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On the surjectivity of the tmf–Hurewicz image of A_1

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Let A_1 be any spectrum in the class of finite spectra whose mod 2 cohomology is isomorphic to $\mathcal{A}(1)$ as a module over the subalgebra $\mathcal{A}(1)$ of the Steenrod algebra; let tmf be the connective spectrum of topological modular forms. We prove that the tmf–Hurewicz image of A_1 is surjective.

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Introduction

In [5], Davis and Mahowald constructed a class of connective finite spectra whose mod 2 cohomology is isomorphic to $\mathcal{A}(1)$ as a module over the subalgebra $\mathcal{A}(1)$ generated by Sq^1 and Sq^2 of the Steenrod algebra. This class of spectra has four different homotopy types, denoted by $A_1[ij]$ for $i, j \in \{0, 1\}$; see the introduction of Bhattacharya, Egger and Mahowald [2] for an explanation of the notation. We write A_1 to refer to any of the $A_1[ij]$, and call each of them a version of A_1. The spectrum A_1 is constructed via three cofiber sequences, starting from the sphere spectrum S^0, as follows. Let $V(0)$ be the mod 2 Moore spectrum, ie the cofiber of multiplication by two on S^0. Next, let Y be the cofiber of multiplication by η, the first Hopf element, on $V(0)$. Davis and Mahowald show that Y admits v_1–self-maps $\Sigma^2 Y \to Y$, that is maps which induce multiplication with v_1. The generic notation A_1 denotes the cofiber of any of them. In fact, there are eight homotopy classes of v_1, giving rise to four different homotopy types of A_1. We note further that the spectra $A_1[00]$ and $A_1[11]$ are Spanier–Whitehead self-dual, and $A_1[10]$ and $A_1[01]$ are Spanier–Whitehead dual to each other.

Let tmf be the ring spectrum of connective topological modular forms. This spectrum is constructed using a certain sheaf in ring spectra on the étale site of the moduli stack of elliptic curves, hence the name; see Behrens [1] for the construction. The spectrum tmf plays an important role in investigating chromatic level two in chromatic homotopy theory. The tmf–Hurewicz map of A_1, $H\colon A_1 \to \mathrm{tmf} \wedge A_1$, is given by smashing A_1

with the unit of tmf. Our goal is to study the induced map in homotopy of H:

$$H_*\colon \pi_*(A_1) \to \pi_*(\mathrm{tmf} \wedge A_1). \tag{1}$$

Closely related to the homomorphism (1) is the edge homomorphism of the topological duality spectral sequence aiming at analyzing the $K(2)$–localization of A_1, where $K(2)$ is the second Morava K–theory at the prime 2. This is, in fact, our initial motivation for studying (1); see Section 2.3 for further discussion. From another perspective, the map (1) is the edge homomorphism of the tmf–based Adams spectral sequence for A_1. This is an upper half-plane spectral sequence converging to $\pi_*(A_1)$ starting with the E_1–term:

$$\mathrm{E}_1^{n,t} = \pi_t(\mathrm{tmf}^{\wedge(n+1)} \wedge A_1) \Rightarrow \pi_{t-n}(A_1).$$

We now state our main theorem, as well as a consequence:

Theorem A (Theorem 17) *The* tmf*–Hurewicz homomorphism*

$$\pi_*(A_1) \to \pi_*(\mathrm{tmf} \wedge A_1)$$

is surjective for all versions of A_1*.*

Theorem B (Theorem 26) *The edge homomorphism of the topological duality resolution*

$$\pi_*(E_C^{h\mathbb{S}_2^1} \wedge A_1) \to \pi_*(E_C^{hG_{24}} \wedge A_1)$$

is surjective.

Remark 1 The target of the tmf–Hurewicz homomorphism is explicitly known. In fact, we explicitly computed the homotopy groups of $\pi_*(E_C^{hG_{24}} \wedge A_1)$ in [12] and showed, in [12, Theorem 5.3.20], that the natural homomorphism $\mathrm{tmf} \wedge A_1 \to E_C^{hG_{24}} \wedge A_1$ is an isomorphism in nonnegative stem, up to an action of the Galois group of $\mathbb{F}_4$ over $\mathbb{F}_2$. Furthermore, the spectrum $E_C^{hG_{24}}$ is homotopy equivalent to $L_{K(2)}\mathrm{tmf}$ up to the Galois action. Thus, Theorem A asserts that all elements of $\pi_*(\mathrm{tmf} \wedge A_1)$ lift to v_2–periodic elements of $\pi_*(A_1)$, elements which are not annihilated by v_2–self-maps of A_1. It is then interesting to be able to locate the corresponding v_2–periodic families in $\pi_*(S^0)$, as done by Hopkins and Mahowald in [9], using the generalized Moore spectra.

Here is the overview of the paper. The principal method for proving the main result is to analyze the map of the Adams spectral sequences

$$\begin{tikzcd}
\mathrm{Ext}^{s,t}_{\mathcal{A}}(\mathbb{F}_2, H_*(A_1)) \arrow[r, Rightarrow] \arrow[d] & \pi_{t-s}(A_1) \arrow[d, "H_*"] \\
\mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(\mathbb{F}_2, H_*(A_1)) \arrow[r, Rightarrow] & \pi_{t-s}(\mathrm{tmf} \wedge A_1)
\end{tikzcd}$$

and then to show that permanent cycles in the lower spectral sequence lift to permanent cycles in the upper one. In the first section we recall the main tool — The Davis–Mahowald spectral sequence — used to study the Ext–group over subalgebras of the Steenrod algebra. The construction of the Davis–Mahowald spectral sequence is more general and is of independent interest. In the second section we prove the main theorems. First we prove the algebraic version of the theorem, that is, the surjectivity of the induced map at the E_2–term of the Adams spectral sequences, then we prove the topological statement based on a technical result, Proposition 13. The latter involves, to some extent, a refined analysis of the E_2–term of the Adams spectral sequence for A_1, on which the action of $g \in \mathrm{Ext}^{4,24}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ as well as the notion of g–weak divisibility (see Definition 12) play a key role.

Conventions Unless otherwise stated, all spectra are completed at the prime 2. $\mathrm{H}^*(X)$ and $\mathrm{H}_*(X)$ denote the mod 2 (co)homology of the spectrum X. Given a graded Hopf algebra A over a field k and M a graded A–comodule, $\mathrm{Ext}^{s,t}_A(M)$ denotes $\mathrm{Ext}^{s,t}_A(k, M) = \mathrm{Ext}^{s}_A(k, \Sigma^t M)$. The difference $t - s$ is referred to as the stem of the classes in $\mathrm{Ext}^{s,t}_A(M)$.

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1 Background and tools

1.1 The dual of the Steenrod algebra

Recall that the Steenrod algebra $\mathcal{A}$ is a cocommutative graded Hopf algebra over $\mathbb{F}_2$, generated by the Steenrod squares Sq^i in degree i for $i \geq 0$, which are subject to the Adem relations

$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum_{i=0}^{\lfloor a/2 \rfloor} \binom{b-i-1}{a-2i} \mathrm{Sq}^{a+b-i} \mathrm{Sq}^i$$

for all $a, b > 0$ and $a < 2b$. In [11], Milnor determines the dual $\mathcal{A}_*$ of the Steenrod algebra, which is a commutative Hopf algebra over $\mathbb{F}_2$. As an algebra, $\mathcal{A}_*$ is a polynomial algebra generated by the elements ξ_i for $i \geq 0$ in degree $2^i - 1$ with $\xi_0 = 1$:

$$\mathcal{A}_* = \mathbb{F}_2[\xi_1, \xi_2, \ldots].$$

The coproduct or the diagonal is given by

$$\Delta(\xi_k) = \sum_{i=0}^{k} \xi_i^{2^{k-i}} \otimes \xi_{k-i}.$$

Subalgebras of $\mathcal{A}$ For $n \geq 1$, denote by $\mathcal{A}(n)$ the subalgebra of $\mathcal{A}$ generated by Sq^i for $0 \leq i \leq n$. The dual $\mathcal{A}(n)_*$ of $\mathcal{A}(n)$ is the quotient of $\mathcal{A}_*$ by the ideal $(\xi_1^{2^{n+1}}, \xi_2^{2^n}, \dots, \xi_n^4, \xi_{n+1}^2, \xi_{n+2}, \dots)$:

$$\mathcal{A}(n)_* = \mathbb{F}_2[\xi_1, \xi_2, \dots, \xi_{n+1}]/(\xi_1^{2^{n+1}}, \xi_2^{2^n}, \dots, \xi_n^4, \xi_{n+1}^2).$$

Notation If $A \to B$ is a map of Hopf algebras, then $A \,\square_B\, k$ denotes the group of primitives of A viewed as a B–comodule via the given map.

The main tool we use to prove Theorem A is the Adams spectral sequence, which will be abbreviated to ASS in the sequel. More precisely, one has the following theorem, due to Hopkins and Mahowald, whose proof can be found in [10].

Theorem 2 *There is an isomorphism of $\mathcal{A}_*$–comodule algebras*:

$$\mathrm{H}_*\mathrm{tmf} \cong \mathcal{A}_* \,\square_{\mathcal{A}(2)_*}\, \mathbb{F}_2.$$

As a consequence, if X is a connective spectrum, then the Adams spectral sequence for $\mathrm{tmf} \wedge X$ reads as

$$\mathrm{E}_2^{s,t} \cong \mathrm{Ext}_{\mathcal{A}_*}^{s,t}(\mathbb{F}_2, (\mathcal{A}_* \,\square_{\mathcal{A}(2)_*}\, \mathbb{F}_2) \otimes \mathrm{H}_*X) \Rightarrow \pi_{t-s}(\mathrm{tmf} \wedge X).$$

By the change-of-rings isomorphism the E_2–term of this spectral sequence is isomorphic to $\mathrm{Ext}_{\mathcal{A}(2)_*}^{s,t}(\mathrm{H}_*X)$. Furthermore, the tmf–Hurewicz map of A_1 induces a map of Adams spectral sequences

$$\begin{tikzcd}
\mathrm{Ext}_{\mathcal{A}_*}^{s,t}(\mathbb{F}_2, \mathrm{H}_*(A_1)) \arrow[r, "H_*"] \arrow[d, Rightarrow] & \mathrm{Ext}_{\mathcal{A}(2)_*}^{s,t}(\mathbb{F}_2, \mathrm{H}_*(A_1)) \arrow[d, Rightarrow] \\
\pi_{t-s}(A_1) \arrow[r, "H_*"] & \pi_{t-s}(\mathrm{tmf} \wedge A_1)
\end{tikzcd}$$

where the map in the E_2–terms is induced the natural projection of Hopf algebras $\mathcal{A}_* \to \mathcal{A}(2)_*$. To analyze $\mathrm{Ext}_{\mathcal{A}_*}^{s,t}(\mathbb{F}_2, \mathrm{H}_*(A_1))$ as well as $\mathrm{Ext}_{\mathcal{A}(2)_*}^{s,t}(\mathbb{F}_2, \mathrm{H}_*(A_1))$, we use the Davis–Mahowald spectral sequence, which is reviewed in the next section.

1.2 The Davis–Mahowald spectral sequence

Initially, the Davis–Mahowald spectral sequence was used by Davis and Mahowald in [6] to calculate Ext–groups over the subalgebra $\mathcal{A}(2)$. In [12, Section 2.1], we established a slight generalization of this spectral sequence. Let us recall this construction.

Let A be a commutative Hopf algebra over a field k of characteristic 2. Let E and P be the graded exterior algebra and the polynomial algebra on a k–vector space V such that V lives in degree 1, respectively. Let E_i and P_i denote the subspaces of elements of homogeneous degree i of E and P, respectively. Suppose that E has the structure of an A–comodule algebra such that $k \oplus V$ is a sub-A–comodule of E. Since P_1 sits in a short exact sequence

$$0 \to k \to k \oplus E_1 \xrightarrow{p} P_1, \tag{2}$$

P_1 admits a unique structure of an A–comodule, making p a map of A–comodules. Then P admits the structure of an A–comodule algebra:

Lemma 3 *If $P_1^{\otimes n}$ is equipped with the usual structure of an A–comodule of a tensor product, then P_n admits a unique structure of an A–comodule making the multiplication $P_1^{\otimes n} \to P_n$ a map of A–comodules.*

Proof The canonical map $P_1^{\otimes n} \to P_n$ is surjective and its kernel is spanned by elements of the form $y_1 \otimes y_2 \otimes \cdots \otimes y_n - y_{\sigma_1} \otimes y_{\sigma(2)} \otimes \cdots \otimes y_{\sigma(n)}$, where σ is a permutation of the set $\{1, 2, \ldots, n\}$. Then, since A is commutative, we see that the kernel is stable under the coaction of A. The lemma follows. □

We define the cochain complex $(E \otimes P, d)$:

(i) $(E \otimes P)_{-1} = k$.

(ii) $(E \otimes P)_m = E \otimes P_m$, for $m \geq 0$.

(iii) $d\colon k = (E \otimes P)_{-1} \to E = (E \otimes P)_0$ is the unit of E.

(iv) $d\big(\prod_{j=1}^{n} x_{i_j} \otimes z\big) = \sum_{t=1}^{n} \prod_{j \neq t} x_{i_j} \otimes p(x_{i_t})z$, where $x_{i_j} \in E_1$, $z \in P_m$ and p is the projection of (2).

Proposition 2.1.5 of [12] shows that $(E \otimes P, d)$ is a differential graded algebra which is an exact sequence of A–comodules. As a consequence, there is a spectral sequence of algebras

$$\mathrm{Ext}_A^s(k, E \otimes P_n) \Rightarrow \mathrm{Ext}_A^{s+n}(k, k). \tag{3}$$

Furthermore, if M is an A–comodule, there is a spectral sequence of modules over (3):

$$\mathrm{Ext}^s_A(k, E \otimes P_n \otimes M) \Rightarrow \mathrm{Ext}^{s+n}_A(k, M). \tag{4}$$

The Davis–Mahowald spectral sequence, or DMSS for short, appears in this paper in the following form. Let A and B be commutative graded Hopf algebras over $\mathbb{F}_2$ together with a map of Hopf algebras $A \to B$. Then A can be considered as a left B–comodule algebra. The group of primitives $A \,\square_B\, k$ inherits the structure of an A–comodule algebra from A, ie the inclusion of the subgroup $A \,\square_B\, k \to A$ is a map of A–comodule algebras. If it turns out that $E := A \,\square_B\, k$ is an exterior algebra, then we are often in a situation to construct the Davis–Mahowald spectral sequence. Precisely, it can be constructed when E is generated by the k–vector space V, as an exterior algebra, and $k \oplus V$ is a sub-A–comodule of E. In this situation, by the change-of-rings theorem, the E_1–term of the latter is isomorphic to

$$\mathrm{E}_1^{s,t,n} = \mathrm{Ext}^{s,t}_A(k, E \otimes P_n) \cong \mathrm{Ext}^{s,t}_B(k, P_n).$$

This allows us to reduce the problem of computing Ext–groups over A to Ext–groups over B, which is often simpler.

Example 4 Consider the natural projection of commutative Hopf algebras

$$\mathcal{A}(n)_* \to \mathcal{A}(n-1)_*.$$

Let ζ_i denote the conjugate of ξ_i. Then

$$\Delta(\zeta_k) = \sum_{i=0}^{k} \zeta_i \otimes \zeta_{k-i}^{2^i}. \tag{5}$$

The subalgebra $\mathcal{A}(n)_*$ is then isomorphic to

$$\mathbb{F}_2[\zeta_1, \zeta_2, \dots, \zeta_{n+1}]/(\zeta_1^{2^{n+1}}, \zeta_2^{2^n}, \dots, \zeta_n^4, \zeta_{n+1}^2).$$

It is straightforward to show that

$$\mathcal{A}(n)_* \,\square_{\mathcal{A}(n-1)_*}\, \mathbb{F}_2 \cong E(\zeta_1^{2^n}, \zeta_2^{2^{n-1}}, \dots, \zeta_{n+1}),$$

the exterior algebra on $\mathbb{F}_2\{\zeta_1^{2^n}, \zeta_2^{2^{n-1}}, \dots, \zeta_{n+1}\}$. Let x_k denote $\zeta_k^{2^{n+1-k}}$. Using (5), we see that the $\mathcal{A}(n)_*$–comodule structure of $\mathcal{A}(n)_* \,\square_{\mathcal{A}(n-1)_*}\, \mathbb{F}_2$ is given by

$$\Delta(x_k) = \sum_{i=0}^{i=k} \zeta_i^{2^{n+1-k}} \otimes x_{k-i},$$

where $x_0 = 1$ by convention. Thus, the Davis–Mahowald spectral sequence reads as

$$\mathrm{E}_1^{s,t,n} = \mathrm{Ext}^{s,t}_{\mathcal{A}(n-1)_*}(P_n) \Rightarrow \mathrm{Ext}^{s+n,t}_{\mathcal{A}(n)_*}(\mathbb{F}_2).$$

Remark 5 This example is an important tool used to perform the calculation of $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1))$ in [12, Section 3]. We will apply the DMSS for other examples appearing in the proof of Proposition 14.

2 The tmf–Hurewicz image of A_1

The Hurewicz map $A_1 \to \mathrm{tmf} \wedge A_1$ is a map of modules over the ring spectrum $A_1 \wedge DA_1$, hence $H_*\colon \pi_*(A_1) \to \pi_*(\mathrm{tmf} \wedge A_1)$ is a map of modules over $\pi_*(A_1 \wedge DA_1)$. We recollect some elements of the latter, important to this work. Let us fix an element of $\pi_{20}(S^0)$, denoted by $\bar{\kappa}$, which is detected by $g \in \mathrm{Ext}^{4,24}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ in the ASS. It generates a subgroup of order 8 in $\pi_{20}(S^0)$. Next, let $\nu \in \pi_3(S^0)$ be the third Hopf element. The induced map in homotopy of the unit $S^0 \to A_1 \wedge DA_1$ sends $\nu \in \pi_3 S^0$ and $\bar{\kappa} \in \pi_{20} S^0$ to nontrivial elements, denoted by the same names. This is due to the fact that ν and $\bar{\kappa}$ are sent nontrivially to $\pi_*(\mathrm{tmf} \wedge A_1)$ via the composite

$$S^0 \to A_1 \wedge DA_1 \to A_1 \xrightarrow{H} \mathrm{tmf} \wedge A_1,$$

where the middle map is induced by the projection $DA_1 \to S^0$ on the top cell of DA_1. According to [2], A_1 has a v_2^{32}–self-map $v_2^{32}\colon \Sigma^{192} A_1 \to A_1$. Its adjoint $S^{192} \to A_1 \wedge DA_1$ represents an element of $\pi_{192}(A_1 \wedge DA_1)$, also denoted by v_2^{32}.

Let $(\bar{\kappa}, \nu)$ be the ideal of $\pi_*(S^0)$ generated by $\bar{\kappa}$ and ν. Consider the commutative diagram

$$\begin{tikzcd} \pi_*(A_1) \arrow[r, "H_*"] \arrow[d] & \pi_*(\mathrm{tmf} \wedge A_1) \arrow[d] \\ \pi_*(A_1)/(\bar{\kappa}, \nu) \arrow[r] & \pi_*(\mathrm{tmf} \wedge A_1)/(\bar{\kappa}, \nu) \end{tikzcd}$$

We see that the upper horizontal map is surjective if and only if the lower is surjective. In fact, this is an easy consequence of $\pi_*(A_1)$ being bounded below. To prove that the lower map is surjective we can proceed as follows. For any $\bar{x} \in \pi_*(\mathrm{tmf} \wedge A_1)/(\bar{\kappa}, \nu)$, first, lift $\bar{x}$ to an element $x \in \pi_*(\mathrm{tmf} \wedge A_1)$, then, find a class that detects x in the ASS for $\mathrm{tmf} \wedge A_1$ and finally, show that class lifts to a permanent cycle in the ASS for A_1.

The first and the second steps follow from the analysis of the ASS for $\mathrm{tmf} \wedge A_1$ in [12, Section 3]. We recall here relevant information. Let $\mathbb{F}_2[\nu, g, w_2]/(\nu^3, g\nu)$ be the subalgebra of $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$ generated by ν, g and w_2, with $|\nu| = (1,4)$, $|g| = (4,24)$ and $|w_2| = (8,56)$. The classes ν and g lift to classes of the same name in $\mathrm{Ext}^{4,24}_{\mathcal{A}_*}(\mathbb{F}_2)$ and converge to ν and $\bar{\kappa}$, respectively, in the ASS for S^0.

Proposition 6 [12, Theorem 3.2.5] $\mathrm{Ext}^{s,t}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1))$ *is a direct sum of cyclic modules as a module over* $\mathbb{F}_2[\nu, g, w_2]/(\nu^3, g\nu)$, *and is generated by the following classes with respective annihilator ideals*:

$$\begin{array}{cccccccc} e_{0,0} & e_{1,5} & e_{1,6} & e_{2,11} & e_{3,15} & e_{3,17} & e_{4,21} & e_{4,23} \\ (0) & (\nu^2) & (0) & (\nu^2) & (\nu^2) & (0) & (\nu^2) & (0) \end{array}$$

$$\begin{array}{cccccccc} e_{6,30} & e_{6,32} & e_{7,36} & e_{7,38} & e_{8,42} & e_{9,47} & e_{9,48} & e_{10,53} \\ (\nu) & (\nu) & (\nu) & (\nu) & (\nu) & (\nu) & (\nu) & (\nu) \end{array}$$

Here $e_{s,t}$ *is the unique nontrivial class of* $\mathrm{Ext}^{s,t+s}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1))$.

According to [12, Theorem 5.3.20], $\pi_*(\mathrm{tmf} \wedge A_1)$ is Δ^8–periodic. More precisely, multiplication by $\Delta^8 \in \pi_{192}(\mathrm{tmf})$ induces an isomorphism

$$\pi_k(\mathrm{tmf} \wedge A_1) \to \pi_{k+192}(\mathrm{tmf} \wedge A_1)$$

for $k \geq 0$. The same property holds for multiplication by v_2^{32}.

Proposition 7 *Multiplication by* $v_2^{32} \in \pi_{192}(A_1 \wedge DA_1)$ *induces an isomorphism* $\pi_k(\mathrm{tmf} \wedge A_1) \to \pi_{k+192}(\mathrm{tmf} \wedge A_1)$ *for* $k \geq 0$.

Proof Let $F_{\mathrm{tmf}}(-,-)$ denote the function spectrum in the category of tmf–modules, so that $\mathrm{Id}_{\mathrm{tmf}} \wedge v_2^{32}$ and $\Delta^8 \wedge \mathrm{Id}_{A_1}$ represent elements of $\pi_{192}(F_{\mathrm{tmf}}(\mathrm{tmf} \wedge A_1, \mathrm{tmf} \wedge A_1))$. Via the natural equivalence

$$F_{\mathrm{tmf}}(\mathrm{tmf} \wedge A_1, \mathrm{tmf} \wedge A_1) \simeq \mathrm{tmf} \wedge A_1 \wedge DA_1,$$

denote by v_2^{32} and Δ^8 the images of $\mathrm{Id}_{\mathrm{tmf}} \wedge v_2^{32}$ and $\Delta^8 \wedge \mathrm{Id}_{A_1}$, respectively, by the induced homomorphism on homotopy groups. Since $\Delta^8 \wedge \mathrm{Id}_{A_1}$ induces an isomorphism on homotopy groups of nonnegative stems and $\pi_*(\mathrm{tmf} \wedge A_1)$ is finite in each stem, it suffices to show that there is a positive integer n such that

$$(\Delta^8)^n = (v_2^{32})^n \in \pi_{192n}(\mathrm{tmf} \wedge A_1 \wedge DA_1).$$

Since Δ^8 induces an isomorphism $\pi_*(\mathrm{tmf} \wedge A_1) \to \pi_{*+192}(\mathrm{tmf} \wedge A_1)$ for $* \geq 0$, by the five lemma, multiplication by Δ^8 induces an isomorphism

$$\pi_*(\mathrm{tmf} \wedge A_1 \wedge DA_1) \to \pi_{*+192}(\mathrm{tmf} \wedge A_1 \wedge DA_1) \quad \text{for } * \geq 0. \tag{6}$$

By the construction in [2], a v_2^{32}–self-map of A_1 is detected, in the E_2–term of the ASS for $A_1 \wedge DA_1$, by a class that is sent to a class detecting Δ^8 in the ASS for $\mathrm{tmf} \wedge A_1 \wedge DA_1$. This means that the difference $\Delta^8 - v_2^{32}$ is detected in an Adams filtration greater than 32, which is the Adams filtration of Δ^8 and of v_2^{32}. Together

$(0,0)$	$(5,1)$	$(6,1)$	$(11,2)$	$(15,3)$	$(17,3)$	$(21,4)$	$(23,4)$
$e_{0,0}$	$e_{1,5}$	$e_{1,6}$	$e_{2,11}$	$e_{3,15}$	$e_{3,17}$	$e_{4,21}$	$e_{4,23}$
$(30,6)$	$(32,6)$	$(36,7)$	$(38,7)$	$(42,8)$	$(47,9)$	$(48,9)$	$(53,10)$
$e_{6,30}$	$e_{6,32}$	$e_{7,36}$	$e_{7,38}$	$e_{8,42}$	$e_{9,47}$	$e_{49,8}$	$e_{10,53}$
$(48,8)$	$(53,9)$	$(54,9)$	$(59,10)$	$(63,11)$	$(65,11)$	$(69,12)$	$(71,12)$
$w_2e_{0,0}$	$w_2e_{1,5}$	$w_2e_{1,6}$	$w_2e_{2,11}$	$w_2e_{3,15}$	$w_2e_{3,17}$	$w_2e_{4,21}$	$w_2e_{4,23}$
$(78,14)$	$(80,14)$	$(84,15)$	$(86,15)$	$(90,16)$	$(95,17)$	$(96,17)$	$(101,18)$
$w_2e_{6,30}$	$w_2e_{6,32}$	$w_2e_{7,36}$	$w_2e_{7,38}$	$w_2e_{8,42}$	$w_2e_{9,47}$	$w_2e_{9,48}$	$w_2e_{10,53}$

Table 1: List M, generators of $\pi_*(\mathrm{tmf} \wedge A_1)/(\bar{\kappa}, \nu)$ as an $\mathbb{F}_2[\Delta^8]$–module for the non-self-dual versions $A_1[00]$ and $A_1[11]$.

with the isomorphism (6), the difference $\Delta^8 - v_2^{32}$ is equal to $\Delta^8 x$ for some element $x \in \pi_0(\mathrm{tmf} \wedge A_1 \wedge DA_1)$, which must be detected in a positive filtration of the ASS, because the only nonzero class of $\mathrm{Ext}^{0,0}_{\mathcal{A}(2)}(\mathrm{H}_*(A_1 \wedge DA_1))$ is the unit. As a consequence, x is nilpotent, as the ASS for $\mathrm{tmf} \wedge A_1 \wedge DA_1$ has a vanishing line parallel to that of the ASS for $\mathrm{tmf} \wedge A_1$; see Proposition 8. Furthermore, $\Delta^8 - v_2^{32}$ has finite order and Δ^8 is in the center of $\pi_*(\mathrm{tmf} \wedge A_1 \wedge DA_1)$. Therefore, by using the binomial formula, we see that $(\Delta^8 + v_2^{32} - \Delta^8)^{2^k}$ is equal to $(\Delta^8)^{2^k}$ for k large enough. □

As an immediate consequence of the above proposition, multiplication by v_2^{32} induces an isomorphism $\pi_k(\mathrm{tmf} \wedge A_1)/(\nu, \bar{\kappa}) \to \pi_{k+192}(\mathrm{tmf} \wedge A_1)/(\nu, \bar{\kappa})$ for $k \geq 0$. Moreover, it follows from the proof of [12, Theorem 5.3.20] that we can identify a set of generators of $\pi_*(\mathrm{tmf} \wedge A_1)/(\bar{\kappa}, \nu)$ for $0 \leq * < 192$ as $\mathbb{Z}_2$–modules. Therefore, this set generates $\pi_*(\mathrm{tmf} \wedge A_1)/(\bar{\kappa}, \nu)$ as a $\mathbb{Z}_2[v_2^{32}]$–module. We give in Tables 1 and 2 a list of generators

$(99,17)$	$(104,18)$	$(105,18)$	$(110,19)$	$(114,20)$	$(116,20)$
$\nu w_2^2 e_{0,0}$	$\nu w_2^2 e_{1,5}$	$\nu w_2^2 e_{1,6}$	$\nu w_2^2 e_{2,11}$	$\nu w_2^2 e_{3,15}$	$\nu w_2^2 e_{3,17}$
$(120,21)$	$(122,21)$	$(147,25)$	$(152,26)$	$(153,26)$	
$\nu w_2^2 e_{4,21}$	$\nu w_2^2 e_{4,23}$	$\nu w_2^3 e_{0,0}$	$\nu w_2^3 e_{1,5}$	$\nu w_2^3 e_{1,6}$	
$(158,27)$	$(162,28)$	$(164,28)$	$(168,29)$	$(170,29)$	
$\nu w_2^3 e_{2,11}$	$\nu w_2^3 e_{3,15}$	$\nu w_2^3 e_{3,17}$	$\nu w_2^3 e_{4,21}$	$\nu w_2^3 e_{4,23}$	

Table 2: List N, generators of $\pi_*(\mathrm{tmf} \wedge A_1)/(\bar{\kappa}, \nu)$ as an $\mathbb{F}_2[\Delta^8]$–module for the non-self-dual versions $A_1[00]$ and $A_1[11]$.

$(0,0)$	$(5,1)$	$(6,1)$	$(11,2)$	$(15,3)$	$(17,3)$	$(21,4)$	$(23,4)$
$e_{0,0}$	$e_{1,5}$	$e_{1,6}$	$e_{2,11}$	$e_{3,15}$	$e_{3,17}$	$e_{4,21}$	$e_{4,23}$
$(30,6)$	$(32,6)$	$(36,7)$	$(38,7)$	$(42,8)$	$(47,9)$	$(48,9)$	$(53,10)$
$e_{6,30}$	$e_{6,32}$	$e_{7,36}$	$e_{7,38}$	$e_{8,42}$	$e_{9,47}$	$e_{49,8}$	$e_{10,53}$
$(48,8)$	$(53,9)$	$(54,9)$	$(59,10)$	$(63,11)$	$(65,11)$	$(69,12)$	
$w_2e_{0,0}$	$w_2e_{1,5}$	$w_2e_{1,6}$	$w_2e_{2,11}$	$w_2e_{3,15}$	$w_2e_{3,17}$	$w_2e_{4,21}$	
$(74,13)$	$(78,14)$	$(80,14)$	$(84,15)$	$(90,16)$	$(95,17)$		
$\nu w_2e_{4,23}$	$w_2e_{6,30}$	$w_2e_{6,32}$	$w_2e_{7,36}$	$w_2e_{8,42}$	$w_2e_{9,47}$		

Table 3: List P, generators of $\pi_*(\mathrm{tmf}\wedge A_1)/(\bar{\kappa},\nu)$ as an $\mathbb{F}_2[\Delta^8]$–module for the self-dual versions $A_1[01]$ and $A_1[10]$.

of the non-self-dual versions $A_1[00]$ and $A_1[11]$, and in Tables 3 and 4 a list of generators of the self-dual versions $A_1[01]$ and $A_1[10]$. This distinction is because the proof that they lift to permanent cycles in the ASS for A_1 is different for different versions; see Propositions 19, 22 and 23. We denote by M, N, P and Q the sets of generators listed in Tables 1, 2, 3, and 4, respectively. In these tables, the pairs of integers indicate the bidegree (t,s) of the corresponding generators living in $\mathrm{Ext}^{s,t+s}$.

We now proceed to prove that all classes in these tables lift to permanent cycles in the ASS for A_1. There are two main steps. First, we show that the induced map on the E_2–terms of the Hurewicz map

$$H_*\colon \mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathrm{H}_*A_1)\to \mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathrm{H}_*A_1)$$

is surjective. This implies, in particular, that the classes in $M\cup N$ and $P\cup Q$ lift to the E_2–term of the ASS for A_1. Second, we show that $\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathrm{H}_*A_1)$ has a certain

$(96,16)$	$(101,17)$	$(105,18)$	$(110,19)$	$(111,19)$	$(116,20)$
$w_2^2e_{0,0}$	$w_2^2e_{1,5}$	$\nu w_2^2e_{1,6}$	$\nu w_2^2e_{2,11}$	$w_2^2e_{3,15}$	$\nu w_2^2e_{3,17}$
$(120,21)$	$(122,21)$	$(126,22)$	$(147,25)$	$(152,26)$	$(153,26)$
$\nu w_2^2e_{4,21}$	$\nu w_2^2e_{4,23}$	$w_2^2e_{6,30}$	$\nu w_2^3e_{0,0}$	$\nu w_2^3e_{1,5}$	$\nu w_2^3e_{1,6}$
$(158,27)$	$(162,28)$	$(164,28)$	$(168,29)$	$(170,29)$	
$\nu w_2^3e_{2,11}$	$\nu w_2^3e_{3,15}$	$\nu w_2^3e_{3,17}$	$\nu w_2^3e_{4,21}$	$\nu w_2^3e_{4,23}$	

Table 4: List Q, generators of $\pi_*(\mathrm{tmf}\wedge A_1)/(\bar{\kappa},\nu)$ as an $\mathbb{F}_2[\Delta^8]$–module for the self-dual versions $A_1[01]$ and $A_1[10]$.

structure (see Proposition 13) that allows us to rule out nontrivial differentials on lifts of the classes in $M \cup N$ and $P \cup Q$ in the ASS for A_1.

2.1 The algebraic tmf–Hurewicz homomorphism

Proposition 8 (vanishing line) *For $n \geq 0$ or $n = \infty$, $\mathrm{Ext}^{s,t}_{\mathcal{A}(n)_*}(\mathrm{H}_* A_1)$ has vanishing line $t - s < f(s)$, where $f(s) = 5s - 4$ if $s \leq 6$ and $f(s) = 5s$ if $s > 6$, ie*

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(n)_*}(\mathrm{H}_* A_1) = 0 \quad \text{if } t - s < f(s).$$

Proof The statement for $n = 0, 1$ follows from the fact that $\mathrm{H}_* A_1$ is $\mathcal{A}(0)_*$– and $\mathcal{A}(1)_*$–cofree. The statement for $n = 2$ follows from the explicit structure of $\mathrm{Ext}^{s,t}_{\mathcal{A}(2)_*}(\mathrm{H}_* A_1)$. Now suppose $n \geq 3$. Set $\Gamma = \mathcal{A}(n)_* \,\square_{\mathcal{A}(2)_*} \mathbb{F}_2$ and note that Γ is an $\mathcal{A}(n)_*$–comodule algebra. The unit $\mathbb{F}_2 \to \mathcal{A}(n)_* \,\square_{\mathcal{A}(2)_*} \mathbb{F}_2$ is a map of Γ–comodules. Denote by $\overline{\Gamma}$ the quotient of Γ, so that we have the short exact sequence of $\mathcal{A}(n)_*$–comodules

$$0 \to \overline{\Gamma}^{\otimes r} \to \Gamma \otimes \overline{\Gamma}^{\otimes r} \to \overline{\Gamma}^{\otimes r+1} \to 0$$

for $r \geq 0$. Splicing these together, we get a long exact sequence of $\mathcal{A}(n)_*$–comodules

$$0 \to \mathbb{F}_2 \to \Gamma \to \Gamma \otimes \overline{\Gamma} \to \cdots \to \Gamma \otimes \overline{\Gamma}^{\otimes r} \to \cdots,$$

which gives rise to a spectral sequence converging to $\mathrm{Ext}^{*,*}_{\mathcal{A}(n)_*}(\mathrm{H}_* A_1)$ with E_1–term isomorphic to $\mathrm{Ext}^{*,*}_{\mathcal{A}(n)_*}(\Gamma \otimes \overline{\Gamma}^{\otimes r} \otimes \mathrm{H}_* A_1)$:

$$\mathrm{E}_1^{s,t,r} = \mathrm{Ext}^{s,t}_{\mathcal{A}(n)_*}(\Gamma \otimes \overline{\Gamma}^{\otimes r} \otimes \mathrm{H}_* A_1) \Rightarrow \mathrm{Ext}^{s+r,t}_{\mathcal{A}(n)_*}(\mathbb{F}_2, \mathrm{H}_* A_1). \tag{7}$$

By the change-of-rings isomorphism,

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(n)_*}(\Gamma \otimes \overline{\Gamma}^{\otimes r} \otimes \mathrm{H}_* A_1) \cong \mathrm{Ext}^{s,t}_{\mathcal{A}(2)_*}(\overline{\Gamma}^{\otimes r} \otimes \mathrm{H}_* A_1).$$

We see that $\overline{\Gamma}^{\otimes r}$ is $(8r-1)$–connected because $\overline{\Gamma}$ is 7–connected. Together with the fact that

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(2)_*}(\mathrm{H}_* A_1) = 0 \quad \text{if } t - s < f(s),$$

we obtain that

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(2)_*}(\overline{\Gamma}^{\otimes r} \otimes \mathrm{H}_* A_1) = 0$$

if $t - s < f(s) + 8r$ or equivalently if $t - (s+r) < f(s+r) + 2r$. We can now conclude by using the spectral sequence (7). $\square$

Proposition 9 (approximation lemma) *Let $m \geq n$ be two nonnegative integers, or $m = \infty$. The restriction homomorphism*

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(m)_*}(\mathrm{H}_* A_1) \to \mathrm{Ext}^{s,t}_{\mathcal{A}(n)_*}(\mathrm{H}_* A_1),$$

where $\mathcal{A}(\infty)_* := \mathcal{A}_*$, is an isomorphism if $t - s < f(s-1) + 2^{n+1} - 1$ and is an epimorphism if $t - s < f(s) + 2^{n+1}$, where $f(s)$ is as in Proposition 8.

This is a well-known consequence of the vanishing line property and is discussed in [13, Lemma 3.4.9]. We give a proof here for completeness.

Proof Let $\Gamma = \mathcal{A}(m)_* \square_{\mathcal{A}(n)_*} \mathbb{F}_2$ and $\overline{\Gamma} = \mathrm{coker}(\mathbb{F}_2 \to \Gamma)$. The restriction homomorphism is the composite

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(m)_*}(\mathrm{H}_* A_1) \to \mathrm{Ext}^{s,t}_{\mathcal{A}(m)_*}(\Gamma \otimes \mathrm{H}_* A_1) \cong \mathrm{Ext}^{s,t}_{\mathcal{A}(n)_*}(\mathrm{H}_* A_1),$$

where the first map is induced by the unit $\mathbb{F}_2 \to \Gamma$ and the second is the change-of-rings isomorphism. The short exact sequence of $\mathcal{A}(m)_*$–comodules $\mathbb{F}_2 \to \Gamma \to \overline{\Gamma}$ gives rise to a long exact sequence

$$\mathrm{Ext}^{s-1,t}_{\mathcal{A}(m)_*}(\overline{\Gamma} \otimes \mathrm{H}_* A_1) \to \mathrm{Ext}^{s,t}_{\mathcal{A}(m)_*}(\mathrm{H}_* A_1) \to \mathrm{Ext}^{s,t}_{\mathcal{A}(n)_*}(\mathrm{H}_* A_1) \to \mathrm{Ext}^{s,t}_{\mathcal{A}(m)_*}(\overline{\Gamma} \otimes \mathrm{H}_* A_1).$$

Since $\overline{\Gamma}$ is 2^{n+1}–connected and $\mathrm{Ext}^{s,t}_{\mathcal{A}(m)_*}(\mathbb{F}_2, \mathrm{H}_* A_1)$ has the vanishing line $t - s < f(s)$,

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(m)_*}(\overline{\Gamma} \otimes \mathrm{H}_* A_1) = 0$$

if $t - s < f(s) + 2^{n+1}$, hence the surjectivity of the respective restriction homomorphism, and

$$\mathrm{Ext}^{s-1,t}_{\mathcal{A}(m)_*}(\overline{\Gamma} \otimes \mathrm{H}_* A_1) = \mathrm{Ext}^{s,t}_{\mathcal{A}(m)_*}(\overline{\Gamma} \otimes \mathrm{H}_* A_1) = 0$$

if $t - s < f(s-1) + 2^{n+1} - 1$, hence the bijectivity of the respective restriction homomorphism. □

Corollary 10 *The restriction map $\mathrm{Ext}^{s,t}_{\mathcal{A}_*}(\mathrm{H}_* A_1) \to \mathrm{Ext}^{s,t}_{\mathcal{A}(2)_*}(\mathrm{H}_* A_1)$ is an epimorphism if $t - s < 5s + 8$ and is an isomorphism if $t - s < 5s + 2$.*

Proposition 11 *The restriction map $\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathrm{H}_*(A_1)) \to \mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1))$ is an epimorphism.*

Proof The restriction map $\mathrm{Res}\colon \mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathrm{H}_*(A_1)) \to \mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathrm{H}^*(A_1))$ is a map of modules over $\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathrm{H}_*(A_1 \wedge DA_1))$. This module structure comes from the fact that A_1 is a module over the ring spectrum $A_1 \wedge DA_1$. It is proved in [2, Lemma 3.6] that the class $w_2 \in \mathrm{Ext}^{8,56}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1 \wedge DA_1))$ lifts to $\mathrm{Ext}^{8,56}_{\mathcal{A}_*}(\mathrm{H}_*(A_1 \wedge DA_1))$ — the notation in that article for w_2 is $b^4_{3,0}$. In particular, the restriction map Res is a map of modules over the subalgebra R generated by g, ν and w_2. By Proposition 6, the classes $e_{s,t}$ where

$$(s,t) \in \big\{(0,0),(1,5),(1,6),(2,11),(3,15),(3,17),(4,21),(4,23),(6,30),\\ (6,32),(7,36),(7,38),(8,42),(9,47),(9,48),(10,53)\big\}$$

generate $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathrm{H}^*(A_1)$ as a module over R. These classes live in the region $\{t-s<5s+8\}$, hence lift to $\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathrm{H}^*(A_1)$ by Proposition 9. □

This theorem shows that all the classes of $M\cup N$ and $P\cup Q$ lift to $\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathrm{H}_*(A_1))$. In the following part, we prove that the latter lift to permanent cycles.

2.2 The topological tmf–Hurewicz homomorphism

The key step is to understand the action of $g\in \mathrm{Ext}^{4,24}_{\mathcal{A}_*}(\mathbb{F}_2)$ on $\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathrm{H}_*(A_1))$. To this end, we introduce the notion of weak divisibility, which plays an important role in the proof by induction of Proposition 14.

Definition 12 Let M be an A–module and $g\in A$ be a nonnilpotent element. An element $x\in M$ is said to be weakly g–divisible if and only if there exists an $n\in\mathbb{N}$ and $y\in M$ such that $g^{n+1}y=g^nx$. Otherwise stated, x is weakly g–divisible if and only if there exists a g–torsion element δ such that $x+\delta$ is g–divisible.

Proposition 13 *The group* $\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathrm{H}^*(A_1))$ *has the following properties*:

(i) *All classes of* $\mathrm{Ext}^{s,t}_{\mathcal{A}_*}(\mathrm{H}^*A_1)$ *in the region*

$$F=\{s\geq 18,5s\leq t-s\leq 5s+6\}\cup\{s\geq 27,5s\leq t-s\leq 5s+14\}$$

are g*–free and are divisible by* g.

(ii) *Any class* x *of* $\mathrm{Ext}^{s,t}_{\mathcal{A}_*}(\mathrm{H}^*A_1)$ *in the region*

$$D=\{s\geq 21,5s\leq t-s\leq 5s+12\}\cup\{s\geq 30,5s\leq t-s\leq 5s+20\}$$

is weakly g*–divisible*, *ie there is a class* y *and a nonnegative integer* n *such that* $g^{n+1}y=g^nx$.

Let us explain what we need this proposition for. We want to show that the classes of $M\cup N$ and $P\cup Q$ lift to permanent cycles in the ASS for A_1. While it is straightforward to show it for the classes of M and P by sparseness and the approximation lemma, the classes of N and Q might support more differentials. Proposition 13 describes the structure of $\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathrm{H}^*(A_1))$ exactly in the zone where the differentials on the classes

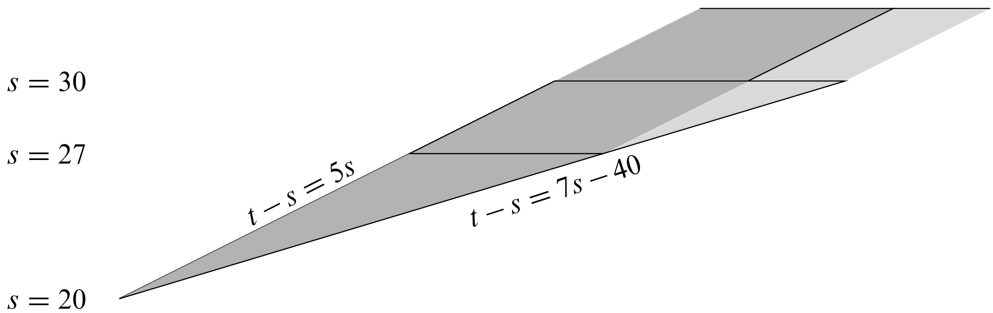

Figure 1: The dark region is associated to S_1 or R_1, and the light to S_2 or R_2.

of N and P arrive. In particular, we will see that the targets of these differentials are g–free, a fact which is established in Lemmas 20 and 21. This, together with the g–linearity of the differentials, allows us to rule out potential differentials. Because the classes involved in the statement of this proposition live in the region where there is an isomorphism $\mathrm{Ext}^{s,t}_{\mathcal{A}_*}(\mathrm{H}^*(A_1)) \cong \mathrm{Ext}^{s,t}_{\mathcal{A}(4)_*}(\mathrm{H}^*(A_1))$ by Proposition 9, it suffices to prove that $\mathrm{Ext}^{*,*}_{\mathcal{A}(4)_*}(\mathrm{H}^* A_1)$ has the required properties. We prove a stronger statement:

Proposition 14 *The group* $\mathrm{Ext}^{*,*}_{\mathcal{A}(4)_*}(\mathrm{H}^*(A_1))$ *has the following properties*:

(i) *All classes in the region*

$$S_1 = \{20 \le s \le 27, 5s \le t-s \le 7s-40\} \cup \{s \ge 27, 5s \le t-s \le 5s+14\}$$

are g–free and are divisible by g. All classes in the region

$$S_2 = \{27 \le s \le 30, 5s+14 \le t-s \le 7s-40\} \cup \{s \ge 30, 5s+14 \le t-s \le 5s+20\}$$

are weakly divisible by g.

(ii) *All classes in the region*

$$T_1 = \{15 \le s \le 18, 5s \le t-s \le 7s-30\} \cup \{s \ge 18, 5s \le t-s \le 5s+6\}$$

are g–free and are divisible by g. All classes in the region

$$T_2 = \{18 \le s \le 21, 5s+6 \le t-s \le 7s-30\} \cup \{s \ge 21, 5s+6 \le t-s \le 5s+12\}$$

are weakly divisible by g.

Before proving this theorem, let us explain the strategy of the proof. Observe that there is a sequence of extensions of commutative Hopf algebras

$$B_{i+1} \,\square_{B_i}\, \mathbb{F}_2 \to B_{i+1} \to B_i \quad \text{for } 0 \le i \le 8$$

in which each $B_{i+1}\,\square_{B_i}\,\mathbb{F}_2$ is isomorphic to an exterior algebra $\Lambda(h_i)$ on one generator h_i of degree at least 8, $B_0=\mathcal{A}(2)_*$ and $B_9=\mathcal{A}(4)_*$. We can then deduce information on $\mathrm{Ext}^{*,*}_{\mathcal{A}(4)_*}(\mathrm{H}_*A_1)$ from $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathrm{H}_*A_1)$ by a sequence of Davis–Mahowald spectral sequences

$$\mathrm{E}_1^{s,t,\sigma}=\bigoplus_{\sigma\geq 0}\mathrm{Ext}^{s,t-|h_i|\sigma}_{B_i}(\mathrm{H}_*A_1\otimes\mathbb{F}_2\{h_i^\sigma\})\Rightarrow \mathrm{Ext}^{s+\sigma,t}_{B_{i+1}}(\mathrm{H}_*A_1).$$

By the calculation of $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathrm{H}_*A_1)$, we see that the classes of $E_1^{s,t,\sigma}$ in the regions S_1 and S_2 have the desired properties. Using this as the base case, we prove by induction that each $\mathrm{Ext}^{s+\sigma,t}_{B_{i+1}}(\mathrm{H}_*A_1)$ has the desired properties. To this end, we first prove, by induction on r, that the E_r–term of the Davis–Mahowald spectral sequence has similar properties in the appropriate regions and then make sure that extensions cannot prevent the target of the spectral sequence from having the desired properties, where the fact that the degree of each h_i is at least 8 becomes crucial.

Proof We have that

$$\begin{aligned}\mathcal{A}(4)_*&=\mathbb{F}_2[\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5]/(\zeta_1^{32},\zeta_2^{16},\zeta_3^{8},\zeta_4^{4},\zeta_5^{2}),\\ \mathcal{A}(2)_*&=\mathbb{F}_2[\zeta_1,\zeta_2,\zeta_3]/(\zeta_1^{8},\zeta_2^{4},\zeta_3^{2}).\end{aligned}$$

From this, we can construct a sequence of maps of commutative Hopf algebras $(B_{i+1}\to B_i)$ with $0\leq i\leq 9$, $B_0=A(2)_*$ and $B_9=A(4)_*$ such that for each i, $B_{i+1}\,\square_{B_i}\,\mathbb{F}_2=\Lambda(h_i)$ is an exterior algebra on one generator h_i of degree at least 8. Informally, we start with $B_0=A(2)_*$ and successively join ζ_4, ζ_3^2, ζ_2^4, ζ_1^8, ζ_5, ζ_4^2, ζ_3^4, ζ_2^8s and ζ_1^{16}. Namely,

$$\begin{gathered}B_1=\mathbb{F}_2[\zeta_1,\zeta_2,\zeta_3,\zeta_4]/(\zeta_1^{8},\zeta_2^{4},\zeta_3^{2},\zeta_4^{2}),\quad B_2=\mathbb{F}_2[\zeta_1,\zeta_2,\zeta_3,\zeta_4]/(\zeta_1^{8},\zeta_2^{4},\zeta_3^{4},\zeta_4^{2}),\\ B_3=\mathbb{F}_2[\zeta_1,\zeta_2,\zeta_3,\zeta_4]/(\zeta_1^{8},\zeta_2^{8},\zeta_3^{4},\zeta_4^{2}),\quad B_4=\mathbb{F}_2[\zeta_1,\zeta_2,\zeta_3,\zeta_4]/(\zeta_1^{16},\zeta_2^{8},\zeta_3^{4},\zeta_4^{2}),\\ B_5=\mathbb{F}_2[\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5]/(\zeta_1^{16},\zeta_2^{8},\zeta_3^{4},\zeta_4^{2},\zeta_5^{2}),\\ B_6=\mathbb{F}_2[\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5]/(\zeta_1^{16},\zeta_2^{8},\zeta_3^{4},\zeta_4^{4},\zeta_5^{2}),\\ B_7=\mathbb{F}_2[\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5]/(\zeta_1^{16},\zeta_2^{8},\zeta_3^{8},\zeta_4^{4},\zeta_5^{2}),\\ B_8=\mathbb{F}_2[\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5]/(\zeta_1^{16},\zeta_2^{16},\zeta_3^{8},\zeta_4^{4},\zeta_5^{2}).\end{gathered}$$

We will prove by induction on i that $\mathrm{Ext}^{*,*}_{B_i}(\mathrm{H}_*A_1)$ has the property (i). The proof of (ii) works similarly; see Remark 15. First, we can directly check that

$$\mathrm{Ext}^{*,*}_{B_0}(\mathrm{H}_*A_1)=\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathrm{H}_*A_1)$$

satisfies (i) by inspecting its structure, which is shown in Proposition 6. Suppose that $\mathrm{Ext}_{B_i}^{*,*}(\mathrm{H}_*A_1)$ satisfies (i). Consider the Davis–Mahowald spectral sequence

$$\mathrm{E}_1^{s,t,\sigma}=\bigoplus_{\sigma\geq 0}\mathrm{Ext}_{B_i}^{s,t}(\mathrm{H}_*A_1\otimes\mathbb{F}_2\{h_i^\sigma\})\Rightarrow\mathrm{Ext}_{B_{i+1}}^{s+\sigma,t}(\mathrm{H}_*A_1),\tag{8}$$

and let the differential d_r go from $\mathrm{E}_r^{s,t,\sigma}$ to $\mathrm{E}_r^{s-r+1,t,\sigma+r}$. Since h_i is a B_i–primitive, we have that

$$\mathrm{E}_1^{s,t,\sigma}=\bigoplus_{\sigma\geq 0}\mathrm{Ext}_{B_i}^{s,t-d\sigma}(\mathrm{H}_*A_1)\otimes\mathbb{F}_2\{h_i^\sigma\},$$

where $d=|h_i|$. We will prove by induction on $r\geq 1$ that each $\mathrm{E}_r^{s,t,\sigma}$–term of the DMSS (8) has:

(a) All classes in the region

$$\begin{aligned}R_1=\{20\leq s+\sigma\leq 27,\,5(s+\sigma)\leq t-s-\sigma\leq 7(s+\sigma)-40\}\\ \cup\{s+\sigma\geq 27,\,5(s+\sigma)\leq t-s-\sigma\leq 5(s+\sigma)+14\}\end{aligned}$$

are g–free and are divisible by g.

(b) All classes in the region

$$\begin{aligned}R_2=\{27\leq s+\sigma\leq 30,\,5(s+\sigma)+14\leq t-s-\sigma\leq 7(s+\sigma)-40\}\\ \cup\{s+\sigma\geq 30,\,5(s+\sigma)+14\leq t-s-\sigma\leq 5(s+\sigma)+20\}\end{aligned}$$

are weakly divisible by g.

A similar proof to that of Proposition 8 shows that $\mathrm{Ext}_{B_i}^{s,t}(\mathrm{H}_*A_1)$ has the same vanishing line, and so

$$\begin{aligned}\mathrm{E}_1^{s,t,\sigma}=0\quad&\text{if } s+\sigma>6,\quad t-(s+\sigma)<5(s+\sigma),\\ &\text{or if } s+\sigma\leq 6,\quad t-(s+\sigma)<5(s+\sigma)-4.\end{aligned}\tag{9}$$

The $\mathrm{E}_1^{s,t,\sigma}$–term is spanned by classes $x\otimes h_i^\sigma$ with $x\in\mathrm{Ext}_{B_i}^{s,t-d\sigma}(\mathrm{H}_*A_1)$. For degree reasons ($d=|h_i|\geq 8$) and by (9), classes $x\otimes h_i^\sigma$ living in R_1 and R_2 are nontrivial only if x lies in S_1 and $S_1\cup S_2$, respectively. Then together with the induction hypothesis, it is straightforward to see that the E_1–term of the spectral sequence (8) has the properties (a) and (b). Suppose that the E_r–term of (8) has those properties. Let $x\in\mathrm{E}_r^{s,t,\sigma}$ represent a class $[x]$ of $\mathrm{E}_{r+1}^{s,t,\sigma}$.

Step 1 Suppose $[x]$ lives in R_1 and $[x]$ is g–torsion. Because R_1 is stable under multiplication by g, we can assume that $g[x]=0$. Then there exists $y\in\mathrm{E}_r^{s+4+r-1,t+24,\sigma-r}$

such that $d_r(y) = gx$. By an inspection on degrees, we see that y belongs to the region $R_1 \cup R_2$. By the induction hypothesis, y is weakly divisible by g, so there is a z and an integer n such that $g^{n+1}z = g^n y$. It follows that

$$g^{n+1}d_r(z) = d_r(g^{n+1}z) = d_r(g^n y) = g^n d_r(y) = g^n gx = g^{n+1}x.$$

However, $d_r(z) - x$ lies in R_1, which consists only of g–free classes, hence $d_r(z) = x$, and so $[x] = 0$. Therefore, all classes in R_1 of E_{r+1} are g–free.

Step 2 Suppose $[x]$ belongs to the region R_1. By the induction hypothesis, there exists $y \in \mathrm{E}_r^{s-4,t-24,\sigma}$ such that $gy = x$. We claim that y is a d_r–cycle. We have that

$$gd_r(y) = d_r(gy) = d_r(x) = 0.$$

Moreover, $d_r(y) \in E_r^{s-r-3,t-24,\sigma+r}$, which belongs to R_1, hence is g–free. We conclude that $d_r(y) = 0$. Thus, $[x]$ is divisible by g.

Steps 1 and 2 show that the E_{r+1}–term has the property (a).

Step 3 Now suppose that $[x]$ belongs to the region R_2. Then x is weakly divisible by g, so there is a class $z \in \mathrm{E}_r^{s-4,t-24,\sigma}$ and an integer n such that $g^{n+1}z = g^n x$. We claim that z is a d_r–cycle. Since x is a d_r–cycle, we have

$$g^{n+1}d_r(z) = d_r(g^{n+1}z) = d_r(g^n x) = g^n d_r(x) = 0.$$

Moreover, $d_r(z) \in \mathrm{E}_r^{s-4-r+1,t-24,\sigma+r}$ which belongs to R_1, hence $d_r(z)$ is g–free, and so $d_r(z) = 0$. Therefore, we obtain that $g^{n+1}[z] = g^n[x]$, hence the E_{r+1}–term has the property (b).

Step 4 It is now straightforward to see that the E_∞–term also has the properties (a) and (b). To finish the proof, we will show that the target of the spectral sequence (8) has the property (i). Let

$$\cdots \subset F_\sigma \subset F_{\sigma-1} \subset \cdots \subset F_1 \subset F_0 = \mathrm{Ext}_{B_{i+1}}^{*,*}(\mathbb{F}_2, \mathrm{H}_* A_1)$$

be the filtration of $\mathrm{Ext}_{B_{i+1}}^{*,*}(\mathrm{H}_* A_1)$ associated to the Davis–Mahowald spectral sequence. A class belongs to F_σ only if it is represented in the E_1–term by a class of the form $x \otimes h_i^\sigma$, where $x \in \mathrm{Ext}_{B_i}^{s,t}(\mathrm{H}_* A_1)$ — here $t - s \geq 5s - 4$ because of (9). Such a class has bidegree $(s+\sigma, t+d\sigma)$, and so has the topological degree $t - s + (d-1)\sigma$. Because $d \geq 8$, the latter exceeds $5(s+\sigma) + 20$ for σ sufficiently large. However, any class in $S_1 \cup S_2$ has bidegree (t, s) satisfying $t - s \leq 5s + 20$. This means that there is an integer m such that all classes of $S_1 \cup S_2$ belongs to $F_0 \backslash F_m \cup \{0\}$. From this, it is straightforward to verify that the properties (a) and (b) of the E_∞–term imply the property (i) of $\mathrm{Ext}^{*,*}_{B_{i+1}}(\mathbb{F}_2, H_* A_1)$. □

Remark 15 The proof of (i) uses a double induction. What makes both base cases work is the fact that $\mathrm{Ext}^{s,t}_{\mathcal{A}(2)_*}(H_* A_1)$ has the required properties and that the slope of each h_i is lower than $\frac{1}{7}$, which is exactly the slope of the lower line limiting the region in question. What makes the inductive step work is self-explained by the choice of the regions: relevant classes lie in the relevant regions. What makes the target of the DMSS have the required properties is that the slope of h_i is lower then the slope of the vanishing line. The regions T_1 and T_2 are chosen to have all of these features, hence the proof of (ii) is similar to that of (i).

We need the following lemma on the E_2–term of the Adams spectral sequence for S^0, which necessitates a calculation of the Ext group up to stem 43; see [14, Theorem 4.42].

Lemma 16 *The class ν is annihilated by g^2, so is g–torsion in the E_2–term of the ASS for S^0.*

Theorem 17 *The induced map in homotopy of the Hurewicz map $A_1 \to \mathrm{tmf} \wedge A_1$ is surjective.*

Proof The map $H_*: \pi_*(A_1) \to \pi_*(\mathrm{tmf} \wedge A_1)$ is a map of $\pi_*(A_1 \wedge DA_1)$–modules. In particular, it is a map of modules over the subalgebra R of $\pi_*(A_1 \wedge DA_1)$ generated by ν, $\bar{\kappa}$ and v_2^{32}. Therefore, we only need to prove that a set of generators of $\pi_*(\mathrm{tmf} \wedge A_1)$ as an R–module belongs to the image of H. Because of Proposition 11, we can choose lifts of classes of $M \cup N$ and $P \cup Q$ to $\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(H_* A_1)$ such that classes which are divisible by ν lift to classes which are divisible by ν. We fix such a choice of lifting and also call them M, N, P and Q. We will prove that all classes of $M \cup N$ and $P \cup Q$ are permanent cycles in the ASS for A_1; then they must survive to the E_∞–term because their images in the ASS for $\mathrm{tmf} \wedge A_1$ do. By comparing the bidegree of the classes of $M \cup N$ and $P \cup Q$ and the vanishing line of the E_2–term, we see that the differentials supported by the classes of $M \cup N$ and $P \cup Q$ of length greater than 4 are trivial. The theorem now follows from Propositions 19, 22, and 23 below. □

Remark 18 We draw particular attention to the choices made for the lifts of the classes of $M \cup N$ and $P \cup Q$ because we will use them in an essential way in the proof of Propositions 22 and 23. The fact that the classes which are divisible by ν are g–torsion follows from Lemma 16.

Proposition 19 *All classes in* M *of* $A_1[00]$ *and* $A_1[11]$ *and in* P *of* $A_1[01]$ *and* $A_1[10]$ *are permanent cycles in the respective ASS.*

Proof Inspection of bidegrees together with the vanishing line Proposition 8 show that there can only be nontrivial differentials d_2 on classes of M and P, and moreover these differentials hit the region where there is an isomorphism between $\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathrm{H}_*(A_1))$ and $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1))$. However, all classes of M and P are permanent cycles in the ASS for $\mathrm{tmf} \wedge A_1$. Therefore, the differentials d_2 on the classes in M and P in the ASS for A_1 are trivial. □

Lemma 20 (a) *The target groups for* d_3 *on classes in* N *are* g*–free. More precisely,* $\mathrm{E}_3^{s,t}$ *is* g*–free if*

$$(s,t) \in F_3 := \{s \geq 23, 5s \leq t-s \leq 5s+1\} \cup \{s \geq 28, 5s \leq t-s \leq 5s+9\}.$$

(b) *Suppose* $s \geq 30$ *and* $t-s \leq 5s+20$*, and let* $s \in \mathrm{E}_3^{s,t}$*. Then* x *is weakly divisible by* g*, ie there exists an integer* n *and a class* $y \in E_3^{s-4,t-24}$ *such that* $g^{n+1}y = g^n x$*.*

Proof (a) The differential d_2–arriving in $g\mathrm{E}_2^{s,t}$ with $(s,t) \in F_3$ starts in $\mathrm{E}_2^{s',t'}$ with

$$(s',t') = (s+2, t+23), \quad \text{and so} \quad (t'-s') = t-s+21.$$

Then we have

$$s' \geq 25, \quad 5s' = 5s+10 \leq t-s+10 \leq 5s+11 = 5s'+1,$$

and

$$s' \geq 30, \quad 5s' = 5s+10 \leq t-s+10 \leq 5s+19 = 5s'+9.$$

So (s',t') belongs to

$$\{s' \geq 25, 5s'+11 \leq t'-s' \leq 5s'+12\} \cup \{s' \geq 30, 5s'+11 \leq t'-s' \leq 5s'+20\}.$$

In this region, Proposition 13 guarantees that all classes are weakly divisible by g and this implies that $\mathrm{E}_3^{s,t}$ is g–free if $(s,t) \in F_3$. In fact, suppose x is a class which lies in F_3 and which is g–torsion. The region F_3 is stable under multiplication by g, so we can assume that $gx = 0$. Let a be a representative of x. Then there exists $b \in \mathrm{E}_2$ such that $d_2(b) = ga$. By the argument above, we know that a is weakly divisible by g, so there exists an integer $n \geq 0$ and a class c at the E_2–term such that $g^n a = g^{n+1}c$. Then we have

$$g^{n+1}d_2(c) = d_2(g^{n+1}c) = d_2(g^n b) = g^n d_2(b) = g^n ga = g^{n+1}a.$$

However, F_3 belongs to the region which is g–free at the E_2–term, hence $d_2(c) = a$, which means that $x = 0$ at the E_3–term.

(b) We represent x by a d_2–cycle a. By part (ii) of Proposition 13, there exists $b \in \mathrm{E}_2^{s-4,t-24}$ and an integer n such that $g^n a = g^{n+1} b$. It is enough to show that b is a d_2–cycle. We first note that, since a is a d_2–cycle, we have

$$g^{n+1} d_2(b) = d_2(g^{n+1} b) = d_2(g^n a) = 0.$$

Therefore it is enough to show that $d_2(b)$ is g–free. In fact, $d_2(b)$ is a class in $\mathrm{E}_2^{s',t'}$ with $s' = s - 2$ and

$$t' - s' = t - s - 19 \leq 5s + 20 - 19 = 5(s-2) + 11 = 5s' + 11,$$

so it is g–free by Proposition 13(ii) □

Lemma 21 *The target groups for the differential d_4 on classes in N are g–free. More precisely*, $\mathrm{E}_4^{s,t}$ *is g–free if*

$$(s,t) \in F_4 := \{s \geq 29, 5s \leq t - s \leq 5s + 4\}.$$

Proof The differential d_3 arriving in $g\mathrm{E}_3^{s,t}$ with $(s,t) \in F_4$ starts in $E_3^{s',t'}$ with

$$s' = s + 1, \quad t' - s' = t - s + 21.$$

Then we have

$$s' \geq 30, \quad 5s' + 16 \leq t' - s' \leq 5s' + 20.$$

By Lemma 20, all classes in such bidegrees are weakly divisible by g and this implies that $\mathrm{E}_4^{s,t}$ is g–free if $(s,t) \in F_4$. In fact, suppose x is a class which lies in F_4 and which is g–torsion. Because F_4 is stable under multiplication by g, we can assume that $gx = 0$. Let $a \in \mathrm{E}_3^{s,t}$ be a representative of x. Then there exists $b \in \mathrm{E}_3$ such that $d_3(b) = ga$. By the argument above, b is weakly divisible by g, so there is a nonnegative integer n and a class $c \in \mathrm{E}_3$ such that $g^{n+1} c = g^n b$. Then we have

$$g^{n+1} d_3(c) = d_3(g^{n+1} c) = d_3(g^n b) = g^n d_3(b) = g^n g a = g^{n+1} a.$$

However, F_4 belongs to the region where g acts freely at the E_3–term by Lemma 20(i), hence $d_3(c) = a$ and so $x = 0$ at the E_4–term. □

Proposition 22 *The differentials d_2, d_3 and d_4 on the classes in N for $A_1[00]$ and $A_1[11]$ are trivial.*

Proof All classes of N are divisible by ν, so are g–torsion in the E_2–term, hence are g–torsion at all terms. It is then enough to show that the target groups of differentials d_2, d_3 and d_4 on the classes in N are g–free at the E_2, E_3 and E_4–terms, respectively. In fact, the target groups for the differential d_2 on the classes in N lie in the region

$$\{s \geq 19, 5s \leq t-s \leq 5s+6\} \cup \{s \geq 27, 5s \leq t-s \leq 5s+14\},$$

consisting only of g–free classes, by Proposition 13(i). Next, a potential nontrivial differential d_3 or d_4 on the classes in N has its target in the region F_3 or F_4, respectively, which is g–free by Lemma 20 or Lemma 21, respectively. □

Proposition 23 *The differentials* d_2, d_3 *and* d_4 *on the classes in* Q *for* $A_1[10]$ *and* $A_1[01]$ *are trivial.*

Proof In this proof, A_1 denotes the self-dual versions $A_1[10]$ and $A_1[01]$. The same argument as in the proof of Proposition 22 shows that the differentials d_2, d_3 and d_4 on the classes in N which are divisible by ν are trivial. Consider the four other classes in N,

$$w_2^2 e_{0,0}, \quad w_2^2 e_{1,5}, \quad w_2^2 e_{3,15}, \quad w_2^2 e_{6,30}. \tag{10}$$

These classes are g–free at the E_2–term. We now show that their g–multiple towers are truncated by differentials d_2 in the ASS for A_1. In fact, by [12, Theorem 4.0.3], the following differentials d_2 happen in the ASS for $\mathrm{tmf} \wedge A_1$:

$$d_2(w_2 e_{10,53}) = g^5 e_{0,0}, \quad d_2(w_2 e_{7,38}) = g^4 e_{1,5},$$
$$d_2(w_2 e_{9,48}) = g^4 e_{3,15}, \quad d_2(w_2 e_{4,23}) = g^2 e_{6,30}.$$

Since the targets of these differentials live on the line $t-s = 5s$, where the E_2–term of the ASS for A_1 and that for $\mathrm{tmf} \wedge A_1$ are isomorphic, the same differentials happen in the ASS for A_1. Moreover, the class w_2^2 is a cycle for the differential d_2 in the ASS for $A_1 \wedge DA_1$ as shown the proof of [2, Lemma 3.10]; see the remark below. Therefore, by the Leibniz rule, the following differentials d_2 happen in the ASS for A_1:

$$d_2(w_2^3 e_{10,53}) = g^5 w_2^2 e_{0,0}, \quad d_2(w_2^3 e_{7,38}) = g^4 w_2^2 e_{1,5},$$
$$d_2(w_2^3 e_{8,48}) = g^4 w_2^2 e_{3,15}, \quad d_2(w_2^3 e_{4,23}) = g^2 w_2^2 e_{6,30}.$$

It follows that the targets of the differentials d_2 on the classes of (10) are g–torsion. Moreover, the differentials d_2 arrive in $E_2^{s,t}$ with $s \geq 18$ and $5s \leq t-s \leq 5s+6$, which consists only of g–free classes by Proposition 13. Thus, the classes of (10) are d_2–cycles and become g–torsions in the E_3–term.

The differentials d_3 on the classes of (10) arrive in $\mathrm{E}_3^{s,t}$ with $s \geq 19$ and $t-s = 5s$. For these bidegrees, there is an isomorphism at the E_2–term of the ASS for A_1 and that for $\mathrm{tmf} \wedge A_1$; in particular, the related Ext–groups are isomorphic to $\mathbb{F}_2$ and are generated by $g^4 e_{15}$, $g^5 e_0$, $g^4 e_{30}$ and $g^5 e_5$. However, in the ASS for $\mathrm{tmf} \wedge A_1$, by [12, Theorem 4.0.3], these are hit by differentials d_2:

$$d_2(w_2 e_{10,53}) = g^5 e_0, \qquad d_2(g w_2 e_{7,38}) = g^5 e_{1,5},$$
$$d_2(w_2 e_{9,48}) = g^4 e_{15}, \quad d_2(g^2 w_2 e_{4,23}) = g^4 e_{6,30}.$$

Because of Proposition 11 and the naturality of the ASS, $\mathrm{E}_3^{s,t} = 0$ for $s \geq 19$ and $t-s = 5s$ in the ASS for A_1. Thus, the differentials d_3 on the classes of (10) are trivial.

Finally, the differential d_4 on the classes of (10) land above the vanishing line, hence are trivial. □

Remark 24 In the proof of [2, Lemma 3.10], it is implicit that w_2^2 — which is $b_{3,0}^8$ in their notation — is a d_2–cycle. This amounts to showing that w_2 commutes with the class R in the statement of their Lemma 3.10. This class R is detected in a positive tmf–filtration of the algebraic tmf spectral sequence converging to $\mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathrm{H}_*(A_1 \wedge DA_1))$. More precisely, R is detected in the group

$$\bigoplus_{\substack{n \geq 1 \\ i_1,\dots,i_n \geq 1}} \mathrm{Ext}_{\mathcal{A}(2)}^{10-n,57-8(i_1+i_2+\cdots+i_n)}(\mathrm{H}_*(A_1 \wedge DA_1) \wedge bo_{i_1} \wedge bo_{i_2} \wedge \cdots \wedge bo_{i_n});$$

see [2] for the notation. By considering the vanishing line of $\mathrm{Ext}_{\mathcal{A}(2)}^{*,*}(\mathrm{H}_*(A_1))$, the only potential nontrivial summand is $\mathrm{Ext}_{\mathcal{A}(2)}^{9,57-8i_1}(\mathrm{H}_*(A_1 \wedge DA_1) \wedge bo_{i_1})$ with $i_1 \geq 1$. Then by using Robert Bruner's ext software [4] one can check that this group is trivial. This means that R is in fact trivial.

Remark 25 To illustrate the proof of Proposition 23, we give some examples and more details.

(a) First, a differential d_3 or d_4 on the first five classes in N listed in Table 2 has target living above the vanishing line, so it is trivial.

(b) The other classes in N might support nontrivial differentials d_3. For example, a differential d_3 on the class $\nu w_2^2 e_{17}$ arrives in $\mathrm{E}_3^{s,t}$ with $s = 23$ and $t-s = 115 = 5s$; $\mathrm{E}_3^{s,t}$ is g–free by Lemma 20. The worst case is the class $\nu w_2^3 e_{23}$ on which a differential d_3 lives in $\mathrm{E}_3^{s,t}$ with $s = 32$ and $t-s = 169 = 5s+9$, which is g–free by Lemma 20.

(c) Only the last eight classes in N as listed in Table 2 might support nontrivial differentials d_4. These classes lie in $\mathrm{E}_4^{s,t}$ with $s \geq 25$ and $5s \leq t-s \leq 5s+25$. Then d_4 on these arrives in $\mathrm{E}_4^{s',t'}$ with $s' = s+4$ and $t' = t+3$, and so

$$s' \geq 29 \quad \text{and} \quad 5s' \leq t'-s' \leq 5s'+4.$$

This region consists only of g–free classes by Lemma 21

2.3 The edge homomorphism of the topological duality spectral sequence

In this last section, we prove Theorem B. Let us restate it here.

Theorem 26 *The edge homomorphism of the topological duality spectral sequence*

$$\pi_*(E_C^{h\mathbb{S}_C^1} \wedge A_1) \to \pi_*(E_C^{hG_{24}} \wedge A_1)$$

is surjective. Therefore, all differentials starting from the 0*–line of the topological duality spectral sequence are trivial.*

Here, E_C is the Lubin–Tate spectrum associated to the formal completion F_C of the supersingular elliptic curve C given by $y^2 + y = x^3$ defined over $\mathbb{F}_4$. Let $\mathbb{S}_C$ be the automorphism group of $\mathbb{S}_C$. The group $\mathbb{S}_C^1$ is defined to be the kernel of the reduced determinant map $\mathbb{S}_C \to \mathbb{Z}_2$. The automorphism group of C is isomorphic to $G_{24} := Q_8 \rtimes C_3$, where Q_8 is the quaternion group and C_3 is the cyclic group of order 3, and G_{24} naturally embeds into $\mathbb{S}_C^1$. We refer the reader to [3] for the construction of the topological duality resolution and to [12] for more motivations on the study of this spectral sequence for A_1.

Proof The duality spectral sequence has four lines and converges to $\pi_*(E_C^{h\mathbb{S}_C^1} \wedge A_1)$. Its edge homomorphism $E_C^{h\mathbb{S}_C^1} \to E_C^{hG_{24}}$ is induced by the inclusion of subgroup $G_{24} \to \mathbb{S}_C^1$. It suffices to prove that the map

$$\pi_*(E_C^{h\mathbb{S}_C} \wedge A_1) \to \pi_*(E_C^{hG_{24}} \wedge A_1)$$

induced by the inclusion of the subgroup $G_{24} \to \mathbb{S}_C$ is surjective. Let Gal denote the Galois group of $\mathbb{F}_4$ over $\mathbb{F}_2$. It acts on $\mathbb{S}_C$ and G_{24}, because C is already defined over $\mathbb{F}_2$. By [3, Lemma 1.37], there are homotopy equivalences

$$\mathrm{Gal}_+ \wedge E_C^{h(\mathbb{S}_C \rtimes \mathrm{Gal})} \simeq E_C^{h\mathbb{S}_C}$$

and

$$\mathrm{Gal}_+ \wedge E_C^{h(G_{24} \rtimes \mathrm{Gal})} \simeq E_C^{hG_{24}}$$

that fit into a commutative diagram

$$\begin{tikzcd}
\mathrm{Gal}_+ \wedge E_C^{h(\mathbb{S}_C \rtimes \mathrm{Gal})} \arrow[r] \arrow[d] & \mathrm{Gal}_+ \wedge E_C^{h(G_{24} \rtimes \mathrm{Gal})} \arrow[d] \\
E_C^{h\mathbb{S}_C} \arrow[r] & E_C^{hG_{24}}
\end{tikzcd}$$

where horizontal maps are induced by respective inclusions of subgroups. By Devinatz and Hopkins [7],

$$L_{K(2)} S^0 \simeq E_C^{h(\mathbb{S}_C \rtimes \mathrm{Gal})},$$

and so the map $E_C^{h(\mathbb{S}_C \rtimes \mathrm{Gal})} \to E_C^{h(G_{24} \rtimes \mathrm{Gal})}$ is identified with $L_{K(2)} S^0 \to E_C^{h(G_{24} \rtimes \mathrm{Gal})}$, the unit map. By [8], the latter factorizes through the homotopy equivalence

$$L_{K(2)} \mathrm{tmf} \simeq E_C^{h(G_{24} \rtimes \mathrm{Gal})}.$$

Therefore, it is enough to show that the map $L_{K(2)} A_1 \to L_{K(2)}(\mathrm{tmf} \wedge A_1)$ induces a surjection on homotopy. In fact, it fits in the commutative diagram:

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d] & \mathrm{tmf} \wedge A_1 \arrow[d] \\
{[(v_2^{32})^{-1}]} A_1 \arrow[r] \arrow[d] & {[(v_2^{32})^{-1}]}(\mathrm{tmf} \wedge A_1) \arrow[d] \\
L_{K(2)} A_1 \arrow[r] & L_{K(2)}(\mathrm{tmf} \wedge A_1)
\end{tikzcd} \tag{11}$$

By [12, Theorem 5.1.1], the natural map $[(\Delta^8)^{-1}](\mathrm{tmf} \wedge A_1) \to L_{K(2)}(\mathrm{tmf} \wedge A_1)$ is a homotopy equivalence. In addition, by the proof of Proposition 7, v_2^{32} is equal to Δ^8 up to some power. Therefore, the map $[(v_2^{32})^{-1}](\mathrm{tmf} \wedge A_1) \to L_{K(2)}(\mathrm{tmf} \wedge A_1)$ in (11) is a homotopy equivalence. On the other hand, the induced map in homotopy of the middle map of (11) is identified with a direct limit of (1):

$$(v_2^{32})^{-1} \pi_*(A_1) \to (v_2^{32})^{-1} \pi_*(\mathrm{tmf} \wedge A_1).$$

Hence it is a surjection, because (1) is. Therefore, the composite

$$[(v_2^{32})^{-1}] A_1 \to [(v_2^{32})^{-1}](\mathrm{tmf} \wedge A_1) \to L_{K(2)}(\mathrm{tmf} \wedge A_1)$$

induces a surjection in homotopy, hence so does the map

$$L_{K(2)} A_1 \to L_{K(2)}(\mathrm{tmf} \wedge A_1),$$

because of (11). □

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