Low-complexity linear parameter-varying approximations of incompressible Navier-Stokes equations for truncated state-dependent Riccati feedback

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Abstract: Nonlinear feedback design via state-dependent Riccati equations is well established but unfeasible for large-scale systems because of computational costs. If the system can be embedded in the class of linear parameter-varying (LPV) systems with the parameter dependency being affine-linear, then the nonlinear feedback law has a series expansion with constant and precomputable coefficients. In this work, we propose a general method to approximating nonlinear systems such that the series expansion is possible and efficient even for high-dimensional systems. We lay out the stabilization of incompressible Navier-Stokes equations as application, discuss the numerical solution of the involved matrix-valued equations, and confirm the performance of the approach in a numerical example.

1 Introduction

Nonlinear feedback design for large-scale systems is challenging, as both the complexity induced by nonlinearities and the huge computational tasks caused by the system's size have to be resolved. The commonly used methods of *backstepping* [22], feedback linearization [29, Ch. 5.3], or *sliding mode* control [16] require structural assumptions and, thus, may not be accessible to a general computational framework. The both holistic and general approach via the Hamilton-Jacobi-Bellman (HJB) equations, however, is only feasible for very moderate system sizes or calls for model order reduction; see, e.g., [14] for a relevant discussion and an application in fluid flow control. As an alternative to reducing the system's size, one may consider approximations to the solution of the HJB equations of lower complexity. For that, for example, truncated polynomial expansions [13] or suboptimal solutions via the so called state-dependent Riccati equation (SDRE) [2] are considered. Here, we will follow on recent developments [1] on series expansions of the SDRE approximation to the HJB solution that can mitigate the still high computational costs of repeatedly solving highdimensional Riccati equations.

As the general setup, we consider the input-affine system

$$\dot{v}(t) = f(v(t)) + Bu(t), \quad y(t) = Cv(t),$$
 (1)

where for time t > 0, $x(t) \in \mathbb{R}^n$ denotes the state, $u(t) \in \mathbb{R}^p$ and $y(t) \in \mathbb{R}^q$ denote the input and output, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a possibly nonlinear function, and where B and C are linear input and output operators. Under the mild condition that f is Lipshitz continuous and f(0) = 0, one can factorize the nonlinearity f(v) = A(v)v with a state-dependent coefficient matrix A(v) and bring the system (1) into state-dependent coefficient (SDC) form:

$$\dot{v}(t) = A(v(t))v(t) + Bu(t), \quad y(t) = Cv(t);$$
 (2)

see, e.g., [8, Eq. (7)]. For such systems, one can define a feedback by

$$u(t) = -B^{\mathsf{T}} P(v(t))v(t),$$

where P solves the SDRE

$$A(v)^{\mathsf{T}}P + PA(v) - PBB^{\mathsf{T}}P = -C^{\mathsf{T}}C; \qquad (3)$$

see [2] for general principles and [8] for a proof of performance beyond an asymptotic smallness condition. Because of its nonlinear and, possibly high-dimensional nature, a solve of the SDRE (3) comes at high costs that make the SDRE approach unfeasible for large systems; see [8] for an example illustrating how the effort grows with the system's dimension.

If, however, the factorization f(v) = A(v)v is affinelinear with respect to a parametrization $\rho(v) \in \mathbb{R}^m$ of v, i.e., it can be represented as

$$A(v) = A_0 + \sum_{k=1}^{m} \rho_k(v) A_k,$$
(4)

then the solution P(v) of the SDRE has a *first-order* approximation

$$P(v) \approx P_0 + \sum_{k=1}^m \rho_k(v) L_k$$

where P_0 and L_k , k = 1, ..., m can be precomputed by one Riccati and m Lyapunov equations; see [1,4]. In this work we propose a general approach for controller design that bases on approximative representations as in (4) for which we employ

- an SDC representation of the nonlinear system,
- an approximative parametrization $\hat{\rho}(v) \in \mathbb{R}^r$, where approximative means that r is chosen small such that an exact reconstruction of v from $\hat{\rho}(v)$ might not be possible, and
- an affine-linear approximation of the coefficient (4).

Also, we discuss how such an approach can be realized for flow control problems modelled by semi-discrete Navier-Stokes equations (NSE). For this, we rely on

- the coordinates provided by a proper orthogonal decomposition (POD) of the velocity states; see [24] for an introduction,
- the quadratic structure of the nonlinearity in the incompressible NSE,
- implicit treatment of the incompressibility constraint, and, importantly,
- low-rank solves for and low-rank representations of the solutions to the high-dimensional Riccati and Lyapunov equations.

We note that with this line of arguments, the feedback design by truncated SDRE approximations can be made feasible for, say, finite element approximations of general nonlinear partial differential equations (PDEs).

Apart from the proposed algorithmic advances and numerical insights into the feedback approximation, we expand here on the work of [1] insofar as the parametrization

step lifts fundamental structural assumptions on the problem class. A related approach, though with updates that require the solutions of nonlinear matrix equations, can be found in [15] based on the expansion of nonlinear systems into Volterra series [27]. Furthermore, we note that, with the explicit low-complexity parametrization of the nonlinearity in an otherwise linear problem formulation, the difficulties of exponentially growing dimensions that come with tensor expansions for general nonlinearities are mitigated; see [23] for a recent discussion regarding model order reduction.

The overall procedure is explained in detail as follows. In Section 2, we explain how a low-complexity linear parameter-varying approximation can be obtained by parametrizing the state of an SDC system. The formulas for expanding the SDRE solution and state the constituting equations for the coefficients of the expansion are recalled in Section 3. Section 4 lays out how POD can be used to realize a low-complexity affine-linear parameter-varying approximation of incompressible Navier-Stokes equations and in Section 5 we briefly describe the concepts for solving the high-dimensional matrix equations. In Section 6, we provide the results of numerical experiments to show the applicability of the approach and to compare with plain static feedback. The paper is concluded in Section 7.

2 Low-complexity linear parameter-varying approximations

In this section, we consider now systems in SDC form (2). If the system state v(t) is encoded into time-varying parameters $\rho(t) = \mu(v(t)) \in \mathbb{R}^m$, with $m \leq n$, and $v(t) = \nu(\rho(t))$, where μ and ν are the corresponding encoding and decoding maps, then the SDC representation (2) can be formulated as a linear parameter-varying (LPV) system via

$$\dot{v}(t) = \widetilde{A}(\rho(t))v(t) + Bu(t), \quad y(t) = Cx(t), \quad (5)$$

where $\widetilde{A}(\rho) := A(\nu(\rho))$. Such an embedding of a nonlinear system into the class of LPV systems is typically called *quasi LPV* system; see, e.g., [21]. Here we will focus on affine-linear LPV representations, where \widetilde{A} depends affine-linearly on ρ so that (5) can be realized as

$$\dot{v}(t) = \left(\widetilde{A}_0 + \sum_{k=1}^m \rho_k(t)\widetilde{A}_k\right)v(t) + Bu(t), \quad y(t) = Cv(t),$$

where ρ_k is the k-th component of ρ and where \widetilde{A}_0 and $\widetilde{A}_k \in \mathbb{R}^{n \times n}$ are constant, $k = 1, \ldots, m$.

If the state v is not exactly parametrized but only approximated with less degrees of freedom in $\hat{\rho}(t) = \hat{\mu}(v(t)) \in \mathbb{R}^r$ such that $r \ll m$, with an inexact reconstruction

$$v(t) \approx \tilde{v}(t) = \hat{\nu}(\hat{\rho}(t)) = \hat{\nu}(\hat{\mu}(v(t))), \tag{6}$$

then an LPV approximation of (1) is given by

$$\hat{v}(t) = \hat{A}(\hat{\rho}(t))\hat{v}(t) + Bu(t), \quad \hat{y}(t) = C\hat{v}(t), \quad (7)$$

with the approximated system matrix $\widehat{A}(\hat{\rho}) := A(\hat{\nu}(\hat{\rho}))$ and the new system state $\hat{\nu}(t) \in \mathbb{R}^n$. Note that the state $\hat{\nu}(t)$ is of *full* dimension *n*, as our reduction efforts will target the structure of the model rather than the dimension of the states. Nonetheless, the ideas and techniques of approximate low-dimensional parametrizations of states and estimates on approximation errors of standard model order reduction (MOR) schemes readily apply.

3 Series expansions of state-dependent Riccati equations

The theory of first-order approximations for a single parameter dependency by [4] has been extended in [1] to the multivariate case. We briefly recall the relevant formulas.

To prepare the argument, we assume that v is parametrized through ρ and consider the dependency of the SDRE solution P on the current value of ρ , i.e., $P(\cdot) = P(\rho(\cdot))$. Then, the multivariate Taylor expansion of Pabout $\rho_0 = 0$ up to order K reads

$$P(\rho) \approx P(0) + \sum_{1 \le |\alpha| \le K} \rho^{(\alpha)} P_{\alpha}, \tag{8}$$

where $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ is a multiindex with $|\alpha| := \sum_{i=1}^m \alpha_i$, where $\rho^{(\alpha)} := \rho_1^{\alpha_1} \rho_2^{\alpha_2} \cdots \rho_m^{\alpha_m}$, and where, importantly, P_{α} are constant matrices given by

$$P_{\alpha} := \frac{1}{\alpha_1! \alpha_2! \cdots \alpha_m!} \frac{\partial^{|\alpha|}}{\partial_{\rho_1}^{\alpha_1} \partial_{\rho_2}^{\alpha_2} \cdots \partial_{\rho_m}^{\alpha_m}} P(0).$$

In particular, the expansion up to order one (i.e., the associated first-order approximation) can be written as

$$P(\rho) \approx P(0) + \sum_{|\alpha|=1} \rho^{(\alpha)} P_{\alpha} =: P_0 + \sum_{k=1}^m \rho_k L_k.$$
 (9)

Substituting P in the SDRE (3) by its series expansion (8) and considering the affine-linear dependency of A on ρ yields

$$\left(\sum_{k=0}^{m} \rho_k A_k\right)^{\mathsf{T}} \left(\sum_{|\alpha| \le K} \rho^{(\alpha)} P_\alpha\right) + \left(\sum_{|\alpha| \le K} \rho^{(\alpha)} P_\alpha\right) \left(\sum_{k=0}^{m} \rho_k A_k\right) - \left(\sum_{|\alpha| \le K} \rho^{(\alpha)} P_\alpha\right) B B^{\mathsf{T}} \left(\sum_{|\alpha| \le K} \rho^{(\alpha)} P_\alpha\right) = -C^{\mathsf{T}} C,$$

where, for compactness of the expression, we introduce $\rho_0 = 1$ and use the relevant conventions for $\alpha = 0 \in \mathbb{N}^m$. By matching the coefficients for $\rho^{(\alpha)}$, we obtain equa-

By matching the coefficients for $\rho^{(\alpha)}$, we obtain equations for the matrices of the first-order approximation (9) as

$$A_0^{\mathsf{I}} P_0 + P_0 A_0 - P_0 B B^{\mathsf{I}} P_0 = -C^{\mathsf{I}} C, \qquad (10)$$

for P_0 and

$$(A_0 - BB^{\mathsf{T}}P_0)^{\mathsf{T}} L_k + L_k (A_0 - BB^{\mathsf{T}}P_0)$$

= $- (A_k^{\mathsf{T}}P_0 + P_0A_k)$ (11)

for L_k , with $k = 1, \ldots, m$; see also [1].

4 Approximations of Navier-Stokes equations through linear affine parametrizations

As the standard model for incompressible flows we consider spatially discretized Navier-Stokes equations (NSE). After a shift of variables that eliminates constant nonzero Dirichlet boundary conditions, the semi-discrete NSEs in the variables of the velocity $v(t) \in \mathbb{R}^n$ and pressure $p \in \mathbb{R}^{n_p}$ with control input u can be written as

$$M\dot{v} = \widetilde{N}(v, v) + \widetilde{A}v + J^{\mathsf{T}}p + \widetilde{B}u, \qquad (12a)$$

$$0 = Jv. \tag{12b}$$

At least for theoretical considerations, the incompressibility constraint (12b) can be resolved and the velocity v can be determined by the equivalent projected equations

$$M\dot{v} = N(v,v) + Av + Bu, \tag{13}$$

where $N(v, v) = \Pi^{\mathsf{T}} \widetilde{N}(v, v)$, where A and B denote $\Pi^{\mathsf{T}} \widetilde{A}$ and $\Pi^{\mathsf{T}} \widetilde{B}$, respectively, and where

$$\Pi := I_n - M^{-1} J^{\mathsf{T}} \left(J M^{-1} J^{\mathsf{T}} \right)^{-1} J$$
 (14)

is the so-called *discrete Leray projector*; see [20] for properties of Π and formulations in the coordinates of the subspace spanned by Π^{T} and see [7] where we have proven that the SDRE feedback based on (13) is equivalent to that of (12).

By the homogeneity in the boundary conditions, the nonlinearity N(v, v) that models the convection is linear in both arguments (see, e.g., [5] for explicit formulas of $N(\cdot, \cdot)$ in a spatial discretization) so that both

$$N_1(v) \colon w \mapsto N(v, w)$$
 and $N_2(v) \colon w \mapsto N(w, v)$

can be realized as state-dependent coefficient matrix and so that for any blending parameter $\lambda \in \mathbb{R}$, an SDC representation is given as

$$N(v,v) = \lambda N_1(v)v + (1-\lambda)N_2(v)v =: N_\lambda(v)v.$$
(15)

Even more, if in an approximative parametrization as in (6), the decoding is linear, then the induced LPV approximation is *affine-linear*; see [18, Rem. 2].

In fact, let $V_r \in \mathbb{R}^{n \times r}$ be the matrix of r POD modes designed to best approximate the velocity in an r-dimensional subspace of \mathbb{R}^n , then with

$$\hat{\rho}(t) := V_r^{\mathsf{T}} v(t) \text{ and } \tilde{v}(t) = V_r \hat{\rho}(t)$$

and the approximation property of the POD basis, we obtain

$$v(t) \approx \tilde{v}(t) = V_r \hat{\rho}(v(t)) = \sum_{i=1}^r \hat{\rho}_i(v(t)) \hat{v}_i$$

where \hat{v}_i , for i = 1, ..., r are the columns of $V_r \in \mathbb{R}^{n \times r}$. By the linearity of $v \to N_\lambda(v)$, the SDC representation (15) is readily approximated by

$$N(v)v \approx N(\tilde{v})v = \left(\sum_{i=1}^{r} \hat{\rho}_i \hat{N}_i\right)v \tag{16}$$

with $\widehat{N}_i := N_\lambda(\widehat{v}_i), \ i = 1, \dots, r.$

From the orthogonality of the POD basis it follows that $\tilde{v}(t) = V_r V_r^{\mathsf{T}} v(t) \to v(t)$ uniformly with $r \to n, r \leq n$, which can be translated into convergence of the LPV approximations

$$\left(\sum_{i=1}^{\hat{r}} \hat{\rho}_i \widehat{N}_i\right) \to \left(\sum_{i=1}^{r} \hat{\rho}_i \widehat{N}_i\right) \to \left(\sum_{i=1}^{n} \hat{\rho}_i \widehat{N}_i\right) = N(v),$$

for $\hat{r} \to r \to n$ and $\hat{r} \leq r \leq n$. Practically, in view of computing approximations to the SDRE solution and the associated feedback law, this means that the series expansion in (9) can be augmented or reduced by simply adding or discarding parameters and the corresponding factors L_k .

5 Numerical handling of high-dimensional matrix equations

First, we note that for systems like the spatially discretized NSEs (13), a mass matrix M needs to be incorporated. Since M is typically symmetric positive definite and, thus, invertible, such systems are readily transformed into the standard form of (2). In practice, however, it is beneficial to consider formulations of the Riccati and Lyapunov equations (10) and (11) without the explicit inversion:

$$A_0^{\mathsf{T}} P_0 M + M^{\mathsf{T}} P_0 A_0 - M^{\mathsf{T}} P_0 B B^{\mathsf{T}} P_0 M = -C^{\mathsf{T}} C, \quad (17)$$

and

$$A_{0,\mathsf{cl}}^{\mathsf{T}}L_{k}M + M^{\mathsf{T}}L_{k}A_{0,\mathsf{cl}} = -\left(M^{\mathsf{T}}P_{0}A_{k} + A_{k}^{\mathsf{T}}P_{0}M\right),$$
(18)

for k = 1, ..., m, where $A_{0,cl} := A_0 - BB^{\mathsf{T}}P_0M$ is the closed-loop system matrix corresponding to (17); see, for example, [6]. Although, these formulations can cover also problems where the mass matrix is not invertible as they appear in differential-algebraic equations like the Navier-Stokes equations in the original formulation (12), for the presentation it is convenient to refer to the projected system (13). In practice, the system matrices in (13) and the

involved projection Π from (14) are realized only implicitly during the application of iterative matrix equation solvers for (17) and (18) like the low-rank ADI; see [9].

Another general problem occurring in the presence of high-dimensional systems is the consumption of computational resources such as time and memory. In particular, it is typically not possible to even store the large-scale dense solutions $P_0 \in \mathbb{R}^{n \times n}$ and $L_k \in \mathbb{R}^{n \times n}$, $k = 1, \ldots, m$. An established approach to handle this problem is the use of low-rank factorizations. The stabilizing solution of the Riccati equation (17) namely P_0 is known to be positive semi-definite such that in the case of small numbers of inputs and outputs, $p, q \ll n$, it can be well represented by a low-rank Cholesky factorization, i.e., $P_0 \approx Z_0 Z_0^{\mathsf{T}}$ with $Z_0 \in \mathbb{R}^{n \times \ell_0}$ and $\ell_0 \ll n$. One can find various numerical methods in the literature to efficiently compute these low-rank factors of Riccati equations without ever forming the full solution P_0 ; see [10] for an overview. In our numerical experiments, we rely on the low-rank Newton-Kleinman-ADI method [11].

Considering the indefinite right-hand side of (18), one needs to assume that the L_k 's are generically indefinite. Nonetheless, the low-rank factorization $P_0 \approx Z_0 Z_0^{\mathsf{T}}$ yields the low-rank indefinite factorization

$$\begin{split} M^{\mathsf{T}} P_0 A_k &+ A_k^{\mathsf{T}} P_0 M \\ &\approx \begin{bmatrix} M^{\mathsf{T}} Z_0 & A_k^{\mathsf{T}} Z_0 \end{bmatrix} \begin{bmatrix} 0 & I_{\ell_0} \\ I_{\ell_0} & 0 \end{bmatrix} \begin{bmatrix} Z_0^{\mathsf{T}} M \\ Z_0^{\mathsf{T}} A_k \end{bmatrix} = C_k S_k C_k^{\mathsf{T}}. \end{split}$$

Based on this factorization in the right-hand side, it is possible to similarly approximate the solution to (18) as $L_k \approx Z_k D_k Z_k^{\mathsf{T}}$, for $k = 1, \ldots, m$, where D_k is a symmetric but possibly indefinite matrix. Here, we are using the LDL^{T} -factorized low-rank ADI method [25].

6 Numerical experiments

The code, raw data and results of the presented numerical experiments are available at [19]. For the solution of Riccati and Lyapunov equations in MATLAB 9.9.0.1467703 (R2020b), we used the solver implementations from MOR-LAB version 5.0 [12] and M-M.E.S.S. version 2.2 [28]. For the simulation part, we resort to our Python interface [17] between *Scipy* and the finite element toolbox FEniCS [26].

We consider the stabilization of the flow in the wake of a 2D cylinder through two control inlets at the periphery of the cylinder. Measurement outputs are defined as averaged velocities over a small neighborhood of three sensor points in the wake; see [5] on technical details, the detailed control setup, and on how the Dirichlet control is relaxed as penalized Robin boundary conditions.

As for the numerical setup, we consider here the Reynoldsnumber 60 and start from the associated non-zero steady state, which is to be stabilized; see Figure 1 for the basic geometry of the example and snapshots of the



Figure 1: Snapshots of the flow in the unstable steady state and in the fully developed periodic vortex shedding regime.

steady state and periodic regime that develops if no stabilization is employed. For the spatial discretization, we use quadratic-linear *Taylor-Hood* finite elements on a nonuniform mesh that leads to a system of size 57 000. For the time integration we use an implicit-explicit Euler time stepping method that in particular treats the linear part and the incompressibility constraint implicitly, whereas the nonlinear part and the feedback is treated explicitly in time. Generally, we are concerned with a system of type (12) with $\begin{bmatrix} v(t) & p(t) \end{bmatrix}^{\mathsf{T}} \in \mathbb{R}^{57\,000}$ with the input $u(t) \in \mathbb{R}^2$ and the output $y(t) \in \mathbb{R}^6$ extracted from the velocity state by a linear output operator C as y(t) = Cv(t).

The basic procedure of the simulations comprises the following steps:

- (0) Compute the steady state for u = 0 to be used as reference for the stabilization, for the shift of the system that removes nonzero boundary conditions, and as the starting value for the closed-loop simulations.
- Perform open-loop simulations to collect data for the POD basis for the affine-linear LPV approximation (16).
- (2) Compute the Riccati solution P_0 and the Lyapunov solutions L_k via (17) and (18).
- (3) Close the loop with the nonlinear feedback law

$$u(t) = -B^{\mathsf{T}}\left(P_0 + \sum_{k=1}^r \hat{\rho}_k(v(t))L_k\right) Mv(t).$$
 (19)

In the presented numerical study, the relevant steps were realized as follows. To acquire the data for the POD basis, we take 401 snapshots of the velocity equally distributed on the time interval [0, 0.5] for the test signal

$$u(t) = \begin{bmatrix} \sin(t) & 0 \end{bmatrix}^{\mathsf{T}},\tag{20}$$

and define V_r as the matrix of the r leading left singular vectors with respect to the weighted inner-product

induced by the mass matrix M of the finite element discretization; see [3]. Then the LPV approximation (15) is computed for the NSE with $\lambda = 0.75$. In the following, we denote the feedback definition by the truncated SDRE series approximation (19) of parameter dimension r by **xSDRE-r** and the classical linear-quadratic regulator feedback, which is readily defined as $u(t) = -B^{\mathsf{T}} P_0 M v(t)$ by LQR (which is equivalent to **xSDRE-0**; cf. the feedback law (19)).

To trigger the instabilities in the closed-loop simulations, we apply the test signal (20) on a short time $[0, t_c]$ before we *switch on* the feedback at $t = t_c$. In this way, the system will deviate from the linearization point. For t_c too large, the state may have left the region of attraction for which the linear LQR or the SDRE-based controller will stabilize the nonlinear system.

In our experiments, we employed *Tikhonov regularization* in the form of an $\alpha \in \{1, 10^3\}$, which is included by replacing the original input matrix *B* by the scaled version $\check{B} := \frac{1}{\sqrt{\alpha}}B$ in the definition of the SDRE (3) as well as in the solved Riccati and Lyapunov equations (17) and (18). Consequently, the corresponding feedback needs to be scaled as well, e.g., in the LQR case as

$$u(t) = -\frac{1}{\alpha} B^{\mathsf{T}} \breve{P}_0 M v(t),$$

where P_0 solves the Riccati equation with B. Also, we used bisection of the time domain to identify a t_c close to a critical value that marks the performance region of the controllers. For both cases of α , we found that the xSDRE-r approach enlarges the domain of attraction of the LQR-controller. We illustrate this finding by plotting the outputs of the closed-loop systems as measured for the LQR-feedback and the xSDRE-r-feedback for various r. Apart from the qualitative differences in the performance, e.g., stabilization is achieved or not, in these setups, a quantitative effect of the reduction parameter r as part of the nonlinear feedback definition becomes evident. In fact, for the case of $\alpha = 10^3$ and $t_c = 0.125$, the plain LQRfeedback did not stabilize the system, while additional modes in the feedback design improved the performance steadily, until with r = 10 stabilization was achieved; see Figure 2 with the norms of the feedback actions over time for this setup.

For smaller regularization parameter $\alpha = 1$ that enables larger control actions, the overall region of performance is extended at the expense of a more sensitive control regime. Here, an illustrative t_c could be identified at t = 0.65 with the LQR-feedback being not stabilizing in contrast to the xSDRE-10-feedback; see Figure 3. However, with xSDRE-2 was stabilizing while xSDRE-5 failing to do so, a clear trend for improvement in performance for larger r could not be observed.

Overall, we can state that the truncated SDRE approach, with an almost negligible overhead in the simulation phase, is capable to improve the feedback per-



Figure 2: Norms of feedback for different r and the corresponding outputs for the LQR and the xSDRE-10-feedback for the case of $\alpha = 10^3$ and $t_c = 0.125$. A continuous performance improvement for xSDRE-r can be observed with stabilization of the system at xSDRE-10, while the classical LQR-feedback fails.



Figure 3: Norms of feedback for different r and the corresponding outputs for the LQR and the xSDRE-10-feedback for the case of $\alpha = 1$ and $t_c = 0.65$. The xSDRE-10-feedback is able to stabilize the system in contrast to the classical LQR approach.

formance if compared to the classical LQR feedback. In particular, for larger regularization parameters the difference in the region of attraction was rather small but the smooth relation between the additional complexity r and the performance turned out to be a useful hyperparameter in the feedback design. In the regime with less regularization, the **xSDRE-r** approach proved to give decisive advantages concerning the domain of performance at the extra cost of finding a suitable r.

7 Conclusion

In this work, we have presented a general framework that uses the embedding of nonlinear systems in the class of LPV systems, POD for reduction of the complexity of the parameter dimension, and the quadratic structure of the convection term in the Navier-Stokes equations to make the nonlinear feedback design through truncated expansions of the SDRE applicable. With state-of-the-art matrix equations solvers, computational feasibility of this approach was achieved too. As illustrated by a numerical example, this generic nonlinear approach provides a measurable improvement over the closely related classical LQR approach with a small additional computational effort at runtime.

Certainly, through the dependency on the POD basis for the parametrization, this approach is problem-specific. We would argue, however, that POD is a rather standard and generally applicable way to define parametrizations so that our proposed method enjoys a similarly general scope.

The potentials and needs for future work are manifold. Firstly, it would be interesting to investigate expansions of second order. Secondly, we have not paid particular attention to the definition of the coordinates for the affine LPV approximation and simply resorted to POD. Since a small dimension r is most important, in particular if one wants to consider a second-order expansions of the SDRE, other, possibly nonlinear parametrizations might be considered. For the analysis of the approximation, suitable measures for the residual, e.g., in the SDRE approximation, need to be derived together with formulas for feasible evaluations in the large-scale systems case.

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