Theoretical groundwork supporting the precessing-spin two-body dynamics of the effective-one-body waveform models SEOBNRv5

Mohammed Khalil,[1,2,3,*] Alessandra Buonanno,[2,3] Héctor Estellés,[2] Deyan P. Mihaylov,[2] Serguei Ossokine,[2] Lorenzo Pompili,[2] and Antoni Ramos-Buades[2]

[1]*Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, Ontario N2L 2Y5, Canada*
[2]*Max Planck Institute for Gravitational Physics (Albert Einstein Institute), Am Mühlenberg 1, Potsdam 14476, Germany*
[3]*Department of Physics, University of Maryland, College Park, Maryland 20742, USA*

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Waveform models are essential for gravitational-wave (GW) detection and parameter estimation of coalescing compact-object binaries. More accurate models are required for the increasing sensitivity of current and future GW detectors. The effective-one-body (EOB) formalism combines the post-Newtonian (PN) and small mass-ratio approximations with numerical-relativity results, and produces highly accurate inspiral-merger-ringdown waveforms. In this paper, we derive the analytical precessing-spin two-body dynamics for the SEOBNRv5 waveform model, which has been developed for the upcoming LIGO-Virgo-KAGRA observing run. We obtain an EOB Hamiltonian that reduces to the exact Kerr Hamiltonian in the test-mass limit. It includes the full 4PN precessing-spin information, and is valid for generic compact objects (i.e., for black holes or neutron stars). We also build an efficient and accurate EOB Hamiltonian that includes partial precessional effects, notably orbit-averaged in-plane spin effects for circular orbits, and derive 4PN-expanded precessing-spin equations of motion, consistent with such an EOB Hamiltonian. The results were used to build the computationally efficient precessing-spin multipolar SEOBNRv5PHM waveform model.

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I. INTRODUCTION

Since 2015, gravitational-wave (GW) observations [1–3] by the LIGO and Virgo detectors [4,5] have significantly improved our understanding of binary black holes (BHs) and neutron stars (NSs), and their astrophysical formation channels [6–8]. Making these detections and inferring the properties of GW sources require accurate waveform models, and the accuracy requirements for these models will increase significantly [9] with upgrades to current GW detectors [10], and with future detectors in space and on the ground, such as LISA [11], the Einstein Telescope [12], and Cosmic Explorer [13,14].

Numerical relativity (NR) simulations [15–17] provide very accurate waveforms, but they are computationally expensive, which makes it important to develop waveform models that combine analytical approximation methods with NR results to produce longer waveforms and to cover the entire parameter space of binary systems. The most commonly used approaches for GW sources of ground-based detectors are post-Newtonian (PN), NR surrogate, phenomenological, and effective-one-body (EOB) waveform models.

The PN approximation is a small-velocity and weak-field expansion (see, e.g., Refs. [18–24] for reviews), and PN-based Taylor-expanded waveform models [25–45] produce fast-to-evaluate waveforms, but are only accurate for the early inspiral. NR surrogate models [46–54] interpolate NR waveforms, which is possible with the recent increase in the number of NR catalogs [55–63]; hence, they are very accurate, but they are limited to regions of the parameter space for which NR simulations exist. Phenomenological models [64–81] combine PN and EOB waveforms for the inspiral with fits to NR results for the late inspiral and merger-ringdown parts of the waveform.

The EOB formalism [82–86] combines information from several analytical approximation methods with NR results. It maps the dynamics of a compact binary to that of a test mass or test spin in a deformed Schwarzschild or Kerr background, with the deformation parameter being the

*mkhalil@perimeterinstitute.ca

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symmetric mass ratio, implying that it contains the exact strong-field test-body limit. EOB waveform models have been constructed for nonspinning [84,87–94], spinning [85,86,95–115], and eccentric binaries [116–121]. In addition, tidal effects [122–126] and information from the post-Minkowskian [127–132] and small mass-ratio approximations [133–139] have been incorporated in EOB models. To reduce the computational cost of EOB waveforms, surrogate or reduced-order frequency-domain models have been developed in Refs. [140–149].

EOB waveform models consist of three main components: (i) the Hamiltonian, which describes the conservative binary dynamics, and from which one obtains the equations of motion (EOMs); (ii) the inspiral-merger-ringdown waveform, which resums PN information for the inspiral and includes functional fits to NR results for the plunge and merger-ringdown parts; and (iii) the radiation reaction (RR) force, which is computed from the inspiral waveform modes and is added to the EOMs to account for the energy and angular momentum losses due to the emitted GWs.

Two main families of EOB waveform models exist: SEOBNR (e.g., see Refs. [107,108,111]) and TEOBResumS (e.g., see Refs. [113,115,150]). In this paper, we derive the analytical precessing-spin two-body dynamics of the SEOBNRv5 waveform model,[1] which has been developed for the upcoming LIGO-Virgo-KAGRA observing run (O4) [151]. This waveform model has been built in Python language and is publicly available. Details of the model are provided in Ref. [152] for the software (pySEOBNR), in Ref. [153] for the aligned-spin model (SEOBNRv5HM), in Ref. [154] for the precessing-spin model (SEOBNRv5PHM), and in Ref. [155] for the inclusion of second-order gravitational self-force results in the nonspinning dissipative sector of the SEOBNRv5 dynamics.

This paper is organized as follows. In Sec. II, we derive a Hamiltonian, $H_{\rm EOB}^{\rm prec}$, that includes the full 4PN precessing-spin information, and reduces in the test-mass limit to the exact Hamiltonian of a nonspinning point particle in Kerr background. The Hamiltonian is valid for generic orbits (inclined, circular, or eccentric), and for generic compact objects (BHs or NSs), since we include the spin-multipole constants, which account for the tidal deformability of the compact object due to its spin. As realized in Refs. [109–111,156,157], solving the EOMs when using the full precessing-spin EOB Hamiltonian can be computationally expensive. Therefore, to develop a more efficient model, we first build a simpler EOB Hamiltonian that includes partial precessional effects, $H_{\rm EOB}^{\rm pprec}$. Notably, we incorporate in such Hamiltonian in-plane spin effects only for circular orbits, and average them over an orbit, while neglecting fourth order spin terms.

[1]SEOBNRv5 is publicly available through the Python package pySEOBNR git.ligo.org/waveforms/software/pyseobnr. Stable versions of pySEOBNR are published through the Python Package Index (PyPI), and can be installed via pip install pyseobnr.

Then, building on previous studies [79,114,115,158], which employed an aligned-spin orbital dynamics in the coprecessing frame [31,159–161] and PN-expanded precessing-spin equations, we derive in Sec. III PN-expanded, orbit-averaged, precessing-spin equations for quasicircular orbits, and couple them consistently with $H_{\rm EOB}^{\rm pprec}$, which is not restricted to aligned spins. We include in the PN-expanded EOMs, the spin-orbit (SO) and spin-spin (SS) couplings to next-to-next-to-leading order (NNLO), which generalizes some results in the literature [39,114,158,162] to higher PN orders for the SS coupling. Furthermore, even for the SO contributions, our results for the EOMs employ a different gauge and spin-supplementary condition (SSC), to be consistent with the Hamiltonian, which leads to some differences compared with previous results in the literature.

We summarize our results in Sec. IV, and include a few Appendixes with more details about some aspects of the calculations. We provide our results as Mathematica files in the Supplemental Material [163].

A. Notation

We use geometric units in which $c = G = 1$.

We consider a binary with masses m_1 and m_2, with $m_1 \geq m_2$, and define the total mass M, reduced mass μ, symmetric mass ratio ν, antisymmetric mass ratio δ, and relative masses $X_{\rm i}$ as follows:

$$M \equiv m_1 + m_2, \qquad \mu \equiv \frac{m_1 m_2}{M}, \qquad \nu \equiv \frac{\mu}{M},$$
$$\delta \equiv \frac{m_1 - m_2}{M}, \qquad X_{\rm i} \equiv \frac{m_{\rm i}}{M}, \tag{1}$$

where i = 1, 2.

We denote the spin vector of each body by $\boldsymbol{S}_{\rm i}$, and define the dimensionless spins $\boldsymbol{\chi}_{\rm i}$ as

$$\boldsymbol{\chi}_{\rm i} \equiv \frac{\boldsymbol{a}_{\rm i}}{m_{\rm i}} \equiv \frac{\boldsymbol{S}_{\rm i}}{m_{\rm i}^2}, \tag{2}$$

along with the intermediate definition for $\boldsymbol{a}_{\rm i}$. The spin magnitudes $\chi_{\rm i}$ vary between -1 and 1, with positive spins being in the direction of the orbital angular momentum. We define the following combinations of $\boldsymbol{a}_{\rm i}$:

$$\boldsymbol{a}_\pm \equiv \boldsymbol{a}_1 \pm \boldsymbol{a}_2. \tag{3}$$

The spin quadrupole, octupole, and hexadecapole constants are denoted as $C_{{\rm i}ES^2}$, $C_{{\rm i}BS^3}$, and $C_{{\rm i}ES^4}$, respectively. These constants equal 1 for BHs, but are greater than 1 for NSs. We define

$$\tilde{C}_{{\rm i}ES^2} \equiv C_{{\rm i}ES^2} - 1,$$
$$\tilde{C}_{{\rm i}BS^3} \equiv C_{{\rm i}BS^3} - 1,$$
$$\tilde{C}_{{\rm i}ES^4} \equiv C_{{\rm i}ES^4} - 1, \tag{4}$$

such that expressions for BHs can be easily recovered by setting $\tilde{C}_{\dots} \to 0$. To simplify some expressions, we define the following combinations of spins and multipole constants:

$$\begin{aligned} C_{\pm}^{a^2} &\equiv \tilde{C}_{1\mathrm{ES}^2} a_1^2 \pm \tilde{C}_{2\mathrm{ES}^2} a_2^2, \\ C_{\pm}^{n\cdot a^2} &\equiv \tilde{C}_{1\mathrm{ES}^2} (\boldsymbol{n}\cdot\boldsymbol{a}_1)^2 \pm \tilde{C}_{2\mathrm{ES}^2} (\boldsymbol{n}\cdot\boldsymbol{a}_2)^2, \\ C_{\pm}^{a^3} &\equiv \tilde{C}_{1\mathrm{BS}^3} a_1^3 \pm \tilde{C}_{2\mathrm{BS}^3} a_2^3, \\ C_{\pm}^{a^4} &\equiv \tilde{C}_{1\mathrm{ES}^4} a_1^4 \pm \tilde{C}_{2\mathrm{ES}^4} a_2^4, \end{aligned} \quad (5)$$

which are zero for BHs (see below the definition of $\boldsymbol{n}$).

In the binary's center of mass, we denote the relative position and momentum vectors, $\boldsymbol{r}$ and $\boldsymbol{p}$, with

$$\boldsymbol{p}^2 = p_r^2 + \frac{L^2}{r^2}, \qquad p_r = \boldsymbol{n}\cdot\boldsymbol{p}, \qquad \boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{p}, \quad (6)$$

where $\boldsymbol{n} \equiv \boldsymbol{r}/r$, $\boldsymbol{L}$ is the orbital angular momentum with magnitude L, and $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}_1 + \boldsymbol{S}_2$ is the total angular momentum. For precessing spins, we use the spherical-coordinates phase-space variables $\{r, \theta, \phi, p_r, p_\theta, p_\phi\}$, where θ is the polar angle, ϕ is the azimuthal angle, and p_ϕ and p_θ are their conjugate momenta. For equatorial orbits (aligned spins), the angular momentum reduces to $L = p_\phi$.

For precessing spins, we use two orthonormal frames: $\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{\lambda}\}$ and $\{\boldsymbol{l}_{\mathrm{N}}, \boldsymbol{n}, \boldsymbol{\lambda}_{\mathrm{N}}\}$. In both frames, $\boldsymbol{n}$ is the unit vector in the direction of $\boldsymbol{r}$. The vector $\boldsymbol{l}$ is the direction of $\boldsymbol{L}$, while $\boldsymbol{l}_{\mathrm{N}}$ is the direction of $\boldsymbol{L}_{\mathrm{N}} \equiv \mu \boldsymbol{r} \times \mathbf{v}$, where $\mathbf{v} \equiv \dot{\boldsymbol{r}}$ is the velocity with $\dot{} \equiv d/dt$ being the time derivative. The other unit vectors are defined by $\boldsymbol{\lambda} \equiv \boldsymbol{l} \times \boldsymbol{n}$ and $\boldsymbol{\lambda}_{\mathrm{N}} \equiv \boldsymbol{l}_{\mathrm{N}} \times \boldsymbol{n}$.

The orbital angular frequency is denoted Ω, and we define the velocity parameter $v \equiv (M\Omega)^{1/3}$. We also often use $u \equiv M/r$ instead of r.

II. HAMILTONIAN

In the EOB formalism, the Hamiltonian H_{EOB}, describing the conservative binary dynamics, is related to an effective Hamiltonian H_{eff}, describing the dynamics of a test body in a deformed BH background, with ν being the deformation, via the energy map [82]

$$H_{\mathrm{EOB}} = M\sqrt{1 + 2\nu\left(\frac{H_{\mathrm{eff}}}{\mu} - 1\right)}. \quad (7)$$

For nonspinning binaries, in the $\nu \to 0$ limit, H_{eff} reduces to the Hamiltonian of a (nonspinning) test mass in a Schwarzschild background. The nonspinning EOB Hamiltonian was first derived in Refs. [82,83] with 2PN information, and then extended to 3PN [84] and 4PN [93], with partial information at 5PN [164–166] and 6PN [167,168].

For spinning binaries, one can follow two strategies: either map the spinning binary dynamics into that of a *test mass* or a *test spin* in a deformed Kerr background. Indeed, the first spinning EOB Hamiltonian [85] was constructed based on the Hamiltonian for the geodesic motion of a test mass in Kerr spacetime, while including leading-order (LO) SO and SS effects. This was later extended to the next-to-leading order (NLO) [96] and NNLO [101] SO levels, in addition to the NLO SS level for aligned [102,169,170] and precessing spins [103], then to NNLO SS for aligned spins and circular orbits [112], which was used to build the (publicly available) `TEOBResumS` waveform model [114,115]. The complete 4PN conservative dynamics for precessing spins and generic orbits was incorporated in EOB Hamiltonians in Ref. [104], and the 4.5PN SO dynamics in Refs. [171,172].

The second strategy, which maps the spinning binary dynamics into that of a test spin, was first developed in Ref. [173] (to pole-dipole order) with NLO SO and LO SS corrections [99], and then extended to NNLO SO in Ref. [100]. Such a Hamiltonian is applicable for generic (precessing) spins, and reproduces (resums) spin-orbit couplings at all PN orders in the test-body limit, which makes it more complicated than Hamiltonians based on the dynamics of a test mass. The test-spin dynamics was augmented to quadrupolar order in Ref. [174], and the EOB Hamiltonian was extended to 4PN order in Ref. [104].

The first `SEOBNR` waveform model developed for aligned-spin BHs [97] used an effective Hamiltonian for a test mass in a deformed Kerr spacetime. Subsequently, the `SEOBNRv1` [105], `SEOBNRv2` [106], `SEOBNRv3` [109,110], and `SEOBNRv4` [107,108,111] models, publicly available in the LIGO Algorithm Library [175], employed an effective Hamiltonian for a test spin in a deformed Kerr background. Here, to build the `SEOBNRv5` model [153,154], we take the effective Hamiltonian to be a deformation of the test-mass Kerr Hamiltonian. The masses of the background BH and test mass are identified to be $M = m_1 + m_2$ and $\mu = m_1 m_2/M$, respectively, while the Kerr spin $\boldsymbol{a}$ is mapped to be

$$\boldsymbol{a} = \boldsymbol{a}_1 + \boldsymbol{a}_2 \equiv \boldsymbol{a}_+. \quad (8)$$

An advantage of this map, besides its simplicity, is that the Kerr Hamiltonian reproduces all even-in-spin leading PN orders for binary BHs [176].

To include PN information in the EOB Hamiltonian, we write an ansatz for the coefficients of H_{eff} and solve for the unknowns such that H_{EOB} is related to a PN-expanded Hamiltonian H_{PN} in another gauge by a canonical transformation. To obtain that transformation, we write an ansatz for a generating function $\mathcal{G}$, perform the transformation using Poisson brackets, such that

$$H_{\mathrm{EOB}} = H_{\mathrm{PN}} + \{\mathcal{G}, H_{\mathrm{PN}}\} + \frac{1}{2!}\{\mathcal{G}, \{\mathcal{G}, H_{\mathrm{PN}}\}\} + \ldots, \quad (9)$$

where each bracket introduces a factor of $1/c^2$, and finally match the right- and left-hand sides of the above equation to

TABLE I. Summary of the Hamiltonians and their relations to each other. For each of the effective Hamiltonians, the corresponding EOB Hamiltonian is obtained via the energy map in Eq. (7).

Symbol	Equation	Description
$H^{\rm Kerr}$	(13)	Kerr Hamiltonian for a (nonspinning) test mass in a *generic* orbit
$H^{\rm Kerr\ eq}$	(16)	Kerr Hamiltonian for a (nonspinning) test mass in an *equatorial* orbit
$H^{\rm Schw}$	(18)	Schwarzschild Hamiltonian for a test mass
$H^{\rm noS}_{\rm eff}$	(19)	Effective Hamiltonian for *nonspinning* binaries; it reduces to $H^{\rm Schw}$ when $\nu \to 0$
$H^{\rm align}_{\rm eff}$	(26)	Effective Hamiltonian for *aligned-spin* binaries; it reduces to $H^{\rm noS}_{\rm eff}$ in the zero-spin limit and to $H^{\rm Kerr\ eq}$ when $\nu \to 0$
$H^{\rm prec}_{\rm eff}$	(35)	Effective Hamiltonian for *precessing-spin* binaries with full precessional (prec) effects; it reduces to $H^{\rm align}_{\rm eff}$ for aligned spins and to $H^{\rm Kerr}$ when $\nu \to 0$
$H^{\rm pprec}_{\rm eff}$	(47)	Effective Hamiltonian for precessing-spin binaries with *partial precessional* (pprec) effects; it reduces to $H^{\rm align}_{\rm eff}$ for aligned spins and, when PN expanded, agrees with $H^{\rm prec}_{\rm eff}$ to $\mathcal{O}(S^3)$ (included), with orbit-averaged in-plane-spin effects for circular orbits ($p_r = 0$)

solve for the unknown coefficients in $H_{\rm eff}$ and the generating function (see, e.g., Refs. [103,104] for more details).

We include in the Hamiltonian all 4PN information for precessing spins, in addition to most of the 5PN nonspinning contributions [164–166]. Many studies have contributed to deriving the 4PN conservative dynamics, for nonspinning binaries [93,177–190], at the SO level [191–204], SS [205–217], and at higher orders in the spins [176,218,219]. We start from the 4PN precessing-spin Hamiltonian in the gauge of Ref. [219], then perform a canonical transformation to EOB coordinates. Thus, our EOB Hamiltonian includes the NNLO SO and SS information, as well as the LO cubic- and quartic-in-spin contributions.

In the following subsections, we begin by reviewing the Kerr Hamiltonian, and by building on it, we construct the effective Hamiltonian, which we first present for nonspinning binaries, then for aligned and precessing spins. We end this section by describing the differences between our EOB Hamiltonian and others in the literature. In Sec. III A, we obtain a more computationally efficient precessing-spin Hamiltonian, albeit with partial precession effects. More specifically, when PN is expanded, such a simplified Hamiltonian reduces to the (PN-expanded) precessing-spin Hamiltonian at cubic-in-spin order, with orbit-averaged in-plane-spin effects for circular orbits. For convenience, Table I summarizes the Hamiltonians used in this paper.

A. Kerr Hamiltonian

In Boyer-Lindquist coordinates (t, r, θ, ϕ), the (inverse) Kerr metric $g^{\mu\nu}_{\rm Kerr}$ can be expressed by the line element (see, e.g., Refs. [174,220])

$$\begin{aligned} \mathrm{d}s^2 &= g^{\mu\nu}_{\rm Kerr}\partial_\mu\partial_\nu \\ &= -\frac{\Lambda}{\Delta\Sigma}\partial_t^2 + \frac{\Delta}{\Sigma}\partial_r^2 + \frac{1}{\Sigma}\partial_\theta^2 + \frac{\Sigma - 2Mr}{\Sigma\Delta\sin^2\theta}\partial_\phi^2 - \frac{4Mra}{\Sigma\Delta}\partial_t\partial_\phi, \end{aligned} \tag{10}$$

where M is the mass of the BH, a is its spin, and

$$\begin{aligned} &\Sigma \equiv r^2 + a^2\cos^2\theta, \qquad \Delta \equiv r^2 - 2Mr + a^2, \\ &\Lambda \equiv (r^2 + a^2)^2 - a^2\Delta\sin^2\theta. \end{aligned} \tag{11}$$

The Kerr Hamiltonian for a nonspinning test mass $H^{\rm Kerr}$ can be obtained by solving the mass-shell constraint $g^{\mu\nu}_{\rm Kerr}p_\mu p_\nu = -\mu^2$ for $H^{\rm Kerr}$, where μ is the mass of the test mass and $p_\mu = (-H^{\rm Kerr}, p_r, p_\theta, p_\phi)$.

Instead of using components to express the Kerr Hamiltonian, we transform to a three-vector notation, following Ref. [103], by treating the Boyer-Lindquist coordinates as spherical coordinates, with $\boldsymbol{r} = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and $\boldsymbol{a} = (0, 0, a)$, in addition to writing the momentum components in terms of the momentum vector $\boldsymbol{p}$ using

$$\begin{aligned} &p_r = \boldsymbol{n}\cdot\boldsymbol{p}, \qquad p_\phi = L_z = (\boldsymbol{r}\times\boldsymbol{p})_z, \\ &\frac{p_\theta^2}{r^2} = \boldsymbol{p}^2 - p_r^2 - \frac{p_\phi^2}{r^2\sin^2\theta}. \end{aligned} \tag{12}$$

The Kerr Hamiltonian can then be written as [103,104]

$$\begin{aligned} H^{\rm Kerr} = \frac{2Mr}{\Lambda}\boldsymbol{L}\cdot\boldsymbol{a} + [A^{\rm Kerr}(\mu^2 + B^{\rm Kerr}_{np}(\boldsymbol{n}\cdot\boldsymbol{p})^2 \\ + B^{\rm Kerr}_{p}\boldsymbol{p}^2 + B^{\rm Kerr}_{npa}(\boldsymbol{n}\times\boldsymbol{p}\cdot\boldsymbol{a})^2)]^{1/2}. \end{aligned} \tag{13}$$

The first term in Eq. (13) only contains odd-in-spin contributions, while the square root is the even-in-spin part, with

$$\begin{aligned} &A^{\rm Kerr} = \frac{\Delta\Sigma}{\Lambda}, \qquad B^{\rm Kerr}_{np} = \frac{r^2}{\Sigma}\left[\frac{\Delta}{r^2} - 1\right], \\ &B^{\rm Kerr}_{p} = \frac{r^2}{\Sigma}, \qquad B^{\rm Kerr}_{npa} = -\frac{r^2}{\Sigma\Lambda}(\Sigma + 2Mr), \end{aligned} \tag{14}$$

and

$$\Sigma = r^2 + (\boldsymbol{n}\cdot\boldsymbol{a})^2, \qquad \Delta = r^2 - 2Mr + a^2,$$
$$\Lambda = (r^2+a^2)^2 - \Delta a^2 + \Delta(\boldsymbol{n}\cdot\boldsymbol{a})^2, \tag{15}$$

where we used $a\cos\theta = \boldsymbol{n}\cdot\boldsymbol{a}$, $a^2\sin^2\theta = a^2 - (\boldsymbol{n}\cdot\boldsymbol{a})^2$, and $a^2 p_\phi^2/r^2 = (\boldsymbol{n}\times\boldsymbol{p}\cdot\boldsymbol{a})^2$.

For equatorial orbits, the Kerr Hamiltonian reduces to

$$H^{\text{Kerr eq}} = \frac{2Mp_\phi a}{r^3 + a^2(r+2M)} + \left[A^{\text{Kerr eq}}\left(\mu^2 + p^2 + B_{np}^{\text{Kerr eq}} p_r^2 + B_{npa}^{\text{Kerr eq}}\frac{p_\phi^2 a^2}{r^2}\right)\right]^{1/2}, \tag{16}$$

where

$$A^{\text{Kerr eq}} = \frac{1-2M/r + a^2/r^2}{1+(1+2M/r)a^2/r^2},$$
$$B_{np}^{\text{Kerr eq}} = \frac{a^2}{r^2} - \frac{2M}{r},$$
$$B_{npa}^{\text{Kerr eq}} = -\frac{1+2M/r}{r^2 + a^2(1+2M/r)}. \tag{17}$$

In the zero-spin limit, we obtain the Schwarzschild Hamiltonian

$$H^{\text{Schw}} = \sqrt{\left(1-\frac{2M}{r}\right)\left[\mu^2 + \left(1-\frac{2M}{r}\right)p_r^2 + \frac{p_\phi^2}{r^2}\right]}. \tag{18}$$

B. Effective Hamiltonian for nonspinning binaries

The effective Hamiltonian for nonspinning (noS) binaries can be expressed as

$$H_{\text{eff}}^{\text{noS}} = \sqrt{A_{\text{noS}}\left[\mu^2 + A_{\text{noS}}\bar{D}_{\text{noS}} p_r^2 + \frac{p_\phi^2}{r^2} + Q_{\text{noS}}\right]}, \tag{19}$$

where $Q_{\text{noS}}(r, p_r)$ is at least quartic in p_r. In the test-mass limit, we have

$$A_{\text{noS}}(r) \overset{\nu=0}{\rightarrow} 1 - \frac{2M}{r}, \qquad \bar{D}_{\text{noS}}(r) \overset{\nu=0}{\rightarrow} 1,$$
$$Q_{\text{noS}}(r, p_r) \overset{\nu=0}{\rightarrow} 0, \tag{20}$$

and the effective Hamiltonian reduces to Eq. (18). For the potentials A_{noS}, $\bar{D}_{\text{noS}}$, and Q_{noS}, we use the results of Ref. [165] (see Table IV there), which are missing two quadratic-in-ν coefficients in A_{noS} and $\bar{D}_{\text{noS}}$ at 5PN.

The 5PN Taylor-expanded potential A_{noS} is given by

$$A_{\text{noS}}^{\text{Tay}} = 1 - 2u + 2\nu u^3 + \nu\left(\frac{94}{3} - \frac{41\pi^2}{32}\right)u^4 + \left[\nu\left(\frac{2275\pi^2}{512} - \frac{4237}{60} + \frac{128\gamma_E}{5} + \frac{256\ln 2}{5}\right) + \left(\frac{41\pi^2}{32} - \frac{221}{6}\right)\nu^2 + \frac{64}{5}\nu\ln u\right]u^5 + \left[\nu a_6 - \nu\left(\frac{144\nu}{5} + \frac{7004}{105}\right)\ln u\right]u^6, \tag{21}$$

where $u \equiv M/r$, $\gamma_E \simeq 0.5772$ is the Euler gamma constant, and we replaced the coefficient of u^6 in A_{noS}, except for the log part, by the parameter a_6, which is calibrated to quasicircular NR simulations. Note that we pull out a factor of ν from a_6 compared with its definition in Ref. [165]. Then, we perform a (1,5) Padé resummation of $A_{\text{noS}}^{\text{Tay}}(u)$, while treating $\ln u$ as a constant, i.e., we use

$$A_{\text{noS}} = P_5^1[A_{\text{noS}}^{\text{Tay}}(u)]. \tag{22}$$

The Padé resummation of A_{noS} was first introduced in Ref. [84] at 3PN order to ensure the presence of an innermost stable circular orbit in the EOB dynamics for any mass ratio. It was then adopted in the initial nonspinning and spinning `EOBNR` models (e.g., see Refs. [87,91,97]), and in all `TEOBResumS` models (e.g., see Refs. [88,96,113,115,150]).

The 5PN potential $\bar{D}_{\text{noS}}$ reads as

$$\bar{D}_{\text{noS}}^{\text{Tay}} = 1 + 6\nu u^2 + (52\nu - 6\nu^2)u^3 + \left[\nu\left(-\frac{533}{45} - \frac{23761\pi^2}{1536} + \frac{1184\gamma_E}{15} - \frac{6496\ln 2}{15} + \frac{2916\ln 3}{5}\right) + \left(\frac{123\pi^2}{16} - 260\right)\nu^2 + \frac{592\nu}{15}\ln u\right]u^4 + \left(-\frac{3392\nu^2}{15} - \frac{1420\nu}{7}\right)u^5\ln u$$
$$+ \left[\nu\left(\frac{294464}{175} - \frac{2840\gamma_E}{7} - \frac{63707\pi^2}{512} + \frac{120648\ln 2}{35} - \frac{19683\ln 3}{7}\right) + \left(\frac{1069}{3} - \frac{205\pi^2}{16}\right)\nu^3 + \left(d_5^{\nu^2} - \frac{6784\gamma_E}{15} + \frac{67736}{105} + \frac{58320\ln 3}{7} - \frac{326656\ln 2}{21}\right)\nu^2\right]u^5, \tag{23}$$

where we set the remaining unknown coefficient $d_5^{\nu^2}$ to zero, but it can be determined in the future from PN calculations, or replaced by a calibration parameter to NR results for eccentric orbits. To improve agreement with NR, we perform a (2,3) Padé resummation of $\bar{D}_{\text{noS}}^{\text{Tay}}(u)$, i.e.,

$$\bar{D}_{\text{noS}} = P_3^2[\bar{D}_{\text{noS}}^{\text{Tay}}(u)]. \tag{24}$$

The 5.5PN contributions to A_{noS} and $\bar{D}_{\mathrm{noS}}$ are known [93,165]; however, since we Padé resum these potentials, we find it more convenient to stop at 5PN.

For Q_{noS}, we use the full 5.5PN expansion, which is also expanded in eccentricity to $\mathcal{O}(p_r^8)$, and it reads as [165,168]

$$
\begin{aligned}
Q_{\mathrm{noS}} = {} & \frac{p_r^4}{\mu^2}\left\{2(4-3\nu)\nu u^2 + u^3\left[10\nu^3 - 83\nu^2 + \nu\left(-\frac{5308}{15} + \frac{496256\ln 2}{45} - \frac{33048\ln 3}{5}\right)\right] + u^4\left[\left(640 - \frac{615\pi^2}{32}\right)\nu^3\right.\right. \\
& + \nu^2\left(\frac{31633\pi^2}{512} - \frac{1184\gamma}{5} + \frac{150683}{105} + \frac{33693536\ln 2}{105} - \frac{6396489\ln 3}{70} - \frac{9765625\ln 5}{126}\right) \\
& \left. + \nu\left(\frac{1295219}{350} - \frac{93031\pi^2}{1536} + \frac{10856\gamma_E}{105} - \frac{40979464}{315}\ln 2 + \frac{14203593\ln 3}{280} + \frac{9765625\ln 5}{504}\right) + \nu\left(\frac{5428}{105} - \frac{592\nu}{5}\right)\ln u\right] \\
& \left. + \frac{88703\pi\nu u^{9/2}}{1890}\right\} + \frac{p_r^6}{\mu^4}\left\{u^2\left[6\nu^3 - \frac{27\nu^2}{5} + \nu\left(-\frac{827}{3} - \frac{2358912}{25}\ln 2 + \frac{1399437\ln 3}{50} + \frac{390625\ln 5}{18}\right)\right]\right. \\
& + u^3\left[-14\nu^4 + 116\nu^3 + \nu^2\left(\frac{159089}{75} - \frac{4998308864\ln 2}{1575} + \frac{26171875\ln 5}{18} - \frac{45409167\ln 3}{350}\right)\right. \\
& \left.\left. + \nu\left(\frac{2613083}{1050} + \frac{6875745536\ln 2}{4725} - \frac{23132628\ln 3}{175} - \frac{101687500\ln 5}{189}\right)\right] - \frac{2723471\pi\nu u^{7/2}}{756000}\right\} \\
& + \frac{p_r^8}{\mu^6}\left\{u\nu\left(-\frac{35772}{175} + \frac{21668992\ln 2}{45} + \frac{6591861\ln 3}{350} - \frac{27734375\ln 5}{126}\right) + u^2\left[-6\nu^4 + \frac{24\nu^3}{7}\right.\right. \\
& + \nu^2\left(\frac{870976}{525} + \frac{703189497728\ln 2}{33075} + \frac{332067403089\ln 3}{39200} - \frac{13841287201\ln 7}{4320} - \frac{468490234375\ln 5}{42336}\right) \\
& \left. + \nu\left(\frac{5790381}{2450} - \frac{16175693888\ln 2}{1575} + \frac{875090984375\ln 5}{169344} + \frac{13841287201\ln 7}{17280} - \frac{393786545409\ln 3}{156800}\right)\right] \\
& \left. + \frac{5994461\pi\nu u^{5/2}}{12700800}\right\}.
\end{aligned} \tag{25}
$$

C. Effective Hamiltonian for aligned spins

For aligned spins, the effective Hamiltonian reduces to the equatorial Kerr Hamiltonian (16) in the test-mass limit. To include PN information for arbitrary mass ratios, we use the following ansatz:

$$
\begin{aligned}
H_{\mathrm{eff}}^{\mathrm{align}} = {} & \frac{M p_\phi (g_{a_+} a_+ + g_{a_-}\delta a_-) + \mathrm{SO}_{\mathrm{calib}} + G_{a^3}^{\mathrm{align}}}{r^3 + a_+^2(r+2M)} \\
& + \left[A^{\mathrm{align}}\left(\mu^2 + \frac{p_\phi^2}{r^2} + (1 + B_{np}^{\mathrm{align}})p_r^2\right.\right. \\
& \left.\left. + B_{npa}^{\mathrm{Kerr\ eq}}\frac{p_\phi^2 a_+^2}{r^2} + Q^{\mathrm{align}}\right)\right]^{1/2},
\end{aligned} \tag{26}
$$

where the gyrogravitomagnetic factors[2] g_{a_+} and g_{a_-} include the SO corrections, $\mathrm{SO}_{\mathrm{calib}}$ is a calibration term to NR results, and $G_{a^3}^{\mathrm{align}}$ contains cubic-in-spin corrections. The nonspinning and SS contributions are included in A^{align}, B_{np}^{align}, and Q^{align}, while the quartic-in-spin corrections are added in A^{align}. The potential $B_{npa}^{\mathrm{Kerr\ eq}}$ is kept the same as in the Kerr Hamiltonian for equatorial orbits.

In some papers [96,101], the gyrogravitomagnetic factors in the SO part of the Hamiltonian were chosen to be in a gauge such that they are functions of $1/r$ and p_r^2 only, but other papers [99,100] made different choices. For SEOBNRv5, we find better results when using a gauge in which g_{a_+} and g_{a_-} depend on $1/r$ and L^2/r^2, but not on p_r^2, such that

$$
\begin{aligned}
g_{a_+}^{3.5\mathrm{PN}} = {} & \frac{7}{4} + \left[\tilde{L}^2 u^2\left(-\frac{45\nu}{32} - \frac{15}{32}\right) + u\left(\frac{23\nu}{32} - \frac{3}{32}\right)\right] \\
& + \left[\tilde{L}^4 u^4\left(\frac{345\nu^2}{256} + \frac{75\nu}{128} + \frac{105}{256}\right)\right. \\
& + \tilde{L}^2 u^3\left(-\frac{1591\nu^2}{768} - \frac{267\nu}{128} + \frac{59}{256}\right) \\
& \left. + u^2\left(\frac{109\nu^2}{192} - \frac{177\nu}{32} - \frac{5}{64}\right)\right],
\end{aligned} \tag{28a}
$$

[2] The SO part of EOB Hamiltonians is often expressed in terms of $S \equiv S_1 + S_2$ and $S_* \equiv S_1 m_2/m_1 + S_2 m_1/m_2$, i.e., $H_{\mathrm{SO}} \propto (g_S S + g_{S_*} S_*) p_\phi / r^3$. The relation between the gyrogravitomagnetic factors in this case and our definition in Eq. (26) is that

$$
g_{a_+} = \frac{1}{2}(g_S + g_{S_*}), \qquad g_{a_-} = \frac{1}{2}(g_S - g_{S_*}). \tag{27}
$$

$$g_{a_-}^{3.5\mathrm{PN}} = \frac{1}{4} + \left[\tilde{L}^2 u^2 \left(\frac{15}{32} - \frac{9\nu}{32}\right) + u\left(\frac{11\nu}{32} + \frac{3}{32}\right)\right] + \left[\tilde{L}^4 u^4 \left(\frac{75\nu^2}{256} - \frac{45\nu}{128} - \frac{105}{256}\right) + \tilde{L}^2 u^3 \left(-\frac{613\nu^2}{768} - \frac{35\nu}{128} - \frac{59}{256}\right) + u^2 \left(\frac{103\nu^2}{192} - \frac{\nu}{32} + \frac{5}{64}\right)\right], \tag{28b}$$

where the square brackets collect different PN orders, and we defined $\tilde{L} \equiv L/(M\mu)$. These PN expressions were obtained by canonically transforming the 3.5PN results of, e.g., Ref. [203].

The 4.5PN SO coupling was derived in Refs. [171,172,221,222], and can be included in the effective Hamiltonian. However, we found that using a calibration term at 5.5PN had a small effect on the dynamics, and thus only included the 3.5PN information with a 4.5PN calibration term of the form

$$\mathrm{SO_{calib}} = \nu d_{\mathrm{SO}} \frac{M^4}{r^3} p_\phi a_+. \tag{29}$$

For completeness, we write the 4.5PN part in terms of L^2/r^2 instead of p_r^2, which we obtained by canonically transforming Eq. (5.6) of Ref. [172], leading to

$$g_{a_+}^{4.5\mathrm{PN}} = g_{a_+}^{3.5\mathrm{PN}} + \left\{\tilde{L}^6 u^6 \left(-\frac{5425\nu^3}{4096} - \frac{1785\nu^2}{2048} - \frac{1715\nu}{4096} - \frac{1575}{4096}\right) + \tilde{L}^4 u^5 \left(\frac{75187\nu^3}{20480} + \frac{37603\nu^2}{10240} + \frac{3717\nu}{4096} - \frac{1023}{4096}\right) + \tilde{L}^2 u^4 \left(\frac{209}{1024} - \frac{15093\nu^3}{5120} + \frac{80189\nu^2}{7680} - \frac{13059\nu}{1024}\right) + u^3 \left[\frac{1079\nu^3}{2048} - \frac{24131\nu^2}{3072} + \left(\frac{487\pi^2}{384} - \frac{525331}{18432}\right)\nu - \frac{175}{2048}\right]\right\}, \tag{30a}$$

$$g_{a_-}^{4.5\mathrm{PN}} = g_{a_+}^{3.5\mathrm{PN}} + \left\{\tilde{L}^6 u^6 \left(-\frac{1225\nu^3}{4096} + \frac{525\nu^2}{2048} + \frac{1785\nu}{4096} + \frac{1575}{4096}\right) + \tilde{L}^4 u^5 \left(\frac{26491\nu^3}{20480} + \frac{4801\nu^2}{10240} - \frac{4843\nu}{20480} + \frac{1023}{4096}\right) + \tilde{L}^2 u^4 \left(-\frac{9549\nu^3}{5120} - \frac{1777\nu^2}{7680} - \frac{28883\nu}{5120} - \frac{209}{1024}\right) + u^3 \left[\frac{1823\nu^3}{2048} - \frac{1025\nu^2}{3072} - \left(\frac{5\pi^2}{384} + \frac{50215}{18432}\right)\nu + \frac{175}{2048}\right]\right\}. \tag{30b}$$

For the cubic-in-spin term $G_{a^3}^{\mathrm{align}}$ in Eq. (26), we obtain

$$G_{a^3}^{\mathrm{align}} = \frac{M p_\phi}{r^2}\left[-\frac{a_+^3}{4} + \frac{\delta}{4} a_- a_+^2 + C_+^{a^3} - \frac{3}{2} a_- C_-^{a^2} + \frac{3}{8} C_+^{a^2}(\delta a_- + 3a_+)\right], \tag{31}$$

where only the first two terms contribute for BHs. The coefficients $C_\pm^{\cdot}$ are defined in Eq. (5).

As mentioned, we include SS and S^4 PN information in the effective Hamiltonian (26) through the following ansatz [cf. Eq. (17)]:

$$A^{\mathrm{align}} = \frac{a_+^2/r^2 + A_{\mathrm{noS}} + A_{\mathrm{SS}}^{\mathrm{align}} + A_{S^4}^{\mathrm{align}}}{1 + (1 + 2M/r)a_+^2/r^2},$$
$$B_{np}^{\mathrm{align}} = -1 + \frac{a_+^2}{r^2} + A_{\mathrm{noS}} \bar{D}_{\mathrm{noS}} + B_{np,\mathrm{SS}}^{\mathrm{align}},$$
$$Q^{\mathrm{align}} = Q_{\mathrm{noS}} + Q_{\mathrm{SS}}^{\mathrm{align}}, \tag{32}$$

where the nonspinning contributions A_{noS}, $\bar{D}_{\mathrm{noS}}$, and Q_{noS} are given by Eqs. (22), (24), and (25), respectively.

For the SS contributions, we obtain

$$A_{\mathrm{SS}}^{\mathrm{align}} = -\frac{M C_+^{a^2}}{r^3} + \frac{M^2}{r^4}\left[\frac{9a_+^2}{8} - \frac{5}{4}\delta a_- a_+ + a_-^2\left(\frac{\nu}{2} + \frac{1}{8}\right) - \delta C_-^{a^2} - C_+^{a^2}\right] + \frac{M^3}{r^5}\left[a_+^2\left(-\frac{175\nu}{64} - \frac{225}{64}\right) + \delta a_- a_+\left(\frac{117}{32} - \frac{39\nu}{16}\right) + a_-^2\left(\frac{21\nu^2}{16} - \frac{81\nu}{64} - \frac{9}{64}\right) - \frac{51}{28}\delta C_-^{a^2} + \left(\frac{207\nu}{28} - \frac{51}{28}\right)C_+^{a^2}\right], \tag{33a}$$

$$B_{np,\mathrm{SS}}^{\mathrm{align}} = \frac{M}{r^3}\left[a_+^2\left(3\nu + \frac{45}{16}\right) - \frac{21}{8}\delta a_- a_+ + a_-^2\left(\frac{3\nu}{4} - \frac{3}{16}\right) + (3\nu - 3)C_+^{a^2}\right] + \frac{M^2}{r^4}\left[a_+^2\left(-\frac{1171\nu}{64} - \frac{861}{64}\right) + \delta a_- a_+\left(\frac{13\nu}{16} + \frac{449}{32}\right) + a_-^2\left(\frac{\nu^2}{16} + \frac{115\nu}{64} - \frac{37}{64}\right) + \left(6\nu - \frac{19}{4}\right)\delta C_-^{a^2} + \left(\frac{111\nu}{4} - \frac{3}{4}\right)C_+^{a^2}\right], \tag{33b}$$

$$Q_{\rm SS}^{\rm align} = \frac{M p_r^4}{\mu^2 r^3}\left[a_+^2\left(\frac{25}{32} - 5\nu^2 + \frac{165\nu}{32}\right) + \delta a_- a_+\left(\frac{45\nu}{8} - \frac{5}{16}\right) + a_-^2\left(-\frac{15\nu^2}{8} + \frac{75\nu}{32} - \frac{15}{32}\right) + \left(\frac{35\nu}{4} - 5\nu^2\right)C_+^{a^2}\right]. \tag{33c}$$

The quartic-in-spin contribution in A is given by

$$\begin{aligned} A_{\rm S^4}^{\rm align} = \frac{M}{r^5}\bigg[&-\frac{3}{4}C_+^{a^4} - \frac{3}{2}a_+ C_+^{a^3} + \frac{3}{2}a_- C_-^{a^3} \\ &+ \left(-\frac{9}{8}a_-^2 - \frac{5a_+^2}{8}\right)C_+^{a^2} - \frac{9}{8}(C_+^{a^2})^2 \\ &+ \frac{9}{8}(C_-^{a^2})^2 + \frac{9}{4}a_- a_+ C_-^{a^2}\bigg], \end{aligned} \tag{34}$$

which vanishes for BHs since the Kerr Hamiltonian with the mapping (8) reproduces it [176].

D. Effective Hamiltonian for precessing spins

For precessing spins, we derive an effective Hamiltonian that reduces to the Kerr Hamiltonian in Eq. (13), and includes higher PN information through the following ansatz:

$$\begin{aligned} H_{\rm eff}^{\rm prec} = \frac{Mr}{\Lambda}&[\boldsymbol{L}\cdot(g_{a_+}\boldsymbol{a}_+ + g_{a_-}\delta\boldsymbol{a}_-) + {\rm SO}_{\rm calib} + G_{a^3}^{\rm prec}] \\ &+ [A^{\rm prec}(\mu^2 + B_p^{\rm prec}\boldsymbol{p}^2 + B_{np}^{\rm prec}(\boldsymbol{n}\cdot\boldsymbol{p})^2 \\ &+ B_{npa}^{\rm Kerr}(\boldsymbol{n}\times\boldsymbol{p}\cdot\boldsymbol{a}_+)^2 + Q^{\rm prec})]^{1/2}. \end{aligned} \tag{35}$$

Similarly to the aligned-spin case, g_{a_+} and g_{a_-} include the SO corrections, ${\rm SO}_{\rm calib}$ is an NR calibration term, and $G_{a^3}^{\rm prec}$ contains S^3 corrections. The nonspinning and SS contributions are included in $A^{\rm prec}$, $B_p^{\rm prec}$, $B_{np}^{\rm prec}$, and $Q^{\rm prec}$, while the S^4 corrections are added in $A^{\rm prec}$. The potential $B_{npa}^{\rm Kerr}$ is the same as in the Kerr Hamiltonian.

The gyrogravitomagnetic factors g_{a_+} and g_{a_-} are given by Eq. (28), the same as in the aligned-spin case, since they are independent of spin. The calibration term is also similar, except for adding a dot product

$${\rm SO}_{\rm calib} = \nu d_{\rm SO}\frac{M^4}{r^3}\boldsymbol{L}\cdot\boldsymbol{a}_+. \tag{36}$$

For the cubic-in-spin term $G_{a^3}^{\rm prec}$, we obtain

$$\begin{aligned} G_{a^3}^{\rm prec} = \boldsymbol{L}\cdot\boldsymbol{a}_+\bigg\{&\frac{L^2}{\mu^2 r^3}\left[\frac{\delta}{2}(\boldsymbol{n}\cdot\boldsymbol{a}_-)(\boldsymbol{n}\cdot\boldsymbol{a}_+) - \frac{(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2}{4}\right] + \frac{p_r^2}{\mu^2 r}\left[\frac{5}{4}(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2 - \delta\frac{3}{2}(\boldsymbol{n}\cdot\boldsymbol{a}_-)(\boldsymbol{n}\cdot\boldsymbol{a}_+)\right] \\ &+ \frac{M}{r^2}\bigg[-\frac{a_+^2}{4} + (\boldsymbol{n}\cdot\boldsymbol{a}_+)^2 + \delta\frac{5}{24}(\boldsymbol{a}_+\cdot\boldsymbol{a}_-) - \delta\frac{5}{3}(\boldsymbol{n}\cdot\boldsymbol{a}_-)(\boldsymbol{n}\cdot\boldsymbol{a}_+) + \frac{1}{8}a_1^2(3\delta\tilde{C}_{1{\rm ES}^2} + 4\tilde{C}_{1{\rm BS}^3}) \\ &+ \frac{1}{8}a_2^2(4\tilde{C}_{2{\rm BS}^3} - 3\tilde{C}_{2{\rm ES}^2}\delta) - \frac{3}{8}(\boldsymbol{a}_1\cdot\boldsymbol{a}_2)(\tilde{C}_{1{\rm ES}^2}(\delta-3) - \tilde{C}_{2{\rm ES}^2}(\delta+3)) + (\boldsymbol{n}\cdot\boldsymbol{a}_1)^2\left(-\frac{3}{8}\tilde{C}_{1{\rm ES}^2}(2\delta+3) - \frac{5\tilde{C}_{1{\rm BS}^3}}{2}\right) \\ &+ \frac{3}{4}(\boldsymbol{n}\cdot\boldsymbol{a}_1)(\boldsymbol{n}\cdot\boldsymbol{a}_2)(\tilde{C}_{1{\rm ES}^2}(2\delta-7) - \tilde{C}_{2{\rm ES}^2}(2\delta+7)) + \frac{1}{8}(\boldsymbol{n}\cdot\boldsymbol{a}_2)^2(\tilde{C}_{2{\rm ES}^2}(6\delta-9) - 20\tilde{C}_{2{\rm BS}^3})\bigg]\bigg\} \\ + \boldsymbol{L}\cdot\boldsymbol{a}_-\bigg\{&\frac{p_r^2\delta(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2}{4\mu^2 r} - \frac{L^2\delta(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2}{4\mu^2 r^3} + \frac{M}{r^2}\bigg[\frac{\delta a_+^2}{24} + \frac{2}{3}\delta(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2 + \frac{1}{8}a_1^2(4\tilde{C}_{1{\rm BS}^3} - 3\tilde{C}_{1{\rm ES}^2}) \\ &+ \frac{1}{8}a_2^2(3\tilde{C}_{2{\rm ES}^2} - 4\tilde{C}_{2{\rm BS}^3}) - \frac{3}{8}(\boldsymbol{a}_1\cdot\boldsymbol{a}_2)(\tilde{C}_{1{\rm ES}^2}(\delta-3) + \tilde{C}_{2{\rm ES}^2}(\delta+3)) + (\boldsymbol{n}\cdot\boldsymbol{a}_1)^2\left(-\frac{3}{8}\tilde{C}_{1{\rm ES}^2}(\delta-6) - \frac{5\tilde{C}_{1{\rm BS}^3}}{2}\right) \\ &+ \frac{3}{4}(\boldsymbol{n}\cdot\boldsymbol{a}_1)(\boldsymbol{n}\cdot\boldsymbol{a}_2)(\tilde{C}_{1{\rm ES}^2}(2\delta-7) + \tilde{C}_{2{\rm ES}^2}(2\delta+7)) + (\boldsymbol{n}\cdot\boldsymbol{a}_2)^2\left(\frac{5\tilde{C}_{2{\rm BS}^3}}{2} - \frac{3}{8}\tilde{C}_{2{\rm ES}^2}(\delta+6)\right)\bigg]\bigg\}. \end{aligned} \tag{37}$$

We include SS and S^4 PN information in the effective Hamiltonian (35) through the following ansatz for the potentials [cf. Eq. (14)]:

$$\begin{aligned} A^{\rm prec} &= \frac{[a_+^2/r^2 + A_{\rm noS} + A_{\rm SS}^{\rm prec} + A_{\rm S^4}^{\rm prec}][1 + (\boldsymbol{n}\cdot\boldsymbol{a}_+)^2/r^2 + A_{\rm SS}^{\rm in\ plane} + A_{\rm S^4}^{\rm in\ plane}]}{1 + a_+^2/r^2 + 2Ma_+^2/r^3 + (\boldsymbol{n}\cdot\boldsymbol{a}_+)^2/r^2 - 2M(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2/r^3 + a_+^2(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2/r^4}, \\ B_p^{\rm prec} &= \frac{1}{1 + (\boldsymbol{n}\cdot\boldsymbol{a}_+)^2/r^2 + B_{p,{\rm SS}}^{\rm in\ plane}}, \end{aligned}$$

TABLE II. Summary of the main differences of the SEOBNRv5 Hamiltonian derived here, which builds on the results of Refs. [103,104], compared with that of SEOBNRv4 and TEOBResumS.

	SEOBNRv5	SEOBNRv4 [99,100,107,111]	TEOBResumS [102,112,113]
Nonspinning part	4PN with partial 5PN in A_{noS} and $\bar{D}_{\mathrm{noS}}$, 5.5PN in Q_{noS}	4PN in A_{noS}, 3PN in $\bar{D}_{\mathrm{noS}}$ and Q_{noS}	4PN with partial 5PN in A_{noS}, 3PN in $\bar{D}_{\mathrm{noS}}$ and Q_{noS}
A_{noS} resummation	(1,5) Padé	Horizon factorization and log resummation	(1,5) Padé
$\bar{D}_{\mathrm{noS}}$ resummation	(2,3) Padé	Log	Taylor expanded ($D_{\mathrm{noS}} \equiv 1/\bar{D}_{\mathrm{noS}}$ is inverse-Taylor resummed)
Hamiltonian in the $\nu \to 0$ limit	Reduces to Kerr Hamiltonian for a *test mass* in a generic orbit	Reduces to Kerr Hamiltonian for a *test spin*, to linear order in spin, in a generic orbit	The A potential reduces to Kerr, but not the full Hamiltonian
Spin-orbit part	3.5PN, in (r, L^2) gauge, Taylor expanded	3.5PN, added in the spin map	3.5PN, in (r, p_r^2) gauge, inverse-Taylor resummed
Higher-order spin information	NNLO SS (4PN), LO S^3 (3.5PN), LO S^4 (4PN)	LO SS (2PN)	NNLO SS (4PN) for circular orbits
Precessing-spin Hamiltonian	Yes	Yes	No
Spin-multipole constants included	Yes	No	Yes (in the SS contributions for circular orbits)

$$B_{np}^{\mathrm{prec}} = \frac{-1 + a_+^2/r^2 + A_{\mathrm{noS}}\bar{D}_{\mathrm{noS}} + B_{np,\mathrm{SS}}^{\mathrm{prec}} + B_{np,\mathrm{SS}}^{\text{in plane}}}{1 + (\boldsymbol{n}\cdot\boldsymbol{a}_+)^2/r^2},$$
$$Q^{\mathrm{prec}} = Q_{\mathrm{noS}} + Q_{\mathrm{SS}}^{\mathrm{prec}} + Q_{\mathrm{SS}}^{\text{in plane}}, \tag{38}$$

where the nonspinning contributions A_{noS}, $\bar{D}_{\mathrm{noS}}$, and Q_{noS} are given by Eqs. (22), (24), and (25), respectively, while $A_{\mathrm{SS}}^{\mathrm{prec}}$, $B_{np,\mathrm{SS}}^{\mathrm{prec}}$, and $Q_{\mathrm{SS}}^{\mathrm{prec}}$ are the same as in Eq. (33) except for replacing a_+a_- by $\boldsymbol{a}_+\cdot\boldsymbol{a}_-$. The other terms $A_{\mathrm{SS}}^{\text{in plane}}$, $B_{p,\mathrm{SS}}^{\text{in plane}}$, $B_{np,\mathrm{SS}}^{\text{in plane}}$, $Q_{\mathrm{SS}}^{\text{in plane}}$, and $A_{\mathrm{S}^4}^{\text{in plane}}$ only contain in-plane spin components, which vanish in the aligned-spin case.

The spin-spin contributions read as

$$A_{\mathrm{SS}}^{\mathrm{prec}} = -\frac{MC_+^{a^2}}{r^3} + \frac{M^2}{r^4}\left[\frac{9a_+^2}{8} - \frac{5}{4}\delta\boldsymbol{a}_-\cdot\boldsymbol{a}_+ + a_-^2\left(\frac{\nu}{2}+\frac{1}{8}\right) - \delta C_-^{a^2} - C_+^{a^2}\right] + \frac{M^3}{r^5}\left[a_+^2\left(-\frac{175\nu}{64} - \frac{225}{64}\right)\right.$$
$$\left. + \delta\boldsymbol{a}_-\cdot\boldsymbol{a}_+\left(\frac{117}{32} - \frac{39\nu}{16}\right) + a_-^2\left(\frac{21\nu^2}{16} - \frac{81\nu}{64} - \frac{9}{64}\right) - \frac{51}{28}\delta C_-^{a^2} + \left(\frac{207\nu}{28} - \frac{51}{28}\right)C_+^{a^2}\right], \tag{39a}$$

$$A_{\mathrm{SS}}^{\text{in plane}} = \frac{3MC_+^{n\cdot a^2}}{r^3} + \frac{M^2}{r^4}\left[\frac{33}{8}\delta(\boldsymbol{n}\cdot\boldsymbol{a}_-)(\boldsymbol{n}\cdot\boldsymbol{a}_+) + \left(-\frac{\nu}{2} - \frac{3}{8}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_-)^2 + \left(\frac{7\nu}{4} - \frac{15}{4}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2\right.$$
$$\left. + 3\delta C_-^{n\cdot a^2} + (3\nu + 6)C_+^{n\cdot a^2}\right] + \frac{M^3}{r^5}\left[\delta(17\nu + 8)(\boldsymbol{n}\cdot\boldsymbol{a}_-)(\boldsymbol{n}\cdot\boldsymbol{a}_+) + \left(-\frac{41\nu^2}{8} + \frac{551\nu}{32} - \frac{219}{64}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_-)^2\right.$$
$$\left. + \left(\frac{1771\nu}{96} - \frac{11\nu^2}{8} - \frac{293}{64}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2 + \delta\left(\frac{81\nu}{16} + \frac{1245}{224}\right)C_-^{n\cdot a^2} + \left(\frac{3555}{224} - \frac{13\nu^2}{16} + \frac{515\nu}{56}\right)C_+^{n\cdot a^2}\right], \tag{39b}$$

$$B_{p,\mathrm{SS}}^{\text{in plane}} = \frac{M}{r^3}\left[-\frac{3}{4}\delta(\boldsymbol{n}\cdot\boldsymbol{a}_-)(\boldsymbol{n}\cdot\boldsymbol{a}_+) + \left(\frac{3\nu}{4} - \frac{3}{16}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_-)^2 + \left(\frac{7\nu}{4} + \frac{15}{16}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2 + (3\nu - 3)C_+^{n\cdot a^2}\right]$$
$$+ \frac{M^2}{r^4}\left[\delta\left(\frac{49\nu}{4} + \frac{43}{8}\right)\boldsymbol{n}\cdot\boldsymbol{a}\_\boldsymbol{n}\cdot\boldsymbol{a}_+ + \left(\frac{545\nu}{32} - \frac{19\nu^2}{8} - \frac{219}{64}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_-)^2 + \left(\frac{805\nu}{96} - \frac{11\nu^2}{8} - \frac{125}{64}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2\right.$$
$$\left. + \delta\left(\frac{81\nu}{16} - \frac{189}{32}\right)C_-^{n\cdot a^2} + \left(-\frac{13\nu^2}{16} + \frac{203\nu}{8} - \frac{51}{32}\right)C_+^{n\cdot a^2}\right], \tag{39c}$$

$$B^{\rm prec}_{np,\rm SS} = \frac{M}{r^3}\left[a_+^2\left(3\nu+\frac{45}{16}\right)-\frac{21}{8}\delta\boldsymbol{a}_-\cdot\boldsymbol{a}_+ + a_-^2\left(\frac{3\nu}{4}-\frac{3}{16}\right)+(3\nu-3)C_+^{a^2}\right]+\frac{M^2}{r^4}\left[a_+^2\left(-\frac{1171\nu}{64}-\frac{861}{64}\right)\right.$$
$$\left.+\delta\boldsymbol{a}_-\cdot\boldsymbol{a}_+\left(\frac{13\nu}{16}+\frac{449}{32}\right)+a_-^2\left(\frac{\nu^2}{16}+\frac{115\nu}{64}-\frac{37}{64}\right)+\left(6\nu-\frac{19}{4}\right)\delta C_-^{a^2}+\left(\frac{111\nu}{4}-\frac{3}{4}\right)C_+^{a^2}\right], \tag{39d}$$

$$B^{\rm in\ plane}_{np,\rm SS} = \frac{M}{r^3}\left[\frac{45}{8}\delta(\boldsymbol{n}\cdot\boldsymbol{a}_-)(\boldsymbol{n}\cdot\boldsymbol{a}_+)+\left(-\frac{15\nu}{4}-\frac{45}{8}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2\right]+\frac{M^2}{r^4}\left[\delta\left(\frac{129\nu}{4}-\frac{17}{8}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_-)(\boldsymbol{n}\cdot\boldsymbol{a}_+)\right.$$
$$+\left(-\frac{33\nu^2}{4}+\frac{981\nu}{16}-\frac{165}{16}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_-)^2+\left(-\frac{11\nu^2}{2}+\frac{1901\nu}{48}+\frac{199}{16}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2+\delta\left(\frac{9\nu}{4}-\frac{75}{8}\right)C_-^{n\cdot a^2}$$
$$\left.+\left(-\frac{13\nu^2}{4}+\frac{37\nu}{4}+\frac{39}{8}\right)C_+^{n\cdot a^2}\right], \tag{39e}$$

$$Q^{\rm prec}_{\rm SS} = \frac{Mp_r^4}{\mu^2r^3}\left[a_+^2\left(-5\nu^2+\frac{165\nu}{32}+\frac{25}{32}\right)+\delta\boldsymbol{a}_-\cdot\boldsymbol{a}_+\left(\frac{45\nu}{8}-\frac{5}{16}\right)+a_-^2\left(-\frac{15\nu^2}{8}+\frac{75\nu}{32}-\frac{15}{32}\right)\right.$$
$$\left.+\left(\frac{35\nu}{4}-5\nu^2\right)C_+^{a^2}\right], \tag{39f}$$

$$Q^{\rm in\ plane}_{\rm SS} = \frac{Mp_r^4}{\mu^2r^3}\left[\delta\left(\frac{35}{16}-\frac{273\nu}{8}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_-)(\boldsymbol{n}\cdot\boldsymbol{a}_+)+\left(\frac{105\nu^2}{8}-\frac{525\nu}{32}+\frac{105}{32}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_-)^2\right.$$
$$\left.+\left(\frac{119\nu^2}{4}-\frac{2849\nu}{96}-\frac{175}{32}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2+\left(35\nu^2-\frac{245\nu}{4}\right)C_+^{n\cdot a^2}\right]$$
$$+\frac{Mp_r^3}{\mu^2r^3}\left[\delta\left(\frac{69\nu}{8}-\frac{5}{8}\right)(\boldsymbol{p}\cdot\boldsymbol{a}_-)(\boldsymbol{n}\cdot\boldsymbol{a}_+)+\left(-\frac{59\nu^2}{4}+\frac{341\nu}{24}+\frac{25}{8}\right)(\boldsymbol{p}\cdot\boldsymbol{a}_+)(\boldsymbol{n}\cdot\boldsymbol{a}_+)\right.$$
$$+\delta\left(\frac{69\nu}{8}-\frac{5}{8}\right)(\boldsymbol{p}\cdot\boldsymbol{a}_+)(\boldsymbol{n}\cdot\boldsymbol{a}_-)+\left(-\frac{15\nu^2}{2}+\frac{75\nu}{8}-\frac{15}{8}\right)(\boldsymbol{p}\cdot\boldsymbol{a}_-)(\boldsymbol{n}\cdot\boldsymbol{a}_-)$$
$$\left.+(35\nu-20\nu^2)(\tilde{C}_{1\rm ES^2}(\boldsymbol{n}\cdot\boldsymbol{a}_1)(\boldsymbol{p}\cdot\boldsymbol{a}_1)+\tilde{C}_{2\rm ES^2}(\boldsymbol{n}\cdot\boldsymbol{a}_2)(\boldsymbol{p}\cdot\boldsymbol{a}_2))\right], \tag{39g}$$

while the quartic-in-spin contributions in $A^{\rm prec}$ are given by

$$A^{\rm prec}_{\rm S^4} = \frac{M}{r^5}\left\{-\frac{1}{4}a_1^2a_2^2[\tilde{C}_{1\rm ES^2}(3\tilde{C}_{2\rm ES^2}+2)+2\tilde{C}_{2\rm ES^2}]-\frac{3}{2}(\boldsymbol{a}_1\cdot\boldsymbol{a}_2)^2[\tilde{C}_{1\rm ES^2}\tilde{C}_{2\rm ES^2}+\tilde{C}_{1\rm ES^2}+\tilde{C}_{2\rm ES^2}]\right.$$
$$\left.+a_1^2(\boldsymbol{a}_1\cdot\boldsymbol{a}_2)(\tilde{C}_{1\rm ES^2}-3\tilde{C}_{1\rm BS^3})+\frac{1}{4}a_1^4(2\tilde{C}_{1\rm ES^2}-3\tilde{C}_{1\rm ES^4})+1\leftrightarrow 2\right\}, \tag{40a}$$

$$A^{\rm in\ plane}_{\rm S^4} = \frac{M}{r^5}\left\{(\boldsymbol{n}\cdot\boldsymbol{a}_1)^4\left(\frac{21\tilde{C}_{1\rm ES^2}}{2}-\frac{35\tilde{C}_{1\rm ES^4}}{4}\right)+(\boldsymbol{n}\cdot\boldsymbol{a}_2)(\boldsymbol{n}\cdot\boldsymbol{a}_1)^3(21\tilde{C}_{1\rm ES^2}-35\tilde{C}_{1\rm BS^3})\right.$$
$$+(\boldsymbol{n}\cdot\boldsymbol{a}_1)^2\left[(\boldsymbol{n}\cdot\boldsymbol{a}_2)^2\left(-\frac{105}{4}\tilde{C}_{1\rm ES^2}\tilde{C}_{2\rm ES^2}-21\tilde{C}_{1\rm ES^2}-21\tilde{C}_{2\rm ES^2}\right)+(\boldsymbol{a}_1\cdot\boldsymbol{a}_2)(15\tilde{C}_{1\rm BS^3}-12\tilde{C}_{1\rm ES^2})\right.$$
$$\left.+a_1^2\left(\frac{15\tilde{C}_{1\rm ES^4}}{2}-9\tilde{C}_{1\rm ES^2}\right)+\frac{15}{4}a_2^2\tilde{C}_{1\rm ES^2}\tilde{C}_{2\rm ES^2}+\frac{3a_2^2\tilde{C}_{1\rm ES^2}}{2}+3a_2^2\tilde{C}_{2\rm ES^2}\right]$$
$$+(\boldsymbol{n}\cdot\boldsymbol{a}_1)a_1^2(\boldsymbol{n}\cdot\boldsymbol{a}_2)(15\tilde{C}_{1\rm BS^3}-6\tilde{C}_{1\rm ES^2})+a_1^2(\boldsymbol{n}\cdot\boldsymbol{a}_2)^2\left(\frac{15\tilde{C}_{1\rm ES^2}\tilde{C}_{2\rm ES^2}}{4}+3\tilde{C}_{1\rm ES^2}+\frac{3\tilde{C}_{2\rm ES^2}}{2}\right)$$
$$\left.+(\boldsymbol{n}\cdot\boldsymbol{a}_1)(\boldsymbol{n}\cdot\boldsymbol{a}_2)(\boldsymbol{a}_1\cdot\boldsymbol{a}_2)\left(15\tilde{C}_{1\rm ES^2}\tilde{C}_{2\rm ES^2}+\frac{27\tilde{C}_{1\rm ES^2}}{2}+\frac{27\tilde{C}_{2\rm ES^2}}{2}\right)+1\leftrightarrow 2\right\}, \tag{40b}$$

which are zero for BHs.

E. Hamiltonian in tortoise coordinates

EOB waveform models often use the tortoise coordinate p_{r_*} instead of p_r, since it improves stability of the EOMs near the event horizon [97,223]. In the nonspinning case, the tortoise coordinate r_* is defined by

$$\frac{dr_*}{dr} = \frac{1}{\xi(r)}, \qquad \xi(r) \equiv A_{\rm noS}(r)\sqrt{\bar{D}_{\rm noS}(r)}, \tag{41}$$

and the conjugate momentum p_{r_*} is given by

$$p_{r_*} = p_r \xi(r). \tag{42}$$

The nonspinning effective Hamiltonian in Eq. (19) can be written in terms of p_{r_*} as

$$H_{\rm eff}^{\rm noS} = \sqrt{p_{r_*}^2 + A_{\rm noS}(r)[\mu^2 + p_\phi^2/r^2 + Q_{\rm noS}(r, p_{r_*})]}, \tag{43}$$

where we obtain $Q_{\rm noS}(r, p_{r_*})$ from Eq. (25) by converting p_r to p_{r_*} using Eq. (42); then PN expands to 5.5PN.

For both aligned and precessing spins, a convenient choice for $\xi(r)$ is

$$\xi(r) = \frac{\sqrt{\bar{D}_{\rm noS}}(A_{\rm noS} + a_+^2/r^2)}{1 + a_+^2/r^2}, \tag{44}$$

which is similar to what was used for ξ in SEOBNRv4 [97,105] except for the different resummation and PN orders in $A_{\rm noS}$ and $\bar{D}_{\rm noS}$. In the $\nu \to 0$ limit, ξ reduces to the Kerr value $(dr/dr_*)_{\rm Kerr} = (r^2 - 2Mr + a_+^2)/(r^2 + a_+^2)$.

The PN expansion of $\xi(r)$ is given by

$$\xi(r) \simeq 1 - \frac{2M}{r} + \frac{3\nu M^2}{r^2} + \frac{2Ma^2}{r^3} + \ldots, \tag{45}$$

which equals 1 at LO, while the spin contribution enters at 3PN. Hence, we can directly replace p_r by p_{r_*} in the 3.5PN S^3 and 4PN SS contributions in the Hamiltonian.

F. Comparison with other models

In this section, we obtained an EOB Hamiltonian that reduces in the $\nu \to 0$ limit to the Kerr Hamiltonian for a nonspinning test mass in a generic orbit. The nonspinning part of the Hamiltonian contains 4PN and partial 5PN results, which are Padé resummed. The Hamiltonian also includes the full 4PN precessing-spin contributions, which are the same PN information included in the Hamiltonians derived in Ref. [104], which extended the results of Ref. [103] to higher orders; however, we use different resummations/factorizations from those employed in the above references.

Table II summarizes the main features of the SEOBNRv5 Hamiltonian, and compares it to two other waveform models: SEOBNRv4 [99,100,107,111] and TEOBResumS [102,112,113].

III. COMPUTATIONALLY EFFICIENT PRECESSING-SPIN DYNAMICS

In Sec. II D, we derived an effective Hamiltonian for precessing spins that reduces to the Kerr Hamiltonian for generic orbits. The EOMs from that Hamiltonian read as

$$\dot{\boldsymbol{r}} = \frac{\partial H_{\rm EOB}^{\rm prec}}{\partial \boldsymbol{p}}, \qquad \dot{\boldsymbol{p}} = -\frac{\partial H_{\rm EOB}^{\rm prec}}{\partial \boldsymbol{r}} + \boldsymbol{\mathcal{F}},$$
$$\dot{\boldsymbol{S}}_{\rm i} = \frac{\partial H_{\rm EOB}^{\rm prec}}{\partial \boldsymbol{S}_{\rm i}} \times \boldsymbol{S}_{\rm i} + \dot{\boldsymbol{S}}_{\rm i}^{\rm RR}, \tag{46}$$

where $\boldsymbol{\mathcal{F}}$ is the RR force, and $\dot{\boldsymbol{S}}_{\rm i}^{\rm RR}$ is the RR contribution to the spin-evolution equations, which starts at $\mathcal{O}(v^{11}S^2)$ [224,225] and is thus neglected in the order we consider here.

These equations are computationally expensive to evolve numerically. Therefore, we simplify them such that we can solve the two-body dynamics more efficiently without losing much accuracy when describing the precessional effects. It was shown in Refs. [31,159–161] that precessing-spin waveforms can be built starting from aligned-spin waveforms in the coprecessing frame, in which the z axis remains perpendicular to the instantaneous orbital plane, and then applying a suitable rotation to the inertial frame. The precessing-spin SEOBNRv3 and SEOBNRv4 models employed the full EOB precessing-spin Hamiltonian [99,100] to evolve the dynamics in the coprecessing frame. To build the precessing-spin TEOBResumS model and speed up the computational time, Refs. [114,115] used an aligned-spin EOB Hamiltonian when evolving the EOMs in the coprecessing frame. Also, the IMRPhenomT model [79] is built using a purely aligned-spin dynamics in the coprecessing frame.

Here, to improve the accuracy in describing precession effects, we find it important to incorporate at least partial precessing-spin information in the Hamiltonian used in the coprecessing frame, as studies in Ref. [154] have demonstrated. To do that, we first obtain a precessing-spin Hamiltonian simpler than the full one derived in Sec. II D, such that it reduces to $H_{\rm eff}^{\rm align}$ for aligned spins, and only includes the in-plane spin components for circular orbits ($p_r = 0$). Then, we orbit average the in-plane spin components in the Hamiltonian, and use it to evolve the EOMs for the dynamical variables r, p_r, ϕ, and p_ϕ, while the evolution equations for the spin and angular momentum vectors are computed in a PN-expanded, orbit-averaged form for quasicircular orbits. The procedure to obtain the PN-expanded EOMs, and the appropriate dynamical variables, is similar to what was used in Refs. [79,114,115,158], but we include higher PN orders in the EOMs, and derive them from the SEOBNRv5 EOB Hamiltonian, employing a different gauge and SSC. Other differences in the waveform model from previous work are described in Ref. [154].

In the following subsections, we present the Hamiltonian, then derive the PN-expanded EOMs for precessing spins, in EOB coordinates, up to NNLO SO (3.5PN) and NNLO SS (4PN).

A. Hamiltonian with partial precessing-spin dynamics

The precessing-spin Hamiltonian presented in Sec. II D reduces to the exact Kerr Hamiltonian in Eq. (13) for generic orbits. Here, we consider a simpler Hamiltonian that starts with an ansatz similar to the aligned-spin Hamiltonian in Eqs. (26) and (32), then complement it with precessing-spin corrections for circular orbits only (i.e., we do not include in-plane spin terms proportional to p_r). Thus, this Hamiltonian reduces to $H_{\rm eff}^{\rm align}$ from Sec. II C for aligned spins, and to $H^{\rm Kerr\ eq}$ for $\nu \to 0$, but does not reduce to the full precessing-spin Hamiltonian $H_{\rm eff}^{\rm prec}$ and the Kerr Hamiltonian for generic orbits.

We use the following ansatz for the pprec effective Hamiltonian [cf. Eqs. (26) and (35)]

$$H_{\rm eff}^{\rm pprec} = \frac{M\boldsymbol{L}\cdot(g_{a_+}\boldsymbol{a}_+ + g_{a_-}\delta\boldsymbol{a}_-) + {\rm SO}_{\rm calib} + \langle G_{a^3}^{\rm pprec}\rangle}{r^3 + a_+^2(r+2M)} + \left[A^{\rm pprec}\left(\mu^2 + B_p^{\rm pprec}\frac{\boldsymbol{L}^2}{r^2} + (1+B_{np}^{\rm pprec})(\boldsymbol{n}\cdot\boldsymbol{p})^2 + B_{npa}^{\rm Kerr\ eq}\frac{(\boldsymbol{L}\cdot\boldsymbol{a}_+)^2}{r^2} + Q^{\rm pprec}\right)\right]^{1/2}, \tag{47}$$

where the gyrogravitomagnetic factors and SO calibration term are the same as in Eqs. (28) and (36), with the same value of $d_{\rm SO}$ as the aligned-spin model.

The SS corrections are added such that [cf. Eqs. (32) and (38)]

$$\begin{aligned}
A^{\rm pprec} &= \frac{a_+^2/r^2 + A_{\rm noS} + A_{\rm SS}^{\rm prec} + A_{S^4}^{\rm prec} + \langle \tilde{A}_{\rm SS}^{\rm in\ plane}\rangle}{1 + (1+2M/r)a_+^2/r^2},\\
B_p^{\rm pprec} &= 1 + \langle \tilde{B}_{p,\rm SS}^{\rm in\ plane}\rangle,\\
B_{np}^{\rm pprec} &= -1 + a_+^2/r^2 + A_{\rm noS}\bar{D}_{\rm noS} + B_{np,\rm SS}^{\rm prec},\\
Q^{\rm pprec} &= Q_{\rm noS} + Q_{\rm SS}^{\rm prec},
\end{aligned} \tag{48}$$

where the nonspinning contributions are given by Eqs. (21)–(25), the SS corrections $A_{\rm SS}^{\rm prec}$, $B_{np,\rm SS}^{\rm prec}$, and $Q_{\rm SS}^{\rm prec}$ are the same as in Eq. (39), and the S[4] term $A_{S^4}^{\rm prec}$ is given by Eq. (40a). The purely in-plane SS contributions are included in $\tilde{A}_{\rm SS}^{\rm in\ plane}$ and $\tilde{B}_{p,\rm SS}^{\rm in\ plane}$, which are obtained by writing an ansatz with unknown coefficients and matching it to a 4PN-expanded precessing-spin Hamiltonian with $p_r = 0$. We indicate with $\langle..\rangle$ the result of orbit averaging the in-plane spin components, as explained in Sec. III B below. We do not include in-plane S^4 corrections for simplicity, and because it is not straightforward to consistently orbit average the S^4 terms in the Hamiltonian. Thus, the partial precessing-spin Hamiltonian agrees with the full precessing-spin Hamiltonian $H_{\rm eff}^{\rm prec}$ from Sec. II D when PN expanded to 4PN and to $\mathcal{O}(S^3)$ (included) for circular orbits.

For the terms $\tilde{A}_{\rm SS}^{\rm in\ plane}$ and $\tilde{B}_{p,\rm SS}^{\rm in\ plane}$, we obtain

$$\begin{aligned}
\tilde{A}_{\rm SS}^{\rm in\ plane} = {}& \frac{M}{r^3}[2(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2 + 3C_+^{n\cdot a^2}] + \frac{M^2}{r^4}\left[\frac{33}{8}\delta\boldsymbol{n}\cdot\boldsymbol{a}_+\boldsymbol{n}\cdot\boldsymbol{a}_- + \left(-\frac{\nu}{2}-\frac{3}{8}\right)\left(\boldsymbol{n}\cdot\boldsymbol{a}_-\right)^2 + \left(\frac{7\nu}{4}-\frac{31}{4}\right)\left(\boldsymbol{n}\cdot\boldsymbol{a}_+\right)^2\right.\\
&\left. + 3\delta C_-^{n\cdot a^2} + 3\nu C_+^{n\cdot a^2}\right] + \frac{M^3}{r^5}\left[\delta\left(17\nu - \frac{1}{4}\right)\boldsymbol{n}\cdot\boldsymbol{a}_+\boldsymbol{n}\cdot\boldsymbol{a}_- + \left(-\frac{41\nu^2}{8} + \frac{583\nu}{32} - \frac{171}{64}\right)\left(\boldsymbol{n}\cdot\boldsymbol{a}_-\right)^2\right.\\
&\left. + \left(\frac{187}{64} - \frac{11\nu^2}{8} + \frac{1435\nu}{96}\right)\left(\boldsymbol{n}\cdot\boldsymbol{a}_+\right)^2 + \delta\left(\frac{81\nu}{16} - \frac{99}{224}\right)C_-^{n\cdot a^2} + \left(\frac{867}{224} - \frac{13\nu^2}{16} + \frac{179\nu}{56}\right)C_+^{n\cdot a^2}\right],
\end{aligned} \tag{49a}$$

$$\begin{aligned}
\tilde{B}_{p,\rm SS}^{\rm in\ plane} = {}& -\frac{(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2}{r^2} + \frac{M}{r^3}\left[\frac{3}{4}\delta\boldsymbol{n}\cdot\boldsymbol{a}_+\boldsymbol{n}\cdot\boldsymbol{a}_- + \left(\frac{3}{16} - \frac{3\nu}{4}\right)\left(\boldsymbol{n}\cdot\boldsymbol{a}_-\right)^2 + \left(-\frac{7\nu}{4} - \frac{15}{16}\right)\left(\boldsymbol{n}\cdot\boldsymbol{a}_+\right)^2 + (3-3\nu)C_+^{n\cdot a^2}\right]\\
& + \frac{M^2}{r^4}\left[\delta\left(-\frac{49\nu}{4} - \frac{43}{8}\right)\boldsymbol{n}\cdot\boldsymbol{a}_+\boldsymbol{n}\cdot\boldsymbol{a}_- + \left(\frac{19\nu^2}{8} - \frac{545\nu}{32} + \frac{219}{64}\right)\left(\boldsymbol{n}\cdot\boldsymbol{a}_-\right)^2 + \left(\frac{11\nu^2}{8} - \frac{805\nu}{96} + \frac{125}{64}\right)\left(\boldsymbol{n}\cdot\boldsymbol{a}_+\right)^2\right.\\
&\left. + \delta\left(\frac{189}{32} - \frac{81\nu}{16}\right)C_-^{n\cdot a^2} + \left(\frac{13\nu^2}{16} - \frac{203\nu}{8} + \frac{51}{32}\right)C_+^{n\cdot a^2}\right],
\end{aligned} \tag{49b}$$

and for the cubic-in-spin term $G_{a^3}^{\rm pprec}$, we get

$$\begin{aligned}
G_{a^3}^{\rm pprec} = {}& \boldsymbol{L}\cdot\boldsymbol{a}_+\left\{\frac{L^2}{\mu^2 r^3}\left[\frac{\delta}{2}(\boldsymbol{n}\cdot\boldsymbol{a}_-)(\boldsymbol{n}\cdot\boldsymbol{a}_+) - \frac{(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2}{4}\right] + \frac{M}{r^2}\left[-\frac{a_+^2}{4} - \frac{3}{4}(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2 + \delta\frac{5}{24}(\boldsymbol{a}_+\cdot\boldsymbol{a}_-)\right.\right.\\
&\left.\left. - \delta\frac{5}{3}(\boldsymbol{n}\cdot\boldsymbol{a}_-)(\boldsymbol{n}\cdot\boldsymbol{a}_+) + \frac{1}{8}a_1^2(3\delta\tilde{C}_{\rm 1ES^2} + 4\tilde{C}_{\rm 1BS^3}) + \frac{1}{8}a_2^2(4\tilde{C}_{\rm 2BS^3} - 3\tilde{C}_{\rm 2ES^2}\delta)\right.\right.
\end{aligned}$$

$$-\frac{3}{8}(\boldsymbol{a}_1\cdot\boldsymbol{a}_2)(\tilde{C}_{1\mathrm{ES}^2}(\delta-3)-\tilde{C}_{2\mathrm{ES}^2}(\delta+3))+\frac{3}{4}(\boldsymbol{n}\cdot\boldsymbol{a}_1)(\boldsymbol{n}\cdot\boldsymbol{a}_2)(\tilde{C}_{1\mathrm{ES}^2}(2\delta-7)-\tilde{C}_{2\mathrm{ES}^2}(2\delta+7))$$
$$+(\boldsymbol{n}\cdot\boldsymbol{a}_1)^2\left(-\frac{3}{8}\tilde{C}_{1\mathrm{ES}^2}(2\delta+3)-\frac{5\tilde{C}_{1\mathrm{BS}^3}}{2}\right)+\frac{1}{8}(\boldsymbol{n}\cdot\boldsymbol{a}_2)^2(\tilde{C}_{2\mathrm{ES}^2}(6\delta-9)-20\tilde{C}_{2\mathrm{BS}^3})\Bigg]\Bigg\}$$
$$+\delta\boldsymbol{L}\cdot\boldsymbol{a}_-\Bigg\{-\frac{L^2(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2}{4\mu^2r^3}+\frac{M}{r^2}\Bigg[\frac{a_+^2}{24}+\frac{5}{12}(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2+\frac{1}{8}a_1^2(4\tilde{C}_{1\mathrm{BS}^3}-3\tilde{C}_{1\mathrm{ES}^2})+\frac{1}{8}a_2^2(3\tilde{C}_{2\mathrm{ES}^2}-4\tilde{C}_{2\mathrm{BS}^3})$$
$$-\frac{3}{8}(\boldsymbol{a}_1\cdot\boldsymbol{a}_2)(\tilde{C}_{1\mathrm{ES}^2}(\delta-3)+\tilde{C}_{2\mathrm{ES}^2}(\delta+3))+(\boldsymbol{n}\cdot\boldsymbol{a}_1)^2\left(-\frac{3}{8}\tilde{C}_{1\mathrm{ES}^2}(\delta-6)-\frac{5\tilde{C}_{1\mathrm{BS}^3}}{2}\right)$$
$$+\frac{3}{4}(\boldsymbol{n}\cdot\boldsymbol{a}_1)(\boldsymbol{n}\cdot\boldsymbol{a}_2)(\tilde{C}_{1\mathrm{ES}^2}(2\delta-7)+\tilde{C}_{2\mathrm{ES}^2}(2\delta+7))+(\boldsymbol{n}\cdot\boldsymbol{a}_2)^2\left(\frac{5\tilde{C}_{2\mathrm{BS}^3}}{2}-\frac{3}{8}\tilde{C}_{2\mathrm{ES}^2}(\delta+6)\right)\Bigg]\Bigg\}. \tag{50}$$

B. Orbit averaging the precessing-spin contributions

To simplify the EOMs, we remove the explicit dependence of the Hamiltonian on the $\boldsymbol{n}\cdot\boldsymbol{a}_\mathrm{i}$ terms by taking their orbit average. Since the spin-precession timescale ($\sim v^{-5}$) is larger than the orbital timescale ($\sim v^{-3}$), orbit averaging the in-plane spin contributions is expected to provide a good approximation for the dynamics.

We define the unit vectors $(\boldsymbol{l}_\mathrm{N},\boldsymbol{n},\boldsymbol{\lambda}_\mathrm{N})$ in Cartesian coordinates such that $\boldsymbol{l}_\mathrm{N}$ is aligned with the z axis, and hence the vector components are given by

$$\begin{aligned}\boldsymbol{l}_\mathrm{N}&=(0,0,1),\\ \boldsymbol{n}&=(\cos\phi,\sin\phi,0),\\ \boldsymbol{\lambda}_\mathrm{N}&\equiv\boldsymbol{l}_\mathrm{N}\times\boldsymbol{n}=(-\sin\phi,\cos\phi,0),\end{aligned}\tag{51}$$

where ϕ is the orbital phase. When neglecting RR effects, an orbit average yields

$$\begin{aligned}\langle n^i\rangle&=\frac{1}{2\pi}\int_0^{2\pi}n^i\mathrm{d}\phi=0=\langle\lambda_\mathrm{N}^i\rangle,\\ \langle n^in^j\rangle&=\langle\lambda_\mathrm{N}^i\lambda_\mathrm{N}^j\rangle=\frac{1}{2}(\delta^{ij}-l_\mathrm{N}^il_\mathrm{N}^j),\end{aligned}\tag{52}$$

which lead to the following relations for the spin dot products:

$$\begin{aligned}\langle(\boldsymbol{n}\cdot\boldsymbol{S}_\mathrm{i})\boldsymbol{n}\rangle&=\langle(\boldsymbol{\lambda}_\mathrm{N}\cdot\boldsymbol{S}_\mathrm{i})\boldsymbol{\lambda}_\mathrm{N}\rangle=\frac{1}{2}[\boldsymbol{S}_\mathrm{i}-(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_\mathrm{i})\boldsymbol{l}_\mathrm{N}],\\ \langle(\boldsymbol{n}\cdot\boldsymbol{S}_\mathrm{i})^2\rangle&=\langle(\boldsymbol{\lambda}_\mathrm{N}\cdot\boldsymbol{S}_\mathrm{i})^2\rangle=\frac{1}{2}[S_\mathrm{i}^2-(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_\mathrm{i})^2],\\ \langle(\boldsymbol{n}\cdot\boldsymbol{S}_1)(\boldsymbol{n}\cdot\boldsymbol{S}_2)\rangle&=\langle(\boldsymbol{\lambda}_\mathrm{N}\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_\mathrm{N}\cdot\boldsymbol{S}_2)\rangle\\ &=\frac{1}{2}[\boldsymbol{S}_1\cdot\boldsymbol{S}_2-(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_2)].\end{aligned}\tag{53}$$

The Hamiltonian from Sec. III A depends on $(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2$, $(\boldsymbol{n}\cdot\boldsymbol{a}_-)^2$, and $(\boldsymbol{n}\cdot\boldsymbol{a}_+)(\boldsymbol{n}\cdot\boldsymbol{a}_-)$, and can be made a function of only $\boldsymbol{a}_\pm$ and $\boldsymbol{l}_\mathrm{N}$ using the following orbit-averaged expressions:

$$\begin{aligned}(\boldsymbol{n}\cdot\boldsymbol{a}_\pm)^2&\simeq\frac{1}{2}[a_\pm^2-(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{a}_\pm)^2],\\ (\boldsymbol{n}\cdot\boldsymbol{a}_+)(\boldsymbol{n}\cdot\boldsymbol{a}_-)&\simeq\frac{1}{2}[\boldsymbol{a}_+\cdot\boldsymbol{a}_--(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{a}_+)(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{a}_-)].\end{aligned}\tag{54}$$

When taking the orbit average, we neglect RR since the Hamiltonian encodes the conservative dynamics, and the RR timescale ($\sim v^{-8}$) is much larger than the spin-precession timescale. We account for dissipative effects in the EOMs through the RR force and the orbital-frequency evolution equation, as described below.

When restricting to binary black holes, the explicit expression of the partial-precessing Hamiltonian as a function of $\boldsymbol{a}_\pm$ and $\boldsymbol{l}_\mathrm{N}$ is given in Appendix A of Ref. [154].

C. Equations of motion

The "Newtonian" angular-momentum vector $\boldsymbol{L}_\mathrm{N}$ is perpendicular to the instantaneous orbital plane, since it is defined by

$$\boldsymbol{L}_\mathrm{N}\equiv\mu\boldsymbol{r}\times\mathbf{v},\tag{55}$$

where $\mathbf{v}\equiv\dot{\boldsymbol{r}}$ is the velocity. We use a coprecessing frame aligned with the orthonormal unit vectors $(\boldsymbol{l}_\mathrm{N},\boldsymbol{n},\boldsymbol{\lambda}_\mathrm{N})$, with $\boldsymbol{l}_\mathrm{N}$ being the direction of $\boldsymbol{L}_\mathrm{N}$. Since $\boldsymbol{l}_\mathrm{N}$ is perpendicular to $\boldsymbol{r}$ and $\mathbf{v}$, we can write the velocity as

$$\mathbf{v}=\dot{r}\boldsymbol{n}+r\Omega\boldsymbol{\lambda}_\mathrm{N},\tag{56}$$

which can be considered as a definition for the orbital frequency Ω, implying that $\Omega=|\boldsymbol{n}\times\mathbf{v}|/r$.

The Hamiltonian is expressed in terms of the canonical angular momentum $\boldsymbol{L}\equiv\boldsymbol{r}\times\boldsymbol{p}$. We denote the $\boldsymbol{L}$-based unit vectors by $(\boldsymbol{l},\boldsymbol{n},\boldsymbol{\lambda})$, where $\boldsymbol{l}$ is the direction of $\boldsymbol{L}$ and $\boldsymbol{\lambda}\equiv\boldsymbol{l}\times\boldsymbol{n}$, and then express the EOMs derived from the Hamiltonian in terms of $\boldsymbol{l}_\mathrm{N}$.

In the coprecessing frame, the partial precessing-spin dynamics can be approximated by the following EOMs:

$$\begin{aligned}\dot{r}&=\frac{\partial H_\mathrm{EOB}^\mathrm{pprec}}{\partial p_r},&\dot{\phi}&=\frac{\partial H_\mathrm{EOB}^\mathrm{pprec}}{\partial p_\phi},\\ \dot{p}_r&=-\frac{\partial H_\mathrm{EOB}^\mathrm{pprec}}{\partial r}+\mathcal{F}_r,&\dot{p}_\phi&=\mathcal{F}_\phi,\end{aligned}\tag{57}$$

where $H_{\rm EOB}^{\rm pprec}$ is related to $H_{\rm eff}^{\rm pprec}$ from Eq. (47) through Eq. (7), and the quantities $\langle G_{a^3}^{\rm pprec}\rangle$, $\langle \tilde{A}_{\rm SS}^{\rm in\ plane}\rangle$, and $\langle \tilde{B}_{p,\rm SS}^{\rm in\ plane}\rangle$ in $H_{\rm eff}^{\rm pprec}$ can be expressed in terms of $\boldsymbol{a}_+$, $\boldsymbol{a}_-$ and $\boldsymbol{l}_{\rm N}$ once we replace the $\boldsymbol{n}\cdot\boldsymbol{a}_\pm$ terms by their orbit average using Eq. (54). The explicit expressions are given in Appendix A of Ref. [154].

At each time step, we also evolve the PN-expanded equations for the spins and angular momentum, given by

$$\begin{aligned}\dot{\boldsymbol{S}}_{\rm i} &= \boldsymbol{\Omega}_{S_{\rm i}} \times \boldsymbol{S}_{\rm i},\\ \boldsymbol{L} &= \boldsymbol{L}(\boldsymbol{l}_{\rm N}, v, \boldsymbol{S}_{\rm i}),\\ \dot{\boldsymbol{l}}_{\rm N} &= \dot{\boldsymbol{l}}_{\rm N}(\boldsymbol{l}_{\rm N}, v, \boldsymbol{S}_{\rm i}),\end{aligned} \tag{58}$$

where $v \equiv (M\Omega)^{1/3}$, and $\boldsymbol{\Omega}_{S_{\rm i}} \equiv \partial H_{\rm EOB}^{\rm prec}/\partial \boldsymbol{S}_{\rm i}$ is the spin-precession frequency, computed in a PN expansion from the precessing-spin EOB Hamiltonian. These equations are derived in the following subsections to NNLO SS in an orbit average for quasicircular orbits.

Equations (57) and (58) can be solved simultaneously for the dynamical variables. Alternatively, they can be decoupled by computing the orbital frequency used in Eq. (58) in a PN expansion, which can be expressed as

$$\dot{v} = \left[\frac{\dot{E}(v)}{\mathrm{d}E(v)/\mathrm{d}v}\right]_{\text{PN-expanded}}, \tag{59}$$

where $E(v)$ is the energy of the binary system and $\dot{E}(v)$ is the rate of energy loss.

Using the EOB orbital frequency, obtained by solving Eq. (57), can lead to slightly more accurate results when solving Eq. (58) than the PN-expanded frequency from Eq. (59). However, the `SEOBNRv5PHM` waveform model [154] uses the PN-expanded frequency since decoupling Eqs. (57) and (58) makes it possible to use the postadiabatic approximation [92,226–228], which improves the computational efficiency of the model.

In Sec. III F below, we obtain $\dot{v}$ in a PN-expanded form, including the NNLO SS contribution that was recently derived in the flux in Ref. [229]. In all equations derived in this section, we include PN orders up to NNLO SS, which implies different powers in v for each quantity depending on the LO, as summarized in Table III.

Some precessing-spin waveform models, such as Refs. [109,111,230], considered a coprecessing frame adapted to the orbital angular momentum $\boldsymbol{L}$, instead of $\boldsymbol{L}_{\rm N}$. Therefore, for completeness, we also provide in Appendix C, the EOMs expressed in terms of $\boldsymbol{l}$.

D. Angular momentum vector

To obtain the angular momentum unit vector $\boldsymbol{l}$ in terms of $\boldsymbol{l}_{\rm N}$, we first use the EOMs (46), and the definition of $\boldsymbol{L}_{\rm N}$ from Eq. (55), to get $\boldsymbol{l}_{\rm N}$ in a PN expansion, i.e.,

TABLE III. Orders in v at which the nonspinning, SO, and SS contributions first enter r, $\boldsymbol{L}$, $\dot{\boldsymbol{S}}_{\rm i}$, $\dot{\boldsymbol{l}}_{\rm N}$, and $\dot{v}$ for quasicircular orbits. The last column indicates the highest power in v we include in each quantity.

Quantity	LO S^0	LO SO	LO SS	Highest order
r	v^0	v^3	v^4	v^8
$\boldsymbol{L}$	v^{-1}	v^2	v^3	v^7
$\dot{\boldsymbol{S}}_{\rm i}$	...	v^5	v^6	v^{10}
$\dot{\boldsymbol{l}}_{\rm N}$	...	v^6	v^7	v^{11}
$\dot{v}$	v^9	v^{12}	v^{13}	v^{17}

$$\boldsymbol{l}_{\rm N} \equiv \frac{\boldsymbol{L}_{\rm N}}{|\boldsymbol{L}_{\rm N}|} = \frac{\mu}{|\boldsymbol{L}_{\rm N}|}\boldsymbol{r} \times \frac{\partial H_{\rm EOB}^{\rm prec}}{\partial \boldsymbol{p}}, \tag{60}$$

where we use the Hamiltonian before taking the orbit average of the in-plane spin terms.

Then, we specialize to circular orbits, which are defined by $p_r = 0$ and $\dot{p}_r = 0$. To obtain r and L as functions of v for circular orbits, we solve

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}\cdot\boldsymbol{p}) &= \boldsymbol{r}\cdot\dot{\boldsymbol{p}} + \dot{\boldsymbol{r}}\cdot\boldsymbol{p} = 0,\\ v^3 &= \frac{M}{r}|\boldsymbol{n}\times\mathbf{v}|,\end{aligned} \tag{61}$$

for $r(\boldsymbol{l},\boldsymbol{\lambda},\boldsymbol{n},\boldsymbol{S}_{\rm i},v)$ and $L(\boldsymbol{l},\boldsymbol{\lambda},\boldsymbol{n},\boldsymbol{S}_{\rm i},v)$, perturbatively in a PN expansion, after using the EOMs (46) without RR.

We substitute that solution for r and L in Eq. (60), and replace $\boldsymbol{\lambda}$ using

$$\begin{aligned}\boldsymbol{S}_{\rm i} &= (\boldsymbol{n}\cdot\boldsymbol{S}_{\rm i})\boldsymbol{n} + (\boldsymbol{\lambda}\cdot\boldsymbol{S}_{\rm i})\boldsymbol{\lambda} + (\boldsymbol{l}\cdot\boldsymbol{S}_{\rm i})\boldsymbol{l}\\ &= (\boldsymbol{n}\cdot\boldsymbol{S}_{\rm i})\boldsymbol{n} + (\boldsymbol{\lambda}_{\rm N}\cdot\boldsymbol{S}_{\rm i})\boldsymbol{\lambda}_{\rm N} + (\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_{\rm i})\boldsymbol{l}_{\rm N},\end{aligned} \tag{62}$$

which implies that

$$\begin{aligned}&(\boldsymbol{\lambda}\cdot\boldsymbol{S}_{\rm i})^2 + (\boldsymbol{l}\cdot\boldsymbol{S}_{\rm i})^2 = (\boldsymbol{\lambda}_{\rm N}\cdot\boldsymbol{S}_{\rm i})^2 + (\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_{\rm i})^2,\\ &(\boldsymbol{\lambda}\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}\cdot\boldsymbol{S}_2) + (\boldsymbol{l}\cdot\boldsymbol{S}_1)(\boldsymbol{l}\cdot\boldsymbol{S}_2)\\ &\quad = (\boldsymbol{\lambda}_{\rm N}\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_{\rm N}\cdot\boldsymbol{S}_2) + (\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_2).\end{aligned} \tag{63}$$

That way, the right-hand side of Eq. (60) only depends on $\boldsymbol{l}$, $\boldsymbol{l}_{\rm N}$, $\boldsymbol{\lambda}_{\rm N}$, v, and the spins.

To solve Eq. (60) for $\boldsymbol{l}(\boldsymbol{l}_{\rm N},\boldsymbol{\lambda}_{\rm N},\boldsymbol{S}_{\rm i},v)$, we expand it in spin, such that

$$\boldsymbol{l} \equiv \boldsymbol{l}_{\rm N} + \boldsymbol{l}_{\rm SO} + \boldsymbol{l}_{\rm SS}, \tag{64a}$$

since $\boldsymbol{l}$ is in the same direction as $\boldsymbol{l}_{\rm N}$ for nonspinning binaries, while $\boldsymbol{l}_{\rm SO}$ and $\boldsymbol{l}_{\rm SS}$ are the SO and SS contributions. Solving order by order in spin, we obtain

$$\boldsymbol{l}_{\rm SO} = \frac{(\boldsymbol{\lambda}_{\rm N}\cdot\boldsymbol{S}_1)\boldsymbol{\lambda}_{\rm N}}{M\mu}\left\{-\frac{v^3}{2}(3X_2+\nu) + v^5\left[\frac{\nu^2}{8} - \frac{9\nu}{8} + \left(\frac{3\nu}{2}+\frac{9}{8}\right)X_2\right] + v^7\left[\frac{\nu^3}{48} + \frac{9\nu^2}{4} - \frac{27\nu}{16} + \left(-\frac{\nu^2}{2} + \frac{15\nu}{4} + \frac{27}{16}\right)X_2\right]\right\} + 1\leftrightarrow 2, \tag{64b}$$

$$\begin{aligned}\boldsymbol{l}_{\rm SS} = {}& \frac{v^4}{M^2\mu^2}\boldsymbol{\lambda}_{\rm N}[(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_{\rm N}\cdot\boldsymbol{S}_1)(X_2-\nu) + \nu(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_{\rm N}\cdot\boldsymbol{S}_2)]\\ &+ \frac{v^6}{M^2\mu^2}\left\{\boldsymbol{\lambda}_{\rm N}(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_{\rm N}\cdot\boldsymbol{S}_1)\left[\frac{\nu^2}{6} + \frac{5\nu}{2} + \left(-\frac{11\nu}{3} - \frac{5}{2}\right)X_2\right] + \boldsymbol{\lambda}_{\rm N}(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_{\rm N}\cdot\boldsymbol{S}_2)\left(-\frac{7\nu^2}{6} - 4\nu\right)\right.\\ &\left.+ \boldsymbol{l}_{\rm N}(\boldsymbol{\lambda}_{\rm N}\cdot\boldsymbol{S}_1)^2\left[-\frac{\nu^2}{8} + \frac{9\nu}{8} + \left(-\frac{3\nu}{4} - \frac{9}{8}\right)X_2\right] + \boldsymbol{l}_{\rm N}(\boldsymbol{\lambda}_{\rm N}\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_{\rm N}\cdot\boldsymbol{S}_2)\left(-\frac{\nu^2}{8} - \frac{3}{2}\nu\right)\right\}\\ &+ \frac{v^8}{M^2\mu^2}\left\{\boldsymbol{\lambda}_{\rm N}(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_{\rm N}\cdot\boldsymbol{S}_1)\left[-\frac{211\nu^3}{144} - \frac{41\nu^2}{16} - \frac{9\nu}{16} + \left(-\frac{479\nu^2}{144} + 2\nu + \frac{9}{16}\right)X_2\right]\right.\\ &+ \boldsymbol{l}_{\rm N}(\boldsymbol{\lambda}_{\rm N}\cdot\boldsymbol{S}_1)^2\left[\frac{\nu^3}{16} - \frac{45\nu^2}{16} - \frac{27\nu}{16} + \left(\frac{15\nu^2}{16} + \frac{9\nu}{8} + \frac{27}{16}\right)X_2\right]\\ &+ \boldsymbol{\lambda}_{\rm N}(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_{\rm N}\cdot\boldsymbol{S}_2)\left[-\frac{179\nu^3}{144} + \frac{9\nu^2}{32} - \frac{3\nu}{8} + \left(\frac{3\nu^2}{8} + \frac{3\nu}{2}\right)X_2\right]\\ &\left.+ \boldsymbol{l}_{\rm N}(\boldsymbol{\lambda}_{\rm N}\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_{\rm N}\cdot\boldsymbol{S}_2)\left[\frac{\nu^3}{16} + \frac{69\nu^2}{32} + \frac{9\nu}{8}\right]\right\} + 1\leftrightarrow 2,\end{aligned} \tag{64c}$$

which is independent of the spin-quadrupole constants. Substituting $\boldsymbol{l}(\boldsymbol{l}_{\rm N})$ in the solution of Eq. (61) yields $r(\boldsymbol{l}_{\rm N}, \boldsymbol{\lambda}_{\rm N}, \boldsymbol{n}, \boldsymbol{S}_{\rm i}, v)$ and $L(\boldsymbol{l}_{\rm N}, \boldsymbol{\lambda}_{\rm N}, \boldsymbol{n}, \boldsymbol{S}_{\rm i}, v)$, which are given in Appendix A.

Finally, we use these relations to obtain $\boldsymbol{L} = L\boldsymbol{l}$ and take its orbit average using Eq. (53), leading to

$$\boldsymbol{L} \equiv \frac{\mu M}{v}(\bar{\boldsymbol{L}}_{S^0} + \bar{\boldsymbol{L}}_{\rm SO} + \bar{\boldsymbol{L}}_{S_1S_2} + \bar{\boldsymbol{L}}_{S^2} + \bar{\boldsymbol{L}}_{S^2\tilde{C}}), \tag{65a}$$

$$\begin{aligned}\bar{\boldsymbol{L}}_{S^0} = {}& \boldsymbol{l}_{\rm N}\left\{1 + \left(\frac{\nu}{6}+\frac{3}{2}\right)v^2 + \left(\frac{\nu^2}{24} - \frac{19\nu}{8} + \frac{27}{8}\right)v^4 + \left[\frac{7\nu^3}{1296} + \frac{31\nu^2}{24} + \left(\frac{41\pi^2}{24} - \frac{6889}{144}\right)\nu + \frac{135}{16}\right]v^6\right.\\ &+ v^8\left[-\frac{55\nu^4}{31104} - \frac{215\nu^3}{1728} + \left(\frac{356035}{3456} - \frac{2255\pi^2}{576}\right)\nu^2 + \nu\left(\frac{98869}{5760} - \frac{128\gamma_E}{3} - \frac{6455\pi^2}{1536} - \frac{256}{3}\ln 2 - \frac{128\ln v}{3}\right)\right.\\ &\left.\left.+ \frac{2835}{128}\right]\right\},\end{aligned} \tag{65b}$$

$$\begin{aligned}\bar{\boldsymbol{L}}_{\rm SO} = {}& \frac{v^3}{M\mu}\left[(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)\boldsymbol{l}_{\rm N}\left(-\frac{7\nu}{12} - \frac{7X_2}{4}\right) + \boldsymbol{S}_1\left(-\frac{\nu}{4} - \frac{3X_2}{4}\right)\right]\\ &+ \frac{v^5}{M\mu}\left\{(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)\boldsymbol{l}_{\rm N}\left[\frac{11\nu^2}{144} - \frac{55\nu}{16} + \left(\frac{55\nu}{24} - \frac{33}{16}\right)X_2\right] + \boldsymbol{S}_1\left[\frac{\nu^2}{48} - \frac{15\nu}{16} + \left(\frac{5\nu}{8} - \frac{9}{16}\right)X_2\right]\right\}\\ &+ \frac{v^7}{M\mu}\left\{(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)\boldsymbol{l}_{\rm N}\left[\frac{5\nu^3}{96} + \frac{275\nu^2}{32} - \frac{405\nu}{32} + \left(-\frac{25\nu^2}{32} + \frac{195\nu}{8} - \frac{135}{32}\right)X_2\right]\right.\\ &\left.+ \boldsymbol{S}_1\left[\frac{\nu^3}{96} + \frac{55\nu^2}{32} - \frac{81\nu}{32} + \left(-\frac{5\nu^2}{32} + \frac{39\nu}{8} - \frac{27}{32}\right)X_2\right]\right\} + 1\leftrightarrow 2,\end{aligned} \tag{65c}$$

$$\bar{\boldsymbol{L}}_{S_1S_2} = \frac{\nu v^4}{(M\mu)^2}\left\{\boldsymbol{l}_{\rm N}[2(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_2) - (\boldsymbol{S}_1\cdot\boldsymbol{S}_2)] + \frac{(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)\boldsymbol{S}_2}{2} + \frac{(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_2)\boldsymbol{S}_1}{2}\right\}$$
$$+ \frac{\nu v^6}{(M\mu)^2}\left\{\boldsymbol{l}_{\rm N}\left[(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_2)\left(\frac{13\nu}{36} - \frac{7}{6}\right) + \frac{2\nu(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)}{3}\right] + \boldsymbol{S}_2(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)\left(\frac{5}{4} - \frac{7\nu}{24}\right) + \boldsymbol{S}_1(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_2)\left(\frac{5}{4} - \frac{7\nu}{24}\right)\right\}$$
$$+ \frac{\nu v^8}{(M\mu)^2}\left\{\boldsymbol{l}_{\rm N}\left[(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_2)\left(-\frac{361\nu^2}{432} + \frac{361\nu}{288} + \frac{15}{4}\right) + \left(-\frac{5\nu^2}{72} - \frac{245\nu}{24} - \frac{5}{4}\right)(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\right]\right.$$
$$\left. + \boldsymbol{S}_2(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)\left(-\frac{223}{288}\nu^2 - \frac{349\nu}{64} + \frac{15}{8}\right) + \boldsymbol{S}_1(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_2)\left(-\frac{223}{288}\nu^2 - \frac{349\nu}{64} + \frac{15}{8}\right)\right\}, \tag{65d}$$

$$\bar{\boldsymbol{L}}_{S^2} = \frac{v^4}{(M\mu)^2}\left\{\boldsymbol{l}_{\rm N}\left[(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)^2(X_2-\nu) + S_1^2\left(\frac{\nu}{2} - \frac{X_2}{2}\right)\right] + (\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)\boldsymbol{S}_1\left(\frac{X_2}{2} - \frac{\nu}{2}\right)\right\}$$
$$+ \frac{v^6}{(M\mu)^2}\left\{\boldsymbol{l}_{\rm N}\left\{(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)^2\left[\frac{121\nu^2}{72} + \frac{35\nu}{8} + \left(\frac{11\nu}{2} - \frac{35}{8}\right)X_2\right] + S_1^2\left[-\frac{\nu^2}{3} - \nu + \left(1 - \frac{5\nu}{3}\right)X_2\right]\right\}\right.$$
$$\left. + \boldsymbol{S}_1(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)\left[\frac{5\nu^2}{24} - \frac{11\nu}{8} + \left(\frac{11}{8} - \frac{\nu}{2}\right)X_2\right]\right\}$$
$$+ \frac{v^8}{(M\mu)^2}\left\{\boldsymbol{l}_{\rm N}(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)^2\left[-\frac{505\nu^3}{864} + \frac{347\nu^2}{96} + \frac{111\nu}{32} + \left(-\frac{2833\nu^2}{288} + \frac{199\nu}{16} - \frac{111}{32}\right)X_2\right]\right.$$
$$+ \boldsymbol{l}_{\rm N}S_1^2\left[\frac{5\nu^3}{144} + \frac{275\nu^2}{48} - \frac{15\nu}{16} + \left(\frac{295\nu^2}{144} - \frac{455\nu}{48} + \frac{15}{16}\right)X_2\right]$$
$$\left. + \boldsymbol{S}_1(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)\left[-\frac{235\nu^3}{288} + \frac{563\nu^2}{96} - \frac{21\nu}{32} + \left(-\frac{113\nu^2}{32} - \frac{7\nu}{3} + \frac{21}{32}\right)X_2\right]\right\} + 1 \leftrightarrow 2, \tag{65e}$$

$$\bar{\boldsymbol{L}}_{S^2\tilde{C}} = \frac{\tilde{C}_{1{\rm ES}^2}}{(M\mu)^2}\boldsymbol{l}_{\rm N}\left\{v^4\left[(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)^2\left(\frac{3X_2}{2} - \frac{3\nu}{2}\right) + S_1^2\left(\frac{\nu}{2} - \frac{X_2}{2}\right)\right]\right.$$
$$+ v^6\left[(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)^2(\nu^2 - 3\nu + (3\nu+3)X_2) + S_1^2\left(-\frac{\nu^2}{3} + \nu + (-\nu-1)X_2\right)\right]$$
$$+ v^8\left[(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)^2\left(-\frac{5\nu^3}{48} + \frac{1475\nu^2}{112} - \frac{135\nu}{16} + \left(-\frac{65\nu^2}{16} + \frac{55\nu}{16} + \frac{135}{16}\right)X_2\right)\right.$$
$$\left.\left. + S_1^2\left(\frac{5\nu^3}{144} - \frac{1475\nu^2}{336} + \frac{45\nu}{16} + \left(\frac{65\nu^2}{48} - \frac{55\nu}{48} - \frac{45}{16}\right)X_2\right)\right]\right\} + 1 \leftrightarrow 2. \tag{65f}$$

For aligned spins, $L(v)$ is gauge invariant, and our result agrees with the literature (e.g., with Refs. [202,219,229,231]). However, for precessing spins, $\boldsymbol{L}$ is gauge dependent, and our result disagrees with Refs. [114,158,202], even at LO SO, because these references used the covariant (Tulczyjew-Dixon) SSC [191,232], while we use the canonical Newton-Wigner (NW) SSC [233,234], since we are working in a Hamiltonian formalism [173,174]. Appendix B shows how to transform between our result and that of Refs. [114,158,202] at LO SO.

E. Spin-evolution equations

We obtain the spin-precession frequency $\boldsymbol{\Omega}_{S_i}$ by differentiating the Hamiltonian with respect to the spin vector. Then, we take the circular-orbit limit by setting $p_r = 0$ and replacing r and L by Eqs. (A1) and (A2). Finally, averaging the spin components over an orbit using Eq. (53) yields

$$\dot{\boldsymbol{S}}_1 = \boldsymbol{\Omega}_{S_1} \times \boldsymbol{S}_1, \tag{66a}$$

$$\begin{aligned}\boldsymbol{\Omega}_{S_1} =& \frac{\boldsymbol{l}_\mathrm{N}}{M}\left\{v^5\left(\frac{3X_2}{2}+\frac{\nu}{2}\right)+v^7\left[\left(\frac{9}{8}-\frac{5\nu}{4}\right)X_2-\frac{\nu^2}{24}+\frac{15\nu}{8}\right]+v^9\left[\left(\frac{5\nu^2}{16}-\frac{39\nu}{4}+\frac{27}{16}\right)X_2-\frac{\nu^3}{48}-\frac{55\nu^2}{16}+\frac{81\nu}{16}\right]\right\}\\
&+\frac{v^6}{M^2\mu}\left\{\boldsymbol{l}_\mathrm{N}\left[\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1\left(\frac{3\nu}{2}-\frac{3X_2}{2}\right)-\frac{3\nu}{2}\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_2\right]+\frac{\nu}{2}\boldsymbol{S}_2\right\}\\
&+\frac{v^8}{M^2\mu}\left\{\boldsymbol{l}_\mathrm{N}\left[\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1\left(-\frac{17\nu^2}{12}-\frac{9\nu}{4}+\left(\frac{9}{4}-\frac{15\nu}{4}\right)X_2\right)+\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_2\left(\frac{\nu^2}{12}-\frac{\nu}{2}\right)\right]-\frac{\nu^2}{4}\boldsymbol{S}_2\right\}\\
&+\frac{v^{10}}{M^2\mu}\left\{\boldsymbol{l}_\mathrm{N}\left[\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1\left(\frac{121\nu^3}{144}-\frac{91\nu^2}{16}-\frac{27\nu}{16}+\left(\frac{385\nu^2}{48}-\frac{97\nu}{16}+\frac{27}{16}\right)X_2\right)\right.\right.\\
&\left.\left.+\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_2\left(\frac{103\nu^3}{144}+\frac{139\nu^2}{48}-\frac{9\nu}{4}\right)\right]+\left(\frac{\nu^3}{48}+\frac{49\nu^2}{16}+\frac{3\nu}{8}\right)\boldsymbol{S}_2\right\},\\
&+\frac{\tilde{C}_{1\mathrm{ES}^2}}{M^2\mu}\boldsymbol{l}_\mathrm{N}(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)\left\{v^6\left(\frac{3\nu}{2}-\frac{3X_2}{2}\right)+v^8\left[-\frac{3\nu^2}{4}+\frac{9\nu}{4}+\left(-\frac{9\nu}{4}-\frac{9}{4}\right)X_2\right]\right.\\
&\left.+v^{10}\left[\frac{\nu^3}{16}-\frac{885\nu^2}{112}+\frac{81\nu}{16}+\left(\frac{39\nu^2}{16}-\frac{33\nu}{16}-\frac{81}{16}\right)X_2\right]\right\},\end{aligned}\tag{66b}$$

and similarly $\dot{\boldsymbol{S}}_2=\boldsymbol{\Omega}_{S_2}\times\boldsymbol{S}_2$, with $\boldsymbol{\Omega}_{S_2}$ given by Eq. (66b) after exchanging the two bodies' labels $1\leftrightarrow 2$. The SO and LO SS parts of the spin-precession frequency agree with the orbit-averaged results given by Eqs. (1)–(5) of Ref. [114], but the NLO and NNLO SS terms do not agree with Refs. [158,235] because of the different gauge.

F. Evolution of the orbital frequency

The evolution equation for the orbital frequency is given by Eq. (59) in terms of the energy loss and the derivative of the binding energy. The circular-orbit binding energy can be obtained from the Hamiltonian (minus the rest mass) by setting $p_r=0$; replacing r, L, and $\boldsymbol{l}$ by Eqs. (A1), (A2), and (64); and then taking the orbit average. This leads to

$$E(v)\equiv-\frac{\mu v^2}{2}(\bar{E}^{S^0}+\bar{E}_\mathrm{SO}+\bar{E}_{S_1S_2}+\bar{E}_{S^2}+\bar{E}_{S^2\tilde{C}}),\tag{67a}$$

$$\bar{E}_{S^0}=1+\left(-\frac{\nu}{12}-\frac{3}{4}\right)v^2+\left(-\frac{\nu^2}{24}+\frac{19\nu}{8}-\frac{27}{8}\right)v^4+\left[-\frac{35\nu^3}{5184}-\frac{155\nu^2}{96}+\left(\frac{34445}{576}-\frac{205\pi^2}{96}\right)\nu-\frac{675}{64}\right]v^6,\tag{67b}$$

$$\begin{aligned}\bar{E}_\mathrm{SO}=&\frac{\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1}{M\mu}\left\{v^3\left(\frac{2\nu}{3}+2X_2\right)+v^5\left[-\frac{\nu^2}{9}+5\nu+\left(3-\frac{10\nu}{3}\right)X_2\right]+v^7\left[-\frac{\nu^3}{12}-\frac{55\nu^2}{4}+\frac{81\nu}{4}+\left(\frac{5\nu^2}{4}-39\nu+\frac{27}{4}\right)X_2\right]\right\}\\
&+1\leftrightarrow 2,\end{aligned}\tag{67c}$$

$$\begin{aligned}\bar{E}_{S_1S_2}=&\frac{\nu}{(M\mu)^2}\left\{v^4[-3(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_2)+(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)]+v^6\left[(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_2)\left(\frac{5\nu}{18}-\frac{5}{3}\right)-\frac{5\nu}{6}(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\right]\right.\\
&\left.+v^8\left[(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_2)\left(\frac{721\nu^2}{216}+\frac{973\nu}{72}-\frac{21}{2}\right)+(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\left(\frac{7\nu^2}{72}+\frac{343\nu}{24}+\frac{7}{4}\right)\right]\right\},\end{aligned}\tag{67d}$$

$$\begin{aligned}\bar{E}_{S^2}=&\frac{v^4}{(M\mu)^2}\left[(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)^2\left(\frac{3\nu}{2}-\frac{3X_2}{2}\right)+S_1^2\left(\frac{X_2}{2}-\frac{\nu}{2}\right)\right]\\
&+\frac{v^6}{(M\mu)^2}\left\{(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)^2\left[-\frac{85\nu^2}{36}-\frac{15\nu}{4}+\left(\frac{15}{4}-\frac{25\nu}{4}\right)X_2\right]+S_1^2\left[\frac{5\nu^2}{12}+\frac{5\nu}{4}+\left(\frac{25\nu}{12}-\frac{5}{4}\right)X_2\right]\right\}\\
&+\frac{v^8}{(M\mu)^2}\left\{S_1^2\left[-\frac{7\nu^3}{144}-\frac{385\nu^2}{48}+\frac{21\nu}{16}+\left(-\frac{413\nu^2}{144}+\frac{637\nu}{48}-\frac{21}{16}\right)X_2\right]\right.\\
&\left.+(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)^2\left[\frac{847\nu^3}{432}-\frac{637\nu^2}{48}-\frac{63\nu}{16}+\left(\frac{2695\nu^2}{144}-\frac{679\nu}{48}+\frac{63}{16}\right)X_2\right]\right\}+1\leftrightarrow 2,\end{aligned}\tag{67e}$$

$$\bar{E}_{S^2\tilde{C}} = \frac{\tilde{C}_{1\mathrm{ES}^2}}{(M\mu)^2}\left\{v^4\left[(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)^2\left(\frac{3\nu}{2}-\frac{3X_2}{2}\right)+S_1^2\left(\frac{X_2}{2}-\frac{\nu}{2}\right)\right]\right.$$
$$+v^6\left[(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)^2\left(-\frac{5\nu^2}{4}+\frac{15\nu}{4}+\left(-\frac{15\nu}{4}-\frac{15}{4}\right)X_2\right)+S_1^2\left(\frac{5\nu^2}{12}-\frac{5\nu}{4}+\left(\frac{5\nu}{4}+\frac{5}{4}\right)X_2\right)\right]$$
$$+v^8\left[S_1^2\left(-\frac{7\nu^3}{144}+\frac{295\nu^2}{48}-\frac{63\nu}{16}+\left(-\frac{91\nu^2}{48}+\frac{77\nu}{48}+\frac{63}{16}\right)X_2\right)\right.$$
$$\left.\left.+(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)^2\left(\frac{7\nu^3}{48}-\frac{295\nu^2}{16}+\frac{189\nu}{16}+\left(\frac{91\nu^2}{16}-\frac{77\nu}{16}-\frac{189}{16}\right)X_2\right)\right]\right\}+1\leftrightarrow 2. \tag{67f}$$

Note that we did not include the 4PN nonspinning contribution in the binding energy to keep it at the same order as the energy flux, which is known to the 3.5PN order [236]. The nonspinning and SO parts agree with Eqs. (233) and (415) of Ref. [19], while the SS part agrees in the aligned-spin limit with, e.g., Refs. [219,231].

The NNLO SO contribution to the energy flux was derived in Ref. [162], while the NNLO SS (4PN beyond the LO) contribution was derived in Ref. [229], though the SS tail contribution at 3.5PN was obtained for aligned spins only. The result in Ref. [229] is expressed in terms of gauge-dependent quantities. Therefore, we use their EOMs to obtain the circular-orbit energy flux as a function of v, and orbit average the in-plane spin components, leading to

$$\dot{E} \equiv -\frac{32\nu^2 v^{10}}{5}(\dot{\bar{E}}_{S^0}+\dot{\bar{E}}_{\mathrm{SO}}+\dot{\bar{E}}_{S_1S_2}+\dot{\bar{E}}_{S^2}+\dot{\bar{E}}_{S^2\tilde{C}}), \tag{68a}$$

$$\dot{\bar{E}}_{S^0} = 1+v^2\left(-\frac{35\nu}{12}-\frac{1247}{336}\right)+4\pi v^3+v^4\left(\frac{65\nu^2}{18}+\frac{9271\nu}{504}-\frac{44711}{9072}\right)+\pi v^5\left(-\frac{583\nu}{24}-\frac{8191}{672}\right)$$
$$+v^6\left[-\frac{775\nu^3}{324}-\frac{94403\nu^2}{3024}-\frac{134543\nu}{7776}+\pi^2\left(\frac{41\nu}{48}+\frac{16}{3}\right)-\frac{1712\ln v}{105}-\frac{1712\gamma_E}{105}+\frac{6643739519}{69854400}-\frac{3424\ln 2}{105}\right]$$
$$+\pi v^7\left(\frac{193385\nu^2}{3024}+\frac{214745\nu}{1728}-\frac{16285}{504}\right), \tag{68b}$$

$$\dot{\bar{E}}_{\mathrm{SO}} = \frac{\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1}{M\mu}\left\{v^3\left(-\frac{3\nu}{2}-\frac{5X_2}{4}\right)+v^5\left(\frac{157\nu^2}{18}-\frac{23\nu}{8}+\left(\frac{43\nu}{4}-\frac{13}{16}\right)X_2\right)+\pi v^6\left(-\frac{17\nu}{3}-\frac{31X_2}{6}\right)\right.$$
$$+v^7\left[-\frac{1117\nu^3}{54}+\frac{625\nu^2}{189}+\frac{180955\nu}{13608}+\left(-\frac{1501\nu^2}{36}+\frac{1849\nu}{126}+\frac{9535}{336}\right)X_2\right]$$
$$\left.+\pi v^8\left[\frac{21241\nu^2}{336}-\frac{10069\nu}{672}+\left(\frac{130583\nu}{2016}-\frac{7163}{672}\right)X_2\right]\right\}+1\leftrightarrow 2, \tag{68c}$$

$$\dot{\bar{E}}_{S_1S_2} = \frac{\nu}{(M\mu)^2}\left\{v^4\left[\frac{289}{48}(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_2)-\frac{103}{48}(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\right]\right.$$
$$+v^6\left[(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_2)\left(-\frac{2023\nu}{72}-\frac{5647}{168}\right)+(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\left(\frac{821\nu}{72}+\frac{2123}{84}\right)\right]$$
$$+v^8\left[(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_2)\left(\frac{2161\nu^2}{48}+\frac{60241\nu}{252}+\frac{107771}{1512}\right)+(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\left(-\frac{4405\nu^2}{144}-\frac{194687\nu}{1008}-\frac{895429}{9072}\right)\right]$$
$$\left.+\frac{63\pi}{4}v^7(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_2)\right\}, \tag{68d}$$

$$\begin{aligned}\dot{\bar{E}}_{S^2} =& \frac{v^4}{(M\mu)^2}\left[(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)^2\left(\frac{287X_2}{96}-\frac{287\nu}{96}\right)+S_1^2\left(\frac{89\nu}{96}-\frac{89X_2}{96}\right)\right]\\ &+\frac{v^6}{(M\mu)^2}\left\{(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)^2\left[\frac{2621\nu^2}{144}+\frac{1255\nu}{56}-\left(\frac{461\nu}{72}+\frac{1255}{56}\right)X_2\right]+S_1^2\left[\left(\frac{185\nu}{72}+\frac{801}{56}\right)X_2-\frac{727\nu^2}{144}-\frac{801\nu}{56}\right]\right\}\\ &+\frac{v^8}{(M\mu)^2}\left\{(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)^2\left[-\frac{5615\nu^3}{96}-\frac{62031\nu^2}{448}-\frac{250813\nu}{6048}+\left(-\frac{11903\nu^2}{288}+\frac{202963\nu}{1344}+\frac{250813}{6048}\right)X_2\right]\right.\\ &\left.+S_1^2\left[\frac{3371\nu^3}{288}+\frac{406253\nu^2}{4032}+\frac{963901\nu}{18144}+\left(\frac{439\nu^2}{96}-\frac{389723\nu}{4032}-\frac{963901}{18144}\right)X_2\right]\right\}\\ &+\frac{\pi v^7}{(M\mu)^2}(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)^2\left(\frac{65X_2}{8}-\frac{65\nu}{8}\right)+1\leftrightarrow 2, \end{aligned}\tag{68e}$$

$$\begin{aligned}\dot{\bar{E}}_{S^2\tilde{C}} =& \frac{\tilde{C}_{1{\rm ES}^2}}{(M\mu)^2}\left\{v^4[(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)^2(3X_2-3\nu)+S_1^2(\nu-X_2)]\right.\\ &+v^6\left[(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)^2\left(\frac{129\nu^2}{8}+\frac{837\nu}{112}+\left(-\frac{135\nu}{16}-\frac{837}{112}\right)X_2\right)+S_1^2\left(-\frac{43\nu^2}{8}-\frac{279\nu}{112}+\left(\frac{45\nu}{16}+\frac{279}{112}\right)X_2\right)\right]\\ &+v^8\left[(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)^2\left(-\frac{81\nu^3}{2}-\frac{41191\nu^2}{672}+\frac{74911\nu}{3024}+\left(-\frac{209\nu^2}{48}+\frac{46801\nu}{672}-\frac{74911}{3024}\right)X_2\right)\right.\\ &\left.+S_1^2\left(\frac{27\nu^3}{2}+\frac{41191\nu^2}{2016}-\frac{74911\nu}{9072}+\left(\frac{209\nu^2}{144}-\frac{46801\nu}{2016}+\frac{74911}{9072}\right)X_2\right)\right]\\ &\left.+\pi v^7(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)^2(8X_2-8\nu)\right\}+1\leftrightarrow 2, \end{aligned}\tag{68f}$$

where the SS tail part [$\mathcal{O}(v^7)$ beyond the LO] is only known for aligned spins, so we expressed it in terms of $\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_{\rm i}$ as an approximation for the precessing case, which would also depend on S_1^2 and $\boldsymbol{S}_1\cdot\boldsymbol{S}_2$.

Inserting E and $\dot{E}$ in Eq. (59) and PN expanding yield

$$\dot{v} \equiv \frac{32\nu v^9}{5M}(\dot{\bar{v}}_{S^0}+\dot{\bar{v}}_{\rm SO}+\dot{\bar{v}}_{S_1S_2}+\dot{\bar{v}}_{S^2}+\dot{\bar{v}}_{S^2\tilde{C}}),\tag{69a}$$

$$\begin{aligned}\dot{\bar{v}}_{S^0} =& 1+\left(-\frac{11\nu}{4}-\frac{743}{336}\right)v^2+4\pi v^3+\left(\frac{59\nu^2}{18}+\frac{13661\nu}{2016}+\frac{34103}{18144}\right)v^4+\pi\left(-\frac{189\nu}{8}-\frac{4159}{672}\right)v^5\\ &+v^6\left[\frac{541\nu^2}{896}-\frac{5605\nu^3}{2592}-\frac{56198689\nu}{217728}+\pi^2\left(\frac{451\nu}{48}+\frac{16}{3}\right)-\frac{1712\ln v}{105}-\frac{1712\gamma_E}{105}+\frac{16447322263}{139708800}-\frac{3424\ln 2}{105}\right]\\ &+\pi\left(\frac{91495\nu^2}{1512}+\frac{358675\nu}{6048}-\frac{4415}{4032}\right)v^7, \end{aligned}\tag{69b}$$

$$\begin{aligned}\dot{\bar{v}}_{\rm SO} =& \frac{\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1}{M\mu}\left\{v^3\left(-\frac{19\nu}{6}-\frac{25X_2}{4}\right)+v^5\left[\frac{79\nu^2}{6}-\frac{21611\nu}{1008}+\left(\frac{281\nu}{8}-\frac{809}{84}\right)X_2\right]+\pi v^6\left(-\frac{37\nu}{3}-\frac{151X_2}{6}\right)\right.\\ &+v^7\left[-\frac{10819\nu^3}{432}+\frac{40289\nu^2}{288}-\frac{1932041\nu}{18144}+\left(-\frac{2903\nu^2}{32}+\frac{257023\nu}{1008}-\frac{1195759}{18144}\right)X_2\right]\\ &\left.+\pi v^8\left[\frac{34303\nu^2}{336}-\frac{46957\nu}{504}+\left(\frac{50483\nu}{224}-\frac{1665}{28}\right)X_2\right]\right\}+1\leftrightarrow 2, \end{aligned}\tag{69c}$$

$$\dot{\bar{v}}_{S_1S_2} = \frac{\nu}{(M\mu)^2}\left\{ v^4\left[\frac{721(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_2)}{48} - \frac{247(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)}{48}\right] + v^6\left[(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_2)\left(\frac{14433}{224} - \frac{11779\nu}{288}\right) + \left(\frac{6373\nu}{288} + \frac{16255}{672}\right)(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\right] + \pi v^7\left[\frac{207(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_2)}{4} - 12(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\right] + v^8\left[\left(-\frac{162541\nu^2}{3456} - \frac{195697\nu}{896} - \frac{9355721}{72576}\right)(\boldsymbol{S}_1\cdot\boldsymbol{S}_2) + (\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_2)\left(\frac{33163\nu^2}{3456} - \frac{10150387\nu}{24192} + \frac{21001565}{24192}\right)\right]\right\}, \tag{69d}$$

$$\dot{\bar{v}}_{S^2} = \frac{v^4}{(M\mu)^2}\left[(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)^2\left(\frac{719X_2}{96} - \frac{719\nu}{96}\right) + S_1^2\left(\frac{233\nu}{96} - \frac{233X_2}{96}\right)\right] + \frac{v^6}{(M\mu)^2}\left\{(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)^2\left[\frac{25373\nu^2}{576} + \frac{2185\nu}{448} + \left(\frac{19423\nu}{576} - \frac{2185}{448}\right)X_2\right] + S_1^2\left[-\frac{6011\nu^2}{576} - \frac{8503\nu}{448} + \left(\frac{8503}{448} - \frac{1177\nu}{576}\right)X_2\right]\right\} + \frac{\pi v^7}{(M\mu)^2}\left[(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)^2\left(\frac{209X_2}{8} - \frac{209\nu}{8}\right) + S_1^2(6\nu - 6X_2)\right] + \frac{v^8}{(M\mu)^2}\left\{(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)^2\left[\left(\frac{11888267}{48384} - \frac{2392243\nu^2}{6912} + \frac{4063301\nu}{16128}\right)X_2 - \frac{869429\nu^3}{6912} + \frac{14283281\nu^2}{48384} - \frac{11888267\nu}{48384}\right] + S_1^2\left[\frac{138323\nu^3}{6912} + \frac{711521\nu^2}{5376} + \frac{8207303\nu}{145152} + \left(\frac{250693\nu^2}{6912} - \frac{812353\nu}{5376} - \frac{8207303}{145152}\right)X_2\right]\right\} + 1 \leftrightarrow 2, \tag{69e}$$

$$\dot{\bar{v}}_{S^2\tilde{C}} = \frac{\tilde{C}_{1\rm ES^2}}{(M\mu)^2}\left\{ v^4\left[(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)^2\left(\frac{15X_2}{2} - \frac{15\nu}{2}\right) + S_1^2\left(\frac{5\nu}{2} - \frac{5X_2}{2}\right)\right] + v^6\left[(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)^2\left(\frac{129\nu^2}{4} - \frac{1977\nu}{224} + \left(\frac{1977}{224} - \frac{73\nu}{16}\right)X_2\right) + S_1^2\left(-\frac{43\nu^2}{4} + \frac{659\nu}{224} + \left(\frac{73\nu}{48} - \frac{659}{224}\right)X_2\right)\right] + v^8\left[(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)^2\left(-\frac{1567\nu^3}{24} + \frac{29329\nu^2}{224} - \frac{597271\nu}{6048} + \left(-\frac{5675\nu^2}{96} - \frac{1517\nu}{168} + \frac{597271}{6048}\right)X_2\right) + S_1^2\left(\frac{1567\nu^3}{72} - \frac{29329\nu^2}{672} + \frac{597271\nu}{18144} + \left(\frac{5675\nu^2}{288} + \frac{1517\nu}{504} - \frac{597271}{18144}\right)X_2\right)\right] + \pi v^7[(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)^2(26X_2 - 26\nu) + S_1^2(6\nu - 6X_2)]\right\} + 1 \leftrightarrow 2. \tag{69f}$$

The SO and LO SS parts of $\dot{v}$ agree with, e.g., Eq. (A1) of Ref. [39].

G. Evolution of the angular momentum vector

To obtain the PN expansion for $\dot{\boldsymbol{l}}_{\rm N}$, we start from the equation for the total angular momentum $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}_1 + \boldsymbol{S}_2$. We first neglect RR, and in the following subsection compute the RR contribution. Setting $\dot{\boldsymbol{J}} = 0$ yields

$$\dot{\boldsymbol{L}} + \dot{\boldsymbol{S}}_1 + \dot{\boldsymbol{S}}_2 = 0, \tag{70}$$

where $\dot{\boldsymbol{S}}_{\rm i}$ is given by Eq. (66), while $\dot{\boldsymbol{L}}$ can be computed by taking the time derivative of Eq. (65).

Solving Eq. (70) for $\dot{\boldsymbol{l}}_{\rm N}$ yields[3]

$$\dot{\boldsymbol{l}}_{\rm N} \equiv \dot{\boldsymbol{l}}_{\rm N}^{\rm SO} + \dot{\boldsymbol{l}}_{\rm N}^{S_1S_2} + \dot{\boldsymbol{l}}_{\rm N}^{S^2} + \dot{\boldsymbol{l}}_{\rm N}^{S^2\tilde{C}}, \tag{71a}$$

[3]To solve Eq. (70), we split $\dot{\boldsymbol{l}}_{\rm N}$ and $\dot{\boldsymbol{S}}_{\rm i}$ into SO and SS contributions, such that $\dot{\boldsymbol{l}}_{\rm N} \equiv \dot{\boldsymbol{l}}_{\rm N}^{\rm SO} + \dot{\boldsymbol{l}}_{\rm N}^{\rm SS}$ and $\dot{\boldsymbol{S}}_{\rm i} \equiv \dot{\boldsymbol{S}}_{\rm i}^{\rm SO} + \dot{\boldsymbol{S}}_{\rm i}^{\rm SS}$, and then solve order by order in spin for $\dot{\boldsymbol{l}}_{\rm N}^{\rm SO}$ and $\dot{\boldsymbol{l}}_{\rm N}^{\rm SS}$. When performing this calculation, several simplifications can be done: $\dot{\boldsymbol{S}}_{\rm i}$ is perpendicular to $\boldsymbol{S}_{\rm i}$, leading to $\boldsymbol{S}_1\cdot\dot{\boldsymbol{S}}_1 = 0 = \boldsymbol{S}_2\cdot\dot{\boldsymbol{S}}_2$, and since $\dot{\boldsymbol{S}}_{\rm i}^{\rm SO}$ is perpendicular to $\boldsymbol{l}_{\rm N}$, we get $\boldsymbol{l}_{\rm N}\cdot\dot{\boldsymbol{S}}_{\rm i}^{\rm SO} = 0$.

$$\dot{\boldsymbol{l}}_{\rm N}^{\rm SO} = \frac{\boldsymbol{l}_{\rm N}\times\boldsymbol{S}_1}{M^2\mu}\bigg\{v^6\left(-\frac{\nu}{2}-\frac{3X_2}{2}\right)+v^8\left[\frac{\nu^2}{4}-\frac{9\nu}{4}+\left(\frac{9\nu}{4}+\frac{9}{4}\right)X_2\right] + v^{10}\left[-\frac{\nu^3}{48}+\frac{81\nu^2}{16}-\frac{27\nu}{16}+\left(-\frac{21\nu^2}{16}+\frac{63\nu}{16}+\frac{27}{16}\right)X_2\right]\bigg\}+1\leftrightarrow 2, \tag{71b}$$

$$\begin{aligned}\dot{\boldsymbol{l}}_{\rm N}^{S_1S_2} = \frac{\nu}{M^3\mu^2}\bigg\{&\frac{3}{2}v^7[(\boldsymbol{l}_{\rm N}\times\boldsymbol{S}_1)(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_2)+(\boldsymbol{l}_{\rm N}\times\boldsymbol{S}_2)(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)]\\
&+v^9\bigg[(\boldsymbol{l}_{\rm N}\times\boldsymbol{S}_1)(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_2)\left(-\frac{5\nu}{4}-\frac{15X_2}{8}-\frac{21}{4}\right)+(\boldsymbol{l}_{\rm N}\times\boldsymbol{S}_2)(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)\left(-\frac{5\nu}{4}+\frac{15X_2}{8}-\frac{57}{8}\right)\\
&-\frac{5}{8}\delta\boldsymbol{l}_{\rm N}(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1\times\boldsymbol{S}_2)+\frac{3}{8}\delta(\boldsymbol{S}_1\times\boldsymbol{S}_2)\bigg]\\
&+v^{11}\bigg\{(\boldsymbol{l}_{\rm N}\times\boldsymbol{S}_1)(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_2)\left[-\frac{\nu^2}{6}+\frac{25\nu}{4}+\left(\frac{71\nu}{32}+\frac{9}{32}\right)X_2+\frac{15}{16}\right]\\
&+(\boldsymbol{l}_{\rm N}\times\boldsymbol{S}_2)(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)\left[-\frac{\nu^2}{6}+\frac{271\nu}{32}+\left(-\frac{71\nu}{32}-\frac{9}{32}\right)X_2+\frac{39}{32}\right]\\
&+\frac{\delta}{96}(89\nu-27)\boldsymbol{l}_{\rm N}(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1\times\boldsymbol{S}_2)-\frac{9}{32}\delta(2\nu+1)(\boldsymbol{S}_1\times\boldsymbol{S}_2)\bigg\}\bigg\},\end{aligned} \tag{71c}$$

$$\dot{\boldsymbol{l}}_{\rm N}^{S^2} = \frac{(\boldsymbol{l}_{\rm N}\times\boldsymbol{S}_1)(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)}{M^3\mu^2}\bigg\{v^7\left(\frac{3X_2}{2}-\frac{3\nu}{2}\right)+v^9[2\nu^2+9\nu+(3\nu-9)X_2] + v^{11}\left[-\frac{23\nu^3}{16}-\frac{157\nu^2}{16}-\frac{93\nu}{16}+\left(-\frac{439\nu^2}{48}+4\nu+\frac{93}{16}\right)X_2\right]\bigg\}+1\leftrightarrow 2, \tag{71d}$$

$$\dot{\boldsymbol{l}}_{\rm N}^{S^2\tilde{C}} = \frac{\tilde{C}_{1{\rm ES}^2}}{M^3\mu^2}(\boldsymbol{l}_{\rm N}\times\boldsymbol{S}_1)(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)\bigg\{v^7\left(\frac{3X_2}{2}-\frac{3\nu}{2}\right)+v^9\left[\frac{11\nu^2}{8}+\frac{9\nu}{8}+\left(\frac{11\nu}{4}-\frac{9}{8}\right)X_2\right] + v^{11}\left[-\frac{43\nu^3}{96}+\frac{1077\nu^2}{224}+\frac{27\nu}{32}+\left(-\frac{479\nu^2}{96}+\frac{3\nu}{2}-\frac{27}{32}\right)X_2\right]\bigg\}+1\leftrightarrow 2, \tag{71e}$$

which agrees up to NLO SO [i.e., to $\mathcal{O}(v^8)$] with Eq. (4c) of Ref. [114] provided that one uses the coefficients of $\boldsymbol{L}(\boldsymbol{l}_{\rm N})$ from Eq. (65), instead of those in Ref. [114] because of the different SSC. Note that $\dot{\boldsymbol{l}}_{\rm N}$ has a component parallel to $\boldsymbol{l}_{\rm N}$, which enters at NLO and NNLO S_1S_2, and is given by

$$\dot{\boldsymbol{l}}_{\rm N}\cdot\boldsymbol{l}_{\rm N} = \frac{\nu\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1\times\boldsymbol{S}_2}{M^3\mu^2}\left[-\frac{\delta}{4}v^9+\frac{\delta}{96}(35\nu-54)v^{11}\right]. \tag{72}$$

H. Radiation-reaction contribution to $\dot{\boldsymbol{l}}_{\rm N}$

When computing $\dot{\boldsymbol{l}}_{\rm N}$, RR enters through $\dot{v}$, which is given by Eq. (69), and from the nonzero $\dot{\boldsymbol{J}}$, which is given by

$$\begin{aligned}\dot{\boldsymbol{J}} &= \dot{\boldsymbol{L}}+\dot{\boldsymbol{S}}_1+\dot{\boldsymbol{S}}_2\\
&= \dot{\boldsymbol{r}}\times\boldsymbol{p}+\boldsymbol{r}\times\dot{\boldsymbol{p}}+\dot{\boldsymbol{S}}_1+\dot{\boldsymbol{S}}_2\\
&= \boldsymbol{r}\times\boldsymbol{\mathcal{F}}+\dot{\boldsymbol{S}}_1^{\rm RR}+\dot{\boldsymbol{S}}_2^{\rm RR},\end{aligned} \tag{73}$$

where we used the EOMs (46) to relate $\dot{\boldsymbol{J}}$ to the RR force $\boldsymbol{\mathcal{F}}$. Since we are working to NNLO SS [i.e. to $\mathcal{O}(v^{10})$] in $\dot{\boldsymbol{L}}$ and $\dot{\boldsymbol{S}}_{\rm i}$, we only need $\dot{\boldsymbol{J}}$ to $\mathcal{O}(v^3)$ beyond its LO, which is $\mathcal{O}(v^7)$, and we can neglect the RR contribution to $\dot{\boldsymbol{S}}_{\rm i}$ because it starts at $\mathcal{O}(v^{11}S^2)$ [224,225].

The RR force $\boldsymbol{\mathcal{F}}$ for circular orbits in the SEOBNR waveform models is chosen to be in a gauge such that [86,111][4]

$$\boldsymbol{\mathcal{F}} = \frac{\dot{E}}{\Omega L}\boldsymbol{p}. \tag{74}$$

[4]This relation was derived in Ref. [86] for precessing spins, and is given at LO SO by Eq. (3.27) there, which includes an extra term depending on $(\boldsymbol{p}\cdot\boldsymbol{S}_{\rm i})\boldsymbol{L}$ that averages to zero over an orbit.

Using the energy loss from Eq. (68) and expanding to LO SO for circular orbits, we get

$$\mathcal{F} \simeq -\frac{32}{5}\nu^2 v^9 \boldsymbol{\lambda}\left\{1 + v^2\left(-\frac{13\nu}{4} - \frac{1247}{336}\right) + 4\pi v^3 + \frac{v^3}{M\mu}\left[\boldsymbol{l}\cdot\boldsymbol{S}_1\left(-\frac{4\nu}{3} - \frac{3X_2}{4}\right) + 1 \leftrightarrow 2\right]\right\}, \tag{75}$$

where we did not write the $\boldsymbol{n}$ component of $\mathcal{F}$ since it does not contribute to $\dot{\boldsymbol{J}}$ and is proportional to p_r. Then, from Eq. (73), and using Eq. (64) to replace $\boldsymbol{l} = \boldsymbol{n}\times\boldsymbol{\lambda}$ by $\boldsymbol{l}_{\rm N}$, we obtain

$$\dot{\boldsymbol{J}} = -\frac{32}{5}M\nu^2 v^7\left\{\boldsymbol{l}_{\rm N}\left[1 + v^2\left(-\frac{35\nu}{12} - \frac{1247}{336}\right) + 4\pi v^3\right] + \frac{v^3}{M\mu}\left[\boldsymbol{l}_{\rm N}(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)\left(-\frac{5}{4}\nu - \frac{1}{2}X_2\right) + \boldsymbol{S}_1\left(-\frac{\nu}{4} - \frac{3X_2}{4}\right) + 1 \leftrightarrow 2\right]\right\}. \tag{76}$$

Following similar steps as in the previous subsection, except for including $\dot{\boldsymbol{J}}$ and $\dot{v}$, we obtain the following RR contribution to $\dot{\boldsymbol{l}}_{\rm N}$:

$$\dot{\boldsymbol{l}}_{\rm N}^{\rm RR} = -\frac{64}{5}\frac{v^8}{M}\left\{\nu\boldsymbol{l}_{\rm N}\left[1 + v^2\left(-\frac{37\nu}{12} - \frac{1751}{336}\right) + 4\pi v^3\right] + \frac{v^3}{M\mu}\left[\boldsymbol{l}_{\rm N}(\boldsymbol{l}_{\rm N}\cdot\boldsymbol{S}_1)\left(\frac{9\nu^2}{8} - \frac{19\nu}{12} + 5X_2\nu - \frac{25}{8}X_2\right) + \boldsymbol{S}_1\left(\frac{\nu^2}{8} + \frac{3\nu X_2}{8}\right) + 1 \leftrightarrow 2\right]\right\}. \tag{77}$$

We do not include this RR contribution in the `SEOBNRv5PHM` waveform model [154], but we checked that it has a negligible effect on the dynamics.

IV. CONCLUSIONS

In this paper, we derived an aligned-spin Hamiltonian (Sec. II C), which is used in the `SEOBNRv5HM` waveform model [153], and a full precessing-spin Hamiltonian (Sec. II D) that reduces in the test-mass limit to the exact Kerr Hamiltonian for generic orbits. The Hamiltonians include the nonspinning part at 4PN order, with partial 5PN and 5.5PN results, in addition to the full 4PN spin information (NNLO SO, NNLO SS, LO S^3, LO S^4). The full 5PN spin contributions (NNNLO SO and SS, NLO S^3 and S^4) to the conservative dynamics are known from the recent work in Refs. [171,172,221,222,237–242], but we leave their inclusion in the Hamiltonian for future work. Our results include the spin-multipole constants, and are thus valid for NSs, though one also needs to include dynamical tidal effects, which can be included as was done in `SEOBNRv4T` [124].

Furthermore, we derived (in Sec. III) a simpler precessing-spin Hamiltonian, $H_{\rm EOB}^{\rm pprec}$, and PN-expanded EOMs, which orbit average the in-plane spin components, and are used in the computationally efficient `SEOBNRv5PHM` waveform model [154]. We included in the EOMs the NNLO SO and SS contributions in a gauge consistent with our EOB Hamiltonian and the NW (canonical) SSC. Extending the precessing-spin EOMs to include LO S^3 and LO S^4 is straightforward, but the equations become lengthy, and would likely have a smaller effect than the error introduced due to orbit averaging the SS contributions. It would still be interesting to compute those higher-order spin contributions and quantify their effect on the dynamics. It is also important to extend the RR force and waveform modes for precessing spins beyond the LO SO and SS contributions derived in Refs. [243,244].

The results obtained in this paper have contributed to improving the accuracy of `SEOBNRv5` waveform models, as detailed in Refs. [153,154]. For example, Ref. [154] demonstrated that using the partially precessing Hamiltonian $H_{\rm EOB}^{\rm pprec}$, and comparing the waveforms to a set of highly precessing NR simulations, led to 100% (86.4%) of cases with a maximum unfaithfulness below 3% (1%), while using the aligned-spin Hamiltonian $H_{\rm EOB}^{\rm align}$ led to 95.8% (75.4%) of cases below 3% (1%). Furthermore, we generally find that `SEOBNRv5` waveform models provide noticeable improvements in accuracy compared with the previous version of the model, `SEOBNRv4`, and to other `IMRPhenom` and `TEOBResumS` models (see for example Fig. 9 of Ref. [153] and Fig. 4 of Ref. [154]). These results highlight the importance of including and resumming analytical PN information in waveform models.

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APPENDIX A: ANGULAR MOMENTUM AND SEPARATION FOR CIRCULAR ORBITS

In this Appendix, we write r and L for circular orbits and precessing spins, which are obtained by solving Eq. (61) and replacing $\boldsymbol{l}$ by $\boldsymbol{l}(\boldsymbol{l}_\mathrm{N})$ from Eq. (64).

For $L(\boldsymbol{l}_\mathrm{N}, \boldsymbol{n}, \boldsymbol{\lambda}_\mathrm{N}, \boldsymbol{S}_\mathrm{i}, v)$, we get

$$L \equiv \frac{M\mu}{v}(\bar{L}_{S^0} + \bar{L}_{\mathrm{SO}} + \bar{L}_{S_1S_2} + \bar{L}_{S^2} + \bar{L}_{S^2\tilde{C}}), \tag{A1a}$$

$$\begin{aligned}\bar{L}_{S^0} &= 1 + v^2\left(\frac{\nu}{6}+\frac{3}{2}\right) + v^4\left(\frac{\nu^2}{24}-\frac{19\nu}{8}+\frac{27}{8}\right) + v^6\left[\frac{7\nu^3}{1296}+\frac{31\nu^2}{24}+\left(\frac{41\pi^2}{24}-\frac{6889}{144}\right)\nu+\frac{135}{16}\right]\\ &\quad + v^8\left[\frac{2835}{128}+\nu\left(\frac{98869}{5760}-\frac{128\gamma_E}{3}-\frac{6455\pi^2}{1536}-\frac{256\ln 2}{3}-\frac{128\ln v}{3}\right)+\left(\frac{356035}{3456}-\frac{2255\pi^2}{576}\right)\nu^2\right.\\ &\quad \left.-\frac{215\nu^3}{1728}-\frac{55\nu^4}{31104}\right],\end{aligned} \tag{A1b}$$

$$\begin{aligned}\bar{L}_{\mathrm{SO}} &= \frac{\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1}{M\mu}\left\{v^3\left(-\frac{5\nu}{6}-\frac{5X_2}{2}\right)+v^5\left[\frac{7\nu^2}{72}-\frac{35\nu}{8}+\left(\frac{35\nu}{12}-\frac{21}{8}\right)X_2\right]\right.\\ &\quad \left.+v^7\left[\frac{\nu^3}{16}+\frac{165\nu^2}{16}-\frac{243\nu}{16}+\left(-\frac{15\nu^2}{16}+\frac{117\nu}{4}-\frac{81}{16}\right)X_2\right]\right\}+1\leftrightarrow 2,\end{aligned} \tag{A1c}$$

$$\begin{aligned}\bar{L}_{S_1S_2} &= \frac{\nu}{M^2\mu^2}\left\{2v^4[(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_2)-(\boldsymbol{n}\cdot\boldsymbol{S}_1)(\boldsymbol{n}\cdot\boldsymbol{S}_2)]\right.\\ &\quad + v^6\left[(\boldsymbol{n}\cdot\boldsymbol{S}_1)(\boldsymbol{n}\cdot\boldsymbol{S}_2)\left(\frac{16}{3}-\frac{2\nu}{3}\right)+(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_2)\left(\frac{4\nu}{9}+\frac{4}{3}\right)+(\boldsymbol{\lambda}_\mathrm{N}\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_\mathrm{N}\cdot\boldsymbol{S}_2)\left(\frac{9\nu}{4}-\frac{7}{3}\right)\right]\\ &\quad + v^8\left[(\boldsymbol{n}\cdot\boldsymbol{S}_1)(\boldsymbol{n}\cdot\boldsymbol{S}_2)\left(-\frac{205\nu^2}{72}-\frac{5315\nu}{144}+\frac{15}{4}\right)+(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_2)\left(-\frac{265\nu^2}{108}-\frac{715\nu}{36}+\frac{25}{4}\right)\right.\\ &\quad \left.\left.+(\boldsymbol{\lambda}_\mathrm{N}\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_\mathrm{N}\cdot\boldsymbol{S}_2)\left(\frac{21\nu^2}{8}+\frac{235\nu}{18}-4\right)\right]\right\},\end{aligned} \tag{A1d}$$

$$\begin{aligned}\bar{L}_{S^2} &= \frac{v^4}{M^2\mu^2}(\nu-X_2)[(\boldsymbol{n}\cdot\boldsymbol{S}_1)^2-(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)^2]+\frac{v^6}{M^2\mu^2}\left\{(\boldsymbol{n}\cdot\boldsymbol{S}_1)^2\left[-\frac{5\nu^2}{3}-\frac{23\nu}{3}+\left(\frac{23}{3}-8\nu\right)X_2\right]\right.\\ &\quad \left.+(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)^2\left[\frac{14\nu^2}{9}+2\nu+\left(\frac{10\nu}{3}-2\right)X_2\right]+(\boldsymbol{\lambda}_\mathrm{N}\cdot\boldsymbol{S}_1)^2\left[\frac{9\nu^2}{8}+\frac{109\nu}{24}+\left(\frac{65\nu}{12}-\frac{109}{24}\right)X_2\right]\right\}\\ &\quad +\frac{v^8}{M^2\mu^2}\left\{(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)^2\left[-\frac{295\nu^3}{216}+\frac{365\nu^2}{24}+\frac{15\nu}{8}+\left(-\frac{815\nu^2}{72}+\frac{5\nu}{8}-\frac{15}{8}\right)X_2\right]\right.\\ &\quad +(\boldsymbol{n}\cdot\boldsymbol{S}_1)^2\left[-\frac{185\nu^3}{144}+\frac{2275\nu^2}{144}-\frac{95\nu}{16}+\left(\frac{175\nu^2}{48}-\frac{985\nu}{36}+\frac{95}{16}\right)X_2\right]\\ &\quad \left.+(\boldsymbol{\lambda}_\mathrm{N}\cdot\boldsymbol{S}_1)^2\left[\frac{21\nu^3}{16}-\frac{55\nu^2}{36}+\frac{65\nu}{16}+\left(-\frac{13\nu^2}{36}+\frac{1237\nu}{144}-\frac{65}{16}\right)X_2\right]\right\}+1\leftrightarrow 2,\end{aligned} \tag{A1e}$$

$$\bar{L}_{S^2\tilde{C}} = \frac{\tilde{C}_{1\mathrm{ES}^2}}{M^2\mu^2}\bigg\{ v^4[(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)^2(X_2-\nu) + (\boldsymbol{n}\cdot\boldsymbol{S}_1)^2(2\nu-2X_2) + (\boldsymbol{\lambda}_\mathrm{N}\cdot\boldsymbol{S}_1)^2(X_2-\nu)]$$
$$+ v^6\bigg[(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)^2\left(\frac{2\nu^2}{3}-2\nu+(2\nu+2)X_2\right) + (\boldsymbol{n}\cdot\boldsymbol{S}_1)^2\left(-\frac{4\nu^2}{3}+4\nu+(-4\nu-4)X_2\right)$$
$$+ (\boldsymbol{\lambda}_\mathrm{N}\cdot\boldsymbol{S}_1)^2\left(\frac{2\nu^2}{3}-2\nu+(2\nu+2)X_2\right)\bigg]$$
$$+ v^8\bigg[(\boldsymbol{n}\cdot\boldsymbol{S}_1)^2\left(\frac{5\nu^3}{36}-\frac{1475\nu^2}{84}+\frac{45\nu}{4}+\left(\frac{65\nu^2}{12}-\frac{55\nu}{12}-\frac{45}{4}\right)X_2\right)$$
$$+ (\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)^2\left(-\frac{5\nu^3}{72}+\frac{1475\nu^2}{168}-\frac{45\nu}{8}+\left(-\frac{65\nu^2}{24}+\frac{55\nu}{24}+\frac{45}{8}\right)X_2\right)$$
$$+ (\boldsymbol{\lambda}_\mathrm{N}\cdot\boldsymbol{S}_1)^2\left(-\frac{5\nu^3}{72}+\frac{1475\nu^2}{168}-\frac{45\nu}{8}+\left(-\frac{65\nu^2}{24}+\frac{55\nu}{24}+\frac{45}{8}\right)X_2\right)\bigg]\bigg\} + 1\leftrightarrow 2, \tag{A1f}$$

which agrees for aligned spins with, e.g., Eq. (8.24) of Ref. [231] and Eq. (5.2) of Ref. [219].

For $r(\boldsymbol{l}_\mathrm{N},\boldsymbol{n},\boldsymbol{\lambda}_\mathrm{N},\boldsymbol{S}_\mathrm{i},v)$, we obtain

$$r \equiv \frac{M}{v^2}(\bar{r}_{S^0}+\bar{r}_\mathrm{SO}+\bar{r}_{S_1S_2}+\bar{r}_{S^2}+\bar{r}_{S^2\tilde{C}}), \tag{A2a}$$

$$\bar{r}_{S^0} = 1 + v^2\frac{\nu}{3} + v^4\left(\frac{\nu^2}{9}-\frac{5\nu}{4}\right) + v^6\left[\frac{2\nu^3}{81}+\frac{11\nu^2}{12}+\left(\frac{41\pi^2}{48}-\frac{1585}{72}\right)\nu\right]$$
$$+ v^8\left[\left(\frac{544}{9}-\frac{451\pi^2}{192}\right)\nu^2 + \nu\left(\frac{153211}{2880}-\frac{64\gamma_E}{3}-\frac{11375\pi^2}{3072}-\frac{128}{3}\ln 2-\frac{64\ln v}{3}\right)\right], \tag{A2b}$$

$$\bar{r}_\mathrm{SO} = \frac{\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1}{M\mu}\bigg\{ v^3\left(-\frac{\nu}{6}-\frac{X_2}{2}\right) + v^5\left[-\frac{19\nu^2}{48}+\frac{3\nu}{16}+\left(\frac{\nu}{8}-\frac{3}{16}\right)X_2\right]$$
$$+ v^7\left[-\frac{47\nu^3}{3456}-\frac{61\nu^2}{192}-\frac{5\nu}{128}+\left(\frac{907\nu^2}{1152}+\frac{139\nu}{32}+\frac{5}{128}\right)X_2\right]\bigg\} + 1\leftrightarrow 2, \tag{A2c}$$

$$\bar{r}_{S_1S_2} = \frac{\nu}{M^2\mu^2}\bigg\{-2v^4(\boldsymbol{n}\cdot\boldsymbol{S}_1)(\boldsymbol{n}\cdot\boldsymbol{S}_2)$$
$$+ v^6\left[(\boldsymbol{n}\cdot\boldsymbol{S}_1)(\boldsymbol{n}\cdot\boldsymbol{S}_2)\left(\frac{85}{24}-\frac{2\nu}{3}\right) + (\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_2)\left(\frac{5\nu}{18}-\frac{8}{3}\right) + (\boldsymbol{\lambda}_\mathrm{N}\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_\mathrm{N}\cdot\boldsymbol{S}_2)\left(\nu-\frac{8}{3}\right)\right]$$
$$+ v^8\bigg[(\boldsymbol{n}\cdot\boldsymbol{S}_1)(\boldsymbol{n}\cdot\boldsymbol{S}_2)\left(-\frac{59\nu^2}{48}-\frac{1115\nu}{96}-\frac{87}{32}\right) + (\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_2)\left(\frac{5\nu^2}{12}+\frac{55\nu}{24}+\frac{3}{2}\right)$$
$$+ (\boldsymbol{\lambda}_\mathrm{N}\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_\mathrm{N}\cdot\boldsymbol{S}_2)\left(\frac{35\nu^2}{24}+\frac{183\nu}{16}+\frac{15}{4}\right)\bigg]\bigg\}, \tag{A2d}$$

$$\bar{r}_{S^2} = \frac{v^4}{M^2\mu^2}(\nu-X_2)(\boldsymbol{n}\cdot\boldsymbol{S}_1)^2 + \frac{v^6}{M^2\mu^2}\bigg\{(\boldsymbol{n}\cdot\boldsymbol{S}_1)^2\left[\frac{\nu^2}{12}-\frac{49\nu}{12}+\left(\frac{49}{12}-\frac{25\nu}{6}\right)X_2\right] + (\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)^2\left[\frac{29\nu^2}{36}+\frac{11\nu}{4}+\left(\frac{11\nu}{6}-\frac{11}{4}\right)X_2\right]$$
$$+ (\boldsymbol{\lambda}_\mathrm{N}\cdot\boldsymbol{S}_1)^2\left[\frac{\nu^2}{2}+\frac{17\nu}{6}+\left(\frac{7\nu}{3}-\frac{17}{6}\right)X_2\right]\bigg\}$$
$$+ \frac{v^8}{M^2\mu^2}\bigg\{(\boldsymbol{n}\cdot\boldsymbol{S}_1)^2\left[-\frac{101\nu^3}{32}+\frac{585\nu^2}{32}-\frac{67\nu}{32}+\left(-\frac{595\nu^2}{96}-\frac{163\nu}{8}+\frac{67}{32}\right)X_2\right]$$
$$+ (\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)^2\left[\frac{5\nu^3}{24}-\frac{47\nu^2}{24}-3\nu+\left(-\frac{13\nu^2}{8}-\frac{25\nu}{24}+3\right)X_2\right]$$
$$+ (\boldsymbol{\lambda}_\mathrm{N}\cdot\boldsymbol{S}_1)^2\left[\frac{35\nu^3}{48}-\frac{105\nu^2}{16}-\frac{75\nu}{16}+\left(\frac{9\nu^2}{16}+\frac{15\nu}{8}+\frac{75}{16}\right)X_2\right]\bigg\} + 1\leftrightarrow 2, \tag{A2e}$$

$$\bar{r}_{S^2\tilde{C}} = \frac{\tilde{C}_{1\mathrm{ES}^2}}{M^2\mu^2}\left\{ v^4\left[(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)^2\left(\frac{X_2}{2}-\frac{\nu}{2}\right)+(\boldsymbol{n}\cdot\boldsymbol{S}_1)^2(\nu-X_2)+(\boldsymbol{\lambda}_\mathrm{N}\cdot\boldsymbol{S}_1)^2\left(\frac{X_2}{2}-\frac{\nu}{2}\right)\right]\right.$$
$$+v^6\left[(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)^2\left(\frac{\nu^2}{2}+\frac{5\nu X_2}{6}\right)+(\boldsymbol{n}\cdot\boldsymbol{S}_1)^2\left(\frac{\nu^2}{2}-\frac{3\nu}{2}+\left(\frac{3}{2}-\frac{19\nu}{6}\right)X_2\right)+(\boldsymbol{\lambda}_\mathrm{N}\cdot\boldsymbol{S}_1)^2\left(\frac{\nu^2}{2}+\frac{5\nu X_2}{6}\right)\right]$$
$$+v^8\left[(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)^2\left(\frac{325\nu^2}{84}+\left(-2\nu^2-\frac{5\nu}{6}\right)X_2\right)+(\boldsymbol{\lambda}_\mathrm{N}\cdot\boldsymbol{S}_1)^2\left(\frac{325\nu^2}{84}+\left(-2\nu^2-\frac{5\nu}{6}\right)X_2\right)\right.$$
$$\left.\left.+(\boldsymbol{n}\cdot\boldsymbol{S}_1)^2\left(-\frac{45\nu^3}{32}+\frac{2297\nu^2}{672}+\frac{69\nu}{32}+\left(\frac{11\nu^2}{32}-\frac{43\nu}{12}-\frac{69}{32}\right)X_2\right)\right]\right\}+1\leftrightarrow 2. \tag{A2f}$$

APPENDIX B: SSC TRANSFORMATION FOR THE ANGULAR MOMENTUM

The transformation from the canonical NW SSC to the covariant SSC is given by the center-of-mass shift [247]

$$\boldsymbol{x}_{\mathrm{i(NW)}} \to \boldsymbol{x}_{\mathrm{i(cov)}} + \frac{1}{2m_\mathrm{i}}\mathbf{v}_{\mathrm{i(cov)}}\times\boldsymbol{S}_\mathrm{i}, \tag{B1}$$

where $\boldsymbol{x}_\mathrm{i}$ and $\mathbf{v}_\mathrm{i}\equiv\dot{\boldsymbol{x}}_\mathrm{i}$ are the position and velocity vectors of each body, with $\mathrm{i}=1,2$.

In the NW SSC, the vector $\boldsymbol{L}_{\mathrm{N(NW)}}=\mu\boldsymbol{r}_{\mathrm{(NW)}}\times\mathbf{v}_{\mathrm{(NW)}}$, where $\boldsymbol{r}\equiv\boldsymbol{x}_1-\boldsymbol{x}_2$ and $\mathbf{v}\equiv\mathbf{v}_1-\mathbf{v}_2$ are the relative position and velocity, respectively. Transforming to the covariant SSC leads to

$$\begin{aligned}\boldsymbol{L}_{\mathrm{N(NW)}} &= \mu\boldsymbol{r}_{\mathrm{(cov)}}\times\mathbf{v}_{\mathrm{(cov)}}\\ &\quad+\frac{X_2^2}{2}\left[\left(\frac{M}{r}+\mathbf{v}^2\right)\boldsymbol{S}_1-(\mathbf{v}\cdot\boldsymbol{S}_1)\mathbf{v}-\frac{M}{r}(\boldsymbol{n}\cdot\boldsymbol{S}_1)\boldsymbol{n}\right]\\ &\quad+\frac{X_1^2}{2}\left[\left(\frac{M}{r}+\mathbf{v}^2\right)\boldsymbol{S}_2-(\mathbf{v}\cdot\boldsymbol{S}_2)\mathbf{v}-\frac{M}{r}(\boldsymbol{n}\cdot\boldsymbol{S}_2)\boldsymbol{n}\right],\end{aligned} \tag{B2}$$

where all quantities on the right-hand side are in the covariant SSC, but we dropped the label in the SO part to simplify the notation. Dividing $\boldsymbol{L}_{\mathrm{N(NW)}}$ by its magnitude, using $\boldsymbol{r}=r\boldsymbol{n}$ and $\mathbf{v}=r\Omega\boldsymbol{\lambda}_\mathrm{N}=rv^3\boldsymbol{\lambda}_\mathrm{N}/M$ for circular orbits, and taking an orbit average yields

$$\begin{aligned}\boldsymbol{l}_{\mathrm{N(NW)}} &= \boldsymbol{l}_{\mathrm{N(cov)}}+\frac{v^3}{2M\mu}\{X_2^2[\boldsymbol{S}_1-(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)\boldsymbol{l}_\mathrm{N}]\\ &\quad+X_1^2[\boldsymbol{S}_2-(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_2)\boldsymbol{l}_\mathrm{N}]\}.\end{aligned} \tag{B3}$$

Our result for $\boldsymbol{L}(\boldsymbol{l}_\mathrm{N})$ in Eq. (65) is in the NW SSC, while Eq. (A5) of Ref. [114] is in the covariant SSC; the difference up to LO SO is given by

$$\begin{aligned}\boldsymbol{L}^{\mathrm{Eq.\,(65)}}_{\mathrm{(NW)}}-\boldsymbol{L}^{\mathrm{Ref.\,[114]}}_{\mathrm{(cov)}} &= -\frac{v^2}{2}\{X_2^2[\boldsymbol{S}_1-(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_1)\boldsymbol{l}_\mathrm{N}]\\ &\quad+X_1^2[\boldsymbol{S}_2-(\boldsymbol{l}_\mathrm{N}\cdot\boldsymbol{S}_2)\boldsymbol{l}_\mathrm{N}]\}\\ &\quad+\mathcal{O}(v^3).\end{aligned} \tag{B4}$$

The leading PN order of $\boldsymbol{L}(\boldsymbol{l}_\mathrm{N})$ in Eq. (65) is $\mu M\boldsymbol{l}_\mathrm{N}/v$, and v is invariant under an SSC transformation; hence, the difference is due to the SSC transformation of $\boldsymbol{l}_\mathrm{N}$. Indeed, we see from Eqs. (B4) and (B3) that accounting for that transformation cancels the difference between our result and that of Ref. [114].

APPENDIX C: EQUATIONS OF MOTION IN TERMS OF $\boldsymbol{l}$

In Sec. III, we derived the PN-expanded EOMs for precessing spins by using a frame adapted to the vector $\boldsymbol{L}_\mathrm{N}\equiv\mu\boldsymbol{r}\times\mathbf{v}$. In this Appendix, we include the corresponding equations in a frame adapted to the orbital angular momentum $\boldsymbol{L}\equiv\boldsymbol{r}\times\boldsymbol{p}$, since one can define a coprecessing frame to be aligned with $\boldsymbol{l}$, as in the `SEOBNRv4PHM` waveform model [111] for example.

The EOMs in this case are given by Eq. (57) and

$$\begin{aligned}\dot{\boldsymbol{S}}_\mathrm{i} &= \boldsymbol{\Omega}_{S_\mathrm{i}}\times\boldsymbol{S}_\mathrm{i},\\ \dot{\boldsymbol{l}} &= \dot{\boldsymbol{l}}(\boldsymbol{l},\boldsymbol{S}_\mathrm{i},v),\\ \dot{v} &= \frac{\dot{E}(v)}{\mathrm{d}E(v)/\mathrm{d}v}.\end{aligned} \tag{C1}$$

The effective Hamiltonian is the same as in Sec. III A, except that we replace the $\boldsymbol{n}\cdot\boldsymbol{a}_\pm$ terms by the following orbit average

$$\begin{aligned}(\boldsymbol{n}\cdot\boldsymbol{a}_\pm)^2 &\simeq \frac{1}{2}[a_\pm^2-(\boldsymbol{l}\cdot\boldsymbol{a}_\pm)^2],\\ (\boldsymbol{n}\cdot\boldsymbol{a}_+)(\boldsymbol{n}\cdot\boldsymbol{a}_-) &\simeq \frac{1}{2}[\boldsymbol{a}_+\cdot\boldsymbol{a}_- -(\boldsymbol{l}\cdot\boldsymbol{a}_+)(\boldsymbol{l}\cdot\boldsymbol{a}_-)],\end{aligned} \tag{C2}$$

i.e., in terms of $\boldsymbol{l}$, instead of $\boldsymbol{l}_\mathrm{N}$ as in Eq. (54).

The spin-precession frequency $\boldsymbol{\Omega}_{S_\mathrm{i}}$ can be directly computed by taking the derivative of the Hamiltonian with respect to the spin vector, then orbit averaging the in-plane spin components, leading to

$$\Omega_{S_1} = \frac{l}{M}\left\{v^5\left(\frac{3X_2}{2}+\frac{\nu}{2}\right)+v^7\left[\left(\frac{9}{8}-\frac{5\nu}{4}\right)X_2-\frac{\nu^2}{24}+\frac{15\nu}{8}\right]+v^9\left[\left(\frac{5\nu^2}{16}-\frac{39\nu}{4}+\frac{27}{16}\right)X_2-\frac{\nu^3}{48}-\frac{55\nu^2}{16}+\frac{81\nu}{16}\right]\right\}$$
$$+\frac{v^6}{M^2\mu}\left\{l\left[l\cdot S_1\left(\frac{3\nu}{2}-\frac{3X_2}{2}\right)-\frac{3\nu}{2}l\cdot S_2\right]+\frac{\nu}{2}S_2\right\}$$
$$+\frac{v^8}{M^2\mu}\left\{l\left[l\cdot S_1\left(-\frac{37\nu^2}{24}-\frac{9\nu}{8}+\left(\frac{9}{8}-\frac{9\nu}{2}\right)X_2\right)+l\cdot S_2\left(-\frac{\nu^2}{24}-2\nu\right)\right]+S_2\left(\frac{3\nu}{2}-\frac{\nu^2}{8}\right)\right\}$$
$$+\frac{v^{10}}{M^2\mu}\left\{l\left[l\cdot S_1\left(\frac{127\nu^3}{144}-\frac{17\nu^2}{2}-\frac{27\nu}{16}+\left(\frac{53\nu^2}{6}-\frac{25\nu}{4}+\frac{27}{16}\right)X_2\right)\right.\right.$$
$$\left.\left.+l\cdot S_2\left(\frac{109\nu^3}{144}+\frac{443\nu^2}{96}-\frac{27\nu}{8}\right)\right]+\left(-\frac{\nu^3}{48}+\frac{43\nu^2}{32}+\frac{3\nu}{2}\right)S_2\right\},$$
$$+\frac{\tilde{C}_{1\mathrm{ES}^2}}{M^2\mu}l(l\cdot S_1)\left\{v^6\left(\frac{3\nu}{2}-\frac{3X_2}{2}\right)+v^8\left[-\frac{3\nu^2}{4}+\frac{9\nu}{4}+\left(-\frac{9\nu}{4}-\frac{9}{4}\right)X_2\right]\right.$$
$$\left.+v^{10}\left[\frac{\nu^3}{16}-\frac{885\nu^2}{112}+\frac{81\nu}{16}+\left(\frac{39\nu^2}{16}-\frac{33\nu}{16}-\frac{81}{16}\right)X_2\right]\right\}, \quad \text{(C3)}$$

and similarly for Ω_{S_2}.

To compute the equation for $\dot{l}$, we first need the orbit-averaged angular momentum, which can be obtained by solving Eq. (61) for r and L, leading to

$$L \equiv \frac{M\mu}{v}(\bar{L}^{S^0}+\bar{L}_{\mathrm{SO}}+\bar{L}_{S_1S_2}+\bar{L}_{S^2}+\bar{L}_{S^2\tilde{C}}), \quad \text{(C4a)}$$

$$\bar{L}_{\mathrm{SO}} = \frac{l\cdot S_1}{M\mu}\left\{v^3\left(-\frac{5\nu}{6}-\frac{5X_2}{2}\right)+v^5\left[\frac{7\nu^2}{72}-\frac{35\nu}{8}+\left(\frac{35\nu}{12}-\frac{21}{8}\right)X_2\right]\right.$$
$$\left.+v^7\left[\frac{\nu^3}{16}+\frac{165\nu^2}{16}-\frac{243\nu}{16}+\left(-\frac{15\nu^2}{16}+\frac{117\nu}{4}-\frac{81}{16}\right)X_2\right]\right\}+1\leftrightarrow 2, \quad \text{(C4b)}$$

$$\bar{L}_{S_1S_2} = \frac{\nu}{M^2\mu^2}\left\{v^4[3(l\cdot S_1)(l\cdot S_2)-S_1\cdot S_2]+v^6\left[(l\cdot S_1)(l\cdot S_2)\left(\frac{5\nu}{72}+\frac{29}{6}\right)+S_1\cdot S_2\left(\frac{3\nu}{8}-\frac{7}{2}\right)\right]\right.$$
$$\left.+v^8\left[(l\cdot S_1)(l\cdot S_2)\left(-\frac{539\nu^2}{216}-\frac{4181\nu}{288}+\frac{99}{8}\right)+S_1\cdot S_2\left(\frac{\nu^2}{24}-\frac{171\nu}{32}-\frac{49}{8}\right)\right]\right\}, \quad \text{(C4c)}$$

$$\bar{L}_{S^2} = \frac{v^4}{M^2\mu^2}\left[(l\cdot S_1)^2\left(\frac{3X_2}{2}-\frac{3\nu}{2}\right)+S_1^2\left(\frac{\nu}{2}-\frac{X_2}{2}\right)\right]$$
$$+\frac{v^6}{M^2\mu^2}\left\{(l\cdot S_1)^2\left[\frac{293\nu^2}{144}+\frac{27\nu}{16}+\left(\frac{47\nu}{8}-\frac{27}{16}\right)X_2\right]+S_1^2\left[-\frac{23\nu^2}{48}+\nu\left(\frac{5}{16}-\frac{61X_2}{24}\right)-\frac{5X_2}{16}\right]\right\}$$
$$+\frac{v^8}{M^2\mu^2}\left\{(l\cdot S_1)^2\left[-\frac{629\nu^3}{432}+\frac{1315\nu^2}{96}+\frac{9\nu}{4}+\left(-\frac{4189\nu^2}{288}+\frac{1039\nu}{96}-\frac{9}{4}\right)X_2\right]\right.$$
$$\left.+S_1^2\left[\frac{13\nu^3}{144}+\frac{145\nu^2}{96}-\frac{3\nu}{8}+\left(\frac{929\nu^2}{288}-\frac{979\nu}{96}+\frac{3}{8}\right)X_2\right]\right\}+1\leftrightarrow 2, \quad \text{(C4d)}$$

$$\begin{aligned}\bar{L}_{S^2\tilde{C}} = \frac{\tilde{C}_{1\mathrm{ES}^2}}{M^2\mu^2}\bigg\{ & v^4\left[(\boldsymbol{l}\cdot\boldsymbol{S}_1)^2\left(\frac{3X_2}{2}-\frac{3\nu}{2}\right)+\boldsymbol{S}_1^2\left(\frac{\nu}{2}-\frac{X_2}{2}\right)\right] \\ & + v^6\left[(\boldsymbol{l}\cdot\boldsymbol{S}_1)^2(\nu^2+3\nu(X_2-1)+3X_2)+\boldsymbol{S}_1^2\left(-\frac{\nu^2}{3}+\nu(1-X_2)-X_2\right)\right] \\ & + v^8\bigg[(\boldsymbol{l}\cdot\boldsymbol{S}_1)^2\left(-\frac{5\nu^3}{48}+\nu^2\left(\frac{1475}{112}-\frac{65X_2}{16}\right)+\nu\left(\frac{55X_2}{16}-\frac{135}{16}\right)+\frac{135X_2}{16}\right) \\ & + \boldsymbol{S}_1^2\left(\frac{5\nu^3}{144}+\nu^2\left(\frac{65X_2}{48}-\frac{1475}{336}\right)+\nu\left(\frac{45}{16}-\frac{55X_2}{48}\right)-\frac{45X_2}{16}\right)\bigg]\bigg\} + 1\leftrightarrow 2,\end{aligned} \tag{C4e}$$

where the nonspinning part is the same as in Eq. (A1). Taking the time derivative of $\boldsymbol{L}=L\boldsymbol{l}$, and then solving $\dot{\boldsymbol{J}}=\dot{\boldsymbol{L}}+\dot{\boldsymbol{S}}_1+\dot{\boldsymbol{S}}_2$ for $\dot{\boldsymbol{l}}$ yields

$$\dot{\boldsymbol{l}} \equiv \dot{\boldsymbol{l}}_{\mathrm{SO}}+\dot{\boldsymbol{l}}_{S_1S_2}+\dot{\boldsymbol{l}}_{S^2}+\dot{\boldsymbol{l}}_{S^2\tilde{C}}+\dot{\boldsymbol{l}}_{\mathrm{RR}}, \tag{C5a}$$

$$\begin{aligned}\dot{\boldsymbol{l}}_{\mathrm{SO}} = \frac{\boldsymbol{l}\times\boldsymbol{S}_1}{M^2\mu}\bigg\{ & v^6\left(-\frac{\nu}{2}-\frac{3X_2}{2}\right)+v^8\left[\frac{\nu^2}{8}-\frac{9\nu}{8}+\left(\frac{3\nu}{2}+\frac{9}{8}\right)X_2\right] \\ & + v^{10}\left[\frac{\nu^3}{48}+\frac{9\nu^2}{4}-\frac{27\nu}{16}+\left(-\frac{\nu^2}{2}+\frac{15\nu}{4}+\frac{27}{16}\right)X_2\right]\bigg\} + 1\leftrightarrow 2,\end{aligned} \tag{C5b}$$

$$\begin{aligned}\dot{\boldsymbol{l}}_{S_1S_2} = \frac{\nu}{M^3\mu^2}\bigg\{ & \frac{3}{2}v^7[(\boldsymbol{l}\times\boldsymbol{S}_1)(\boldsymbol{l}\cdot\boldsymbol{S}_2)+(\boldsymbol{l}\times\boldsymbol{S}_2)(\boldsymbol{l}\cdot\boldsymbol{S}_1)] \\ & + v^9\left[(\boldsymbol{l}\times\boldsymbol{S}_1)(\boldsymbol{l}\cdot\boldsymbol{S}_2)\left(-\frac{5\nu}{8}-\frac{21}{4}\right)+(\boldsymbol{l}\times\boldsymbol{S}_2)(\boldsymbol{l}\cdot\boldsymbol{S}_1)\left(-\frac{5\nu}{8}-\frac{21}{4}\right)-\frac{\delta}{4}\boldsymbol{l}(\boldsymbol{l}\cdot\boldsymbol{S}_1\times\boldsymbol{S}_2)\right] \\ & + v^{11}\bigg\{(\boldsymbol{l}\times\boldsymbol{S}_1)(\boldsymbol{l}\cdot\boldsymbol{S}_2)\left[-\frac{9\nu^2}{16}+\frac{241\nu}{32}+\left(-\frac{3\nu}{8}-\frac{3}{2}\right)X_2+\frac{15}{16}\right] \\ & + (\boldsymbol{l}\times\boldsymbol{S}_2)(\boldsymbol{l}\cdot\boldsymbol{S}_1)\left[-\frac{9\nu^2}{16}+\frac{229\nu}{32}+\left(\frac{3\nu}{8}+\frac{3}{2}\right)X_2-\frac{9}{16}\right]+\frac{\delta}{48}(22\nu+27)\boldsymbol{l}(\boldsymbol{l}\cdot\boldsymbol{S}_1\times\boldsymbol{S}_2)\bigg\}\bigg\},\end{aligned} \tag{C5c}$$

$$\begin{aligned}\dot{\boldsymbol{l}}_{S^2} = \frac{(\boldsymbol{l}\times\boldsymbol{S}_1)(\boldsymbol{l}\cdot\boldsymbol{S}_1)}{M^3\mu^2}\bigg\{ & v^7\left(\frac{3X_2}{2}-\frac{3\nu}{2}\right)+v^9\left[\frac{11\nu^2}{8}+\frac{57\nu}{8}+\left(\frac{7\nu}{4}-\frac{57}{8}\right)X_2\right] \\ & + v^{11}\left[-\frac{43\nu^3}{48}-\frac{153\nu^2}{16}-\frac{45\nu}{16}+\left(-\frac{289\nu^2}{48}+\frac{27\nu}{4}+\frac{45}{16}\right)X_2\right]\bigg\} + 1\leftrightarrow 2,\end{aligned} \tag{C5d}$$

$$\begin{aligned}\dot{\boldsymbol{l}}_{S^2\tilde{C}} = \frac{\tilde{C}_{1\mathrm{ES}^2}}{M^3\mu^2}(\boldsymbol{l}\times\boldsymbol{S}_1)(\boldsymbol{l}\cdot\boldsymbol{S}_1)\bigg\{ & v^7\left(\frac{3X_2}{2}-\frac{3\nu}{2}\right)+v^9(\nu^2+2\nu X_2) \\ & + v^{11}\left[-\frac{\nu^3}{6}+\frac{159\nu^2}{56}+\left(\frac{21\nu}{8}-\frac{17\nu^2}{6}\right)X_2\right]\bigg\} + 1\leftrightarrow 2,\end{aligned} \tag{C5e}$$

$$\begin{aligned}\dot{\boldsymbol{l}}_{\mathrm{RR}} = -\frac{64}{5}\frac{v^8}{M}\boldsymbol{l}\bigg\{ & \nu+v^2\nu\left(-\frac{37\nu}{12}-\frac{1751}{336}\right)+4\pi\nu v^3+\frac{v^3}{M\mu}\bigg[(\boldsymbol{l}\cdot\boldsymbol{S}_1)\left(\frac{7\nu^2}{4}-\frac{19\nu}{12}+\left(\frac{55\nu}{8}-\frac{25}{8}\right)X_2\right) \\ & + (\boldsymbol{l}\cdot\boldsymbol{S}_2)\left(\frac{7\nu^2}{4}-\frac{19\nu}{12}+\left(\frac{55\nu}{8}-\frac{25}{8}\right)X_1\right)\bigg]\bigg\}.\end{aligned} \tag{C5f}$$

The evolution of the orbital frequency can be obtained by replacing $\boldsymbol{l}_{\mathrm{N}}$ in Eq. (69) by its PN expansion in terms of $\boldsymbol{l}$, resulting in

$$\dot{v} \equiv \frac{32\nu v^9}{5M}(\dot{\bar{v}}_{S^0} + \dot{\bar{v}}_{\rm SO} + \dot{\bar{v}}_{S_1S_2} + \dot{\bar{v}}_{S^2} + \dot{\bar{v}}_{S^2\tilde{C}}), \tag{C6a}$$

$$\begin{aligned}\dot{\bar{v}}_{\rm SO} = \frac{\boldsymbol{l}\cdot\boldsymbol{S}_1}{M\mu}\bigg\{ & v^3\left(-\frac{19\nu}{6}-\frac{25X_2}{4}\right) + v^5\left[\frac{79\nu^2}{6}-\frac{21611\nu}{1008}+\left(\frac{281\nu}{8}-\frac{809}{84}\right)X_2\right] + \pi v^6\left(-\frac{37\nu}{3}-\frac{151X_2}{6}\right) \\ & + v^7\left[-\frac{10819\nu^3}{432}+\frac{40289\nu^2}{288}-\frac{1932041\nu}{18144}+\left(-\frac{2903\nu^2}{32}+\frac{257023\nu}{1008}-\frac{1195759}{18144}\right)X_2\right] \\ & + \pi v^8\left[\frac{34303\nu^2}{336}-\frac{46957\nu}{504}+\left(\frac{50483\nu}{224}-\frac{1665}{28}\right)X_2\right]\bigg\} + 1\leftrightarrow 2,\end{aligned} \tag{C6b}$$

$$\begin{aligned}\dot{\bar{v}}_{S_1S_2} = & \frac{\nu v^4}{M^2\mu^2}\left[\frac{721}{48}(\boldsymbol{l}\cdot\boldsymbol{S}_1)(\boldsymbol{l}\cdot\boldsymbol{S}_2)-\frac{247}{48}(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\right] \\ & + \frac{\nu v^6}{M^2\mu^2}\left[(\boldsymbol{l}\cdot\boldsymbol{S}_1)(\boldsymbol{l}\cdot\boldsymbol{S}_2)\left(\frac{17415}{224}-\frac{11323\nu}{288}\right)+\left(\frac{5917\nu}{288}+\frac{7309}{672}\right)(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\right] \\ & + \frac{\nu\pi v^7}{M^2\mu^2}\left[\frac{207}{4}(\boldsymbol{l}\cdot\boldsymbol{S}_1)(\boldsymbol{l}\cdot\boldsymbol{S}_2)-12(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\right] + \frac{\nu v^8}{M^2\mu^2}\left[\left(-\frac{138421\nu^2}{3456}-\frac{1203227\nu}{8064}-\frac{1623071}{10368}\right)(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\right. \\ & \left. + (\boldsymbol{l}\cdot\boldsymbol{S}_1)(\boldsymbol{l}\cdot\boldsymbol{S}_2)\left(\frac{9043\nu^2}{3456}-\frac{11824525\nu}{24192}+\frac{21670157}{24192}\right)\right],\end{aligned} \tag{C6c}$$

$$\begin{aligned}\dot{\bar{v}}_{S^2} = & \frac{v^4}{M^2\mu^2}\left[(\boldsymbol{l}\cdot\boldsymbol{S}_1)^2\left(\frac{719X_2}{96}-\frac{719\nu}{96}\right)+S_1^2\left(\frac{233\nu}{96}-\frac{233X_2}{96}\right)\right] \\ & + \frac{v^6}{M^2\mu^2}\left\{(\boldsymbol{l}\cdot\boldsymbol{S}_1)^2\left[\frac{25829\nu^2}{576}+\frac{85\nu}{448}+\left(\frac{21691\nu}{576}-\frac{85}{448}\right)X_2\right]+S_1^2\left[-\frac{6467\nu^2}{576}-\frac{6403\nu}{448}+\left(\frac{6403}{448}-\frac{3445\nu}{576}\right)X_2\right]\right\} \\ & + \frac{\pi v^7}{M^2\mu^2}\left[(\boldsymbol{l}\cdot\boldsymbol{S}_1)^2\left(\frac{209X_2}{8}-\frac{209\nu}{8}\right)+S_1^2(6\nu-6X_2)\right] \\ & + \frac{v^8}{M^2\mu^2}\left\{(\boldsymbol{l}\cdot\boldsymbol{S}_1)^2\left[-\frac{893549\nu^3}{6912}+\frac{16130213\nu^2}{48384}-\frac{12067655\nu}{48384}+\left(-\frac{2540311\nu^2}{6912}+\frac{3888965\nu}{16128}+\frac{12067655}{48384}\right)X_2\right]\right. \\ & \left. + S_1^2\left[\frac{162443\nu^3}{6912}+\frac{1518919\nu^2}{16128}+\frac{8745467\nu}{145152}+\left(\frac{398761\nu^2}{6912}-\frac{754241\nu}{5376}-\frac{8745467}{145152}\right)X_2\right]\right\} + 1\leftrightarrow 2,\end{aligned} \tag{C6d}$$

$$\begin{aligned}\dot{\bar{v}}_{S^2\tilde{C}} = \frac{\tilde{C}_{1{\rm ES}^2}}{M^2\mu^2}\bigg\{ & v^4\left[(\boldsymbol{l}\cdot\boldsymbol{S}_1)^2\left(\frac{15X_2}{2}-\frac{15\nu}{2}\right)+S_1^2\left(\frac{5\nu}{2}-\frac{5X_2}{2}\right)\right] \\ & + v^6\left[(\boldsymbol{l}\cdot\boldsymbol{S}_1)^2\left(\frac{129\nu^2}{4}-\frac{1977\nu}{224}+\left(\frac{1977}{224}-\frac{73\nu}{16}\right)X_2\right)+S_1^2\left(-\frac{43\nu^2}{4}+\frac{659\nu}{224}+\left(\frac{73\nu}{48}-\frac{659}{224}\right)X_2\right)\right] \\ & + v^8\left[(\boldsymbol{l}\cdot\boldsymbol{S}_1)^2\left(-\frac{1567\nu^3}{24}+\frac{29329\nu^2}{224}-\frac{597271\nu}{6048}+\left(-\frac{5675\nu^2}{96}-\frac{1517\nu}{168}+\frac{597271}{6048}\right)X_2\right)\right. \\ & \left. + S_1^2\left(\frac{1567\nu^3}{72}-\frac{29329\nu^2}{672}+\frac{597271\nu}{18144}+\left(\frac{5675\nu^2}{288}+\frac{1517\nu}{504}-\frac{597271}{18144}\right)X_2\right)\right] \\ & + \pi v^7[(\boldsymbol{l}\cdot\boldsymbol{S}_1)^2(26X_2-26\nu)+S_1^2(6\nu-6X_2)]\bigg\} + 1\leftrightarrow 2,\end{aligned} \tag{C6e}$$

where the nonspinning part is the same as in Eq. (69b).

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