 Ω-classes as a sum over stable graphs

By expanding the exponential (2.1), we find an expression for the Ω-classes as a sum over decorated stable graphs in which vertices, leaves, and edges carry cohomology classes multiplied by sums of products of Bernoulli polynomials. However, a correct expansion of (2.1) into an expression in terms of stable graphs has to carefully take into account all possible self-intersections and re-expand each self-intersected edge into the Chern class of its normal bundle. The result of this procedure is written down in clean form in the following statement.

COROLLARY 2.3 ([JPPZ17, Corollary 4]). *The class $\Omega^{[x]}_{g,n}(r,s;a)$ is equal to*

$$\sum_{\Gamma\in\mathsf{G}_{g,n}}\sum_{w\in\mathsf{W}_{\Gamma,r,s}}\frac{r^{2g-1-h^1(\Gamma)}}{|\,\mathrm{Aut}\,(\Gamma)|}\xi_{\Gamma,*}\prod_{v\in V(\Gamma)}\exp\left(\sum_{m\geqslant 1}\frac{(-x)^m B_{m+1}(s/r)}{m(m+1)}\kappa_m(v)\right)$$

$$\times\prod_{\substack{e\in E(\Gamma)\\ e=(h,h')}}\frac{1-\exp\left(-\sum_{m\geqslant 1}\frac{(-x)^m B_{m+1}(w(h)/r)}{m(m+1)}\big((\psi_h)^m-(-\psi_{h'})^m\big)\right)}{\psi_h+\psi_{h'}}$$

$$\times\prod_{i=1}^{n}\exp\left(-\sum_{m\geqslant 1}\frac{(-x)^m B_{m+1}(a_i/r)}{m(m+1)}\psi_i^m\right).$$

Here $\mathsf{G}_{g,n}$ is the finite set of stable graphs of genus g with n legs, $\mathsf{W}_{\Gamma,r,s}$ is the finite set of half-edges decorations with an integer in $\{0,\dots,r-1\}$ in such a way that the leaf i is decorated by a_i, decorations of half-edges forming the same edge $e \in E(\Gamma)$ sum up to r, and locally on each vertex $v \in V(\Gamma)$ the sum of all decorations is congruent to $(2g-2+n)s$ modulo r.

2.2 Riemann–Roch for Ω-classes

The Riemann–Roch theorem for an rth root L of $\omega_{\log}^{\otimes s}(-\sum_i a_i p_i)$ provides the following relation:

$$\frac{(2g-2+n)s-\sum_i a_i}{r} - g + 1 = h^0(C,L) - h^1(C,L)\,.$$

In some cases, that is, for particular parametrisations of r, s, a_i and for topologies (g,n), it can happen that either h^0 or h^1 vanishes, turning the derived pushforward $R^\bullet\pi_*\mathcal{L}$ into a vector bundle. If that happens, the Riemann–Roch formula provides a bound for the complex cohomological degree of Ω:

$$[\deg_{\mathbb{C}} = k].\Omega_{g,n}(r,s;a) = 0 \quad \text{for } k > \operatorname{rank}(R^{\bullet}\pi_*\mathcal{L})\,,$$

where $[\deg_{\mathbb{C}} = k].\Omega$ extracts the homogeneous component of complex cohomological degree k from Ω. These relations are non-trivial whenever the rank is less than $3g-3+n$.

(1) One of these instances is provided in genus zero by the following result of Jarvis, Kimura, and Vaintrob.

Theorem 2.4 ([JKV01, Proposition 4.4]). *Let $g = 0$, $n \geqslant 3$, $s = 0$, and consider a_i all strictly positive except for at most a single a_j which can be positive, or zero, or equal to -1. Then no rth root of $\omega_{\log}^{\otimes s}\big(-\sum_i a_i p_i\big)$ does has global sections; that is, we have $h^0 = 0$.*

Under the condition of the theorem above, the rank of $R^{\bullet}\pi_*\mathcal{L}$ equals h^1, and therefore one gets

$$[\deg_{\mathbb{C}} = k].\Omega_{0,n}(r,0;a) = 0 \quad \text{for } k > \frac{\sum_i a_i}{r} - 1\,.$$

(2) Another instance is provided by the negative s case with the a_i all positive. Let $s = -s'$ for s' a positive integer. In this case, $\omega_{C,\log}^{\otimes -s'}\big(-\sum a_i\big)$ has strictly negative degree on all connected components of every stratum, therefore implying again that $h^0 = 0$. Thus, the rank equals h^1, and one gets

$$[\deg_{\mathbb{C}} = k].\Omega_{g,n}(r,s;a) = 0 \quad \text{for } k > \frac{(2g-2+n)s' + r(g-1) + \sum_i a_i}{r}\,,$$

which is interesting when $(2g-2+n)s' + r(g-1) + \sum_i a_i < (3g-3+n)r$. For instance, if $r = 2$ and all $a_i = s' = 1$, one gets that $\Omega_{g,n}(2,-1;1^n)$ has top complex degree equal to $2g-2+n$, and that top degree in fact defines (up to prefactors) a cohomological field theory $\Theta_{g,n}$ with beautiful properties [CGG22, Nor17] (its intersection numbers are generated by topological recursion in the sense of Eynard and Orantin, the associated partition function is a solution of the Korteweg–de Vries hierarchy which arises from the Brézin–Gross–Witten random matrix model, and it is related to the volumes of the moduli spaces of super Riemann surfaces).

3. Proofs

In this section we provide the proofs of the statements in the introduction.

Proof of Theorem 1.2. Firstly, let us recall a generalised Gauss–Bonnet formula expressing the orbifold Euler characteristic of certain open orbifolds as integrals of the Chern class of the logarithmic cotangent bundle. A proof of the formula can be found in [CMZ22].

Proposition 3.1 (Gauss–Bonnet for open orbifolds). *Let $\overline{M}$ be a compact smooth k-dimensional orbifold, and let D be a normal crossing divisor and $\overline{M} = M \setminus D$. Then the orbifold Euler characteristic of M can be computed as*

$$\chi(M) = (-1)^k \int_{\overline{M}} c_k\big(\Omega^1_{\overline{M}}(\log D)\big)\,,$$

where $c_k\big(\Omega^1_{\overline{M}}(\log D)\big)$ is the kth Chern class of the logarithmic cotangent bundle.

Let us apply the above proposition to compute the Euler characteristic $\chi_{g,n} = \chi(\mathcal{M}_{g,n})$. The fibre of the logarithmic cotangent bundle of $\overline{\mathcal{M}}_{g,n}$ over a curve $(C,p_1,\ldots,p_n)$ is given by the space of quadratic differentials on C with simple poles at the marked points, that is,

$$H^0\big(C, \omega_C^{\otimes 2}\big(\textstyle\sum_i p_i\big)\big).$$

On the other hand, consider the Ω-class with parameters $r = 1$ and $s = -1$ and all $a_i = 0$. As explained in Section 2.2, the class Ω is the Chern class of an actual bundle whose fibre over a curve $(C, p_1, \dots, p_n)$ is isomorphic to $H^1\big(C, (\omega_{C,\log})^{-1}\big)$. By Serre duality,

$$H^1\big(C, (\omega_{C,\log})^{-1}\big) \cong H^0\big(C, \omega_C^{\otimes 2}\big(\textstyle\sum_i p_i\big)\big)^{\vee}\,.$$

Thus, we deduce Proposition 1.5:

$$\chi_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}(1, -1; 0)\,.$$

Specialising formula (2.1), we get $\Omega_{g,n}(1, -1; 0) = \Lambda(-1) \exp\big(-\sum_{m \geqslant 1} (1/m)\kappa_m\big)$. This is a simple consequence of the identity $B_m(-1) = B_m + (-1)^m m$, together with Mumford's formula for λ-classes. Here $\Lambda(-1) = \sum_{i=1}^{g} (-1)^i \lambda_i$ is the total Chern class of the dual of the Hodge bundle. We can convert the evaluation of the above class into a combination of simple Hodge integrals using Lemma A.1: the values $v_k = -1$ are given by $u_m = -1/m$ through equation (A.1). In turn, we find

$$\begin{aligned}\chi_{g,n} &= \int_{\overline{\mathcal{M}}_{g,n}} \Lambda(-1) \exp\Bigg(-\sum_{m \geqslant 1} \frac{1}{m}\kappa_m\Bigg) \\ &= \int_{\overline{\mathcal{M}}_{g,n}} \Lambda(-1) + \sum_{\ell \geqslant 1} \frac{(-1)^\ell}{\ell!} \sum_{\mu_1, \dots, \mu_\ell \geqslant 1} \int_{\overline{\mathcal{M}}_{g,n+\ell}} \Lambda(-1) \prod_{j=1}^{\ell} \psi_{n+j}^{\mu_j + 1}\,.\end{aligned}$$

Notice that the sum over ℓ terminates at $\ell = 3g - 3 + n$, and the sum over the μ is also finite since we have $\mu_1 + \cdots + \mu_\ell \leqslant 3g - 3 + n$. We also observe that the first summand vanishes for degree reasons, unless $(g, n) = (0, 3)$ or $(1, 1)$. In these cases,

$$\int_{\overline{\mathcal{M}}_{0,3}} \Lambda(-1) = \int_{\overline{\mathcal{M}}_{0,3}} 1 = 1\,, \qquad \int_{\overline{\mathcal{M}}_{1,1}} \Lambda(-1) = -\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1 = -\frac{1}{24}\,.$$

Collapsing geometric series into their compact form allows a re-arranging of signs leading to the statement

$$\begin{aligned}\chi_{g,n} &= \sum_{\ell \geqslant 0} \frac{(-1)^\ell}{\ell!} \int_{\overline{\mathcal{M}}_{g,n+\ell}} \frac{\Lambda(-1)}{\prod_{j=1}^{\ell} 1 - \psi_{n+j}} \psi_{n+1}^2 \cdots \psi_{n+\ell}^2 \\ &= (-1)^{3g-3+n} \sum_{\ell \geqslant 0} \frac{1}{\ell!} \int_{\overline{\mathcal{M}}_{g,n+\ell}} \frac{\Lambda(1)}{\prod_{j=1}^{\ell} 1 + \psi_{n+j}} \psi_{n+1}^2 \cdots \psi_{n+\ell}^2\,.\end{aligned}$$

This concludes the proof of Theorem 1.2. □

Proof of Corollary 1.3. Thanks to the intersection-theoretic expression of the Euler characteristic, together with an explicit formula for Hodge integrals due to Dubrovin–Yang–Zagier (see [DYZ17, Section 1.3]), we are able to give a new proof of the Harer–Zagier formula.

CLAIM 1. The Euler characteristic satisfies[1] $\chi_{g,n+1} = -(2g - 2 + n)\chi_{g,n}$.

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[1] The relation $\chi_{g,n+1} = -(2g-2+n)\chi_{g,n}$ easily follows from a short exact sequence involving mapping class groups

Indeed, denoting the forgetful morphism by $\pi\colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$, we have

$$\begin{aligned} \chi_{g,n+1} &= \int_{\overline{\mathcal{M}}_{g,n+1}} \Lambda(-1) \exp\Biggl(-\sum_{m \geqslant 1} \frac{1}{m} \kappa_m \Biggr) \\ &= \int_{\overline{\mathcal{M}}_{g,n+1}} \pi^*\Lambda(-1) \exp\Biggl(-\sum_{m \geqslant 1} \frac{1}{m} (\pi^*\kappa_m + \psi_{n+1}^m) \Biggr) \\ &= \int_{\overline{\mathcal{M}}_{g,n+1}} \exp\Biggl(-\sum_{m \geqslant 1} \frac{1}{m} \psi_{n+1}^m \Biggr) \pi^*\Lambda(-1) \pi^* \exp\Biggl(-\sum_{m \geqslant 1} \frac{1}{m} \kappa_m \Biggr) \\ &= \int_{\overline{\mathcal{M}}_{g,n+1}} (1 - \psi_{n+1}) \pi^*\Lambda(-1) \pi^* \exp\Biggl(-\sum_{m \geqslant 1} \frac{1}{m} \kappa_m \Biggr). \end{aligned}$$

Here we used the property $\pi^*\Lambda(-1) = \Lambda(-1)$ and the relation $\pi^*\kappa_m = \kappa_m - \psi_{n+1}^m$, together with the fact that π^* is a ring homomorphism. Now applying the projection formula and the relation $\pi_*\psi_{n+1} = 2g - 2 + n$, we find

$$\begin{aligned} \chi_{g,n+1} &= -\int_{\overline{\mathcal{M}}_{g,n}} (\pi_*\psi_{n+1}) \Lambda(-1) \exp\Biggl(-\sum_{m \geqslant 1} \frac{1}{m} \kappa_m \Biggr) \\ &= -(2g - 2 + n) \int_{\overline{\mathcal{M}}_{g,n}} \Lambda(-1) \exp\Biggl(-\sum_{m \geqslant 1} \frac{1}{m} \kappa_m \Biggr) = -(2g - 2 + n) \chi_{g,n} \,. \end{aligned}$$

Claim 2. The Harer–Zagier relation holds true.

Indeed, as a consequence of Claim 1, we just have to compute $\chi_{0,3}$, $\chi_{1,1}$ and $\chi_{g,0}$ for $g \geqslant 2$.

The genus zero case is straightforward. Since $\chi_{0,3} = 1$, we have $\chi_{0,n} = (-1)^{n-3}(n-3)!$. The genus one case follows from the computation

$$\chi_{1,1} = \int_{\overline{\mathcal{M}}_{1,1}} \Lambda(-1) - \int_{\overline{\mathcal{M}}_{1,2}} \Lambda(-1) \psi_2^2 = -\frac{1}{12} \,,$$

which implies $\chi_{1,n} = (-1)^n (n-1)!/12$.

The genus $g \geqslant 2$ case instead relies on a beautiful chain of results, which we summarise here. The main equation from which it follows is the Toda equation for simple Hurwitz numbers, which was conjectured by Pandharipande [Pan00] and proved shortly after by Okounkov [Oko00]. On the other hand, simple Hurwitz numbers are expressed in terms of Hodge integrals via the well-known ELSV formula [ELSV01]. The Toda equation for Hurwitz numbers with only simple ramifications has been conveniently rearranged into a simpler quadratic equation by Dubrovin, Yang, and Zagier, which is then employed to understand two different asymptotic behaviours of the generating series of these Hurwitz numbers [DYZ17, Section 1.3]. Combining the ELSV formula and the asymptotic behaviour of the generating series results in the following new identity involving Hodge integrals:

$$\sum_{\ell \geqslant 1} \frac{1}{\ell!} \sum_{\mu_1, \ldots, \mu_\ell \geqslant 1} \int_{\overline{\mathcal{M}}_{g,\ell}} \Lambda(-1) \prod_{i=1}^{\ell} \psi_i^{\mu_i + 1} = \frac{B_{2g}}{2g(2g-2)} \,.$$

(see [HZ86, Section 6]). It moreover follows from the Givental action of the translation T, given in explicit form in the introduction. We mention here yet another (substantially equivalent) form of the same argument, that uses the particular intersection-theoretic expression for $\chi_{g,n}$.

Conveniently enough, this is exactly the relation needed to conclude the proof, as the left-hand side equals $\chi_{g,0}$ by Theorem 1.2. □

4. Properties, symmetries, and parameter shifts of the Ω-CohFT

Mainly within the past five years, applications of Ω-classes in enumerative geometry have been blooming in the literature. Interestingly enough, these applications arise from quite different contexts and with different motivations. A complete list of recent papers employing Ω-classes is out of the scope of this work and, as far as we know, would likely be outdated soon. Instead, the aim of this section is to investigate, prove, extend, and collect properties of Ω-classes as a reference tool for interested mathematicians in the field.

THEOREM 4.1. *Fix integers $g, n \geqslant 0$ such that $2g-2+n>0$. Let r and s be integers with r positive and $1 \leqslant a_1, \dots, a_n \leqslant r$ integers satisfying the modular constraint*

$$a_1 + a_2 + \cdots + a_n \equiv (2g-2+n)s \pmod{r}\,.$$

The Ω-classes satisfy the following properties:

(i) *Shift of s:*

$$\Omega^{[x]}(r, s+r; a_1, \dots, a_n) = \Omega^{[x]}(r, s; a_1, \dots, a_n) \cdot \exp\left(\sum_{m \geqslant 1} \frac{(-x)^m}{m} \left(\frac{s}{r}\right)^m \kappa_m \right).$$

(ii) *Shift of a_i:*

$$\Omega^{[x]}(r, s; a_1, \dots, a_i + r, \dots, a_n) = \Omega^{[x]}(r, s; a_1, \dots, a_n) \cdot \left(1 + x\frac{a_i}{r}\psi_i\right).$$

(iii) *Zero and r symmetry:*

$$\Omega(r, 0; a_1, \dots, a_n) = \Omega(r, r; a_1, \dots, a_n)\,,$$
$$\Omega(r, s; a_1, \dots, 0, \dots, a_n) = \Omega(r, s; a_1, \dots, r, \dots, a_n)\,.$$

(iv) *Pullback property:*

$$\Omega(r, s; a_1, \dots, a_n, s) = \pi^* \Omega(r, s; a_1, \dots, a_n)\,,$$

where $\pi \colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the forgetful map.

(v) (String equation). *For formal variables $x_1, \dots, x_{n+1}$, we have*

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \frac{\Omega(r, s; a_1, \dots, a_n, a_{n+1} = s)}{\prod_{i=1}^{n+1}(1 - x_i\psi_i)} \Bigg|_{x_{n+1}=0} = (x_1 + \cdots + x_n) \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Omega(r, s; a_1, \dots, a_n)}{\prod_{i=1}^{n}(1 - x_i\psi_i)}\,.$$

(vi) (Dilaton equation). *For formal variables $x_1, \dots, x_{n+1}$, we have*

$$\frac{\partial}{\partial x_{n+1}} \int_{\overline{\mathcal{M}}_{g,n+1}} \frac{\Omega(r, s; a_1, \dots, a_n, a_{n+1} = s)}{\prod_{i=1}^{n+1}(1 - x_i\psi_i)} \Bigg|_{x_{n+1}=0} = (2g-2+n) \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Omega(r, s; a_1, \dots, a_n)}{\prod_{i=1}^{n}(1 - x_i\psi_i)}\,.$$

Iterating the first two properties above, one finds:

(I) *Multiple shifts of* s*:*

$$\begin{aligned}&\Omega^{[x]}(r, s+Nr; a_1, \dots, a_n)\\ &\qquad = \Omega^{[x]}(r, s; a_1, \dots, a_n) \cdot \exp\left(\sum_{m \geqslant 1} \frac{(-x)^m}{m} p_m\left(\frac{s}{r}, \dots, \frac{s}{r} + N - 1\right) \kappa_m \right),\end{aligned}$$

where p_m *is the power-sum symmetric polynomial of degree* m*.*

(II) *Multiple shifts of* a_i*:*

$$\Omega^{[x]}(r, s; a_1, \dots, a_i + Nr, \dots, a_n) = \Omega^{[x]}(r, s; a_1, \dots, a_n) \cdot \prod_{t=0}^{N-1} \left(1 + x\left(\frac{a_i}{r} + t\right)\psi_i\right).$$

Remark 4.2. Property (i) was employed in [CMS19, Appendix A] for the case $r = 1$; we extend it to general r. Property (ii) was observed by Chiodo via geometric arguments for $s < 0$; we extend it to general s. Property (iii) is obvious; we list it here for completeness. Property (iv) was proved in [LPSZ17] for $0 \leqslant s \leqslant r$; we extend it to an arbitrary integer s, and it constitutes the strongest statement of the list. Properties (v) and (vi) have been proved in [DLN16] for $s = 0$ via topological recursion techniques; we extend them here to an arbitrary integer s. The other properties are new, to the best of our knowledge.

Remark 4.3. Another interesting property, which only holds for $r = 1$, is a relation between two different parametrisations of Ω-classes, which we refer to as *Segre and Chern*:[2]

$$\Omega^{[-x]}(1, 1-s; 0, \dots, 0) = \left(\Omega^{[x]}(1, s; 0, \dots, 0)\right)^{-1}.$$

It has been proved and employed in [CMS19].

Proof of Theorem 4.1. Most equations can be proved exploiting properties of Bernoulli polynomials tuned in the right way. We proceed by grouping similar properties.

Proof of properties (I) *and* (II)*, which specialise to properties* (i) *and* (ii)*.* Let us recall a few basic facts from the theory of symmetric functions. Let p_m, σ_l, and h_l be the following three bases of symmetric polynomials: power sums, elementary symmetric polynomials, and complete homogeneous polynomials. Explicitly, for a set of variables $X = (X_1, \dots, X_N)$, we have

$$p_m(X) = \sum_{i=1}^{N} X_i^m, \qquad \sigma_l(X) = \sum_{1 \leqslant i_1 < \cdots < i_l \leqslant N} X_{i_1} \cdots X_{i_l}, \qquad h_l(X) = \sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_l \leqslant N} X_{i_1} \cdots X_{i_l}.$$

The generating series of the σ_l and of the h_l read

$$\sum_{l \geqslant 0} \sigma_l(X) u^l = \prod_{i=1}^{N} (1 + X_i u), \qquad \sum_{l \geqslant 0} h_l(X) u^l = \prod_{i=1}^{N} \frac{1}{(1 - X_i u)} \tag{4.1}$$

---

[2] The relation that one might expect from Serre duality applied to an rth root of $\omega_{\log}^{\otimes s}(-\sum_i a_i p_i)$, that is,

$$\Omega^{[-x]}(r, r-s; r-a_1, \dots, r-a_n) = \left(\Omega^{[x]}(r, s; a_1, \dots, a_n)\right)^{-1},$$

is in fact false. As an explicit counterexample, in $(g, n) = (1, 2)$, we have $\Omega^{[x]}(2, 1; 0, 2)\Omega^{[-x]}(2, 1; 2, 0) = 1 - \frac{3}{4}x^2\kappa_2$. However, experimentally we find the vanishing $[\deg_{\mathbb{C}} = k].\Omega^{[-x]}(r, r-s; r-a_1, \dots, r-a_n)\Omega^{[x]}(r, s; a_1, \dots, a_n) = 0$ for k odd.

and are related to the power sums by Newton's identities:

$$\exp\left(\sum_{m\geqslant 1}\frac{(-1)^{m+1}}{m}p_m u^m\right)=\sum_{l\geqslant 0}\sigma_l u^l\,,\quad \exp\left(\sum_{m\geqslant 1}\frac{1}{m}p_m u^m\right)=\sum_{l\geqslant 0}h_l u^l\,. \tag{4.2}$$

Let us, moreover, recall that the Bernoulli polynomials satisfy $B_{m+1}(x+1)=B_{m+1}(x)+(m+1)x^m$ for any non-negative integer m and any complex variable x. For a positive integer N, we can iterate this property N times to obtain

$$B_{m+1}(x+N)=B_{m+1}(x)+(m+1)p_m(x,x+1,\dots,x+N-1)\,.$$

We can apply the property above for $x=s/r$, obtaining

$$\frac{B_{m+1}\big((s+Nr)/r\big)}{m(m+1)}=\frac{B_{m+1}(s/r)}{m(m+1)}+\frac{1}{m}p_m\Big(\frac{s}{r},\frac{s}{r}+1,\dots,\frac{s}{r}+N-1\Big).$$

As a consequence, we find that the Ω-classes before pushforward to $\overline{\mathcal{M}}_{g,n}$ (that is, $\Omega'_{g,n}(r,s;a)$ on $\overline{\mathcal{M}}^{r,s}_{g,a}$) satisfy the shifting property

$$\Omega'^{[x]}_{g,n}(r,s+Nr;a)=\Omega'^{[x]}_{g,n}(r,s;a)\cdot\exp\left(\sum_{m\geqslant 1}\frac{(-x)^m}{m}p_m\Big(\frac{s}{r},\frac{s}{r}+1,\dots,\frac{s}{r}+N-1\Big)\kappa_m\right).$$

Notice that we can now apply the pushforward ϵ_* on both sides and obtain the statement (I): in fact, writing the above class as a sum over stable graphs Γ (cf. Section 2), we see that ϵ_* simply acts by multiplication of the factor $r^{2g-1-h^1(\Gamma)}$ which for fixed g depends only on the first Betti number $h^1(\Gamma)$ of each of the stable graphs produced, which is left unchanged by the decoration of ψ- or κ-classes. In other words, we find

$$\Omega^{[x]}_{g,n}(r,s+Nr;a)=\Omega^{[x]}_{g,n}(r,s;a)\cdot\exp\left(\sum_{m\geqslant 1}\frac{(-x)^m}{m}p_m\Big(\frac{s}{r},\frac{s}{r}+1,\dots,\frac{s}{r}+N-1\Big)\kappa_m\right)$$

in $H^{\mathrm{even}}(\overline{\mathcal{M}}_{g,n})$. This proves property (I), which restricts to property (i) for $N=1$. Property (II) is proved similarly, but this time employing instead equations (4.1) and (4.2) for elementary symmetric polynomials. Property (II) specialises to property (ii).

Proof of property (iii). Property (iii) is obvious from equation (2.1) and the identity for Bernoulli polynomials $B_{m+1}(1)=B_{m+1}(0)=B_{m+1}$.

Proof of property (iv). Whenever s lies within the range $0\leqslant s<r$, by [LPSZ17] the Ω-classes form a CohFT with flat unit, which can be restated precisely as

$$\pi^*\Omega_{g,n}(r,s;a_1,\dots,a_n)=\Omega_{g,n+1}(r,s;a_1,\dots,a_n,s)\,.$$

The case $s=r$ is handled by property (iii). We need to perform the extension of s outside the range $[0,r]$ and show that the statement keeps holding true. For this purpose, we start with s outside the range, and we shift s by adding or subtracting r the required number of times, controlling the shift process by property (i). At this point, we perform the pullback of the correction produced and recognise that it gets perfectly reabsorbed this time by means of property (ii):

$$\Omega_{g,n}(r,s;a) = \begin{cases} \Omega_{g,n}(r,\langle s\rangle\,;a)\exp\left(\displaystyle\sum_{m\geqslant 1}\frac{(-1)^m}{m}p_m\left(\frac{\langle s\rangle}{r},\frac{\langle s\rangle}{r}+1,\dots,\frac{s}{r}-1\right)\kappa_m\right) & \text{if } s\geqslant r, \\ \Omega_{g,n}(r,\langle s\rangle\,;a)\exp\left(-\displaystyle\sum_{m\geqslant 1}\frac{1}{m}p_m\left(1-\frac{\langle s\rangle}{r},2-\frac{\langle s\rangle}{r},\dots,-\frac{s}{r}\right)\kappa_m\right) & \text{if } s<0\,. \end{cases}$$

Here we wrote $s = r[s] + \langle s\rangle$ for the Euclidean division of s by r. Let us focus on the case $s \geqslant r$. Pulling back by the forgetful map, we find

$$\begin{aligned} &\pi^*\Omega_{g,n}(r,s;a_1,\dots,a_n) \\ &\quad= \pi^*\Omega_{g,n}(r,\langle s\rangle\,;a_1,\dots,a_n)\exp\left(\sum_{m\geqslant 1}\frac{(-1)^m}{m}p_m\left(\frac{\langle s\rangle}{r},\frac{\langle s\rangle}{r}+1,\dots,\frac{s}{r}-1\right)\pi^*\kappa_m\right) \\ &\quad= \Omega_{g,n+1}(r,\langle s\rangle\,;a_1,\dots,a_n,\langle s\rangle)\exp\left(\sum_{m\geqslant 1}\frac{(-1)^m}{m}p_m\left(\frac{\langle s\rangle}{r},\frac{\langle s\rangle}{r}+1,\dots,\frac{s}{r}-1\right)\left(\kappa_m-\psi_{n+1}^m\right)\right). \end{aligned}$$

Here we used the fact that π^* is a ring homomorphism, together with the flat unit property for $\Omega_{g,n}(r,\langle s\rangle)$ and the relation $\pi^*\kappa_m = \kappa_m - \psi_{n+1}^m$. We can now absorb the exponential of κ-classes, shifting the Ω-class from $\langle s\rangle$ back to s, then apply Newton's identity (4.2) and the generating series for elementary symmetric polynomials (4.1):

$$\begin{aligned} &\pi^*\Omega_{g,n}(r,s;a_1,\dots,a_n) \\ &\quad= \Omega_{g,n+1}(r,s;a_1,\dots,a_n,\langle s\rangle)\exp\left(\sum_{m\geqslant 1}\frac{(-1)^{m+1}}{m}p_m\left(\frac{\langle s\rangle}{r},\frac{\langle s\rangle}{r}+1,\dots,\frac{s}{r}-1\right)\psi_{n+1}^m\right) \\ &\quad= \Omega_{g,n+1}(r,s;a_1,\dots,a_n,\langle s\rangle)\sum_{l\geqslant 0}\sigma_l\left(\frac{\langle s\rangle}{r},\frac{\langle s\rangle}{r}+1,\dots,\frac{s}{r}-1\right)\psi_{n+1}^l \\ &\quad= \Omega_{g,n+1}(r,s;a_1,\dots,a_n,\langle s\rangle)\prod_{t=1}^{[s]}\left(1+\left(\frac{s}{r}-t\right)\psi_{n+1}\right) \\ &\quad= \Omega_{g,n+1}(r,s;a_1,\dots,a_n,s)\,. \end{aligned}$$

In the last equation we again used property (ii). This proves the case $s > r$. The case $s < 0$ is obtained by applying Newton's identity for the generating series for complete homogeneous polynomials h_l. This concludes the proof of property (iv).

Proof of properties (v) and (vi): String and dilaton equations. These are essentially corollaries of property (iv) applied in the standard way of proving string and dilaton equations through the projection formula. For instance, the dilaton equation can be proved as

$$\begin{aligned} &\frac{\partial}{\partial x_{n+1}}\int_{\overline{\mathcal{M}}_{g,n+1}}\frac{\Omega(r,s;a_1,\dots,a_n,s)}{\prod_{i=1}^{n+1}(1-x_i\psi_i)}\Bigg|_{x_{n+1}=0} = \int_{\overline{\mathcal{M}}_{g,n+1}}\frac{\Omega(r,s;a_1,\dots,a_n,s)}{\prod_{i=1}^{n}(1-x_i\psi_i)}\psi_{n+1} \\ &\quad= \int_{\overline{\mathcal{M}}_{g,n+1}}\pi^*\left[\Omega(r,s;a_1,\dots,a_n)\right]\prod_{i=1}^{n}\sum_{d_i\geqslant 0}x_i^{d_i}\psi_i^{d_i}\cdot\psi_{n+1} \end{aligned}$$

$$
\begin{aligned}
&= \int_{\overline{\mathcal{M}}_{g,n+1}} \pi^*\big[\Omega(r,s;a_1,\dots,a_n)\big] \prod_{i=1}^{n} \sum_{d_i \geqslant 0} x_i^{d_i} (\pi^*\psi_i - D_{i,n+1})^{d_i} \cdot \psi_{n+1} \\
&= \int_{\overline{\mathcal{M}}_{g,n+1}} \pi^*\big[\Omega(r,s;a_1,\dots,a_n)\big] \prod_{i=1}^{n} \sum_{d_i \geqslant 0} x_i^{d_i} \big(\pi^*\big(\psi_i^{d_i}\big) - D_{i,n+1}\pi^*\big(\psi_i^{d_i-1}\big)\big) \cdot \psi_{n+1} \\
&= \int_{\overline{\mathcal{M}}_{g,n+1}} \pi^*\big[\Omega(r,s;a_1,\dots,a_n)\big] \prod_{i=1}^{n} \sum_{d_i \geqslant 0} x_i^{d_i} \pi^*\big(\psi_i^{d_i}\big) \cdot \psi_{n+1} \\
&= \int_{\overline{\mathcal{M}}_{g,n+1}} \pi^*\left[\Omega(r,s;a_1,\dots,a_n) \prod_{i=1}^{n} \sum_{d_i \geqslant 0} x_i^{d_i} \psi_i^{d_i}\right] \cdot \psi_{n+1} = \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Omega(r,s;a_1,\dots,a_n)}{\prod_{i=1}^{n}(1 - x_i\psi_i)} \pi_*\psi_{n+1} \\
&= (2g-2+n) \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Omega(r,s;a_1,\dots,a_n)}{\prod_{i=1}^{n}(1 - x_i\psi_i)} \,.
\end{aligned}
$$

Here $D_{i,n+1}$ is the divisor given by the locus of curves with a rational component attached by a single node and containing the two marked points i and $n+1$, satisfying the constraints $D_{i,n+1}D_{j,n+1} = D_{i,n+1}\psi_{n+1} = 0$. Here the convention that negative powers of ψ-classes are zero is used. This concludes the proof of property (vi). The string equation or property (v) is proved in a similar way. This concludes the proof of the theorem. ☐

4.1 Some vanishing of the Ω-integrals

As an application of the properties above, we provide several vanishing results for integrals of Ω-classes with weighted ψ-classes. Again, we will write $s = r[s] + \langle s \rangle$ for the Euclidean division of an integer s by a natural number r.

THEOREM 4.4. *Fix integers $g, n \geqslant 0$ such that $2g-2+n > 0$. Let r and s be integers with r positive, and $1 \leqslant a_1, \dots, a_n \leqslant r$ integers satisfying the modular constraint*

$$
ca_1 + a_2 + \cdots + a_n \equiv (2g-2+n)s \pmod{r} \,.
$$

We have

$$
\int_{\overline{\mathcal{M}}_{g,n+1}} \Omega^{[x]}_{g,n+1}(r,s;a_1,\dots,a_n,s) = 0 \quad \textit{for any } s \in \mathbb{Z} \,.
$$

Proof. The proof is an immediate consequence of property (iv) of Theorem 4.1, as the pullback π^* preserves the cohomological degree, whereas the target moduli space is higher in dimension. ☐

By employing properties (I) and (II) of Theorem 4.1, we obtain the following vanishing properties.

COROLLARY 4.5. (i) *If $s \geqslant r$, we have*

$$
\int_{\overline{\mathcal{M}}_{g,n+1}} \Omega^{[x]}_{g,n+1}(r,s;a_1,\dots,a_n,\langle s \rangle) \prod_{t=1}^{[s]} \left(1 + \left(\frac{s}{r} - t\right) \psi_{n+1} x\right) = 0 \,,
$$

$$
\int_{\overline{\mathcal{M}}_{g,n+1}} \Omega^{[x]}_{g,n+1}(r,\langle s \rangle;a_1,\dots,a_n,s) \exp\left(\sum_{m \geqslant 1} \frac{(-1)^m}{m} p_m\left(\frac{\langle s \rangle}{r}, \frac{\langle s \rangle}{r} + 1, \dots, \frac{s}{r} - 1\right) \kappa_m x^m\right) = 0 \,.
$$

(ii) *If $s < 0$, we have*

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \Omega^{[x]}_{g,n+1}(r,s;a_1,\dots,a_n,\langle s\rangle) \prod_{t=0}^{-[s]-1} \left(1 + \left(\frac{s}{r} + t\right)\psi_{n+1}x\right)^{-1} = 0\,,$$

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \Omega^{[x]}_{g,n+1}(r,\langle s\rangle\,;a_1,\dots,a_n,s) \times \exp\left(-\sum_{m\geqslant 1}\frac{(-1)^m}{m}p_m\left(\frac{\langle s\rangle}{r},\frac{\langle s\rangle}{r}+1,\dots,\frac{s}{r}-1\right)\kappa_m x^m\right) = 0\,.$$

Remark 4.6. The statements of Corollary 4.5 can be expressed in terms of the generalised Stirling numbers s and S of first and second type, respectively (see, for example, [Cha02]):

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \Omega_{g,n+1}(r,s;a_1,\dots,a_n,\langle s\rangle)\sum_{m\geqslant 0}(-1)^{[s]}\mathsf{s}\left([s],[s]-m,\frac{\langle s\rangle}{r}\right)\psi_{n+1}^m = 0 \qquad \text{for } s > r\,,$$

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \Omega_{g,n+1}(r,s;a_1,\dots,a_n,\langle s\rangle)\sum_{m\geqslant 0}\mathsf{S}\left([s]+m,[s],\frac{\langle s\rangle}{r}\right)\psi_{n+1}^m = 0 \qquad \text{for } s < 0\,.$$

By [Cha02], we have the following expression in terms of the usual Stirling numbers $\mathsf{s}(a,b)$ and $\mathsf{S}(a,b)$ of first and second type, respectively:

$$(-1)^k\mathsf{s}\big(k,k-m,x\big) = \sum_{i=0}^{m}\binom{k+i-m}{i}\mathsf{s}(k,k-m+i)x^i \qquad \text{for } s > r\,,$$

$$\mathsf{S}\big(k+m,k,x\big) = \sum_{i=0}^{m}\binom{m-k-1}{i}(-1)^i\mathsf{S}(m-i-k,-k)x^i \qquad \text{for } s < 0\,.$$

5. A curious coincidence

We conclude the paper with a question raised by Rahul Pandharipande. In [FP00, Theorem 3] the following evaluation is established. For $g \geqslant 2$,

$$\int_{[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,d)]^{\mathrm{vir}}} c_{\mathrm{top}}\big(R^1\pi_*\mu^*(\mathcal{O}(-1)\oplus\mathcal{O}(-1))\big) = \frac{d^{2g-3}}{2g(2g-2)!}|B_{2g}| = \frac{d^{2g-3}}{(2g-3)!}|\chi_{g,0}|\,.$$

This formula has an application in the Gromov–Witten theory of Calabi–Yau 3-folds of multiple covers of a fixed rational curve with normal bundle $N = \mathcal{O}(-1)\oplus\mathcal{O}(-1)$. We refer to the original paper for more details. The question is whether the appearance of $\chi_{g,0}$ has some deeper geometric meaning.

One can see that, since integrating over the moduli space of stable maps with n fixed points produces an extra global factor of d^n, a straightforward rescaling confirms that the statement above holds for general n in the following sense:

$$\int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d)]^{\mathrm{vir}}} c_{\mathrm{top}}\big(R^1\pi_*\mu^*(\mathcal{O}(-1)\oplus\mathcal{O}(-1))\big)\prod_{i=1}^{n}\mathrm{ev}_i^*([\mathrm{pt}]) = \frac{d^{2g-3+n}}{(2g-3+n)!}|\chi_{g,n}|\,. \tag{5.1}$$

On the other hand, by Lemma 1.4, we have

$$\frac{1}{d^g}\int_{\overline{\mathcal{M}}_{g,n}}\Omega^{[-d]}(1,-1;0,\dots,0) = \frac{1}{d^g}\int_{\overline{\mathcal{M}}_{g,n}}\Lambda(d)\exp\left(-\sum_{m=1}^{\infty}(-d)^m\frac{\kappa_m}{m}\right) = d^{2g-3+n}|\chi_{g,n}|\,. \tag{5.2}$$

It would be insightful if there existed some connection, already at the level of classes, between the integrands in the left-hand sides of equations (5.1) and (5.2). One way to go about it could be to choose a good parametrisation for the virtual localisation calculation.

Appendix. A technical lemma

The following lemma appears in some form in the literature; see for instance [Pix13, Lemma 2.3]. We nevertheless provide a proof here for the reader's convenience.

Lemma A.1. *Let $g \geqslant 0$ and $n > 0$ be integers such that $2g-2+n > 0$. Then for any class $\alpha \in H^{\mathrm{even}}\big(\overline{\mathcal{M}}_{g,n}\big)$ and any sequence $(u_m)_{m \geqslant 1}$ of complex numbers, we find*

$$\begin{aligned}\int_{\overline{\mathcal{M}}_{g,n}} & \alpha \cdot \prod_{i=1}^{n} \psi_i^{d_i} \exp\Big(\sum_{m \geqslant 1} u_m \kappa_m\Big) \\ &= \int_{\overline{\mathcal{M}}_{g,n}} \alpha \cdot \prod_{i=1}^{n} \psi_i^{d_i} + \sum_{\ell \geqslant 1} \frac{1}{\ell!} \sum_{\mu_1,\dots,\mu_\ell \geqslant 1} \int_{\overline{\mathcal{M}}_{g,n+\ell}} \pi_\ell^* \alpha \cdot \prod_{i=1}^{n} \psi_i^{d_i} \prod_{j=1}^{\ell} v_{\mu_j} \psi_{n+j}^{\mu_j+1} ,\end{aligned}$$

where the map $\pi_\ell \colon \overline{\mathcal{M}}_{g,n+\ell} \to \overline{\mathcal{M}}_{g,n}$ is the morphism forgetting the last ℓ marked points, and the sequence $(v_k)_{k \geqslant 1}$ is obtained from $(u_m)_{m \geqslant 1}$ from the expansion

$$\exp\Big(-\sum_{m \geqslant 1} u_m x^m\Big) = 1 - \sum_{k \geqslant 1} v_k x^k . \tag{A.1}$$

Proof. Let us first recall that multi-indexes κ-classes and single-index κ-classes are related by

$$\exp\Big(\sum_{m \geqslant 1} u_m \kappa_m\Big) = 1 + \sum_{\ell \geqslant 1} \frac{1}{\ell!} \sum_{\mu_1,\dots,\mu_\ell \geqslant 1} \Big(\prod_{j=1}^{\ell} v_{\mu_j}\Big) \kappa_{\mu_1,\dots,\mu_\ell} ,$$

where u_m and v_k are related by the expansion (A.1). As by definition

$$\kappa_{\mu_1,\dots,\mu_\ell} = \pi_{\ell,*}\big(\psi_{n+1}^{\mu_1+1} \cdots \psi_{n+\ell}^{\mu_\ell+1}\big) ,$$

the projection formula implies that

$$\begin{aligned}\int_{\overline{\mathcal{M}}_{g,n}} & \alpha \cdot \prod_{i=1}^{n} \psi_i^{d_i} \exp\Big(\sum_{m \geqslant 1} u_m \kappa_m\Big) \\ &= \int_{\overline{\mathcal{M}}_{g,n}} \alpha \cdot \prod_{i=1}^{n} \psi_i^{d_i} + \sum_{\ell \geqslant 1} \frac{1}{\ell!} \sum_{\mu_1,\dots,\mu_\ell \geqslant 1} \int_{\overline{\mathcal{M}}_{g,n+\ell}} \pi_\ell^* \Big(\alpha \cdot \prod_{i=1}^{n} \psi_i^{d_i}\Big) \prod_{j=1}^{\ell} v_{\mu_j} \psi_{n+j}^{\mu_j+1} .\end{aligned}$$

Observe that

$$\pi_\ell^* \Big(\alpha \cdot \prod_{i=1}^{n} \psi_i^{d_i}\Big) = \pi_\ell^* x \alpha \cdot \prod_{i=1}^{n} \big(\pi_\ell^* \psi_i\big)^{d_i} ,$$

and moreover recall that if $\pi \colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the forgetful map forgetting the $(n+1)$th marked point, then $\pi^*(\psi_i) = \psi_i - D_{i,n+1}$ on $\overline{\mathcal{M}}_{g,n+1}$, where $D_{i,n+1}$ is the Poincaré dual of the divisor represented by the curves with a single node separating a rational component with exactly two leaves marked i and $n+1$ from the other component. As a consequence, $\pi_\ell^*(\psi_i) = \psi_i - \bar{D}_{i,n+1}$, where $\bar{D}_{i,n+1}$ is the Poincaré dual of the divisor represented by the curves with a single node separating a rational component decorated by the leaf i and leaves in a non-empty subset $I \subset \{n+1, \ldots, n+\ell\}$ from the other component, summing over all such subsets I. The important observation is the following: because we are interested in κ-classes, all the new leaves $n+1, \ldots, n+\ell$ obtained by applying the projection formula are decorated by a ψ-class to the power at least one. By a dimension argument, a rational component attached to a single node and decorated by the leaves i and I has dimension $|I|-1$, whereas the ψ-classes decorating that component have total degree at least $|I|$. This proves that the terms involving the classes $\bar{D}_{i,n+1}$ vanish in the integral, therefore proving the statement. □

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