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Oriented and unitary equivariant bordism of surfaces

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Fix a finite group G. We study $\Omega_2^{\mathrm{SO},G}$ and $\Omega_2^{U,G}$, the unitary and oriented bordism groups of smooth G–equivariant compact surfaces, respectively, and we calculate them explicitly. Their ranks are determined by the possible representations around fixed points, while their torsion subgroups are isomorphic to the direct sum of the Bogomolov multipliers of the Weyl groups of representatives of conjugacy classes of all subgroups of G. We present an alternative proof of the fact that surfaces with free actions which induce nontrivial elements in the Bogomolov multiplier of the group cannot equivariantly bound. This result permits us to show that the 2–dimensional SK–groups (Schneiden und Kleben, or "cut and paste") of the classifying spaces of a finite group can be understood in terms of the bordism group of free equivariant surfaces modulo the ones that bound arbitrary actions.

55N22, 57R75, 57R77, 57R85

1 Introduction

Equivariant bordism groups have been a subject of ongoing research since the 1960s. Conner, Floyd, Landweber, Stong, Smith and tom Dieck, among others, laid the foundations for the extraordinary homology and cohomology theories obtained from equivariant bordism, and found many interesting properties of these groups. Given a finite group G, a particularly important problem is the explicit calculation of the oriented and complex G–equivariant bordism groups of a point, since they provide the coefficients for the theories. This turns out to be a complicated task.

Explicit calculations of the equivariant bordism groups for finite abelian groups (see Landweber [19], Ossa [26] and Stong [34]) led some to expect that, at least in the unitary case, equivariant bordism groups are always a free module over the unitary bordism ring for any finite group G; see Rowlett [28, page 1], May [21, Chapter XXVIII.5] and Greenlees and May [12, Conjecture 1.2]. This belief was confirmed for general abelian groups (see Löffler [20] and [21, Chapter XXVIII, Theorem 5.1]) and for metacyclic groups [28], and therefore it was conjectured that for any finite group this was the case. This conjecture remained dormant for some years and it was recalled Uribe in his 2018 ICM Lecture [35], where he named it "the evenness conjecture in equivariant unitary bordism".

When the evenness conjecture holds true for a group G, it implies that the G–equivariant unitary bordism ring is torsion-free. In particular, any unitary manifold with a free action of a finite group that generates a torsion class in the unitary bordism group of free actions would bound equivariantly. This has always been the first step for proving the evenness conjecture, namely, to construct explicit equivariant manifolds whose boundaries are the desired generators of the equivariant unitary bordism groups of free actions.

In the case of surfaces, the evenness conjecture would imply that all oriented surfaces with orientation-preserving free actions bound equivariantly (note that if an oriented surface with orientation-preserving free action does not bound equivariantly, then the class of the difference of this surface with G–times the quotient surface induces a nontrivial torsion class in the reduced G–equivariant unitary bordism group). Domínguez and Segovia [9] showed that indeed this is the case for abelian, dihedral, symmetric and alternating groups. Nevertheless, it fails to be true in general. It has been recently shown that there is an obstruction class for an oriented surface with an orientation-preserving free action to bound equivariantly (see Samperton [29; 30]), and this obstruction class lies in the Bogomolov multiplier of the group; see Bogomolov [3] and Kunyavskiĭ [18]. The Bogomolov multiplier of a finite group consists of the classes of the Schur multiplier $H^2(G,\mathbb{C}^*)$ that vanish once restricted to any abelian subgroup; the homological version of the Bogomolov multiplier is the quotient of the second integral homology of the group by the classes generated by 2–dimensional tori; see Moravec [22]. This result implies that indeed there are torsion classes in the equivariant unitary bordism groups, and therefore that the evenness conjecture in equivariant unitary bordism is false in general. The evenness conjecture might then be restated instead as a classification question, namely *which finite groups satisfy the evenness conjecture in equivariant unitary bordism*?

We focus on the calculation of the oriented and the unitary G–equivariant bordism groups for compact surfaces. We use the fixed-point construction methods developed by Rowlett [27] to determine the rank of the equivariant bordism groups, and then use the explicit generators of the equivariant bordism groups for adjacent families in dimension 3 in order to determine which equivariant surfaces bound. In Theorem 4.3 we present a generalization to all finite groups of the result shown by Samperton in [29] which states that the obstruction class for equivariantly bounding an oriented surface with free action is the element in the Bogomolov multiplier of the group that the surface defines. The Conner–Floyd spectral sequence will then allow us to determine the torsion group in the equivariant bordism group of surfaces. Our main result is:

Theorem 4.4 *Let G be a finite group and $\mathrm{Tor}_{\mathbb{Z}}(\Omega_2^G)$ the torsion subgroup of the unitary or oriented G–equivariant bordism of surfaces Ω_2^G. Then there is a canonical isomorphism*

$$\bigoplus_{(K)} \widetilde{B}_0(W_K) \cong \mathrm{Tor}_{\mathbb{Z}}(\Omega_2^G),$$

where (K) runs over all conjugacy classes of subgroups of G, $W_K = N_G K/K$ and $\widetilde{B}_0(W_K)$ is the homology version of the Bogomolov multiplier of the group W_K.

With the torsion group in hand, we describe explicitly in Theorem 4.5 the G–equivariant bordism groups of surfaces, unitary and oriented.

Since there are infinitely many groups with nontrivial Bogomolov multipliers, we conclude that there are infinitely many groups which do not satisfy the evenness conjecture in equivariant unitary bordism. On the other hand, there are also infinitely many groups G whose G–equivariant unitary bordism group of surfaces is a free abelian group, thus implying that these groups may still satisfy the evenness conjecture for equivariant unitary bordism.

We use our previous calculations to interpret which equivariant surfaces bound in terms of the SK–relation (cutting and pasting from the German *Schneiden und Kleben*). The study of invariants under cutting and pasting started with the characterization by Jänich [15; 14] of invariants with the additive properties of the Euler characteristic and the signature, and it was further developed with the introduction of the SK–groups of a space by Karras, Kreck, Neumann and Ossa [17]. The SK–groups of a space can be understood as the groups of equivalence classes of manifolds with continuous maps to the space subject to the equivalence relation given by cutting and pasting. The 2–dimensional SK–groups of BG can be understood in terms of cutting and pasting surfaces with free G–actions. The SK–groups of BG were studied in [17] and were identified by Neumann in [24, Theorem 2] with the second integral homology group of BG modulo the toral classes (as far as we know this is the first reference where the homological Bogomolov multiplier appears).

We conclude with the study of two explicit groups, of order 64 and 243, whose Bogomolov multipliers are nontrivial. We sketch why both groups possess nontrivial Bogomolov multipliers and give explicit homomorphisms from the fundamental group of a genus-2 surface to both groups that define the desired surfaces with free actions that do not bound equivariantly. These constructions allow us to give explicit generators for the torsion subgroup of the equivariant unitary bordism groups for both groups.

2 Preliminaries

2.1 Equivariant bordism

Let G be a finite group and consider compact manifolds endowed with smooth actions of the group G preserving either the orientation or the unitary (tangentially stable almost complex) structure.

Recall that a tangentially stable almost complex G–structure over the G–manifold M consists of a G–equivariant complex vector bundle ξ over M such that $TM \oplus \mathbb{R}^k \cong \xi$ as G–equivariant real vector bundles and k is some natural number; here G acts trivially on the stabilized part $\mathbb{R}^k$. Two tangentially stable almost complex structures are identified if they become isomorphic as complex vector bundles after stabilization with further G–trivial $\mathbb{C}$ summands.

With this definition at hand, if K is a subgroup of G, then the fixed-point set M^K is endowed with a canonical tangential stable almost complex W_K–structure with $W_K := N_G K/K$. This follows from the isomorphism of W_K–equivariant real bundles

$$\xi^K \cong (TM \oplus \mathbb{R}^k)^K \cong (TM|_{M^K})^K \oplus \mathbb{R}^k = T(M^K) \oplus \mathbb{R}^k \tag{1}$$

and the fact that ξ^K becomes a W_K–equivariant complex vector bundle over M^K.

Now, as $N_G K$–equivariant real vector bundles, we have the isomorphism

$$\xi|_{M^K} \cong TM|_{M^K} \oplus \mathbb{R}^k \cong T(M^K) \oplus \nu(M^K, M) \oplus \mathbb{R}^k \cong \xi^K \oplus \nu(M^K, M), \tag{2}$$

where $\nu(M^K, M)$ denotes the normal bundle of the embedding $M^K \hookrightarrow M$. Since both $\xi|_{M^K}$ and ξ^K are $N_G K$–equivariant complex vector bundles over M^K, the normal bundle $\nu(M^K, M)$ is naturally endowed with the structure of an $N_G K$–equivariant complex vector bundle. The fact that the normal bundles of the fixed points M^K are endowed with complex structures plays an important role in the study of tangentially stable almost complex G–structures.

Tangentially stable almost complex G–structures are also called G–equivariant unitary structures, and the equivalence classes of manifolds under the bordism relation in the realm of G–equivariant unitary structures is called the G–equivariant unitary bordism group.

Following the notation of Stong [34], denote by Ω_*^G either the bordism ring $\Omega_*^{\mathrm{SO},G}$ of G–equivariant oriented manifolds or the bordism ring $\Omega_*^{U,G}$ of G–equivariant unitary (tangentially stable almost complex) manifolds. Whenever the upper script SO or U is not specified, it means that the construction and results apply to both homology theories.

For the explicit definitions of both unitary and oriented equivariant bordism rings see [34, Section 2], and for the properties of the tangentially stable almost complex manifolds defining the unitary equivariant bordism groups, including the ones presented above, see [21, XXVIII, Section 3; 13, Section 2; 2, Section 5].

2.2 Equivariant bordism for families

The study of the equivariant bordism groups led Conner and Floyd to restrict their attention to manifolds with prescribed isotropy groups [4; 5]. The allowed isotropy groups are therefore organized in families of subgroups of G which are closed under conjugation and under taking subgroups. For any such family of subgroups $\mathcal{F}$ there is a classifying G–space $E\mathcal{F}$ for actions whose isotropy groups lie on $\mathcal{F}$. This G–space is characterized by its properties on fixed-point sets, namely, the fixed-point set $E\mathcal{F}^H$ is contractible whenever $H \in \mathcal{F}$ and empty otherwise. The construction of $E\mathcal{F}$ can be carried out in such a way that an inclusion of families $\mathcal{F}' \subset \mathcal{F}$ induces a G–cofibration $E\mathcal{F}' \to E\mathcal{F}$ [8, Section 1.6].

The equivariant bordism groups $\Omega_*^G\{\mathcal{F}, \mathcal{F}'\}$ for a pair of families $\mathcal{F}' \subset \mathcal{F}$ are the bordism groups of G–equivariant compact manifolds with boundary $(M, \partial M)$ such that the isotropy groups of M lie in $\mathcal{F}$ and the isotropy groups of its boundary ∂M lie in $\mathcal{F}'$. Following [7, page 310] one may define the bordism of groups for a pair of G–spaces (X, A) and a pair of families by

$$\Omega_*^G\{\mathcal{F}, \mathcal{F}'\}(X, A) := \Omega_*^G(X \times E\mathcal{F}, X \times E\mathcal{F}' \cup A \times E\mathcal{F}), \tag{3}$$

or, equivalently, using a more geometrical description [34].

2.3 Long exact sequence for families

Whenever three families are related by the inclusions $\mathcal{F}'' \subset \mathcal{F}' \subset \mathcal{F}$ there is induced a long exact sequence in bordism [5, Theorem 5.1]

$$\cdots \to \Omega_*^G\{\mathcal{F}', \mathcal{F}''\} \to \Omega_*^G\{\mathcal{F}, \mathcal{F}''\} \to \Omega_*^G\{\mathcal{F}, \mathcal{F}'\} \xrightarrow{\partial} \Omega_{*-1}^G\{\mathcal{F}', \mathcal{F}''\} \to \cdots . \tag{4}$$

2.4 Conner–Floyd spectral sequence

More generally, associated to the families $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k = \mathcal{F}$ there is a spectral sequence converging to $\Omega_n^G\{\mathcal{F}\}$, whose filtration is

$$F_p\Omega_n^G\{\mathcal{F}\} := \operatorname{Im}(\Omega_n^G\{\mathcal{F}_p\} \to \Omega_n^G\{\mathcal{F}\}). \tag{5}$$

This spectral sequence is usually called the *Conner–Floyd spectral sequence*, its first page is given by

$$E^1_{p,q} \cong \Omega^G_{p+q}\{\mathcal{F}_p, \mathcal{F}_{p-1}\}, \tag{6}$$

and the differentials are induced by the boundary maps. The first page of this spectral sequence might be difficult to calculate, but whenever the pair of families $\mathcal{F}_{p-1} \subset \mathcal{F}_p$ are adjacent (see below for the definition), fixed-point methods together with the classification of the normal bundles can make them computable in terms of nonequivariant bordism groups.

2.5 Equivariant bordism for adjacent families

A pair of families $\mathcal{F}' \subset \mathcal{F}$ are called *adjacent* whenever they differ by the conjugacy class (K) of a subgroup K, in other words $\mathcal{F} - \mathcal{F}' = (K)$. A manifold $(M, \partial M)$ in $\Omega_n^G\{\mathcal{F}, \mathcal{F}'\}$ is cobordant to the G–equivariant tubular neighborhood of the fixed-point set of all the subgroups of G conjugate to K (all isotropy groups in the complement of the tubular neighborhood belong to $\mathcal{F}'$; the explicit bordism can be found in [5, Lemma 5.2]). The fixed points M^K of K become a free $W_K := N_G K/K$ space and the G–equivariant tubular neighborhood can be reconstructed from a specific W_K–equivariant twisted bundle over M^K by extending the $N_G K$ space to a G space. Hence, if M^K is of dimension $n-k$ and M^K/W_K is connected, its tubular neighborhood can be recovered from a map $M^K \to C_{N_G K,K}(k)$ where $C_{N_G K,K}(k)$ is a W_K–space which classifies the $N_G K$–equivariant tubular neighborhoods of rank k around K–fixed points [35, (2.5)]. In the unitary case there is a decomposition in terms of nonequivariant unitary bordism groups [35, Theorem 2.8]

$$\Omega_n^{U,G}\{\mathcal{F}, \mathcal{F}'\} := \bigoplus_{2k \le n} \Omega^U_{n-2k}(C_{N_G K,K}(k) \times_{W_K} EW_K), \tag{7}$$

and a similar one in the case of oriented bordisms [1, Theorem 2.11] This localization theorem will become very useful in what follows once we apply it for the study of the equivariant bordism groups of surfaces.

2.6 G–fixed points

For every subgroup K of G denote by $\mathcal{A}K$ the family of all subgroups of K and its conjugates in G, and denote by $\mathcal{P}K$ the family $\mathcal{A}K-(K)$ of all *proper* subgroups of K, and its conjugates in G. The localization map at the fixed points of the whole group action

$$\Omega_*^G \to \Omega_*^G\{\mathcal{A}G, \mathcal{P}G\} \tag{8}$$

together with the decomposition into nonequivariant bordisms groups presented in (7), has been a powerful tool for determining the equivariant bordism groups for abelian groups (see for instance [6; 20; 13]). In this particular case, the bordism groups $\Omega_*^G\{\mathcal{A}G, \mathcal{P}G\}$ are isomorphic to the nonequivariant bordism groups of products of complex Grassmannians in the unitary case, and of products of real, complex and quaternionic Grassmannians in the oriented case.

2.7 Rowlett spectral sequence

We still need another spectral sequence suited for understanding the equivariant bordism groups of pairs of families. This spectral sequence was constructed by Rowlett in [27, Proposition 2.1], whence its name. Consider a pair of families $\mathcal{F}' \subset \mathcal{F}$ that are also families of subgroups of the normal subgroup K of G and $(M, \partial M)$ in $\Omega_n^G\{\mathcal{F}, \mathcal{F}'\}$. Then it is easy to see that the classifying map $M/K \to EW_K$ of the free $W_K = G/K$ action of the quotient induces an isomorphism of bordism groups $\Omega_*^G\{\mathcal{F}, \mathcal{F}'\} \xrightarrow{\cong} \Omega_*^G\{\mathcal{F}, \mathcal{F}'\}(EW_K)$ by mapping M to the composition $M \to M/K \to EW_K$; the inverse is simply induced by the map $EW_K \to *$. The space EW_K can be constructed as a CW–complex whose n–skeleton $(EW_K)^n$ is constructed from $(EW_K)^{n-1}$ by attaching a finite number of copies of $W_K \times B^n$ with W_K acting trivially on the n–dimensional balls. One may filter $\Omega_*^G\{\mathcal{F}, \mathcal{F}'\}(EW_K)$ by the images under the inclusion of the skeletons $\Omega_*^G\{\mathcal{F}, \mathcal{F}'\}((EW_K)^n)$, and therefore one obtains a spectral sequence converging to $\Omega_*^G\{\mathcal{F}, \mathcal{F}'\}$ whose first page becomes

$$E^1_{p,q} \cong \Omega^G_{p+q}\{\mathcal{F}, \mathcal{F}'\}((EW_K)^p, (EW_K)^{p-1}) \cong H_p((EW_K)^p, (EW_K)^{p-1}) \otimes_{W_K} \Omega_q^K\{\mathcal{F}, \mathcal{F}'\}, \tag{9}$$

and whose second page is

$$E^2_{p,q} \cong H_p(W_K, \Omega_q^K\{\mathcal{F}, \mathcal{F}'\}), \tag{10}$$

where the action of an element of W_K on a K–manifold M consists of the same manifold M endowed with the conjugate K–action. The zeroth column consists of the W_K–coinvariants

$$E^2_{0,q} \cong (\Omega_q^K\{\mathcal{F}, \mathcal{F}'\})_{W_K}, \tag{11}$$

and the edge homomorphism

$$\Omega_q^K\{\mathcal{F}, \mathcal{F}'\} \cong E^1_{0,q} \to E^2_{0,q} \to E^\infty_{0,q} \to \Omega_q^G\{\mathcal{F}, \mathcal{F}'\} \tag{12}$$

is simply the extension homomorphism factorizing through the coinvariants

$$\Omega_q^K\{\mathcal{F}, \mathcal{F}'\} \to (\Omega_q^K\{\mathcal{F}, \mathcal{F}'\})_{W_K} \to \Omega_q^G\{\mathcal{F}, \mathcal{F}'\} \quad \text{given by } M \mapsto M \times_K G. \tag{13}$$

In characteristic zero the spectral sequence collapses on the zeroth column of the second page. Since in characteristic zero the invariants and the coinvariants are isomorphic, we conclude that the extension homomorphism induces an isomorphism

$$\Omega_*^K\{\mathcal{F},\mathcal{F}'\}^{W_K}\otimes\mathbb{Q}\xrightarrow{\cong}\Omega_*^G\{\mathcal{F},\mathcal{F}'\}\otimes\mathbb{Q}. \tag{14}$$

In order to find the torsion classes in Ω_*^G we will construct the inverse map of the isomorphism (14) for every pair of adjacent families of groups. This map will be simply given by the localization at fixed points and will be the subject of the next section.

3 Localization at fixed points

For every subgroup K of G let us define the fixed-point homomorphism

$$f_K\circ r_K^G\colon\Omega_*^G\to\Omega_*^K\{\mathcal{A}K,\mathcal{P}K\} \tag{15}$$

as the composition of the restriction homomorphism $r_K^G\colon\Omega_*^G\to\Omega_*^K$ with the localization at K–fixed points

$$f_K\colon\Omega_*^K\to\Omega_*^K\{\mathcal{A}K,\mathcal{P}K\}. \tag{16}$$

The composition $f_K\circ r_K^G$ takes a G–manifold and maps it to the tubular neighborhood N of the K–invariant points M^K. Since on the complement of N in M there are no points with isotropy K, the tubular neighborhood N and M become cobordant in $\Omega_*^K\{\mathcal{A}K,\mathcal{P}K\}$ [5, Lemma 5.2]. Since N_GK acts on the normal bundle N of M^K, the localization at K–fixed points lands in the W_K–fixed submodule. Therefore the fixed-point homomorphism becomes

$$f_K\circ r_K^G\colon\Omega_*^G\to\Omega_*^K\{\mathcal{A}K,\mathcal{P}K\}^{W_K}. \tag{17}$$

Also, for every pair of families of subgroups in G, we have the localized fixed-point homomorphism

$$\phi_*\colon\Omega_*^G\{\mathcal{F},\mathcal{F}'\}\to\bigoplus_{(K)\subset\mathcal{F}-\mathcal{F}'}\Omega_*^K\{\mathcal{A}K,\mathcal{P}K\}^{W_K}. \tag{18}$$

This homomorphism applied to the pair of adjacent families $\{\mathcal{A}K,\mathcal{P}K\}$, composed with the edge homomorphism of the Rowlett spectral sequence (13), gives us the maps

$$\Omega_*^K\{\mathcal{A}K,\mathcal{P}K\}_{W_A}\to\Omega_*^G\{\mathcal{A}K,\mathcal{P}K\}\xrightarrow{\phi}\Omega_*^K\{\mathcal{A}K,\mathcal{P}K\}^{W_A}. \tag{19}$$

In characteristic zero, this composition is an isomorphism and therefore we obtain the isomorphism

$$\phi_*\colon\Omega_*^G\{\mathcal{A}K,\mathcal{P}K\}\otimes\mathbb{Q}\xrightarrow{\cong}\Omega_*^K\{\mathcal{A}K,\mathcal{P}K\}^{W_A}\otimes\mathbb{Q}, \tag{20}$$

which becomes the inverse of the map in (14) for adjacent families.

Applying the Conner–Floyd spectral sequence, we see that the fixed-point homomorphism (18) in characteristic zero becomes an isomorphism, and therefore we quote:

Theorem 3.1 [27, Theorem 1.1] *The fixed-point homomorphism in characteristic zero is an isomorphism*

$$\phi_*\otimes\mathbb{Q}\colon\Omega_*^G\otimes\mathbb{Q}\xrightarrow{\cong}\bigoplus_{(K)}\Omega_*^K\{\mathcal{A}K,\mathcal{P}K\}^{W_K}\otimes\mathbb{Q}. \tag{21}$$

We would like to remark that the rational isomorphism obtained in Theorem 3.1 by localizing on fixed points holds in general for any rational G–equivariant homology theory whose coefficients form a rational G–Mackey functor [11, Theorem A.16; 32, Corollary 3.4.28].

3.1 Kernel of fixed-point homomorphism

In the unitary case, the equivariant bordism group $\Omega_*^{U,K}\{\mathcal{A}K, \mathcal{P}K\}$ is isomorphic to the unitary bordism group of a disjoint union of products of complex Grassmannians [35, Theorem 2.8]. Therefore, the group $\Omega_*^{U,K}\{\mathcal{A}K, \mathcal{P}K\}$ is a free Ω_*^U–module on even-dimensional generators. Hence, by Theorem 3.1, we obtain the following result:

Lemma 3.2 *The group of torsion elements in $\Omega_*^{U,G}$ is isomorphic to the kernel of the fixed-point homomorphism ϕ of* (18):

$$\mathrm{Tor}_{\mathbb{Z}}(\Omega_*^{U,G}) = \mathrm{Ker}(\phi_*^U). \tag{22}$$

Whenever a group G satisfies the evenness conjecture in equivariant unitary bordism, the fixed-point homomorphism ϕ_*^U is automatically a monomorphism. This is the case for abelian [20] and metacyclic [28] groups. In the next section we will show that there are groups G such that the kernel of the fixed-point homomorphism is not trivial in dimension 2, thus defining torsion elements in $\Omega_2^{U,G}$. This fact refutes the evenness conjecture in the general case.

In the oriented case there are many torsion classes in the bordism ring Ω_*^{SO}, all of order 2 [36; 33]. Therefore we will be mainly interested in the torsion classes of the equivariant bordism group $\Omega_*^{\mathrm{SO},G}$ which are trivial under the fixed-point homomorphism ϕ_*^{SO}.

A very interesting and more general question associated to the equivariant oriented case is the following:

> *Are there G–equivariant oriented manifolds whose bordism class vanishes under the fixed-point homomorphism ϕ^{SO} which do not bound equivariantly?*

In the next section we answer this question for dimension 2. The 3–dimensional case (with its interesting application to Chern–Simons theory) remains open for the interested reader.

Note that the equivariant bordism group $\Omega_*^{\mathrm{SO},K}\{\mathcal{A}K, \mathcal{P}K\}$ is in general more difficult to calculate than the unitary one. On the one hand, the fixed-point set M^K need not be orientable, and on the other, the normal bundles are classified by products of real, complex and quaternionic Grassmannians.

Since we are mainly interested in the 2– and 3–dimensional bordism groups, we know that all fixed points are of real codimension 0 or 2 in the unitary case because the normal bundles are endowed with a complex structure, see (2), and 0, 2 or 3 in the oriented case, because there are no 1–dimensional real representations preserving the orientation. Here the real codimension of the fixed points matches the real dimension of the representation of the respective isotropy group.

In the case that the fixed points are of real codimension 2, the normal bundle is of complex dimension 1 in the unitary case and of real dimension 2 in the oriented case. Since the 2–dimensional oriented

representations can be parametrized by the 1–dimensional complex representations, we may denote by $\mathrm{Irr}^1_{\mathbb{C}}(K)$ the set of 1–dimensional nontrivial irreducible complex representations of the group K. The complex conjugation map on $\mathrm{Irr}^1_{\mathbb{C}}(K)$ acts freely on the representations of complex type $\mathrm{Irr}^1_{\mathbb{C}}(K)_{\mathbb{C}}$ and acts trivially on the representations of real type $\mathrm{Irr}^1_{\mathbb{C}}(K)_{\mathbb{R}}$. Denote by $\mathrm{Irr}^1_{\mathbb{C}}(K)_{\mathbb{C}}/\mathrm{conj}$ the quotient of representations of complex type by complex conjugation and by $\mathrm{Irr}^3_{\mathbb{R},\mathrm{SO}}(K)$ the set of 3–dimensional irreducible real representations of K in the category of oriented representations.

Proposition 3.3 *Let K be a finite group. Then the relative oriented equivariant bordism groups are*

$$\Omega_2^{\mathrm{SO},K}\{\mathcal{A}K,\mathcal{P}K\} = \Bigg(\bigoplus_{\mathrm{Irr}^1_{\mathbb{C}}(K)_{\mathbb{C}}/\mathrm{conj}} \mathbb{Z}\Bigg) \oplus \Bigg(\bigoplus_{\mathrm{Irr}^1_{\mathbb{C}}(K)_{\mathbb{R}}} \mathbb{Z}/2\Bigg), \tag{23}$$

$$\Omega_3^{\mathrm{SO},K}\{\mathcal{A}K,\mathcal{P}K\} = \bigoplus_{\mathrm{Irr}^3_{\mathbb{R},\mathrm{SO}}(K)} \mathbb{Z}/2, \tag{24}$$

and the relative equivariant unitary bordism groups are

$$\Omega_2^{U,K}\{\mathcal{A}K,\mathcal{P}K\} = \Omega_2^U \oplus \bigoplus_{\mathrm{Irr}^1_{\mathbb{C}}(K)} \mathbb{Z}, \tag{25}$$

$$\Omega_3^{U,K}\{\mathcal{A}K,\mathcal{P}K\} = 0. \tag{26}$$

Proof Let us begin with the relative oriented equivariant bordism groups. Any manifold M in $\Omega_*^{\mathrm{SO}.K}\{\mathcal{A}K,\mathcal{P}K\}$ is equivalent in the bordism group to the normal bundle N around the fixed-point set M^K [5, Lemma 5.2]. Whenever M is connected, of dimension 2 and $M \neq M^K$, this normal bundle is classified by a map

$$M^K \to \bigsqcup_{\mathrm{Irr}^1_{\mathbb{C}}(K)} BU(1), \tag{27}$$

where the K action on the bundle around the point is encoded by the irreducible representation (here we are using that $\mathrm{SO}(2) \cong U(1)$). Note that whenever V is a nontrivial 1–dimensional complex representation, the unit ball $B(\mathbb{R} \oplus V)$ bounds the union of $B(V)$ and $B(\overline{V})$, where $\overline{V}$ denotes the representation V with reverse orientation. This implies that in the relative oriented bordism group $\Omega_2^{K,\mathrm{SO}}\{\mathcal{A}K,\mathcal{P}K\}$ we have the equation $B(V) + B(\overline{V}) = 0$. Hence whenever V is of complex type, and therefore V is not isomorphic to $\overline{V}$, the relative oriented bordism group $\Omega_2^{K,\mathrm{SO}}\{\mathcal{A}K,\mathcal{P}K\}$ counts the difference between the number of K–fixed points with normal bundle isomorphic to $\overline{V}$ and the number of K–fixed points with normal bundle isomorphic to V; these are the integral invariants. If V is of real type, and hence V is isomorphic to $\overline{V}$, the ball $B(\mathbb{R} \oplus V)$ bounds $B(V)$ twice, and the relative oriented bordism group $\Omega_2^{K,\mathrm{SO}}\{\mathcal{A}K,\mathcal{P}K\}$ counts the parity of the number of points with normal bundle isomorphic to V; these are the $\mathbb{Z}/2$ invariants. This argument proves (23).

For the 3–dimensional case, the codimension-2 fixed points become circles, and since $\Omega_1^{\mathrm{SO}}(BU(1)) = 0$, we conclude that we only need to focus our attention on the isolated points of the K action. Around each isolated fixed point of the action we obtain a 3–dimensional real and oriented representation V

of K. This representation is irreducible in the category of oriented representations even though it may be not irreducible as a real representation. Note that the splitting of the representation as the product of two nonoriented representations implies that one must be a sign representation and the other must factor through a dihedral representation in $O(2)$. Hence the product of these two representations will be equivalent to a representation that factors through an oriented dihedral representation in $\mathrm{SO}(3)$ which is irreducible in the category of oriented representations. Now, the unit ball $B(\mathbb{R}\oplus V)$ bounds $B(V)$ twice because V and $\overline{V}$ are isomorphic. Therefore we can conclude that the isomorphism (24) counts the parity of the number of fixed points of K with the prescribed representation on its normal bundle.

The relative unitary bordism groups are much simpler. The 3–dimensional case (26) is trivial because both Ω_3^U and $\Omega_1^U(BU(1))$ are trivial. The 2–dimensional case (25) detects half of the first Chern number of the surface whenever the action is trivial, and it counts the number of fixed points with prescribed representation on their normal bundle. Here we are using that the isomorphism $\Omega_2^U \xrightarrow{\cong} \mathbb{Z}$ is given by the assignment $[\Sigma]\mapsto \frac{1}{2}c_1(\Sigma)$ where $c_1(\Sigma)$ is the first Chern number of the surface. □

As a consequence of the previous result, the 2–dimensional bordism classes of interest have no isolated fixed points for any subgroup K of G.

Corollary 3.4 *The torsion subgroups of both unitary and oriented equivariant bordism of surfaces are respectively isomorphic to the kernels of the associated fixed-point homomorphism*,

$$\mathrm{Tor}_{\mathbb{Z}}(\Omega_2^{U,G}) = \mathrm{Ker}(\phi_2^U) \quad \textit{and} \quad \mathrm{Tor}_{\mathbb{Z}}(\Omega_2^{\mathrm{SO},G}) = \mathrm{Ker}(\phi_2^{\mathrm{SO}}). \tag{28}$$

Therefore the equivariant bordism groups $\mathrm{Ker}(\phi_2^{\mathrm{SO}})$ *and* $\mathrm{Ker}(\phi_2^U)$ *are generated by G–surfaces without isolated K–fixed points for any subgroup K of G*; *in the unitary case it is moreover required that the surfaces have trivial first Chern number.*

Proof Proposition 3.3 shows that the relative oriented and unitary bordism groups $\Omega_2^K\{\mathcal{A}K,\mathcal{P}K\}$ are torsion-free for all subgroups K of G, except in the oriented case whenever K has 1–dimensional complex representations of real type; such representations come from nontrivial elements in $\mathrm{Hom}(K,\mathbb{Z}/2)$. Whenever a closed oriented surface Σ has one K–fixed point whose normal bundle has the structure of a nontrivial element in $\mathrm{Hom}(K,\mathbb{Z}/2)$, the connected component of such a K–fixed point has an induced action of $\mathbb{Z}/2$. Since the Euler characteristic of the connected component is even, the number of fixed points of this $\mathbb{Z}/2$–action must also be even. Hence the original action of K on this connected component must have an even number of fixed points, and all of them will have isomorphic complex representation of real type on the normal bundles.

The previous argument shows that the image of the fixed-point homomorphism is torsion-free in both oriented and unitary cases. Therefore by Theorem 3.1, we can conclude that the torsion classes are generated by G–equivariant manifolds without isolated K–fixed points for any subgroup K of G, and in the unitary case it is furthermore required that the underlying surface has trivial first Chern number. □

The presence of the platonic groups A_4, $\mathfrak{S}_4$, A_5 or the dihedral groups D_{2k} as subgroups of a general group G makes the understanding of the bordism group $\Omega_3^{\mathrm{SO},G}$ more interesting. We need first a definition:

Definition 3.5 Let M be a G–manifold (oriented or unitary). Define the *ramification locus* of the G–action as the space

$$\overline{M} := \bigcup_{\substack{K\subset G\\ K\neq\{1\}}} M^K, \tag{29}$$

where M^K denotes the space of fixed points of the subgroup K.

Let us start with the dihedral groups:

Proposition 3.6 *The equivariant bordism groups $\Omega_3^{\mathrm{SO},D_{2k}}$ are generated by equivariant manifolds whose fixed points are all of codimension 0 or empty. In particular, the fixed-point homomorphism ϕ_3^{SO} is trivial.*

Proof Take M a closed oriented D_{2k}–equivariant manifold such that M/D_{2k} is connected. Let us first assume that no element in D_{2k} besides the identity acts trivially (we could always take the induced action on M of the group D_{2k}/L, where L is the subgroup that acts trivially and consider M as a D_{2k}/L–equivariant manifold). Hence the ramification locus $\overline{M}$ is the union of 1–dimensional and 0–dimensional manifolds.

Whenever the fixed-point set $M^{D_{2k}}$ is nonempty, it will consist of a finite number of isolated points. We will argue that the number of fixed points with isomorphic normal representations is even, thus implying that the image of the localization map (16) at D_{2k}–fixed points

$$f_{D_{2k}}\colon \Omega_3^{\mathrm{SO},D_{2k}} \to \Omega_3^{\mathrm{SO},D_{2k}}\{\mathcal{A}D_{2k}, \mathcal{P}D_{2k}\} \tag{30}$$

is trivial, and moreover that the fixed-point set $M^{D_{2k}}$ could be removed with an equivariant cobordism by attaching handles around pairs of fixed points with isomorphic normal representation.

If x belongs to $M^{D_{2k}}$, we claim that there is another fixed point $x'\in M^{D_{2k}}$, such that both have isomorphic representations of D_{2k} on their normal neighborhoods. The reason for this is the following. Consider the class $[x]\in\overline{M}/D_{2k}$ on the quotient of the ramification locus $\overline{M}$. The connected component of the fixed-point set of the cyclic subgroup $\mathbb{Z}/k$ around x defines a path on the quotient $\overline{M}/D_{2k}$ with the class $[x]$ at one end. Since $\overline{M}/D_{2k}$ is compact, the other end of this path ends at the class of the point $[x']$, where we have chosen x' to be on the same connected component as x on the fixed-point set $M^{\mathbb{Z}/k}$. The D_{2k} representations around x and x' are isomorphic because their restrictions to the group $\mathbb{Z}/k$ give representations with opposite orientations.

Note that whenever $k>2$, the points x and x' are different. When $k=2$ it could be the case that $x=x'$, and if this were the case, around $[x]$ in $\overline{M}/D_{2k}$ we would have a loop (the path we defined above from x to $x'=x$) and an extra path leaving from it. Following this third path from x, we will reach another point x'', which will be different from x.

We just have shown that the fixed points in $M^{D_{2k}}$ come in pairs with isomorphic representations. If the isomorphic representation is V and $B(V)$ denotes the unit ball in V, this pair of points could removed by the bordism that adds the handle $[0,1]\times B(V)$ on the normal neighborhoods of the pair of points.

The previous construction could be carried out on all the fixed points of the conjugacy classes of subgroups which are of dihedral type, and therefore we see that M is equivariantly cobordant to a manifold M' whose fixed points of its dihedral subgroups are empty. Hence the ramification locus $\overline{M'}$ is a 1–dimensional manifold, and therefore $\phi_3^{\mathrm{SO}}([M]) = \phi_3^{\mathrm{SO}}([M']) = 0$.

We could then choose as generators of $\Omega_3^{\mathrm{SO},D_{2k}}$ manifolds without 0– and 1–dimensional fixed points. □

Propositions 3.3 and 3.6 imply that the fixed-point homomorphism ϕ_3^{SO} is trivial on subgroups isomorphic to cyclic or dihedral groups. Nevertheless, the fixed-point homomorphism may be nontrivial when evaluated on subgroups isomorphic to the platonic groups A_4, $\mathfrak{S}_4$ and A_5. To understand the image of ϕ_3^{SO} for the platonic groups, we first need to define the *blowup* of a representation.

Definition 3.7 Let V be a finite-dimensional real G–representation. The *blowup* $\gamma(V)$ of V is the total space of the bundle of real lines $\mathbb{P}(V)$ of V,

$$\gamma(V) := \{(v, L) \in V \times \mathbb{P}(V) \mid v \in L\}, \tag{31}$$

endowed with the natural G action: $g\cdot(v,L) := (gv, gL)$. Denote by $B(\gamma(V))$ and $S(\gamma(V))$ the unit ball and sphere bundles of $\gamma(V)$, respectively.

Note that the sphere bundle of $\gamma(V)$ and the sphere of the representation $S(V)$ are canonically isomorphic:

$$\rho\colon S(V) \xrightarrow{\cong} S(\gamma(V)), \quad v \mapsto (v, \langle v\rangle). \tag{32}$$

So one may glue $B(\overline{V})$, where $\overline{V}$ is V with the opposite orientation, to $B(\gamma(V))$ along their boundary,

$$Y(V) := B(\overline{V}) \cup_\rho B(\gamma(V)), \tag{33}$$

thus constructing a closed oriented G–manifold.

What is interesting about the blowup is that, for faithful 3–dimensional oriented real representations V, the blowup $\gamma(V)$ only contains points with cyclic or dihedral isotropy groups. This is a key fact that will be used in what follows.

Proposition 3.8 *Let G be a finite subgroup of* $\mathrm{SO}(3)$. *Then the fixed-point homomorphism ϕ_3^{SO} is only nontrivial on subgroups isomorphic to the platonic groups A_4, $\mathfrak{S}_4$ and A_5. Moreover, its restriction*

$$\phi_3^{\mathrm{SO}}\colon \Omega_3^{\mathrm{SO},G} \to \bigoplus_{\substack{(K)\\ K \text{ platonic}}} \Omega_3^{\mathrm{SO},K}\{\mathcal{A}K, \mathcal{P}K\}^{W_K} \tag{34}$$

is surjective.

Proof Let (K) be a conjugacy class of subgroups of G with K isomorphic to any of the platonic groups A_4, $\mathfrak{S}_4$ or A_5. Denote by V_K the 3–dimensional real representation induced by the symmetries

of the respective platonic solid. Note that V_K is isomorphic to the representation with the reverse orientation $\overline{V_K}$, and therefore the closed oriented K–manifold $Y(V_K)$ defined in (33) is diffeomorphic to $B(V_K) \cup_\rho B(\gamma(V_K))$. Note furthermore that $\Omega_3^K\{\mathcal{A}K, \mathcal{P}K\} \cong \mathbb{Z}/2$ since V_K is the only irreducible representation of dimension 3.

The localization map at K–fixed points of (16)

$$f_K \colon \Omega_3^{\mathrm{SO},K} \to \Omega_3^K\{\mathcal{A}K, \mathcal{P}K\} \cong \mathbb{Z}/2 \tag{35}$$

maps $Y(V_K)$ to the normal bundle of its K–fixed points $Y(V_K)^K$. Since the blowup $\gamma(V_K)$ has no K–fixed points, $Y(V_K)^K = B(V_K)^K$ and the fixed-point set consists of only one point. Hence $f_K(Y(V_K)) = B(V_K)$ with $[B(V_K)]$ the generator of the group $\Omega_3^{\mathrm{SO},K}\{\mathcal{A}K, \mathcal{P}K\}$.

The commutativity of the diagram

$$\begin{array}{ccc} \Omega_3^{\mathrm{SO},K} & \xrightarrow{\quad i_K^G \quad} & \Omega_3^{\mathrm{SO},G} \\ \Big\downarrow{\scriptstyle f_K} & & \Big\downarrow{\scriptstyle f_K \circ r_K^G} \\ \Omega_3^{\mathrm{SO},K}\{\mathcal{A}K, \mathcal{P}K\} & \xrightarrow[i_K^{N_G K}]{} & \Omega_3^{\mathrm{SO},K}\{\mathcal{A}K, \mathcal{P}K\}^{W_K} \end{array} \tag{36}$$

where $i_H^L \colon \Omega_*^H \to \Omega_*^L$ given by $[M] \mapsto [L \times_H M]$ is the induction map for the inclusion of groups $H \subset L$, implies that the manifold $f_K \circ r_K^G(G \times_K Y(V_K))$ generates the group $\Omega_3^{\mathrm{SO},K}\{\mathcal{A}K, \mathcal{P}K\}^{W_K}$.

Note that whenever $K \subsetneqq K'$, we have $(G \times_K Y(V_K))^{K'} = \varnothing$. Therefore we conclude that the images under ϕ_3^{SO} of the G–manifolds $G \times_K Y(V_K)$, where (K) runs over the conjugacy classes of platonic subgroups of G, provide the desired surjectivity. □

Let us see the previous result in an example. Let $G = A_5$ and take the A_5–manifolds $Y(V_{A_5})$ and $A_5 \times_{A_4} Y(V_{A_4})$ in $\Omega_3^{\mathrm{SO},A_5}$. The images under ϕ_3^{SO} of these two manifolds in

$$\Omega_3^{\mathrm{SO},A_5}\{\mathcal{A}A_5, \mathcal{P}A_5\} \oplus \Omega_3^{\mathrm{SO},A_4}\{\mathcal{A}A_4, \mathcal{P}A_4\} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \tag{37}$$

are $(1,1)$ and $(0,1)$, respectively. The surjectivity of (34) follows.

3.2 Surfaces without isolated fixed points for any subgroup

Let $\mathcal{F}$ be a family of subgroups in G. Then denote by $\overline{\Omega}_2^G\{\mathcal{F}\}$ the subgroup of $\Omega_2^G\{\mathcal{F}\}$ generated by manifolds without isolated K–fixed points for all $K \in \mathcal{F}$, and whose underlying first Chern number is zero in the unitary case. Since Corollary 3.4 also implies that

$$\overline{\Omega}_2^G\{\mathcal{F}\} = \mathrm{Ker}(\phi_2|_{\Omega_2^G\{\mathcal{F}\}}) = \mathrm{Tor}_{\mathbb{Z}}(\Omega_2^G\{\mathcal{F}\}), \tag{38}$$

we may study the properties of $\overline{\Omega}_2^G$ restricted to families.

Lemma 3.9 *Let $\{\mathcal{F}, \mathcal{F}'\}$ be an adjacent pair of families differing by the conjugacy class (K) of the subgroup $K \subset G$. Then the canonical map of bordism groups for families $\overline{\Omega}_2^G\{\mathcal{F}'\} \to \overline{\Omega}_2^G\{\mathcal{F}\}$ fits into the split exact sequence*

$$\overline{\Omega}_2^G\{\mathcal{F}'\} \to \overline{\Omega}_2^G\{\mathcal{F}\} \to \widetilde{\Omega}_2(BW_K) \to 0, \tag{39}$$

with $\widetilde{\Omega}_2$ the reduced bordism groups.

Proof A generator in $\overline{\Omega}_2^G\{\mathcal{F}\}$ not in the image of $\overline{\Omega}_2^G\{\mathcal{F}'\}$ is represented by a G–connected manifold M such that the fixed-point set M^K is a closed nonempty surface without boundary, and such that there is a G–equivariant homomorphism $G \times_{N_G K} M^K \xrightarrow{\cong} M$ given by $[(g, m)] \mapsto gm$. The closed surface M^K is endowed with a free action of the group W_K, thus producing a unique map up to homotopy $M^K/W_K \to BW_K$. The induction map

$$\widetilde{\Omega}_2(BW_K) \to \overline{\Omega}_2^G\{\mathcal{F}\} \quad \text{given by } L \mapsto G \times_{N_G K} L \tag{40}$$

produces the desired section.

For the unitary case we need only to see that the first Chern number of M is zero, if and only if the first Chern number of M^K is zero, if and only if the first Chern number of M^K/W_K is zero.

Here we have used the isomorphism

$$\widetilde{\Omega}_2^U(BW_K) \cong \operatorname{Ker}(\Omega_2^U(BW_K) \to \Omega_2^U), \tag{41}$$

where the forgetful map $\Omega_2^U(BW_K) \to \Omega_2^U$ simply takes a framed bordism $[\Sigma \to BW_K]$ and maps it to $[\Sigma]$. The kernel consists of framed surfaces whose underlying first Chern number is zero. In the oriented case $\widetilde{\Omega}_2^{\mathrm{SO}}(BW_K) = \Omega_2^{\mathrm{SO}}(BW_K)$. □

4 Bounding equivariant surfaces

In this section we present our main result, which is the calculation of the groups $\overline{\Omega}_2^G$. To do this we use the Conner–Floyd spectral sequence of Section 2.4 associated to the families of subgroups

$$\{1\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_l = \mathcal{A}G, \tag{42}$$

where all the pairs are adjacent, ie $\mathcal{F}_j - \mathcal{F}_{j-1} = (K_j)$ for some conjugacy class of subgroups (K_j), and such that the conjugacy classes (K_j) span all conjugacy classes of subgroups of G (and hence $l+1$ is the number of conjugacy classes of subgroups of G).

We may filter the group $\overline{\Omega}_2^G$ by the subgroups

$$F_p\overline{\Omega}_2^G := \operatorname{Im}(\overline{\Omega}_2^G\{\mathcal{F}_p\} \to \overline{\Omega}_2^G) \tag{43}$$

whose associated graded groups are the quotients

$$\operatorname{Gr}_p \overline{\Omega}_2^G = F_p\overline{\Omega}_2^G / F_{p-1}\overline{\Omega}_2^G. \tag{44}$$

The commutative diagram with exact rows

$$\begin{array}{ccccc} \Omega_3^G\{\mathcal{A}G, \mathcal{F}_{p-1}\} & \longrightarrow & \overline{\Omega}_2^G\{\mathcal{F}_{p-1}\} & \longrightarrow & \overline{\Omega}_2^G \\ \downarrow & & \downarrow & & \| \\ \Omega_3^G\{\mathcal{A}G, \mathcal{F}_p\} & \longrightarrow & \overline{\Omega}_2^G\{\mathcal{F}_p\} & \longrightarrow & \overline{\Omega}_2^G \end{array} \tag{45}$$

together with the result of Lemma 3.9 implies that the following sequence is exact:

$$\Omega_3^G\{\mathcal{A}G, \mathcal{F}_p\} \xrightarrow{\partial} \widetilde{\Omega}_2(BW_{K_p}) \to \mathrm{Gr}_p\, \overline{\Omega}_2^G \to 0. \tag{46}$$

We therefore need to understand the image of the boundary map

$$\Omega_3^G\{\mathcal{A}G, \mathcal{F}_p\} \xrightarrow{\partial} \widetilde{\Omega}_2(BW_{K_p}) \tag{47}$$

in order to determine the groups $\mathrm{Gr}_p\, \overline{\Omega}_2^G$.

Note that the image of the boundary map (47) is equivalent to the image of the boundary map

$$\Omega_3^{W_{K_p}}\{\mathcal{A}W_{K_p}, \{1\}\} \xrightarrow{\partial} \widetilde{\Omega}_2(BW_{K_p}). \tag{48}$$

This follows from the fact that the manifolds of interest will have trivial actions of the groups in the conjugacy class (K_p), and then one follows the same argument as presented in Lemma 3.9. Therefore we obtain the following result:

Lemma 4.1 *Consider the associated graded groups* $\mathrm{Gr}_*\, \overline{\Omega}_2^G$ *of* $\overline{\Omega}_2^G$ *induced by the families of subgroups presented in* (42)*. Then*

$$\mathrm{Gr}_p\, \overline{\Omega}_2^G \cong \mathrm{Coker}(\Omega_3^{W_{K_p}}\{\mathcal{A}W_{K_p}, \{1\}\} \xrightarrow{\partial} \widetilde{\Omega}_2(BW_{K_p})). \tag{49}$$

Hence we need to understand which surfaces with free actions equivariantly bound.

4.1 Bounding free actions on surfaces

It turns out that the only free actions on surfaces that equivariantly bound are those on which the quotient surface is a torus. This result is originally due to the second author [29; 30] whenever the group G does not contain any subgroup isomorphic to the platonic groups $A_4, \mathfrak{S}_4, A_5$ or to the dihedral groups D_{2k}, and it motivated our investigation. Here we will produce an alternative proof, generalizing it for all finite groups. Let us first recall the definition of the Bogomolov multiplier of a finite group.

The cohomology group $H^2(G, \mathbb{C}^*)$ determines the isomorphism classes of central $\mathbb{C}^*$ group extensions of G, and therefore complex irreducible projective representations of the group G define elements in $H^2(G, \mathbb{C}^*)$. Schur [31] extensively studied this cohomology group, and therefore it was called the *Schur multiplier* of G [16].

Bogomolov [3] defined the subgroup $B_0(G)$ of the Schur multiplier consisting of all elements which vanish when restricted to all its abelian subgroups:

$$B_0(G) = \bigcap_{A \subset G \text{ abelian}} \mathrm{Ker}(\mathrm{res}_A^G : H^2(G, \mathbb{C}^*) \to H^2(A, \mathbb{C}^*)). \tag{50}$$

The interest in this group comes from, among other things, a result Bogomolov [3, Theorem 3.1] which states that whenever the field of G–invariants $\mathbb{C}[G]^G$ of the rational field $\mathbb{C}[G]$ is rational over $\mathbb{C}$, the Bogomolov multiplier of the group G vanishes.

Using the fact that for finite groups $H^2(G,\mathbb{C}^*)\cong H_2(G,\mathbb{Z})$, Moravec [22] showed that the Bogomolov multiplier group $B_0(G)$ is isomorphic to the group

$$\widetilde{B}_0(G) := H_2(G,\mathbb{Z})/M_0(G), \tag{51}$$

where $M_0(G)$ is the subgroup of $H_2(G,\mathbb{Z})$ generated by the images

$$\mathrm{Im}(H_2(\mathbb{Z}\times\mathbb{Z},\mathbb{Z})\to H_2(G,\mathbb{Z})) \tag{52}$$

of all homomorphisms $\mathbb{Z}\times\mathbb{Z}\to G$. This homology version of the Bogomolov multiplier was then used to calculate $B_0(G)$ for several types of finite groups [22].

In this homological form, the Bogomolov multiplier appeared much earlier in [24] in connection with SK–groups (cutting and pasting of manifolds) and in [25] as SK_1 in algebraic K-theory.

Using now the fact that there are canonical isomorphisms

$$\widetilde{\Omega}_2^U(BG)\xrightarrow{\cong}\Omega_2^{\mathrm{SO}}(BG)\xrightarrow{\cong}H_2(BG,\mathbb{Z}), \tag{53}$$

we present a generalization of a result which was established by the second author in [30]. First we need a lemma:

Lemma 4.2 *Let Σ be an oriented surface with free G–action that bounds equivariantly. Then Σ can be extended to an oriented G–manifold whose ramification locus is a 1–dimensional manifold (all the isotropy groups are all cyclic).*

Proof Let M be an oriented G–manifold whose boundary is the surface with free G–action Σ. Take a point x in the ramification locus $\overline{M}$ and denote by G_x its isotropy group. Since the G–action is free on the boundary, the action of G_x on the normal neighborhood of x must induce an injective homomorphism $G_x\to\mathrm{SO}(3)$. Hence G_x must be isomorphic to a cyclic group, a dihedral group or any of the platonic groups A_4, $\mathfrak{S}_4$ or A_5. Whenever G_x is cyclic, x is a smooth point in the ramification locus $\overline{M}$, because locally G_x acts by rotations. Whenever G_x is neither trivial nor cyclic, x is a singular point on the ramification locus. Simply note that the irreducible and oriented 3–dimensional representations of the dihedral and the platonic groups have the origin as a singular point. Therefore the obstruction for the ramification locus $\overline{M}$ to be a 1–dimensional manifold is the presence of points whose isotropy groups are isomorphic to the dihedral or the platonic groups (A_4, $\mathfrak{S}_4$ and A_5). Our goal is to modify M to build a new manifold without any such isotropies.

We briefly outline the overall strategy of our desingularizing process. There are three steps:

(i) Perform the blowup construction on the normal neighborhoods of the points whose isotropies are isomorphic to either A_5, $\mathfrak{S}_4$ or A_4; this produces a new manifold M' with the same boundary as M and no points with A_5, $\mathfrak{S}_4$ or A_4 isotropy.

(ii) In M', "cancel" as many pairs of distinct orbits of a given dihedral isotropy type possible; our cancellation method results in a manifold M'' that is equivariantly cobordant to M', relative to the boundary Σ. By canceling as many pairs as possible, we guarantee that for the action of G on M'', a given conjugacy class of a dihedral subgroup of G occurs on at most one orbit in M''.

(iii) The final step is the hardest. If x is a point in M'' with dihedral isotropy $G_x \leq G$ that is maximal, we show that the action of G on $G\cdot x$ possesses an "involutive" element g such that $y = gx \neq x$, $g^2x = x$, $G_y = G_x$ and g commutes with a preferred rotation $\sigma \in G_x$. We then classify the possibilities for $\langle G_x, g\rangle$, and build an appropriate equivariant handle that desingularizes the orbit $G\cdot x$. Inductively applying this construction to all dihedrally stabilized points, we arrive at our desired manifold M'''. In fact, this is oversimplifying; we must return to (ii) once at some point in this process, but the basic idea is as described.

Let us expand on (i). Take a point $x \in M$ whose isotropy G_x is isomorphic to A_5 (we will start with the larger isotropy first). Let N_x be a normal G_x–neighborhood of x such that $N_x \cap g\cdot N_x = \varnothing$ for all $g \in G - G_x$, and let

$$\sigma\colon B(V_{G_x}) \xRightarrow{\cong} N_x \tag{54}$$

be a G_x–equivariant diffeomorphism with V_{G_x} the faithful representation of G_x around x. Take $G\cdot N_x$ as a G–equivariant neighborhood around the orbit $G\cdot x$ and note that σ induces a G–equivariant diffeomorphism $G\times_{G_x} B(V_{G_x}) \xRightarrow{\cong} G\cdot N_x$. Construct the blowup $B(\gamma(V_{G_x}))$ presented in Definition 3.7 and note that the sphere bundles are G_x–diffeomorphic to the boundary of N_x:

$$S(\gamma(V_{G_x})) \cong S(V_{G_x}) \xRightarrow{\cong} \partial N_x. \tag{55}$$

Cut $G\cdot N_x$ from M and glue $G\times_{G_x} B(\gamma(V_{G_x}))$ along the boundary using the diffeomorphism σ. Define the new G–manifold

$$M' := (M - G\cdot N_x) \cup_{\partial(G\cdot N_x)} G\times_{G_x} B(\gamma(V_{G_x})) \tag{56}$$

and note that M' has the same boundary as M, but with the property that inside ∂N_x there are no more points with isotropy isomorphic to A_5. Cutting and pasting the blowups for every point with isotropy isomorphic to A_5 produces a manifold without points whose isotropy is isomorphic to A_5. Then a similar blowup procedure is carried out for points with isotropy isomorphic to $\mathfrak{S}_4$, and then to points with isotropy isomorphic to A_4. The resulting manifold M' has the same boundary as M, but it does not contain points with isotropy isomorphic to A_5, $\mathfrak{S}_4$ or A_4. The only isotropies that appear on M' are cyclic or dihedral groups. This concludes (i). (We note for the interested reader that M and M' are not necessarily relatively cobordant, even though they do have the same boundary.)

For (ii), suppose x and y are two points in M' with equal dihedral stabilizers $G_x = G_y$ such that x and y are not in the same G orbit, but the representation of G_x on a regular neighborhood of x is equivalent to the representation of $G_y = G_x$ on a regular neighborhood of y. Call this representation V. Choose local charts around x and y such that the angle of rotations of the elements in G_x agree in both

charts to the angles of rotations in V. Now simply attach an equivariant 4–dimensional handle of the form $[0,1]\times(G\times_{G_x} B(V))$ to the equivariant neighborhood of $\{x,y\}$. Note that the only points in this handle with noncyclic isotropy are those in $G\cdot\{x,y\}$. Thus, the cobordant 3–manifold (where the open G–equivariant regular neighborhood of $G\cdot\{x,y\}$ is replaced with the vertical boundary of our handle) has fewer points with isotropy isomorphic to dihedral groups. Iterate this procedure, attaching such an equivariant handle anytime we see a pair x and y with the same isotropy group $G_x=G_y$, isomorphic local representations, and $y\notin G\cdot x$, until there are no more such pairs. Call the resulting manifold M''. Of course, since these handle attachments occur away from $\partial M'=\Sigma$, M'' still has boundary Σ. More precisely, M' and M'' are equivariantly bordant relative to Σ.

Step (iii) is the most involved. Let x be a point in M'' with dihedral isotropy $G_x\cong D_{2k}$. Define the set Λ_x of points y in M'' such that $G_y=G_x$ and whose local representations on regular neighborhoods around y and x, respectively, are isomorphic. By construction, if y is any other point in this set, then $y\in G\cdot x$. In fact, $\Lambda_x=N_G(G_x)\cdot x$ and Λ_x is bijective with the group $N_G(G_x)/G_x$. Our goal now is to build a G–handle that allows us to cancel the singularities in the *single* orbit $G\cdot x$ with one another in pairs in a G–equivariant fashion. In particular, we will need to show that $|G\cdot x|$ is even and admits a G–invariant matching.

Let us first show that there is an element $g\in N_G(G_x)$ such that its projection on $N_G(G_x)/G_x$ has order two. This g will allow us to define $y:=gx$ such that $x=gy$. We proceed by induction down the subgroup lattice of G and through the different isomorphism classes of faithful local representations of these point stabilizers. Let x be any point in M'' whose stabilizer $G_x\cong D_{2k}$ is maximal among all dihedral point stabilizers in M'' (with respect to subgroup inclusion) and consider the restricted action of just G_x on M''.

If $k>2$, let $\sigma\in G_x$ be an element of order k and take its fixed-point set $(M'')^\sigma$. Note that this fixed-point set is a disjoint union of embedded circles. The group $\mathbb{Z}_2\cong G_x/\langle\sigma\rangle$ acts on $(M'')^\sigma$ and the set of fixed points is precisely Λ_x. The Euler characteristic of $(M'')^\sigma$ being zero implies that Λ_x has an even number of points. Since $|\Lambda_x|=|N_{G_x}(G_x)/G_x|$, the group $N_{G_x}(G_x)/G_x$ has an element of order 2, and therefore we may choose $g\in N_{G_x}(G_x)$ as a lift of this element of order 2. If $y:=gx$, then by construction $x=gy$.

If $k=2$, then there is only one isomorphism class of local representations. Let

$$\Gamma_{G_x}=\{p\in M''\mid \mathrm{Stab}_{G_x}(p)\neq\{1\}\}$$

be the ramification locus of this action. Then Γ_{G_x} is a properly embedded topological graph. Because the action of G_x on $\partial M''=\Sigma$ is free, $\Gamma_{G_x}\cap\Sigma$ is empty, and so in particular every vertex in this graph has valence 6. The quotient graph Γ_{G_x}/G_x resides in the quotient manifold M''/G_x, and its vertices are in bijection with the vertices of Γ_{G_x}, and hence in bijection with the points in Λ_x. The vertices of Γ_{G_x}/G_x all have valence 3. Since twice the number of edges equals three times the number of vertices, we see that each connected component of Γ_{G_x}/G_x has an even number of vertices, and hence the same is true for each connected component of Γ_{G_x}. Note that this implies $|\Lambda_x|=|N_{G_x}(G_x)/G_x|$ is even. Now, as in the previous paragraph, we again choose $g\in N_{G_x}(G_x)$ lifting an element of order 2 in $N_{G_x}(G_x)/G_x$ and take $y:=gx$ with $x=gy$.

In both cases $k > 2$ and $k = 2$, note that $g^2 \in G_x$, and therefore the conjugation action of g on G_x squares to an inner automorphism of G_x. This is especially helpful when $k = 2$, ie when $G_x \cong D_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, since in this case $\mathrm{Inn}(G_x) = 0$, and we can conclude that g acts on G_x by an automorphism of order either 1 or 2 (never 3).

We now specify a preferred "rotation subgroup generator" of G_x. When $k = 2$ and g conjugates G_x by a nontrivial automorphism (necessarily of order 2, as just discussed above), then we take σ in G_x to be the unique nontrivial element of G_x fixed by conjugation with g. If g conjugates G_x trivially and there is not a loop in Γ_{G_x}/G_x at x, then we take σ to be an arbitrarily chosen element of G_x; if there is a loop at x, then we take σ to correspond to the unique element in G_x that does not stabilize points in the preimage of the interior of the loop (here the preimage can be taken with respect to $\Gamma_{G_x} \to \Gamma_{G_x}/G_x$). When $k > 2$, our preferred σ is given essentially for free: pick either one of the two nontrivial elements of G_x with minimal (unsigned) rotation angle (in its action on N_x) and call it σ.

With these choices for σ, in either the $k = 2$ or $k > 2$ case we may parametrize G_x as $G_x = \langle \sigma, \alpha \mid \sigma^k = \alpha^2 = 1, \alpha\sigma\alpha = \sigma^{-1} \rangle \cong D_{2k}$ for some arbitrarily chosen "reflection" α in G_x. We also know that g commutes with σ whenever $k = 2$ (because of how we picked σ), but when $k > 2$ it may be that $g\sigma g^{-1} = \sigma^{-1}$ since the local representations around x and y are isomorphic. If this were the case, replace g by αg and note that αg commutes with σ. Therefore we have found $g \in N_G(G_x)$ with $gx = y \neq x$, $gy = x$ and $g\sigma g^{-1} = \sigma$.

This in turn implies the following essential facts:

- When $k > 2$, no matter how σ conjugates G_x, we must have that $g^2 = \sigma^l$ for some $0 \leq l \leq k-1$.
- When $k = 2$, σ must conjugate $G_x = \{1, \sigma, \alpha, \sigma\alpha\} \cong D_4$ by an automorphism that leaves σ invariant. Thus σ either commutes with all of G_x, or else swaps α and $\sigma\alpha$.
  - If g swaps α and $\sigma\alpha$, notice that g does not commute with α or $\alpha\sigma$, and hence g^2 (which *does* commute with g) cannot equal α or $\alpha\sigma$. In other words, when g acts nontrivially on G_x, then we must have $g^2 = 1$ or $g^2 = \sigma$.
  - If g commutes with all of G_x, then in principle g^2 might equal any element of D_4. We will see below that in fact the only possibility is $g^2 = 1 \in D_4$.

Consider the group

$$K := \mathrm{Stab}_G\{x, y\} = \mathrm{Stab}_{N_G(G_x)}\{x, y\} = \langle \sigma, \alpha, g \rangle. \tag{57}$$

Notice that by construction, if h is any element of G such that $hx = y$, then in fact $h \in K$. (This is because $hx = gx$ implies $g^{-1}h \in G_x = G_y$, and hence $g^{-1}h \in G_x$ and so $h \in gG_x \subset K$.) In other words, we can build a G–equivariant matching on the orbit $G \cdot x$ by taking $\{x, y\}$ to be one pair in the matching, and inducing up to the entire orbit; the stabilizer of any edge in this matching is then conjugate to G_x. Therefore, if we can build a K–equivariant handle that allows us to desingularize the action of K on $N_x \sqcup N_y$, then we may induce this to a well-defined G–equivariant handle that desingularizes the action of G on the entire orbit $G \cdot N_x$.

We will now classify the possibilities for how K acts on $N_x \sqcup N_y$, and build nonsingular handles for each possibility. This will involve some casework, some of which depends on the integer $k \geq 2$ such that $G_x \cong D_{2k}$, and our success depends critically on the established fact that g commutes with σ.

Notice that K sits in an exact sequence

$$1 \to G_x \to K \to K/G_x \to 1, \tag{58}$$

where $K/G_x = \langle g \bmod G_x \rangle = \mathbb{Z}_2$. Recall that equivalence classes of such extensions can be placed in (noncanonical) bijection with the following pairs of data: homomorphisms $f\colon \mathbb{Z}_2 \to \mathrm{Out}(G_x)$ such that a certain canonically associated class in $H^3(\mathbb{Z}_2; Z(G_x))$ vanishes, together with a class $\omega \in H^2_f(\mathbb{Z}_2; Z(G_x))$, where $\mathbb{Z}_2$ acts on the coefficients $Z(G_x)$ in a manner induced by f.

However, not all homomorphisms $f\colon \mathbb{Z}_2 \to \mathrm{Out}(G_x)$ will be pertinent to our situation, because (except in the case $k = 2$ and g commutes with G_x), we already know that $g^2 = \sigma^l$ for some $0 \leq l \leq k-1$. Let us use some group cohomology to constrain the possibilities for K when $k > 2$.

If k is odd, then $Z(G_x) = \{0\}$, and so $H^2(K/G_x; Z(G_x)) = \{0\}$ and there is only one thing K could possibly be given that $g^2 = \sigma^l$, namely

$$K = \langle \sigma, \alpha, g \mid \sigma^k = \alpha^2 = 1, \alpha\sigma\alpha = \sigma^{-1}, g\sigma g^{-1} = \sigma, g\alpha g^{-1} = \alpha\sigma^l, g^2 = \sigma^l \rangle \tag{59}$$

where $0 \leq l \leq k-1$.

If $k > 2$ is even, then $Z(G_x) = \langle \sigma^{k/2} \rangle \cong \mathbb{Z}_2$ and the action of K/G_x on the coefficients $\mathbb{Z}_2$ must be trivial no matter what l is; therefore $H^2(K/G_x; Z(G_x)) \cong \mathbb{Z}_2$ and we should expect two nonequivalent extensions for a given l. These are precisely:

$$K_* = \langle \sigma, \alpha, g \mid \sigma^k = \alpha^2 = 1, \alpha\sigma\alpha = \sigma^{-1}, g\sigma g^{-1} = \sigma, g\alpha g^{-1} = \alpha\sigma^l, g^2 = \sigma^l \rangle,$$
$$K_\dagger = \langle \sigma, \alpha, g \mid \sigma^k = \alpha^2 = 1, \alpha\sigma\alpha = \sigma^{-1}, g\sigma g^{-1} = \sigma, g\alpha g^{-1} = \alpha\sigma^{l+(k/2)}, g^2 = \sigma^l \rangle,$$

where $0 \leq l \leq k-1$.

If $k = 2$, rather than use group cohomology to give an upper bound on the possibilities for K, we simply list the six known possibilities so far:

$$K_1 = \langle \sigma, \alpha, g \mid \sigma^2 = \alpha^2 = 1, \alpha\sigma\alpha = \sigma,\ g\sigma g^{-1} = \sigma,\ g\alpha g^{-1} = \alpha\sigma,\ g^2 = 1 \rangle$$
$$= \langle \alpha, g\alpha \mid (g\alpha)^4 = \alpha^2 = 1,\ \alpha(g\alpha)\alpha = (g\alpha)^{-1} \rangle \cong D_8,$$
$$K_2 = \langle \sigma, \alpha, g \mid \sigma^2 = \alpha^2 = 1, \alpha\sigma\alpha = \sigma,\ g\sigma g^{-1} = \sigma,\ g\alpha g^{-1} = \alpha,\ g^2 = \sigma \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4,$$
$$K_3 = \langle \sigma, \alpha, g \mid \sigma^2 = \alpha^2 = 1, \alpha\sigma\alpha = \sigma,\ g\sigma g^{-1} = \sigma,\ g\alpha g^{-1} = \alpha,\ g^2 = \alpha \rangle \cong K_2,$$
$$K_4 = \langle \sigma, \alpha, g \mid \sigma^2 = \alpha^2 = 1, \alpha\sigma\alpha = \sigma,\ g\sigma g^{-1} = \sigma,\ g\alpha g^{-1} = \alpha,\ g^2 = \sigma\alpha \rangle \cong K_2,$$
$$K_5 = \langle \sigma, \alpha, g \mid \sigma^2 = \alpha^2 = 1, \alpha\sigma\alpha = \sigma,\ g\sigma g^{-1} = \sigma,\ g\alpha g^{-1} = \alpha\sigma,\ g^2 = \sigma \rangle = K_* \text{ (for } l = 1) \cong D_8,$$
$$K_6 = \langle \sigma, \alpha, g \mid \sigma^2 = \alpha^2 = 1, \alpha\sigma\alpha = \sigma,\ g\sigma g^{-1} = \sigma,\ g\alpha g^{-1} = \alpha,\ g^2 = 1 \rangle = K_* \text{ (for } l = 0) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Each group has order 8 as needed, so no further constraints are required to specify K, beyond which of the two possible automorphisms fixing σ we have g act on G_x by, and which element $\sigma^2 \in G_x$ is.

All of the above listed possibilities for K are based on naive algebra. An algebraic classification of the possibilities for K is not immediately equivalent to a geometric classification of the different possible *faithful representations* of these K with $K \to \mathrm{Iso}_+(B^3 \sqcup B^3)$, which are, after all, what we need to desingularize. In particular, we will see that K_1 and K_2 (and therefore K_3 and K_4) in the case $k = 2$ have *no* faithful representations into $\mathrm{Iso}_+(B^3 \sqcup B^3)$.

To understand how these possible abstract structures of the extension K relate to the geometry of the action of K on $N_x \sqcup N_y \cong B^3 \sqcup B^3$, we parametrize so that G_x acts on each copy of $B^3 \subset \mathbb{R}^3$ in the same standard way:

$$\sigma(x, y, z) = \Big(\cos\Big(\frac{2\pi l}{k}\Big)x - \sin\Big(\frac{2\pi l}{k}\Big)y, \sin\Big(\frac{2\pi l}{k}\Big)x + \cos\Big(\frac{2\pi l}{k}\Big)y, z\Big), \tag{60}$$

$$\alpha(x, y, z) = (x, -y, -z). \tag{61}$$

Here $0 < l < k$ and l is coprime with k.

Note that this standard action of D_{2k} on B^3 is unique up to a sign, meaning any two faithful representations $D_{2k} \to \mathrm{SO}(3)$ that have σ acting by rotation angle $\pm 2\pi l/k$ are related by a conjugacy in $\mathrm{SO}(3)$. With this, we see that the equivalence class of an isometric action of K on $N_x \sqcup N_y$ — *when it exists* — is entirely determined (in the relevant sense, namely, up to conjugation by $\mathrm{Iso}_+(N_x \sqcup N_y)$) by the representation of G_x on either component, and the diffeomorphism affected when g swaps N_x and N_y.

We will now show that none of K_1–K_4 in the $k = 2$ case above are geometrically realizable. It suffices to show this for K_1 and K_2. In both cases, we assume without loss of generality that σ and α act on $B^3 \sqcup B^3$ in the standard way shown above.

For K_1 the only possibilities for g are $g(x, y, z) = (-y, x, z)$, $g(x, y, z) = (y, -x, z)$, $g(x, y, z) = (-y, -x, -z)$ or $g(x, y, z) = (y, x, -z)$, since g must leave the z–axis fixed and swap the x– and y–axes. The first two contradict $g^2 = 1$. The second two contradict that g commutes with α.

For K_2, since g commutes with all three generators, it leaves each axis fixed, and the possible actions are exactly $g(x, y, z) = (x, -y, -z)$, $g(x, y, z) = (-x, y, -z)$ or $g(x, y, z) = (-x, -y, z)$. None of these squares to σ.

Finally, we will show that all remaining K are geometrically realizable while simultaneously achieving our most important goal: a description of the desingularizing handle we are after for each possibility.

For each of them, we may define a faithful 4–dimensional real representation of $K = \langle \sigma, \alpha, g \rangle$ as follows:

$$\sigma(x, y, z, t) = \Big(\cos\Big(\frac{2\pi l}{k}\Big)x - \sin\Big(\frac{2\pi l}{k}\Big)y, \sin\Big(\frac{2\pi l}{k}\Big)x + \cos\Big(\frac{2\pi l}{k}\Big)y, z, t\Big), \tag{62}$$

$$\alpha(x, y, z, t) = (x, -y, -z, t), \tag{63}$$

$$g(x, y, z, t) = \Big(\cos\Big(\frac{\pi j}{k}\Big)x - \sin\Big(\frac{\pi j}{k}\Big)y, \sin\Big(\frac{\pi j}{k}\Big)x + \cos\Big(\frac{\pi j}{k}\Big)y, -z, -t\Big). \tag{64}$$

Here $j = l$ if k is odd, and $j = l$ or $l + \big(\frac{1}{2}k\big)$ if $k > 2$ is even and $K = K_*$ or $K = K_\dagger$, respectively. For the groups K_5 and K_6 we take $j = 1$ and $j = 2$, respectively. Note that we are continuing to assume (without loss of generality) that σ is the element in G_x that acts on N_x as rotation by the angle $2\pi l/k$.

Clearly (x, y, z, t) is a fixed point of g only if $z = t = 0$. In the two cases with $k = 2$, for K_5 we have

$$g(x, y, z, t) = (-y, x, -z, -t) \tag{65}$$

and for the group K_6 we have

$$g(x, y, z, t) = (-x, -y, -z, -t), \tag{66}$$

so other than the zero point $(0, 0, 0, 0)$, g has no fixed points at all in $\mathbb{R}^4$. Thus $(x, y, 0, 0) \neq (0, 0, 0, 0)$ is a fixed point of g only if $k > 2$ and $j = l = 0$. Nontrivial powers of σ never share a nonzero fixed point with g, as σ acts freely on the plane $z = t = 0$. We conclude that any nonzero point in the $z = t = 0$ plane has either a trivial stabilizer, a cyclic stabilizer generated by an element of G_x, or a stabilizer of the form $\langle \sigma^p \alpha, g \rangle \cong D_4$ for some $0 \leq p \leq k - 1$, and moreover, this third case can only occur when $k > 2$ and $j = l = 0$. This last fact — that noncycle stabilizers occur in this plane only when $k > 2$ — is essential to the remainder of our argument.

Now consider the action of K on the unit sphere $S^3 \subset \mathbb{R}^4$, ie on

$$S^3 = \{(x, y, z, t) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + t^2 = 1\}. \tag{67}$$

The isotropy group of both $(0, 0, 0, -1)$, and $(0, 0, 0, 1)$ is the dihedral group $G_x = \langle \sigma, \alpha \rangle \cong D_{2k}$ and g swaps these two points. Moreover, every other point in $S^3 \setminus \{(0, 0, 0, -1), (0, 0, 0, 1)\}$ has either trivial or cyclic isotropy, with some minor exceptions: when $k > 2$ and $j = l = 0$ there are points in $S^3 \setminus \{(0, 0, 0, -1), (0, 0, 0, 1)\}$ with isotropy isomorphic to D_4. Denote by W a small K–equivariant ball around the union of $(0, 0, 0, -1)$ and $(0, 0, 0, 1)$ and remove it from S^3. Attach the G–equivariant handle $G \times_K (S^3 \setminus W)$ to the boundary of the G–equivariant normal neighborhood of $G \cdot x$ on M. For $k > 2$, the resulting 3–manifold has fewer points with dihedral isotropy D_{2k}, although it may create new points with D_4 isotropy. Attach the handles inductively for all dihedral isotropies isomorphic to D_{2k} with $k > 2$ (picking maximal such isotropies at every step) and all isomorphism classes of faithful irreducible representations, and arrive at a manifold whose isotropies are only cyclic or dihedral of order 4. Now repeat step (ii) of the proof to arrive at a manifold with only cyclic isotropies and dihedral isotropies of order 4 with, moreover, the property that for any x and y with $G_x = G_y \cong D_4$, we know x and y are in the same G orbit. Finally, return to step (iii) and desingularize any remaining D_4 isotropies as we did in the case with $k > 2$. Since the handles we have constructed for the D_4 singularities have no noncyclic isotropies on their interior, attaching them to the remaining D_4 singularities gives our final manifold M''' with only cyclic isotropies. □

We are now ready to show which surfaces with free G–actions bound equivariantly.

Theorem 4.3 *Let G be a finite group. Then the oriented and unitary equivariant bordism of surfaces with free G–actions fits into the exact sequence*

$$\Omega_3^G\{\mathcal{A}G, \{1\}\} \xrightarrow{\partial} \widetilde{\Omega}_2(BG) \to \widetilde{B}_0(G) \to 0. \tag{68}$$

Proof Let us first show that the image of the boundary map consists of toral classes in $H_2(BG,\mathbb{Z})$, that is, homology classes coming from the image of maps of tori $S^1\times S^1\to BG$.

Let M be a 3–dimensional G–manifold (oriented or unitary) whose boundary ∂M has a free G–action; in the oriented case take M as shown in Lemma 4.2. Note that if the ramification locus $\overline{M}$ is not empty, then it is a smooth oriented 1–dimensional manifold; in the unitary case this follows from the fact that fixed points of all nontrivial subgroups can only have complex codimension 0 or 1.

If $\overline{M}$ is empty then M has a free G action and therefore the boundary surface $(\partial M)/G$ bounds. If $\overline{M}$ is not empty we may consider the G–equivariant tubular neighborhood N of $\overline{M}$ in M. The manifolds N and M define the same bordism class, since on $M-N$ the action of G is free, and therefore ∂M and ∂N are cobordant. The tubular neighborhood N is homeomorphic to the unit ball bundle $B\nu$ of the normal bundle ν of $\overline{M}$ in M. The sphere bundle $S\nu$ defines the S^1–principal bundle $S^1\to S\nu\to\overline{M}$, and since every circle bundle over the circle is topologically a torus, the sphere bundle $S\nu$ is homeomorphic to a disjoint union of 2–dimensional tori. Hence ∂N is a disjoint union of 2–dimensional tori, and its quotient $\partial N/G$ is a torus (since M/G is connected and $\chi(\partial N/G)=\chi(\partial N)/|G|$). Hence we have now proved that the image of the boundary map ∂ of (68) consists only of toral classes in $H_2(BG,\mathbb{Z})$.

Now let us show the converse, namely that any toral class in $H_2(BG,\mathbb{Z})$ lies in the image of the boundary map of (68). Take any toral class defined by a homomorphism $\varphi\colon\mathbb{Z}\times\mathbb{Z}\to G$ and denote by $A:=\varphi(\mathbb{Z}\times\{0\})$ and $C:=\varphi(\{0\}\times\mathbb{Z})$ the cyclic subgroups of G that define the toral class. Denote by $a:=\varphi(1,0)$ and $c:=\varphi(0,1)$ the generators of A and C, respectively.

Let N_GA be the normalizer of A in G and note that C is a subgroup of the normalizer. Denote by ι and $\bar\iota$ the homomorphism $\iota\colon\mathbb{Z}\to N_GA$, $\iota(n):=c^n$ and the homomorphism to the quotient $\bar\iota\colon\mathbb{Z}\to W_A$. Consider the irreducible representation $\rho\colon A\to U(1)$, $\rho(a):=e^{2\pi i/|A|}$, and define the $U(1)$ extension of W_A by the exact sequence of groups

$$U(1)\to U(1)\times_A N_GA\to W_A, \tag{69}$$

where $U(1)\times_A N_GA$ is defined by the equivalence relation $(\lambda\rho(\alpha),g)\sim(\lambda,\alpha g)$ for all $\alpha\in A$, $\lambda\in U(1)$ and $g\in N_GA$.

Consider the homomorphism $\tilde\iota\colon\mathbb{Z}\to U(1)\times_A N_GA$, $\tilde\iota(n):=[(1,\iota(c^n))]$, and note that its classifying map

$$B\tilde\iota\colon S^1\to B(U(1)\times_A N_GA) \tag{70}$$

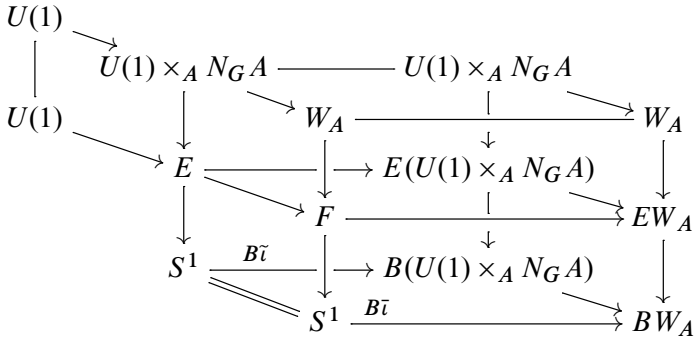
factors through the classifying map $B\bar\iota\colon S^1\to BW_A$.

Let $E:=(B\tilde\iota)^*E(U(1)\times_A N_GA)$ be the pullback of the universal bundle and note two things. First, E is a principal $U(1)\times_A N_GA$–bundle over the circle S^1, and therefore it is a surface. Second, the canonical homomorphism

$$N_GA\to U(1)\times_A N_GA\quad\text{given by}\quad g\mapsto[(1,g)] \tag{71}$$

induces a free action of $N_G A$ on E. Now it is straightforward to notice that the homology class of the surface $E/N_G A \to BN_G A \to BG$ in $\widetilde{\Omega}_2(BG)$ agrees with the homology class defined by $B\varphi_*[S^1 \times S^1]$.

We still need to show that the surface E equivariantly bounds. Take the quotient $F = E/U(1)$ and note that F is homeomorphic to $(B\bar{\iota})^* EW_A$; hence F is the principal W_A–bundle over the circle that $\bar{\iota}$ defines (see the following commutative diagram):

(72)


Then E is a principal $U(1)$–bundle over F, and therefore we may take the associated complex vector bundle

$$\mathbb{C} \to \mathbb{C} \times_{U(1)} E \to F. \tag{73}$$

The unit bundle $D(\mathbb{C} \times_{U(1)} E)$ is a unitary manifold endowed with the action of $N_G A$, whose boundary, the sphere bundle $S(\mathbb{C} \times_{U(1)} E)$, is homeomorphic to E:

$$\partial(D(\mathbb{C} \times_{U(1)} E)) = S(\mathbb{C} \times_{U(1)} E) \cong E. \tag{74}$$

Therefore we have just proved that

$$[G \times_{N_G A} D(\mathbb{C} \times_{U(1)} E)] \xrightarrow{\partial} [E/N_G A] = B\varphi_*[S^1 \times S^1], \tag{75}$$

thus showing that any toral class in $\widetilde{\Omega}_2(BG)$ equivariantly bounds. □

Now we can put the pieces together to understand the torsion of the equivariant bordism group of surfaces.

4.2 Torsion of the equivariant bordism group of surfaces

By Corollary 3.4,

$$\overline{\Omega}_2^G = \mathrm{Ker}(\phi_2) = \mathrm{Tor}_{\mathbb{Z}}(\Omega_2^G). \tag{76}$$

Let us now determine explicitly these torsion subgroups.

Theorem 4.4 *Let G be a finite group. Then there is a canonical isomorphism*

$$\bigoplus_{(K)} \widetilde{B}_0(W_K) \cong \mathrm{Tor}_{\mathbb{Z}}(\Omega_2^G), \tag{77}$$

where (K) runs over all conjugacy classes of subgroups of G, $W_K = N_G K/K$ and $\widetilde{B}_0(W_K)$ is the homology version of the Bogomolov multiplier of the group W_K.

Proof Denote by $\mathrm{Gr}_*\,\mathrm{Tor}_{\mathbb{Z}}(\Omega_2^G)$ the associated graded groups of the G–equivariant, unitary or oriented, bordism groups of surfaces that are induced by the Conner–Floyd spectral sequence of the families of subgroups of (42). Lemma 4.1 and Theorem 4.3 imply that

$$\mathrm{Gr}_p\,\mathrm{Tor}_{\mathbb{Z}}(\Omega_2^G) \cong \widetilde{B}_0(W_{K_p}), \tag{78}$$

and since all consecutive pairs of families are adjacent, we obtain the graded isomorphism

$$\mathrm{Gr}_*\,\mathrm{Tor}_{\mathbb{Z}}(\Omega_2^G) \cong \bigoplus_{(K)} \widetilde{B}_0(W_K). \tag{79}$$

Now, for a fixed conjugacy class of subgroups (K), the canonical map

$$\widetilde{\Omega}_2(BW_K) \to \Omega_2^G \quad \text{given by } \ \Sigma/W_K \mapsto G \times_{N_G K} \Sigma, \tag{80}$$

which sends the quotient space of a surface Σ by the free W_K–action to the surface with G–action whose isotropy groups lie in (K), factors through $\widetilde{B}_0(W_K)$, thus producing a canonical homomorphism

$$\widetilde{B}_0(W_K) \to \Omega_2^G. \tag{81}$$

Bundling up all these homomorphisms we obtain a canonical map

$$\bigoplus_{(K)} \widetilde{B}_0(W_K) \to \mathrm{Tor}_{\mathbb{Z}}(\Omega_2^G) \tag{82}$$

which becomes an isomorphism since it is compatible with the graded isomorphism (79). □

In particular, if G is a group whose Bogomolov multipliers vanish for all groups W_K with K a nontrivial subgroup, then $\mathrm{Tor}_{\mathbb{Z}}(\Omega_2^G) \cong \widetilde{B}_0(G)$. This is the case whenever G is one of the smallest p–groups with nontrivial Bogomolov multiplier. In the last section we present two p–groups of this kind.

We are now ready to provide an explicit calculation of the unitary and oriented equivariant bordism group of surfaces. Assembling Theorems 3.1 and 4.4, and Proposition 3.3, we obtain the following result:

Theorem 4.5 *Let G be a finite group. Then the unitary and oriented equivariant bordism of surfaces canonically decompose as follows*:

$$\Omega_2^{U,G} \cong \bigoplus_{(K)} \left(\widetilde{B}_0(W_K) \oplus \Omega_2^U \oplus \Big(\bigoplus_{\mathrm{Irr}^1_{\mathbb{C}}(K)} \mathbb{Z} \Big)^{W_K} \right), \tag{83}$$

$$\Omega_2^{\mathrm{SO},G} \cong \bigoplus_{(K)} \left(\widetilde{B}_0(W_K) \oplus \Big(\bigoplus_{\mathrm{Irr}^1_{\mathbb{C}}(K)_{\mathbb{C}}/\mathrm{conj}} \mathbb{Z} \Big)^{W_K} \right). \tag{84}$$

Here (K) runs over the conjugacy classes of subgroups of G, W_K is the Weyl group $N_G K/K$, $\mathrm{Irr}^1_{\mathbb{C}}(K)$ is the set of 1–dimensional nontrivial irreducible complex representations of K endowed with the natural W_K action, and $\mathrm{Irr}^1_{\mathbb{C}}(K)_{\mathbb{C}}/\mathrm{conj}$ denotes the representations of complex type modulo complex conjugation.

5 2–Dimensional SK–groups of classifying spaces

Jänich in [15; 14] started the study of the characterization of invariants with the additivity property of the Euler characteristic and the signature under cutting and pasting of manifolds.

Karras and Kreck in their diploma thesis extended the ideas of Jänich to cutting and pasting in the bundle situation. The book [17] presented and simplified these results with the definition of the SK–groups of a space (cutting and pasting groups from the German *Schneiden und Kleben*). Later Neumann [24] completely calculated the 2–dimensional SK–groups of a space in terms of what is now known as the Bogomolov multiplier of its fundamental group. We recall in this section the main results of [17; 24] that allow us to relate the SK–relation with the equivariant bordism relation on surfaces with free actions.

The Schneiden und Kleben groups $\mathrm{SK}_*(X)$ of a space are defined as the Grothendieck group of the semigroups obtained by defining the SK–equivalence on the class of continuous maps from oriented n–dimensional manifolds to X [17].

The SK–relation is defined as follows: given (M_i, f_i) with $f_i\colon M_i \to X$, we say that (M_1, f_1) and (M_2, f_2) are related by cutting and pasting along ∂N if $M_1 = N \cup_\phi -N'$, $M_2 = N \cup_\psi -N'$ and there are homotopies $f_1|_N \simeq f_2|_N$ and $f_1|_{N'} \simeq f_2|_{N'}$.

The Schneiden und Kleben bordism groups $\overline{\mathrm{SK}}_n(X)$ of a space are defined as the quotient of the oriented bordism groups by the equivalence relation generated by the SK–relation:

$$\overline{\mathrm{SK}}_*(X) = \Omega^{\mathrm{SO}}_*(X)/\sim. \tag{85}$$

The group $\overline{\mathrm{SK}}_2(BG)$ can be interpreted as the bordism group of surfaces with free G–actions modulo the SK–relations.

The following results summarize the main properties of the SK–relation [17, Lemmas 1.5 and 1.6].

(i) Any $f\colon S^1 \to X$ is zero in $\mathrm{SK}_1(X)$.

(ii) If M fibers over S^n with fiber F, then for any $f\colon M \to X$, in $\mathrm{SK}_*(X)$,

$$[M, f] = [S^n, *][F, f|_F]. \tag{86}$$

(iii) If $[M_2, f_2]$ is obtained from $[M_1, f_1]$ by surgery of type $(k+1, n-k)$, then in $\mathrm{SK}_*(X)$

$$[M_1, f_1] + [S^n, *] = [M_2, f_2] + [S^k \times S^{n-1}, *]. \tag{87}$$

Now, if I_* denotes the subgroup of $\mathrm{SK}_*(X)$ generated by the spheres with constant maps to X, which is isomorphic to the integers, we have:

Theorem 5.1 [17, Theorem 1.1] *For a connected space X, there is the exact sequence*

$$0 \to I_* \to \mathrm{SK}_*(X) \to \overline{\mathrm{SK}}_*(X) \to 0, \tag{88}$$

which is moreover split. The map $\frac{1}{2}(\chi - \tau)\colon \mathrm{SK}_n(X) \to \mathbb{Z} \cong I_n$ gives the splitting.

The groups $\overline{\mathrm{SK}}_*(X)$ fit into short exact sequences whose middle terms are the oriented bordism groups.

Theorem 5.2 [17, Theorem 1.2] *Let $F_n(X)$ be the submodule of $\Omega_n^{\mathrm{SO}}(X)$ generated by all elements which have a representative that fibers over S^1. Then $F_n(X)$ fits into the short exact sequence*

$$0 \to F_*(X) \to \Omega_*^{\mathrm{SO}}(X) \to \overline{\mathrm{SK}}_*(X) \to 0. \tag{89}$$

This theorem follows from the observations that any manifold that fibers over S^1 gives a class that is zero in $\overline{\mathrm{SK}}_*(X)$, and that the kernel of the homomorphism $\Omega_*^{\mathrm{SO}}(X) \to \overline{\mathrm{SK}}_*(X)$ consists of mapping tori. The key lemma for the opposite inclusion asserts that if (M_1, f_1) is obtained from (M_2, f_2) by cutting and pasting along N, then in $\Omega_*^{\mathrm{SO}}(X)$ the class of $(N \cup_\phi -N', f_1) - (N \cup_\psi -N', f_2)$ is equal to the mapping torus of the diffeomorphism of ∂N, $\phi^{-1} \circ \psi$. Any mapping torus fibers over S^1 and any fibration over S^1 is a mapping torus.

In dimensions 0 and 1 the groups $\overline{\mathrm{SK}}_n(X)$ are trivial. In dimension 2 the oriented manifolds that fiber over the circle are tori. Therefore by Theorem 5.2 we obtain the following result:

Theorem 5.3 [24, Theorem 2] *Let G be a discrete group. Then the 2–dimensional $\overline{\mathrm{SK}}$–group of BG is isomorphic to the Bogomolov multiplier of G:*

$$\overline{\mathrm{SK}}_2(BG) \cong \widetilde{B}_0(G). \tag{90}$$

Reinterpreting the SK–groups of BG in view of our previous results, we know by Theorem 4.3 that an element of $\overline{\mathrm{SK}}_2(BG)$ is zero whenever the associated G–cover of the surface is the boundary of a 3–dimensional manifold with a G–action. By Theorem 5.2, $\mathrm{SK}_2(BG) \cong \mathbb{Z} \oplus \widetilde{B}_0(G)$, and therefore a surface $\Sigma \to BG$ is zero in the group $\mathrm{SK}_2(BG)$ whenever the Euler characteristic of Σ is 0 and the G–cover $\widetilde{\Sigma}$ of Σ is the boundary of a 3–dimensional manifold with a G–action.

It would be interesting to explore the relation of this work with the higher-dimensional SK–groups of classifying spaces.

6 Small groups with nontrivial Bogomolov multiplier

We conclude this work by presenting some explicit examples of groups with nontrivial Bogomolov multiplier which induce nontrivial torsion subgroups in the equivariant bordism groups of surfaces. Some of the calculations were done with the help of the *Homological Algebra Programming* package for GAP [10].

6.1 2–Group of size 64

The smallest groups with nontrivial Bogomolov multiplier are 2–groups of order 64. There are nine of them, and all are in the same isoclinism class. By [23, Theorem 1.2] they all have isomorphic Bogomolov multipliers, and in this case it is the group $\mathbb{Z}/2$. Among the nine isoclinic groups we chose to study the group

$$C_8 \rtimes Q_8, \tag{91}$$

which is the semidirect product of the group of quaternions Q_8 with the cyclic group C_8 of order 8; this group is denoted by

$$\texttt{SmallGroup(64,182)}$$

in the GAP small groups library. Consider the presentations $Q_8 = \langle a, b \mid a^2 = b^2, aba^{-1} = b^{-1}\rangle$ and $C_8 = \langle c \mid c^8 = 1\rangle$, and the action of Q_8 on C_8 given by the equations

$$ac = c^3, \quad bc = c^5 \quad \text{and} \quad (ab)c = c^7. \tag{92}$$

Since $H^2(C_8, \mathbb{C}^*) = 0 = H^2(Q_8, \mathbb{C}^*)$, we know by the Lyndon–Hochschild spectral sequence that

$$H^2(C_8 \rtimes Q_8, \mathbb{C}^*) \cong H^1(Q_8, H^1(C_8, \mathbb{C}^*)). \tag{93}$$

Define $\widehat{C}_8 := \mathrm{Hom}(C_8, \mathbb{C}^*) = H^1(C_8, \mathbb{C}^*)$ and let $\widehat{C}_8 = \langle \rho \mid \rho^8 = 1\rangle$ with $\rho(c) = e^{2\pi i/8}$. Take the first two terms of the complex $C^*(Q_8, \widehat{C}_8)$,

$$\widehat{C}_8 \xrightarrow{\delta} \mathrm{Map}(Q_8, \widehat{C}_8), \tag{94}$$

and note that

$$\delta(\rho^k)(a^{\pm}) = \rho^{-2k}, \quad \delta(\rho^k)(b^{\pm}) = \rho^{4k}, \quad \delta(\rho^k)((ab)^{\pm}) = \rho^{2k} \quad \text{and} \quad \delta(\rho^k)(a^2) = \rho^0. \tag{95}$$

On the other hand, take the 1–cocycle $F\colon Q_8 \to \widehat{C}_8$ defined by the equations

$$F(a^{\pm}) = \rho^2, \quad F(b^{\pm}) = \rho^0, \quad F((ab)^{\pm}) = \rho^2 \quad \text{and} \quad F(a^2) = \rho^0, \tag{96}$$

and note that F does not bound but $F^2 = \delta(\rho^2)$. We have therefore that

$$H^1(Q_8, \widehat{C}_8) \cong \langle [F] \mid [F^2] = 0\rangle \cong \mathbb{Z}/2. \tag{97}$$

Now any abelian subgroup of $C_8 \rtimes Q_8$ splits as a semidirect product of abelian groups $C \rtimes A$ with $C \subset C_8$ and $A \subset Q_8$. Since A can only be $\mathbb{Z}/4$ or $\mathbb{Z}/2$, it is now straightforward to check that $[F]|_{C\rtimes A} = 0$. Hence $[F]$ is the generator of the Bogomolov multiplier of $C_8 \rtimes Q_8$ and

$$\overline{\Omega}_2^{U, C_8\rtimes Q_8} \cong \overline{\Omega}_2^{\mathrm{SO}, C_8\rtimes Q_8} \cong \mathbb{Z}/2. \tag{98}$$

Finally, with the explicit description of F we can define a surface Σ_2 of genus 2 which defines the generator of $\widetilde{\Omega}_2^U(B(C_8 \rtimes Q_8))$. Consider the presentation of the fundamental group of the surface

$$\pi_1(\Sigma_2) = \langle x, y, z, w \mid [x, y][z, w] = 1\rangle \tag{99}$$

and define the assignment

$$\Phi\colon \pi_1(\Sigma_2) \to C_8 \rtimes Q_8, \qquad x \mapsto a, \quad y \mapsto c, \quad z \mapsto ab, \quad w \mapsto c, \tag{100}$$

which induces a surjective homomorphism since

$$\Phi([x, y][z, w]) = aca^{-1}c^{-1}(ab)c(ab)^{-1}c^{-1} = c^3c^{-1}c^7c^{-1} = c^0. \tag{101}$$

The homomorphism Φ induces a map $B\Phi\colon \Sigma_2 \to B(C_8 \rtimes Q_8)$, and from the construction above of F, we deduce that $B\Phi_*[\Sigma_2]$ generates the group $H_2(B(C_8 \rtimes Q_8), \mathbb{Z})$.

Hence the surface

$$\widetilde{\Sigma} := (B\Phi)^* E(C_8 \rtimes Q_8) \tag{102}$$

is a unitary surface with a free action of $C_8 \rtimes Q_8$ which does not equivariantly bound.

By Theorem 4.4, the class of $\widetilde{\Sigma}$ is the generator of the torsion subgroup of $\Omega_2^{\mathrm{SO},C_8\rtimes Q_8}$:

$$\mathrm{Tor}_{\mathbb{Z}}\, \Omega_2^{\mathrm{SO},C_8\rtimes Q_8} = \langle [\widetilde{\Sigma}] \rangle \cong \mathbb{Z}/2. \tag{103}$$

To make sure that the first Chern number vanishes, we take the bordism class

$$[\widetilde{\Sigma}] - [(C_8 \rtimes Q_8) \times \Sigma_2] \in \overline{\Omega}_2^{U,C_8\rtimes Q_8} \cong \mathbb{Z}/2, \tag{104}$$

and by Theorem 4.4 we conclude that this is indeed the generator of the torsion subgroup of $\Omega_2^{U,C_8\rtimes Q_8}$:

$$\mathrm{Tor}_{\mathbb{Z}}\, \Omega_2^{U,C_8\rtimes Q_8} = \langle [\widetilde{\Sigma}] - [(C_8 \rtimes Q_8) \times \Sigma_2] \rangle \cong \mathbb{Z}/2. \tag{105}$$

6.2 3–Group of size 243

The smallest 3–groups with nontrivial Bogomolov multiplier are of order 243, and the three of them are isoclinic with Bogomolov multiplier the group $\mathbb{Z}/3$. We chose to study the group

$$G := (C_9 \rtimes C_9) \rtimes C_3, \tag{106}$$

which is defined by the presentation

$$G = \langle a, b, c \mid a^3 = c^3,\ a^9 = b^9 = 1,\ [a,b] = c^8 b^6,\ [b,c] = a^3,\ [a,c] = b^3 c^6 \rangle. \tag{107}$$

The left C_9 is generated by c, the right C_9 by b, and the C_3 by ab; their corresponding actions are

$$bcb^{-1} = c^4, \quad (ab)b(ab)^{-1} = c^8 b^7 \quad \text{and} \quad (ab)c(ab)^{-1} = cb^3. \tag{108}$$

This group corresponds to the small group

`SmallGroup(243,30)`

in the small groups library of GAP [10].

The second page of the Lyndon–Hochschild spectral sequence has for terms

$$H^2(C_9 \rtimes C_9, \mathbb{C}^*)^{C_3} = 0, \quad H^1(C_3, H^1(C_9 \rtimes C_9, \mathbb{C}^*)) = \mathbb{Z}/3 \quad \text{and} \quad H^2(C_3, \mathbb{C}^*) = 0, \tag{109}$$

where the middle term encodes the information of the Bogomolov multiplier.

Consider the surface Σ_2 of genus 2 as in (99), and define the assignment

$$\Phi\colon \pi_1(\Sigma_2) \to (C_9 \rtimes C_9) \rtimes C_3, \qquad x \mapsto a, \quad y \mapsto b^6, \quad z \mapsto c, \quad w \mapsto b, \tag{110}$$

which induces a surjective homomorphism since $[a, b^6] = a^3$, $[c, b] = a^6$ and

$$\Phi([x,y][z,w]) = [a,b^6][c,b] = 1. \tag{111}$$

The map $B\Phi\colon \Sigma_2 \to B((C_9 \rtimes C_9) \rtimes C_3)$ generates the Bogomolov multiplier, and therefore the surface $\widetilde{\Sigma} := (B\Phi)^* E((C_9 \rtimes C_9) \rtimes C_3)$ generates the torsion subgroup of the equivariant oriented bordism group of surfaces

$$\operatorname{Tor}_{\mathbb{Z}} \Omega_2^{\mathrm{SO},(C_9 \rtimes C_9) \rtimes C_3} = \langle [\widetilde{\Sigma}] \rangle \cong \mathbb{Z}/3. \tag{112}$$

In the unitary case,

$$\operatorname{Tor}_{\mathbb{Z}} \Omega_2^{U,(C_9 \rtimes C_9) \rtimes C_3} = \langle [\widetilde{\Sigma}] - [(C_9 \rtimes C_9) \rtimes C_3 \times \Sigma_2] \rangle \cong \mathbb{Z}/3. \tag{113}$$

Then $\widetilde{\Sigma}$ is a surface of genus 486 with a free action of $(C_9 \rtimes C_9) \rtimes C_3$ which does not equivariantly bound.

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References

[1] **A Ángel**, *A spectral sequence for orbifold cobordism*, from "Algebraic topology: old and new", Banach Center Publ. 85, Polish Acad. Sci. Inst. Math., Warsaw (2009) 141–154 MR Zbl

[2] **A Ángel**, **J M Gómez**, **B Uribe**, *Equivariant complex bundles, fixed points and equivariant unitary bordism*, Algebr. Geom. Topol. 18 (2018) 4001–4035 MR Zbl

[3] **F A Bogomolov**, *The Brauer group of quotient spaces of linear representations*, Izv. Akad. Nauk SSSR Ser. Mat. 51 (1987) 485–516 MR Zbl In Russian; translated in Math. USSR-Izv. 30 (1988) 455–485

[4] **P E Conner**, **E E Floyd**, *Differentiable periodic maps*, Ergebnisse der Math. 33, Academic, New York (1964) MR Zbl

[5] **P E Conner**, **E E Floyd**, *Maps of odd period*, Ann. of Math. 84 (1966) 132–156 MR Zbl

[6] **T tom Dieck**, *Bordism of G–manifolds and integrality theorems*, Topology 9 (1970) 345–358 MR Zbl

[7] **T tom Dieck**, *Orbittypen und äquivariante Homologie, I*, Arch. Math. (Basel) 23 (1972) 307–317 MR Zbl

[8] **T tom Dieck**, *Transformation groups*, de Gruyter Stud. Math. 8, de Gruyter, Berlin (1987) MR Zbl

[9] **J E Domínguez**, **C Segovia**, *Extending free actions of finite groups on surfaces*, Topology Appl. 305 (2022) art. id. 107898 MR Zbl

[10] **GAP Group**, *GAP*: *groups*, *algorithms*, *and programming* (2021) Version 4.11.1 Available at `https://www.gap-system.org`

[11] **J P C Greenlees**, **J P May**, *Generalized Tate cohomology*, Mem. Amer. Math. Soc. 543, Amer. Math. Soc., Providence, RI (1995) MR Zbl

[12] **J P C Greenlees**, **J P May**, *Localization and completion theorems for MU–module spectra*, Ann. of Math. 146 (1997) 509–544 MR Zbl

[13] **B Hanke**, *Geometric versus homotopy theoretic equivariant bordism*, Math. Ann. 332 (2005) 677–696 MR Zbl

[14] **K Jänich**, *Charakterisierung der Signatur von Mannigfaltigkeiten durch eine Additivitätseigenschaft*, Invent. Math. 6 (1968) 35–40 MR Zbl

[15] **K Jänich**, *On invariants with the Novikov additive property*, Math. Ann. 184 (1969) 65–77 MR Zbl

[16] **G Karpilovsky**, *The Schur multiplier*, Lond. Math. Soc. Monogr. (N.S.) 2, Oxford Univ. Press (1987) MR Zbl

[17] **U Karras**, **M Kreck**, **W D Neumann**, **E Ossa**, *Cutting and pasting of manifolds*: SK–*groups*, Math. Lect. Ser. 1, Publish or Perish, Boston, MA (1973) MR Zbl

[18] **B Kunyavskiĭ**, *The Bogomolov multiplier of finite simple groups*, from "Cohomological and geometric approaches to rationality problems" (F Bogomolov, Y Tschinkel, editors), Progr. Math. 282, Birkhäuser, Boston, MA (2010) 209–217 MR Zbl

[19] **P S Landweber**, *Equivariant bordism and cyclic groups*, Proc. Amer. Math. Soc. 31 (1972) 564–570 MR Zbl

[20] **P Löffler**, *Bordismengruppen unitärer Torusmannigfaltigkeiten*, Manuscripta Math. 12 (1974) 307–327 MR Zbl

[21] **J P May**, *Equivariant homotopy and cohomology theory*, CBMS Reg. Conf. Ser. Math. 91, Amer. Math. Soc., Providence, RI (1996) MR Zbl

[22] **P Moravec**, *Unramified Brauer groups of finite and infinite groups*, Amer. J. Math. 134 (2012) 1679–1704 MR Zbl

[23] **P Moravec**, *Unramified Brauer groups and isoclinism*, Ars Math. Contemp. 7 (2014) 337–340 MR Zbl

[24] **W D Neumann**, *Manifold cutting and pasting groups*, Topology 14 (1975) 237–244 MR Zbl

[25] **R Oliver**, SK_1 *for finite group rings, II*, Math. Scand. 47 (1980) 195–231 MR Zbl

[26] **E Ossa**, *Unitary bordism of abelian groups*, Proc. Amer. Math. Soc. 33 (1972) 568–571 MR Zbl

[27] **R J Rowlett**, *The fixed-point construction in equivariant bordism*, Trans. Amer. Math. Soc. 246 (1978) 473–481 MR Zbl

[28] **R J Rowlett**, *Bordism of metacyclic group actions*, Michigan Math. J. 27 (1980) 223–233 MR Zbl

[29] **E Samperton**, *Schur-type invariants of branched G–covers of surfaces*, from "Topological phases of matter and quantum computation" (P Bruillard, C Ortiz Marrero, J Plavnik, editors), Contemp. Math. 747, Amer. Math. Soc., Providence, RI (2020) 173–197 MR Zbl

[30] **E G Samperton**, *Free actions on surfaces that do not extend to arbitrary actions on* 3*–manifolds*, C. R. Math. Acad. Sci. Paris 360 (2022) 161–167 MR Zbl

[31] **I Schur**, *Über die Darstellung der endlichen Gruppen durch gebrochen lineare Substitutionen*, J. Reine Angew. Math. 127 (1904) 20–50 MR Zbl

[32] **S Schwede**, *Global homotopy theory*, New Math. Monogr. 34, Cambridge Univ. Press (2018) MR Zbl

[33] **R E Stong**, *Notes on cobordism theory*: *mathematical notes*, Princeton Univ. Press (1968) MR Zbl

[34] **R E Stong**, *Complex and oriented equivariant bordism*, from "Topology of manifolds" (J C Cantrell, C H Edwards, Jr, editors), Markham, Chicago, IL (1970) 291–316 MR Zbl

[35] **B Uribe**, *The evenness conjecture in equivariant unitary bordism*, from "Proceedings of the International Congress of Mathematicians, II" (B Sirakov, P N de Souza, M Viana, editors), World Sci., Hackensack, NJ (2018) 1217–1239 MR Zbl

[36] **C T C Wall**, *Determination of the cobordism ring*, Ann. of Math. 72 (1960) 292–311 MR Zbl

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