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Holographic torus correlators of stress tensor in $\text{AdS}_3/\text{CFT}_2$

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Abstract: In this work, we investigate holographic correlators of the stress tensor of a conformal field theory (CFT) on a torus within the context of $\text{AdS}_3/\text{CFT}_2$. To compute the correlators of the stress tensor, we employ the Einstein-Hilbert theory of gravity and perturbatively solve Einstein's equation in the bulk. In addition, we present an explicit prescription for developing a recurrence relation that simplifies the computation of higher point correlators. Our results show that the correlators and recurrence relation are consistent with known results in CFTs. Additionally, in line with the proposed cutoff-AdS/$T\bar{T}$-CFT holography, we extend our computation program to investigate holographic torus correlators at a finite cutoff in the AdS_3 and derive a parallel recurrence relation associated with higher point correlators.

Keywords: AdS-CFT Correspondence, Field Theories in Lower Dimensions, Gauge-Gravity Correspondence, Conformal and W Symmetry

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1 Introduction

Understanding nonperturbative effects in the presence of strong coupling is one of the most difficult problems in modern physics. Analytical results in strong coupling are rare and extremely difficult to obtain. A remarkable tool for strongly coupled QFT is offered by the holographic principle [1, 2]. The Anti-de Sitter gravity/conformal field theory (AdS/CFT) correspondence [3–5] provides a rare window to gain analytical insight into strongly coupled physics. In the most helpful limit to exploit this correspondence, we use the weakly coupled bulk description to study the physics of strong coupling by performing gravitational perturbative calculations.

On the other hand, while the AdS/CFT correspondence is a powerful tool, we have yet to fully harness its computational power for the most fundamental observables of conformal field theories (CFTs) — the correlators of local operators. Stress tensor correlators provide critical information about the energy, momentum, and stress distribution of a system. Previous studies on holographic correlators of the stress tensor have mainly focused on CFTs with trivial topology, with holographic computations done in pure AdS space [6–9], even in $\mathrm{AdS}_3/\mathrm{CFT}_2$ [10]. The study of holographic field theories on manifolds with nontrivial topologies [11], e.g., tori or higher genus Riemann surfaces, is important as it provides a way to study the holographic principle in curved spacetimes and understand the behavior of conformal field theories in curved spacetimes (to study the holographic principle and behavior of CFTs in curved spacetimes). This has important implications for the study of quantum gravity and the AdS/CFT correspondence. However, investigating the correlators in CFTs with nontrivial topology through the variational principle requires solving Einstein's equation in a nontrivial bulk background geometry. While near-boundary solutions are well-understood [12–16], solving the global boundary value problem is generally very difficult without the full symmetry of pure AdS. This is even true for linearized equations and has been discussed in previous literature [17–20]. Moreover, the higher-point stress tensor correlators in strongly coupled CFTs with nontrivial topology are still not well understood. Therefore, explicit results from holographic computations are highly desirable.

We address (the) long-standing problem in AdS_3/CFT_2 by computing the holographic torus correlators of the stress tensor via Einstein's equation in the thermal AdS background. We then propose a prescription, which applies to any Riemann surface, for computing n-point correlators by deriving a recurrence relation. Our results are consistent with the corresponding CFT data [21], providing a non-trivial check of AdS_3/CFT_2 for correlators and Ward identities in torus CFTs. We extend our program of computing holographic stress tensor correlators to the cutoff-AdS/$T\bar{T}$-CFT holography, which is a valuable tool for exploring and better understanding holography in asymptotically AdS and non-AdS spacetimes in a controlled setting. The $T\bar{T}$-deformation [22, 23] of holographic CFTs has been proposed as a way to move the conformal boundary to a finite cutoff in the bulk AdS space [24]. While stress tensor correlators of $T\bar{T}$-deformed CFTs have been studied in the complex plane [25, 26] and on the torus [21, 27], and holographic correlators in the complex plane were investigated in [25, 28], we present the first computation of holographic correlators on a torus at a finite cutoff in thermal AdS_3. We also obtain a recurrence relation of correlators similar to that in the case of CFTs.

2 Holographic prescription

In holographic computation, we use the Fefferman-Graham coordinates near the conformal boundary [14]. This allows us to express the bulk metric in a simple form as following

$$ds^2 = \frac{dr^2}{r^2} + \frac{1}{r^2} g_{ij}(x,r) dx^i dx^j. \tag{2.1}$$

In dimension three, the Fefferman-Graham series of the metric truncates as Banados space-time [29]

$$g_{ij}(x,r) = g_{ij}^{(0)}(x) + g_{ij}^{(2)}(x) r^2 + g_{ij}^{(4)}(x) r^4, \tag{2.2}$$

and Einstein's equation is reduced to one equation that determines $g^{(4)}$ in terms of $g^{(0)}$ and $g^{(2)}$:

$$g_{ij}^{(4)} = \frac{1}{4} g_{ik}^{(2)} g^{(0)kl} g_{lj}^{(2)}, \tag{2.3}$$

and another two equations

$$\nabla^{(0)i} g_{ij}^{(2)} = \nabla_j^{(0)} g^{(2)i}{}_i, \tag{2.4}$$

$$g^{(2)i}{}_i = -\frac{1}{2} R[g^{(0)}] \tag{2.5}$$

where the covariant derivative and raising (lowering) indices are all with respect to the metric $g^{(0)}$. If the holographic field theory lives on a cutoff surface $r = r_c$ as the boundary, the background metric of the field theory γ is identified with the induced metric h on the boundary by

$$\gamma_{ij} = r_c^2 h_{ij}. \tag{2.6}$$

By taking the functional derivative of the gravity on-shell action [15, 30, 31] with respect to the boundary metric, we can identify the one-point correlator of the stress tensor with the Brown-York tensor on the boundary

$$\langle T_{ij} \rangle = -\frac{1}{8\pi G}(K_{ij} - K h_{ij} + h_{ij}), \tag{2.7}$$

where G is Newton constant and K_{ij} is the extrinsic curvature. From (2.4) and (2.5) (or from the Gauss-Codazzi equation in general dimensions), we have the conservation equation

$$\nabla^i \langle T_{ij} \rangle = 0, \tag{2.8}$$

and the trace relation

$$\langle T \rangle = \frac{R}{16\pi G} + 4\pi G r_c^2 (\langle T^{ij} \rangle \langle T_{ij} \rangle - \langle T \rangle^2). \tag{2.9}$$

For CFTs, the cutoff surface is at the conformal boundary $r_c = 0$, and the trace relation reduces to the Weyl anomaly

$$\langle T \rangle = \frac{R}{16\pi G}. \tag{2.10}$$

For $T\bar{T}$-deformed CFTs, the cutoff location r_c is related to the $T\bar{T}$ deformation parameter μ by [24]

$$\mu = 16\pi G r_c^2. \tag{2.11}$$

To obtain multi-point correlators, it suffices to compute the varied one-point correlator for a variation of the boundary metric. Equations (2.8) and (2.10)(or (2.9)) serve as our basis for this computation, but they don't fully determine the one-point correlator, or in the bulk language, the near-boundary solution to Einstein's equation is (understandably) not unique. The remaining information, which in our case is two constants of integration, must be drawn from the global geometry of the bulk space by requiring that the near-boundary solution reconstructed from the one-point correlator can be extended to be a global solution. We will show in our case that this condition, dubbed as the global regularity condition, fixes the two constants.

There is another way to determine the two constants of integration, which naturally leads to a recurrence relation for the holographic higher-point correlators. These two constants represent the one-point-averaged correlators, which correspond to changes in lower-point correlators due to global metric variations. On a Riemann surface, global metric variations are related to the differentiation with respect to the moduli. This concept has previously been used to investigate the stress tensor insertion Ward identities (see references [32] and [33]). In particular, for a torus with metric $ds^2 = dz d\bar{z}$ and modular parameter τ, if we let the global metric vary as $\delta\gamma_{\bar{z}\bar{z}}(z) = \alpha$ and $\delta\gamma_{zz}(z) = \bar{\alpha}$, the new metric is given by:

$$\begin{aligned} ds^2 &= dzd\bar{z} + \bar{\alpha}dz^2 + \alpha d\bar{z}^2, \\ &= (1+\alpha+\bar{\alpha})d(z+\alpha(\bar{z}-z))d(\bar{z}+\bar{\alpha}(z-\bar{z}))+O(\alpha^2). \end{aligned} \tag{2.12}$$

One can perform a Weyl transformation by multiplying a global factor $1-\alpha-\bar{\alpha}$ and a coordinate transformation $\phi : z' = z+\alpha(\bar{z}-z), \bar{z}' = \bar{z}+\bar{\alpha}(z-\bar{z})$ to equation (2.12). We obtain a torus with the Euclidean metric $ds^2 = dz'd\bar{z}'$ and a new modular parameter $\tau' = \tau + \alpha(\bar{\tau}-\tau)$. Therefore, variations of modular parameter of the torus are equivalent to global variations of the metric and coordinate transformations. For a general correlator $\langle O\rangle$, we have:

$$(\bar{\tau}-\tau)\partial_\tau\langle O\rangle = \mathcal{L}_{(z-\bar{z})\partial_z}\langle O\rangle + \int_{\mathrm{T}^2} d^2z \left(\frac{\delta\langle O\rangle}{\delta\gamma_{\bar{z}\bar{z}}(z)} - \frac{\delta\langle O\rangle}{\delta\gamma_{z\bar{z}}(z)}\right), \tag{2.13}$$

$$(\tau-\bar{\tau})\partial_{\bar{\tau}}\langle O\rangle = \mathcal{L}_{(\bar{z}-z)\partial_{\bar{z}}}\langle O\rangle + \int_{\mathrm{T}^2} d^2z \left(\frac{\delta\langle O\rangle}{\delta\gamma_{zz}(z)} - \frac{\delta\langle O\rangle}{\delta\gamma_{z\bar{z}}(z)}\right). \tag{2.14}$$

Here $\mathcal{L}$ denotes the Lie derivative. The explicit realization of [32, 33] is crucial for deriving the holographic recurrence relation of higher-point stress tensor correlators, discussed later in this letter.

3 Torus correlators in holographic CFT

We start by computing the holographic torus correlators of stress tensor on the conformal boundary (for CFTs) from the thermal AdS_3. Other classical gravity saddles (real smooth ones) with the torus conformal boundary are classified in [34] (first considered in [35]). They can all be obtained from the thermal AdS_3 by modular transformations. The thermal AdS_3 is a solid torus with the metric

$$ds^2 = d\rho^2 + \cosh^2\rho dt^2 + \sinh^2\rho d\phi^2, \tag{3.1}$$

or in the form of the Fefferman-Graham series

$$ds^2 = \frac{dr^2}{r^2} + \frac{1}{r^2}\Big[dzd\bar{z} - r^2\pi^2(dz^2+d\bar{z}^2) + r^4\pi^4 dzd\bar{z}\Big], \tag{3.2}$$

with

$$r = \frac{1}{\pi e^\rho},\ z = \frac{\phi+it}{2\pi},\ \bar{z} = \frac{\phi-it}{2\pi}, \tag{3.3}$$

where $z,\bar{z}$ are doubly periodically identified $(z,\bar{z}) \sim (z+1,\bar{z}+1) \sim (z+\tau,\bar{z}+\bar{\tau})$. The conformal boundary at $\rho=\infty$ or $r=0$ is a torus with two periods 1 and τ and the Euclidean boundary metric $\gamma_{ij}dx^idx^j = dzd\bar{z}$. We read off one-point correlators from the bulk geometry

$$\langle T_{zz}\rangle = -\frac{\pi}{8G},\quad \langle T_{\bar{z}\bar{z}}\rangle = -\frac{\pi}{8G},\quad \langle T_{z\bar{z}}\rangle = 0. \tag{3.4}$$

To compute the holographic correlators, we take a variation of the boundary metric

$$\delta\gamma_{ij}dx^i dx^j = \epsilon f_{ij}(z,\bar{z})dx^i dx^j. \tag{3.5}$$

The variation $\delta\gamma_{ij}$ induces a bulk metric variation $\delta g_{\mu\nu}$ through Einstein's equations. The resulting variation of one-point correlators can be formally expressed by powers of the infinitesimal parameter ϵ

$$\sum_{n=1}^{\infty} \epsilon^n T_{ij}^{[n]}. \tag{3.6}$$

From (2.8) and (2.10), we can order-by-order solve $T_{ij}^{[n]}$ to compute $n+1$ point correlators. For the first-order, we find

$$\begin{aligned}
T_{z\bar{z}}^{[1]} &= \frac{1}{16\pi G}(-2\pi^2(f_{zz}+f_{\bar{z}\bar{z}}) + \partial_{\bar{z}}^2 f_{zz} - 2\partial_z\partial_{\bar{z}} f_{z\bar{z}} + \partial_z^2 f_{\bar{z}\bar{z}}),\\
T_{zz}^{[1]}(z) &= \frac{1}{16\pi G}\Bigg[(-\partial_z\partial_{\bar{z}} f_{zz} + 2\partial_z^2 f_{z\bar{z}})(z) + C^{[1]}\\
&\qquad + \frac{1}{\pi}\int_{\mathrm{T}^2} d^2w G_\tau(z-w)(-4\pi^2\partial_w - \partial_w^3) f_{\bar{z}\bar{z}}(w)\Bigg],\\
T_{\bar{z}\bar{z}}^{[1]}(z) &= \text{c.c. of } T_{zz}^{[1]}(z)
\end{aligned} \tag{3.7}$$

where "c.c." denotes complex conjugate, $C^{[1]}, \bar{C}^{[1]}$ are constants of integration, and

$$G_\tau(z) = \zeta_\tau(z) - 2\zeta_\tau\left(\frac{1}{2}\right)z + \frac{2\pi i}{\mathrm{Im}\tau}\mathrm{Im}z \tag{3.8}$$

is a Green's function on a torus with $\zeta_\tau(z)$ being the Weierstrass Zeta function. More details can be found in appendix A.

As we discussed earlier, any choice of constants corresponds to a near-boundary solution to Einstein's equation in its Fefferman-Graham coordinates. Although the Fefferman-Graham coordinates of the varied bulk metric may be different from the ρ,ϕ,t or $r,z,\bar{z}$ coordinates of the solid torus, we can make them coincide in the region $\rho\in(0,\infty)$ by using a boundary-preserving diffeomorphism. Therefore, a generic bulk metric solution in the region $\rho\in(0,\infty)$ for a varied boundary metric is given by a Fefferman-Graham series in ρ,ϕ,t from equation (3.7), plus a change by a boundary-preserving diffeomorphism. To satisfy the global regularity condition, the metric must be regular at $\rho=0$. We leave the details of the computation to appendix B, but to the first order, we have the following condition:

$$\begin{aligned}
\int_{\mathrm{T}^2} d^2z[g_{zz}^{(2)[1]} - g_{\bar{z}\bar{z}}^{(2)[1]} + 2\pi^2(g_{zz}^{(0)[1]} - g_{\bar{z}\bar{z}}^{(0)[1]})] &= 0,\\
\int_{\mathrm{T}^2} d^2z[g_{zz}^{(2)[1]} + 2g_{z\bar{z}}^{(2)[1]} + g_{\bar{z}\bar{z}}^{(2)[1]}] &= 0
\end{aligned} \tag{3.9}$$

where $g^{(0)[1]}$ and $g^{(2)[1]}$ are the first-order variations of $g^{(0)}$ and $g^{(2)}$, respectively. The above conditions determine the constants as follows:

$$C^{[1]} = \frac{4\pi^2}{\mathrm{Im}\tau}\int_{\mathrm{T}^2} d^2z f_{\bar{z}\bar{z}},\ \bar{C}^{[1]} = \frac{4\pi^2}{\mathrm{Im}\tau}\int_{\mathrm{T}^2} d^2z f_{zz}. \tag{3.10}$$

The constants can be also obtained by (2.13) and (2.14). Finally, we can obtain two-point correlators as follows:

$$\begin{aligned}
\langle T_{zz}(z)T_{zz}(w)\rangle &= \frac{1}{32\pi^2 G}\left(\wp''_\tau(z-w)+4\pi^2\wp_\tau(z-w)+8\pi^2\zeta_\tau\left(\frac{1}{2}\right)\right),\\
\langle T_{zz}(z)T_{\bar{z}\bar{z}}(w)\rangle &= -\frac{1}{32\pi G}\partial_z\partial_{\bar{z}}\delta(z-w),\\
\langle T_{z\bar{z}}(z)T_{zz}(w)\rangle &= \frac{1}{32\pi G}(-2\pi^2+\partial_z^2)\delta(z-w),\\
\langle T_{z\bar{z}}(z)T_{z\bar{z}}(w)\rangle &= -\frac{1}{32\pi G}\partial_z\partial_{\bar{z}}\delta(z-w),
\end{aligned} \tag{3.11}$$

where $\delta(z-w)$ is the delta function and $\wp_\tau(z)=-\zeta'_\tau(z)$ is the Weierstrass P function. The two-point correlator coincides with the one derived in the field theory, cf. equation (58) in [21].

Alternatively, the constants can be derived from (2.13) and (2.14) in the form of a recurrence relation. We begin by turning on a boundary Euclidean metric variation $\delta\gamma_{\bar{z}\bar{z}}=F$ while keeping other components fixed, then we get the holographic Virasoro Ward identity [36, 37] from (2.8) and (2.10):

$$\partial_{\bar{z}}\langle T_{zz}\rangle - 2\partial_z F\langle T_{zz}\rangle - F\partial_z\langle T_{zz}\rangle + \frac{1}{16\pi G}\partial_z^3 F = 0. \tag{3.12}$$

Taking the n-th functional derivative with respect to F and evaluating at $F=0$, we find

$$\begin{aligned}
&\partial_{\bar{z}}\langle T_{zz}(z)T_{zz}(z_1)\dots T_{zz}(z_n)\rangle\\
&-\sum_{i=1}^n \partial_z\delta(z-z_i)\langle T_{zz}(z)T_{zz}(z_1)\dots\hat{T}_{zz}(z_i)\dots T_{zz}(z_n)\rangle\\
&-\frac{1}{2}\sum_{i=1}^n \delta(z-z_i)\partial_z\langle T_{zz}(z)T_{zz}(z_1)\dots\hat{T}_{zz}(z_i)\dots T_{zz}(z_n)\rangle\\
&+\frac{1}{32\pi G}\delta_{n,1}\partial_z^3\delta(z-z_1)=0.
\end{aligned} \tag{3.13}$$

Where $\delta_{n,1}$ is the discrete delta function and $\hat{T}_{zz}(z_i)$ means to drop the i-th operator. Solving with Green's function on the torus, we have

$$\begin{aligned}
&\langle T_{zz}(z)T_{zz}(z_1)\dots T_{zz}(z_n)\rangle\\
&=\frac{1}{\pi}\sum_{i=1}^n\left[\partial_z G_\tau(z-z_i)\langle T_{zz}(z_1)\dots T_{zz}(z_n)\rangle - \frac{1}{2}G_\tau(z-z_i)\partial_{z_i}\langle T_{zz}(z_1)\dots T_{zz}(z_n)\rangle\right]\\
&\quad-\frac{1}{32\pi^2 G}\delta_{n,1}\partial_z^3 G_\tau(z-z_1)+\frac{1}{\mathrm{Im}\tau}\int_{\mathrm{T}^2} d^2v\langle T_{zz}(v)T_{zz}(z_1)\dots T_{zz}(z_n)\rangle.
\end{aligned} \tag{3.14}$$

The last term of one-point-averaged correlators, corresponding to the constants of integration, can be obtained from (2.13) by setting $O=T_{zz}(z_1)\dots T_{zz}(z_n)$ with the last term vanishing for CFTs. Then we obtain the recurrence relation

$$\begin{aligned}
&\langle T_{zz}(z)T_{zz}(z_1)\dots T_{zz}(z_n)\rangle \\
&= -i\partial_\tau \langle T_{zz}(z_1)\dots T_{zz}(z_n)\rangle + \frac{1}{32\pi^2 G}\delta_{n,1}\wp_\tau''(z-z_1) \\
&\quad -\frac{1}{2\pi}\sum_{i=1}^{n}\Bigg[2\left(\wp_\tau(z-z_i)+2\zeta_\tau\left(\frac{1}{2}\right)\right)\langle T_{zz}(z_1)\dots T_{zz}(z_n)\rangle \\
&\quad +\left(\zeta_\tau(z-z_i)-2\zeta_\tau\left(\frac{1}{2}\right)(z-z_i)\right)\partial_{z_i}\langle T_{zz}(z_1)\dots T_{zz}(z_n)\rangle\Bigg],
\end{aligned} \tag{3.15}$$

which is consistent with the field theory derivation, cf. equation (57) in [21]. This recurrence relation recovers the two-point correlators for thermal AdS$_3$ (3.11) and provides an efficient way to compute higher-point correlators. For example, the three-point correlator is given by

$$\begin{aligned}
&\langle T_{zz}(z_1)T_{zz}(z_2)T_{zz}(z_3)\rangle \\
&= -\frac{1}{64\pi^3 G}\Bigg[12\wp_\tau(z_1-z_2)\wp_\tau(z_2-z_3)\wp_\tau(z_3-z_1) \\
&\qquad +4\pi^2\Big(\wp_\tau(z_1-z_2)\wp_\tau(z_2-z_3)+\wp_\tau(z_2-z_3)\wp_\tau(z_3-z_1)+\wp_\tau(z_3-z_1)\wp_\tau(z_1-z_2)\Big) \\
&\qquad +\left(16\pi^2\zeta_\tau\left(\frac{1}{2}\right)-g_{2,\tau}\right)\Big(\wp_\tau(z_1-z_2)+\wp_\tau(z_2-z_3)+\wp_\tau(z_3-z_1)\Big)\Bigg] \\
&\quad +\frac{1}{320\pi^3 G}\left(-4\left(g_{2,\tau}+60\pi^2\zeta_\tau\left(\frac{1}{2}\right)\right)\zeta_\tau\left(\frac{1}{2}\right)-i\pi\partial_\tau g_{2,\tau}+18g_{3,\tau}\right).
\end{aligned} \tag{3.16}$$

We have used the identities

$$\wp_\tau''(z) = 6\wp_\tau^2(z) - \frac{1}{2}g_{2,\tau} \tag{3.17}$$

and

$$2\pi i\partial_\tau \wp_\tau(z) = \left(\zeta_\tau(z)-2\zeta_\tau\left(\frac{1}{2}\right)z\right)\wp_\tau'(z)+\frac{1}{3}\wp_\tau''(z)-4\zeta_\tau\left(\frac{1}{2}\right)\wp_\tau(z)-\frac{1}{6}g_{2,\tau} \tag{3.18}$$

with $g_{2,\tau}$ and $g_{3,\tau}$ being the invariants

$$\begin{aligned}
g_{2,\tau} &= 60\sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)}\frac{1}{(m+n\tau)^4}, \\
g_{3,\tau} &= 140\sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)}\frac{1}{(m+n\tau)^6}.
\end{aligned} \tag{3.19}$$

The recurrence relation (3.15) can be employed to compute the correlators for any dominant gravity saddle in the path integral or to calculate the exact correlators from a complete partition function if it is available. However, this task is considerably more intricate and challenging, as demonstrated by prior research such as [34, 38].

4 $T\bar{T}$-deformed Torus correlators

In this section, we compute the holographic stress tensor correlators on a torus at a finite cutoff (for $T\bar{T}$-deformed CFTs). We start by embedding a torus as a cutoff surface into the thermal AdS$_3$

$$ds^2 = d\rho^2 + \pi^2 e^{2\rho}[dZd\bar{Z} - e^{-2\rho}(dZ^2 + d\bar{Z}^2) + e^{-4\rho}dZd\bar{Z}]. \tag{4.1}$$

By taking the periods in Z to be 1 and $\Omega = \frac{\tau + e^{-2\rho_c}\bar{\tau}}{1+e^{-2\rho_c}}$ and defining the coordinates for torus

$$z = \frac{Z - e^{-2\rho_c}\bar{Z}}{1 - e^{-2\rho_c}}, \; \bar{z} = \frac{\bar{Z} - e^{-2\rho_c}Z}{1 - e^{-2\rho_c}} \tag{4.2}$$

and the Fefferman-Graham radial coordinate

$$r = \frac{1}{\pi e^{\rho}(1 - e^{-2\rho_c})}, \tag{4.3}$$

we get a torus at $\rho = \rho_c$ with periods 1 and τ in the $z, \bar{z}$ coordinates, and a field theory background metric $\gamma_{ij}dx^i dx^j = r_c^2 h_{ij}dx^i dx^j = dzd\bar{z}$. We read off one-point correlators from the bulk geometry

$$\begin{aligned} \langle T_{zz}\rangle &= -\frac{\pi}{8G}\frac{1 - e^{-2\rho_c}}{1 + e^{-2\rho_c}}, \; \langle T_{\bar{z}\bar{z}}\rangle = -\frac{\pi}{8G}\frac{1 - e^{-2\rho_c}}{1 + e^{-2\rho_c}} \\ \langle T_{z\bar{z}}\rangle &= \frac{\pi}{8G}\frac{e^{-2\rho_c} - e^{-4\rho_c}}{1 + e^{-2\rho_c}}. \end{aligned} \tag{4.4}$$

As in the previous section, we can solve the varied one-point correlator order by order from (2.8) and (2.9) for a variation of the field theory metric $\epsilon f_{ij}dx^i dx^j$. Leaving the computation of $T_{ij}^{[1]}$ to appendix C, we obtain the two-point correlator

$$\begin{aligned} &\langle T_{zz}(z)T_{zz}(w)\rangle \\ &= \frac{1}{16\pi G}\Bigg\{\frac{1}{(1+e^{-2\rho_c})^4}\Bigg[2\pi(\wp_\Omega(Z-W) + e^{-8\rho_c}\overline{\wp_\Omega(Z-W)}) \\ &\quad + \frac{1}{2\pi}(\wp''_\Omega(Z-W) + e^{-8\rho_c}\overline{\wp''_\Omega(Z-W)}) + 4\pi\left(\zeta_\Omega\left(\frac{1}{2}\right) + e^{-8\rho_c}\zeta_{\bar{\Omega}}\left(\frac{1}{2}\right)\right)\Bigg] \\ &\quad - \frac{1}{(1-e^{-4\rho_c})^3}\Big[2\pi^2 e^{-2\rho_c}(1-e^{-2\rho_c})^2(1+e^{-4\rho_c}) \\ &\quad + (2e^{-2\rho_c} - 3e^{-6\rho_c} + 2e^{-10\rho_c})\partial_z^2 - (e^{-4\rho_c} + e^{-8\rho_c})\partial_z\partial_{\bar{z}} + e^{-6\rho_c}\partial_{\bar{z}}^2\Big]\delta(z-w)\Bigg\}. \end{aligned} \tag{4.5}$$

To relate to the $T\bar{T}$-deformed CFT, we have $\rho_c = \sinh^{-1}\frac{1}{2\pi r_c}$ and r_c in turn is related to the $T\bar{T}$ deformation parameter μ by (2.11). As a cross-check, we can compute the generating functional I from the one-point correlators (4.4) (as a special case of (2.13) and (2.14)) by

$$\begin{aligned} i\partial_\tau I &= \langle T_{zz}\rangle - \langle T_{z\bar{z}}\rangle \\ -i\partial_{\bar{\tau}} I &= \langle T_{\bar{z}\bar{z}}\rangle - \langle T_{z\bar{z}}\rangle \end{aligned} \tag{4.6}$$

We obtain

$$
\begin{aligned}
I &= \frac{i\pi}{8G}(1-e^{-2\rho_c})(\tau-\bar{\tau}) \\
&= \frac{i}{\mu}(\tau-\bar{\tau})\left(\sqrt{1+\frac{\pi\mu c}{6}}-1\right)
\end{aligned}
\tag{4.7}
$$

It satisfies the $T\bar{T}$ flow equation for partition function [27, 39, 40] with CFT limit $I = \frac{i\pi}{12}c(\tau-\bar{\tau})$, cf. equation (2.14) in [40] with the identification of the deformation parameter $\mu = -2\lambda^2$. A recurrence algorithm to compute higher-point correlators can be derived in a way similar to the previous section, but it does not have a simple form like (3.15). So we leave it to appendix C.

5 Conclusions and perspectives

In this letter, we investigate the holographic correlators of the stress tensor on the conformal boundary for CFTs and at a finite cutoff for $T\bar{T}$-deformed CFTs on torus topology. First, we solve Einstein's equation with a torus boundary to complete a direct calculation of the correlators. Then, we develop a holographic recurrence algorithm to calculate higher-point correlators of the stress tensor. The resulting recurrence relation for holographic CFTs is identical to that found in standard CFTs, which provides an explicit check of the $\mathrm{AdS}_3/\mathrm{CFT}_2$. The recurrence relation in $T\bar{T}$-deformed CFTs can also be obtained using this algorithm and the low point correlators coincide with that obtained by different perspectives in the literature. Moreover, the recurrence algorithm can be naturally generalized to higher-genus Riemann surfaces.

It would be interesting to extend our computation of holographic correlators to other operators and higher dimensions. Exact results could contribute significantly to the understanding of CFTs with non-trivial topology, in terms of OPEs, conformal blocks [41–48] and possible bootstrap programs [49] on the cylinder or the other topologies. With a proper prescription for analytic continuation to Minkowski signature, we can also obtain exact results for holographic transport coefficients, as was done in [50] and numerous subsequent works.

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A Green's function on torus

In this appendix, we briefly introduce the Green's functions on a torus with a modular parameter τ. The defining equations for the Green's functions $G_\tau(z,w)$ and $\tilde{G}_\tau(z,w)$ are

$$\frac{1}{\pi}\partial_{\bar{z}}G_\tau(z,w)=\delta(z-w)-\frac{1}{\text{Im}\tau} \tag{A.1}$$

and

$$\frac{1}{\pi}\partial_z\partial_{\bar{z}}\tilde{G}_\tau(z,w)=\delta(z-w)-\frac{1}{\text{Im}\tau}, \tag{A.2}$$

where $\delta(z-w)$ is the delta function with respect to the measure $d^2z=\frac{i}{2}dz\wedge d\bar{z}$. Green's functions can be rewritten as $G_\tau(z-w)$ and $\tilde{G}_\tau(z-w)$ by translational invariance. For Green's function $G_\tau(z)$, since it takes the form $\frac{1}{z}$ in the complex plane, it is tempting to represent Green's function on the torus as a formal series

$$\sum_{(m,n)\in\mathbb{Z}^2}\frac{1}{z-(m+n\tau)} \tag{A.3}$$

with manifest double periodicity. To make the formal series convergent, it's natural to add the holomorphic terms $\sum\limits_{(m,n)\in\mathbb{Z}^2\backslash(0,0)}\frac{1}{(m+n\tau)}-\frac{z}{(m+n\tau)^2}$ and we obtain the Weierstrass Zeta function

$$\zeta_\tau(z)=\frac{1}{z}+\sum_{(m,n)\in\mathbb{Z}^2\backslash(0,0)}\left(\frac{1}{z-(m+n\tau)}+\frac{1}{(m+n\tau)}+\frac{z}{(m+n\tau)^2}\right). \tag{A.4}$$

It's straightforward to prove $\zeta_\tau(z+1)-\zeta_\tau(z)$ and $\zeta_\tau(z+\tau)-\zeta_\tau(z)$ are independent on z, so we can restore the double periodicity by adding a linear function, obtaining the Green's function on the torus

$$G_\tau(z)=\zeta_\tau(z)-2\zeta_\tau\left(\frac{1}{2}\right)z+\frac{2\pi i}{\text{Im}\tau}\text{Im}z. \tag{A.5}$$

Similarly, we have

$$\tilde{G}_\tau(z)=\log(|\sigma_\tau(z)|^2)-\zeta_\tau\left(\frac{1}{2}\right)z^2-\overline{\zeta_\tau\left(\frac{1}{2}\right)}\bar{z}^2-\frac{2\pi}{\text{Im}\tau}(\text{Im}z)^2 \tag{A.6}$$

where

$$\sigma_\tau(z)=z\prod_{(m,n)\in\mathbb{Z}^2\backslash(0,0)}\left(1-\frac{z}{m+n\tau}\right)e^{\frac{z}{m+n\tau}+\frac{z^2}{2(m+n\tau)^2}} \tag{A.7}$$

is the Weierstrass sigma function, with its log derivative being the Weierstrass zeta function.

B Global regularity condition

As discussed in section 3, we can make the Fefferman-Graham coordinates of the varied bulk metric (Fefferman-Graham coordinates always exist near the conformal boundary, see [16] for example) coincide with the torus coordinates ρ, ϕ, t by a diffeomorphism in the region $\rho \in (0, \infty)$. So the varied bulk metric in this region is given by a Fefferman-Graham series in ρ, ϕ, t, determined from (3.7), plus a change by a boundary preserving diffeomorphism. We characterize the diffeomorphism by a vector expanded in powers of ϵ

$$V = \sum_{n=1}^{\infty} \epsilon^n V^{[n]} \tag{B.1}$$

To the first order, the varied bulk metric is given by

$$ds^2 = (1 + \epsilon \mathcal{L}_{V^{[1]}})(d\rho^2 + \cosh^2 \rho dt^2 + \sinh^2 \rho d\phi^2) + \epsilon g_{ij}^{FG[1]} dx^i dx^j, \tag{B.2}$$

where $g^{FG[1]}$ is the first-order variation of the bulk metric in its Feffermen-Graham coordinates, given by

$$\begin{aligned}
{g^{FG[1]}}_{zz} &= {g^{(2)[1]}}_{zz} - e^{-2\rho} {g^{(2)[1]}}_{z\bar{z}} + \pi^2 e^{2\rho} g^{(0)[1]}_{zz} - \pi^2 e^{-2\rho} g^{(0)[1]}_{\bar{z}\bar{z}}, \\
{g^{FG[1]}}_{z\bar{z}} &= -\frac{1}{2} e^{-2\rho} {g^{(2)[1]}}_{zz} - \frac{1}{2} e^{-2\rho} {g^{(2)[1]}}_{\bar{z}\bar{z}} + {g^{(2)[1]}}_{z\bar{z}} + \pi^2 (e^{2\rho} - e^{-2\rho}) g^{(0)[1]}_{z\bar{z}}, \\
{g^{FG[1]}}_{\bar{z}\bar{z}} &= {g^{(2)[1]}}_{\bar{z}\bar{z}} - e^{-2\rho} {g^{(2)[1]}}_{z\bar{z}} + \pi^2 e^{2\rho} g^{(0)[1]}_{\bar{z}\bar{z}} - \pi^2 e^{-2\rho} g^{(0)[1]}_{zz}.
\end{aligned} \tag{B.3}$$

We require the metric to be regular at $\rho = 0$, that is, its components in the (t, x, y) coordinates

$$t = t, \ x = \rho \cos \phi, \ y = \rho \sin \phi \tag{B.4}$$

which properly covers $\rho = 0$, are regular. We have the transformation equation of the components

$$\begin{aligned}
g_{tt} &= g_{tt}, \\
g_{t\rho} &= g_{tx} \cos \phi + g_{ty} \sin \phi, \\
g_{t\phi} &= \rho(-g_{tx} \sin \phi + g_{ty} \cos \phi), \\
g_{\rho\rho} &= g_{xx} \cos^2 \phi + 2 g_{xy} \cos \phi \sin \phi + g_{yy} \sin^2 \phi, \\
g_{\phi\phi} &= \rho^2 (g_{xx} \sin^2 \phi - 2 g_{xy} \cos \phi \sin \phi + g_{yy} \cos^2 \phi), \\
g_{\rho\phi} &= \rho[-(g_{xx} - g_{yy}) \cos \phi \sin \phi + g_{xy}(\cos^2 \phi - \sin^2 \phi)].
\end{aligned} \tag{B.5}$$

Take the $\rho \to 0$ limit, the components in (t, x, y) coordinates on the right-hand side should

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go to a limit that can only depend on t with a period $\mathrm{Im}\tau$

$$\begin{aligned}
\lim_{\rho\to 0} g_{tt} &= g^*_{tt}(t),\\
\lim_{\rho\to 0} g_{t\rho} &= g^*_{tx}(t)\cos\phi + g^*_{ty}(t)\sin\phi,\\
\lim_{\rho\to 0} \frac{g_{t\phi}}{\rho} &= -g^*_{tx}(t)\sin\phi + g^*_{ty}(t)\cos\phi,\\
\lim_{\rho\to 0} g_{\rho\rho} &= g^*_{xx}(t)\cos^2\phi + 2g^*_{xy}(t)\cos\phi\sin\phi + g^*_{yy}(t)\sin^2\phi,\\
\lim_{\rho\to 0} \frac{g_{\phi\phi}}{\rho^2} &= g^*_{xx}(t)\sin^2\phi - 2g^*_{xy}(t)\cos\phi\sin\phi + g^*_{yy}(t)\cos^2\phi,\\
\lim_{\rho\to 0} \frac{g_{\rho\phi}}{\rho} &= -(g^*_{xx}(t) - g^*_{yy}(t))\cos\phi\sin\phi + g^*_{xy}(t)(\cos^2\phi - \sin^2\phi).
\end{aligned} \tag{B.6}$$

Now we impose these conditions on the varied bulk metric in (B.2), and we find

$$\begin{aligned}
&\lim_{\rho\to 0}(2\cosh\rho\sinh\rho V^{[1]^\rho} + 2\cosh^2\rho\partial_t V^{[1]^t} + g^{FG[1]}_{tt}) = g^{*[1]}{}_{tt}(t),\\
&\lim_{\rho\to 0}(\partial_t V^{[1]^\rho} + \cosh^2\rho\partial_\rho V^{[1]^t}) = g^{*[1]}{}_{tx}(t)\cos\phi + g^{*[1]}{}_{ty}(t)\sin\phi,\\
&\lim_{\rho\to 0}\frac{\cosh^2\rho\partial_\phi V^{[1]^t} + \sinh^2\rho\partial_t V^{[1]^\phi} + g^{FG[1]}_{t\phi}}{\rho} = -g^{*[1]}{}_{tx}(t)\sin\phi + g^{*[1]}{}_{ty}(t)\cos\phi,\\
&\lim_{\rho\to 0} 2\partial_\rho V^{[1]^\rho} = g^{*[1]}{}_{xx}(t)\cos^2\phi + 2g^{*[1]}{}_{xy}(t)\cos\phi\sin\phi + g^{*[1]}{}_{yy}(t)\sin^2\phi,\\
&\lim_{\rho\to 0}\frac{2\cosh\rho\sinh\rho V^{[1]^\rho} + 2\sinh^2\rho\partial_\phi V^{[1]^\phi} + g^{FG[1]}_{\phi\phi}}{\rho^2} = g^{*[1]}{}_{xx}(t)\sin^2\phi - 2g^{*[1]}{}_{xy}(t)\cos\phi\sin\phi\\
&\qquad\qquad + g^{*[1]}{}_{yy}(t)\cos^2\phi,\\
&\lim_{\rho\to 0}\frac{\partial_\phi V^{[1]^\rho} + \sinh^2\rho\partial_\rho V^{[1]^\phi}}{\rho} = -(g^{*[1]}{}_{xx}(t) - g^{*[1]}{}_{yy}(t))\cos\phi\sin\phi + g^{*[1]}{}_{xy}(t)(\cos^2\phi - \sin^2\phi).
\end{aligned} \tag{B.7}$$

In addition, we have the power series expansion of (B.3) at $\rho = 0$

$$\begin{aligned}
g^{FG[1]}_{\phi\phi} &= g^{FG[1]}_{1\phi\phi}\rho + g^{FG[1]}_{2\phi\phi}\rho^2 + \mathrm{O}(\rho^3)\\
&= \left[\frac{1}{2\pi^2}(g^{(2)[1]}{}_{zz} + 2g^{(2)[1]}{}_{z\bar z} + g^{(2)[1]}{}_{\bar z\bar z}) + g^{(0)[1]}_{zz} + 2g^{(0)[1]}_{z\bar z} + g^{(0)[1]}_{\bar z\bar z}\right]\rho\\
&\quad - \frac{1}{2\pi^2}(g^{(2)[1]}{}_{zz} + 2g^{(2)[1]}{}_{z\bar z} + A_{\bar z\bar z})\rho^2 + \mathrm{O}(\rho^3),\\
g^{FG[1]}_{t\phi} &= g^{FG[1]}_{0t\phi} + \mathrm{O}(\rho^2) = \frac{i}{4\pi^2}(g^{(2)[1]}{}_{zz} - g^{(2)[1]}{}_{\bar z\bar z}) + \frac{i}{2}(g^{(0)[1]}_{zz} - g^{(0)[1]}_{\bar z\bar z}) + \mathrm{O}(\rho^2),\\
g^{FG[1]}_{tt} &= g^{FG[1]}_{0tt} + \mathrm{O}(\rho) = -\frac{1}{2\pi^2}(g^{(2)[1]}{}_{zz} - 2g^{(2)[1]}{}_{z\bar z} + g^{(2)[1]}{}_{\bar z\bar z}) + \mathrm{O}(\rho).
\end{aligned} \tag{B.8}$$

Integrating the third equation in (B.7) over the torus, we find

$$\int_{\mathrm{T}^2} d^2z\, g^{FG[1]}_{0t\phi} = 0. \tag{B.9}$$

From the fourth equation in (B.7) we know $V^{[1]\rho}$ can be linearly approximated near $\rho = 0$

$$V^{[1]\rho} = a_0 + a_1\rho + \mathrm{o}(\rho). \tag{B.10}$$

Then we subtract the fifth equation from the fourth and integrate it over the torus. We find

$$\int_{\mathrm{T}^2} d^2z\, g^{FG[1]}_{2\phi\phi} = 0. \tag{B.11}$$

Plugging (B.8) into (B.9) and (B.11), we obtain the global regularity condition (3.9).

C Computation of holographic correlators for $T\bar{T}$-deformed CFT

As in the section 3, to compute holographic correlators we solve the varied one-point correlator order by order from (2.8) and (2.9). For the first order, we get

$$\partial_{\bar z}T^{[1]}{}_{zz}+\partial_z T^{[1]}{}_{z\bar z} = \frac{\pi}{8G}\frac{1-e^{-2\rho_c}}{1+e^{-2\rho_c}}(-\partial_z f_{zz}-3\partial_z f_{\bar z\bar z}+2e^{-2\rho_c}\partial_{\bar z} f_{zz}+2e^{-2\rho_c}\partial_z f_{z\bar z}), \tag{C.1}$$

$$\partial_z T^{[1]}{}_{\bar z\bar z}+\partial_{\bar z} T^{[1]}{}_{z\bar z} = \frac{\pi}{8G}\frac{1-e^{-2\rho_c}}{1+e^{-2\rho_c}}(-\partial_{\bar z} f_{\bar z\bar z}-3\partial_{\bar z} f_{zz}+2e^{-2\rho_c}\partial_z f_{\bar z\bar z}+2e^{-2\rho_c}\partial_{\bar z} f_{z\bar z}), \tag{C.2}$$

and

$$\begin{aligned}&(1+e^{-4\rho_c})T^{[1]}{}_{z\bar z}+e^{-2\rho_c}(T^{[1]}{}_{zz}+T^{[1]}{}_{\bar z\bar z})\\ &=-\frac{\pi(1-e^{-2\rho_c})^2}{8G}(f_{zz}+f_{\bar z\bar z}+2e^{-2\rho_c}f_{z\bar z})+\frac{1-e^{-4\rho_c}}{16\pi G}(\partial_{\bar z}^2 f_{zz}-2\partial_z\partial_{\bar z}f_{z\bar z}+\partial_z^2 f_{\bar z\bar z}).\end{aligned} \tag{C.3}$$

Adding ∂_z (C.1) and $\partial_{\bar z}$ (C.2), and then plugging in (C.3), we get an equation of $T^{[1]}{}_{z\bar z}$

$$\begin{aligned}&(1-e^{-2\rho_c})^2\partial_Z\partial_{\bar Z}T_{1z\bar z}\\ &=-\frac{\pi(1-e^{-2\rho_c})^2}{8G}\partial_z\partial_{\bar z}(f_{zz}+f_{\bar z\bar z}+2e^{-2\rho_c}f_{z\bar z})\\ &\quad+\frac{1-e^{-4\rho_c}}{16\pi G}\partial_z\partial_{\bar z}(\partial_{\bar z}^2 f_{zz}-2\partial_z\partial_{\bar z}f_{z\bar z}+\partial_z^2 f_{\bar z\bar z})\\ &\quad-\frac{\pi e^{-2\rho_c}}{8G}\frac{1-e^{-2\rho_c}}{1+e^{-2\rho_c}}\Big[-\partial_z^2 f_{zz}-3\partial_z^2 f_{\bar z\bar z}-\partial_{\bar z}^2 f_{\bar z\bar z}-3\partial_{\bar z}^2 f_{zz}\\ &\quad+2e^{-2\rho_c}(\partial_z\partial_{\bar z}f_{zz}+\partial_z^2 f_{z\bar z}+\partial_z\partial_{\bar z}f_{\bar z\bar z}+\partial_{\bar z}^2 f_{z\bar z})\Big].\end{aligned} \tag{C.4}$$

We solve this equation with the Green's function $\tilde{G}_\Omega$ on torus (see appendix A)

$$\begin{aligned}T^{[1]}{}_{z\bar z}(z) = \frac{1}{8\pi^2 G}\int_{\mathrm{T}^2} d^2W\tilde{G}_\Omega(Z-W)\Big[&-\pi^2\partial_w\partial_{\bar w}(f_{zz}+f_{\bar z\bar z}+2e^{-2\rho_c}f_{z\bar z})\\ &+\frac{1+e^{-2\rho_c}}{2(1-e^{-2\rho_c})}\partial_w\partial_{\bar w}(\partial_{\bar w}^2 f_{zz}-2\partial_w\partial_{\bar w}f_{z\bar z}+\partial_w^2 f_{\bar z\bar z})\\ &-\frac{\pi^2 e^{-2\rho_c}}{1-e^{-4\rho_c}}(-\partial_w^2 f_{zz}-3\partial_w^2 f_{\bar z\bar z}-\partial_{\bar w}^2 f_{\bar z\bar z}-3\partial_{\bar w}^2 f_{zz}\\ &+2e^{-2\rho_c}(\partial_w\partial_{\bar w}f_{zz}+\partial_w^2 f_{z\bar z}+\partial_w\partial_{\bar w}f_{\bar z\bar z}+\partial_{\bar w}^2 f_{z\bar z}))\Big](w)+\frac{1}{8\pi G}D^{[1]},\end{aligned} \tag{C.5}$$

and $T^{[1]}{}_{zz}, T^{[1]}{}_{\bar{z}\bar{z}}$ follow from (C.2) and (C.3)

$$T^{[1]}{}_{zz} = \frac{1}{\pi}\int_{\mathrm{T}^2} d^2w G_\tau(z-w)\Bigg[\frac{\pi}{8G}\frac{1-e^{-2\rho_c}}{1+e^{-2\rho_c}}(-\partial_w f_{zz} - 3\partial_w f_{\bar{z}\bar{z}} \\ + 2e^{-2\rho_c}\partial_{\bar{w}} f_{zz} + 2e^{-2\rho_c}\partial_w f_{z\bar{z}}) - \partial_w T^{[1]}{}_{z\bar{z}}\Bigg] + \frac{E^{[1]}}{8\pi G}, \tag{C.6}$$

$$T^{[1]}{}_{\bar{z}\bar{z}} = \frac{1}{\pi}\int_{\mathrm{T}^2} d^2w \overline{G_\tau(z-w)}\Bigg[\frac{\pi}{8G}\frac{1-e^{-2\rho_c}}{1+e^{-2\rho_c}}(-\partial_{\bar{w}} f_{\bar{z}\bar{z}} - 3\partial_{\bar{w}} f_{zz} \\ + 2e^{-2\rho_c}\partial_w f_{\bar{z}\bar{z}} + 2e^{-2\rho_c}\partial_{\bar{w}} f_{z\bar{z}}) - \partial_{\bar{w}} T^{[1]}{}_{z\bar{z}}\Bigg] + \frac{\bar{E}^{[1]}}{8\pi G}, \tag{C.7}$$

where $D^{[1]}, E^{[1]}, \bar{E}^{[1]}$ are constants of integration. The global regularity condition reads in the present context as

$$\int_{\mathrm{T}^2} d^2Z\Big[g^{(2)[1]}{}_{ZZ} - g^{(2)[1]}{}_{\bar{Z}\bar{Z}} + 2\pi^2(1-e^{-2\rho_c})^2(g^{(0)[1]}{}_{ZZ} - g^{(0)[1]}{}_{\bar{Z}\bar{Z}})\Big] = 0, \tag{C.8}$$

$$\int_{\mathrm{T}^2} d^2Z\Big[g^{(2)[1]}{}_{ZZ} + 2g^{(2)[1]}{}_{Z\bar{Z}} + g^{(2)[1]}{}_{\bar{Z}\bar{Z}}\Big] = 0. \tag{C.9}$$

In addition, by integrating (C.3) over the torus we find

$$(1+e^{-4\rho_c})D^{[1]} + e^{-2\rho_c}(E^{[1]} + \bar{E}^{[1]}) = -\pi^2(1-e^{-2\rho_c})^2\frac{1}{\mathrm{Im}\tau}\int_{\mathrm{T}^2} d^2z(f_{zz} + f_{\bar{z}\bar{z}} + 2e^{-2\rho_c}f_{z\bar{z}}). \tag{C.10}$$

We determine the constants from the three equations above

$$D^{[1]} = -\frac{\pi^2(1-e^{-2\rho_c})}{(1+e^{-2\rho_c})^3}\Bigg[(1+4e^{-2\rho_c}+e^{-4\rho_c})\frac{1}{\mathrm{Im}\tau}\int_{\mathrm{T}} d^2z(f_{zz}+f_{\bar{z}\bar{z}}) \\ + 2e^{-2\rho_c}(1-2e^{-2\rho_c}-e^{-4\rho_c})\frac{1}{\mathrm{Im}\tau}\int_{\mathrm{T}} d^2z f_{z\bar{z}}\Bigg],$$

$$E^{[1]} = \frac{2\pi^2(1-e^{-2\rho_c})}{(1+e^{-2\rho_c})^3}\Bigg[(1+e^{-2\rho_c}+e^{-4\rho_c})\frac{1}{\mathrm{Im}\tau}\int_{\mathrm{T}^2} d^2z f_{\bar{z}\bar{z}} \\ + (2e^{-4\rho_c}+e^{-6\rho_c})\frac{1}{\mathrm{Im}\tau}\int_{\mathrm{T}^2} d^2z f_{zz} - 2e^{-2\rho_c}\frac{1}{\mathrm{Im}\tau}\int_{\mathrm{T}^2} d^2z f_{z\bar{z}}\Bigg],$$

$$\bar{E}^{[1]} = \frac{2\pi^2(1-e^{-2\rho_c})}{(1+e^{-2\rho_c})^3}\Bigg[(1+e^{-2\rho_c}+e^{-4\rho_c})\frac{1}{\mathrm{Im}\tau}\int_{\mathrm{T}^2} d^2z f_{zz} \\ + (2e^{-4\rho_c}+e^{-6\rho_c})\frac{1}{\mathrm{Im}\tau}\int_{\mathrm{T}^2} d^2z f_{\bar{z}\bar{z}} - 2e^{-2\rho_c}\frac{1}{\mathrm{Im}\tau}\int_{\mathrm{T}^2} d^2z f_{z\bar{z}}\Bigg]. \tag{C.11}$$

We can compute all two-point correlators by taking the functional derivative of $T^{[1]}_{ij}$ with respect to f_{ij}. As in section 3, the constants can also be determined from (2.13) and (2.14).

Setting $O = T_{zz}$ we obtain

$$(\bar{\tau}-\tau)\partial_\tau \langle T_{zz}(w)\rangle = (w-\bar{w})\partial_w \langle T_{zz}(w)\rangle + 2\langle T_{zz}(w)\rangle + \int_{\mathrm{T}^2} d^2z \left(\frac{\delta \langle T_{zz}(w)\rangle}{\delta \gamma_{\bar{z}\bar{z}}(z)} - \frac{\delta \langle T_{zz}(w)\rangle}{\delta \gamma_{z\bar{z}}(z)}\right),$$
$$(\tau-\bar{\tau})\partial_{\bar{\tau}} \langle T_{zz}(w)\rangle = (\bar{w}-w)\partial_{\bar{w}} \langle T_{zz}(w)\rangle - 2\langle T_{z\bar{z}}(w)\rangle + \int_{\mathrm{T}^2} d^2z \left(\frac{\delta \langle T_{zz}(w)\rangle}{\delta \gamma_{zz}(z)} - \frac{\delta \langle T_{zz}(w)\rangle}{\delta \gamma_{z\bar{z}}(z)}\right). \tag{C.12}$$

In addition, we take a functional derivative with respect to $\gamma_{\bar{z}\bar{z}}$ in (2.9), and integrate over one point

$$\int_{\mathrm{T}^2} d^2z \frac{\delta \langle T_{z\bar{z}}(z)\rangle}{\delta \gamma_{\bar{z}\bar{z}}(w)} - \langle T_{zz}(w)\rangle$$
$$+ 8\pi G r_c^2 \int_{\mathrm{T}^2} d^2z \left[2\langle T_{z\bar{z}}(z)\rangle \frac{\delta \langle T_{z\bar{z}}(z)\rangle}{\delta \gamma_{\bar{z}\bar{z}}(w)} - \langle T_{zz}(z)\rangle \frac{\delta \langle T_{\bar{z}\bar{z}}(z)\rangle}{\delta \gamma_{\bar{z}\bar{z}}(w)} - \langle T_{\bar{z}\bar{z}}(z)\rangle \frac{\delta \langle T_{zz}(z)\rangle}{\delta \gamma_{\bar{z}\bar{z}}(w)}\right] = 0. \tag{C.13}$$

The three equations above determine the constants $D^{[1]}, E^{[1]}, \bar{E}^{[1]}$ as one-point-averaged correlators.

A recurrence algorithm for holographic correlators similar to (3.15) can be derived, but it takes a much more complicated form. Turning on a variation of the metric $\gamma_{\bar{z}\bar{z}} = F$, the varied one-point correlator is solved as

$$\begin{aligned} &T_{z\bar{z}}^{[n]}(z) \\ &= \frac{1}{\pi}\int_{\mathrm{T}^2} d^2W \tilde{G}_\Omega(Z-W) \\ &\times \Bigg(-\frac{e^{-2\rho_c}}{(1-e^{-2\rho_c})^2}(\partial(3\partial F T_{zz}^{[n-1]} + 2F\partial T_{zz}^{[n-1]}) + \bar{\partial}(\bar{\partial} F T_{zz}^{[n-1]}) + 2\partial\bar{\partial}(F T_{z\bar{z}}^{[n-1]})) \\ &+ \frac{1-e^{-4\rho_c}}{(1-e^{-2\rho_c})^2}\partial\bar{\partial}\left(F T_{zz}^{[n-1]} + \frac{\partial^2 F}{16\pi G} - 8\pi G r_c^2 \sum_{m=1}^{n-1}(T_{z\bar{z}}^{[m]} T_{z\bar{z}}^{[n-m]} - T_{zz}^{[m]} T_{\bar{z}\bar{z}}^{[n-m]})\right)\Bigg)(w) + \frac{D^{[n]}}{8\pi G}, \end{aligned} \tag{C.14}$$

and

$$T_{zz}^{[n]}(z) = \frac{1}{\pi}\int_{\mathrm{T}^2} d^2w G_\tau(z-w)(-\partial T_{z\bar{z}}^{[n]} + 3\partial F T_{zz}^{[n-1]} + 2F\partial T_{zz}^{[n-1]})(w) + \frac{E^{[n]}}{8\pi G}, \tag{C.15}$$
$$T_{\bar{z}\bar{z}}^{[n]}(z) = \frac{1}{\pi}\int_{\mathrm{T}^2} d^2w \overline{G_\tau(z-w)}(-\bar{\partial} T_{z\bar{z}}^{[n]} + \bar{\partial} F T_{zz}^{[n-1]} + 2\partial(F T_{z\bar{z}}^{[n-1]}))(w) + \frac{\bar{E}^{[n]}}{8\pi G}, \tag{C.16}$$

where $D^{[n]}$, $E^{[n]}$ and $\bar{E}^{[n]}$ are constants of integration. To fix $D^{[n]}$, $E^{[n]}$ and $\bar{E}^{[n]}$, we set $O = T_{zz}(z_1)\dots T_{zz}(z_n)$ in (2.13) and (2.14), and obtain

$$\begin{aligned} &(\bar{\tau}-\tau)\partial_\tau \langle T_{zz}(z_1)\dots T_{zz}(z_n)\rangle \\ &= \sum_{i=1}^{n}[(z_i-\bar{z}_i)\partial_{z_i}\langle T_{zz}(z_1)\dots T_{zz}(z_n)\rangle + 2\langle T_{zz}(z_1)\dots T_{zz}(z_n)\rangle] \\ &\quad + \int_{\mathrm{T}^2} d^2z \left[\frac{\delta \langle T_{zz}(z_1)\dots T_{zz}(z_n)\rangle}{\delta \gamma_{\bar{z}\bar{z}}(z)} - \frac{\delta \langle T_{zz}(z_1)\dots T_{zz}(z_n)\rangle}{\delta \gamma_{z\bar{z}}(z)}\right], \end{aligned} \tag{C.17}$$

$$
\begin{aligned}
&(\tau-\bar{\tau})\partial_{\bar{\tau}}\langle T_{zz}(z_1)\dots T_{zz}(z_n)\rangle \\
&=\sum_{i=1}^{n}[(\bar{z}_i-z_i)\partial_{\bar{z}_i}\langle T_{zz}(z_1)\dots T_{zz}(z_n)\rangle-2\langle T_{z\bar{z}}(z_i)T_{zz}(z_1)\dots\hat{T}_{zz}(z_i)\dots T_{zz}(z_n)]\\
&\quad+\int_{\mathrm{T}^2}d^2z\left[\frac{\delta\langle T_{zz}(z_1)\dots T_{zz}(z_n)\rangle}{\delta\gamma_{zz}(z)}-\frac{\delta\langle T_{zz}(z_1)\dots T_{zz}(z_n)\rangle}{\delta\gamma_{z\bar{z}}(z)}\right],
\end{aligned}\tag{C.18}
$$

where $\hat{T}_{zz}(z_i)$ means dropping the i-th operator. In addition, we take n times of functional derivatives with respect to $\gamma_{\bar{z}\bar{z}}(z_1),\dots,\gamma_{\bar{z}\bar{z}}(z_n)$ in (2.9), and integrate over one point to get

$$
\begin{aligned}
&\int_{\mathrm{T}^2}d^2z\langle T_{z\bar{z}}(z)T_{zz}(z_1)\dots T_{zz}(z_n)\rangle\\
&=-\frac{e^{-2\rho_c}}{1+e^{-4\rho_c}}\left(\int_{\mathrm{T}^2}d^2z\langle T_{zz}(z)T_{zz}(z_1)\dots T_{zz}(z_n)\rangle+\int_{\mathrm{T}^2}d^2z\langle T_{\bar{z}\bar{z}}(z)T_{zz}(z_1)\dots T_{zz}(z_n)\rangle\right)\\
&\quad+\frac{1-e^{-4\rho_c}}{1+e^{-4\rho_c}}\left(\frac{n}{2}\langle T_{zz}(z_1)\dots T_{zz}(z_n)\rangle-\int_{\mathrm{T}^2}d^2zM(z,z_1,\dots,z_n)\right),
\end{aligned}\tag{C.19}
$$

where we denote

$$
\begin{aligned}
&M(z,z_1,\dots,z_n)\\
&=\sum_{m=1}^{n-1}\sum_{\sigma}\frac{8\pi Gr_c^2}{m!(n-m)!}\Big[\langle T_{z\bar{z}}(z)T_{zz}(z_{\sigma(1)})\dots T_{zz}(z_{\sigma(m)})\rangle\langle T_{z\bar{z}}(z)T_{zz}(z_{\sigma(m+1)})\dots T_{zz}(z_{\sigma(n)})\rangle\\
&\quad-\langle T_{zz}(z)T_{zz}(z_{\sigma(1)})\dots T_{zz}(z_{\sigma(m)})\rangle\langle T_{\bar{z}\bar{z}}(z)T_{zz}(z_{\sigma(m+1)})\dots T_{zz}(z_{\sigma(n)})\rangle\Big]
\end{aligned}\tag{C.20}
$$

with σ running over all permutations of $(1,\dots,n)$.

With the constants of integration determined by (C.17), (C.18) and (C.19), we obtain the following recurrence relations for correlators

$$
\begin{aligned}
&\langle T_{z\bar{z}}(z)T_{zz}(z_1)\dots T_{zz}(z_n)\rangle\\
&=-\frac{1}{\pi}\frac{1}{1-e^{-4\rho_c}}\sum_{i=1}^{n}\Bigg\{\Bigg[\frac{e^{-2\rho_c}}{2(1+e^{-2\rho_c})}\big((\partial_{Z_i}+e^{-2\rho_c}\partial_{\bar{Z}_i})\tilde{G}_\Omega(Z-Z_i)\partial_{z_i}+(e^{-2\rho_c}\partial_{Z_i}+\partial_{\bar{Z}_i})\tilde{G}_\Omega(Z-Z_i)\partial_{\bar{z}_i}\big)\\
&\quad-\frac{1}{2(1+e^{-2\rho_c})^2}\big(-2(e^{-2\rho_c}+e^{-6\rho_c})\partial_Z^2-4e^{-6\rho_c}\partial_{\bar{Z}}^2\big)\tilde{G}_\Omega(Z-Z_i)\Bigg]\langle T_{zz}(z_1)\dots T_{zz}(z_n)\rangle\\
&\quad+\frac{e^{-2\rho_c}}{(1+e^{-2\rho_c})^2}(e^{-2\rho_c}\partial_Z^2+e^{-2\rho_c}\partial_{\bar{Z}}^2)\tilde{G}_\Omega(Z-Z_i)\langle T_{z\bar{z}}(z_i)T_{zz}(z_1)\dots\hat{T}_{zz}(z_i)\dots T_{zz}(z_n)\rangle\Bigg\}\\
&\quad+\sum_{i=1}^{n}\Bigg[\frac{1-8e^{-4\rho_c}-e^{-8\rho_c}}{2(1-e^{-4\rho_c})^2}\langle T_{zz}(z_1)\dots T_{zz}(z_n)\rangle\\
&\quad-\frac{e^{-2\rho_c}+e^{-6\rho_c}}{(1-e^{-4\rho_c})^2}\langle T_{z\bar{z}}(z_i)T_{zz}(z_1)\dots\hat{T}_{zz}(z_i)\dots T_{zz}(z_n)\rangle\Bigg]\left(\delta(z-z_i)-\frac{1}{\mathrm{Im}\tau}\right)\\
&\quad+\frac{\delta_{n,1}}{32\pi^2G}\Bigg\{\frac{e^{-2\rho_c}}{(1+e^{-2\rho_c})^4}(\partial_Z^4+e^{-4\rho_c}\partial_{\bar{Z}}^4)\tilde{G}_\Omega(Z-Z_1)\\
&\quad+\frac{\pi}{(1-e^{-4\rho_c})^3}\Big[(1+e^{-12\rho_c})\partial_z^2+(e^{-4\rho_c}+e^{-8\rho_c})\partial_{\bar{z}}^2+(e^{-2\rho_c}-6e^{-6\rho_c}+e^{-10\rho_c})\partial_z\partial_{\bar{z}}\Big]\delta(z-z_1)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{\pi}\int_{\mathrm{T}^2} d^2w\tilde{G}_\Omega(Z-W)\partial_w\partial_{\bar{w}}M(w,z_1,\ldots,z_n)\\
&+\frac{1}{(1+e^{-2\rho_c})^2\mathrm{Im}\tau}\Bigg\{-(1-e^{-4\rho_c})\int_{\mathrm{T}^2} d^2wM(w,z_1,\ldots,z_n)\\
&-e^{-2\rho_c}\sum_{i=1}^n\langle T_{z\bar{z}}(z_i)T_{zz}(z_1)\ldots\hat{T}_{zz}(z_i)\ldots T_{zz}(z_n)\rangle\\
&-\frac{1}{2}e^{-2\rho_c}\Bigg[(\bar{\tau}-\tau)\partial_\tau+(\tau-\bar{\tau})\partial_{\bar{\tau}}+\sum_{i=1}^n(\bar{z}_i-z_i)\partial_{z_i}\\
&+\sum_{i=1}^n(z_i-\bar{z}_i)\partial_{\bar{z}_i}-n(4+e^{2\rho_c}-e^{-2\rho_c})\Bigg]\langle T_{zz}(z_1)\ldots T_{zz}(z_n)\rangle\Bigg\}. \qquad \text{(C.21)}
\end{aligned}
$$

$$
\begin{aligned}
&\langle T_{zz}(z)T_{zz}(z_1)\ldots T_{zz}(z_n)\rangle\\
&=\frac{1}{\pi}\frac{1}{1-e^{-4\rho_c}}\sum_{i=1}^n\Bigg\{\Bigg[-\frac{1}{2(1+e^{-2\rho_c})}\big((\partial_Z+e^{-6\rho_c}\partial_{\bar{Z}})\tilde{G}_\Omega(Z-Z_i)\partial_{z_i}\\
&+(e^{-2\rho_c}\partial_Z+e^{-4\rho_c}\partial_{\bar{Z}})\tilde{G}_\Omega(Z-Z_i)\partial_{\bar{z}_i}\big)\\
&+\frac{1}{(1+e^{-2\rho_c})^2}\big((1+e^{-4\rho_c})\partial_Z^2+2e^{-8\rho_c}\partial_{\bar{Z}}^2\big)\tilde{G}_\Omega(Z-Z_i)\Bigg]\langle T_{zz}(z_1)\ldots T_{zz}(z_n)\rangle\\
&+\frac{1}{(1+e^{-2\rho_c})^2}(e^{-2\rho_c}\partial_Z^2+e^{-6\rho_c}\partial_{\bar{Z}}^2)\tilde{G}_\Omega(Z-Z_i)\langle T_{z\bar{z}}(z_i)T_{zz}(z_1)\ldots\hat{T}_{zz}(z_i)\ldots T_{zz}(z_n)\rangle\Bigg\}\\
&-\frac{\delta_{n,1}}{32\pi^2G}\frac{1}{(1+e^{-2\rho_c})^4}(\partial_Z^4+e^{-8\rho_c}\partial_{\bar{Z}}^4)\tilde{G}_\Omega(Z-Z_1)+\frac{1}{\pi}\int_{\mathrm{T}^2} d^2w\tilde{G}_\Omega(Z-W)\partial_w^2M(w,z_1,\ldots,z_n)\\
&+\sum_{i=1}^n\Bigg[\frac{e^{-2\rho_c}+3e^{-6\rho_c}}{(1-e^{-4\rho_c})^2}\langle T_{zz}(z_1)\ldots T_{zz}(z_n)\rangle\\
&+\frac{2e^{-4\rho_c}}{(1-e^{-4\rho_c})^2}\langle T_{z\bar{z}}(z_i)T_{zz}(z_1)\ldots\hat{T}_{zz}(z_i)\ldots T_{zz}(z_n)\rangle\Bigg]\left(\delta(z-z_i)-\frac{1}{\mathrm{Im}\tau}\right)\\
&-\frac{\delta_{n,1}}{16\pi G}\frac{1}{(1-e^{-4\rho_c})^3}\Big[(2e^{-2\rho_c}-3e^{-6\rho_c}+2e^{-10\rho_c})\partial_z^2-(e^{-4\rho_c}+e^{-8\rho_c})\partial_z\partial_{\bar{z}}+e^{-6\rho_c}\partial_{\bar{z}}^2\Big]\delta(z-z_1)\\
&+\frac{1}{(1+e^{-2\rho_c})^2\mathrm{Im}\tau}\Bigg\{-(1-e^{-4\rho_c})\int_{\mathrm{T}^2} d^2wM(w,z_1,\ldots,z_n)\\
&-e^{-2\rho_c}\sum_{i=1}^n\langle T_{z\bar{z}}(z_i)T_{zz}(z_1)\ldots\hat{T}_{zz}(z_i)\ldots T_{zz}(z_n)\rangle\\
&+\frac{1}{2}\Bigg[(1+e^{-2\rho_c}+e^{-4\rho_c})(\bar{\tau}-\tau)\partial_\tau-e^{-2\rho_c}(\tau-\bar{\tau})\partial_{\bar{\tau}}+(1+e^{-2\rho_c}+e^{-4\rho_c})\sum_{i=1}^n(\bar{z}_i-z_i)\partial_{z_i}\\
&-e^{-2\rho_c}\sum_{i=1}^n(z_i-\bar{z}_i)\partial_{\bar{z}_i}-2n(1+e^{-2\rho_c}+2e^{-4\rho_c})\Bigg]\langle T_{zz}(z_1)\ldots T_{zz}(z_n)\rangle\Bigg\}. \qquad \text{(C.22)}
\end{aligned}
$$

$$
\begin{aligned}
&\langle T_{\bar{z}\bar{z}}(z)T_{zz}(z_1)\ldots T_{zz}(z_n)\rangle \\
&=\frac{1}{\pi}\frac{1}{1-e^{-4\rho_c}}\sum_{i=1}^{n}\Bigg\{\Bigg[-\frac{1}{2(1+e^{-2\rho_c})}\big((e^{-4\rho_c}\partial_Z+e^{-2\rho_c}\partial_{\bar{Z}})\tilde{G}_\Omega(Z-Z_i)\partial_{z_i} \\
&\quad+(e^{-6\rho_c}\partial_Z+\partial_{\bar{Z}})\tilde{G}_\Omega(Z-Z_i)\partial_{\bar{z}_i}\big) \\
&\quad+\frac{1}{(1+e^{-2\rho_c})^2}\big((e^{-4\rho_c}+e^{-8\rho_c})\partial_Z^2+2e^{-4\rho_c}\partial_{\bar{Z}}^2\big)\tilde{G}_\Omega(Z-Z_i)\Bigg]\langle T_{zz}(z_1)\ldots T_{zz}(z_n)\rangle \\
&\quad+\frac{1}{(1+e^{-2\rho_c})^2}(e^{-6\rho_c}\partial_Z^2+e^{-2\rho_c}\partial_{\bar{Z}}^2)\tilde{G}_\Omega(Z-Z_i)\langle T_{z\bar{z}}(z_i)T_{zz}(z_1)\ldots\hat{T}_{zz}(z_i)\ldots T_{zz}(z_n)\rangle\Bigg\} \\
&\quad-\frac{\delta_{n,1}}{32\pi^2G}\frac{e^{-4\rho_c}}{(1+e^{-2\rho_c})^4}(\partial_Z^4+\partial_{\bar{Z}}^4)\tilde{G}_\Omega(Z-Z_1)+\frac{1}{\pi}\int_{\mathrm{T}^2}d^2w\tilde{G}_\Omega(Z-W)\partial_{\bar{w}}^2M(w,z_1,\ldots,z_n) \\
&\quad+\sum_{i=1}^{n}\Bigg[\frac{e^{-2\rho_c}+3e^{-6\rho_c}}{(1-e^{-4\rho_c})^2}\langle T_{zz}(z_1)\ldots T_{zz}(z_n)\rangle \\
&\quad-\frac{1-4e^{-4\rho_c}+e^{-8\rho_c}}{(1-e^{-4\rho_c})^2}\langle T_{z\bar{z}}(z_i)T_{zz}(z_1)\ldots\hat{T}_{zz}(z_i)\ldots T_{zz}(z_n)\rangle\Bigg]\left(\delta(z-z_i)-\frac{1}{\mathrm{Im}\tau}\right) \\
&\quad-\frac{\delta_{n,1}}{32\pi G}\frac{1}{(1-e^{-4\rho_c})^3}\Big[(e^{-2\rho_c}+e^{-10\rho_c})(\partial_z^2+\partial_{\bar{z}}^2)+(1-3e^{-4\rho_c}-3e^{-8\rho_c}+e^{-12\rho_c})\partial_z\partial_{\bar{z}}\Big]\delta(z-z_1) \\
&\quad+\frac{1}{(1+e^{-2\rho_c})^2\mathrm{Im}\tau}\Bigg\{-(1-e^{-4\rho_c})\int_{\mathrm{T}^2}d^2wM(w,z_1,\ldots,z_n) \\
&\quad+(1+e^{-2\rho_c}+e^{-4\rho_c})\sum_{i=1}^{n}\langle T_{z\bar{z}}(z_i)T_{zz}(z_1)\ldots\hat{T}_{zz}(z_i)\ldots T_{zz}(z_n)\rangle \\
&\quad+\frac{1}{2}\Bigg[(1+e^{-2\rho_c}+e^{-4\rho_c})(\tau-\bar{\tau})\partial_{\bar{\tau}}-e^{-2\rho_c}(\bar{\tau}-\tau)\partial_\tau+(1+e^{-2\rho_c}+e^{-4\rho_c})\sum_{i=1}^{n}(z_i-\bar{z}_i)\partial_{\bar{z}_i} \\
&\quad-e^{-2\rho_c}\sum_{i=1}^{n}(\bar{z}_i-z_i)\partial_{z_i}+2n(e^{-2\rho_c}-e^{-4\rho_c})\Bigg]\langle T_{zz}(z_1)\ldots T_{zz}(z_n)\rangle\Bigg\}.
\end{aligned}
\tag{C.23}
$$

By taking the limit $\rho_c\to\infty$, the second one reproduces the recurrence relation (3.15) in the main text for holographic CFTs.

References

[1] G. 't Hooft, *Dimensional reduction in quantum gravity*, *Conf. Proc. C* **930308** (1993) 284 [gr-qc/9310026] [INSPIRE].

[2] L. Susskind, *The World as a hologram*, *J. Math. Phys.* **36** (1995) 6377 [hep-th/9409089] [INSPIRE].

[3] J.M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200] [INSPIRE].

[4] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett. B* **428** (1998) 105 [hep-th/9802109] [INSPIRE].

[5] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150] [INSPIRE].

[6] H. Liu and A.A. Tseytlin, *On four point functions in the CFT/AdS correspondence*, *Phys. Rev. D* **59** (1999) 086002 [hep-th/9807097] [INSPIRE].

[7] E. D'Hoker et al., *Graviton and gauge boson propagators in AdS_{d+1}*, *Nucl. Phys. B* **562** (1999) 330 [hep-th/9902042] [INSPIRE].

[8] G. Arutyunov and S. Frolov, *Three point Green function of the stress energy tensor in the AdS/CFT correspondence*, *Phys. Rev. D* **60** (1999) 026004 [hep-th/9901121] [INSPIRE].

[9] S. Raju, *Four Point Functions of the Stress Tensor and Conserved Currents in AdS_4/CFT_3*, *Phys. Rev. D* **85** (2012) 126008 [arXiv:1201.6452] [INSPIRE].

[10] A. Bagchi, D. Grumiller and W. Merbis, *Stress tensor correlators in three-dimensional gravity*, *Phys. Rev. D* **93** (2016) 061502 [arXiv:1507.05620] [INSPIRE].

[11] K. Krasnov, *Holography and Riemann surfaces*, *Adv. Theor. Math. Phys.* **4** (2000) 929 [hep-th/0005106] [INSPIRE].

[12] C. Fefferman and C.R. Graham, *Conformal invariants*, *Astérisque* **S131** (1985) 95.

[13] M. Henningson and K. Skenderis, *The Holographic Weyl anomaly*, *JHEP* **07** (1998) 023 [hep-th/9806087] [INSPIRE].

[14] K. Skenderis and S.N. Solodukhin, *Quantum effective action from the AdS/CFT correspondence*, *Phys. Lett. B* **472** (2000) 316 [hep-th/9910023] [INSPIRE].

[15] S. de Haro, S.N. Solodukhin and K. Skenderis, *Holographic reconstruction of space-time and renormalization in the AdS/CFT correspondence*, *Commun. Math. Phys.* **217** (2001) 595 [hep-th/0002230] [INSPIRE].

[16] C. Fefferman and C.R. Graham, *The ambient metric*, *Ann. Math. Stud.* **178** (2011) 1 [arXiv:0710.0919] [INSPIRE].

[17] C.R. Graham and J.M. Lee, *Einstein metrics with prescribed conformal infinity on the ball*, *Adv. Math.* **87** (1991) 186 [INSPIRE].

JHEP06(2023)116

[18] M.T. Anderson, *On the structure of conformally compact Einstein metrics*, math/0402198.

[19] M.T. Anderson, *Geometric aspects of the AdS/CFT correspondence*, *IRMA Lect. Math. Theor. Phys.* **8** (2005) 1 [hep-th/0403087] [INSPIRE].

[20] M.T. Anderson, *Einstein metrics with prescribed conformal infinity on 4-manifolds*, *Geom. Funct. Anal.* **18** (2008) 305.

[21] S. He and Y. Sun, *Correlation functions of CFTs on a torus with a $T\overline{T}$ deformation*, *Phys. Rev. D* **102** (2020) 026023 [arXiv:2004.07486] [INSPIRE].

[22] A.B. Zamolodchikov, *Expectation value of composite field $T\overline{T}$ in two-dimensional quantum field theory*, hep-th/0401146 [INSPIRE].

[23] F.A. Smirnov and A.B. Zamolodchikov, *On space of integrable quantum field theories*, *Nucl. Phys. B* **915** (2017) 363 [arXiv:1608.05499] [INSPIRE].

[24] L. McGough, M. Mezei and H. Verlinde, *Moving the CFT into the bulk with $T\overline{T}$*, *JHEP* **04** (2018) 010 [arXiv:1611.03470] [INSPIRE].

[25] P. Kraus, J. Liu and D. Marolf, *Cutoff AdS_3 versus the $T\overline{T}$ deformation*, *JHEP* **07** (2018) 027 [arXiv:1801.02714] [INSPIRE].

[26] O. Aharony and T. Vaknin, *The $T\overline{T}$ deformation at large central charge*, *JHEP* **05** (2018) 166 [arXiv:1803.00100] [INSPIRE].

[27] S. He, Y. Sun and Y.-X. Zhang, *$T\overline{T}$-flow effects on torus partition functions*, *JHEP* **09** (2021) 061 [arXiv:2011.02902] [INSPIRE].

[28] Y. Li and Y. Zhou, *Cutoff AdS_3 versus $T\overline{T}$ CFT_2 in the large central charge sector: correlators of energy-momentum tensor*, *JHEP* **12** (2020) 168 [arXiv:2005.01693] [INSPIRE].

[29] M. Banados, *Three-dimensional quantum geometry and black holes*, *AIP Conf. Proc.* **484** (1999) 147 [hep-th/9901148] [INSPIRE].

[30] V. Balasubramanian and P. Kraus, *A Stress tensor for Anti-de Sitter gravity*, *Commun. Math. Phys.* **208** (1999) 413 [hep-th/9902121] [INSPIRE].

[31] R. Emparan, C.V. Johnson and R.C. Myers, *Surface terms as counterterms in the AdS/CFT correspondence*, *Phys. Rev. D* **60** (1999) 104001 [hep-th/9903238] [INSPIRE].

[32] D. Friedan and S.H. Shenker, *The Analytic Geometry of Two-Dimensional Conformal Field Theory*, *Nucl. Phys. B* **281** (1987) 509 [INSPIRE].

[33] T. Eguchi and H. Ooguri, *Conformal and Current Algebras on General Riemann Surface*, *Nucl. Phys. B* **282** (1987) 308 [INSPIRE].

[34] A. Maloney and E. Witten, *Quantum Gravity Partition Functions in Three Dimensions*, *JHEP* **02** (2010) 029 [arXiv:0712.0155] [INSPIRE].

[35] J.M. Maldacena and A. Strominger, *AdS_3 black holes and a stringy exclusion principle*, *JHEP* **12** (1998) 005 [hep-th/9804085] [INSPIRE].

[36] A.M. Polyakov, *Quantum Gravity in Two-Dimensions*, *Mod. Phys. Lett. A* **2** (1987) 893 [INSPIRE].

[37] M. Banados and R. Caro, *Holographic ward identities: Examples from 2+1 gravity*, *JHEP* **12** (2004) 036 [hep-th/0411060] [INSPIRE].

[38] X. Yin, *Partition Functions of Three-Dimensional Pure Gravity*, *Commun. Num. Theor. Phys.* **2** (2008) 285 [arXiv:0710.2129] [INSPIRE].

[39] J. Cardy, *The $T\overline{T}$ deformation of quantum field theory as random geometry*, *JHEP* **10** (2018) 186 [arXiv:1801.06895] [INSPIRE].

[40] S. Datta and Y. Jiang, *$T\bar{T}$ deformed partition functions*, *JHEP* **08** (2018) 106 [arXiv:1806.07426] [INSPIRE].

[41] L. Hadasz, Z. Jaskolski and P. Suchanek, *Recursive representation of the torus 1-point conformal block*, *JHEP* **01** (2010) 063 [arXiv:0911.2353] [INSPIRE].

[42] P. Kraus et al., *Witten Diagrams for Torus Conformal Blocks*, *JHEP* **09** (2017) 149 [arXiv:1706.00047] [INSPIRE].

[43] Y. Gobeil, A. Maloney, G.S. Ng and J.-Q. Wu, *Thermal Conformal Blocks*, *SciPost Phys.* **7** (2019) 015 [arXiv:1802.10537] [INSPIRE].

[44] A.L. Fitzpatrick and K.-W. Huang, *Universal Lowest-Twist in CFTs from Holography*, *JHEP* **08** (2019) 138 [arXiv:1903.05306] [INSPIRE].

[45] Y.-Z. Li, Z.-F. Mai and H. Lü, *Holographic OPE Coefficients from AdS Black Holes with Matters*, *JHEP* **09** (2019) 001 [arXiv:1905.09302] [INSPIRE].

[46] L.F. Alday, M. Kologlu and A. Zhiboedov, *Holographic correlators at finite temperature*, *JHEP* **06** (2021) 082 [arXiv:2009.10062] [INSPIRE].

[47] R. Karlsson, A. Parnachev, V. Prilepina and S. Valach, *Thermal stress tensor correlators, OPE and holography*, *JHEP* **09** (2022) 234 [arXiv:2206.05544] [INSPIRE].

[48] M. Pavlov, *Global torus blocks in the necklace channel*, arXiv:2302.10153 [INSPIRE].

[49] L. Iliesiu et al., *The Conformal Bootstrap at Finite Temperature*, *JHEP* **10** (2018) 070 [arXiv:1802.10266] [INSPIRE].

[50] G. Policastro, D.T. Son and A.O. Starinets, *The Shear viscosity of strongly coupled $N = 4$ supersymmetric Yang-Mills plasma*, *Phys. Rev. Lett.* **87** (2001) 081601 [hep-th/0104066] [INSPIRE].

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