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ON FULL ASYMPTOTICS OF REAL ANALYTIC TORSIONS FOR COMPACT LOCALLY SYMMETRIC ORBIFOLDS

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We consider a certain sequence of flat vector bundles on a compact locally symmetric orbifold, and we evaluate explicitly the associated asymptotic Ray–Singer real analytic torsion. The basic idea is to computing the heat trace via Selberg's trace formula, so that a key point in this paper is to evaluate the orbital integrals associated with nontrivial elliptic elements. For that purpose, we deduce a geometric localization formula, so that we can rewrite an elliptic orbital integral as a sum of certain identity orbital integrals associated with the centralizer of that elliptic element. The explicit geometric formula of Bismut for semisimple orbital integrals plays an essential role in these computations.

1. Introduction

Let (Z, g^{TZ}) be a closed Riemannian manifold of dimension m, and let $F \to Z$ be a complex vector bundle equipped with a Hermitian metric h^F and a flat connection $\nabla^{F,f}$. Let $(\Omega^\bullet(Z,F), d^{Z,F})$ be the associated de Rham complex valued in F. It is equipped with an L_2-metric induced by g^{TZ}, h^F. Let $\boldsymbol{D}^{Z,F,2}$ be the corresponding de Rham–Hodge Laplacian. The real analytic torsion $\mathcal{T}(Z,F)$ is a real-valued (graded) spectral invariant of $\boldsymbol{D}^{Z,F,2}$ introduced by Ray and Singer [1971; 1973]. When Z is odd-dimensional and $(F, \nabla^{F,f})$ is acyclic, this invariant does not depend on the metric data g^{TZ}, h^F. Ray and Singer also conjectured that, for a unitarily flat vector bundle F (i.e., $\nabla^{F,f} h^F = 0$), this invariant coincides with the Reidemeister torsion, a topological invariant associated with $(F, \nabla^{F,f}) \to Z$. This conjecture was later proved by Cheeger [1979] and Müller [1978]. Using the Witten deformation, Bismut and Zhang [1991; 1992] gave an extension of the Cheeger–Müller theorem for arbitrary flat vector bundles.

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If Z is a compact orbifold, and if F is a flat orbifold vector bundle on Z, the Ray–Singer analytic torsion $\mathcal{T}(Z,F)$ extends naturally to this case (see Definition 2.2.3). In particular, if F is acyclic, and if Z and all the singular strata have odd dimensions, then $\mathcal{T}(Z,F)$ is independent of the metric data; see [Shen and Yu 2022, Corollary 4.9]. We refer to [Ma 2005; Shen and Yu 2022] for more details.

We consider a certain sequence of (acyclic) flat vector bundles $\{F_d\}_{d\in\mathbb{N}}$ on a compact locally symmetric space Z, and we study the asymptotic behavior of $\mathcal{T}(Z,F_d)$ as $d\to+\infty$. When Z is a manifold, such question was already studied by Müller [2012], by Bismut, Ma and Zhang [Bismut et al. 2011; 2017] and by Müller and Pfaff [2013b; 2013a]. In particular, Bismut, Ma and Zhang [Bismut et al. 2011; 2017] worked on the manifolds which are more general than locally symmetric manifolds. When Z is a compact hyperbolic orbifold, such question was studied by Fedosova [2015] using the method of harmonic analysis. Here, we consider this question for an arbitrary compact locally symmetric orbifold (of noncompact type).

Let G be a connected linear reductive Lie group equipped with a Cartan involution $\theta\in\mathrm{Aut}(G)$ and an invariant nondegenerate symmetric bilinear form B. Let $K\subset G$ be the fixed-point set of θ, which is a maximal compact subgroup of G. Put

$$X=G/K. \tag{1.0.1}$$

Then X is a Riemannian symmetric space with the Riemannian metric induced from B. For convenience, we also assume that G has a compact center; then X is of noncompact type.

Now let $\Gamma\subset G$ be a cocompact discrete subgroup. Set

$$Z=\Gamma\backslash X. \tag{1.0.2}$$

Then Z is a compact locally symmetric space. In general, Z is an orbifold. Let ΣZ denote the orbifold resolution of the singular points in Z whose connected components correspond exactly to the nontrivial elliptic conjugacy classes of Γ.

Since G has compact center, the compact form U of G exists and is a connected compact linear Lie group. If (E,ρ^E,h^E) is a unitary (analytic) representation of U, then it extends uniquely to a representation of G by a unitary trick. In this way, $F=G\times_K E$ is a vector bundle on X equipped with an invariant flat connection $\nabla^{F,f}$ (see Section 3.4 and (4.1.8)) and a unimodular Hermitian metric h^F induced by h^E. Moreover, $(F,\nabla^{F,f},h^F)$ descends to a flat Hermitian orbifold vector bundle on Z, which is still denoted by $(F,\nabla^{F,f},h^F)$. Let $\boldsymbol{D}^{Z,F,2}$ denote the corresponding de Rham–Hodge Laplacian.

The fundamental rank $\delta(G)$ (or $\delta(X)$) of G (or X) is the difference of the complex ranks of G and of K. As we will see in Theorem 4.1.4, if $\delta(G)\neq 1$, we always have

$$\mathcal{T}(Z,F)=0. \tag{1.0.3}$$

If F is defined instead by a unitary representation of Γ, this result is obtained by Moscovici and Stanton [1991, Corollary 2.2]. If Γ is torsion-free, with F defined via a representation of G as above, (1.0.3) was proved in [Bismut et al. 2017, Remark 8.7] by using Bismut's formula for orbital integrals [2011, Theorem 6.1.1]; see also [Ma 2019, Theorems 5.4 and 5.5]. A new proof was given in [Müller and Pfaff 2013a, Proposition 4.2] (with a correction given in [Matz and Müller 2023, p. 44]). Note that in [Ma 2019, Remark 5.6], it is indicated that, using essentially Theorem 5.4 of that work, the identity (1.0.3) still holds if Γ is not torsion-free (i.e., Z is an orbifold), which gives us exactly Theorem 4.1.4 in this paper. Due to this vanishing result, we only need to deal with the case $\delta(G)=1$.

We now describe the sequence of flat vector bundles $\{F_d\}_{d\in\mathbb{N}}$ which is concerned here. Note that U contains K as a Lie subgroup. Let T be a maximal torus of K, and let T_U be the maximal torus of U containing T. Let $\mathfrak{u}$ be the Lie algebra of U, and let $\mathfrak{t}_U\subset\mathfrak{u}$ be the Lie algebra of T_U. Let $R(\mathfrak{u},\mathfrak{t}_U)$ be the associated real root system with a system of positive roots $R^+(\mathfrak{u},\mathfrak{t}_U)$. Then let $P_{++}(U)\subset\mathfrak{t}_U^*$ denote the set of (real) dominant weights of U with respect to the above root system. If $\lambda\in P_{++}(U)$, let $(E_\lambda,\rho^{E_\lambda})$ be the irreducible unitary representation of U with the highest weight λ. We extend it to a representation of G. We require λ to be nondegenerate, i.e., as G-representations, $(E_\lambda,\rho^{E_\lambda})$ is not isomorphic to $(E_\lambda,\rho^{E_\lambda}\circ\theta)$. We also take an arbitrary $\lambda_0\in P_{++}(U)$. If $d\in\mathbb{N}$, let (E_d,ρ^{E_d},h^{E_d}) be the unitary representation of U with highest weight $d\lambda+\lambda_0$. By Weyl's dimension formula, $\dim E_d$ is a polynomial in d. This way, we get a sequence of (unimodular) flat vector bundles $\{(F_d,\nabla^{F_d},h^{F_d})\}_{d\in\mathbb{N}}$ on X or on Z.

Note that in Section 8.1 (see also [Bergeron and Venkatesh 2013, Lemma 4.1]), the nondegeneracy of λ implies that, for d large enough,

$$H^\bullet(Z,F_d)=0. \tag{1.0.4}$$

Furthermore, $\dim Z$ is odd when $\delta(G)=1$. Then, for any sufficiently large d, $\mathcal{T}(Z,F_d)$ is independent of the different choices of h^{E_d} (or h^{F_d}).

Let $E[\Gamma]$ be the finite set of elliptic classes in Γ. Set $E^+[\Gamma]=E[\Gamma]\backslash\{1\}$. The first main result in this paper is the following theorem.

Theorem 1.0.1. *Assume that $\delta(G)=1$. There exists a (real) polynomial $P(d)$ in d, and for each $[\gamma]\in E^+[\Gamma]$ there exists a nice exponential polynomial $PE^{[\gamma]}(d)$ in d (i.e., a finite sum of the terms of the form $\alpha d^j e^{2\pi\sqrt{-1}\beta d}$, with $\alpha\in\mathbb{C}$, $j\in\mathbb{N}$, $\beta\in\mathbb{Q}$; see Definition 7.6.1) such that there exists a constant $c>0$ for d large, we have*

$$\mathcal{T}(Z,F_d)=P(d)+\sum_{[\gamma]\in E^+[\Gamma]}PE^{[\gamma]}(d)+\mathcal{O}(e^{-cd}). \tag{1.0.5}$$

Moreover, the degrees of $P(d)$, $PE^{[\gamma]}(d)$ can be determined in terms of λ, λ_0.

For a hyperbolic 3-manifold Z, Müller [2012, Theorem 1.1] computed explicitly the leading term of $\mathcal{T}(Z,F_d)$ as $d\to+\infty$. In [Bismut et al. 2011; 2017], under a more general setting for a closed manifold Z, Bismut, Ma and Zhang [Bismut et al. 2017, Remark 7.8] proved that there exists a constant $c>0$ such that

$$\mathcal{T}(Z,F_d)=\mathcal{T}_{L_2}(Z,F_d)+\mathcal{O}(e^{-cd}), \tag{1.0.6}$$

where $\mathcal{T}_{L_2}(Z,F_d)$ denotes the L_2-torsion [Lott 1992; Mathai 1992] associated with $F_d\to Z$. Moreover, they constructed universally an element $W\in\Omega^\bullet(Z,o(TZ))$ (where $o(TZ)$ denotes the orientation bundle of TZ) such that if $n_0=\deg E_d$, then

$$\mathcal{T}_{L_2}(Z,F_d)=d^{n_0+1}\int_Z W+\mathcal{O}(d^{n_0}). \tag{1.0.7}$$

The integral of W in the right-hand side of (1.0.7) is called a W-invariant. If we specialize (1.0.7) for a compact locally symmetric manifold Z, we get

$$\mathcal{T}_{L_2}(Z, F_d) = d^{n_0+1}\,\mathrm{Vol}(Z)[W]^{\max} + \mathcal{O}(d^{n_0}). \tag{1.0.8}$$

In [Bismut et al. 2017, Section 8.7], the explicit computation on $[W]^{\max}$ was carried out for $G = \mathrm{SL}_2(\mathbb{C})$ to recover [Müller 2012, Theorem 1.1].

We now compare (1.0.5) with (1.0.6). If ignoring that Γ may act on X noneffectively, we can extend the notion of L_2-torsion to the orbifold Z, so that $\mathcal{T}_{L_2}(Z, F_d)$ is still defined in terms of the Γ-trace of the heat operators on X. Then $P(d)$ in (1.0.5) is exactly $\mathcal{T}_{L_2}(Z, F_d)$. But different from (1.0.6), we still have the nontrivial terms $PE^{[\gamma]}(d)$, $[\gamma] \in E^+[\Gamma]$ in (1.0.5). We will see, in a refined version of (1.0.5) stated in Theorem 1.0.2, that $PE^{[\gamma]}(d)$ is essentially a linear combination of certain L_2-torsions for ΣZ associated with $[\gamma]$ and λ, λ_0. Therefore, we can define an L_2-torsion for ΣZ as

$$\widetilde{\mathcal{T}}_{L_2}(\Sigma Z, F_d) = \sum_{[\gamma]\in E^+[\Gamma]} PE^{[\gamma]}(d). \tag{1.0.9}$$

Then, as an analogue to (1.0.6), we restate our Theorem 1.0.1 as follows.

Theorem 1.0.1′. *Assume that Γ acts on X effectively. For $Z = \Gamma\backslash X$, as $d \to +\infty$, we have*

$$\mathcal{T}(Z, F_d) = \mathcal{T}_{L_2}(Z, F_d) + \widetilde{\mathcal{T}}_{L_2}(\Sigma Z, F_d) + \mathcal{O}(e^{-cd}). \tag{1.0.10}$$

Moreover, $\mathcal{T}_{L_2}(Z, F_d)$ is a polynomial in d, and $\widetilde{\mathcal{T}}_{L_2}(\Sigma Z, F_d)$ is a nice exponential polynomial in d. Their leading terms can be determined in terms of W-invariants as in (1.0.8).

To understand better on $\widetilde{\mathcal{T}}_{L_2}(\Sigma Z, F_d)$, we need to recall the results in [Müller and Pfaff 2013a] (also in [Müller and Pfaff 2013b] for the hyperbolic case) for a compact locally symmetric manifold Z. They gave a proof to (1.0.6) using Selberg's trace formula, and then showed that $\mathcal{T}_{L_2}(Z, F_d)$ is a polynomial in d. Theorem 1.0.1′ here is an extension of their results, which shows a nontrivial contribution from ΣZ.

Let us give more detail on the results in [Müller and Pfaff 2013a]. Let $\boldsymbol{D}^{X,F_d,2}$ be the G-invariant Laplacian operator on X which is the lift of $\boldsymbol{D}^{Z,F_d,2}$. For $t > 0$, let $p_t^{X,F_d}(x, x')$ denote the heat kernel of $\frac{1}{2}\boldsymbol{D}^{X,F_d,2}$ with respect to the Riemannian volume element on X. For $t > 0$, the identity orbital integral $\mathcal{I}_X(E_d, t)$ of p_t^{X,F_d} is defined as

$$\mathcal{I}_X(F_d, t) = \mathrm{Tr}_s^{\Lambda^\bullet(T_x^*X)\otimes F_{d,x}}\Big[\Big(N^{\Lambda^\bullet(T_x^*X)} - \frac{m}{2}\Big)p_t^{X,F_d}(x, x)\Big], \tag{1.0.11}$$

where $N^{\Lambda^\bullet(T_x^*X)}$ is the number operator on $\Lambda^\bullet(T_x^*X)$, and the right-hand side of (1.0.11) is independent of the choice of $x \in X$. Let $\mathcal{M}\mathcal{I}_X(F_d, s)$, $s \in \mathbb{C}$, denote the Mellin transform (see (7.2.57)) of $\mathcal{I}_X(F_d, t)$, which is holomorphic at 0. Set

$$\mathcal{P}\mathcal{I}_X(F_d) = \frac{\partial}{\partial s}\Big|_{s=0}\mathcal{M}\mathcal{I}_X(F_d, s). \tag{1.0.12}$$

The L_2-torsion is defined as

$$\mathcal{T}_{L^2}(Z, F_d) = \mathrm{Vol}(Z)\mathcal{P}\mathcal{I}_X(F_d). \tag{1.0.13}$$

Using essentially Harish-Chandra's Plancherel theorem for $\mathcal{I}_X(F_d, t)$, Müller and Pfaff [2013a] managed to show that $\mathcal{P}\mathcal{I}_X(F_d)$ is a polynomial in d (for d large enough). Moreover, if $\lambda_0 = 0$, there exists a constant $C_\lambda \neq 0$ such that

$$\mathcal{P}\mathcal{I}_X(F_d) = C_\lambda d \dim E_d + R(d), \tag{1.0.14}$$

where $R(d)$ is a polynomial in d of degree no greater than $\deg \dim E_d$. They also gave concrete formulae for C_λ in some model cases [Müller and Pfaff 2013a, Corollaries 1.4 and 1.5].

In Section 7.4, we use instead an explicit geometric formula of [Bismut 2011, Theorem 6.1.1] for semisimple orbital integrals to give a different computation on $\mathcal{P}\mathcal{I}_X(F_d)$. In Section 7.5, we verify that our computational results coincide with the ones of [Müller and Pfaff 2013a].

For the orbifold case, i.e., Γ contains nontrivial elliptic elements, a key ingredient to Theorem 1.0.1 is to evaluate explicitly the elliptic orbital integrals associated with $[\gamma] \in E^+[\Gamma]$. For that purpose, we make use of the full power of Bismut's formula [2011, Theorem 6.1.1]. Note that if Z is a hyperbolic orbifold, i.e., $G = \mathrm{Spin}(1, 2n+1)$, the result in Theorem 1.0.1 (or Theorem 1.0.1′) was obtained in [Fedosova 2015, Theorem 1.1], where she evaluated the elliptic orbital integrals using Harish-Chandra's Plancherel theorem.

In fact, we obtain in this paper a refined version of Theorem 1.0.1, where we give more explicit descriptions of the exponential polynomials $PE^{[\gamma]}(d)$ and $\widetilde{\mathcal{T}}_{L_2}(\Sigma Z, F_d)$. Before stating this refined result, we need to introduce some notation and facts.

Fix $k \in T$, and let $X(k)$ denote the fixed-point set of k acting on X. Then $X(k)$ is a connected symmetric space with $\delta(X(k)) = 1$. Let $Z(k)^0$ be the identity component of the centralizer $Z(k)$ of k in G. Then $X(k) = Z(k)^0/K(k)^0$, with $K(k)^0 = Z(k)^0 \cap K$. Let $U(k)$ denote the centralizer of k in U with Lie algebra $\mathfrak{u}(k) \subset \mathfrak{u}$. Then $U(k)^0$ is naturally a compact form of $Z(k)^0$, and the triplet $(X(k), Z(k)^0, U(k)^0)$ becomes a smaller version of (X, G, U), except that $Z(k)^0$ may have noncompact center. Note that T_U is also a maximal torus of $U(k)^0$. We get the splitting of roots

$$R(\mathfrak{u}, \mathfrak{t}_U) = R(\mathfrak{u}(k), \mathfrak{t}_U) \cup R(\mathfrak{u}^\perp(k), \mathfrak{t}_U), \tag{1.0.15}$$

where $\mathfrak{u}^\perp(k)$ is the orthogonal space of $\mathfrak{u}(k)$ in $\mathfrak{u}$ with respect to B. Let $R^+(\mathfrak{u}(k), \mathfrak{t}_U)$, $R^+(\mathfrak{u}^\perp(k), \mathfrak{t}_U)$ be the induced positive roots, and let $\rho_{\mathfrak{u}}$, $\rho_{\mathfrak{u}(k)}$ denote the half of the sum of the roots in $R^+(\mathfrak{u}, \mathfrak{t}_U)$, $R^+(\mathfrak{u}(k), \mathfrak{t}_U)$ respectively.

Let $W(\mathfrak{u}_{\mathbb{C}}, \mathfrak{t}_{U,\mathbb{C}})$ be the Weyl group associated with the pair $(\mathfrak{u}, \mathfrak{t}_U)$. Put

$$W_U^1(k) = \{\omega \in W(\mathfrak{u}_{\mathbb{C}}, \mathfrak{t}_{U,\mathbb{C}}) \mid \omega^{-1}(R^+(\mathfrak{u}(k), \mathfrak{t}_U)) \subset R^+(\mathfrak{u}, \mathfrak{t}_U)\}. \tag{1.0.16}$$

If $\sigma \in W_U^1(k)$, let $\varepsilon(\sigma)$ denote its sign. For $\mu \in P_{++}(U)$, set

$$\varphi_k^U(\sigma, \mu) = \varepsilon(\sigma) \frac{\xi_{\sigma(\mu+\rho_{\mathfrak{u}})+\rho_{\mathfrak{u}}}(k)}{\prod_{\alpha \in R^+(\mathfrak{u}^\perp(k), \mathfrak{t}_U)} (\xi_\alpha(k) - 1)} \in \mathbb{C}^*, \tag{1.0.17}$$

where ξ_α is the character of T_U with (dominant) weight $2\pi\sqrt{-1}\alpha$. It is clear that $\varphi_k^U(\sigma, d\lambda + \lambda_0)$ is an oscillating term of the form $c_1 e^{2\pi\sqrt{-1}c_2 d}$, with $c_1 \in \mathbb{C}^*$, $c_2 \in \mathbb{R}$. If k is of finite order, then $c_2 \in \mathbb{Q}$.

By an equivalent definition of nondegeneracy in Definition 7.3.1, for $\sigma \in W_U^1(k)$, $\sigma\lambda$ is a nondegenerate dominant weight of $U(k)^0$ with respect to $\theta|_{Z(k)^0}$. Let $E_{\sigma,d}^k$ denote the unitary representations of $U(k)^0$ (up to a finite central extension) with highest weight $d\sigma\lambda + \sigma(\lambda_0 + \rho_{\mathrm{u}}) - \rho_{\mathrm{u}(k)}$, $d \in \mathbb{N}$, and let $\{F_{\sigma,d}^k\}_{d\in\mathbb{N}}$ be the corresponding sequence of flat vector bundles on $X(k)$.

Now we state our second main theorem, which refines Theorem 1.0.1.

Theorem 1.0.2. *Assume that* $\delta(G) = 1$.

(1) *If* $\Gamma \subset G$ *is a cocompact discrete subgroup and* $\gamma \in \Gamma$ *is elliptic, let* $S(\gamma)$ *denote the finite subgroup of* $\Gamma \cap Z(\gamma)$ *which acts on* $X(\gamma)$ *trivially. Then there exists a constant* $c > 0$, *and, for each* $[\gamma] \in E^+[\Gamma]$, *there exists a nice exponential polynomial in* d, *denoted by* $\mathcal{PE}_{X,\gamma}(F_d)$, *such that, for* $Z = \Gamma\backslash X$, *as* $d \to +\infty$, *we have*

$$\mathcal{T}(Z, F_d) = \frac{\mathrm{Vol}(Z)}{|S(1)|}\mathcal{PI}_X(F_d) + \sum_{[\gamma]\in E^+[\Gamma]} \frac{\mathrm{Vol}(\Gamma \cap Z(\gamma)\backslash X(\gamma))}{|S(\gamma)|}\mathcal{PE}_{X,\gamma}(F_d) + \mathcal{O}(e^{-cd}). \tag{1.0.18}$$

(2) *Fix an elliptic* $[\gamma] \in E^+[\Gamma]$. *Then* $\mathcal{PE}_{X,\gamma}(F_d)$ *depends only on the conjugacy class of* γ *in* G *and is independent of the lattice* Γ. *If* γ *is conjugate to* $k \in T$ *by an element in* G, *then we have the identity*

$$\mathcal{PE}_{X,\gamma}(F_d) = \sum_{\sigma\in W_U^1(k)} \varphi_k^U(\sigma, d\lambda + \lambda_0)\mathcal{PI}_{X(k)}(F_{\sigma,d}^k), \tag{1.0.19}$$

Theorem 1.0.1 now is just a consequence of (1.0.18). Note that, for $[\gamma] \in E^+[\Gamma]$, the (compact) orbifold $\Gamma \cap Z(\gamma)\backslash X(\gamma)$ represents an orbifold stratum in ΣZ (see (3.4.13), Remark 3.4.3). An important observation on (1.0.18) is that the sequence $\{\mathcal{T}(Z, F_d)\}_{d\in\mathbb{N}}$ encodes the volume information on Z as well as on ΣZ. Moreover, combining (1.0.13), (1.0.18) with (1.0.19), we justify that the quantity $\widetilde{\mathcal{T}}_{L_2}(\Sigma Z, F_d)$ defined by (1.0.9) is indeed a linear combination of L_2-torsions such as $\mathcal{T}_{L^2}(\Gamma \cap Z(\gamma)\backslash X(\gamma), F_{\sigma,d}^{\gamma})$ for ΣZ.

Now we explain our approach to Theorem 1.0.2. Let us start with defining $\mathcal{PE}_{X,\gamma}(F_d)$ and (1.0.18). In fact, $\mathcal{T}(Z, F_d)$ can be rewritten as the derivative at 0 of the Mellin transform of

$$\mathrm{Tr}_s\Big[\Big(N^{\Lambda^\bullet(T^*Z)} - \frac{m}{2}\Big)\exp\Big(-\frac{t\boldsymbol{D}^{Z,F_d,2}}{2}\Big)\Big], \quad t > 0, \tag{1.0.20}$$

where $\mathrm{Tr}_s[\cdot]$ denotes the supertrace with respect to the $\mathbb{Z}_2$-grading on $\Lambda^\bullet(T^*Z)$.

If $\gamma \in G$ is semisimple, let $\mathcal{E}_{X,\gamma}(F_d, t)$ denote the orbital integral (see Section 3.3) of the Schwartz kernel of $(N^{\Lambda^\bullet(T^*X)} - m/2)\exp(-t\boldsymbol{D}^{X,F_d,2}/2)$ associated with γ. Note that in $\mathcal{E}_{X,\gamma}(F_d, t)$, we take the supertrace of the endomorphism on $\Lambda^\bullet(T^*X) \otimes F$ (see (4.1.16)). Moreover, $\mathcal{E}_{X,\gamma}(F_d, t)$ depends only on the conjugacy class of γ in G. Let $\mathcal{ME}_{X,\gamma}(F_d, s)$ denote the Mellin transform of $\mathcal{E}_{X,\gamma}(F_d, t)$, $t > 0$ with appropriate $s \in \mathbb{C}$. If $\gamma = 1$, they are just $\mathcal{I}_X(F_d, t)$, $\mathcal{MI}_X(F_d, s)$ introduced in (1.0.11)–(1.0.12).

We use the notation in Section 3.5. Let $[\Gamma]$ denote the set of the conjugacy classes in Γ. By applying Selberg's trace formula to $Z = \Gamma\backslash X$, we get

$$\mathrm{Tr}_s\Big[\Big(N^{\Lambda^\bullet(T^*Z)} - \frac{m}{2}\Big)\exp\Big(-\frac{t\boldsymbol{D}^{Z,F_d,2}}{2}\Big)\Big] = \sum_{[\gamma]\in[\Gamma]} \frac{\mathrm{Vol}(\Gamma \cap Z(\gamma)\backslash X(\gamma))}{|S(\gamma)|}\mathcal{E}_{X,\gamma}(F_d, t). \tag{1.0.21}$$

Now we compare (1.0.18) with (1.0.21). Then a proof to (1.0.18) mainly includes the following three parts:

(1) We show that if $[\gamma] \in E[\Gamma]$, then $\mathcal{M}\mathcal{E}_{X,\gamma}(F_d, s)$ admits a meromorphic extension to $s \in \mathbb{C}$ which is holomorphic at $s = 0$. Thus we define

$$\mathcal{P}\mathcal{E}_{X,\gamma}(F_d) = \frac{\partial}{\partial s}\Big|_{s=0} \mathcal{M}\mathcal{E}_{X,\gamma}(F_d, s). \tag{1.0.22}$$

Such consideration also holds for an arbitrary elliptic element $\gamma \in G$.

(2) If $\gamma \in \Gamma$ is elliptic, then it is of finite order, and from (1.0.19), we get that $\mathcal{P}\mathcal{E}_{X,\gamma}(F_d)$ is a nice exponential polynomial in d for d large enough.

(3) We prove that all the terms in the sum of (1.0.21) associated with nonelliptic $[\gamma] \in [\Gamma]$ contribute as $\mathcal{O}(e^{-cd})$ in $\mathcal{T}(Z, F_d)$.

Indeed, to handle the contribution of the nonelliptic $[\gamma] \in [\Gamma]$, we use a spectral gap of $\boldsymbol{D}^{Z,F_d,2}$ due to the nondegeneracy of λ. By [Bismut et al. 2011, Théorème 3.2], and [Bismut et al. 2017, Theorem 4.4] which holds for a more general setting (see also [Müller and Pfaff 2013a, Proposition 7.5, Corollary 7.6] for a proof by using representation theory for symmetric spaces), there exist constants $C > 0$, $c > 0$ such that, for $d \in \mathbb{N}$,

$$\boldsymbol{D}^{Z,F_d,2} \geq cd^2 - C. \tag{1.0.23}$$

That also explains (1.0.4) for large d. Part (3) follows essentially from the same arguments as in [Müller and Pfaff 2013a, Section 8] and [Bismut et al. 2017, Sections 6.6, 7.2, Remarks 7.8, 8.15] which makes good use of (1.0.23) and the fact that nonelliptic elements in Γ admit a uniform strictly positive lower bound for their displacement distances on X.

For elliptic $\gamma \in \Gamma$, we apply Bismut's formula [2011, Theorem 6.1.1] to evaluate $\mathcal{E}_{X,\gamma}(F_d, t)$. Then we can write $\mathcal{E}_{X,\gamma}(F_d, t)$ as a Gaussian-like integral with the integrand given as a product of an analytic function determined by the adjoint action of γ on Lie algebras and the character χ_{E_d} of the representation E_d. By coordinating these two factors, especially using all sorts of character formulae for χ_{E_d}, we can integrate it out. We show that $\mathcal{E}_{X,\gamma}(F_d, t)$ is a finite sum of the terms

$$t^{-j-\frac{1}{2}} e^{-t(cd+b)^2} Q(d), \tag{1.0.24}$$

where $j \in \mathbb{N}$, $c \neq 0$, b are real constants, and $Q(d)$ is a nice exponential polynomial in d. It is crucial that $c \neq 0$. Indeed, we will see in Section 7.3 that this quantity c measures the difference between the representations $(E_\lambda, \rho^{E_\lambda})$ and $(E_\lambda, \rho^{E_\lambda} \circ \theta)$.

As a consequence of (1.0.24), $\mathcal{P}\mathcal{E}_{X,\gamma}(F_d)$ in (1.0.22) is well-defined, which is clearly a nice exponential polynomial in d (for d large enough). The details on these computations are carried out in Section 7.2, where we apply the techniques inspired by the computations in Shen's approach [2018, Section 7] to the Fried conjecture and also in its extension to orbifold case in [Shen and Yu 2022].

The formula (1.0.19) gives a new and geometric approach to the above results on $\mathcal{P}\mathcal{E}_{X,\gamma}(F_d)$. It is nicer in the sense that each $\mathcal{P}\mathcal{I}_{X(k)}(F^k_{\sigma,d})$ is already well understood and related to the L_2-torsions for the singular stratum of Z. For proving it, we apply a geometric localization formula for $\mathcal{E}_{X,\gamma}(F_d, t)$ as follows.

Theorem 1.0.3. *Assume that $\delta(G)=1$. We use the same notation as in Theorem 1.0.2. Let $\gamma=k\in T$. Then, for $t>0$, $d\in\mathbb{N}$,*

$$\mathcal{E}_{X,\gamma}(F_d,t)=\sum_{\sigma\in W_U^1(k)}\varphi_k^U(\sigma,d\lambda+\lambda_0)\mathcal{I}_{X(k)}(F_{\sigma,d}^k,t). \tag{1.0.25}$$

After taking the Mellin transform on both sides of (1.0.25), we get exactly (1.0.19). In Theorem 6.0.1, we will show a general version of the above geometric localization formula for $\mathcal{E}_{X,\gamma}(F_d,t)$ associated with any semisimple element $\gamma\in G$.

Our approach to Theorem 1.0.3 is a more delicate application of Bismut's formula [2011, Theorem 6.1.1]. As we said, $\mathcal{E}_{X,\gamma}(F_d,t)$, $\mathcal{I}_{X(k)}(F_{\sigma,d}^k,t)$ are equal to integrals of some integrands involving χ_{E_d}, $\chi_{E_{\sigma,d}^k}$ respectively. To relate the two sides of (1.0.25), we employ a generalized version of the Kirillov character formula (see Theorem 5.4.4), which gives an explicit way of decomposing $\chi_{E_d}|_{U(k)^0}$ into a sum of $\chi_{E_{\sigma,d}^k}$, $\sigma\in W_U^1(k)$. This character formula was proved by Duflo, Heckman and Vergne [Duflo et al. 1984, II.3, Theorem (7)] under a general setting, and we will recall its special case for our need in Section 5.4. Then we expand the integral formula for $\mathcal{E}_{X,\gamma}(F_d,t)$ carefully into a sum of certain integrals involving $\chi_{E_{\sigma,d}^k}$, $\sigma\in W_U^1(k)$, which correspond to $\mathcal{I}_{X(k)}(F_{\sigma,d}^k,t)$ via Bismut's formula. This way, we prove (1.0.25).

Theorem 1.0.3 can be interpreted as follows: the action of elliptic element γ on X could lead to a geometric localization onto its fixed-point set $X(k)$ when we evaluate the orbital integrals. Even though we only prove it for a very restrictive situation, we still expect such phenomenon in general due to a geometric formulation for the semisimple orbital integrals; see [Bismut 2011, Chapter 4].

Finally, we note that in [Bismut et al. 2017, Section 8], the authors explained well how to use Bismut's formula for semisimple orbital integrals to study the asymptotic analytic torsion. Here, we go one step further in that direction to get a refined evaluation on it. Bergeron and Venkatesh [2013] also studied the asymptotic analytic torsion but under a totally different setting. In [Liu 2018; 2021], the asymptotic equivariant analytic torsion for a locally symmetric space was studied, and the oscillating terms also appeared naturally in that case. Moreover, Finski [2018, Theorem 1.5] obtained the full asymptotic expansion of the holomorphic analytic torsions for the tensor powers of a given positive line bundle over a compact complex orbifold.

This paper is organized as follows. In Section 2, we recall the definition of Ray–Singer analytic torsion for compact orbifolds. We also include a brief introduction to the orbifolds at beginning.

In Section 3, we introduce the explicit geometric formula of Bismut for semisimple orbital integrals and the Selberg's trace formula for compact locally symmetric orbifolds. They are the main tools to study the analytic torsions in this paper.

In Section 4, we give a vanishing theorem for $\mathcal{T}(Z,F)$, so that we only need to focus on the case $\delta(G)=1$.

In Section 5, we study the Lie algebra of G provided $\delta(G)=1$. Furthermore, we introduce a generalized Kirillov formula for compact Lie groups.

In Section 6, we prove a general version of Theorem 1.0.3.

In Section 7, given the sequence $\{F_d\}_{d\in\mathbb{N}}$, we compute explicitly $\mathcal{E}_{X,\gamma}(F_d,t)$ in terms of root systems for elliptic γ; in particular, we prove (1.0.24). Then we give the formulae for $\mathcal{P}\mathcal{I}_X(F_d)$, $\mathcal{P}\mathcal{E}_{X,\gamma}(F_d)$.

Finally, in Section 8, we introduce the spectral gap (1.0.23) and we give a proof of Theorem 1.0.2.

In this paper, if V is a real vector space and if E is a complex vector space, we will use the symbol $V \otimes E$ to denote the complex vector space $V \otimes_{\mathbb{R}} E$. If both V and E are complex vector spaces, then $V \otimes E$ is just the usual tensor over $\mathbb{C}$.

2. Ray–Singer analytic torsion

In this section, we recall the definitions of the orbifold and the orbifold vector bundle. We also refer to [Satake 1956; 1957; Adem et al. 2007, Chapter 1] for more details. Then we recall the definition of Ray–Singer analytic torsion for compact orbifolds, where we refer to [Ma 2005; Shen and Yu 2022] for more details. In particular, Shen and Yu [2022] extended many important results on real analytic torsion from the manifold setting to the orbifold setting.

2.1. ***Orbifolds and orbifold vector bundles.*** Let Z be a topological space.

Definition 2.1.1. If U is a connected open subset of Z, an orbifold chart for U is a triple $(\widetilde{U}, \pi_U, G_U)$ such that

- $\widetilde{U}$ is a connected open set of some $\mathbb{R}^m$ and G_U is a finite group acting smoothly and effectively on $\widetilde{U}$ on the left;
- π_U is a continuous surjective $\widetilde{U} \to U$, which is invariant by a G_U-action;
- π_U induces a homeomorphism between $G_U \backslash \widetilde{U}$ and U.

If $V \subset U$ is a connected open subset, an embedding of orbifold chart for the inclusion $i : V \to U$ is an orbifold chart $(\widetilde{V}, \pi_V, G_V)$ for V and an orbifold chart $(\widetilde{U}, \pi_U, G_U)$ for U together with a smooth embedding $\phi_{UV} : \widetilde{V} \to \widetilde{U}$ such that the following diagram commutes:

$$\begin{tikzcd} \widetilde{V} \arrow[r, "\phi_{UV}"] \arrow[d, "\pi_V"'] & \widetilde{U} \arrow[d, "\pi_U"] \\ V \arrow[r, "i"] & U \end{tikzcd} \tag{2.1.1}$$

If U_1, U_2 are two connected open subsets of Z with the charts $(\widetilde{U}_1, \pi_{U_1}, G_{U_1})$, $(\widetilde{U}_2, \pi_{U_2}, G_{U_2})$ respectively, we say that these two orbifold charts are compatible if, for any point $z \in U_1 \cap U_2$, there exists an open connected neighborhood $V \subset U_1 \cap U_2$ of z with an orbifold chart $(\widetilde{V}, \pi_V, G_V)$ such that there exist two embeddings of orbifold charts $\phi_{U_1V} : (\widetilde{V}, \pi_V, G_V) \to (\widetilde{U}_1, \pi_{U_1}, G_{U_1})$, $\phi_{U_2V} : (\widetilde{V}, \pi_V, G_V) \to (\widetilde{U}_2, \pi_{U_2}, G_{U_2})$. In this case, the diffeomorphism $\phi_{U_2V} \circ \phi_{U_1V}^{-1} : \phi_{U_1V}(\widetilde{V}) \to \phi_{U_2V}(\widetilde{V})$ is called a coordinate transformation.

Definition 2.1.2. An orbifold atlas on Z is couple $(\mathcal{U}, \widetilde{\mathcal{U}})$ consisting of a cover $\mathcal{U}$ of open connected subsets of Z and a family of compatible orbifold charts $\widetilde{\mathcal{U}} = \{(\widetilde{U}, \pi_U, G_U)\}_{U \in \mathcal{U}}$.

An orbifold atlas $(\mathcal{V}, \widetilde{\mathcal{V}})$ is called a refinement of $(\mathcal{U}, \widetilde{\mathcal{U}})$ if $\mathcal{V}$ is a refinement of $\mathcal{U}$ and if every orbifold chart in $\widetilde{\mathcal{V}}$ has an embedding into some orbifold chart in $\widetilde{\mathcal{U}}$. Two orbifold atlas are said to be equivalent if they have a common refinement, and the equivalent class of an orbifold atlas is called an orbifold structure on Z.

An orbifold is a second countable Hausdorff space equipped with an orbifold structure. It is said to have dimension m if all the orbifold charts which define the orbifold structure are of dimension m.

If Z, Y are two orbifolds, a smooth map $f : Z \to Y$ is a continuous map from Z to Y such that it lifts locally to an equivariant smooth map from an orbifold chart of Z to any orbifold chart of Y. In this way, we can define the notion of smooth functions and the smooth action of Lie groups.

By [Shen and Yu 2022, Proposition 2.12], if Γ is discrete group acting smoothly and properly discontinuously on the left on an orbifold X, then $Z = \Gamma\backslash X$ has a canonical orbifold structure induced from X.

In the sequel, let Z be an orbifold with an orbifold structure given by $(\mathcal{U}, \widetilde{\mathcal{U}})$. If $z \in Z$, there exists an open connected neighborhood U_z of z with a compatible orbifold chart $(\widetilde{U}_z, G_z, \pi_z)$ such that $\pi_z^{-1}(z)$ contains only one point $x \in \widetilde{U}_z$. Then G_z does not depend on the choice of such open connected neighborhood (up to canonical isomorphisms compatible with the orbifold structure), and G_z is called the local group at z.

Put

$$Z_{\mathrm{reg}} = \{z \in Z \mid G_z = \{1\}\}, \quad Z_{\mathrm{sing}} = \{z \in Z \mid G_z \neq \{1\}\}. \tag{2.1.2}$$

Then Z_{reg} is naturally a smooth manifold. But Z_{sing} is not necessarily an orbifold. Kawasaki [1978, Section 2] explained two different methods to view Z_{sing} as an immersed image of a disjoint union of orbifolds. We just recall that method which appears naturally in Kawasaki's local index theorems for orbifolds [1978; 1979].

If $z \in Z_{\mathrm{sing}}$, let $1 = (h_z^0), (h_z^1), \dots, (h_z^{l_z})$ be the conjugacy classes in G_z. Put

$$\Sigma Z = \{(z, (h_z^j)) \mid z \in Z_{\mathrm{sing}}, j = 1, \dots, l_z\}. \tag{2.1.3}$$

Let $(\widetilde{U}_z, G_z, \pi_z)$ be the local orbifold chart for $z \in Z_{\mathrm{sing}}$ such that $\pi_z^{-1}(z)$ contains only one point. For $j = 1, \dots, l_z$, let $\widetilde{U}_z^{h_z^j} \subset \widetilde{U}_z$ be the fixed-point set of h_z^j, which is a submanifold of $\widetilde{U}_z$. Note that $\widetilde{U}_z^{h_z^j} \subset Z_{\mathrm{sing}}$. Let $Z_{G_z}(h_z^j)$ be the centralizer of h_z^j in G_z. Then $Z_{G_z}(h_z^j)$ acts smoothly on $\widetilde{U}_z^{h_z^j}$. Put

$$K_z^j = \ker(Z_{G_z}(h_z^j) \to \mathrm{Aut}(\widetilde{U}_z^{h_z^j})). \tag{2.1.4}$$

Then $(\widetilde{U}_z^{h_z^j}, Z_{G_z}(h_z^j)/K_z^j, \pi_z^j : \widetilde{U}_z^{h_z^j} \to \widetilde{U}_z^{h_z^j}/Z_{G_z}(h_z^j))$ defines an orbifold chart near $(z, (h_z^j)) \in \Sigma Z$. They form an orbifold structure for ΣZ. Let Z^i, $i = 1, \dots, l$, denote the connected components of the orbifold ΣZ.

The integer $m_z^j = |K_z^j|$ is called the multiplicity of ΣZ in Z at $(z, (h_z^j))$. This defines a function $m : \Sigma Z \to \mathbb{Z}_+$. As explained in [Kawasaki 1978, Section 1], m is locally constant on ΣZ, and let $m_i \in \mathbb{Z}_+$ be the value of m on Z^i for $i = 1, \dots, l$. We call m_i the multiplicity of Z^i in Z. We will put

$$Z^0 = Z, \quad m_0 = 1. \tag{2.1.5}$$

Remark 2.1.3. In Definition 2.1.1, for an orbifold chart, we require the action G_U on $\widetilde{U}$ to be effective. To emphasize this condition, the orbifold defined above is often called an effective orbifold. In fact, we can drop this effectiveness; then we get a general version of the (possibly ineffective) orbifold, for example, using the orbifold groupoid; see [Adem et al. 2007, Definition 1.38]. The point-view of orbifold groupoid provides a unified way to deal with effective and ineffective orbifolds.

As explained in [Adem et al. 2007, Example 2.5], for global quotient groupoids (including all the effective orbifolds and certain ineffective orbifolds), a natural stratification called the inertia groupoid was introduced as an extension of the one $\bigcup_{i=0}^{l} Z^i$ defined in (2.1.3)–(2.1.5). It plays a key role in the study of the geometry of orbifolds. We will go back to this point in Sections 3.4 and 3.5. Through this paper, the terminology orbifold will always refer to the effective one unless otherwise stated.

We say E is an orbifold vector bundle of rank r on Z if there exists a smooth map of orbifolds $\pi : E \to Z$ such that, for any $U \in \mathcal{U}$ and $(\widetilde{U}, G_U, \pi_U) \in \widetilde{\mathcal{U}}$, there exists an orbifold chart $(\widetilde{U}^E, G_U^E, \pi_U^E)$ of E such that $\widetilde{U}^E$ is an vector bundle on $\widetilde{U}$ of rank r equipped an effective action of G_U^E and $\pi_U^E(\widetilde{U}^E) = \pi^{-1}(U)$. Moreover, there exists a surjective group morphism $\psi_U : G_U^E \to G_U$ such that the action of G_U^E on $\widetilde{U}$ is identified via ψ_U with the action of G_U on $\widetilde{U}$. If we have an open embedding $\phi_{UV} : (\widetilde{V}, \pi_V, G_V) \to (\widetilde{U}, \pi_U, G_U)$, we require that it lifts to the open embedding $\phi_{UV}^E : (\widetilde{V}^E, \pi_V^E, G_V^E) \to (\widetilde{U}^E, \pi_U^E, G_U^E)$ of the orbifold charts of E such that $\phi_{UV}^E : \widetilde{V}^E \to \widetilde{U}^E$ is a morphism of vector bundles associated with the open embedding $\phi_{UV} : \widetilde{V} \to \widetilde{U}$. If every $\psi_U : G_U^E \to G_U$ is an isomorphism of groups, we call E a proper orbifold vector bundle on Z.

Note that if E is proper, then the rank of E can be extended to a locally constant function ρ on ΣZ. The orbifold chart of Z^i is given by the triples such as

$$(\widetilde{U}_z^{h_z^j}, Z_{G_z}(h_z^j)/K_z^j, \pi_z^j : \widetilde{U}_z^{h_z^j} \to \widetilde{U}_z^{h_z^j}/Z_{G_z}(h_z^j)).$$

By the above definition of E, we have an orbifold chart $(\widetilde{U}^E, G_U^E = G_U, \pi_U^E)$ such that $\widetilde{U}^E$ is a G_U-equivariant vector bundle on $\widetilde{U}$. Then, for $x \in \widetilde{U}_z^{h_z^j}$, h_z^j acts on the fibers $\widetilde{U}_z^E$ linearly, so that we can set $\rho(z, (h_z^j)) = \mathrm{Tr}^{\widetilde{U}_z^E}[h_z^j]$. Then ρ is really a locally constant function on ΣZ. For $i = 1, \dots, l$, let ρ_i be the value of ρ on the component Z^i. We also put $\rho_0 = r$.

We call $s : Z \to E$ a smooth section of E over Z if it is a smooth map between orbifolds such that $\pi \circ s = \mathrm{Id}_Z$. We will use $C^\infty(Z, E)$ to denote the vector space of smooth sections of E over Z.

Take an orbifold chart $(\widetilde{U}, G_U, \pi_U) \in \widetilde{\mathcal{U}}$ of Z. Then G_U acts canonically on the tangent vector bundle $T\widetilde{U}$ of $\widetilde{U}$. The open embeddings of orbifold charts of Z also lift to the open embeddings of their tangent vector bundles. This way, we get a proper orbifold vector bundle TZ on Z, and the projection $\pi : TZ \to Z$ is just given by the obvious projection $T\widetilde{U} \to \widetilde{U}$. We call TZ the tangent vector bundle of Z. If we equipped TZ with Euclidean metric g^{TZ}, we will call Z a Riemannian orbifold and call g^{TZ} a Riemannian metric of Z.

Let $\Omega^\bullet(Z)$ denote the set of smooth differential forms of Z, which has a $\mathbb{Z}$-graded structure by degrees. The de Rham differential $d^Z : \Omega^\bullet(Z) \to \Omega^{\bullet+1}(Z)$ is given by the family of de Rham differential operators $d^{\widetilde{U}} : \Omega^\bullet(\widetilde{U}) \to \Omega^{\bullet+1}(\widetilde{U})$. Then we can define the de Rham complex $(\Omega^\bullet(Z), d^Z)$ of Z and the associated de Rham cohomology $H^\bullet(Z, \mathbb{R})$. By [Kawasaki 1978, Section 1], there is a natural isomorphism between $H^\bullet(Z, \mathbb{R})$ and the singular cohomology of the underlying topological space Z.

Now let us recall the integrals on Z. Assume that Z is compact. We may take a finite open covering $\{U_i\}_{i \in I}$ of the precompact orbifold charts for Z. Since Z is Hausdorff, there exists a partition of unity subordinate to this open cover. We can find a family of smooth functions $\{\phi_i \in C_c^\infty(Z)\}_{i\in I}$ with values in [0, 1] such that $\mathrm{Supp}(\phi_i) \subset U_i$, and that

$$\sum_{i\in I} \phi_i = 1. \tag{2.1.6}$$

Take $\tilde{\phi}_i = \pi_{U_i}^*(\phi_i) \in C_c^\infty(\widetilde{U}_i)^{G_{U_i}}$.

If $\alpha \in \Omega^m(Z, o(TZ))$, let $\tilde{\alpha}_{U_i}$ be its lift on the chart $(\widetilde{U}_i, \pi_{U_i}, G_{U_i})$. We define

$$\int_Z \alpha = \sum_i \frac{1}{|G_{U_i}|} \int_{\widetilde{U}_i} \tilde{\phi}_i \tilde{\alpha}_{U_i}. \tag{2.1.7}$$

By [Shen and Yu 2022, Section 3.2], if $\alpha \in \Omega^m(Z, o(TZ))$, then α is also integrable on Z_{reg}, so that

$$\int_Z \alpha = \int_{Z_{\mathrm{reg}}} \alpha. \tag{2.1.8}$$

Also if $\alpha \in \Omega^\bullet(Z, o(TZ))$, we have

$$\int_Z d^Z \alpha = 0. \tag{2.1.9}$$

If (Z, g^{TZ}) is a Riemannian orbifold, we can define the integration of functions on Z with respect to the Riemannian volume element. If we have a Hermitian orbifold vector bundle $(F, h^F) \to (Z, g^{TZ})$, one can define the L_2 scalar product for the space of continuous sections of F as usual. Then, after completion, we get the Hilbert space $L^2(Z, F)$.

Chern–Weil theory on the characteristic forms extends to orbifolds, where their constructions are parallel to the case of smooth manifolds. We refer to [Shen and Yu 2022, Section 3.4] for more details. Note that the characteristic forms are not only defined on Z but also defined on ΣZ. The part ΣZ has a nontrivial contribution in Kawasaki's local index theorems for orbifolds [1978; 1979].

Finally, we introduce the orbifold Euler characteristic number of (Z, g^{TZ}) [Satake 1957]. Let $\nabla^{TZ} = \{\nabla^{T\widetilde{U}_i}\}_{U_i\in\mathcal{U}}$ be the Levi-Civita connection on TZ associated with g^{TZ}. The Euler form $e(TZ, \nabla^{TZ}) \in \Omega^m(Z, o(TZ))$ is given by the family of closed forms

$$\{e(\widetilde{U}_i, \nabla^{T\widetilde{U}_i}) \in \Omega^m(\widetilde{U}_i, o(T\widetilde{U}_i))^{G_{U_i}}\}_{U_i\in\mathcal{U}}. \tag{2.1.10}$$

If Z is oriented, then we can view $e(TZ, \nabla^{TZ})$ as a differential form on Z.

If Z is compact, set

$$\chi_{\mathrm{orb}}(Z) = \int_Z e(TZ, \nabla^{TZ}). \tag{2.1.11}$$

By [Satake 1957, Section 3], $\chi_{\mathrm{orb}}(Z)$ is a rational number, and it vanishes when Z is odd-dimensional.

2.2. ***Flat vector bundles and analytic torsions of orbifolds.*** If (F, ∇^F) is an orbifold vector bundle over Z with a connection ∇^F, we call (F, ∇^F) a flat vector bundle if the curvature $R^F = \nabla^{F,2}$ vanishes identically on Z. A detailed discussion for the flat vector bundles on Z is given in [Shen and Yu 2022, Sections 2.3–2.5].

Let (Z, g^{TZ}) be a compact Riemannian orbifold of dimension m. Let (F, ∇^F) be a flat complex orbifold vector bundle of rank r on Z with Hermitian metric h^F. Note that we do not assume that F is proper.

Let $\Omega^\bullet(Z,F)$ be the set of smooth sections of $\Lambda^\bullet(T^*Z)\otimes F$ on Z. Let d^Z be the exterior differential acting on $\Omega^\bullet(Z,\mathbb{R})$.

Definition 2.2.1. For $i=0,1,\dots,m$, if $\alpha\in\Omega^i(Z,\mathbb{R})$, $s\in C^\infty(Z,F)$, the operator $d^{Z,F}$ acting on $\Omega^i(Z,F)$ is defined by

$$d^{Z,F}(\alpha\otimes s)=(d^Z\alpha)\otimes s+(-1)^i\alpha\wedge\nabla^F s\in\Omega^{i+1}(Z,F). \tag{2.2.1}$$

Since $\nabla^{F,2}=0$, then $(\Omega^\bullet(Z,F),d^{Z,F})$ is a complex, which is called the de Rham complex for the flat orbifold vector bundle (F,∇^F) on Z. Let $H^\bullet(Z,F)$ denote the corresponding de Rham cohomology group of Z valued in F, as in the case of closed manifolds, $H^\bullet(Z,F)$ is always finite-dimensional.

Let $\langle\cdot,\cdot\rangle_{\Lambda^\bullet(T^*Z)\otimes F,z}$ be the Hermitian metric on $\Lambda^\bullet(T^*_zZ)\otimes F_z$, $z\in Z$ induced by g^{TZ}_z and h^F_z. Let dv be the Riemannian volume element on Z induced by g^{TZ}. The L_2-scalar product on $\Omega^\bullet(Z,F)$ is given as follows: if $s,s'\in\Omega^\bullet(Z,F)$, then

$$\langle s,s'\rangle_{L^2}=\int_Z\langle s(z),s(z')\rangle_{\Lambda^\bullet(T^*Z)\otimes F,z}\,dv(z). \tag{2.2.2}$$

By (2.1.8), it will be the same if we take the integrals on Z_{reg}.

Let $d^{Z,F,*}$ be the formal adjoint of $d^{Z,F}$ with respect to the above L_2-metric on $\Omega^\bullet(Z,F)$; i.e., for $s,s'\in\Omega^\bullet(Z,F)$,

$$\langle d^{Z,F,*}s,s'\rangle_{L^2}=\langle s,d^{Z,F}s'\rangle_{L^2}. \tag{2.2.3}$$

Then $d^{Z,F,*}$ is a first-order differential operator acting $\Omega^\bullet(Z,F)$ on which decreases the degree by 1.

Definition 2.2.2. The de Rham–Hodge operator $\boldsymbol{D}^{Z,F}$ of $\Omega^\bullet(Z,F)$ is defined as

$$\boldsymbol{D}^{Z,F}=d^{Z,F}+d^{Z,F,*}. \tag{2.2.4}$$

It is a first-order self-adjoint elliptic differential operator acting on $\Omega^\bullet(Z,F)$.

The Hodge Laplacian is

$$\boldsymbol{D}^{F,Z,2}=[d^{Z,F},d^{Z,F,*}]=d^{Z,F}d^{Z,F,*}+d^{Z,F,*}d^{Z,F}. \tag{2.2.5}$$

Here, $[\cdot,\cdot]$ denotes the supercommutator. Then $\boldsymbol{D}^{Z,F,2}$ is a second-order essentially self-adjoint nonnegative elliptic operator, which preserves the degree.

The Hodge decomposition for $\Omega^\bullet(Z,F)$ still holds in this case (see [Ma 2005, Proposition 2.2; Dai and Yu 2017, Proposition 2.1]),

$$\Omega^\bullet(Z,F)=\ker(\boldsymbol{D}^{Z,F,2}|_{\Omega^\bullet(Z,F)})\oplus\mathrm{Im}(d^{Z,F}|_{\Omega^{\bullet-1}(Z,F)})\oplus\mathrm{Im}(d^{Z,F,*}|_{\Omega^{\bullet+1}(Z,F)}). \tag{2.2.6}$$

Then we have the canonical identification of vector spaces,

$$\mathcal{H}^\bullet(Z,F):=\ker\boldsymbol{D}^{Z,F,2}\simeq H^\bullet(Z,F). \tag{2.2.7}$$

Put

$$\chi(Z,F)=\sum_{j=0}^m(-1)^j\dim H^j(Z,F). \tag{2.2.8}$$

If F is proper, recall that the numbers $\rho_i,\ i=0,\dots,l$, are defined in previous subsection as the extension of the rank of F. Then by [Shen and Yu 2022, Theorem 4.3], we have

$$\chi(Z,F)=\sum_{i=0}^{l}\rho_i\frac{\chi_{\mathrm{orb}}(Z_i)}{m_i}. \tag{2.2.9}$$

The right-hand side of (2.2.9) contains the nontrivial contributions from ΣZ.

Let P denote the orthogonal projection from $\Omega^\bullet(Z,F)$ to $\mathcal{H}^\bullet(Z,F)$. Let $\mathcal{H}^\perp$ denote the orthogonal subspace of $\mathcal{H}^\bullet(Z,F)$ in $\Omega^\bullet(Z,F)$, and let $(\boldsymbol{D}^{Z,F,2})^{-1}$ be the inverse of $\boldsymbol{D}^{Z,F,2}$ acting on $\mathcal{H}^\perp$. Let $N^{\Lambda^\bullet(T^*Z)}$ be the number operator on $\Lambda^\bullet(T^*Z)$ which acts on $\Lambda^j(T^*Z)$ by multiplication of j.

For $s\in\mathbb{C}$, $\Re(s)$ is large enough; set

$$\begin{aligned}\vartheta(F)(s)&=-\operatorname{Tr}_s[N^{\Lambda^\bullet(T^*Z)}(\boldsymbol{D}^{Z,F,2})^{-s}]\\&=-\frac{1}{\Gamma(s)}\int_0^{+\infty}\operatorname{Tr}_s[N^{\Lambda^\bullet(T^*Z)}\exp(-t\boldsymbol{D}^{Z,F,2})(1-P)]t^{s-1}\,dt,\end{aligned} \tag{2.2.10}$$

where $\Gamma(s)$ is the gamma function for $s\in\mathbb{C}$. By the short time asymptotic expansions of the heat trace (see [Ma 2005, Proposition 2.1]), $\vartheta(F)(s)$ admits a unique meromorphic extension to $s\in\mathbb{C}$ which is holomorphic at $s=0$.

Definition 2.2.3. Let $\mathcal{T}(g^{TZ},\nabla^F,h^F)\in\mathbb{R}$ be given by

$$\mathcal{T}(g^{TZ},\nabla^F,h^F)=\frac{d}{ds}\Big|_{s=0}\vartheta(F)(s). \tag{2.2.11}$$

The quantity $\mathcal{T}(g^{TZ},\nabla^F,h^F)$ is called Ray–Singer analytic torsion associated with (F,∇^F,h^F).

By [Shen and Yu 2022, Proposition 4.6, Corollary 4.9], for an orientable closed orbifold Z, if m is even and F is unitarily flat, then $\mathcal{T}(g^{TZ},\nabla^F,h^F)=0$; if m is odd and F is acyclic, then $\mathcal{T}(g^{TZ},\nabla^F,h^F)$ is independent of the metrics g^{TZ} and h^F.

Now we explain how to evaluate $\mathcal{T}(g^{TZ},\nabla^F,h^F)$ in practice when F is acyclic. Using the analogous arguments in [Bismut and Zhang 1992, Theorem 7.10, Section XI], as $t\to 0^+$, the heat supertrace $\operatorname{Tr}_s[(N^{\Lambda^\bullet(T^*Z)}-m/2)\exp(-t\boldsymbol{D}^{Z,F,2}/2)]$ either has a leading term as a multiple of $1/\sqrt{t}$ or is a small quantity as $\mathcal{O}(\sqrt{t})$; see [Shen and Yu 2022, equation (4.37)]. To deal with this possible divergent term $1/\sqrt{t}$ in the integral of (2.2.10), we proceed as in the proof of [Bismut and Lott 1995, Theorem 3.29]. For $t>0$, put

$$b_t(g^{TZ},F)=\Big(1+2t\frac{\partial}{\partial t}\Big)\operatorname{Tr}_s\Big[\Big(N^{\Lambda^\bullet(T^*Z)}-\frac{m}{2}\Big)\exp\Big(-\frac{t\boldsymbol{D}^{Z,F,2}}{2}\Big)\Big]. \tag{2.2.12}$$

By [Bismut and Zhang 1992, Theorem 7.10; Bismut and Lott 1995, Theorem 2.13; Shen and Yu 2022, Section 4.3] and since F is acyclic, as $t\to 0$,

$$b_t(g^{TZ},F)=\mathcal{O}(\sqrt{t}); \tag{2.2.13}$$

as $t\to+\infty$,

$$b_t(g^{TZ},F)=\mathcal{O}\Big(\frac{1}{\sqrt{t}}\Big). \tag{2.2.14}$$

By [Bismut and Lott 1995, Theorem 3.29; Shen and Yu 2022, Corollary 4.14], we have

$$\mathcal{T}(g^{TZ}, \nabla^F, h^F) = -\int_0^{+\infty} b_t(g^{TZ}, F)\,\frac{dt}{t}. \tag{2.2.15}$$

One particular case is that if, for $t > 0$, we always have

$$\mathrm{Tr}_s\Big[\Big(N^{\Lambda^\bullet(T^*Z)} - \frac{m}{2}\Big)\exp\Big(-\frac{t\boldsymbol{D}^{Z,F,2}}{2}\Big)\Big] = 0, \tag{2.2.16}$$

then $\mathcal{T}(g^{TZ}, \nabla^F, h^F) = 0$. This holds even for nonacyclic F.

3. Orbital integrals and locally symmetric spaces

In this section, we recall the geometry of the symmetric space X, and we recall an explicit geometric formula for semisimple orbital integrals obtained in [Bismut 2011, Chapter 6] . Then, given a cocompact discrete subgroup $\Gamma \subset G$, we describe the orbifold structure on $Z = \Gamma\backslash X$, and we give Selberg's trace formula for Z.

In this section, G is taken to be a connected linear real reductive Lie group; we do not require that it has a compact center. Then X is a symmetric space which may have de Rham components of both noncompact type and Euclidean type.

3.1. ***Real reductive Lie group.*** Let G be a connected linear real reductive Lie group with Lie algebra $\mathfrak{g}$, and let $\theta \in \mathrm{Aut}(G)$ be a Cartan involution. Let K be the fixed-point set of θ in G. Then K is a maximal compact subgroup of G, and let $\mathfrak{k}$ be its Lie algebra. Let $\mathfrak{p} \subset \mathfrak{g}$ be the eigenspace of θ associated with the eigenvalue -1. The Cartan decomposition of $\mathfrak{g}$ is given by

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}. \tag{3.1.1}$$

Put $m = \dim \mathfrak{p}$, $n = \dim \mathfrak{k}$.

Let B be a G- and θ-invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$, which is positive on $\mathfrak{p}$ and negative on $\mathfrak{k}$. It induces a symmetric bilinear form B^* on $\mathfrak{g}^*$, which extends to a symmetric bilinear form on $\Lambda^\bullet(\mathfrak{g}^*)$. The K-invariant bilinear form $\langle\cdot,\cdot\rangle = -B(\cdot, \theta\,\cdot)$ is a scalar product on $\mathfrak{g}$, which extends to a scalar product on $\Lambda^\bullet(\mathfrak{g}^*)$. We will use $|\cdot|$ to denote the norm under this scalar product.

Let $U\mathfrak{g}$ be the universal enveloping algebra of $\mathfrak{g}$. Let $C^{\mathfrak{g}} \in U\mathfrak{g}$ be the Casimir element associated with B; i.e., if $\{e_i\}_{i=1,\dots,m+n}$ is a basis of $\mathfrak{g}$, and if $\{e_i^*\}_{i=1,\dots,m+n}$ is the dual basis of $\mathfrak{g}$ with respect to B, then

$$C^{\mathfrak{g}} = -\sum e_i^* e_i. \tag{3.1.2}$$

We can identify $U\mathfrak{g}$ with the algebra of left-invariant differential operators over G; then $C^{\mathfrak{g}}$ is a second-order differential operator, which is $\mathrm{Ad}(G)$-invariant. Similarly, let $C^{\mathfrak{k}} \in U\mathfrak{k}$ denote the Casimir operator associated with $(\mathfrak{k}, B|_{\mathfrak{k}})$.

Let $\mathfrak{z}_{\mathfrak{g}} \subset \mathfrak{g}$ be the center of $\mathfrak{g}$. Put

$$\mathfrak{g}_{\mathrm{ss}} = [\mathfrak{g}, \mathfrak{g}]. \tag{3.1.3}$$

Then

$$\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}} \oplus \mathfrak{g}_{\mathrm{ss}}. \tag{3.1.4}$$

They are orthogonal with respect to B.

Let Z_G be the center of G, and let G_{ss} be the closed analytic subgroup of G associated with $\mathfrak{g}_{\mathrm{ss}}$; see [Knapp 2002, Corollary 7.11]. Then G is the commutative product of Z_G and G_{ss}; in particular,

$$G = Z_G^0 G_{\mathrm{ss}}. \tag{3.1.5}$$

Let $i = \sqrt{-1}$ denote one square root of -1. Put

$$\mathfrak{u} = \sqrt{-1}\mathfrak{p} \oplus \mathfrak{k}. \tag{3.1.6}$$

For simplicity, if $a \in \mathfrak{p}$, we write ia or $\sqrt{-1}a \in \sqrt{-1}\mathfrak{p} \subset \mathfrak{u}$ to denote the corresponding vector.

Then $\mathfrak{u}$ is a (real) Lie algebra, which is called the compact form of $\mathfrak{g}$. Then

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{u}_{\mathbb{C}}. \tag{3.1.7}$$

Let $G_{\mathbb{C}}$ be the complexification of G with Lie algebra $\mathfrak{g}_{\mathbb{C}}$, which is closed and linear reductive [Knapp 1986, Proposition 5.6]. Then G is the analytic subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}$. Let $U \subset G_{\mathbb{C}}$ be the analytic subgroup associated with $\mathfrak{u}$. If G has compact center, i.e., $\mathfrak{z}_{\mathfrak{g}} \cap \mathfrak{p} = \{0\}$, then by [Knapp 1986, Proposition 5.3], U is compact; since $G_{\mathbb{C}}$ is closed, U is a maximal compact subgroup of $G_{\mathbb{C}}$.

Definition 3.1.1. An element $\gamma \in G$ is said to be semisimple if there exists $g \in G$ such that

$$\gamma = g(e^a k)g^{-1}, \quad a \in \mathfrak{p}, \quad k \in K, \quad \mathrm{Ad}(k)a = a. \tag{3.1.8}$$

We call $\gamma_h = ge^a g^{-1}$ and $\gamma_e = gkg^{-1}$ the hyperbolic and elliptic parts of γ. These two parts are uniquely determined by γ. If $\gamma_h = 1$, we say γ is elliptic, and if $\gamma_e = 1$ and $\gamma_h \neq 1$, we say γ is hyperbolic.

Let $Z(\gamma)$ be the centralizer of γ in G. If $v \in \mathfrak{g}$, let $Z(v) \subset G$ be the stabilizer of v in G via the adjoint action. Let $\mathfrak{z}(\gamma)$, $\mathfrak{z}(v)$ be the Lie algebras of $Z(\gamma)$, $Z(v)$ respectively. If $\gamma = \gamma_h\gamma_e$ is semisimple as above, by [Eberlein 1996, Theorem 2.19.23; Knapp 2002, Lemma 7.36],

$$Z(\gamma) = Z(\gamma_h) \cap Z(\gamma_e), \quad Z(\gamma_h) = Z(\mathrm{Ad}(g)a). \tag{3.1.9}$$

By [Knapp 2002, Proposition 7.25], $Z(\gamma)$ is reductive (possibly with several connected components). Set

$$\theta_g = C(g)\theta C(g^{-1}). \tag{3.1.10}$$

Then θ_g defines a Cartan involution on $Z(\gamma)$. Let $K(\gamma)$ be the fixed-point set of θ_g in $Z(\gamma)$; then

$$K(\gamma) = Z(\gamma) \cap gKg^{-1}. \tag{3.1.11}$$

Let $Z(\gamma)^0$, $K(\gamma)^0$ be the connected components of the identities of $Z(\gamma)$, $K(\gamma)$ respectively. By [Bismut 2011, Theorem 3.3.1],

$$\frac{Z(\gamma)}{K(\gamma)} = \frac{Z(\gamma)^0}{K(\gamma)^0}. \tag{3.1.12}$$

Moreover, $K(\gamma)$, $K(\gamma)^0$ are maximal compact subgroups of $Z(\gamma)$, $Z(\gamma)^0$ respectively.

Taking the corresponding Lie algebras in (3.1.9), we have

$$\mathfrak{z}(\gamma) = \mathfrak{z}(\gamma_h) \cap \mathfrak{z}(\gamma_e), \quad \mathfrak{z}(\gamma_h) = \mathfrak{z}(\mathrm{Ad}(g)a). \tag{3.1.13}$$

Let $\mathfrak{k}(\gamma) \subset \mathfrak{z}(\gamma)$ be the Lie algebra of $K(\gamma)$. Put

$$\mathfrak{p}(\gamma) = \mathfrak{z}(\gamma) \cap \mathrm{Ad}(g)\mathfrak{p}. \tag{3.1.14}$$

Then the Cartan decomposition of $\mathfrak{z}(\gamma)$ with respect to θ_g is given by

$$\mathfrak{z}(\gamma) = \mathfrak{k}(\gamma) \oplus \mathfrak{p}(\gamma). \tag{3.1.15}$$

Let $B_{\mathfrak{z}(\gamma)}$ denote the restriction of B on $\mathfrak{z}(\gamma) \times \mathfrak{z}(\gamma)$. Then $B_{\mathfrak{z}(\gamma)}$ is invariant under the adjoint action of θ_g on $\mathfrak{z}(\gamma)$. Moreover, $B_{\mathfrak{z}(\gamma)}$ is positive on $\mathfrak{p}(\gamma)$ and negative on $\mathfrak{k}(\gamma)$. The splitting in (3.1.15) is orthogonal with respect to $B_{\mathfrak{z}(\gamma)}$.

3.2. ***Symmetric space.*** Set

$$X = G/K. \tag{3.2.1}$$

Then X is a smooth manifold with the smooth structure induced by G. By definition, X is diffeomorphic to $\mathfrak{p}$.

Let $\omega^{\mathfrak{g}} \in \Omega^1(G, \mathfrak{g})$ be the canonical left-invariant 1-form on G. Then by (3.1.1),

$$\omega^{\mathfrak{g}} = \omega^{\mathfrak{p}} + \omega^{\mathfrak{k}}. \tag{3.2.2}$$

Let $p : G \to X$ denote the obvious projection. Then p is a K-principal bundle over X. Then $\omega^{\mathfrak{k}}$ is a connection form of this principal bundle. The associated curvature form

$$\Omega^{\mathfrak{k}} = d\omega^{\mathfrak{k}} + \tfrac{1}{2}[\omega^{\mathfrak{k}}, \omega^{\mathfrak{k}}] = -\tfrac{1}{2}[\omega^{\mathfrak{p}}, \omega^{\mathfrak{p}}]. \tag{3.2.3}$$

If (E, ρ^E, h^E) is a finite-dimensional unitary or Euclidean representation of K, then $F = G \times_K E$ defines a vector bundle over X equipped with a metric h^F induced by h^E and a unitary or a Euclidean connection ∇^F induced by $\omega^{\mathfrak{k}}$. Note that G acts on $(F, h^F, \nabla^F) \to X$ equivariantly on the left; more precisely, for $\gamma \in G$, $(g, v) \in G \times_K E$, the action of γ on F is represented by

$$\gamma(g, v) = (\gamma g, v) \in G \times_K E. \tag{3.2.4}$$

In particular, we have the identification

$$TX = G \times_K \mathfrak{p}, \tag{3.2.5}$$

where the right-hand side is defined by the adjoint action of K on $\mathfrak{p}$. The bilinear form B restricting to $\mathfrak{p}$ gives a Riemannian metric g^{TX}, and $\omega^{\mathfrak{k}}$ induces the associated Levi-Civita connection ∇^{TX}. Then G acts on (X, g^{TX}) isometrically. Let $d(\cdot,\cdot)$ denote the Riemannian distance on X.

Let $C(G, E)$ denote the set of continuous map from G into E. If $k \in K,\ s \in C(G, E)$, put

$$(k.s)(g) = \rho^E(k)s(gk). \tag{3.2.6}$$

Let $C_K(G, E)$ be the set of K-invariant maps in $C(G, E)$. Let $C(X, F)$ denote the continuous sections of F over X. Then

$$C_K(G, E) = C(X, F). \tag{3.2.7}$$

Also $C_K^\infty(G, E) = C^\infty(X, F)$.

The Casimir operator $C^{\mathfrak{g}}$ acting on $C^\infty(G, E)$ preserves $C^\infty_K(G, E)$, so it induces an operator $C^{\mathfrak{g},X}$ acting on $C^\infty(X, F)$. Let $\Delta^{H,X}$ be the Bochner Laplacian acting on $C^\infty(X, F)$ given by ∇^F, and let $C^{\mathfrak{k},E} \in \operatorname{End}(E)$ be the action of the Casimir $C^{\mathfrak{k}}$ on E via ρ^E. The element $C^{\mathfrak{k},E}$ induces a self-adjoint section of $\operatorname{End}(F)$ over X. Then

$$C^{\mathfrak{g},X} = -\Delta^{H,X} + C^{\mathfrak{k},E}. \tag{3.2.8}$$

Let $C^{\mathfrak{k},\mathfrak{p}} \in \operatorname{End}(\mathfrak{p})$, $C^{\mathfrak{k},\mathfrak{k}} \in \operatorname{End}(\mathfrak{k})$ be the actions of $C^{\mathfrak{k}}$ acting on $\mathfrak{p}$, $\mathfrak{k}$ via the adjoint actions. Moreover, we can also view $C^{\mathfrak{k},\mathfrak{p}}$ as a parallel section of $\operatorname{End}(TX)$.

If $A \in \operatorname{End}(E)$ commutes with K, then it can be viewed a parallel section of $\operatorname{End}(F)$ over X. Let dx be the Riemannian volume element of (X, g^{TX}).

Definition 3.2.1. Let $\mathcal{L}^X_A$ be the Bochner-like Laplacian acting on $C^\infty(X, F)$ given by

$$\mathcal{L}^X_A = \tfrac{1}{2} C^{\mathfrak{g},X} + \tfrac{1}{16}\operatorname{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] + \tfrac{1}{48}\operatorname{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}] + A. \tag{3.2.9}$$

For $t > 0$, $x, x' \in X$, let $p^X_t(x, x')$ denote its heat kernel with respect to dx'.

Since $\mathcal{L}^X_A$ is G-invariant, $p^X_t(x, x')$ lifts to a function $p^X_t(g, g')$ on $G \times G$ valued in $\operatorname{End}(E)$ such that, for $g'' \in G$, $k, k' \in K$,

$$p^X_t(g''g, g''g') = p^X_t(g, g'), \quad p^X_t(gk, g'k') = \rho^E(k^{-1})\, p^X_t(g, g')\, \rho^E(k'). \tag{3.2.10}$$

We set

$$p^X_t(g) = p^X_t(1, g). \tag{3.2.11}$$

Then p^X_t is a $K \times K$-invariant smooth function on G valued in $\operatorname{End}(E)$. We will not distinguish the heat kernel $p^X_t(x, x')$ and the function $p^X_t(g)$ in the sequel.

3.3. ***Bismut's formula for semisimple orbital integrals.*** Let dg be the left-invariant Haar measure on G induced by $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$. Since G is unimodular, dg is also right-invariant. Let dk be the Haar measure on K induced by $-B|_{\mathfrak{k}}$; then

$$dg = dx\, dk. \tag{3.3.1}$$

Now let $\gamma \in G$ be a semisimple element given as in (3.1.8).

By [Eberlein 1996, Definition 2.19.21; Bismut 2011, Theorem 3.1.2], $\gamma \in G$ is semisimple if and only if the displacement function $X \ni x \mapsto d(x, \gamma x)$ on X associated with γ can reach its minimum $m_\gamma \geq 0$ in X. In this case, the minimizing set $X(\gamma)$ of this displacement function is a geodesically convex submanifold of X, and by [Bismut 2011, Theorem 3.3.1],

$$X(\gamma) \simeq \frac{Z(\gamma)^0}{K(\gamma)^0} = \frac{Z(\gamma)}{K(\gamma)}. \tag{3.3.2}$$

Moreover, we have

$$m_\gamma = |a|. \tag{3.3.3}$$

Let dy be the Riemannian volume element of $X(\gamma)$, and let dz be the bi-invariant Haar measure on $Z(\gamma)$ induced by $B_{\mathfrak{z}(\gamma)}$. Let $dk(\gamma)$ be the Haar measure on $K(\gamma)$ such that

$$dz = dy\, dk(\gamma). \tag{3.3.4}$$

Let $\mathrm{Vol}(K(\gamma)\backslash K)$ be the volume of $K(\gamma)\backslash K$ with respect to $dk, dk(\gamma)$. Then we have

$$\mathrm{Vol}(K(\gamma)\backslash K) = \frac{\mathrm{Vol}(K)}{\mathrm{Vol}(K(\gamma))}. \tag{3.3.5}$$

Let dv be the G-left invariant measure on $Z(\gamma)\backslash G$ such that

$$dg = dz\, dv. \tag{3.3.6}$$

By [Bismut 2011, Definition 4.2.2, Proposition 4.4.2], for $t > 0$, the orbital integral

$$\mathrm{Tr}^{[\gamma]}[\exp(-t\mathcal{L}_A^X)] = \frac{1}{\mathrm{Vol}(K(\gamma)\backslash K)} \int_{Z(\gamma)\backslash G} \mathrm{Tr}^E[p_t^X(v^{-1}\gamma v)]\, dv \tag{3.3.7}$$

is well-defined. As indicated by the notation, it only depends on the conjugacy class $[\gamma]$ of γ in G.

Using the theory of hypoelliptic Laplacian and the techniques from local index theory, Bismut obtained an explicit geometric formula for $\mathrm{Tr}^{[\gamma]}[\exp(-t\mathcal{L}_A^X)]$ in [Bismut 2011, Theorem 6.1.1] as well as its extension to the wave operators of $\mathcal{L}_A^X$ [Bismut 2011, Section 6.3]. Now we describe in detail this formula. We may and we will assume that

$$\gamma = e^a k, \quad a \in \mathfrak{p}, \quad k \in K, \quad \mathrm{Ad}(k)a = a. \tag{3.3.8}$$

Put

$$\mathfrak{z}_0 = \mathfrak{z}(a), \quad \mathfrak{p}_0 = \ker \mathrm{ad}(a) \cap \mathfrak{p}, \quad \mathfrak{k}_0 = \ker \mathrm{ad}(a) \cap \mathfrak{k}. \tag{3.3.9}$$

Let $\mathfrak{z}_0^\perp$, $\mathfrak{p}_0^\perp$, $\mathfrak{k}_0^\perp$ be the orthogonal vector spaces to $\mathfrak{z}_0$, $\mathfrak{p}_0$, $\mathfrak{k}_0$ in $\mathfrak{g}, \mathfrak{p}, \mathfrak{k}$ with respect to B. Then

$$\mathfrak{z}_0 = \mathfrak{p}_0 \oplus \mathfrak{k}_0, \quad \mathfrak{z}_0^\perp = \mathfrak{p}_0^\perp \oplus \mathfrak{k}_0^\perp. \tag{3.3.10}$$

By [Bismut 2011, equation (3.3.6)],

$$\mathfrak{z}(\gamma) = \mathfrak{z}_0 \cap \mathfrak{z}(k). \tag{3.3.11}$$

Also $\mathfrak{p}(\gamma)$, $\mathfrak{k}(\gamma)$ are subspaces of $\mathfrak{p}_0$, $\mathfrak{k}_0$ respectively. Let $\mathfrak{z}_0^\perp(\gamma)$, $\mathfrak{p}_0^\perp(\gamma)$, $\mathfrak{k}_0^\perp(\gamma)$ be the orthogonal spaces to $\mathfrak{z}(\gamma)$, $\mathfrak{p}(\gamma)$, $\mathfrak{k}(\gamma)$ in $\mathfrak{z}_0$, $\mathfrak{p}_0$, $\mathfrak{k}_0$. Then

$$\mathfrak{z}_0^\perp(\gamma) = \mathfrak{p}_0^\perp(\gamma) \oplus \mathfrak{k}_0^\perp(\gamma). \tag{3.3.12}$$

Also the action $\mathrm{ad}(a)$ gives an isomorphism between $\mathfrak{p}_0^\perp$ and $\mathfrak{k}_0^\perp$.

For $Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma)$, $\mathrm{ad}(Y_0^{\mathfrak{k}})$ preserves $\mathfrak{p}(\gamma), \mathfrak{k}(\gamma), \mathfrak{p}_0^\perp(\gamma), \mathfrak{k}_0^\perp(\gamma)$, and it is an antisymmetric endomorphism with respect to the scalar product.

Recall that the function $\widehat{A}$ is given by

$$\widehat{A}(x) = \frac{x/2}{\sinh(x/2)}. \tag{3.3.13}$$

Let H be a finite-dimensional Hermitian vector space. If $B \in \operatorname{End}(H)$ is self-adjoint, then

$$\frac{B/2}{\sinh(B/2)}$$

is a self-adjoint positive endomorphism. Put

$$\widehat{A}(B) = \det{}^{\frac{1}{2}}\left[\frac{B/2}{\sinh(B/2)}\right]. \tag{3.3.14}$$

In (3.3.14), the square root is taken to be the positive square root.

If $Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma)$, as explained in [Bismut 2011, p. 105], the following function $A(Y_0^{\mathfrak{k}})$ has a natural square root that is analytic in $Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma)$:

$$A(Y_0^{\mathfrak{k}}) = \frac{1}{\det(1-\operatorname{Ad}(k))|_{\mathfrak{z}_0^{\perp}(\gamma)}} \cdot \frac{\det(1-\exp(-i\operatorname{ad}(Y_0^{\mathfrak{k}}))\operatorname{Ad}(k))|_{\mathfrak{k}_0^{\perp}(\gamma)}}{\det(1-\exp(-i\operatorname{ad}(Y_0^{\mathfrak{k}}))\operatorname{Ad}(k))|_{\mathfrak{p}_0^{\perp}(\gamma)}}. \tag{3.3.15}$$

Its square root is denoted by

$$\left[\frac{1}{\det(1-\operatorname{Ad}(k))|_{\mathfrak{z}_0^{\perp}(\gamma)}} \cdot \frac{\det(1-\exp(-i\operatorname{ad}(Y_0^{\mathfrak{k}}))\operatorname{Ad}(k))|_{\mathfrak{k}_0^{\perp}(\gamma)}}{\det(1-\exp(-i\operatorname{ad}(Y_0^{\mathfrak{k}}))\operatorname{Ad}(k))|_{\mathfrak{p}_0^{\perp}(\gamma)}}\right]^{\frac{1}{2}}. \tag{3.3.16}$$

The value of (3.3.16) at $Y_0^{\mathfrak{k}} = 0$ is taken to be such that

$$\frac{1}{\det(1-\operatorname{Ad}(k))|_{\mathfrak{p}_0^{\perp}(\gamma)}}. \tag{3.3.17}$$

We recall an important function J_γ defined in [Bismut 2011, equation (5.5.5)].

Definition 3.3.1. Let $J_\gamma(Y_0^{\mathfrak{k}})$ be the analytic function of $Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma)$ given by

$$\begin{aligned} J_\gamma(Y_0^{\mathfrak{k}}) = {} & \frac{1}{|\det(1-\operatorname{Ad}(\gamma))|_{\mathfrak{z}_0^{\perp}}|^{\frac{1}{2}}} \frac{\widehat{A}(i\operatorname{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{p}(\gamma)})}{\widehat{A}(i\operatorname{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{k}(\gamma)})} \\ & \cdot \left[\frac{1}{\det(1-\operatorname{Ad}(k))|_{\mathfrak{z}_0^{\perp}(\gamma)}} \frac{\det(1-\exp(-i\operatorname{ad}(Y_0^{\mathfrak{k}}))\operatorname{Ad}(k))|_{\mathfrak{k}_0^{\perp}(\gamma)}}{\det(1-\exp(-i\operatorname{ad}(Y_0^{\mathfrak{k}}))\operatorname{Ad}(k))|_{\mathfrak{p}_0^{\perp}(\gamma)}}\right]^{\frac{1}{2}}. \end{aligned} \tag{3.3.18}$$

By [Bismut 2011, equation (6.1.1)], there exist $C_\gamma > 0$, $c_\gamma > 0$ such that, if $Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma)$,

$$|J_\gamma(Y_0^{\mathfrak{k}})| \le C_\gamma e^{c_\gamma |Y_0^{\mathfrak{k}}|}. \tag{3.3.19}$$

Put $p = \dim \mathfrak{p}(\gamma)$, $q = \dim \mathfrak{k}(\gamma)$. Then $r = \dim \mathfrak{z}(\gamma) = p + q$. By [Bismut 2011, Theorem 6.1.1], for $t > 0$, we have

$$\operatorname{Tr}^{[\gamma]}[\exp(-t\mathcal{L}_A^X)] = \frac{e^{-\frac{|a|^2}{2t}}}{(2\pi t)^{\frac{p}{2}}} \int_{\mathfrak{k}(\gamma)} J_\gamma(Y_0^{\mathfrak{k}}) \operatorname{Tr}^E[\rho^E(k)\exp(-i\rho^E(Y_0^{\mathfrak{k}}) - tA)] e^{-\frac{|Y_0^{\mathfrak{k}}|^2}{2t}} \frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{\frac{q}{2}}}. \tag{3.3.20}$$

Remark 3.3.2. A generalization of Bismut's formula (3.3.20) to the twisted case is obtained in [Liu 2018; 2019]. An extension of this formula for considering arbitrary elements in the center of an enveloping algebra instead of the Casimir operator (3.2.8) was obtained in [Bismut and Shen 2022].

3.4. ***Compact locally symmetric spaces.*** Let Γ be a cocompact discrete subgroup of G. Then Γ acts on X isometrically and properly discontinuously. Then $Z = \Gamma\backslash X$ is compact second countable Hausdorff space.

If $x \in X$, put

$$\Gamma_x = \{\gamma \in \Gamma \mid \gamma x = x\}. \tag{3.4.1}$$

Then Γ_x is a finite subgroup of Γ. Put

$$r_x = \inf_{\gamma \in \Gamma - \Gamma_x} d(x, \gamma x). \tag{3.4.2}$$

Then we always have $r_x > 0$. Set

$$U_x = B\left(x, \frac{r_x}{4}\right) \subset X. \tag{3.4.3}$$

If $x \in X,\ \gamma \in \Gamma$, we have

$$r_{\gamma x} = r_x, \quad U_{\gamma x} = \gamma U_x. \tag{3.4.4}$$

It is clear that $\Gamma_x \backslash U_x$ can identified with a connected open subset of Z.

Set

$$S = \ker(\Gamma \to \mathrm{Diffeo}(X)) = \Gamma \cap \ker(K \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathfrak{p})). \tag{3.4.5}$$

Then S is a finite subgroup of $\Gamma \cap K$ and a normal subgroup of Γ.

Remark 3.4.1. Note that G_{ss} is a connected noncompact simple linear Lie group. Then

$$S = Z_G \cap \Gamma \cap K. \tag{3.4.6}$$

Put

$$\Gamma' = \Gamma / S. \tag{3.4.7}$$

Then Γ' acts on X effectively and we have $Z = \Gamma' \backslash X$.

If $x \in X$, we have

$$S \subset \Gamma_x, \quad \Gamma'_x = \Gamma_x / S. \tag{3.4.8}$$

Then the orbifold charts $(U_x, \Gamma'_x, \pi_x : U_x \to \Gamma'_x \backslash U_x)_{x \in X}$ together with the action of Γ' on these charts give an (effective) orbifold structure for Z, so that $Z = \Gamma \backslash X$ is a compact orbifold with a Riemannian metric g^{TZ} induced by g^{TX}.

By [Selberg 1960, Lemma 1], if $\gamma \in \Gamma$, then γ is semisimple. Let $[\Gamma]$ denote the set of the conjugacy classes of Γ. If $\gamma \in \Gamma$, we say $[\gamma] \in [\Gamma]$ is an elliptic class if γ is elliptic. Let $\mathrm{E}[\Gamma] \subset [\Gamma]$ be the set of elliptic classes. Then $\mathrm{E}[\Gamma]$ is always a finite set. If $\mathrm{E}[\Gamma]$ only contains the trivial conjugacy class $[1]$; i.e., Γ is torsion free, then Z is compact smooth manifold.

Let $[\Gamma']$ be the set of conjugacy classes in Γ', and let $\mathrm{E}[\Gamma']$ denote the set of elliptic classes in $[\Gamma']$. If $\gamma' \in \Gamma'$, let $Z_{\Gamma'}(\gamma')$ denote the centralizer of γ' in Γ', and let $[\gamma']'$ denote the conjugacy class of γ' in Γ'. If $\gamma' \in \Gamma'$ is elliptic, let $X(\gamma')$ be its fixed-point set in X on which $Z_{\Gamma'}(\gamma')$ acts isometrically and properly discontinuously; see [Selberg 1960, Lemma 2]. Note that if $\gamma \in \Gamma$ is a lift of $\gamma' \in \Gamma'$, then $X(\gamma) = X(\gamma')$, and γ is elliptic if and only if γ' is elliptic.

Proposition 3.4.2. *We have*

$$Z_{\mathrm{sing}} = \Gamma'\backslash\Gamma'\Big(\bigcup_{[\gamma']'\in \mathrm{E}[\Gamma']\backslash\{1\}} X(\gamma')\Big) \subset Z. \tag{3.4.9}$$

Moreover, we have

$$\Sigma Z = \bigcup_{[\gamma']'\in \mathrm{E}[\Gamma']\backslash\{1\}} Z_{\Gamma'}(\gamma')\backslash X(\gamma'). \tag{3.4.10}$$

Note that the right-hand side of (3.4.10) *is a disjoint union of compact orbifolds.*

If $\gamma' \in \Gamma'$, *put*

$$S'(\gamma') = \ker(Z_{\Gamma'}(\gamma') \to \mathrm{Diffeo}(X(\gamma'))). \tag{3.4.11}$$

Then $|S'(\gamma')|$ *is the multiplicity of the connected component* $Z_{\Gamma'}(\gamma')\backslash X(\gamma')$ *in* ΣZ.

Proof. Note that $z \in Z$ with a lift $x \in X$ belongs to Z_{sing} if and only if the stabilizer Γ'_x is nontrivial. Thus x is a fixed point of some $\gamma' \in \Gamma'$, from which (3.4.9) follows. By definition in Section 2.1, we get the rest of this proposition. □

Note that $\Gamma\backslash G$ is a compact smooth homogeneous space equipped with a right action of K. Moreover, the action of K is almost free; i.e., for each $\bar{g} \in \Gamma\backslash G$, the stabilizer $K_{\bar{g}}$ is finite. Then the quotient space $(\Gamma\backslash G)/K$ also has a natural orbifold structure, which, after examining the local charts, is equivalent to Z.

Let $d\bar{g}$ be the volume element on $\Gamma\backslash G$ induced by dg. By (3.3.1), we get

$$\mathrm{Vol}(\Gamma\backslash G) = \frac{\mathrm{Vol}(K)}{|S|}\,\mathrm{Vol}(Z). \tag{3.4.12}$$

In the context of geometry, we have many interesting cases where $S = \{1\}$. For instance, given a Riemannian symmetric space (X, g^{TX}) of noncompact type, let $G = \mathrm{Isom}(X)^0$ be the connected component of identity of the Lie group of isometries of X. By [Eberlein 1996, Proposition 2.1.1], G is a semisimple Lie group with trivial center (which might not be linear, but we do not need that linearity for the geometry of Z). We refer to [Eberlein 1996, Chapter 2; Bismut 2011, Chapter 3] for more details. This way, any subgroup of G acts on X effectively. In particular, if Γ is a cocompact discrete subgroup of G, then $Z = \Gamma\backslash X$ is a compact good orbifold with the orbifold fundamental group Γ. By (3.4.10), we have

$$\Sigma Z = \bigcup_{[\gamma]\in E[\Gamma]\backslash\{1\}} \Gamma \cap Z(\gamma)\backslash X(\gamma). \tag{3.4.13}$$

In general, by [Helgason 1978, Chapter V, §4, Theorem 4.1], $G = \mathrm{Isom}(X = G/K)^0$ if and only if K acts on $\mathfrak{p}$ effectively.

Remark 3.4.3. Note that, as mentioned in Remark 2.1.3, when $S \neq \{1\}$, we can also consider $Z = \Gamma\backslash X$ as an ineffective orbifold by taking the action of Γ instead of Γ' on the local charts. This way, the role of the above $Z \cup \Sigma Z$ is replaced by the inertia groupoid defined in [Adem et al. 2007, Example 2.5], which is exactly

$$\bigcup_{[\gamma]\in E[\Gamma]} \Gamma \cap Z(\gamma)\backslash X(\gamma). \tag{3.4.14}$$

It is a very natural object to use in the context here, for instance, for the Selberg's trace formula in the next subsection. In the problems we are concerned with, these two point-views on Z are equivalent.

If $\rho : \Gamma' \to \mathrm{GL}(\mathbb{C}^k)$ is a representation of Γ', which can be viewed as a representation of Γ via the projection $\Gamma \to \Gamma' = \Gamma/S$, then $F = \Gamma' \backslash (X \times \mathbb{C}^k)$ is a proper flat orbifold vector bundle on Z with the flat connection $\nabla^{F,f}$ induced from the exterior differential d^X on $\mathbb{C}^k$-valued functions. By [Shen and Yu 2022, Theorem 2.35], all the proper orbifold vector bundles on Z of rank k come from this.

Now let $\rho : \Gamma \to \mathrm{GL}(\mathbb{C}^k)$ be a representation of Γ; we do not assume that it comes from a representation of Γ'. We still have a flat orbifold vector bundle $(F = \Gamma\backslash(X \times \mathbb{C}^k), \nabla^{F,f})$ on Z, which may not be proper in general. Note that Γ acts on $C^\infty(X, \mathbb{C}^k)$ so that if $\varphi \in C^\infty(X, \mathbb{C}^k)$, $\gamma \in \Gamma$, then

$$(\gamma\varphi)(x) = \rho(\gamma)\varphi(\gamma^{-1}x). \tag{3.4.15}$$

Let $C^\infty(X, \mathbb{C}^k)^\Gamma$ denote the Γ-invariant sections in $C^\infty(X, \mathbb{C}^k)$. Then

$$C^\infty(Z, F) = C^\infty(X, \mathbb{C}^k)^\Gamma. \tag{3.4.16}$$

Definition 3.4.4. Let (V, ρ^V) be the isotypic component of $(\mathbb{C}^k, \rho|_S)$ corresponding to the trivial representation of S on $\mathbb{C}$, i.e., the maximal S-invariant subspace of $\mathbb{C}^k$ via ρ. Set

$$F^{\mathrm{pr}} = \Gamma\backslash(X \times V). \tag{3.4.17}$$

It is clear that F^{pr} is a proper flat orbifold vector bundle on Z.

Proposition 3.4.5. *We have*

$$C^\infty(Z, F) = C^\infty(Z, F^{\mathrm{pr}}). \tag{3.4.18}$$

In particular, if $\rho|_S : S \to \mathrm{GL}(\mathbb{C}^k)$ does not have the isotypic component of the trivial representation of S on $\mathbb{C}$, then

$$C^\infty(Z, F) = \{0\}. \tag{3.4.19}$$

Let (E, ρ^E) be a finite-dimensional complex representation of G. When restricting to Γ, K, we get the corresponding representations of Γ, K respectively, which are still denoted by ρ^E. As discussed in Section 3.2, associated with the K-representation (E, ρ^E) we define a homogeneous vector bundle $F = G \times_K E$ on X. Moreover, G acts on F equivariantly. By taking a Γ-quotient on the left, it descends to an orbifold vector bundle on Z, which we still denote by the same notation.

The map $(g, v) \in G \times_K E \to (pg, \rho^E(g)v) \in X \times E$ gives a canonical trivialization of F over X. This trivialization provides a flat connection $\nabla^{X,F,f}$ for $F \to X$, which is G-invariant. Then it descends to a flat connection $\nabla^{Z,F,f}$ on the orbifold vector bundle F over Z. Moreover, the above trivialization of $F \to X$ implies that the flat orbifold vector bundle $(F, \nabla^{Z,F,f})$ is exactly the one given by $\Gamma\backslash(X \times E)$ with the flat connection $\nabla^{F,f}$ induced by d^X. We will always use the notation $\nabla^{F,f}$ for the above flat connection. By (3.2.7), (3.4.16), we get

$$C^\infty(Z, F) = C_K^\infty(G, E)^\Gamma. \tag{3.4.20}$$

3.5. ***Selberg's trace formula.*** Let Z be the compact locally symmetric space discussed in Section 3.4, and let (F, h^F, ∇^F) be a Hermitian vector bundle on X defined by a unitary representation (E, ρ^E) of K. As said before, (F, h^F, ∇^F) descends to a Hermitian orbifold vector bundle on Z. Recall the Bochner-like Laplacian $\mathcal{L}_A^X$ is defined by (3.2.9). Since it commutes with G, it descends to a Bochner-like Laplacian $\mathcal{L}_A^Z$ acting on $C^\infty(Z, F)$.

Here the convergences of the integrals and infinite sums are already guaranteed by the results in [Bismut 2011, Chapters 2, 4; Shen 2018, Section 4D].

For $t > 0$, let $p_t^Z(z, z')$, $z, z' \in Z$, be the heat kernel of $\mathcal{L}_A^Z$ over Z with respect to dz'. If z, z' are identified with their lifts in X, then

$$p_t^Z(z, z') = \frac{1}{|S|} \sum_{\gamma \in \Gamma} \gamma p_t^X(\gamma^{-1} z, z') = \frac{1}{|S|} \sum_{\gamma \in \Gamma} p_t^X(z, \gamma z')\gamma. \tag{3.5.1}$$

Note that the action of γ on $F_{\gamma^{-1}z}$ or on the metric dual of $F_{z'}$ is given as in (3.2.4).

Since Z is compact, for $t > 0$, $\exp(-t\mathcal{L}_A^Z)$ is trace class. We have

$$\mathrm{Tr}[\exp(-t\mathcal{L}_A^Z)] = \int_Z \mathrm{Tr}^F[p_t^Z(z, z)]\, dz. \tag{3.5.2}$$

Combining (3.2.10), (3.2.11), (3.4.12) and (3.5.1), (3.5.2), and proceeding as in [Bismut 2011, equations (4.8.8)–(4.8.12)], we get

$$\begin{aligned} \mathrm{Tr}[\exp(-t\mathcal{L}_A^Z)] &= \frac{1}{\mathrm{Vol}(K)} \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} \mathrm{Tr}^E[p_t^X(\bar{g}^{-1}\gamma \bar{g})]\, d\bar{g} \\ &= \sum_{[\gamma] \in [\Gamma]} \frac{\mathrm{Vol}(\Gamma \cap Z(\gamma) \backslash Z(\gamma))}{\mathrm{Vol}(K(\gamma))} \mathrm{Tr}^{[\gamma]}[\exp(-t\mathcal{L}_A^X)]. \end{aligned} \tag{3.5.3}$$

Take $\gamma \in \Gamma$. Recall that $X(\gamma) = Z(\gamma)/K(\gamma)$ defined in Section 3.3. Then $K(\gamma)$ acts on $Z(\gamma)$ on the right, which induces an action on $\Gamma \cap Z(\gamma) \backslash Z(\gamma)$ on the right. Set

$$S(\gamma) = \ker(\Gamma \cap Z(\gamma) \to \mathrm{Diffeo}(X(\gamma))). \tag{3.5.4}$$

Then $S(\gamma)$ represents the isotropy group of the principal orbit type for the right action of $K(\gamma)$ on $\Gamma \cap Z(\gamma) \backslash Z(\gamma)$. As in (3.4.12), we have

$$\mathrm{Vol}(\Gamma \cap Z(\gamma) \backslash Z(\gamma)) = \frac{\mathrm{Vol}(K(\gamma))}{|S(\gamma)|} \mathrm{Vol}(\Gamma \cap Z(\gamma) \backslash X(\gamma)). \tag{3.5.5}$$

Theorem 3.5.1. *For $t > 0$, we have the identity*

$$\mathrm{Tr}[\exp(-t\mathcal{L}_A^Z)] = \sum_{[\gamma] \in [\Gamma]} \frac{\mathrm{Vol}(\Gamma \cap Z(\gamma) \backslash X(\gamma))}{|S(\gamma)|} \mathrm{Tr}^{[\gamma]}[\exp(-t\mathcal{L}_A^X)]. \tag{3.5.6}$$

Proof. This is a direct consequence of (3.5.3) and (3.5.5). □

In the case where $S = 1$, the trace formula (3.5.6) shows clearly the different contributions from Z and from each components of ΣZ. Then combining (3.4.10), (3.5.6) with the results in [Bismut 2011, Theorem 7.8.2; Liu 2018, Theorem 7.7.1], we can recover (2.2.9) for Z. If we use the same settings as in [Bismut 2011, Sections 7.1, 7.2] and we use instead the results in Theorem 7.7.1 of that work, then we can recover the Kawasaki's local index theorem [1979] for Z. By taking account of Remarks 2.1.3 and 3.4.3, the above considerations also hold even for $S \neq \{1\}$.

4. Analytic torsions for compact locally symmetric spaces

In this section, we explain how to make use of Bismut's formula (3.3.20) and Selberg's trace formula (3.5.6) to study the analytic torsions of Z. We continue using the same settings as in Section 3. We will see that by a vanishing result on the analytic torsion, only the case $\delta(G) = 1$ remains interesting. For studying this case, more tools will be introduced in Sections 5 and 6.

4.1. ***A vanishing result on the analytic torsions.*** Recall that G is a connected linear real reductive Lie group. Recall that $\mathfrak{z}_{\mathfrak{g}}$ is the center of $\mathfrak{g}$. Set

$$\mathfrak{z}_{\mathfrak{p}} = \mathfrak{z}_{\mathfrak{g}} \cap \mathfrak{p}, \quad \mathfrak{z}_{\mathfrak{k}} = \mathfrak{z}_{\mathfrak{g}} \cap \mathfrak{k}. \tag{4.1.1}$$

Then

$$\mathfrak{z}_{\mathfrak{g}} = \mathfrak{z}_{\mathfrak{p}} \oplus \mathfrak{z}_{\mathfrak{k}}, \quad Z_G = \exp(\mathfrak{z}_{\mathfrak{p}})(Z_G \cap K). \tag{4.1.2}$$

Let T be a maximal torus of K with Lie algebra $\mathfrak{t}$; put

$$\mathfrak{b} = \{ f \in \mathfrak{p} \mid [f, \mathfrak{t}] = 0 \}. \tag{4.1.3}$$

It is clear that

$$\mathfrak{z}_{\mathfrak{p}} \subset \mathfrak{b}. \tag{4.1.4}$$

Put $\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{t}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let H be analytic subgroup of G associated with $\mathfrak{h}$. Then it is also a Cartan subgroup of G; see [Knapp 1986, p. 129 and Theorem 5.22(b)]. Moreover, $\dim \mathfrak{t}$ is just the complex rank of K, and $\dim \mathfrak{h}$ is the complex rank of G.

Definition 4.1.1. Using the above notation, the deficiency of G, or the fundamental rank of G is defined as

$$\delta(G) = \operatorname{rk}_{\mathbb{C}} G - \operatorname{rk}_{\mathbb{C}} K = \dim_{\mathbb{R}} \mathfrak{b}. \tag{4.1.5}$$

The number $m - \delta(G)$ is even.

The following result was proved in [Shen 2018, Proposition 3.3].

Proposition 4.1.2. *If $\gamma \in G$ is semisimple, then*

$$\delta(G) \leq \delta(Z(\gamma)^0). \tag{4.1.6}$$

The two sides of (4.1.6) *are equal if and only if γ can be conjugated into H.*

Recall that $\mathfrak{u} = \sqrt{-1}\mathfrak{p} \oplus \mathfrak{k}$ is the compact form of G, and that $U \subset G_{\mathbb{C}}$ is the analytic subgroup with Lie algebra $\mathfrak{u}$. Let $U\mathfrak{u}$, $U\mathfrak{g}_{\mathbb{C}}$ be the enveloping algebras of $\mathfrak{u}$, $\mathfrak{g}_{\mathbb{C}}$ respectively. Then $U\mathfrak{g}_{\mathbb{C}}$ can be identified with the left-invariant holomorphic differential operators on $G_{\mathbb{C}}$. Let $C^{\mathfrak{u}} \in U\mathfrak{u}$ be the Casimir operator of $\mathfrak{u}$ associated with B. Then

$$C^{\mathfrak{u}} = C^{\mathfrak{g}} \in U\mathfrak{g} \cap U\mathfrak{u} \subset U\mathfrak{g}_{\mathbb{C}}. \tag{4.1.7}$$

In the sequel, we always assume that U is compact; this is the case when G has compact center.

Proposition 4.1.3 (unitary trick). *Assume that U is compact. Then any irreducible finite-dimensional* (*analytic*) *complex representation of U extends uniquely to an irreducible finite-dimensional complex representation of G such that their induced representations of Lie algebras are compatible.*

We now fix a unitary representation (E, ρ^E, h^E) of U, and we extend it to a representation of G, whose restriction to K is still unitary. Put $F = G \times_K E$, with the Hermitian metric h^F induced by h^E. Let ∇^F be the Hermitian connection induced by the connection form $\omega^{\mathfrak{k}}$.

Furthermore, as explained in the last part of Section 3.4, F is equipped with a canonical flat connection $\nabla^{F,f}$ as follows:

$$\nabla^{F,f} = \nabla^F + \rho^E(\omega^{\mathfrak{p}}). \tag{4.1.8}$$

If G has compact center, then $(F, h^F, \nabla^{F,f})$ is a unimodular flat vector bundle.

Let $(\Omega_c^\bullet(X, F), d^{X,F})$ be the (compactly supported) de Rham complex twisted by F. Let $d^{X,F,*}$ be the adjoint operator of $d^{X,F}$ with respect to the L_2 metric on $\Omega_c^\bullet(X, F)$. The de Rham–Hodge operator $\boldsymbol{D}^{X,F}$ of this de Rham complex is given by

$$\boldsymbol{D}^{X,F} = d^{X,F} + d^{X,F,*}. \tag{4.1.9}$$

The Clifford algebras $c(TX)$, $\hat{c}(TX)$ act on $\Lambda^\bullet(T^*X)$. We still use $e_1, \dots, e_m$ to denote an orthonormal basis of $\mathfrak{p}$ or TX, and let $e^1, \dots, e^m$ be the corresponding dual basis of $\mathfrak{p}^*$ or T^*X.

Let $\nabla^{\Lambda^\bullet(T^*X)\otimes F,u}$ be the unitary connection on $\Lambda^\bullet(T^*X) \otimes F$ induced by ∇^{TX} and ∇^F. Then the standard Dirac operator is given by

$$D^{X,F} = \sum_{j=1}^m c(e_j) \nabla_{e_j}^{\Lambda^\bullet(T^*X)\otimes F,u}. \tag{4.1.10}$$

By [Bismut et al. 2017, equation (8.42)], we have

$$\boldsymbol{D}^{X,F} = D^{X,F} + \sum_{j=1}^m \hat{c}(e_j) \rho^E(e_j). \tag{4.1.11}$$

At the same time, as explained in Section 3.2, $C^{\mathfrak{g}}$ descends to an elliptic differential operator $C^{\mathfrak{g},X}$ acting on $C^\infty(X, \Lambda^\bullet(T^*X) \otimes F)$. As in (3.2.9), we put

$$\mathcal{L}^{X,F} = \tfrac{1}{2} C^{\mathfrak{g},X} + \tfrac{1}{16} \operatorname{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] + \tfrac{1}{48} \operatorname{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}]. \tag{4.1.12}$$

For simplicity, we will always put

$$\beta_{\mathfrak{g}} = \tfrac{1}{16} \operatorname{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] + \tfrac{1}{48} \operatorname{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}] \in \mathbb{R}. \tag{4.1.13}$$

By [Bismut et al. 2017, Proposition 8.4], we have

$$\tfrac{1}{2}\boldsymbol{D}^{X,F,2} = \mathcal{L}^{X,F} - \tfrac{1}{2}C^{\mathfrak{g},E} - \beta_{\mathfrak{g}} =: \mathcal{L}^{X,F}_{A}, \tag{4.1.14}$$

where $A = -\frac{1}{2}C^{\mathfrak{g},E} - \beta_{\mathfrak{g}}$.

Let $\gamma \in G$ be a semisimple element. In the sequel, we may assume that

$$\gamma = e^{a}k, \quad a \in \mathfrak{p}, \quad k \in K, \quad \mathrm{Ad}(k)a = a. \tag{4.1.15}$$

We also use the same notation as in Section 3.3.

Recall that $p = \dim \mathfrak{p}(\gamma)$, $q = \dim \mathfrak{k}(\gamma)$. By (3.3.20) and (4.1.14), we have

$$\begin{aligned}\mathrm{Tr}_s^{[\gamma]}\Big[\Big(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2}\Big)\exp\Big(-\frac{t\boldsymbol{D}^{X,F,2}}{2}\Big)\Big] \\ = \frac{e^{-\frac{|a|^2}{2t}}}{(2\pi t)^{\frac{p}{2}}}\exp(t\beta)\int_{\mathfrak{k}(\gamma)} J_\gamma(Y_0^{\mathfrak{k}})\,\mathrm{Tr}_s^{\Lambda^\bullet(\mathfrak{p}^*)}\Big[\Big(N^{\Lambda^\bullet(\mathfrak{p}^*)} - \frac{m}{2}\Big)\mathrm{Ad}(k)\exp(-i\,\mathrm{ad}(Y_0^{\mathfrak{k}}))\Big] \\ \cdot \mathrm{Tr}^{E}\Big[\rho^E(k)\exp\Big(-i\rho^E(Y_0^{\mathfrak{k}}) + \frac{t}{2}C^{\mathfrak{u},E}\Big)\Big]e^{-\frac{|Y_0^{\mathfrak{k}}|^2}{2t}}\frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{\frac{q}{2}}}.\end{aligned} \tag{4.1.16}$$

Now we take a cocompact discrete subgroup $\Gamma \subset G$. Then $Z = \Gamma\backslash X$ is a compact locally symmetric orbifold. We use the same notation as in Sections 3.4 and 3.5. Then we get a flat orbifold vector bundle $(F, \nabla^{F,f}, h^F)$ on Z. Furthermore, $\boldsymbol{D}^{X,F}$ descends to the corresponding de Rham–Hodge operator $\boldsymbol{D}^{Z,F}$ acting on $\Omega^\bullet(Z, F)$. Let $\mathcal{T}(Z, F)$ denote the associated analytic torsion as in Definition 2.2.3, i.e.,

$$\mathcal{T}(Z, F) = \mathcal{T}(g^{TZ}, \nabla^{F,f}, h^F). \tag{4.1.17}$$

As explained in Section 2.2, for computing $\mathcal{T}(Z, F)$, it is enough to evaluate

$$\mathrm{Tr}_s\Big[\Big(N^{\Lambda^\bullet(T^*Z)} - \frac{m}{2}\Big)\exp\Big(-\frac{t\boldsymbol{D}^{Z,F,2}}{2}\Big)\Big], \quad t > 0. \tag{4.1.18}$$

Then we apply Selberg's trace formula in Theorem 3.5.1. We get

$$\begin{aligned}\mathrm{Tr}_s\Big[\Big(N^{\Lambda^\bullet(T^*Z)} - \frac{m}{2}\Big)\exp\Big(-\frac{t\boldsymbol{D}^{Z,F,2}}{2}\Big)\Big] \\ = \sum_{[\gamma]\in[\Gamma]} \frac{\mathrm{Vol}(\Gamma\cap Z(\gamma)\backslash X(\gamma))}{|S(\gamma)|}\,\mathrm{Tr}^{[\gamma]}\Big[\Big(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2}\Big)\exp\Big(-\frac{t\boldsymbol{D}^{X,F,2}}{2}\Big)\Big].\end{aligned} \tag{4.1.19}$$

As in [Bismut et al. 2017, Remark 8.7], by [Ma 2019, Theorems 5.4, 5.5, Remark 5.6], we have the following vanishing theorem on $\mathcal{T}(Z, F)$.

Theorem 4.1.4. *If m is even, or if m is odd and $\delta(G) \geq 3$, then*

$$\mathcal{T}(Z, F) = 0. \tag{4.1.20}$$

Proof. By [Bismut 2011, Theorem 7.9.1; Ma 2019, Theorem 5.4], and using instead (4.1.19), we get that under the assumptions in this theorem, for $t > 0$,

$$\mathrm{Tr}_s\Big[\Big(N^{\Lambda^\bullet(T^*Z)} - \frac{m}{2}\Big)\exp(-t\boldsymbol{D}^{Z,F,2})\Big] = 0. \tag{4.1.21}$$

Then (4.1.20) follows from the definition of $\mathcal{T}(Z, F)$. ☐

Therefore, the only nontrivial case is that $\delta(G)=1$, so that m is odd. If $\gamma\in G$ is of the form (4.1.15), let $\mathfrak{t}(\gamma)\subset\mathfrak{k}(\gamma)$ be a Cartan subalgebra. Put

$$\mathfrak{b}(\gamma)=\{v\in\mathfrak{p}(k)\mid [v,\mathfrak{t}(\gamma)]=0\},\quad \mathfrak{h}(\gamma)_{\mathfrak{p}}=\mathfrak{b}(\gamma)\cap\mathfrak{p}(\gamma). \tag{4.1.22}$$

In particular, $a\in\mathfrak{b}(\gamma)$. Then $\mathfrak{h}(\gamma)=\mathfrak{h}(\gamma)_{\mathfrak{p}}\oplus\mathfrak{t}(\gamma)$ is a Cartan subalgebra of $\mathfrak{z}(\gamma)$.

Recall that H is a maximally compact Cartan subgroup of G. The following result is just an analogue of [Shen 2018, Theorem 4.12; Bismut 2011, Theorem 7.9.1].

Proposition 4.1.5. *If $\delta(G)=1$, if γ is semisimple and cannot be conjugated into H by an element in G, then*

$$\operatorname{Tr}_s^{[\gamma]}\Big[\Big(N^{\Lambda^\bullet(T^*X)}-\frac{m}{2}\Big)\exp\Big(-\frac{t\boldsymbol{D}^{X,F,2}}{2}\Big)\Big]=0. \tag{4.1.23}$$

Proof. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$ containing $\mathfrak{t}(\gamma)$. Then $\mathfrak{b}\subset\mathfrak{b}(\gamma)$. If $a\notin\mathfrak{b}$, then $\dim\mathfrak{b}(\gamma)\geq 2$. Therefore, by [Shen 2018, equation (4-44)], for $Y_0^{\mathfrak{k}}\in\mathfrak{k}(\gamma)$, we have

$$\operatorname{Tr}_s^{\Lambda^\bullet(\mathfrak{p}^*)}\Big[\Big(N^{\Lambda^\bullet(\mathfrak{p}^*)}-\frac{m}{2}\Big)\operatorname{Ad}(k)\exp(-i\operatorname{ad}(Y_0^{\mathfrak{k}}))\Big]=0. \tag{4.1.24}$$

This implies (4.1.23). □

Set

$$\mathfrak{g}'=\mathfrak{z}_{\mathfrak{k}}\oplus\mathfrak{g}_{\mathrm{ss}}. \tag{4.1.25}$$

Then $\mathfrak{g}'$ is an ideal of $\mathfrak{g}$. Let G' be the analytic subgroup of G associated with $\mathfrak{g}'$, which is closed and has a compact center; see [Knapp 2002, Proposition 7.27]. The group K is still a maximal subgroup of G'. Let $U'\subset U$ be the compact form of G' with Lie algebra $\mathfrak{u}'$. Then

$$\mathfrak{u}=\sqrt{-1}\mathfrak{z}_{\mathfrak{p}}\oplus\mathfrak{u}'. \tag{4.1.26}$$

Now we assume that $\delta(G)=1$ and that G has noncompact center, so that $\mathfrak{b}=\mathfrak{z}_{\mathfrak{p}}$ has dimension 1. Then $\delta(G')=0$. Under the hypothesis that U is compact, up to a finite cover, we may write

$$U\simeq\mathbb{S}^1\times U'. \tag{4.1.27}$$

We take $a_1\in\mathfrak{b}$ with $|a_1|=1$. If (E,ρ^E) is an irreducible unitary representation of U, then $\rho^E(a_1)$ acts on E by a real scalar operator. Let $\alpha_E\in\mathbb{R}$ be such that

$$\rho^E(a_1)=\alpha_E\operatorname{Id}_E. \tag{4.1.28}$$

Put $X'=G'/K$. Then X' is an even-dimensional symmetric space (of noncompact type). We identify $\mathfrak{z}_{\mathfrak{p}}$ with a real line $\mathbb{R}$. Then

$$G=\mathbb{R}\times G',\quad X=\mathbb{R}\times X'. \tag{4.1.29}$$

In this case, the evaluation for analytic torsions can be made more explicit. If $\gamma\in G'$, let $X'(\gamma)$ denote the minimizing set of $d_\gamma(\cdot)$ in X', so that

$$X(\gamma)=\mathbb{R}\times X'(\gamma). \tag{4.1.30}$$

Let $[\cdot]^{\max}$ denote the coefficient of a differential form (valued in $o(TX')$) on X' of the corresponding Riemannian volume form. Similarly, for $k \in T$, let $[\cdot]^{\max(k)}$ denote the analogous object on $X'(k)$. The following results are the analogues of [Shen 2018, Proposition 4.14].

Proposition 4.1.6. *Assume that G has noncompact center with $\delta(G) = 1$ and that (E, ρ^E) is irreducible. Then*

$$\operatorname{Tr}_s^{[1]}\Big[\Big(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2}\Big)\exp\Big(-\frac{t\boldsymbol{D}^{X,F,2}}{2}\Big)\Big] = -\frac{e^{-\frac{1}{2}t\alpha_E^2}}{\sqrt{2\pi t}}[e(TX', \nabla^{TX'})]^{\max}\dim E. \tag{4.1.31}$$

If $\gamma = e^a k$ is such that $a \in \mathfrak{b}$, $k \in T$, then

$$\operatorname{Tr}_s^{[\gamma]}\Big[\Big(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2}\Big)\exp\Big(-\frac{t\boldsymbol{D}^{X,F,2}}{2}\Big)\Big] = -\frac{1}{\sqrt{2\pi t}}e^{-\frac{|a|^2}{2t}-\frac{1}{2}t\alpha_E^2}[e(TX'(k), \nabla^{TX'(k)})]^{\max(k)}\operatorname{Tr}^E[\rho^E(k)]. \tag{4.1.32}$$

Proof. Let $C^{\mathfrak{u}'}$ denote the Casimir operator of $\mathfrak{u}'$ associated with $B|_{\mathfrak{u}'}$. Then we have

$$C^{\mathfrak{u}} = -a_1^2 + C^{\mathfrak{u}'}. \tag{4.1.33}$$

Since (E, ρ^E) is an irreducible representation, by (4.1.28) and (4.1.33), we get

$$C^{\mathfrak{u},E} = -\alpha_E^2 + C^{\mathfrak{u}',E}. \tag{4.1.34}$$

Then by (4.1.34) and [Bismut et al. 2017, Theorem 8.5], a modification of the proof of [Shen 2018, Proposition 4.14] proves the identities in our proposition. □

If we assembly the results in Proposition 4.1.6, it is enough to study the corresponding analytic torsions. We will get back to this point in Corollary 7.4.4 for asymptotic analytic torsions.

4.2. ***Symmetric spaces of noncompact type with fundamental rank 1.*** In this subsection, we focus on the case where $\delta(G) = 1$ and G has compact center (i.e., $\mathfrak{z}_{\mathfrak{p}} = 0$), so that X is a symmetric space of noncompact type [Shen 2018, Proposition 6.18]. For simplicity, let us also assume that G is linear semisimple in this subsection.

Note that the rank $\delta(X)$ of X (see [Eberlein 1996, Section 2.7]) is the same as $\delta(G)$. Then $\delta(X) = 1$. By the de Rham decomposition, we can write

$$X = X_1 \times X_2, \tag{4.2.1}$$

where X_1 is an irreducible symmetric space of noncompact type with $\delta(X_1) = 1$, and X_2 is a symmetric space of noncompact type with $\delta(X_2) = 0$.

As in [Bismut 2011, Remark 7.9.2], among the noncompact simple connected real linear groups such that m is odd and $\dim \mathfrak{b} = 1$, there are only $\mathrm{SL}_3(\mathbb{R})$, $\mathrm{SL}_4(\mathbb{R})$, $\mathrm{SL}_2(\mathbb{H})$, and $\mathrm{SO}^0(p,q)$ with pq odd > 1. Also, we have $\mathfrak{sl}_4(\mathbb{R}) = \mathfrak{so}(3,3)$ and $\mathfrak{sl}_2(\mathbb{H}) = \mathfrak{so}(5,1)$. Therefore, X_1 is one of the following cases (see [Shen 2018, Proposition 6.19]):

$$X_1 = \mathrm{SL}_3(\mathbb{R})/\mathrm{SO}(3) \quad \text{or} \quad \mathrm{SO}^0(p,q)/\mathrm{SO}(p+q), \quad \text{with } pq > 1 \text{ odd}. \tag{4.2.2}$$

Since $\delta(G) = 1$, we have the decomposition of Lie algebras

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2, \tag{4.2.3}$$

where

$$\mathfrak{g}_1 = \mathfrak{sl}_3(\mathbb{R}) \quad \text{or} \quad \mathfrak{so}(p,q), \tag{4.2.4}$$

with $pq > 1$ odd, and $\mathfrak{g}_2$ is semisimple with $\delta(\mathfrak{g}_2) = 0$. The Cartan involution θ preserves the splitting (4.2.3); see [Knapp 2002, VII.6, p. 471].

Let G_1 be the identity component of $Z_G(\mathfrak{g}_2)$. Then G_1 is a connected linear semisimple closed subgroup of G with Lie algebra of $\mathfrak{g}_1$. Similarly, we can find a connected linear semisimple closed subgroup G_2 of G with Lie algebra of $\mathfrak{g}_2$ such that we have canonically $G_1 \times G_2 \to G$ a finite central extension. Let θ_j be the induced Cartan involution on $G_j (j = 1, 2)$ from θ. Set $K_j = G_j \cap K$; then

$$X_j = G_j / K_j, \quad j = 1, 2. \tag{4.2.5}$$

Note that in general, G_1 is a just a finite central extension of $\mathrm{SL}_3(\mathbb{R})$ or $\mathrm{SO}^0(p,q)$ ($pq > 1$ odd). The invariant bilinear form B also splits as $B_1 \oplus B_2$ with respect to the splitting (4.2.3).

Remark 4.2.1. Let G_*, $G_{1,*}$, $G_{2,*}$ denote the identity components of the isometry groups of X, X_1, X_2 respectively. Then we have

$$G_* = G_{1,*} \times G_{2,*}. \tag{4.2.6}$$

By [Shen 2018, Proposition 6.19], $G_{1,*} = \mathrm{SL}_3(\mathbb{R})$ or $\mathrm{SO}^0(p,q)$, with $pq > 1$ odd, and $G_{2,*}$ is a semisimple Lie group with Lie algebra $\mathfrak{g}_2$ and trivial center. Also $\delta(G_{2,*}) = 0$. If we consider G_* instead of G, then the factor G_1 is exactly $\mathrm{SL}_3(\mathbb{R})$ or $\mathrm{SO}^0(p,q)$, with $pq > 1$ odd.

Let U_1, U_2 be (connected linear) compact forms of G_1, G_2. Then $U_1 \times U_2$ is a finite central extension of the compact form U of G. Let (E, ρ^E) be an irreducible unitary representation of U, and hence of $U_1 \times U_2$. Then

$$(E, \rho^E) = (E_1, \rho^{E_1}) \otimes (E_2, \rho^{E_2}), \tag{4.2.7}$$

where (E_j, ρ^{E_j}) is an irreducible unitary representation of U_j, $j = 1, 2$. Let F, F_1, F_2 be the homogeneous flat vector bundles on X, X_1, X_2 associated with these representations. Then we have

$$F = F_1 \boxtimes F_2 := \pi_1^*(F_1) \otimes \pi_2^*(F_2), \tag{4.2.8}$$

where π_i denote the projections $X \to X_i$, $i = 1, 2$.

Take $\gamma \in G$. Let $(\gamma_1, \gamma_2) \in G_1 \times G_2$ be one of its lifts. Then γ is semisimple (resp. elliptic) if and only if both γ_1, γ_2 are semisimple (resp. elliptic). Set $m_i = \dim X_i$; then m_1 is odd, and m_2 is even.

Proposition 4.2.2. *If $\gamma \in G$ is semisimple, for $t > 0$, we have*

$$\begin{aligned}\mathrm{Tr}_s^{[\gamma]}\Big[\Big(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2}\Big)\exp\Big(-\frac{t\boldsymbol{D}^{X,F,2}}{2}\Big)\Big] \\ = \mathrm{Tr}_s^{[\gamma_1]}\Big[\Big(N^{\Lambda^\bullet(T^*X_1)} - \frac{m_1}{2}\Big)\exp\Big(-\frac{t\boldsymbol{D}^{X_1,F_1,2}}{2}\Big)\Big] \cdot \mathrm{Tr}_s^{[\gamma_2]}\Big[\exp\Big(-\frac{t\boldsymbol{D}^{X_2,F_2,2}}{2}\Big)\Big].\end{aligned} \tag{4.2.9}$$

Then if γ_2 is nonelliptic,

$$\mathrm{Tr}_s^{[\gamma]}\Big[\Big(N^{\Lambda^\bullet(T^*X)}-\frac{m}{2}\Big)\exp\Big(-\frac{t\boldsymbol{D}^{X,F,2}}{2}\Big)\Big]=0. \tag{4.2.10}$$

If γ_2 is elliptic, *then*

$$\mathrm{Tr}_s^{[\gamma]}\Big[\Big(N^{\Lambda^\bullet(T^*X)}-\frac{m}{2}\Big)\exp\Big(-\frac{t\boldsymbol{D}^{X,F,2}}{2}\Big)\Big]=[e(TX_2(\gamma_2),\nabla^{TX_2(\gamma_2)})]^{\max_2(\gamma_2)}\,\mathrm{Tr}^{E_2}[\rho^{E_2}(\gamma_2)]\\ \cdot\mathrm{Tr}_s^{[\gamma_1]}\Big[\Big(N^{\Lambda^\bullet(T^*X_1)}-\frac{m_1}{2}\Big)\exp\Big(-\frac{t\boldsymbol{D}^{X_1,F_1,2}}{2}\Big)\Big], \tag{4.2.11}$$

where $[\,\cdot\,]^{\max_2(\gamma_2)}$ is taking the coefficient of the Riemannian volume element on $X_2(\gamma_2)$.

Proof. We write

$$N^{\Lambda^\bullet(T^*X)}-\frac{m}{2}=\Big(N^{\Lambda^\bullet(T^*X_1)}-\frac{m_1}{2}\Big)+\Big(N^{\Lambda^\bullet(T^*X_2)}-\frac{m_2}{2}\Big). \tag{4.2.12}$$

Note that, since $\delta(G_1)=1$, by [Bismut 2011, Theorem 7.8.2], we always have

$$\mathrm{Tr}_s^{[\gamma_1]}\Big[\exp\Big(-\frac{t\boldsymbol{D}^{X_1,F_1,2}}{2}\Big)\Big]=0. \tag{4.2.13}$$

Combining the definition of orbital integrals (3.3.7) together with (4.2.12) and (4.2.13), we get (4.2.9).

The identities (4.2.10), (4.2.11) follow from applying the results in [Bismut 2011, Theorem 7.8.2] to $\mathrm{Tr}_s^{[\gamma_2]}[\exp(-t\boldsymbol{D}^{X_2,F_2,2}/2)]$. □

For studying $\mathcal{T}(Z,F)$, Proposition 4.2.2 helps us to reduce the computations on

$$\mathrm{Tr}_s\Big[\Big(N^{\Lambda^\bullet(T^*Z)}-\frac{m}{2}\Big)\exp\Big(-\frac{t\boldsymbol{D}^{Z,F,2}}{2}\Big)\Big]$$

to the model cases listed in (4.2.2). But it is far from enough to get an explicit evaluation. In Sections 5 and 6, we will introduce more tools, which allows us work out a proof to Theorem 1.0.2.

5. Cartan subalgebra and root system of G when $\delta(G)=1$

We use the same notation as in Section 3 and Section 4.1. In Sections 5.1–5.3, we always assume that G is a connected linear real reductive Lie group with compact center and with $\delta(G)=1$. But, as we will see in Remark 5.3.3, the constructions and results in these subsections are still true (most of them are trivial) if U is compact and if G has noncompact center with $\delta(G)=1$.

Section 5.4 is independent from other subsections, where we introduce a generalized Kirillov formula for compact Lie groups.

Recall that T is a maximal torus of K with Lie algebra $\mathfrak{t}\subset\mathfrak{k}$, and that $\mathfrak{b}\subset\mathfrak{p}$ is defined in (4.1.3). Since $\delta(G)=1$, we know $\mathfrak{b}$ is 1-dimensional. We now fix a vector $a_1\in\mathfrak{b}$, $|a_1|=1$. Recall that $\mathfrak{h}=\mathfrak{b}\oplus\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $h^{\mathfrak{g}_{\mathbb{C}}}$ be the Hermitian product on $\mathfrak{g}_{\mathbb{C}}$ induced by the scalar product $-B(\cdot,\theta\,\cdot)$ on $\mathfrak{g}$.

5.1. ***Reductive Lie algebra with fundamental rank 1.*** Since G has compact center, $\mathfrak{b} \not\subset \mathfrak{z}_{\mathfrak{g}}$. Let $Z(\mathfrak{b})$ be the centralizer of $\mathfrak{b}$ in G, and let $Z(\mathfrak{b})^0$ be its identity component with Lie algebra $\mathfrak{z}(\mathfrak{b}) = \mathfrak{p}(\mathfrak{b}) \oplus \mathfrak{k}(\mathfrak{b}) \subset \mathfrak{g}$. Let $\mathfrak{m}$ be the orthogonal subspace of $\mathfrak{b}$ in $\mathfrak{z}(\mathfrak{b})$ (with respect to B) such that

$$\mathfrak{z}(\mathfrak{b}) = \mathfrak{b} \oplus \mathfrak{m}. \tag{5.1.1}$$

Then $\mathfrak{m}$ is a Lie subalgebra of $\mathfrak{z}(\mathfrak{b})$, which is invariant by θ.

Put

$$\mathfrak{p}_{\mathfrak{m}} = \mathfrak{m} \cap \mathfrak{p}, \quad \mathfrak{k}_{\mathfrak{m}} = \mathfrak{m} \cap \mathfrak{k}. \tag{5.1.2}$$

Then

$$\mathfrak{m} = \mathfrak{p}_{\mathfrak{m}} \oplus \mathfrak{k}_{\mathfrak{m}}, \quad \mathfrak{p}(\mathfrak{b}) = \mathfrak{b} \oplus \mathfrak{p}_{\mathfrak{m}}, \quad \mathfrak{k}(\mathfrak{b}) = \mathfrak{k}_{\mathfrak{m}}. \tag{5.1.3}$$

Let $\mathfrak{z}^{\perp}(\mathfrak{b})$, $\mathfrak{p}^{\perp}(\mathfrak{b})$, $\mathfrak{k}^{\perp}(\mathfrak{b})$ be the orthogonal subspaces of $\mathfrak{z}(\mathfrak{b})$, $\mathfrak{p}(\mathfrak{b})$, $\mathfrak{k}(\mathfrak{b})$ in $\mathfrak{g}$, $\mathfrak{p}$, $\mathfrak{k}$ respectively with respect to B. Then

$$\mathfrak{z}^{\perp}(\mathfrak{b}) = \mathfrak{p}^{\perp}(\mathfrak{b}) \oplus \mathfrak{k}^{\perp}(\mathfrak{b}). \tag{5.1.4}$$

Moreover,

$$\mathfrak{p} = \mathfrak{b} \oplus \mathfrak{p}_{\mathfrak{m}} \oplus \mathfrak{p}^{\perp}(\mathfrak{b}), \quad \mathfrak{k} = \mathfrak{k}(\mathfrak{b}) \oplus \mathfrak{k}^{\perp}(\mathfrak{b}). \tag{5.1.5}$$

Let $M \subset Z(\mathfrak{b})^0$ be the analytic subgroup associated with $\mathfrak{m}$. If we identify $\mathfrak{b}$ with $\mathbb{R}$, then

$$Z(\mathfrak{b})^0 = \mathbb{R} \times M. \tag{5.1.6}$$

Then M is a Lie subgroup of $Z(\mathfrak{b})^0$; i.e., it is closed in $Z(\mathfrak{b})^0$. Let K_M be the analytic subgroup of M associated with the Lie subalgebra $\mathfrak{k}_{\mathfrak{m}}$. Since M is reductive, K_M is a maximal compact subgroup of M. Then the splittings in (5.1.3), (5.1.4), (5.1.5) are invariant by the adjoint action of K_M.

Then $\mathfrak{t}$ is Cartan subalgebra of $\mathfrak{k}$, of $\mathfrak{k}_{\mathfrak{m}}$, and of $\mathfrak{m}$. Recall that $\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. We fix $a_1 \in \mathfrak{b}$ such that $B(a_1, a_1) = 1$. The choice of a_1 fixes an orientation of $\mathfrak{b}$. Let $\mathfrak{n} \subset \mathfrak{z}^{\perp}(\mathfrak{b})$ be the direct sum of the eigenspaces of $\mathrm{ad}(a_1)$ with the positive eigenvalues. Set $\bar{\mathfrak{n}} = \theta\mathfrak{n}$. Then

$$\mathfrak{z}^{\perp}(\mathfrak{b}) = \mathfrak{n} \oplus \bar{\mathfrak{n}}. \tag{5.1.7}$$

By [Shen 2018, Section 6A], $\dim \mathfrak{n} = \dim \mathfrak{p} - \dim \mathfrak{p}_{\mathfrak{m}} - 1$. Then $\dim \mathfrak{n}$ is even under our assumption $\delta(G) = 1$. Put

$$l = \tfrac{1}{2} \dim \mathfrak{n}. \tag{5.1.8}$$

By [Shen 2018, Proposition 6.2], there exists $\beta \in \mathfrak{b}^*$ such that if $a \in \mathfrak{b}$, $f \in \mathfrak{n}$, then

$$[a, f] = \beta(a) f, \quad [a, \theta(f)] = -\beta(a)\theta(f). \tag{5.1.9}$$

The map $f \in \mathfrak{n} \mapsto f - \theta(f) \in \mathfrak{p}^{\perp}(\mathfrak{b})$ is an isomorphism of K_M-modules. Similarly, $f \in \mathfrak{n} \mapsto f + \theta(f) \in \mathfrak{k}^{\perp}(\mathfrak{b})$ is also an isomorphism of K_M-modules. Since θ fixes K_M, $\mathfrak{n} \simeq \bar{\mathfrak{n}}$ as K_M-modules via θ.

By [Shen 2018, Proposition 6.3], we have

$$[\mathfrak{n}, \bar{\mathfrak{n}}] \subset \mathfrak{z}(\mathfrak{b}), \quad [\mathfrak{n}, \mathfrak{n}] = [\bar{\mathfrak{n}}, \bar{\mathfrak{n}}] = 0. \tag{5.1.10}$$

Also

$$B|_{\mathfrak{n}\times\mathfrak{n}} = 0, \quad B|_{\bar{\mathfrak{n}}\times\bar{\mathfrak{n}}} = 0. \tag{5.1.11}$$

Then the bilinear form B induces an isomorphism of $\mathfrak{n}^*$ and $\bar{\mathfrak{n}}$ as K_M-modules. Therefore, as K_M-modules, $\mathfrak{n}$ is isomorphic to $\mathfrak{n}^*$.

As a consequence of (5.1.10), we get

$$[\mathfrak{z}(\mathfrak{b}), \mathfrak{z}(\mathfrak{b})], [\mathfrak{z}^\perp(\mathfrak{b}), \mathfrak{z}^\perp(\mathfrak{b})] \subset \mathfrak{z}(\mathfrak{b}), \quad [\mathfrak{z}(\mathfrak{b}), \mathfrak{z}^\perp(\mathfrak{b})] \subset \mathfrak{z}^\perp(\mathfrak{b}). \tag{5.1.12}$$

Then $(\mathfrak{g}, \mathfrak{z}(\mathfrak{b}))$ is a symmetric pair.

If $k \in K_M$, let $M(k)$ be the centralizer of k in M, and let $\mathfrak{m}(k)$ be its Lie algebra. Let $M(k)^0$ be the identity component of $M(k)$. The Cartan involution θ acts on $M(k)$. The associated Cartan decomposition is

$$\mathfrak{m}(k) = \mathfrak{p}_{\mathfrak{m}}(k) \oplus \mathfrak{k}_{\mathfrak{m}}(k), \tag{5.1.13}$$

where $\mathfrak{p}_{\mathfrak{m}}(k) = \mathfrak{p}_{\mathfrak{m}} \cap \mathfrak{m}(k)$, $\mathfrak{k}_{\mathfrak{m}}(k) = \mathfrak{k}_{\mathfrak{m}} \cap \mathfrak{m}(k)$.

Recall that $Z(k)$ is the centralizer of k in G and that $Z(k)^0$ is the identity component of $Z(k)$ with Lie algebra $\mathfrak{z}(k) \subset \mathfrak{g}$. Then

$$M(k) = M \cap Z(k), \quad \mathfrak{m}(k) = \mathfrak{m} \cap \mathfrak{z}(k). \tag{5.1.14}$$

Note that $Z(k)^0$ is still a reductive Lie group equipped with the Cartan involution induced by the action of θ. By the assumption that $\delta(G) = 1$, we have

$$\delta(Z(k)^0) = 1. \tag{5.1.15}$$

In particular,

$$\mathfrak{b} \subset \mathfrak{p}(k). \tag{5.1.16}$$

Set

$$\mathfrak{z}_{\mathfrak{b}}(k) = \mathfrak{z}(\mathfrak{b}) \cap \mathfrak{z}(k), \quad \mathfrak{p}_{\mathfrak{b}}(k) = \mathfrak{p}(\mathfrak{b}) \cap \mathfrak{p}(k), \quad \mathfrak{k}_{\mathfrak{b}}(k) = \mathfrak{k}(\mathfrak{b}) \cap \mathfrak{k}(k). \tag{5.1.17}$$

Then

$$\mathfrak{z}_{\mathfrak{b}}(k) = \mathfrak{b} \oplus \mathfrak{m}(k) = \mathfrak{p}_{\mathfrak{b}}(k) \oplus \mathfrak{k}_{\mathfrak{b}}(k). \tag{5.1.18}$$

We also have the identities

$$\mathfrak{p}_{\mathfrak{b}}(k) = \mathfrak{b} \oplus \mathfrak{p}_{\mathfrak{m}}(k), \quad \mathfrak{k}_{\mathfrak{b}}(k) = \mathfrak{k}_{\mathfrak{m}}(k). \tag{5.1.19}$$

Let $\mathfrak{p}_{\mathfrak{b}}^\perp(k)$, $\mathfrak{k}_{\mathfrak{b}}^\perp(k)$, $\mathfrak{z}_{\mathfrak{b}}^\perp(k)$ be the orthogonal spaces of $\mathfrak{p}_{\mathfrak{b}}(k)$, $\mathfrak{k}_{\mathfrak{b}}(k)$, $\mathfrak{z}_{\mathfrak{b}}(k)$ in $\mathfrak{p}(k)$, $\mathfrak{k}(k)$, $\mathfrak{z}(k)$ with respect to B, so that

$$\mathfrak{p}(k) = \mathfrak{p}_{\mathfrak{b}}(k) \oplus \mathfrak{p}_{\mathfrak{b}}^\perp(k), \quad \mathfrak{k}(k) = \mathfrak{k}_{\mathfrak{b}}(k) \oplus \mathfrak{k}_{\mathfrak{b}}^\perp(k), \quad \mathfrak{z}(k) = \mathfrak{z}_{\mathfrak{b}}(k) \oplus \mathfrak{z}_{\mathfrak{b}}^\perp(k). \tag{5.1.20}$$

Then

$$\mathfrak{z}_{\mathfrak{b}}^\perp(k) = \mathfrak{p}_{\mathfrak{b}}^\perp(k) \oplus \mathfrak{k}_{\mathfrak{b}}^\perp(k) = \mathfrak{z}^\perp(\mathfrak{b}) \cap \mathfrak{z}(k). \tag{5.1.21}$$

Put

$$\mathfrak{n}(k) = \mathfrak{z}(k) \cap \mathfrak{n}, \quad \bar{\mathfrak{n}}(k) = \mathfrak{z}(k) \cap \bar{\mathfrak{n}}. \tag{5.1.22}$$

Then

$$\mathfrak{z}_{\mathfrak{b}}^{\perp}(k) = \mathfrak{n}(k) \oplus \bar{\mathfrak{n}}(k). \tag{5.1.23}$$

By (5.1.17), (5.1.23), we get

$$\mathfrak{z}(k) = \mathfrak{p}_{\mathfrak{b}}(k) \oplus \mathfrak{k}_{\mathfrak{b}}(k) \oplus \mathfrak{n}(k) \oplus \bar{\mathfrak{n}}(k). \tag{5.1.24}$$

Since $\delta(\mathfrak{m}(k)) = 0$, $\dim \mathfrak{n}(k)$ is even. We set

$$l(k) = \tfrac{1}{2} \dim \mathfrak{n}(k). \tag{5.1.25}$$

Let $K_M(k)$ denote the centralizer of k in K_M. The map $f \in \mathfrak{n}(k) \mapsto f - \theta(f) \in \mathfrak{p}_{\mathfrak{b}}^{\perp}(k)$ is an isomorphism of $K_M(k)$-modules, and similarly for $\mathfrak{k}_{\mathfrak{b}}^{\perp}(k)$. Since θ fixes $K_M(k)$, we have $\mathfrak{n}(k) \simeq \bar{\mathfrak{n}}(k)$ as $K_M(k)$-modules via θ.

5.2. *A compact Hermitian symmetric space $Y_{\mathfrak{b}}$.* Recall that $\mathfrak{u} = \sqrt{-1}\mathfrak{p} \oplus \mathfrak{k}$ is the compact form of $\mathfrak{g}$.

Let $\mathfrak{u}(\mathfrak{b}) \subset \mathfrak{u}$, $\mathfrak{u}_{\mathfrak{m}} \subset \mathfrak{u}$ be the compact forms of $\mathfrak{z}(\mathfrak{b})$, $\mathfrak{m}$. Then

$$\mathfrak{u}(\mathfrak{b}) = \sqrt{-1}\mathfrak{b} \oplus \mathfrak{u}_{\mathfrak{m}}, \quad \mathfrak{u}_{\mathfrak{m}} = \sqrt{-1}\mathfrak{p}_{\mathfrak{m}} \oplus \mathfrak{k}_{\mathfrak{m}}. \tag{5.2.1}$$

Since M has compact center, let U_M be the analytic subgroup of U associated with $\mathfrak{u}_{\mathfrak{m}}$. Then U_M is the compact form of M. Let $U(\mathfrak{b}) \subset U$, $A_0 \subset U$ be the connected subgroups of U associated with Lie algebras $\mathfrak{u}(\mathfrak{b})$, $\sqrt{-1}\mathfrak{b}$. Then A_0 is in the center of $U(\mathfrak{b})$. By [Shen 2018, Proposition 6.6], A_0 is closed in U and is diffeomorphic to a circle $\mathbb{S}^1$. Moreover, we have

$$U(\mathfrak{b}) = A_0 U_M. \tag{5.2.2}$$

The bilinear form $-B$ induces an $\mathrm{Ad}(U)$-invariant metric on $\mathfrak{u}$. Let $\mathfrak{u}^{\perp}(\mathfrak{b}) \subset \mathfrak{u}$ be the orthogonal subspace of $\mathfrak{u}(\mathfrak{b})$. Then

$$\mathfrak{u}^{\perp}(\mathfrak{b}) = \sqrt{-1}\mathfrak{p}^{\perp}(\mathfrak{b}) \oplus \mathfrak{k}^{\perp}(\mathfrak{b}). \tag{5.2.3}$$

By (5.1.12), we get

$$[\mathfrak{u}(\mathfrak{b}), \mathfrak{u}(\mathfrak{b})], [\mathfrak{u}^{\perp}(\mathfrak{b}), \mathfrak{u}^{\perp}(\mathfrak{b})] \subset \mathfrak{u}(\mathfrak{b}), \quad [\mathfrak{u}(\mathfrak{b}), \mathfrak{u}^{\perp}(\mathfrak{b})] \subset \mathfrak{u}^{\perp}(\mathfrak{b}). \tag{5.2.4}$$

Then $(\mathfrak{u}, \mathfrak{u}(\mathfrak{b}))$ is a symmetric pair.

Put $a_0 = a_1/\beta(a_1) \in \mathfrak{b}$. Set

$$J = \sqrt{-1}\,\mathrm{ad}(a_0)|_{\mathfrak{u}^{\perp}(\mathfrak{b})} \in \mathrm{End}(\mathfrak{u}^{\perp}(\mathfrak{b})). \tag{5.2.5}$$

By (5.1.9), J is an $U(\mathfrak{b})$-invariant complex structure on $\mathfrak{u}^{\perp}(\mathfrak{b})$ which preserves $B|_{\mathfrak{u}^{\perp}(\mathfrak{b})}$. The spaces $\mathfrak{n}_{\mathbb{C}} = \mathfrak{n} \otimes_{\mathbb{R}} \mathbb{C}$, $\bar{\mathfrak{n}}_{\mathbb{C}} = \bar{\mathfrak{n}} \otimes_{\mathbb{R}} \mathbb{C}$ are exactly the eigenspaces of J associated with eigenvalues $\sqrt{-1}, -\sqrt{-1}$.

The following proposition is just the summary of the results in [Shen 2018, Section 6B].

Proposition 5.2.1. *Set*

$$Y_{\mathfrak{b}} = U/U(\mathfrak{b}). \tag{5.2.6}$$

Then $Y_{\mathfrak{b}}$ is a compact symmetric space, and J induces an integrable complex structure on $Y_{\mathfrak{b}}$ such that

$$T^{(1,0)}Y_{\mathfrak{b}} = U \times_{U(\mathfrak{b})} \mathfrak{n}_{\mathbb{C}}, \quad T^{(0,1)}Y_{\mathfrak{b}} = U \times_{U(\mathfrak{b})} \bar{\mathfrak{n}}_{\mathbb{C}}. \tag{5.2.7}$$

The form $-B(\cdot, J\cdot)$ induces a Kähler form $\omega^{Y_{\mathfrak{b}}}$ on $Y_{\mathfrak{b}}$.

Let $\omega^{\mathfrak{u}}$ be the canonical left-invariant 1-form on U with values in $\mathfrak{u}$. Let $\omega^{\mathfrak{u}(\mathfrak{b})}$ and $\omega^{\mathfrak{u}^{\perp}(\mathfrak{b})}$ be the $\mathfrak{u}(\mathfrak{b})$ and $\mathfrak{u}^{\perp}(\mathfrak{b})$ components of $\omega^{\mathfrak{u}}$, so that

$$\omega^{\mathfrak{u}} = \omega^{\mathfrak{u}(\mathfrak{b})} + \omega^{\mathfrak{u}^{\perp}(\mathfrak{b})}. \tag{5.2.8}$$

Moreover, $\omega^{\mathfrak{u}(\mathfrak{b})}$ defines a connection form on the principal $U(\mathfrak{b})$-bundle $U \to Y_{\mathfrak{b}}$. Let $\Omega^{\mathfrak{u}(\mathfrak{b})}$ be the curvature form. Then

$$\Omega^{\mathfrak{u}(\mathfrak{b})} = -\tfrac{1}{2}[\omega^{\mathfrak{u}^{\perp}(\mathfrak{b})}, \omega^{\mathfrak{u}^{\perp}(\mathfrak{b})}]. \tag{5.2.9}$$

Note that the real tangent bundle of $Y_{\mathfrak{b}}$ is given by

$$TY_{\mathfrak{b}} = U \times_{U(\mathfrak{b})} \mathfrak{u}^{\perp}(\mathfrak{b}). \tag{5.2.10}$$

Then $-B|_{\mathfrak{u}^{\perp}(\mathfrak{b})}$ induces a Riemannian metric $g^{TY_{\mathfrak{b}}}$ on $Y_{\mathfrak{b}}$. The corresponding Levi-Civita connection is induced by $\omega^{\mathfrak{u}(\mathfrak{b})}$.

Recall that the first splitting in (5.2.1) is orthogonal with respect to $-B$. Let $\Omega^{\mathfrak{u}_{\mathfrak{m}}}$ be the $\mathfrak{u}_{\mathfrak{m}}$-component of $\Omega^{\mathfrak{u}(\mathfrak{b})}$. Since the Kähler form $\omega^{Y_{\mathfrak{b}}}$ is invariant under the left action of U on $Y_{\mathfrak{b}}$, we also can view $\omega^{Y_{\mathfrak{b}}}$ as an element in $\Lambda^2(\mathfrak{u}_{\mathfrak{b}}^{\perp})^*)$. By [Shen 2018, equation (6-48)],

$$\Omega^{\mathfrak{u}(\mathfrak{b})} = \beta(a_1)\omega^{Y_{\mathfrak{b}}} \otimes \sqrt{-1}a_1 + \Omega^{\mathfrak{u}_{\mathfrak{m}}}. \tag{5.2.11}$$

Moreover, by [Shen 2018, Proposition 6.9], we have

$$B(\Omega^{\mathfrak{u}(\mathfrak{b})}, \Omega^{\mathfrak{u}(\mathfrak{b})}) = 0, \quad B(\Omega^{\mathfrak{u}_{\mathfrak{m}}}, \Omega^{\mathfrak{u}_{\mathfrak{m}}}) = \beta(a_1)^2\omega^{Y_{\mathfrak{b}},2}. \tag{5.2.12}$$

Remark 5.2.2. By [Shen 2018, Proposition 6.20], if G has compact center, then as symmetric spaces, the Kähler manifold $Y_{\mathfrak{b}}$ is isomorphic either to $\mathrm{SU}(3)/\mathrm{U}(2)$ or to $\mathrm{SO}(p+q)/\mathrm{SO}(p+q-2) \times \mathrm{SO}(2)$ with $pq > 1$ odd. This way, the computations on $Y_{\mathfrak{b}}$ can be made more explicit.

Now we fix $k \in K_M$. Let $U(k)$ be the centralizer of k in U, and let $U(k)^0$ be its identity component. Let $\mathfrak{u}(k)$ be the Lie algebra of $U(k)^0$. Then $\mathfrak{u}(k)$ is the compact form of $\mathfrak{z}(k)$, and $U(k)^0$ is the compact form of $Z(k)^0$.

We will use the same notation as in Section 5.1. Then the compact form of $\mathfrak{m}(k)$ is given by

$$\mathfrak{u}_{\mathfrak{m}}(k) = \sqrt{-1}\mathfrak{p}_{\mathfrak{m}}(k) \oplus \mathfrak{k}_{\mathfrak{m}}(k). \tag{5.2.13}$$

Let $\mathfrak{u}_{\mathfrak{b}}(k)$ be the compact form of $\mathfrak{z}_{\mathfrak{b}}(k)$. Then

$$\mathfrak{u}_{\mathfrak{b}}(k) = \sqrt{-1}\mathfrak{b} \oplus \mathfrak{u}_{\mathfrak{m}}(k). \tag{5.2.14}$$

Let $U_{\mathfrak{b}}(k)$ be the analytic subgroup associated with $\mathfrak{u}_{\mathfrak{b}}(k)$. Then

$$U_{\mathfrak{b}}(k) = U(\mathfrak{b}) \cap U(k)^0. \tag{5.2.15}$$

Set

$$Y_{\mathfrak{b}}(k) = U(k)^0/U_{\mathfrak{b}}(k). \tag{5.2.16}$$

As in Proposition 5.2.1, $Y_{\mathfrak{b}}(k)$ is a connected complex manifold equipped with a Kähler form $\omega^{Y_{\mathfrak{b}}(k)}$.

Let $\mathfrak{u}_{\mathfrak{b}}^{\perp}(k)$ be the orthogonal space of $\mathfrak{u}_{\mathfrak{b}}(k)$ in $\mathfrak{u}(k)$ with respect to B. Then

$$\mathfrak{u}_{\mathfrak{b}}^{\perp}(k) = \sqrt{-1}\mathfrak{p}_{\mathfrak{b}}^{\perp}(k) \oplus \mathfrak{k}_{\mathfrak{b}}^{\perp}(k). \tag{5.2.17}$$

Then the real tangent bundle of $Y_{\mathfrak{b}}(k)$ is given by

$$TY_{\mathfrak{b}}(k) = U(k)^0 \times_{U_{\mathfrak{b}}(k)} \mathfrak{u}_{\mathfrak{b}}^{\perp}(k). \tag{5.2.18}$$

Moreover,

$$T^{(1,0)}Y_{\mathfrak{b}}(k) = U(k)^0 \times_{U_{\mathfrak{b}}(k)} \mathfrak{n}(k)_{\mathbb{C}}, \quad T^{(0,1)}Y_{\mathfrak{b}}(k) = U(k)^0 \times_{U_{\mathfrak{b}}(k)} \bar{\mathfrak{n}}(k)_{\mathbb{C}}. \tag{5.2.19}$$

Let $\Omega^{\mathfrak{u}_{\mathfrak{b}}(k)}$ be the curvature form as in (5.2.9) for the pair $(U(k)^0, U_{\mathfrak{b}}(k))$, which can be viewed as an element in $\Lambda^2(\mathfrak{u}_{\mathfrak{b}}^{\perp}(k)^*) \otimes \mathfrak{u}_{\mathfrak{b}}(k)$. Using the splitting (5.2.14), let $\Omega^{\mathfrak{u}_{\mathfrak{m}}(k)}$ be the $\mathfrak{u}_{\mathfrak{m}}(k)$-component of $\Omega^{\mathfrak{u}_{\mathfrak{b}}(k)}$. Then as in (5.2.11) and (5.2.12), we have

$$\Omega^{\mathfrak{u}_{\mathfrak{b}}(k)} = \beta(a_1)\omega^{Y_{\mathfrak{b}}(k)} \otimes \sqrt{-1}a_1 + \Omega^{\mathfrak{u}_{\mathfrak{m}}(k)}, \tag{5.2.20}$$

$$B(\Omega^{\mathfrak{u}_{\mathfrak{b}}(k)}, \Omega^{\mathfrak{u}_{\mathfrak{b}}(k)}) = 0, \quad B(\Omega^{\mathfrak{u}_{\mathfrak{m}}(k)}, \Omega^{\mathfrak{u}_{\mathfrak{m}}(k)}) = \beta(a_1)^2\omega^{Y_{\mathfrak{b}}(b),2}. \tag{5.2.21}$$

5.3. ***Positive root system and character formula.*** Recall that $\mathfrak{t}$ is Cartan subalgebra of $\mathfrak{k}$, of $\mathfrak{k}_{\mathfrak{m}}$, and of $\mathfrak{m}$. Recall that $\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$, and H is the associated maximally compact Cartan subgroup of G.

Put

$$\mathfrak{t}_U = \sqrt{-1}\mathfrak{b} \oplus \mathfrak{t} \subset \mathfrak{u}. \tag{5.3.1}$$

Then $\mathfrak{t}_U$ is a Cartan subalgebra of $\mathfrak{u}$. Let $T_U \subset U$ be the corresponding maximal torus. Then A_0 is a circle in T_U. Then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{u}_{\mathfrak{m}}$, and the corresponding maximal torus is T.

Let $R(\mathfrak{u}, \mathfrak{t}_U)$ be the real root system for the pair (U, T_U) [Bröcker and tom Dieck 1985, Chapter V]. The root system for the complexified pair $(\mathfrak{u}_{\mathbb{C}}, \mathfrak{t}_{U,\mathbb{C}}) = (\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is given by $2\pi i R(\mathfrak{u}, \mathfrak{t}_U)$. Similarly, let $R(\mathfrak{u}(\mathfrak{b}), \mathfrak{t}_U)$, $R(\mathfrak{u}_{\mathfrak{m}}, \mathfrak{t})$ denote the real root systems for the pairs $(\mathfrak{u}(\mathfrak{b}), \mathfrak{t}_U)$, $(\mathfrak{u}_{\mathfrak{m}}, \mathfrak{t})$. When we embed $\mathfrak{t}^*$ into $\mathfrak{t}_U^*$ by the splitting in (5.3.1),

$$R(\mathfrak{u}(\mathfrak{b}), \mathfrak{t}_U) = R(\mathfrak{u}_{\mathfrak{m}}, \mathfrak{t}). \tag{5.3.2}$$

For a root $\alpha \in R(\mathfrak{u}, \mathfrak{t}_U)$, if $\alpha(\sqrt{-1}a_1) = 0$, then $\alpha \in R(\mathfrak{u}_{\mathfrak{m}}, \mathfrak{t})$. Fix a positive root system $R^+(\mathfrak{u}_{\mathfrak{m}}, \mathfrak{t})$. We get a positive root system $R^+(\mathfrak{u}, \mathfrak{t}_U)$ consisting of an element α such that $\alpha(\sqrt{-1}a_1) > 0$ and the elements in $R^+(\mathfrak{u}_{\mathfrak{m}}, \mathfrak{t})$.

Let $W(\mathfrak{u}, \mathfrak{t}_U)$ denote the algebraic Weyl group associated with $R(\mathfrak{u}, \mathfrak{t}_U)$. If $\omega \in W(\mathfrak{u}, \mathfrak{t}_U)$, let $l(\omega)$ denote the length of ω with respect to $R^+(\mathfrak{u}, \mathfrak{t}_U)$. Set

$$\varepsilon(\omega) = (-1)^{l(\omega)}. \tag{5.3.3}$$

Let $W(U, T_U)$ be the analytic Weyl group. Then $W(\mathfrak{u}, \mathfrak{t}_U) = W(U, T_U)$.

Put

$$W_u = \{\omega \in W(U, T_U) \mid \omega^{-1} \cdot \alpha > 0 \text{ for all } \alpha \in R^+(\mathfrak{u}_{\mathfrak{m}}, \mathfrak{t})\}. \tag{5.3.4}$$

Put

$$\rho_{\mathfrak{u}} = \frac{1}{2} \sum_{\alpha^0 \in R^+(\mathfrak{u},\mathfrak{t}_U)} \alpha^0 \in \mathfrak{t}_U^*, \quad \rho_{\mathfrak{u}_\mathfrak{m}} = \frac{1}{2} \sum_{\alpha^0 \in R^+(\mathfrak{u}_\mathfrak{m},\mathfrak{t})} \alpha^0 \in \mathfrak{t}^*. \tag{5.3.5}$$

Then $\rho_{\mathfrak{u}}|_{\mathfrak{t}} = \rho_{\mathfrak{u}_\mathfrak{m}}$.

Let $P_{++}(U) \subset \mathfrak{t}_U^*$ be the set of dominant weights of (U, T_U) with respect to $R^+(\mathfrak{u}, \mathfrak{t}_U)$. If $\lambda \in P_{++}(U)$, let $(E_\lambda, \rho^{E_\lambda})$ be the irreducible unitary representation of U with the highest weight λ, which by the unitary trick extends to an irreducible representation of G.

By [Warner 1972, Lemmas 1.1.2.15, 2.4.2.1], if $\omega \in W_u$, then $\omega(\lambda + \rho_{\mathfrak{u}}) - \rho_{\mathfrak{u}}$ is a dominant weight for $R^+(\mathfrak{u}(\mathfrak{b}), \mathfrak{t}_U)$. Let $V_{\lambda,\omega}$ denote the representation of $U(\mathfrak{b})$ with the highest weight $\omega(\lambda + \rho_{\mathfrak{u}}) - \rho_{\mathfrak{u}}$.

Recall that $U(\mathfrak{b})$ acts on $\mathfrak{n}_\mathbb{C}$. Let $H^\bullet(\mathfrak{n}_\mathbb{C}, E_\lambda)$ be the Lie algebra cohomology of $\mathfrak{n}_\mathbb{C}$ with coefficients in E_λ; see [Kostant 1961]. By [Warner 1972, Theorem 2.5.1.3], for $i = 0, \dots, 2l$, we have the identification of $U(\mathfrak{b})$-modules

$$H^i(\mathfrak{n}_\mathbb{C}, E_\lambda) \simeq \bigoplus_{\substack{\omega \in W_u \\ l(\omega)=i}} V_{\lambda,\omega}. \tag{5.3.6}$$

By (5.3.6) and the Poincaré duality, we get the following identifications as $U(\mathfrak{b})$-modules:

$$\bigoplus_{i=0}^{2l} (-1)^i \Lambda^i \mathfrak{n}_\mathbb{C}^* \otimes E_\lambda = \bigoplus_{\omega \in W_u} \varepsilon(\omega) V_{\lambda,\omega}. \tag{5.3.7}$$

Note that if we apply the unitary trick, the above identification also holds as $Z(\mathfrak{b})^0$-modules.

Definition 5.3.1. Let $P_0 : \mathfrak{t}_U^* \to \mathfrak{t}^*$ denote the orthogonal projection with respect to $B^*|_{\mathfrak{t}_U^*}$. For $\omega \in W_u$, $\lambda \in P_{++}(U)$, put

$$\eta_\omega(\lambda) = P_0(\omega(\lambda + \rho_{\mathfrak{u}}) - \rho_{\mathfrak{u}}) \in \mathfrak{t}^*. \tag{5.3.8}$$

Note that

$$P_0 \rho_{\mathfrak{u}} = \rho_{\mathfrak{u}_\mathfrak{m}}. \tag{5.3.9}$$

Then

$$\eta_\omega(\lambda) = P_0(\omega(\lambda + \rho_{\mathfrak{u}})) - \rho_{\mathfrak{u}_\mathfrak{m}}. \tag{5.3.10}$$

Proposition 5.3.2. *If $\lambda \in P_{++}(U)$, for $\omega \in W_u$, then $\eta_\omega(\lambda)$ is a dominant weight of (U_M, T) with respect to $R^+(\mathfrak{u}_\mathfrak{m}, \mathfrak{t})$. Moreover, the restriction of the $U(\mathfrak{b})$-representation $V_{\lambda,\omega}$ to the subgroup U_M is irreducible, which has the highest weight $\eta_\omega(\lambda)$.*

Proof. Since $\omega(\lambda + \rho_{\mathfrak{u}}) - \rho_{\mathfrak{u}}$ is analytically integrable, $\eta_\omega(\lambda)$ is also analytically integrable as a weight associated with (U_M, T). By (5.3.2) and the corresponding identification of positive root systems, we know that $\eta_\omega(\lambda)$ is dominant with respect to $R^+(\mathfrak{u}_\mathfrak{m}, \mathfrak{t})$.

Recall that $A_0 \simeq \mathbb{S}^1$ is defined in Section 5.2. By (5.2.2), we get that A_0 acts on $V_{\lambda,\omega}$ as scalars given by its character, and then U_M act irreducibly on $V_{\lambda,\omega}$, which clearly has the highest weight $\eta_\omega(\lambda)$. □

Remark 5.3.3. In general, U is just the analytic subgroup of $G_\mathbb{C}$ with Lie algebra $\mathfrak{u}$. If U is compact but G has noncompact center, i.e., $\mathfrak{z}_\mathfrak{p} = \mathfrak{b}$, then $\mathfrak{n} = \bar{\mathfrak{n}} = 0$, so that $l = 0$. Recall that in this case, G', U' are defined in Section 4.1. Then

$$M = G', \quad U_M = U'. \tag{5.3.11}$$

The compact symmetric space $Y_{\mathfrak{b}}$ now reduces to one point.

Moreover, in (5.3.4), $W_u = \{1\}$, so that $V_{\lambda,\omega}$ becomes just E_λ itself. The identities (5.3.6), (5.3.7) are trivially true; so is Proposition 5.3.2.

5.4. ***Kirillov character formula for compact Lie groups.*** In this subsection, we recall the Kirillov character formula for compact Lie groups. We only use the group U_M as an explanatory example. We fix the maximal torus T and the positive (real) root system $R^+(\mathfrak{u}_\mathfrak{m}, \mathfrak{t})$.

Let $\lambda \in \mathfrak{t}^*$ be a dominant (analytically integrable) weight of U_M with respect to the above root system. Let $(V_\lambda, \rho^{V_\lambda})$ be the irreducible unitary representation of U_M with the highest weight λ.

Put

$$\mathcal{O} = \mathrm{Ad}^*(U_M)(\lambda + \rho_{\mathfrak{u}_\mathfrak{m}}) \subset \mathfrak{u}_\mathfrak{m}^*. \tag{5.4.1}$$

Then $\mathcal{O}$ is an even-dimensional closed manifold.

Since $\lambda + \rho_{\mathfrak{u}_\mathfrak{m}}$ is regular, we have the following identifications of U_M-manifolds:

$$\mathcal{O} \simeq U_M/T. \tag{5.4.2}$$

For $u \in \mathfrak{u}_\mathfrak{m}$, an associated vector field $\tilde{u}$ on $\mathcal{O}$ is defined as follows: if $f \in \mathcal{O}$, then

$$\tilde{u}_f = -\,\mathrm{ad}^*(u) f \in T_f\mathcal{O}. \tag{5.4.3}$$

Such vector fields span the whole tangent space at each point. Let ω_L denote the real 2-form on $\mathcal{O}$ such that if $u, v \in \mathfrak{u}_\mathfrak{m}$, $f \in \mathcal{O}$,

$$\omega_L(\tilde{u}, \tilde{v})_f = -\langle f, [u, v]\rangle. \tag{5.4.4}$$

Then ω_L is a U_M-invariant symplectic form on $\mathcal{O}$. Put $r^+ = \frac{1}{2}\dim \mathfrak{u}_\mathfrak{m}/\mathfrak{t}$. In fact, if we can define an almost complex structure on $T\mathcal{O}$ such that the holomorphic tangent bundle is given by the positive root system $R^+(\mathfrak{u}_\mathfrak{m}, \mathfrak{t})$. Then $(\mathcal{O}, \omega_L)$ become a closed Kähler manifold, and r^+ is its complex dimension.

The Liouville measure on $\mathcal{O}$ is defined as

$$d\mu_L = \frac{(\omega_L)^{r^+}}{(r^+)!}. \tag{5.4.5}$$

It is invariant by the left action of U_M. Let $\mathrm{Vol}_L(\mathcal{O})$ denote the symplectic volume of $\mathcal{O}$ with respect to the Liouville measure. Then we have (see [Berline et al. 1992, Proposition 7.26])

$$\mathrm{Vol}_L(\mathcal{O}) = \Pi_{\alpha^0 \in R^+(\mathfrak{u}_\mathfrak{m},\mathfrak{t})} \frac{\langle \alpha^0, \lambda + \rho_{\mathfrak{u}_\mathfrak{m}}\rangle}{\langle \alpha^0, \rho_{\mathfrak{u}_\mathfrak{m}}\rangle} = \dim V_\lambda. \tag{5.4.6}$$

The second identity is the Weyl dimension formula (see [Knapp 1986, Theorem 4.48]).

By the Kirillov formula (see [Berline et al. 1992, Theorem 8.4]), if $y \in \mathfrak{u}_\mathfrak{m}$, we have

$$\widehat{A}^{-1}(\mathrm{ad}(y)|_{\mathfrak{u}_\mathfrak{m}}) \,\mathrm{Tr}^{V_\lambda}[\rho^{V_\lambda}(e^y)] = \int_{f \in \mathcal{O}} e^{2\pi i \langle f, y\rangle}\, d\mu_L. \tag{5.4.7}$$

To shorten the notation here, if $k \in T$, put $Y = U_M(k)^0$ with Lie algebra $\mathfrak{y} = \mathfrak{u}_\mathfrak{m}(k)$. Then $T \subset Y$, and it also a maximal torus of Y.

In the sequel, we will give a generalized version of (5.4.7) for describing the function $\mathrm{Tr}^{V_\lambda}[\rho^{V_\lambda}(ke^y)]$, with $y \in \mathfrak{y}$.

Let $\mathfrak{q}$ be the orthogonal space of $\mathfrak{y}$ in $\mathfrak{u}_\mathfrak{m}$ with respect to B, so that

$$\mathfrak{u}_\mathfrak{m} = \mathfrak{y} \oplus \mathfrak{q}. \tag{5.4.8}$$

Since the adjoint action of T preserves the splitting in (5.4.8). Then $R(\mathfrak{u}_\mathfrak{m}, \mathfrak{t})$ splits into two disjoint parts

$$R(\mathfrak{u}_\mathfrak{m}, \mathfrak{t}) = R(\mathfrak{y}, \mathfrak{t}) \cup R(\mathfrak{q}, \mathfrak{t}), \tag{5.4.9}$$

where $R(\mathfrak{q}, \mathfrak{t})$ is just the set of real roots for the adjoint action of $\mathfrak{t}$ on $\mathfrak{q}_\mathbb{C}$.

The positive root system $R^+(\mathfrak{u}_\mathfrak{m}, \mathfrak{t})$ induces a positive root system $R^+(\mathfrak{y}, \mathfrak{t})$. Set

$$R^+(\mathfrak{q}, \mathfrak{t}) = R^+(\mathfrak{u}_\mathfrak{m}, \mathfrak{t}) \cap R(\mathfrak{q}, \mathfrak{t}). \tag{5.4.10}$$

Then we have the disjoint union

$$R^+(\mathfrak{u}_\mathfrak{m}, \mathfrak{t}) = R^+(\mathfrak{y}, \mathfrak{t}) \cup R^+(\mathfrak{q}, \mathfrak{t}). \tag{5.4.11}$$

Put

$$\rho_\mathfrak{y} = \frac{1}{2} \sum_{\alpha^0 \in R^+(\mathfrak{y}, \mathfrak{t})} \alpha^0, \quad \rho_\mathfrak{q} = \frac{1}{2} \sum_{\alpha^0 \in R^+(\mathfrak{q}, \mathfrak{t})} \alpha^0. \tag{5.4.12}$$

Then

$$\rho_{\mathfrak{u}_\mathfrak{m}} = \rho_\mathfrak{y} + \rho_\mathfrak{q} \in \mathfrak{t}^*. \tag{5.4.13}$$

Let $\mathcal{C} \subset \mathfrak{t}^*$ denote the Weyl chamber corresponding to $R^+(\mathfrak{u}_\mathfrak{m}, \mathfrak{t})$, and let $\mathcal{C}_0 \subset \mathfrak{t}^*$ denote the Weyl chamber corresponding to $R^+(\mathfrak{y}, \mathfrak{t})$. Then $\mathcal{C} \subset \mathcal{C}_0$.

Let $W(U_M, T)$, $W(Y, T)$ be the Weyl groups associated with the pairs (U_M, T), (Y, T) respectively. Then $W(Y, T)$ is canonically a subgroup of $W(U_M, T)$. Put

$$W^1(k) = \{\omega \in W(U_M, T) \mid \omega(\mathcal{C}) \subset \mathcal{C}_0\}. \tag{5.4.14}$$

Note that the set $W^1(k)$ is similar to the set W_u defined in (5.3.4).

Lemma 5.4.1. *The inclusion $W^1(k) \hookrightarrow W(U_M, T)$ induces a bijection between $W^1(k)$ and the quotient $W(Y, T)\backslash W(U_M, T)$.*

Proof. This lemma follows from $W(Y, T)$ acting simply transitively on the Weyl chambers associated with $(\mathfrak{y}, \mathfrak{t})$. □

Let $\mathcal{O}^k$ denote the fixed-point set of the holomorphic action of k on $\mathcal{O}$. We embed $\mathfrak{y}^*$ in $\mathfrak{u}_\mathfrak{m}^*$ by the splitting (5.4.8). Then

$$\mathcal{O}^k = \mathcal{O} \cap \mathfrak{y}^*. \tag{5.4.15}$$

Lemma 5.4.2 (see [Duflo et al. 1984, I.2, Lemma (7); Bouaziz 1987, Lemmas 6.1.1, 7.2.2]). *As subsets of $\mathfrak{y}^*$, we have the identification*

$$\mathcal{O}^k = \bigcup_{\sigma \in W^1(k)} \mathrm{Ad}^*(Y)(\sigma(\lambda + \rho_{\mathfrak{u}_\mathfrak{m}})) \subset \mathfrak{y}^*, \tag{5.4.16}$$

where the union is disjoint.

For each $\sigma \in W^1(k)$, put

$$\mathcal{O}^k_{\sigma(\lambda+\rho_{\mathfrak{u}_\mathfrak{m}})} = \mathrm{Ad}^*(Y)(\sigma(\lambda+\rho_{\mathfrak{u}_\mathfrak{m}})) \subset \mathfrak{y}^*. \tag{5.4.17}$$

Let $d\mu^k_\sigma$ denote the Liouville measure on $\mathcal{O}^k_{\sigma(\lambda+\rho_{\mathfrak{u}_\mathfrak{m}})}$ as defined in (5.4.5).

If $\delta \in \mathfrak{t}^*$ is (real) analytically integrable, let ξ_δ denote the character of T with differential $2\pi i\delta$. Note that for $\sigma \in W^1(k)$, $\sigma\rho_{\mathfrak{u}_\mathfrak{m}} + \rho_{\mathfrak{u}_\mathfrak{m}}$ is analytically integrable even though $\rho_{\mathfrak{u}_\mathfrak{m}}$ may not be analytically integrable.

Definition 5.4.3. For $\sigma \in W^1(k)$, set

$$\varphi_k(\sigma,\lambda) = \varepsilon(\sigma)\frac{\xi_{\sigma(\lambda+\rho_{\mathfrak{u}_\mathfrak{m}})+\rho_{\mathfrak{u}_\mathfrak{m}}}(k)}{\Pi_{\alpha^0\in R^+(\mathfrak{q},\mathfrak{t})}(\xi_{\alpha^0}(k)-1)}. \tag{5.4.18}$$

Note that if $y \in \mathfrak{y}$, the analytic function

$$\frac{\det(1-e^{\mathrm{ad}(y)}\,\mathrm{Ad}(k))|_{\mathfrak{q}}}{\det(1-\mathrm{Ad}(k))|_{\mathfrak{q}}} \tag{5.4.19}$$

has a square root which is analytic in $y \in \mathfrak{y}$ and equal to 1 at $y=0$. We denote this square root by

$$\left[\frac{\det(1-e^{\mathrm{ad}(y)}\,\mathrm{Ad}(k))|_{\mathfrak{q}}}{\det(1-\mathrm{Ad}(k))|_{\mathfrak{q}}}\right]^{\frac{1}{2}}. \tag{5.4.20}$$

The following theorem is a special case of a generalized Kirillov formula obtained by Duflo, Heckman and Vergne [Duflo et al. 1984, II.3, Theorem (7)]. We will also include a simpler proof for the sake of completeness.

Theorem 5.4.4 (generalized Kirillov formula). *For $y \in \mathfrak{y}$, we have the identity of analytic functions*

$$\begin{aligned}\widehat{A}^{-1}(\mathrm{ad}(y)|_{\mathfrak{y}})\left[\frac{\det(1-e^{\mathrm{ad}(y)}\,\mathrm{Ad}(k))|_{\mathfrak{q}}}{\det(1-\mathrm{Ad}(k))_{\mathfrak{q}}}\right]^{\frac{1}{2}} &\mathrm{Tr}^{V_\lambda}[\rho^{V_\lambda}(ke^y)]\\ &= \sum_{\sigma\in W^1(k)}\varphi_k(\sigma,\lambda)\int_{f\in\mathcal{O}^k_{\sigma(\lambda+\rho_{\mathfrak{u}_\mathfrak{m}})}} e^{2\pi i\langle f,y\rangle}\,d\mu^k_\sigma.\end{aligned} \tag{5.4.21}$$

If $k=1$, (5.4.21) *is reduced to* (5.4.7).

Proof. Let $\mathfrak{t}'$ denote the set of regular element in $\mathfrak{t}$ associated with the root $R(\mathfrak{u}_\mathfrak{m},\mathfrak{t})$, which is an open dense subset of $\mathfrak{t}$. Since both sides of (5.4.21) are analytic and invariant by the adjoint action of Y, we only need to prove (5.4.21) for $y \in \mathfrak{t}'$.

We firstly compute the left-hand side of (5.4.21).

For $y \in \mathfrak{t}'$, we have

$$\widehat{A}^{-1}(\mathrm{ad}(y)|_{\mathfrak{y}}) = \Pi_{\alpha^0\in R^+(\mathfrak{y},\mathfrak{t})}\frac{e^{\pi i\langle\alpha^0,y\rangle}-e^{-\pi i\langle\alpha^0,y\rangle}}{\langle 2\pi i\alpha^0,y\rangle}. \tag{5.4.22}$$

Let $y_0 \in \mathfrak{t}$ be such that $k=\exp(y_0)$. Then

$$\left[\frac{\det(1-e^{\mathrm{ad}(y)}\,\mathrm{Ad}(k))|_{\mathfrak{q}}}{\det(1-\mathrm{Ad}(k))|_{\mathfrak{q}}}\right]^{\frac{1}{2}} = \Pi_{\alpha^0\in R^+(\mathfrak{q},\mathfrak{t})}\frac{e^{\pi i\langle\alpha^0,y+y_0\rangle}-e^{-\pi i\langle\alpha^0,y+y_0\rangle}}{e^{\pi i\langle\alpha^0,y_0\rangle}-e^{-\pi i\langle\alpha^0,y_0\rangle}}. \tag{5.4.23}$$

By the Weyl character formula for (U_M, T), we get

$$\operatorname{Tr}^{V_\lambda}[\rho^{V_\lambda}(ke^y)] = \operatorname{Tr}^{V_\lambda}[\rho^{V_\lambda}(e^{y+y_0})] = \frac{\sum_{\omega\in W(\mathfrak{u}_{\mathfrak{m},\mathbb{C}},\mathfrak{t}_{\mathbb{C}})}\varepsilon(\omega)e^{2\pi i\langle\omega(\lambda+\rho_{\mathfrak{u}_\mathfrak{m}}),y+y_0\rangle}}{\prod_{\alpha^0\in R^+(\mathfrak{u}_\mathfrak{m},\mathfrak{t})}(e^{\pi i\langle\alpha^0,y+y_0\rangle}-e^{-\pi i\langle\alpha^0,y+y_0\rangle})}. \tag{5.4.24}$$

Note that we have $\xi_{\alpha_0}(k)=1$ for $\alpha_0\in R^+(\mathfrak{y},\mathfrak{t})$. Then

$$\prod_{\alpha^0\in R^+(\mathfrak{y},\mathfrak{t})}\frac{e^{\pi i\langle\alpha^0,y+y_0\rangle}-e^{-\pi i\langle\alpha^0,y+y_0\rangle}}{e^{\pi i\langle\alpha^0,y\rangle}-e^{-\pi i\langle\alpha^0,y\rangle}} = e^{-2\pi i\langle\rho_\mathfrak{y},y_0\rangle}. \tag{5.4.25}$$

Combining (5.4.22)–(5.4.25), we get the left-hand side of (5.4.21) is equal the to function

$$\frac{e^{2\pi i\langle\rho_\mathfrak{y},y_0\rangle}}{\prod_{\alpha^0\in R^+(\mathfrak{y},\mathfrak{t})}\langle 2\pi i\alpha^0,y\rangle}\frac{\sum_{\omega\in W(\mathfrak{u}_{\mathfrak{m},\mathbb{C}},\mathfrak{t}_{\mathbb{C}})}\varepsilon(\omega)e^{2\pi i\langle\omega(\lambda+\rho_{\mathfrak{u}_\mathfrak{m}}),y+y_0\rangle}}{\prod_{\alpha^0\in R^+(\mathfrak{q},\mathfrak{t})}(e^{\pi i\langle\alpha^0,y\rangle}-e^{-\pi i\langle\alpha^0,y\rangle})}. \tag{5.4.26}$$

Now we show that the right-hand side of (5.4.21) is also equal to (5.4.26).

Note that, for $\omega\in W(Y,T)$, $\omega\rho_{\mathfrak{u}_\mathfrak{m}}-\rho_{\mathfrak{u}_\mathfrak{m}}$ is analytically integrable. We claim that if $\omega\in W(Y,T)$, then

$$\xi_{\omega\rho_{\mathfrak{u}_\mathfrak{m}}-\rho_{\mathfrak{u}_\mathfrak{m}}}(k) = e^{2\pi i\langle\omega\rho_{\mathfrak{u}_\mathfrak{m}}-\rho_{\mathfrak{u}_\mathfrak{m}},y_0\rangle} = 1. \tag{5.4.27}$$

Actually, we have $\xi_{2\rho_{\mathfrak{u}_\mathfrak{m}}}(k)=\xi_{2\omega\rho_{\mathfrak{u}_\mathfrak{m}}}(k)=1$. Then, after taking the square roots, we get $\xi_{\omega\rho_{\mathfrak{u}_\mathfrak{m}}-\rho_{\mathfrak{u}_\mathfrak{m}}}(k)=\xi_{\omega\rho_{\mathfrak{u}_\mathfrak{m}}-\rho_{\mathfrak{u}_\mathfrak{m}}}(e^{y_0})=\pm 1$. The continuity of the character implies exactly (5.4.27).

As a consequence of (5.4.27), we get that for $\sigma\in W^1(k)$, if $\omega\in W(Y,T)$, then

$$e^{2\pi i\langle\omega\sigma(\lambda+\rho_{\mathfrak{u}_\mathfrak{m}}),y_0\rangle} = e^{2\pi i\langle\sigma(\lambda+\rho_{\mathfrak{u}_\mathfrak{m}}),y_0\rangle}. \tag{5.4.28}$$

For $\sigma\in W^1(k)$, since $\sigma(\lambda+\rho_{\mathfrak{u}_\mathfrak{m}})\in\mathcal{C}_0$ and y is regular, by [Berline et al. 1992, Corollary 7.25], we have

$$\int_{f\in\mathcal{O}^k_{\sigma(\lambda+\rho_{\mathfrak{u}_\mathfrak{m}})}} e^{2\pi i\langle f,y\rangle}\,d\mu^k_\sigma = \frac{1}{\prod_{\alpha^0\in R^+(\mathfrak{y},\mathfrak{t})}\langle 2\pi i\alpha^0,y\rangle}\sum_{\omega\in W(Y,T)}\varepsilon(\omega)e^{2\pi i\langle\omega\sigma(\lambda+\rho_{\mathfrak{u}_\mathfrak{m}}),y\rangle}. \tag{5.4.29}$$

We rewrite $\varphi_k(\sigma,\lambda)$ as

$$\varepsilon(\sigma)\frac{e^{2\pi i\langle\rho_\mathfrak{y},y_0\rangle}}{\prod_{\alpha^0\in R^+(\mathfrak{q},\mathfrak{t})}(e^{\pi i\langle\alpha^0,y\rangle}-e^{-\pi i\langle\alpha^0,y\rangle})}e^{2\pi i\langle\sigma(\lambda+\rho_{\mathfrak{u}_\mathfrak{m}}),y_0\rangle}. \tag{5.4.30}$$

Combining together Lemma 5.4.1 and (5.4.28)–(5.4.30), a direct computation shows that the right-hand side of (5.4.21) is given exactly by (5.4.26). □

Remark 5.4.5. Let C^0 denote the identity component of the center of Y, and let Y_{ss} be the closed analytic subgroup of Y associated with $\mathfrak{y}_{\mathrm{ss}}=[\mathfrak{y},\mathfrak{y}]$. By Weyl's theorem [Knapp 1986, Theorem 4.26], the universal covering group of Y_{ss} is compact, which we denote by $\widetilde{Y}_{\mathrm{ss}}$. Put

$$\widetilde{Y} = C^0\times\widetilde{Y}_{\mathrm{ss}}. \tag{5.4.31}$$

Then $\widetilde{Y}$ is clearly a finite central extension of Y. Let $\widetilde{T}$ be the maximal torus of $\widetilde{Y}$ associated with the Cartan subalgebra $\mathfrak{t}$, which is also a finite extension of T. By [Knapp 1986, Corollary 4.25], the weights

$\rho_{\mathfrak{u}_\mathfrak{m}}$, $\rho_{\mathfrak{y}}$ are analytically integrable with respect to $\widetilde{T}$, since they are algebraically integrable [Knapp 1986, Propositions 4.15, 4.33].

Note that, for $\sigma \in W^1(k)$, $\sigma(\lambda + \rho_{\mathfrak{u}_\mathfrak{m}})$ is regular and positive with respect to $R^+(\mathfrak{y}, \mathfrak{t})$; thus $\sigma(\lambda + \rho_{\mathfrak{u}_\mathfrak{m}}) - \rho_{\mathfrak{y}}$ is nonnegative with respect to $R^+(\mathfrak{y}, \mathfrak{t})$ by the property of $\rho_{\mathfrak{y}}$ [Knapp 1986, Proposition 4.33]. Since now $\sigma(\lambda + \rho_{\mathfrak{u}_\mathfrak{m}}) - \rho_{\mathfrak{y}}$ is also analytically integrable with respect to $\widetilde{T}$, it is a dominant weight for $(\widetilde{Y}, \widetilde{T})$ with respect to $R^+(\mathfrak{y}, \mathfrak{t})$. In this case, let $V^k_{\lambda,\sigma}$ be the irreducible unitary representation of $\widetilde{Y}$ with highest weight $\sigma(\lambda + \rho_{\mathfrak{u}_\mathfrak{m}}) - \rho_{\mathfrak{y}}$. Then by (5.4.7), (5.4.21), we get that, for $y \in \mathfrak{y}$,

$$\left[\frac{\det(1 - e^{\mathrm{ad}(y)}\,\mathrm{Ad}(k))|_{\mathfrak{q}}}{\det(1 - \mathrm{Ad}(k))_{\mathfrak{q}}}\right]^{\frac{1}{2}} \mathrm{Tr}^{V_\lambda}[\rho^{V_\lambda}(ke^y)] = \sum_{\sigma \in W^1(k)} \varphi_k(\sigma, \lambda)\, \mathrm{Tr}^{V^k_{\lambda,\sigma}}[\rho^{V^k_{\lambda,\sigma}}(e^y)]. \tag{5.4.32}$$

6. A geometric localization formula for orbital integrals

Recall that $G_{\mathbb{C}}$ is the complexification of G with Lie algebra $\mathfrak{g}_{\mathbb{C}}$, and that G, U are the analytic subgroups of $G_{\mathbb{C}}$ with Lie algebras $\mathfrak{g}$, $\mathfrak{u}$ respectively. In this section, we always assume that U is compact; we do not require that G have compact center. We need not to assume $\delta(G) = 1$ either.

Under the settings in Section 4.1, for $t > 0$ and semisimple $\gamma \in G$, we set

$$\mathcal{E}_{X,\gamma}(F, t) = \mathrm{Tr}_s^{[\gamma]}\Big[\Big(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2}\Big) \exp\Big(-\frac{t\boldsymbol{D}^{X,F,2}}{2}\Big)\Big]. \tag{6.0.1}$$

The indices X, F in this notation indicate precisely the symmetric space and the flat vector bundle which are concerned with defining the orbital integrals.

If $\gamma \in G$ is semisimple, then there exists a unique elliptic element γ_e and a unique hyperbolic element γ_h in G such that $\gamma = \gamma_e\gamma_h = \gamma_h\gamma_e$. Here, we will show that $\mathcal{E}_{X,\gamma}(F, t)$ becomes a sum of the orbital integrals associated with γ_h, but defined for the centralizer of γ_e instead of G. This suggests that the elliptic part of γ should lead to a localization for the geometric orbital integrals.

We still fix a maximal torus T of K with Lie algebra $\mathfrak{t}$. For simplicity, if $\gamma \in G$ is semisimple, we may and we will assume

$$\gamma = e^a k, \quad k \in T, \quad a \in \mathfrak{p}, \quad \mathrm{Ad}(k^{-1})a = a. \tag{6.0.2}$$

In this case,

$$\gamma_e = k \in T, \quad \gamma_h = e^a. \tag{6.0.3}$$

Recall that $Z(\gamma_e)^0$ is the identity component of the centralizer of γ_e in G. Then

$$\gamma_h \in Z(\gamma_e)^0. \tag{6.0.4}$$

The Cartan involution θ preserves $Z(\gamma_e)^0$ such that $Z(\gamma_e)^0$ is a connected linear reductive Lie group. Then we have the diffeomorphism

$$Z(\gamma_e)^0 = K(\gamma_e)^0 \exp(\mathfrak{p}(\gamma_e)). \tag{6.0.5}$$

It is clear that $\delta(Z(\gamma_e)^0) = \delta(G)$.

Recall that T_U is a maximal torus of U with Lie algebra $\mathfrak{t}_U = \sqrt{-1}\mathfrak{b} \oplus \mathfrak{t} \subset \mathfrak{u}$. Let $R^+(\mathfrak{u}, \mathfrak{t}_U)$ be a positive root system for $R(\mathfrak{u}, \mathfrak{t}_U)$, which is not necessarily the same as in Section 5.3 when $\delta(G) = 1$.

Since U is the compact form of G, $U(\gamma_e)^0$ is the compact form for $Z(\gamma_e)^0$. Moreover, T_U is also a maximal torus of $U(\gamma_e)^0$. Let $R(\mathfrak{u}(\gamma_e), \mathfrak{t}_U)$ be the corresponding real root system with the positive root system $R^+(\mathfrak{u}(\gamma_e), \mathfrak{t}_U) = R(\mathfrak{u}(\gamma_e), \mathfrak{t}_U) \cap R^+(\mathfrak{u}, \mathfrak{t}_U)$. Let $\rho_{\mathfrak{u}}$, $\rho_{\mathfrak{u}(\gamma_e)}$ be the corresponding half sums of positive roots.

Let $\widetilde{U}(\gamma_e)$ be a connected finite covering group of $U(\gamma_e)^0$ such that $\rho_{\mathfrak{u}}$, $\rho_{\mathfrak{u}(\gamma_e)}$ are analytically integrable with respect to the maximal torus $\widetilde{T}_U$ of $\widetilde{U}(\gamma_e)$ associated with $\mathfrak{t}_U$. It always exists by a construction similar to that in Remark 5.4.5.

Let $\widetilde{K}(\gamma_e)$ be the analytic subgroup of $\widetilde{U}(\gamma_e)$ associated with the Lie algebra $\mathfrak{k}(\gamma_e)$. By [Knapp 2002, Proposition 7.12], $\widetilde{U}(\gamma_e)$ has a unique complexification $\widetilde{U}(\gamma_e)_{\mathbb{C}}$ which is a connected linear reductive Lie group. Let $\widetilde{Z}(\gamma_e)$ be the analytic subgroup of $\widetilde{U}(\gamma_e)_{\mathbb{C}}$ associated with $\mathfrak{z}(\gamma_e) \subset \mathfrak{u}(\gamma_e)_{\mathbb{C}} = \mathfrak{z}(\gamma_e)_{\mathbb{C}}$. Then we have the Cartan decomposition

$$\widetilde{Z}(\gamma_e) = \widetilde{K}(\gamma_e) \exp(\mathfrak{p}(\gamma_e)). \tag{6.0.6}$$

We still denote by θ the corresponding Cartan involution on $\widetilde{Z}(\gamma_e)$.

The Lie group $\widetilde{Z}(\gamma_e)$ is a finite covering group of $Z(\gamma_e)^0$. Moreover, we have the identification of symmetric spaces

$$X(\gamma_e) \simeq \widetilde{Z}(\gamma_e)/\widetilde{K}(\gamma_e). \tag{6.0.7}$$

Note that even under an additional assumption that G has compact center, $\widetilde{Z}(\gamma_e)$ may still have noncompact center.

Let λ be a dominant weight for (U, T_U) with respect to $R^+(\mathfrak{u}, \mathfrak{t}_U)$. Let $(E_\lambda, \rho^{E_\lambda})$ be the associated irreducible unitary representation of U. As before, let $(F_\lambda, h^{F_\lambda})$ be the corresponding homogeneous vector bundle on X with the G-invariant flat connection $\nabla^{F_\lambda, f}$. Let $\boldsymbol{D}^{X, F_\lambda, 2}$ denote the associated de Rham–Hodge Laplacian.

Let $W^1_U(\gamma_e) \subset W(U, T_U)$ be the set defined as in (5.4.14) but with respect to the group U and to $\gamma_e = k \in T \subset T_U$. As in Definition 5.4.3, for $\sigma \in W^1_U(\gamma_e)$, set

$$\varphi^U_{\gamma_e}(\sigma, \lambda) = \varepsilon(\sigma) \frac{\xi_{\sigma(\lambda+\rho_{\mathfrak{u}})+\rho_{\mathfrak{u}}}(\gamma_e)}{\prod_{\alpha^0 \in R^+(\mathfrak{u}^\perp(\gamma_e), \mathfrak{t}_U)} (\xi_{\alpha^0}(\gamma_e) - 1)}. \tag{6.0.8}$$

As explained in Remark 5.4.5, if $\sigma \in W^1_U(\gamma_e)$, then $\sigma(\lambda + \rho_{\mathfrak{u}}) - \rho_{\mathfrak{u}(\gamma_e)}$ is a dominant weight of $\widetilde{U}(\gamma_e)$ with respect to $R^+(\mathfrak{u}(\gamma_e), \mathfrak{t}_U)$. Let $E_{\sigma,\lambda}$ be the irreducible unitary representation of $\widetilde{U}(\gamma_e)$ with highest weight $\sigma(\lambda + \rho_{\mathfrak{u}}) - \rho_{\mathfrak{u}(\gamma_e)}$.

We extend $E_{\sigma,\lambda}$ to an irreducible representation of $\widetilde{Z}(\gamma_e)$ by the unitary trick. Then

$$F_{\sigma,\lambda} = \widetilde{Z}(\gamma_e) \times_{\widetilde{K}(\gamma_e)} E_{\sigma,\lambda}$$

is a homogeneous vector bundle on $X(\gamma_e)$ with an invariant flat connection $\nabla^{F_{\sigma,\lambda}, f}$ as explained in Section 4. Let $\boldsymbol{D}^{X(\gamma_e), F_{\sigma,\lambda}, 2}$ denote the associated de Rham–Hodge Laplacian acting on $\Omega^\bullet(X(\gamma_e), F_{\sigma,\lambda})$.

We also view $\gamma_h = e^a$ as a hyperbolic element in $\widetilde{Z}(\gamma_e)$. For $\sigma \in W^1_U(\gamma_e)$, as in (6.0.1), we set

$$\mathcal{E}_{X(\gamma_e),\gamma_h}(F_{\sigma,\lambda}, t) = \mathrm{Tr}_s^{[\gamma_h]}\left[\left(N^{\Lambda^\bullet(T^*X(\gamma_e))} - \frac{p'}{2}\right) \exp\left(-\frac{t\boldsymbol{D}^{X(\gamma_e), F_{\sigma,\lambda}, 2}}{2}\right)\right]. \tag{6.0.9}$$

Note that we use $B|_{\mathfrak{z}(\gamma_e)}$ on $\mathfrak{z}(\gamma_e)$ to define this orbital integral for $\widetilde{Z}(\gamma_e)$.

Set

$$c(\gamma)=\left|\frac{\det(1-\mathrm{Ad}(\gamma_e))|_{\mathfrak{z}^{\perp}(\gamma_e)}}{\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{z}^{\perp}(\gamma_e)}}\right|^{\frac{1}{2}}>0. \tag{6.0.10}$$

In particular, $c(\gamma_e)=1$.

The following theorem is essentially a consequence of the generalized Kirillov formula in Theorem 5.4.4.

Theorem 6.0.1. *Let $\gamma\in G$ be given as in* (6.0.2). *For $t>0$, we have the identity*

$$\mathcal{E}_{X,\gamma}(F_\lambda,t)=c(\gamma)\sum_{\sigma\in W_U^1(\gamma_e)}\varphi^U_{\gamma_e}(\sigma,\lambda)\mathcal{E}_{X(\gamma_e),\gamma_h}(F_{\sigma,\lambda},t). \tag{6.0.11}$$

We call (6.0.11) *a localization formula for the geometric orbital integral.*

Proof. Set $p'=\dim\mathfrak{p}(\gamma_e)=\dim X(\gamma_e)$. At first, if m is even, then p' is even. Then the both sides of (6.0.11) are 0 by [Bismut 2011, Theorem 7.9.1].

If m is odd, then p' is odd, and $\delta(G)=\delta(Z(\gamma_e)^0)$ is odd. If $\delta(G)\geq 3$, then the both sides of (6.0.11) are 0 by [Bismut 2011, Theorem 7.9.1].

Now we consider the case where $\delta(G)=\delta(Z(\gamma_e)^0)=1$. If γ cannot be conjugated into H by an element in G, then γ_h cannot be conjugated into H by an element in $Z(\gamma_e)^0$. Then both sides of (6.0.11) are 0 by Proposition 4.1.5.

Now we assume that $\delta(G)=1$ and $a\in\mathfrak{b}$. Note that $\mathfrak{z}(\gamma)$ is the centralizer of γ_h in $\mathfrak{z}(\gamma_e)$. We will prove (6.0.11) using (4.1.16)

For $y\in\mathfrak{k}(\gamma)$, let $J^{\sim}_{\gamma_h}(y)$ be the function defined in 3.3.1 for $\gamma_h=e^a\in\widetilde{Z}(\gamma_e)$:

$$J^{\sim}_{\gamma_h}(y)=\frac{1}{|\det(1-\mathrm{Ad}(\gamma_h))|_{\mathfrak{z}_0^{\perp}\cap\mathfrak{z}(\gamma_e)}|^{\frac{1}{2}}}\frac{\widehat{A}(i\,\mathrm{ad}(y)|_{\mathfrak{p}(\gamma)})}{\widehat{A}(i\,\mathrm{ad}(y)|_{\mathfrak{k}(\gamma)})}. \tag{6.0.12}$$

The Casimir operator $C^{\mathfrak{u}(\gamma_e),E_{\sigma,\lambda}}$ acts on $E_{\sigma,\lambda}$ by the scalar given by

$$-4\pi^2(|\lambda+\rho_{\mathfrak{u}}|^2-|\rho_{\mathfrak{u}(\gamma_e)}|^2). \tag{6.0.13}$$

Similar to (4.1.13), set

$$\beta_{\mathfrak{z}(\gamma_e)}=\tfrac{1}{16}\mathrm{Tr}^{\mathfrak{p}(\gamma_e)}[C^{\mathfrak{k}(\gamma_e),\mathfrak{p}(\gamma_e)}]+\tfrac{1}{48}\mathrm{Tr}^{\mathfrak{k}(\gamma_e)}[C^{\mathfrak{k}(\gamma_e),\mathfrak{k}(\gamma_e)}]. \tag{6.0.14}$$

Then by [Bismut 2011, Propositions 2.6.1, 7.5.1],

$$2\pi^2|\rho_{\mathfrak{u}(\gamma_e)}|^2=-\beta_{\mathfrak{z}(\gamma_e)}. \tag{6.0.15}$$

By (4.1.16), (6.0.13), (6.0.15), for $\sigma\in W^1_U(\gamma_e)$, we get

$$\begin{aligned}\mathcal{E}_{X(\gamma_e),\gamma_h}(F_{\sigma,\lambda},t)=\frac{e^{-\frac{|a|^2}{2t}}}{(2\pi t)^{\frac{p}{2}}}&\exp(-2\pi^2t|\lambda+\rho_{\mathfrak{u}}|^2)\\ &\cdot\int_{\mathfrak{k}(\gamma)}J^{\sim}_{\gamma_h}(y)\,\mathrm{Tr}_s^{\Lambda^\bullet(\mathfrak{p}(\gamma_e)^*)}\Big[\Big(N^{\Lambda^\bullet(\mathfrak{p}(\gamma_e)^*)}-\frac{p'}{2}\Big)\exp(-i\,\mathrm{ad}(y))\Big]\\ &\qquad\cdot\mathrm{Tr}^{E_{\sigma,\lambda}}[\exp(-i\rho^{E_{\sigma,\lambda}}(y))]e^{-\frac{|y|^2}{2t}}\frac{dy}{(2\pi t)^{\frac{q}{2}}}.\end{aligned} \tag{6.0.16}$$

Note that $\dim \mathfrak{p}^{\perp}(\gamma_e)$ is even. We claim that if $y \in \mathfrak{k}(\gamma)$, then

$$\begin{aligned}\operatorname{Tr}_s^{\Lambda^\bullet(\mathfrak{p}^*)}\Big[\Big(N^{\Lambda^\bullet(\mathfrak{p}^*)} - \frac{m}{2}\Big)\exp(-i\operatorname{ad}(y))\operatorname{Ad}(k^{-1})\Big] \\ = \operatorname{Tr}_s^{\Lambda^\bullet(\mathfrak{p}(\gamma_e)^*)}\Big[\Big(N^{\Lambda^\bullet(\mathfrak{p}(\gamma_e)^*)} - \frac{p'}{2}\Big)e^{-i\operatorname{ad}(y)}\Big]\det(1-e^{-i\operatorname{ad}(y)}\operatorname{Ad}(k^{-1}))|_{\mathfrak{p}^{\perp}(\gamma_e)}.\end{aligned} \tag{6.0.17}$$

Indeed, we can verify (6.0.17) for $y \in \mathfrak{t}$. Since both sides of (6.0.17) are invariant by the adjoint action of $K(\gamma_e)^0$, (6.0.17) holds in full generality.

Also $K(\gamma)^0$ preserves the splitting

$$\mathfrak{p}^{\perp}(\gamma_e) = \mathfrak{p}_0^{\perp}(\gamma) \oplus (\mathfrak{p}^{\perp}(\gamma_e) \cap \mathfrak{p}_0^{\perp}). \tag{6.0.18}$$

The action $\operatorname{ad}(a)$ gives an isomorphism between $\mathfrak{p}^{\perp}(\gamma_e) \cap \mathfrak{p}_0^{\perp}$ and $\mathfrak{k}^{\perp}(\gamma_e) \cap \mathfrak{k}_0^{\perp}$ as $K(\gamma)$-vector spaces.

Note that

$$\mathfrak{z}^{\perp}(\gamma_e) \cap \mathfrak{z}_0^{\perp} = (\mathfrak{p}^{\perp}(\gamma_e) \cap \mathfrak{p}_0^{\perp}) \oplus (\mathfrak{k}^{\perp}(\gamma_e) \cap \mathfrak{k}_0^{\perp}). \tag{6.0.19}$$

Then

$$\begin{aligned}\det(1-e^{-i\operatorname{ad}(y)}\operatorname{Ad}(\gamma_e))|_{\mathfrak{p}^{\perp}(\gamma_e)} \\ = \det(1-e^{-i\operatorname{ad}(y)}\operatorname{Ad}(\gamma_e))|_{\mathfrak{p}_0^{\perp}(\gamma_e)}[\det(1-e^{-i\operatorname{ad}(y)}\operatorname{Ad}(\gamma_e))|_{\mathfrak{z}^{\perp}(\gamma_e)\cap\mathfrak{z}_0^{\perp}}]^{\frac{1}{2}}.\end{aligned} \tag{6.0.20}$$

Here the square root is taken to be positive at $y = 0$.

By Definition 3.3.1 and (6.0.12), for $y \in \mathfrak{k}(\gamma)$,

$$\begin{aligned}J_\gamma(y) = J_{\gamma_h}^{\sim}(y)\frac{1}{|\det(1-\operatorname{Ad}(\gamma))|_{\mathfrak{z}_0^{\perp}\cap\mathfrak{z}^{\perp}(\gamma_e)}|^{\frac{1}{2}}} \\ \cdot\Big[\frac{1}{\det(1-\operatorname{Ad}(\gamma_e))|_{\mathfrak{z}_0^{\perp}(\gamma)}}\frac{\det(1-\exp(-i\operatorname{ad}(y))\operatorname{Ad}(\gamma_e))|_{\mathfrak{k}_0^{\perp}(\gamma)}}{\det(1-\exp(-i\operatorname{ad}(y))\operatorname{Ad}(\gamma_e))|_{\mathfrak{p}_0^{\perp}(\gamma)}}\Big]^{\frac{1}{2}}.\end{aligned} \tag{6.0.21}$$

Combining (6.0.17), (6.0.20) and (6.0.21), we get

$$\begin{aligned}J_\gamma(y)\operatorname{Tr}_s^{\Lambda^\bullet(\mathfrak{p}^*)}\Big[\Big(N^{\Lambda^\bullet(\mathfrak{p}^*)} - \frac{m}{2}\Big)\exp(-i\operatorname{ad}(y))\operatorname{Ad}(\gamma_e)\Big] \\ = c(\gamma)J_{\gamma_h}^{\sim}(y)\operatorname{Tr}_s^{\Lambda^\bullet(\mathfrak{p}(\gamma_e)^*)} \\ \cdot\Big[\Big(N^{\Lambda^\bullet(\mathfrak{p}(\gamma_e)^*)} - \frac{p'}{2}\Big)e^{-i\operatorname{ad}(y)}\Big]\Big[\frac{\det(1-\exp(-i\operatorname{ad}(y))\operatorname{Ad}(\gamma_e))|_{\mathfrak{z}^{\perp}(\gamma_e)}}{\det(1-\operatorname{Ad}(\gamma_e))|_{\mathfrak{z}^{\perp}(\gamma_e)}}\Big]^{\frac{1}{2}}.\end{aligned} \tag{6.0.22}$$

Note that, for $y \in \mathfrak{k}(\gamma)$,

$$\Big[\frac{\det(1-\exp(-i\operatorname{ad}(y))\operatorname{Ad}(\gamma_e))|_{\mathfrak{z}^{\perp}(\gamma_e)}}{\det(1-\operatorname{Ad}(\gamma_e))|_{\mathfrak{z}^{\perp}(\gamma_e)}}\Big]^{\frac{1}{2}} = \Big[\frac{\det(1-\exp(-i\operatorname{ad}(y))\operatorname{Ad}(\gamma_e))|_{\mathfrak{u}^{\perp}(\gamma_e)}}{\det(1-\operatorname{Ad}(\gamma_e))|_{\mathfrak{u}^{\perp}(\gamma_e)}}\Big]^{\frac{1}{2}}. \tag{6.0.23}$$

By (4.1.16), (6.0.13), (6.0.15), (6.0.22) and (6.0.23), we get

$$\begin{aligned}\mathcal{E}_{X,\lambda}(F_\lambda,t)=c(\gamma)&\frac{e^{-\frac{|a|^2}{2t}}}{(2\pi t)^{\frac{p}{2}}}\exp(-2\pi^2t|\lambda+\rho_{\mathfrak{u}}|^2)\\ &\cdot\int_{\mathfrak{k}(\gamma)}J_{\gamma_h}^{\sim}(y)\operatorname{Tr}_s^{\Lambda^\bullet(\mathfrak{p}(\gamma_e)^*)}\Big[\Big(N^{\Lambda^\bullet(\mathfrak{p}(\gamma_e)^*)}-\frac{p'}{2}\Big)e^{-i\,\mathrm{ad}(y)}\Big]\\ &\cdot\Big[\frac{\det(1-e^{-i\,\mathrm{ad}(y)}\operatorname{Ad}(\gamma_e))|_{\mathfrak{u}^\perp(\gamma_e)}}{\det(1-\operatorname{Ad}(\gamma_e))|_{\mathfrak{u}^\perp(\gamma_e)}}\Big]^{\frac{1}{2}}\operatorname{Tr}^{E_\lambda}[\rho^{E_\lambda}(\gamma_e)e^{-i\rho^{E_\lambda}(y)}]e^{-\frac{|y|^2}{2t}}\frac{dy}{(2\pi t)^{\frac{q}{2}}}.\end{aligned}\tag{6.0.24}$$

Then (6.0.11) follows from (5.4.32), (6.0.16) and (6.0.24). □

Remark 6.0.2. A similar consideration can be made for $\operatorname{Tr}_s^{[\gamma]}[\exp(-t\boldsymbol{D}^{X,F_\lambda,2})]$, where (6.0.11) will become an analogue of the index theorem for orbifolds as in (2.2.9). The related computation can be found in [Bismut and Shen 2022, Section 10.4].

7. Full asymptotics of elliptic orbital integrals

In this section, we always assume that $\delta(G)=1$ and that U is compact. We also use the notation and settings as in Sections 5.1, 5.2 and 5.3.

In this section, given a irreducible unitary representation E of U with certain nondegenerate highest weight Λ, and for elliptic γ, we will compute explicitly $\mathcal{E}_{X,\gamma}(F=G\times_K E,t)$ and its Mellin transform in terms of the root systems. Note that, when $\gamma=1$, $\mathcal{E}_{X,\gamma}(F_d,t)$ is already computed in [Bergeron and Venkatesh 2013; Müller and Pfaff 2013a] using the Plancherel formula for identity orbital integral. We here give a different approach via Bismut's formula as in (4.1.16).

Then in Section 7.3, we apply these results to a sequence of flat vector bundles $\{F_d\}_{d\in\mathbb{N}}$ on X defined by a sequence of nondegenerate dominant weights $\Lambda=d\lambda+\lambda_0$. This way, we show that the Mellin transforms of the elliptic orbital integrals are exponential polynomials in d.

7.1. ***Estimates of elliptic orbital integrals for small time t.*** Recall that T is a maximal torus of K, T_U is a maximal torus of U, and $W(U,T_U)$ denotes the (analytic) Weyl group of (U,T_U). The positive root system $R^+(\mathfrak{u},\mathfrak{t}_U)$ is given in Section 5.3. Recall that $P_{++}(U)$ is the set of dominant weights of (U,T_U) with respect to $R^+(\mathfrak{u},\mathfrak{t}_U)$.

Let (E,ρ^E) be the irreducible unitary representation of U associated with the highest weight $\Lambda\in P_{++}(U)$. We will prove our main result of this subsection and next subsection for this (E,ρ^E).

Our homogeneous flat vector bundle concerned here is given by $F=G\times_K E$. Let $\boldsymbol{D}^{X,F,2}$ denote the associated de Rham–Hodge Laplacian.

For $t>0$, if $\gamma\in G$ is semisimple, as in (6.0.1), set

$$\mathcal{E}_{X,\gamma}(F,t)=\operatorname{Tr}_s^{[\gamma]}\Big[\Big(N^{\Lambda^\bullet(T^*X)}-\frac{m}{2}\Big)\exp\Big(-\frac{t\boldsymbol{D}^{X,F,2}}{2}\Big)\Big].\tag{7.1.1}$$

It is clear that $\mathcal{E}_{X,\gamma}(F_d,t)$ only depends on the conjugacy class $[\gamma]$ in G. If $\gamma=1$, we also write

$$\mathcal{I}_X(F,t)=\mathcal{E}_{X,1}(F,t).\tag{7.1.2}$$

In the sequel, we only consider the case of elliptic γ.

By (4.1.16), (6.0.13), (6.0.15), if $\gamma = k \in K$, we have

$$\mathcal{E}_{X,\gamma}(F,t) = \frac{1}{(2\pi t)^{\frac{p}{2}}} \exp(-2\pi^2 t|\Lambda+\rho_{\mathfrak{u}}|^2) \cdot \int_{\mathfrak{k}(\gamma)} J_\gamma(Y_0^{\mathfrak{k}}) \operatorname{Tr}_s^{\Lambda^\bullet(\mathfrak{p}^*)}\Big[\Big(N^{\Lambda^\bullet(\mathfrak{p}^*)} - \frac{m}{2}\Big) \operatorname{Ad}(k)\exp(-i\operatorname{ad}(Y_0^{\mathfrak{k}}))\Big] \cdot \operatorname{Tr}^E[\rho^E(k)\exp(-i\rho^E(Y_0^{\mathfrak{k}}))] e^{-\frac{|Y_0^{\mathfrak{k}}|^2}{2t}} \frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{\frac{q}{2}}}. \tag{7.1.3}$$

By (3.3.18), we have the following formula for $J_\gamma(Y_0^{\mathfrak{k}})$, $Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma)$:

$$J_\gamma(Y_0^{\mathfrak{k}}) = \frac{\widehat{A}(i\operatorname{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{p}(\gamma)})}{\widehat{A}(i\operatorname{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{k}(\gamma)})}\Big[\frac{1}{\det(1-\operatorname{Ad}(k))|_{\mathfrak{z}^\perp(\gamma)}} \frac{\det(1-\exp(-i\operatorname{ad}(Y_0^{\mathfrak{k}}))\operatorname{Ad}(k))|_{\mathfrak{k}^\perp(\gamma)}}{\det(1-\exp(-i\operatorname{ad}(Y_0^{\mathfrak{k}}))\operatorname{Ad}(k))|_{\mathfrak{p}^\perp(\gamma)}}\Big]^{\frac{1}{2}}. \tag{7.1.4}$$

Proposition 7.1.1. *For an elliptic element $\gamma \in G$, there exists a constant $C_\gamma > 0$ (depending on Λ) such that for $t \in]0,1]$*

$$|\sqrt{t}\,\mathcal{E}_{X,\gamma}(F,t)| \le C_\gamma, \quad \Big|\Big(1+2t\frac{\partial}{\partial t}\Big)\mathcal{E}_{X,\gamma}(F,t)\Big| \le C_\gamma\sqrt{t}. \tag{7.1.5}$$

As $t \to 0$, $\mathcal{E}_{X,\gamma}(E,t)$ has the asymptotic expansion in the form of

$$\frac{1}{\sqrt{t}}\sum_{j=0}^{+\infty} a_j^\gamma t^j, \tag{7.1.6}$$

with $a_j^\gamma \in \mathbb{C}$.

Proof. If γ is elliptic, up to a conjugation, we assume that $\gamma = k \in T$. Thus the subgroup H defined in Section 4.1 is also a Cartan subgroup of $Z(\gamma)^0$. Then $\mathfrak{b}(\gamma) = \mathfrak{b}$. Let $\mathfrak{b}^\perp(\gamma)$ be the orthogonal complementary space of $\mathfrak{b}(\gamma)$ in $\mathfrak{p}(\gamma)$, whose dimension is $p-1$. Note that similar estimates have been proved in [Liu 2021, Theorem 4.4.1]; here we only sketch a proof to (7.1.5).

By (7.1.3), we have

$$\mathcal{E}_{X,\gamma}(F,t) = \frac{1}{(2\pi t)^{\frac{p}{2}}} \exp(-2\pi^2 t|\Lambda+\rho_{\mathfrak{u}}|^2) \cdot \int_{\mathfrak{k}(k)} J_k(\sqrt{t}Y_0^{\mathfrak{k}}) \operatorname{Tr}_s^{\Lambda^\bullet(\mathfrak{p}^*)}\Big[\Big(N^{\Lambda^\bullet(\mathfrak{p}^*)} - \frac{m}{2}\Big) \operatorname{Ad}(k)\exp(-i\operatorname{ad}(\sqrt{t}Y_0^{\mathfrak{k}}))\Big] \cdot \operatorname{Tr}^E[\rho^E(k)\exp(-i\rho^E(\sqrt{t}Y_0^{\mathfrak{k}}))] e^{-\frac{|Y_0^{\mathfrak{k}}|^2}{2}} \frac{dY_0^{\mathfrak{k}}}{(2\pi)^{\frac{q}{2}}}, \tag{7.1.7}$$

where the integral is rescaled by $\sqrt{t}$.

In this proof, we denote by C or c a positive constant independent of the variables t and $Y_0^{\mathfrak{k}}$. We use the symbol $\mathcal{O}_{\mathrm{ind}}$ to denote the big-O convention which does not depend on t and $Y_0^{\mathfrak{k}}$.

The same computations as in [Liu 2021, equations (4.4.8)–(4.4.10)] show that, for $Y_0^{\mathfrak{k}} \in \mathfrak{t}$,

$$\begin{aligned} J_k(\sqrt{t}Y_0^{\mathfrak{k}}) &= \frac{1}{\det(1-\operatorname{Ad}(k))|_{\mathfrak{p}^\perp(k)}} + \mathcal{O}_{\mathrm{ind}}(\sqrt{t}|Y_0^{\mathfrak{k}}|e^{C\sqrt{t}|Y_0^{\mathfrak{k}}|}) \\ &\quad\cdot \frac{1}{t^{(p-1)/2}} \operatorname{Tr}_s^{\Lambda^\bullet(\mathfrak{p}^*)}\Big[\Big(N^{\Lambda^\bullet(\mathfrak{p}^*)} - \frac{m}{2}\Big)\rho^{\Lambda^\bullet(\mathfrak{p}^*)}(k)\exp(-i\rho^{\Lambda^\bullet(\mathfrak{p}^*)}(\sqrt{t}Y_0^{\mathfrak{k}}))\Big] \\ &= -\det(i\operatorname{ad}(Y_0^{\mathfrak{k}}))|_{\mathfrak{b}^\perp(k)} \det(1-\operatorname{Ad}(k))|_{\mathfrak{p}^\perp(k)} + \mathcal{O}_{\mathrm{ind}}(\sqrt{t}|Y_0^{\mathfrak{k}}|e^{C\sqrt{t}|Y_0^{\mathfrak{k}}|}). \end{aligned} \tag{7.1.8}$$

Using the adjoint invariance, the further estimates on the above quantities by a function in $|Y_0^{\mathfrak{k}}|$ hold for all $Y_0^{\mathfrak{k}} \in \mathfrak{k}(k)$.

It is clear that

$$|\mathrm{Tr}^E[\rho^E(k)\exp(-i\rho^E(\sqrt{t}Y_0^{\mathfrak{k}}))]| \le C\exp(C\sqrt{t}|Y_0^{\mathfrak{k}}|). \tag{7.1.9}$$

Combining (7.1.8) and (7.1.9), we see that there exists a number $N \in \mathbb{N}$ big enough such that if $t \in]0,1]$,

$$|\sqrt{t}\mathcal{E}_{X,\gamma}(F,t)| \le C'_k \int_{\mathfrak{k}(k)} (1+|Y_0^{\mathfrak{k}}|)^N \exp\left(C|Y_0^{\mathfrak{k}}| - \frac{|Y_0^{\mathfrak{k}}|^2}{2}\right) dY_0^{\mathfrak{k}}. \tag{7.1.10}$$

The second estimate in (7.1.5) can be proved using the same arguments as in [Liu 2021, equations (4.4.24)–(4.4.29)].

The asymptotic expansion in (7.1.6) is just a consequence of (7.1.5) and (7.1.7). □

7.2. ***Elliptic orbital integrals for Hodge Laplacians.*** In this subsection, we explain how to use Bismut's formula (4.1.16) to compute explicitly the expansion of $\mathcal{E}_{X,\gamma}(F,t)$ in $t>0$ when $\gamma \in G$ is elliptic. Then we study the corresponding Mellin transform. After conjugation, we may and we will assume that $\gamma = k \in T$. Then T is also a maximal torus for $K(\gamma)^0$, and $\mathfrak{b}(\gamma) = \mathfrak{b}$.

Recall that $\omega^{Y_{\mathfrak{b}}(\gamma)}$, $\Omega^{\mathfrak{u}_{\mathfrak{b}}(\gamma)}$, $\Omega^{\mathfrak{u}_{\mathfrak{m}}(\gamma)}$ are defined in Section 5.2. Note that $\dim \mathfrak{u}_{\mathfrak{b}}^{\perp}(\gamma) = 4l(\gamma)$. If $\nu \in \Lambda^{\bullet}(\mathfrak{u}_{\mathfrak{b}}^{\perp}(\gamma)^*)$, let $[\nu]^{\max(\gamma)} \in \mathbb{R}$ be such that

$$\nu - [\nu]^{\max(\gamma)} \frac{\omega^{Y_{\mathfrak{b}}(\gamma),2l(\gamma)}}{(2l(\gamma))!} \tag{7.2.1}$$

is of degree strictly smaller than $4l(\gamma)$.

Recall that $-B(\cdot,\theta\cdot)$ is a Euclidean product on $\mathfrak{g}$. Let $\mathfrak{n}^{\perp}(\gamma)$, $\bar{\mathfrak{n}}^{\perp}(\gamma)$ be the orthogonal spaces of $\mathfrak{n}(\gamma)$, $\bar{\mathfrak{n}}(\gamma)$ in $\mathfrak{n}$, $\bar{\mathfrak{n}}$ respectively. As T-modules, $\mathfrak{n}^{\perp}(\gamma) \simeq \bar{\mathfrak{n}}^{\perp}(\gamma)$.

Since $\mathfrak{t} \subset \mathfrak{k}(\gamma) \subset \mathfrak{k}$, $R(\mathfrak{k}(\gamma),\mathfrak{t})$ is a subroot system of $R(\mathfrak{k},\mathfrak{t})$. Let $R^+(\mathfrak{k}(\gamma),\mathfrak{t})$ be the positive root system for $(\mathfrak{k}(\gamma),\mathfrak{t})$ induced by $R^+(\mathfrak{k},\mathfrak{t})$. We use the notation in Sections 5.1, 5.2. Then $\mathfrak{t}$ is a Cartan subalgebra for $\mathfrak{k}_{\mathfrak{m}}(\gamma)$, $\mathfrak{u}_{\mathfrak{m}}(\gamma)$, $\mathfrak{m}(\gamma)$. Let $R(\mathfrak{k}_{\mathfrak{m}}(\gamma),\mathfrak{t})$, $R(\mathfrak{u}_{\mathfrak{m}}(\gamma),\mathfrak{t})$ be the corresponding root systems.

Similar to (5.4.10), we have the disjoint union

$$R(\mathfrak{u}_{\mathfrak{m}}(\gamma),\mathfrak{t}) = R(\sqrt{-1}\mathfrak{p}_{\mathfrak{m}}(\gamma),\mathfrak{t}) \cup R(\mathfrak{k}_{\mathfrak{m}}(\gamma),\mathfrak{t}). \tag{7.2.2}$$

Since $R(\mathfrak{u}_{\mathfrak{m}}(\gamma),\mathfrak{t}) \subset R(\mathfrak{u}_{\mathfrak{m}},\mathfrak{t})$, by intersecting with $R^+(\mathfrak{u}_{\mathfrak{m}},\mathfrak{t})$, we get a positive root system $R^+(\mathfrak{u}_{\mathfrak{m}}(\gamma),\mathfrak{t})$. Moreover,

$$R^+(\mathfrak{u}_{\mathfrak{m}}(\gamma),\mathfrak{t}) = R^+(\sqrt{-1}\mathfrak{p}_{\mathfrak{m}}(\gamma),\mathfrak{t}) \cup R^+(\mathfrak{k}_{\mathfrak{m}}(\gamma),\mathfrak{t}). \tag{7.2.3}$$

Let $\mathrm{Vol}(K/T)$, $\mathrm{Vol}(U_M/T)$ be the Riemannian volumes of K/T, U_M/T with respect to the restriction of $-B$ to $\mathfrak{k}$, $\mathfrak{u}_{\mathfrak{m}}$ respectively. We have explicit formulae for them in terms of the roots; for example,

$$\mathrm{Vol}(U_M,T) = \Pi_{\alpha^0 \in R^+(\mathfrak{u}_{\mathfrak{m}},\mathfrak{t})} \frac{1}{2\pi\langle\alpha^0,\rho_{\mathfrak{u}_{\mathfrak{m}}}\rangle}. \tag{7.2.4}$$

For $\gamma = k \in T$, set

$$c_G(\gamma) = \frac{(-1)^{\frac{p-1}{2}+1}\operatorname{Vol}(K(\gamma)^0/T)|W(U_M(\gamma)^0,T)|}{\operatorname{Vol}(U_M(\gamma)^0/T)|W(K(\gamma)^0,T)|}\frac{1}{\det(1-\operatorname{Ad}(\gamma))|_{\mathfrak{n}^{\perp}(\gamma)}}. \tag{7.2.5}$$

If $\gamma = 1$, we define

$$c_G = c_G(1) = \frac{(-1)^{\frac{m-1}{2}+1}\operatorname{Vol}(K/T)|W(U_M,T)|}{\operatorname{Vol}(U_M/T)|W(K,T)|}. \tag{7.2.6}$$

We will use the same notation as in Sections 5.3 and 5.4. In particular, W_u is defined by (5.3.4) as a subset of $W(U,T_U)$, and $W^1(\gamma)$ is defined by (5.4.14) as a subset of $W(U_M,T)$. As explained in Remark 5.4.5, for $\omega \in W_u$, $\sigma \in W^1(\gamma)$, let $E^{\gamma}_{\omega,\sigma}$ denote the irreducible unitary representation of $Y = U_M(\gamma)^0$ or its finite central extension with highest weight $\sigma(\eta_\omega(\Lambda)+\rho_{\mathfrak{u}_{\mathfrak{m}}})-\rho_{\mathfrak{y}}$.

Definition 7.2.1. For $j = 0,1,\dots,l(\gamma)$, $\omega \in W_u$, $\sigma \in W^1(\gamma)$, set

$$Q^{\gamma}_{j,\omega,\sigma}(\Lambda) = \frac{(-1)^j\beta(a_1)^{2j}}{j!\,(2l(\gamma)-2j)!\,(8\pi^2)^j}\dim E^{\gamma}_{\omega,\sigma}[\omega^{Y_{\mathfrak{b}}(\gamma),2j}\langle\omega(\Lambda+\rho_{\mathfrak{u}}),\Omega^{\mathfrak{u}_{\mathfrak{m}}(\gamma)}\rangle^{2l(\gamma)-2j}]^{\max(\gamma)}. \tag{7.2.7}$$

In particular, if $l(\gamma) \geq 1$, we have

$$\begin{aligned} Q^{\gamma}_{0,\omega,\sigma}(\Lambda) &= \frac{1}{(2l)!}\dim E^{\gamma}_{\omega,\sigma}[\langle\omega(\Lambda+\rho_{\mathfrak{u}}),\Omega^{\mathfrak{u}_{\mathfrak{m}}(\gamma)}\rangle^{2l(\gamma)}]^{\max(\gamma)},\\ Q^{\gamma}_{l(\gamma),\omega,\sigma}(\Lambda) &= \frac{(-1)^{l(\gamma)}\beta(a_1)^{2l(\gamma)}(2l(\gamma)-1)!!}{(4\pi^2)^{l(\gamma)}}\dim E^{\gamma}_{\omega,\sigma}. \end{aligned} \tag{7.2.8}$$

Recall that $a_1 \in \mathfrak{b}$ is such that $B(a_1,a_1)=1$. For $\omega \in W_u$, set

$$b_{\Lambda,\omega} = \langle\omega\cdot(\Lambda+\rho_{\mathfrak{u}}),\sqrt{-1}a_1\rangle \in \mathbb{R}. \tag{7.2.9}$$

Then we have

$$|\eta_\omega(\Lambda)+\rho_{\mathfrak{u}_{\mathfrak{m}}}|^2-|\Lambda+\rho_{\mathfrak{u}}|^2 = -b^2_{\Lambda,\omega}. \tag{7.2.10}$$

Note that $\varphi_\gamma(\sigma,\eta_\omega(\Lambda))$ is defined in Definition 5.4.3.

Theorem 7.2.2. *For $t>0$, we have the identity*

$$\mathcal{E}_{X,\gamma}(F,t) = \frac{c_G(\gamma)}{\sqrt{2\pi t}}\sum_{j=0}^{l(\gamma)}t^{-j}\sum_{\substack{\omega\in W_u\\ \sigma\in W^1(\gamma)}}\varepsilon(\omega)\varphi_\gamma(\sigma,\eta_\omega(\Lambda))e^{-2\pi^2tb^2_{\Lambda,\omega}}Q^{\gamma}_{j,\omega,\sigma}(\Lambda). \tag{7.2.11}$$

Remark 7.2.3. The formula (7.2.11) is compatible with the estimate (7.1.5). For example, we take $\gamma=1$; then $W^1(\gamma)$ reduces to $\{1\}$, the representation $E^{\gamma}_{\omega,\sigma}$ is just $V_{\Lambda,\omega}$ introduced in (5.3.6), and $l(\gamma)=l$, $\varphi_\gamma(\sigma,\eta_\omega(\Lambda))=1$. Then we take the asymptotic expansion of the right-hand side of (7.4.2) as $t\to 0$, the coefficient of $t^{-l-1/2}$ is given by

$$\frac{c_G}{\sqrt{2\pi}}\sum_{\omega\in W_u}\varepsilon(\omega)Q^{\gamma=1}_{l,\omega,1}(\Lambda). \tag{7.2.12}$$

By (5.3.7), if $l \geq 1$, we get

$$\sum_{\omega \in W_u} \varepsilon(\omega) \dim V_{\Lambda,\omega} = \mathrm{Tr}_s^{\Lambda^\bullet(\mathfrak{n}_{\mathbb{C}}^*)}[1] \dim E = 0. \tag{7.2.13}$$

Then by (7.2.8) and (7.2.13), the quantity in (7.2.12) is 0 (provided $l \geq 1$).

Before proving Theorem 7.2.2, we need some preparation work.

Definition 7.2.4. For $y \in \mathfrak{t}$, put

$$\begin{aligned} \pi_{\mathfrak{u}_{\mathfrak{m}}(\gamma)/\mathfrak{t}}(y) &= \prod_{\alpha^0 \in R^+(\mathfrak{u}_{\mathfrak{m}}(\gamma),\mathfrak{t})} \langle 2\pi\sqrt{-1}\alpha^0, y\rangle, \\ \pi_{\sqrt{-1}\mathfrak{p}_{\mathfrak{m}}(\gamma)/\mathfrak{t}}(y) &= \prod_{\alpha^0 \in R^+(\sqrt{-1}\mathfrak{p}_{\mathfrak{m}}(\gamma),\mathfrak{t})} \langle 2\pi\sqrt{-1}\alpha^0, y\rangle, \\ \pi_{\mathfrak{k}_{\mathfrak{m}}(\gamma)/\mathfrak{t}}(y) &= \prod_{\alpha^0 \in R^+(\mathfrak{k}_{\mathfrak{m}}(\gamma),\mathfrak{t})} \langle 2\pi\sqrt{-1}\alpha^0, y\rangle. \end{aligned} \tag{7.2.14}$$

For $y \in \mathfrak{t}$, put

$$\begin{aligned} \sigma_{\mathfrak{u}_{\mathfrak{m}}(\gamma)/\mathfrak{t}}(y) &= \prod_{\alpha^0 \in R^+(\mathfrak{u}_{\mathfrak{m}}(\gamma),\mathfrak{t})} \big(\exp(\langle \pi\sqrt{-1}\alpha^0, y\rangle) - \exp(-\langle \pi\sqrt{-1}\alpha^0, y\rangle)\big), \\ \sigma_{\sqrt{-1}\mathfrak{p}_{\mathfrak{m}}(\gamma)/\mathfrak{t}}(y) &= \prod_{\alpha^0 \in R^+(\sqrt{-1}\mathfrak{p}_{\mathfrak{m}}(\gamma),\mathfrak{t})} \big(\exp(\langle \pi\sqrt{-1}\alpha^0, y\rangle) - \exp(-\langle \pi\sqrt{-1}\alpha^0, y\rangle)\big), \\ \sigma_{\mathfrak{k}_{\mathfrak{m}}(\gamma)/\mathfrak{t}}(y) &= \prod_{\alpha^0 \in R^+(\mathfrak{k}_{\mathfrak{m}}(\gamma)t,\mathfrak{t})} \big(\exp(\langle \pi\sqrt{-1}\alpha^0, y\rangle) - \exp(-\langle \pi\sqrt{-1}\alpha^0, y\rangle)\big). \end{aligned} \tag{7.2.15}$$

We can always extend analytically the above functions to $y \in \mathfrak{t}_{\mathbb{C}}$. If $\gamma = 1$, the above functions become $\pi_{\mathfrak{u}_{\mathfrak{m}}/\mathfrak{t}}(y)$, $\pi_{\sqrt{-1}\mathfrak{p}_{\mathfrak{m}}/\mathfrak{t}}(y)$, $\pi_{\mathfrak{k}_{\mathfrak{m}}/\mathfrak{t}}(y)$, $\sigma_{\mathfrak{u}_{\mathfrak{m}}/\mathfrak{t}}(y)$, $\sigma_{\sqrt{-1}\mathfrak{p}_{\mathfrak{m}}/\mathfrak{t}}(y)$, $\sigma_{\mathfrak{k}_{\mathfrak{m}}/\mathfrak{t}}(y)$.

If the adjoint action of T preserves certain orthogonal splittings of $\mathfrak{u}_{\mathfrak{m}}$, $\mathfrak{u}_{\mathfrak{m}}(\gamma)$, etc., so that we have the corresponding splitting of the root systems, then we can also define the associated π-function or σ-function as above.

It is clear that if $y \in \mathfrak{t}_{\mathbb{C}}$,

$$\begin{aligned} \pi_{\mathfrak{u}_{\mathfrak{m}}(\gamma)/\mathfrak{t}}(y) &= \pi_{\sqrt{-1}\mathfrak{p}_{\mathfrak{m}}(\gamma)/\mathfrak{t}}(y)\pi_{\mathfrak{k}_{\mathfrak{m}}(\gamma)/\mathfrak{t}}(y), \\ \sigma_{\mathfrak{u}_{\mathfrak{m}}(\gamma)/\mathfrak{t}}(y) &= \sigma_{\sqrt{-1}\mathfrak{p}_{\mathfrak{m}}(\gamma)/\mathfrak{t}}(y)\sigma_{\mathfrak{k}_{\mathfrak{m}}/\mathfrak{t}}(y). \end{aligned} \tag{7.2.16}$$

Set

$$\begin{aligned} \mathfrak{k}'_{\mathfrak{m}}(\gamma) &= \mathfrak{k}^\perp(\gamma) \cap \mathfrak{k}_{\mathfrak{m}}, \qquad \mathfrak{p}'_{\mathfrak{m}}(\gamma) = \mathfrak{p}^\perp(\gamma) \cap \mathfrak{p}_{\mathfrak{m}}, \\ \mathfrak{k}''_{\mathfrak{m}}(\gamma) &= \mathfrak{k}^\perp(\gamma) \cap \mathfrak{k}^\perp(\mathfrak{b}), \quad \mathfrak{p}''_{\mathfrak{m}}(\gamma) = \mathfrak{p}^\perp(\gamma) \cap \mathfrak{p}^\perp(\mathfrak{b}). \end{aligned} \tag{7.2.17}$$

Let $\mathfrak{m}^\perp(\gamma)$ be the orthogonal space of $\mathfrak{m}(\gamma)$ in $\mathfrak{m}$ with respect to B. Then

$$\mathfrak{m}^\perp(\gamma) = \mathfrak{p}'_{\mathfrak{m}}(\gamma) \oplus \mathfrak{k}'_{\mathfrak{m}}(\gamma). \tag{7.2.18}$$

We also have

$$\mathfrak{k}_{\mathfrak{m}} = \mathfrak{k}_{\mathfrak{m}}(\gamma) \oplus \mathfrak{k}'_{\mathfrak{m}}(\gamma), \quad \mathfrak{p}_{\mathfrak{m}} = \mathfrak{p}_{\mathfrak{m}}(\gamma) \oplus \mathfrak{p}'_{\mathfrak{m}}(\gamma) \tag{7.2.19}$$

and

$$\mathfrak{k}^{\perp}(\gamma)=\mathfrak{k}'_{\mathfrak{m}}(\gamma)\oplus\mathfrak{k}''_{\mathfrak{m}}(\gamma),\quad \mathfrak{p}^{\perp}(\gamma)=\mathfrak{p}'_{\mathfrak{m}}(\gamma)\oplus\mathfrak{p}''_{\mathfrak{m}}(\gamma). \tag{7.2.20}$$

Set

$$\mathfrak{u}^{\perp}_{\mathfrak{m}}(\gamma)=\sqrt{-1}\mathfrak{p}'_{\mathfrak{m}}(\gamma)\oplus\mathfrak{k}'_{\mathfrak{m}}(\gamma). \tag{7.2.21}$$

Then it is the orthogonal space of $\mathfrak{u}_{\mathfrak{m}}(\gamma)$ in $\mathfrak{u}_{\mathfrak{m}}$ with respect to B.

Lemma 7.2.5. *The following spaces are isomorphic to each other as modules of T by the adjoint actions*:

$$\mathfrak{n}^{\perp}(\gamma)\simeq\bar{\mathfrak{n}}^{\perp}(\gamma)\simeq\mathfrak{k}''_{\mathfrak{m}}(\gamma)\simeq\mathfrak{p}''_{\mathfrak{m}}(\gamma). \tag{7.2.22}$$

Proof. Note that

$$\dim\mathfrak{n}=\dim\mathfrak{k}-\dim\mathfrak{k}_{\mathfrak{m}},\quad \dim\mathfrak{n}(\gamma)=\dim\mathfrak{k}(\gamma)-\dim\mathfrak{k}_{\mathfrak{m}}(\gamma). \tag{7.2.23}$$

Together with the splittings (7.2.19), (7.2.20), we get

$$\dim\mathfrak{k}''_{\mathfrak{m}}(\gamma)=\dim\mathfrak{n}^{\perp}(\gamma). \tag{7.2.24}$$

Similarly, $\dim\mathfrak{p}''_{\mathfrak{m}}(\gamma)=\dim\mathfrak{n}^{\perp}(\gamma)$.

If $f\in\mathfrak{n}^{\perp}(\gamma)$, then $f+\theta(f)\in\mathfrak{k}$; we can verify directly that $f+\theta(f)\in\mathfrak{k}''_{\mathfrak{m}}(\gamma)$. Then the map $f\in\mathfrak{n}^{\perp}(\gamma)\mapsto f+\theta(f)\in\mathfrak{k}''_{\mathfrak{m}}(\gamma)$ defines an isomorphisms of T-modules. Similar for $\mathfrak{n}^{\perp}(\gamma)\simeq\mathfrak{p}''_{\mathfrak{m}}(\gamma)$. □

Since $\gamma=k\in T$, let $y_0\in\mathfrak{t}$ be such that $\exp(y_0)=\gamma$. Note that y_0 is not unique.

Lemma 7.2.6. *If $y\in\mathfrak{t}$ is regular with respect to $R(\mathfrak{k}_{\mathfrak{m}}(\gamma),\mathfrak{t})$, then we have*

$$\begin{aligned}
&J_{\gamma}(y)\operatorname{Tr}_s^{\Lambda^{\bullet}(\mathfrak{p}^*)}\Big[\Big(N^{\Lambda^{\bullet}(\mathfrak{p}^*)}-\frac{m}{2}\Big)\operatorname{Ad}(k)\exp(-\mathrm{i}\operatorname{ad}(y))\Big]\\
&\quad=\frac{(-1)^{\dim\mathfrak{p}_{\mathfrak{m}}(\gamma)/2+1}}{\det(1-\operatorname{Ad}(k))|_{\mathfrak{n}^{\perp}(\gamma)}}\operatorname{Tr}_s^{\Lambda^{\bullet}(\mathfrak{n}^*_{\mathbb{C}})}[e^{-i\operatorname{ad}(y)}\operatorname{Ad}(k)]\\
&\qquad\qquad\cdot\frac{\pi_{\sqrt{-1}\mathfrak{p}_{\mathfrak{m}}(\gamma)/\mathfrak{t}}(iy)}{\pi_{\mathfrak{k}_{\mathfrak{m}}(\gamma)/\mathfrak{t}}(iy)}\frac{\sigma_{\mathfrak{u}_{\mathfrak{m}}(\gamma)/\mathfrak{t}}(iy)\sigma_{\mathfrak{u}^{\perp}_{\mathfrak{m}}(\gamma)/\mathfrak{t}}(-iy+y_0)}{\sigma_{\mathfrak{u}^{\perp}_{\mathfrak{m}}(\gamma)/\mathfrak{t}}(y_0)}.
\end{aligned} \tag{7.2.25}$$

Proof. Using (5.4.23), (7.2.20) and Lemma 7.2.5, we get that, for $y\in\mathfrak{t}$,

$$\begin{aligned}
&\left[\frac{1}{\det(1-\operatorname{Ad}(k))|_{\mathfrak{z}^{\perp}(\gamma)}}\frac{\det(1-e^{-\mathrm{i}\operatorname{ad}(y)}\operatorname{Ad}(k))|_{\mathfrak{k}^{\perp}(\gamma)}}{\det(1-e^{-\mathrm{i}\operatorname{ad}(y)}\operatorname{Ad}(k))|_{\mathfrak{p}^{\perp}(\gamma)}}\right]^{\frac{1}{2}}\\
&\qquad=\frac{(-1)^{\frac{1}{2}\dim\mathfrak{p}'_{\mathfrak{m}}(\gamma)}}{\det(1-\operatorname{Ad}(k))|_{\mathfrak{n}^{\perp}(\gamma)}}\frac{1}{\sigma_{\mathfrak{u}^{\perp}_{\mathfrak{m}}(\gamma)/\mathfrak{t}}(y_0)}\frac{\sigma_{\mathfrak{k}'_{\mathfrak{m}}(\gamma)/\mathfrak{t}}(-iy+y_0)}{\sigma_{\sqrt{-1}\mathfrak{p}'_{\mathfrak{m}}(\gamma)/\mathfrak{t}}(-iy+y_0)}.
\end{aligned} \tag{7.2.26}$$

Recall that in Section 5.1, as K_M-modules, we have the isomorphism

$$\mathfrak{p}\simeq\mathfrak{b}\oplus\mathfrak{p}_{\mathfrak{m}}\oplus\mathfrak{n}. \tag{7.2.27}$$

Note that

$$\operatorname{Ad}(k)=e^{\operatorname{ad}(y_0)}. \tag{7.2.28}$$

If $y\in\mathfrak{t}$, when acting on $\mathfrak{p}$, we have

$$\operatorname{Ad}(k)\exp(-i\operatorname{ad}(y))=\exp(\operatorname{ad}(-iy+y_0)). \tag{7.2.29}$$

Note that $\dim \mathfrak{b} = 1$. Then, for $y \in \mathfrak{t}$, we get

$$\begin{aligned}
\operatorname{Tr}_s^{\Lambda^\bullet(\mathfrak{p}^*)}\Big[\Big(N^{\Lambda^\bullet(\mathfrak{p}^*)} - \frac{m}{2}\Big)\operatorname{Ad}(k)\exp(-\mathrm{i}\operatorname{ad}(y))\Big] \\
&= -\operatorname{Tr}_s^{\Lambda^\bullet(\mathfrak{p}_\mathfrak{m}^*)}[\operatorname{Ad}(k)e^{-i\operatorname{ad}(y)}]\operatorname{Tr}_s^{\Lambda^\bullet(\mathfrak{n}_\mathbb{C}^*)}[\operatorname{Ad}(k)e^{-i\operatorname{ad}(y)}] \\
&= -\det(1-\operatorname{Ad}(k^{-1})e^{i\operatorname{ad}(y)})|_{\mathfrak{p}_\mathfrak{m}}\operatorname{Tr}_s^{\Lambda^\bullet(\mathfrak{n}_\mathbb{C}^*)}[\operatorname{Ad}(k)e^{-i\operatorname{ad}(y)}], \quad (7.2.30)
\end{aligned}$$

where we have the identity

$$\det(1-\operatorname{Ad}(k^{-1})e^{i\operatorname{ad}(y)})|_{\mathfrak{p}_\mathfrak{m}} = (-1)^{\frac{1}{2}\dim\mathfrak{p}_\mathfrak{m}}\sigma_{\sqrt{-1}\mathfrak{p}'_\mathfrak{m}(\gamma)/\mathfrak{t}}(-iy+y_0)^2\sigma_{\sqrt{-1}\mathfrak{p}_\mathfrak{m}(\gamma)/\mathfrak{t}}(iy)^2. \quad (7.2.31)$$

Note that analogous to (7.2.27), we have $\mathfrak{p}(\gamma) \simeq \mathfrak{b} \oplus \mathfrak{p}_\mathfrak{m}(\gamma) \oplus \mathfrak{n}(\gamma)$; using [Bismut 2011, equation (7.5.24)], if $y \in \mathfrak{t}$, we have

$$\begin{aligned}
\widehat{A}(i\operatorname{ad}(y)|_{\sqrt{-1}\mathfrak{p}(\gamma)}) &= \frac{\pi_{\sqrt{-1}\mathfrak{p}_\mathfrak{m}(\gamma)/\mathfrak{t}}(iy)}{\sigma_{\sqrt{-1}\mathfrak{p}_\mathfrak{m}(\gamma)/\mathfrak{t}}(iy)}\widehat{A}(i\operatorname{ad}(y)|_{\mathfrak{n}(\gamma)}), \\
\widehat{A}(i\operatorname{ad}(y)|_{\mathfrak{k}(\gamma)}) &= \frac{\pi_{\mathfrak{k}(\gamma)/\mathfrak{t}}(iy)}{\sigma_{\mathfrak{k}(\gamma)/\mathfrak{t}}(iy)} = \frac{\pi_{\mathfrak{k}_\mathfrak{m}(\gamma)/\mathfrak{t}}(iy)}{\sigma_{\mathfrak{k}_\mathfrak{m}(\gamma)/\mathfrak{t}}(iy)}\widehat{A}(i\operatorname{ad}(y)|_{\mathfrak{n}(\gamma)}).
\end{aligned} \quad (7.2.32)$$

Combining (7.1.4), (7.2.26) and (7.2.30)–(7.2.32), we get (7.2.25). □

Now we prove Theorem 7.2.2.

Proof of Theorem 7.2.2. Put

$$\begin{aligned}
F_\gamma(\Lambda,t) = \frac{1}{(2\pi t)^{\frac{p}{2}}}\int_{\mathfrak{k}(\gamma)} J_\gamma(Y_0^\mathfrak{k})\operatorname{Tr}_s^{\Lambda^\bullet(\mathfrak{p}^*)}\Big[\Big(N^{\Lambda^\bullet(\mathfrak{p}^*)} - \frac{m}{2}\Big)\operatorname{Ad}(k)e^{-i\operatorname{ad}(Y_0^\mathfrak{k})}\Big] \\
\cdot\operatorname{Tr}^E[\rho^E(k)e^{-i\rho^E(Y_0^\mathfrak{k})}]e^{-\frac{|Y_0^\mathfrak{k}|^2}{2t}}\frac{dY_0^\mathfrak{k}}{(2\pi t)^{\frac{q}{2}}}. \quad (7.2.33)
\end{aligned}$$

By (7.1.3), we have

$$\mathcal{E}_{X,\gamma}(F,t) = \exp(-2\pi^2 t|\Lambda+\rho_\mathfrak{u}|^2)F_\gamma(\Lambda,t). \quad (7.2.34)$$

Recall that $r = p+q = \dim_\mathbb{R}\mathfrak{z}(\gamma)$. By the Weyl integration formula,

$$\begin{aligned}
F_\gamma(\Lambda,t) = \frac{\operatorname{Vol}(K(\gamma)^0/T)}{(2\pi t)^{\frac{r}{2}}|W(K(\gamma)^0,T)|}\int_\mathfrak{t}|\pi_{\mathfrak{k}(\gamma)/\mathfrak{t}}(y)|^2 J_\gamma(y)\operatorname{Tr}_s^{\Lambda^\bullet(\mathfrak{p}^*)}\Big[\Big(N^{\Lambda^\bullet(\mathfrak{p}^*)} - \frac{m}{2}\Big)\operatorname{Ad}(k)e^{-i\operatorname{ad}(y)}\Big] \\
\cdot\operatorname{Tr}^E[\rho^E(k)\exp(-i\rho^E(y))]e^{-\frac{|y|^2}{2t}}\,dy. \quad (7.2.35)
\end{aligned}$$

Recall that $l(\gamma) = \frac{1}{2}\dim\mathfrak{n}(\gamma)$. We can verify directly that if $y \in \mathfrak{t}$,

$$\pi_{\mathfrak{k}(\gamma)/\mathfrak{t}}(iy)^2 = (-1)^{l(\gamma)}\pi_{\mathfrak{k}_\mathfrak{m}(\gamma)/\mathfrak{t}}(iy)^2\det(i\operatorname{ad}(y))|_{\mathfrak{n}(\gamma)_\mathbb{C}}. \quad (7.2.36)$$

Moreover, if $y \in \mathfrak{t}$ is such that $\pi_{\mathfrak{u}_\mathfrak{m}(\gamma)/\mathfrak{t}}(y) \neq 0$,

$$\frac{|\pi_{\mathfrak{k}(\gamma)/\mathfrak{t}}(y)|^2}{|\pi_{\mathfrak{u}_\mathfrak{m}(\gamma)/\mathfrak{t}}(y)|^2} = \frac{\pi_{\mathfrak{k}(\gamma)/\mathfrak{t}}(iy)^2}{\pi_{\mathfrak{u}_\mathfrak{m}(\gamma)/\mathfrak{t}}(iy)^2}. \quad (7.2.37)$$

Then by Lemma 7.2.6 and (7.2.5), (7.2.32), (7.2.36), we get

$$\begin{aligned}
&\frac{|\pi_{\mathfrak{k}(\gamma)/\mathfrak{t}}(y)|^2}{|\pi_{\mathfrak{u}_{\mathfrak{m}}(\gamma)/\mathfrak{t}}(y)|^2} J_\gamma(y)\operatorname{Tr}_s^{\Lambda^\bullet(\mathfrak{p}^*)}\Big[\Big(N^{\Lambda^\bullet(\mathfrak{p}^*)}-\frac{m}{2}\Big)\operatorname{Ad}(k)\exp(-\mathrm{i}\operatorname{ad}(y))\Big]\\
&\quad=\frac{(-1)^{l(\gamma)+\frac{1}{2}\dim\mathfrak{p}_{\mathfrak{m}}(\gamma)+1}}{\det(1-\operatorname{Ad}(k))|_{\mathfrak{n}^\perp(\gamma)}}\operatorname{Tr}_s^{\Lambda^\bullet(\mathfrak{n}^*_{\mathbb{C}})}[e^{-i\operatorname{ad}(y)}\operatorname{Ad}(k)]\\
&\qquad\cdot\det(i\operatorname{ad}(y))|_{\mathfrak{n}(\gamma)_{\mathbb{C}}}\widehat{A}^{-1}(i\operatorname{ad}(y)|_{\mathfrak{u}_{\mathfrak{m}}(\gamma)})\left[\frac{\det(1-e^{-i\operatorname{ad}(y)}\operatorname{Ad}(k))|_{\mathfrak{u}^\perp_{\mathfrak{m}}(\gamma)}}{\det(1-\operatorname{Ad}(k))_{\mathfrak{u}^\perp_{\mathfrak{m}}(\gamma)}}\right]^{\frac{1}{2}}.
\end{aligned}\tag{7.2.38}$$

Note that we have the even number

$$p-1=\dim\mathfrak{p}_{\mathfrak{m}}(\gamma)+2l(\gamma).\tag{7.2.39}$$

Now we can rewrite (7.2.35) as

$$\begin{aligned}
F_\gamma(\Lambda,t)=&\frac{(-1)^{\frac{p-1}{2}+1}\operatorname{Vol}(K(\gamma)^0/T)}{(2\pi t)^{\frac{r}{2}}|W(K(\gamma)^0,T)|}\frac{1}{\det(1-\operatorname{Ad}(k))|_{\mathfrak{n}^\perp(\gamma)}}\\
&\cdot\int_{\mathfrak{t}}|\pi_{\mathfrak{u}_{\mathfrak{m}}(\gamma)/\mathfrak{t}}(y)|^2\det(i\operatorname{ad}(y))|_{\mathfrak{n}(\gamma)_{\mathbb{C}}}\cdot\widehat{A}^{-1}(i\operatorname{ad}(y)|_{\mathfrak{u}_{\mathfrak{m}}(\gamma)})\\
&\cdot\left[\frac{\det(1-e^{-i\operatorname{ad}(y)}\operatorname{Ad}(k))|_{\mathfrak{u}^\perp_{\mathfrak{m}}(\gamma)}}{\det(1-\operatorname{Ad}(k))_{\mathfrak{u}^\perp_{\mathfrak{m}}(\gamma)}}\right]^{\frac{1}{2}}\\
&\cdot\operatorname{Tr}_s^{\Lambda^\bullet(\mathfrak{n}^*_{\mathbb{C}})\otimes E}[e^{-i\rho^{\Lambda^\bullet(\mathfrak{n}^*_{\mathbb{C}})\otimes E}(y)}\rho^{\Lambda^\bullet(\mathfrak{n}^*_{\mathbb{C}})\otimes E}(k)]e^{-|y|^2/2t}\,dy.
\end{aligned}\tag{7.2.40}$$

Note that the function in $y\in\mathfrak{t}$

$$\begin{aligned}
\det(i\operatorname{ad}(y))|_{\mathfrak{n}(\gamma)_{\mathbb{C}}}\cdot\widehat{A}^{-1}(i\operatorname{ad}(y)|_{\mathfrak{u}_{\mathfrak{m}}(\gamma)})&\left[\frac{\det(1-e^{-i\operatorname{ad}(y)}\operatorname{Ad}(k))|_{\mathfrak{u}^\perp_{\mathfrak{m}}(\gamma)}}{\det(1-\operatorname{Ad}(k))_{\mathfrak{u}^\perp_{\mathfrak{m}}(\gamma)}}\right]^{\frac{1}{2}}\\
&\cdot\operatorname{Tr}_s^{\Lambda^\bullet(\mathfrak{n}^*_{\mathbb{C}})\otimes E}[e^{-i\rho^{\Lambda^\bullet(\mathfrak{n}^*_{\mathbb{C}})\otimes E}(y)}\rho^{\Lambda^\bullet(\mathfrak{n}^*_{\mathbb{C}})\otimes E}(k)]
\end{aligned}\tag{7.2.41}$$

can be extended directly to a $U_M(\gamma)^0$-invariant function in $y\in\mathfrak{u}_{\mathfrak{m}}(\gamma)$. Since $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{u}_{\mathfrak{m}}(\gamma)$, we can apply the Weyl integration formula for the pair $(\mathfrak{u}_{\mathfrak{m}}(\gamma),\mathfrak{t})$; we get

$$\begin{aligned}
F_\gamma(\Lambda,t)=\frac{c_G(\gamma)}{(2\pi t)^{\frac{r}{2}}}\int_{y\in\mathfrak{u}_{\mathfrak{m}}(\gamma)}&\det(i\operatorname{ad}(y))|_{\mathfrak{n}(\gamma)_{\mathbb{C}}}\cdot\widehat{A}^{-1}(i\operatorname{ad}(y)|_{\mathfrak{u}_{\mathfrak{m}}(\gamma)})\left[\frac{\det(1-e^{-i\operatorname{ad}(y)}\operatorname{Ad}(k))|_{\mathfrak{u}^\perp_{\mathfrak{m}}(\gamma)}}{\det(1-\operatorname{Ad}(k))_{\mathfrak{u}^\perp_{\mathfrak{m}}(\gamma)}}\right]^{\frac{1}{2}}\\
&\cdot\operatorname{Tr}_s^{\Lambda^\bullet(\mathfrak{n}^*_{\mathbb{C}})\otimes E}[e^{-i\rho^{\Lambda^\bullet(\mathfrak{n}^*_{\mathbb{C}})\otimes E}(y)}\rho^{\Lambda^\bullet(\mathfrak{n}^*_{\mathbb{C}})\otimes E}(k)]e^{-\frac{|y|^2}{2t}}\,dy.
\end{aligned}\tag{7.2.42}$$

The constant $c_G(\gamma)$ is defined by (7.2.5).

Note that

$$r=\dim\mathfrak{u}_{\mathfrak{m}}(\gamma)+4l(\gamma)+1.\tag{7.2.43}$$

If $y\in\mathfrak{u}_{\mathfrak{m}}(\gamma)$, then

$$B\Big(y,\frac{\Omega^{\mathfrak{u}_{\mathfrak{m}}(\gamma)}}{2\pi}\Big)\in\Lambda^2(\mathfrak{u}^\perp_{\mathfrak{b}}(\gamma)^*).\tag{7.2.44}$$

If $y \in \mathfrak{u}_{\mathfrak{m}}(\gamma)$, by [Shen 2018, equation (7-27)], we have

$$\frac{\det(i\,\mathrm{ad}(y))|_{\mathfrak{n}(\gamma)_{\mathbb{C}}}}{(2\pi t)^{2l(\gamma)}} = \left[\exp\left(\frac{1}{t}B\left(y, \frac{\Omega^{\mathfrak{u}_{\mathfrak{m}}(\gamma)}}{2\pi}\right)\right)\right]^{\max(\gamma)}. \tag{7.2.45}$$

Combining (7.2.42)–(7.2.45), we get

$$F_\gamma(\Lambda,t)=\frac{c_G(\gamma)}{\sqrt{2\pi t}}\left[\int_{y\in\mathfrak{u}_{\mathfrak{m}}(\gamma)}\widehat{A}^{-1}(i\,\mathrm{ad}(y)|_{\mathfrak{u}_{\mathfrak{m}}(\gamma)})\left[\frac{\det(1-e^{-i\,\mathrm{ad}(y)}\,\mathrm{Ad}(k))|_{\mathfrak{u}_{\mathfrak{m}}^{\perp}(\gamma)}}{\det(1-\mathrm{Ad}(k))_{\mathfrak{u}_{\mathfrak{m}}^{\perp}(\gamma)}}\right]^{\frac{1}{2}}\right.$$
$$\left.\cdot\mathrm{Tr}_s^{\Lambda^\bullet(\mathfrak{n}_{\mathbb{C}}^*)\otimes E}[\rho^{\Lambda^\bullet(\mathfrak{n}_{\mathbb{C}}^*)\otimes E}(e^{-iy}k)]e^{\frac{1}{t}B\left(y,\frac{\Omega^{\mathfrak{u}_{\mathfrak{m}}(\gamma)}}{2\pi}\right)-\frac{|y|^2}{2t}}\frac{dy}{(2\pi t)^{\dim\mathfrak{u}_{\mathfrak{m}}(\gamma)/2}}\right]^{\max(\gamma)}. \tag{7.2.46}$$

By (5.2.21) if $y \in \mathfrak{u}_{\mathfrak{m}}(\gamma)$, then

$$B\left(y, \frac{\Omega^{\mathfrak{u}_{\mathfrak{m}}(\gamma)}}{2\pi}\right) - \frac{|y|^2}{2} = \frac{1}{2}B\left(y + \frac{\Omega^{\mathfrak{u}_{\mathfrak{m}}(\gamma)}}{2\pi}, y + \frac{\Omega^{\mathfrak{u}_{\mathfrak{m}}(\gamma)}}{2\pi}\right) - \frac{\beta(a_1)^2}{8\pi^2}\omega^{Y_{\mathfrak{b}}(\gamma),2}. \tag{7.2.47}$$

Let $\Delta^{\mathfrak{u}_{\mathfrak{m}}(\gamma)}$ be the standard negative Laplace operator on the Euclidean space $(\mathfrak{u}_{\mathfrak{m}}(\gamma), -B|_{\mathfrak{u}_{\mathfrak{m}}(\gamma)})$. Then by considering the heat kernel of $-\Delta^{\mathfrak{u}_{\mathfrak{m}}(\gamma)}$, we can rewrite (7.2.46) as

$$F_\gamma(\Lambda,t) = \frac{c_G(\gamma)}{\sqrt{2\pi t}}\left[\exp\left(-\frac{\beta(a_1)^2\omega^{Y_{\mathfrak{b}}(\gamma),2}}{8\pi^2 t}\right)\right.$$
$$\cdot\exp\left(\frac{t}{2}\Delta^{\mathfrak{u}_{\mathfrak{m}}(\gamma)}\right)\left\{\widehat{A}^{-1}(i\,\mathrm{ad}(y)|_{\mathfrak{u}_{\mathfrak{m}}(\gamma)})\left[\frac{\det(1-e^{-i\,\mathrm{ad}(y)}\,\mathrm{Ad}(k))|_{\mathfrak{u}_{\mathfrak{m}}^{\perp}(\gamma)}}{\det(1-\mathrm{Ad}(k))_{\mathfrak{u}_{\mathfrak{m}}^{\perp}(\gamma)}}\right]^{\frac{1}{2}}\right.$$
$$\left.\left.\cdot\mathrm{Tr}_s^{\Lambda^\bullet(\mathfrak{n}_{\mathbb{C}}^*)\otimes E}[\rho^{\Lambda^\bullet(\mathfrak{n}_{\mathbb{C}}^*)\otimes E}(e^{-iy}k)]\right\}\right|_{y=-\frac{\Omega^{\mathfrak{u}_{\mathfrak{m}}(\gamma)}}{2\pi}}\Bigg]^{\max(\gamma)}. \tag{7.2.48}$$

Recall that $V_{\Lambda,\omega}$ is an irreducible unitary representation of U_M with highest weight $\eta_\omega(\Lambda)$. By (5.3.7), for $y \in \mathfrak{u}_{\mathfrak{m}}(\gamma)$, then

$$\mathrm{Tr}_s^{\Lambda^\bullet(\mathfrak{n}_{\mathbb{C}}^*)\otimes E}[\rho^{\Lambda^\bullet(\mathfrak{n}_{\mathbb{C}}^*)\otimes E}(e^{-iy}k)] = \sum_{\omega\in W_u}\varepsilon(\omega)\,\mathrm{Tr}^{V_{\Lambda,\omega}}[\rho^{V_{\Lambda,\omega}}(e^{-iy}k)]. \tag{7.2.49}$$

Then we apply the generalized Kirillov formula (5.4.21) to each term in the right-hand side of (7.2.49), we conclude that, for $\omega \in W_u$, the function in $y \in \mathfrak{u}_{\mathfrak{m}}(\gamma)$

$$\widehat{A}^{-1}(i\,\mathrm{ad}(y)|_{\mathfrak{u}_{\mathfrak{m}}(\gamma)})\left[\frac{\det(1-e^{-i\,\mathrm{ad}(y)}\,\mathrm{Ad}(k))|_{\mathfrak{u}_{\mathfrak{m}}^{\perp}(\gamma)}}{\det(1-\mathrm{Ad}(k))_{\mathfrak{u}_{\mathfrak{m}}^{\perp}(\gamma)}}\right]^{\frac{1}{2}}\mathrm{Tr}^{V_{\Lambda,\omega}}[\rho^{V_{\Lambda,\omega}}(e^{-iy}k)] \tag{7.2.50}$$

is an eigenfunction of $\Delta^{\mathfrak{u}_{\mathfrak{m}}(\gamma)}$ associated with the eigenvalue $4\pi^2|\eta_\omega(\Lambda)+\rho_{\mathfrak{u}_{\mathfrak{m}}}|^2$. Then the heat operator $\exp\left(\frac{t}{2}\Delta^{\mathfrak{u}_{\mathfrak{m}}(\gamma)}\right)$ acts on the function (7.2.50) as a scalar $e^{2\pi^2 t|\eta_\omega(\Lambda)+\rho_{\mathfrak{u}_{\mathfrak{m}}}|^2}$. By (5.3.8), (5.3.9), for $\omega \in W_u$, we get

$$\eta_\omega(\Lambda) + \rho_{\mathfrak{u}_{\mathfrak{m}}} = P_0(\omega(\Lambda + \rho_{\mathfrak{u}})). \tag{7.2.51}$$

Combing the above computation with the term $e^{-2\pi^2 t|\Lambda+\rho_{\mathfrak{u}}|^2}$ in (7.2.34), by (7.2.10), we get the factor $e^{-2\pi^2 t b_{\Lambda,\omega}^2}$ in (7.2.11).

Now we deal with the main part in (7.2.48) after removing the heat operator $\exp\big(\frac{t}{2}\Delta^{\mathfrak{u}_\mathfrak{m}(\gamma)}\big)$. We will use the same notation as in Section 5.4. The orbit $\mathcal{O}^{\gamma}_{\sigma(\eta_\omega(\Lambda)+\rho_{\mathfrak{u}_\mathfrak{m}})}$ is defined in (5.4.17) equipped with a Liouville measure $d\mu^{\gamma}_{\sigma}$. We claim the identity

$$\begin{aligned}&\Bigg[\exp\bigg(-\frac{\beta(a_1)^2\omega^{Y_\mathfrak{b}(\gamma),2}}{8\pi^2 t}\bigg)\\&\qquad\cdot\bigg\{\widehat{A}^{-1}(i\,\mathrm{ad}(y)|_{\mathfrak{u}_\mathfrak{m}(\gamma)})\bigg[\frac{\det(1-e^{-i\,\mathrm{ad}(y)}\,\mathrm{Ad}(k))|_{\mathfrak{u}^{\perp}_\mathfrak{m}(\gamma)}}{\det(1-\mathrm{Ad}(k))_{\mathfrak{u}^{\perp}_\mathfrak{m}(\gamma)}}\bigg]^{\frac{1}{2}}\mathrm{Tr}_s^{V_{\Lambda,\omega}}[\rho^{V_{\Lambda,\omega}}(e^{-iy}k)]\bigg\}\bigg|_{y=-\frac{\Omega^{\mathfrak{u}_\mathfrak{m}(\gamma)}}{2\pi}}\Bigg]^{\max(\gamma)}\\&=\sum_{\sigma\in W^1(\gamma)}\varphi_\gamma(\sigma,\eta_\omega(\Lambda))\cdot\dim E^{\gamma}_{\omega,\sigma}\bigg[\exp\bigg(-\frac{\beta(a_1)^2\omega^{Y_\mathfrak{b}(\gamma),2}}{8\pi^2 t}-\langle\sigma(\eta_\omega(\Lambda)+\rho_{\mathfrak{u}_\mathfrak{m}}),\Omega^{\mathfrak{u}_\mathfrak{m}(\gamma)}\rangle\bigg)\bigg]^{\max(\gamma)}.\end{aligned}\tag{7.2.52}$$

Indeed, by (5.4.21), we have the following identity as elements in $\Lambda^\bullet(\mathfrak{u}^{\perp}_\mathfrak{b}(\gamma)^*)$:

$$\begin{aligned}&\bigg\{\widehat{A}^{-1}(i\,\mathrm{ad}(y)|_{\mathfrak{u}_\mathfrak{m}(\gamma)})\bigg[\frac{\det(1-e^{-i\,\mathrm{ad}(y)}\,\mathrm{Ad}(k))|_{\mathfrak{u}^{\perp}_\mathfrak{m}(\gamma)}}{\det(1-\mathrm{Ad}(k))_{\mathfrak{u}^{\perp}_\mathfrak{m}(\gamma)}}\bigg]^{\frac{1}{2}}\mathrm{Tr}_s^{V_{\Lambda,\omega}}[\rho^{V_{\Lambda,\omega}}(e^{-iy}k)]\bigg\}\bigg|_{y=-\frac{\Omega^{\mathfrak{u}_\mathfrak{m}(\gamma)}}{2\pi}}\\&\qquad=\sum_{\sigma\in W^1(\gamma)}\varphi_\gamma(\sigma,\eta_\omega(\Lambda))\int_{f\in\mathcal{O}^{\gamma}_{\sigma(\eta_\omega(\Lambda)+\rho_{\mathfrak{u}_\mathfrak{m}})}}e^{-\langle f,\Omega^{\mathfrak{u}_\mathfrak{m}(\gamma)}\rangle}\,d\mu^{\gamma}_{\sigma}.\end{aligned}\tag{7.2.53}$$

Recall that the curvature form $\Omega^{\mathfrak{u}_\mathfrak{b}(\gamma)}$ is invariant by the action of $U_M(\gamma)^0$ on $Y_\mathfrak{b}(\gamma)$. Since a_1 and $\omega^{Y_\mathfrak{b}(\gamma)}$ are invariant by $U_M(\gamma)^0$-action, so is $\Omega^{\mathfrak{u}_\mathfrak{m}(\gamma)}$. Therefore, for $f\in\mathfrak{u}_\mathfrak{m}(\gamma)^*$, $u\in U_M(\gamma)^0$,

$$\begin{aligned}&\bigg[\exp\bigg(-\frac{\beta(a_1)^2\omega^{Y_\mathfrak{b}(\gamma),2}}{8\pi^2 t}\bigg)\exp(-\langle\mathrm{Ad}^*(u)f,\Omega^{\mathfrak{u}_\mathfrak{m}(\gamma)}\rangle)\bigg]^{\max(\gamma)}\\&\qquad=\det\mathrm{Ad}(u)|_{\mathfrak{u}^{\perp}_\mathfrak{b}(\gamma)}\bigg[\exp\bigg(-\frac{\beta(a_1)^2\omega^{Y_\mathfrak{b}(\gamma),2}}{8\pi^2 t}\bigg)\exp(-\langle f,\Omega^{\mathfrak{u}_\mathfrak{m}(\gamma)}\rangle)\bigg]^{\max(\gamma)}.\end{aligned}\tag{7.2.54}$$

Since $U_M(\gamma)^0$ acts on $\mathfrak{u}^{\perp}_\mathfrak{b}(\gamma)$ isometrically with respect to $-B|_{\mathfrak{u}^{\perp}_\mathfrak{b}(\gamma)}$,

$$\det\mathrm{Ad}(u)|_{\mathfrak{u}^{\perp}_\mathfrak{b}(\gamma)}=1.\tag{7.2.55}$$

Then (7.2.52) follows from (5.4.6) and (7.2.53)–(7.2.55).

The right-hand side of (7.2.52) is a polynomial in t^{-1}. Recall that $\dim\mathfrak{u}^{\perp}_\mathfrak{b}(\gamma)=4l(\gamma)$. Then, for each $\sigma\in W^1(\gamma)$, we can rewrite the term $[\cdots]^{\max(\gamma)}$ in the right-hand side of (7.2.52) as follows:

$$\sum_{j=0}^{l(\gamma)}\frac{1}{t^j}\frac{(-1)^j\beta(a_1)^{2j}}{j!(2l(\gamma)-2j)!(8\pi^2)^j}[\omega^{Y_\mathfrak{b}(\gamma),2j}\langle\omega(\Lambda+\rho_\mathfrak{u}),\Omega^{\mathfrak{u}_\mathfrak{m}(\gamma)}\rangle^{2l(\gamma)-2j}]^{\max(\gamma)}.\tag{7.2.56}$$

Finally, putting together (7.2.7), (7.2.34), (7.2.48), (7.2.49), (7.2.52), and (7.2.56), we get (7.2.11). □

The Mellin transform of $\mathcal{E}_{X,\gamma}(F,t)$ (if applicable) is defined by the following formula as a function in $s\in\mathbb{C}$ with $\Re(s)\gg 0$:

$$\mathcal{M}\mathcal{E}_{X,\gamma}(F,s)=-\frac{1}{\Gamma(s)}\int_0^{+\infty}\mathcal{E}_{X,\gamma}(F,t)t^{s-1}\,dt.\tag{7.2.57}$$

If $\mathcal{ME}_{X,\gamma}(F,s)$ admits a meromorphic extension on $\mathbb{C}$ which is holomorphic at $s=0$, we will set

$$\mathcal{PE}_{X,\gamma}(F)=\frac{\partial}{\partial s}\Big|_{s=0}\mathcal{ME}_{X,\gamma}(F,s). \tag{7.2.58}$$

Theorem 7.2.7. *Suppose that the dominant weight Λ is such that, for every $\omega\in W_u$, $b_{\Lambda,\omega}\neq 0$. Then, for $s\in\mathbb{C}$ with $\Re(s)>l(\gamma)+1$, $\mathcal{ME}_{X,\gamma}(F,s)$ is well-defined and holomorphic, which admits a meromorphic extension to $s\in\mathbb{C}$.*

Moreover, we have the identity

$$\begin{aligned}&\mathcal{ME}_{X,\gamma}(F,s)\\ &\quad=-\frac{c_G(\gamma)}{\sqrt{2\pi}}\sum_{j=0}^{l(\gamma)}\frac{\Gamma\big(s-j-\frac{1}{2}\big)}{\Gamma(s)}\Big[\sum_{\substack{\omega\in W_u\\ \sigma\in W^1(\gamma)}}\varepsilon(\omega)\varphi_\gamma(\sigma,\eta_\omega(\Lambda))Q^\gamma_{j,\omega,\sigma}(\Lambda)(2\pi^2b^2_{\Lambda,\omega})^{j+\frac{1}{2}-s}\Big].\end{aligned} \tag{7.2.59}$$

Then $\mathcal{ME}_{X,\gamma}(F,s)$ is holomorphic at $s=0$. We have

$$\begin{aligned}&\mathcal{PE}_{X,\gamma}(F)\\ &\quad=-\frac{c_G(\gamma)}{\sqrt{2}}\sum_{j=0}^{l(\gamma)}\frac{(-4)^{j+1}(j+1)!}{(2j+2)!}\Big[\sum_{\substack{\omega\in W_u\\ \sigma\in W^1(\gamma)}}\varepsilon(\omega)\varphi_\gamma(\sigma,\eta_\omega(\Lambda))Q^\gamma_{j,\omega,\sigma}(\Lambda)(2\pi^2b^2_{\Lambda,\omega})^{j+\frac{1}{2}}\Big].\end{aligned} \tag{7.2.60}$$

Proof. By Theorem 7.2.2, the assumption on Λ implies that $\mathcal{E}_{X,\gamma}(F,t)$ decays exponentially as $t\to+\infty$. By (7.1.6) and (7.2.11), we get (7.2.59). This proves the first part of this theorem.

Equation (7.2.60) is a direct consequence of (7.2.59) by taking its derivative at 0. This completes the proof of our theorem. □

The formula in the right-hand side of (7.2.60) still looks complicated; we can rewrite it in a neat way as follows. We introduce the following functions.

Definition 7.2.8. Let $a^1\in\mathfrak{b}^*$ take value -1 at a_1. Note that $\gamma\in T$. For $\omega\in W_u$, $\sigma\in W^1(\gamma)$, if $\Lambda\in P_{++}(U)$, for $z\in\mathbb{C}$, set

$$P^\gamma_{\omega,\sigma,\Lambda}(z)=\dim E^\gamma_{\omega,\sigma}\cdot\big[\exp\big(\langle\Omega^{\mathfrak{u}_\mathfrak{b}(\gamma)},\sigma(\eta_\omega(\Lambda)+\rho_{\mathfrak{u}_\mathfrak{m}})+z\sqrt{-1}a^1\rangle\big)\big]^{\max(\gamma)}. \tag{7.2.61}$$

Since θ fixes $\Omega^{\mathfrak{u}_\mathfrak{b}(\gamma)}$, by the fact that $\det\theta|_{\mathfrak{u}_\mathfrak{b}^\perp(\gamma)}=1$, we have $P^\gamma_{\omega,\sigma,\Lambda}(z)$ is an even polynomial in z. Moreover, by the dimension formula (5.4.6), the coefficients of z^j, $j\in\mathbb{N}$, in $P^\gamma_{\omega,\sigma,\Lambda}(z)$ are polynomials in Λ. Such polynomials are related to the Plancherel measures in the representation theory.

Lemma 7.2.9. *We have the identity*

$$\sum_{j=0}^{l(\gamma)}\frac{(-4)^{j+1}(j+1)!}{\sqrt{2}(2j+2)!}Q^\gamma_{j,\omega,\sigma}(\Lambda)(2\pi^2(b_{\Lambda,\omega})^2)^{j+\frac{1}{2}}=-2\pi\int_0^{|b_{\Lambda,\omega}|}P^\gamma_{\omega,\sigma,\Lambda}(t)\,dt. \tag{7.2.62}$$

Proof. We have

$$\langle\eta_\omega(\Lambda)+\rho_{\mathfrak{u}_\mathfrak{m}}+z\sqrt{-1}a^1,\Omega^{\mathfrak{u}_\mathfrak{b}(\gamma)}\rangle=z\beta(a_1)\omega^{Y_\mathfrak{b}(\gamma)}+\langle\omega(\Lambda+\rho_\mathfrak{u}),\Omega^{\mathfrak{u}_\mathfrak{m}(\gamma)}\rangle. \tag{7.2.63}$$

Since $P^{\gamma}_{\omega,\sigma,\Lambda}(z)$ is an even function in z,

$$\begin{aligned} P^{\gamma}_{\omega,\sigma,\Lambda}(z) &= \dim E^{\gamma}_{\omega,\sigma}\cdot\frac{1}{(2l(\gamma))!}\Big[\big(z\beta(a_1)\omega^{Y_{\mathfrak{b}}(\gamma)}+\langle\omega(\Lambda+\rho_{\mathfrak{u}}),\Omega^{\mathfrak{u}_{\mathfrak{m}}(\gamma)}\rangle\big)^{2l(\gamma)}\Big]^{\max(\gamma)} \\ &= \dim E^{\gamma}_{\omega,\sigma}\cdot\sum_{j=0}^{l(\gamma)}\frac{\beta(a_1)^{2j}z^{2j}}{(2l(\gamma)-2j)!(2j)!}\Big[\omega^{Y_{\mathfrak{b}}(\gamma),2j}\langle\omega(\Lambda+\rho_{\mathfrak{u}}),\Omega^{\mathfrak{u}_{\mathfrak{m}}(\gamma)}\rangle^{2l(\gamma)-2j}\Big]^{\max(\gamma)}. \end{aligned} \tag{7.2.64}$$

Note that, for $j=0,1,\cdots,l(\gamma)$,

$$\int_0^{|b_{\Lambda,\omega}|} t^{2j}\,dt = \frac{1}{2j+1}|b_{\Lambda,\omega}|^{2j+1}. \tag{7.2.65}$$

Then (7.2.62) is a consequence of (7.2.7), (7.2.64) and (7.2.65). □

As a consequence, we get the following formula for $\mathcal{PE}_{X,\gamma}(F)$.

Theorem 7.2.10. *Suppose that the dominant weight Λ is such that, for every $\omega\in W_u$, $b_{\Lambda,\omega}\neq 0$. Then*

$$\mathcal{PE}_{X,\gamma}(F) = 2\pi c_G(\gamma)\cdot\sum_{\substack{\omega\in W_u\\ \sigma\in W^1(\gamma)}}\varepsilon(\omega)\varphi_\gamma(\sigma,\eta_\omega(\Lambda))\int_0^{|b_{\Lambda,\omega}|}P^{\gamma}_{\omega,\sigma,\Lambda}(t)\,dt. \tag{7.2.66}$$

7.3. ***A family of representations of G.*** We recall a definition of nondegeneracy of λ in [Bismut et al. 2017, Definition 1.13, Proposition 8.12].

Definition 7.3.1. A dominant weight $\Lambda\in P_{++}(U)$ is said to be nondegenerate with respect to the Cartan involution θ if

$$W(U,T_U)\cdot\Lambda\cap\mathfrak{t}^* = \varnothing. \tag{7.3.1}$$

It is equivalent to

$$\mathrm{Ad}^*(U)\Lambda\cap\mathfrak{k}^* = \varnothing. \tag{7.3.2}$$

Note that if such dominant weight exists, we must have $\delta(G)>0$.

Let (E,ρ^E) be the irreducible unitary representation of U with highest weight $\Lambda\in P_{++}(U)$. By the unitary trick, it extends to an irreducible representation of G, which we still denote by (E,ρ^E). Then Λ being nondegenerate is equivalent to saying that (E,ρ^E) is not isomorphic to $(E,\rho^E\circ\theta)$ as G-representations (as in [Müller and Pfaff 2013a]).

Definition 7.3.2. If $\lambda\in\mathfrak{t}^*_U$, for $\omega\in W(U,T_U)$, put

$$a_{\lambda,\omega} = \langle\omega\cdot\lambda,\sqrt{-1}a_1\rangle\in\mathbb{R}. \tag{7.3.3}$$

Recall the real number $b_{\lambda,\omega}$ is already defined by (7.2.9); then $b_{\lambda,\omega}=a_{\lambda,\omega}+a_{\rho_{\mathfrak{u}},\omega}$. In particular, we simply put $a_\lambda=a_{\lambda,1}$, $b_\lambda=b_{\lambda,1}$.

Lemma 7.3.3. *If $\lambda\in P_{++}(U)$ is nondegenerate, then, for $\omega\in W(U,T_U)$, $a_{\lambda,\omega}\neq 0$.*

Now we fix two dominant weights $\lambda, \lambda_0 \in P_{++}(U)$. Let $\{(E_d, \rho^{E_d})\}|_{d\in\mathbb{N}}$ be the sequence of representations of G given by the irreducible unitary representations of U with the highest weights $d\lambda + \lambda_0$, $d \in \mathbb{N}$.

Put $F_d = G \times_K E_d$. Let $\boldsymbol{D}^{X,F_d,2}$ denote the associated de Rham–Hodge Laplacian. For $t > 0$, let $\exp(-t\boldsymbol{D}^{X,F_d,2}/2)$ denote the heat operator associated with $\boldsymbol{D}^{X,F_d,2}/2$. By taking $\Lambda = d\lambda + \lambda_0$, we apply our results in previous subsection to the sequence $\mathcal{E}_{X,\gamma}(F_d, t)$, $d \in \mathbb{N}$.

7.4. ***Asymptotics for identity orbital integrals.*** In this subsection, we specialize our results in Section 7.2 for $\gamma = 1$ and $\Lambda = d\lambda + \lambda_0$. Now the set $W^1(\gamma)$ reduces to $\{1\}$, and $l(\gamma) = l$, $\varphi_\gamma(\sigma, \eta_\omega(\Lambda)) = 1$. We will drop the superscript γ and subscript σ in our notation.

Moreover, for $\omega \in W_u$, the representation $E^{\gamma=1}_{\omega,\sigma=1}$ is just $V_{\Lambda,\omega}$ introduced in (5.3.6), which is the irreducible unitary representation of U_M with highest weight $\eta_\omega(\Lambda)$ given by (5.3.8).

Definition 7.4.1. By taking $\Lambda = d\lambda + \lambda_0$ in (7.2.7), we define the following functions in d: for $j = 0, 1, \dots, l$, $\omega \in W_u$, set

$$\begin{aligned} Q^{\lambda,\lambda_0}_{j,\omega}(d) &= Q_{j,\omega}(d\lambda + \lambda_0) \\ &= \frac{(-1)^j \beta(a_1)^{2j}}{j!(2l-2j)!(8\pi^2)^j} \dim V_{d\lambda+\lambda_0,\omega}[\omega^{Y_{\mathfrak{b}},2j} \langle \omega(d\lambda + \lambda_0 + \rho_{\mathfrak{u}}), \Omega^{\mathfrak{u}_{\mathfrak{m}}}\rangle^{2l-2j}]^{\max}. \end{aligned} \tag{7.4.1}$$

By the Weyl dimension formula, $\dim V_{d\lambda+\lambda_0,\omega}$ is a polynomial in d. Then $Q^{\lambda,\lambda_0}_{j,\omega}(d)$ is a polynomial in d of degree $\leq \frac{1}{2}\dim(\mathfrak{g}/\mathfrak{h}) - 2j$.

By Theorem 7.2.2 and (7.4.1), we get directly the following results.

Theorem 7.4.2. *For $t > 0$, we have the identity*

$$\mathcal{I}_X(F_d, t) = \frac{c_G}{\sqrt{2\pi t}} \sum_{j=0}^{l} t^{-j} \sum_{\omega\in W_u} \varepsilon(\omega) e^{-2\pi^2 t(da_{\lambda,\omega}+b_{\lambda_0,\omega})^2} Q^{\lambda,\lambda_0}_{j,\omega}(d). \tag{7.4.2}$$

Theorem 7.4.3. *Suppose that λ is nondegenerate with respect to θ. For $d \in \mathbb{N}$ large enough and for $s \in \mathbb{C}$ with $\Re(s) \gg 0$, $\mathcal{M}\mathcal{I}_X(F_d, s)$ is well-defined and holomorphic, which admits a unique meromorphic extension to $s \in \mathbb{C}$ and is holomorphic at $s = 0$.*

Moreover, we have the identities

$$\mathcal{M}\mathcal{I}_X(F_d,s) = -\frac{c_G}{\sqrt{2\pi}} \sum_{j=0}^{l} \frac{\Gamma\left(s-j-\frac{1}{2}\right)}{\Gamma(s)} \left[\sum_{\omega\in W_u} \varepsilon(\omega) Q^{\lambda,\lambda_0}_{j,\omega}(d)(2\pi^2(da_{\lambda,\omega}+b_{\lambda_0,\omega})^2)^{j+\frac{1}{2}-s}\right], \tag{7.4.3}$$

$$\mathcal{P}\mathcal{I}_X(F_d) = -\frac{c_G}{\sqrt{2}} \sum_{j=0}^{l} \frac{(-4)^{j+1}(j+1)!}{(2j+2)!} \left[\sum_{\omega\in W_u} \varepsilon(\omega) Q^{\lambda,\lambda_0}_{j,\omega}(d)(2\pi^2(da_{\lambda,\omega}+b_{\lambda_0,\omega})^2)^{j+\frac{1}{2}}\right]. \tag{7.4.4}$$

In particular, the quantity $\mathcal{P}\mathcal{I}_X(F_d)$ is a polynomial in d for d large enough, whose coefficients depend only on the given root system and λ, λ_0, and has degree $\leq \frac{1}{2}\dim(\mathfrak{g}/\mathfrak{h}) + 1$.

Proof. Since λ is nondegenerate, by Lemma 7.3.3, $a_{\lambda,\omega} \neq 0,\ \omega \in W_u$. Then there exists $d_0 \in \mathbb{N}$ such that for $d \geq d_0$, $(da_{\lambda,\omega} + b_{\lambda_0,\omega})^2 > 0$. Then by Theorem 7.2.7, we get first part of this theorem and (7.4.3), (7.4.4).

Note that

$$[(da_{\lambda,\omega} + b_{\lambda_0,\omega})^2]^{\frac{1}{2}} = |da_{\lambda,\omega} + b_{\lambda_0,\omega}|.$$

For $d \gg d_0$,

$$|da_{\lambda,\omega} + b_{\lambda_0,\omega}| = \operatorname{sign}(a_{\lambda,\omega})(da_{\lambda,\omega} + b_{\lambda_0,\omega}).$$

Then we see that $\mathcal{PI}_X(F_d)$ is a polynomial in d for d large enough. □

As explained in Remark 5.3.3, when G has noncompact center with $\delta(G) = 1$ (but U is still assumed to be compact), most of the above computations can be reduce into very simple ones. Recall that $a_\lambda, b_{\lambda_0} \in \mathbb{R}$ are defined in Definition 7.3.2.

Corollary 7.4.4. *Assume that U is compact and that G has noncompact center with $\delta(G) = 1$, and assume that λ is nondegenerate. Then, for $t > 0,\ s \in \mathbb{C}$,*

$$\begin{aligned} \mathcal{I}_X(F_d, t) &= \frac{c_G}{\sqrt{2\pi t}} e^{-2\pi^2 t (da_\lambda + b_{\lambda_0})^2} \dim E_d, \\ \mathcal{MI}_X(F_d, s) &= -\frac{c_G}{\sqrt{2\pi}} \frac{\Gamma\big(s - \frac{1}{2}\big)}{\Gamma(s)} (2\pi^2 (da_\lambda + b_{\lambda_0})^2)^{1/2 - s} \dim E_d. \end{aligned} \tag{7.4.5}$$

Furthermore,

$$\mathcal{PI}_X(F_d) = 2\pi c_G |da_\lambda + b_{\lambda_0}| \dim E_d. \tag{7.4.6}$$

Proof. By the hypothesis, we get that $l = 0$, $W_u = \{1\}$ and $Q_{0,1}^{\lambda,\lambda_0}(d) = \dim E_d$. Then (7.4.5), (7.4.6) are just special cases of (7.4.2), (7.4.3) and (7.4.4).

However, we can prove them more directly using a result of Proposition 4.1.6. It is enough to prove the first identity in (7.4.5). Note that by (5.3.11), we have

$$X' = M/K, \tag{7.4.7}$$

with $\delta(X') = 0$.

By [Müller and Pfaff 2013a, Proposition 5.2] or [Shen 2018, Proposition 4.1], we have

$$[e(TX', \nabla^{TX'})]^{\max} = (-1)^{\frac{m-1}{2}} \frac{|W(U_M, T)|/|W(K, T)|}{\operatorname{Vol}(U_M/K)}. \tag{7.4.8}$$

Then by (7.2.6), we have

$$[e(TX', \nabla^{TX'})]^{\max} = -c_G. \tag{7.4.9}$$

By (4.1.28) and (7.3.3), we have

$$\alpha_{E_d} = -2\pi(da_\lambda + b_{\lambda_0}). \tag{7.4.10}$$

Combing (4.1.31) and (7.4.8) - (7.4.10), we get the first identity in (7.4.5), and hence the other identities. This gives a second proof to this corollary. □

7.5. ***Connection to Müller and Pfaff's results.*** In this subsection, we assume that G has compact center with $\delta(G) = 1$. We explain here how to connect our computations in the previous subsection to the results in [Müller and Pfaff 2013a].

For $\gamma = 1$, $\omega \in W_u$, the function $P^{\gamma}_{\omega,\sigma,\Lambda}$ defined in (7.2.61) now reduces to

$$P_{\omega,\Lambda}(z) = \dim V_{\Lambda,\omega}\big[\exp\big(\langle \eta_\omega(\Lambda) + \rho_{\mathfrak{u}_{\mathfrak{m}}} + z\sqrt{-1}a^1, \Omega^{\mathfrak{u}(\mathfrak{b})}\rangle\big)\big]^{\max}. \tag{7.5.1}$$

We can verify directly that

$$P_{\omega,\Lambda}(z) = \frac{\mathrm{Vol}(U_M/T)}{\mathrm{Vol}(U/T_U)} \Pi_{\alpha^0 \in R^+(\mathfrak{u},\mathfrak{t}_U)} \frac{\langle \alpha^0, \eta_\omega(\Lambda) + \rho_{\mathfrak{u}_{\mathfrak{m}}} + z\sqrt{-1}a^1\rangle}{\langle \alpha^0, \rho_{\mathfrak{u}}\rangle}. \tag{7.5.2}$$

The scalar product in (7.5.2) is taken with respect to $-B|_{\mathfrak{u}}$. Up to a universal constant, $P_{\omega,\Lambda}(z)$ is just the polynomial related to the Plancherel measure of representation $V_{\Lambda,\omega}$ as given in [Müller and Pfaff 2013a, equation (6.10)]. Note that there is no factor $(2\pi)^{2l}$ in (7.5.2) because of our normalization for $[\cdot]^{\max}$.

By Theorem 7.2.10, we have the following result for sufficiently large d.

Corollary 7.5.1. *Suppose that λ is nondegenerate with respect to θ. Then*

$$\mathcal{PI}_X(F_d) = 2\pi c_G \sum_{\omega \in W_u} \varepsilon(\omega) \int_0^{|da_{\lambda,\omega} + b_{\lambda_0,\omega}|} P_{\omega, d\lambda+\lambda_0}(t)\, dt. \tag{7.5.3}$$

By [Müller and Pfaff 2013a, Lemma 6.1], we can get the identity

$$|W(K,T)| = 2|W(K_M,T)|. \tag{7.5.4}$$

Combining (7.2.6), (7.5.2), (7.5.4), we see that the formula in Corollary 7.5.1, is exactly the same formula of [Müller and Pfaff 2013a, Proposition 6.6] for $\mathcal{PI}_X(F_d)$.

Recall that the U-representation E_d has highest weight $d\lambda + \lambda_0 \in P_{++}(U)$. Then by Weyl dimension formula, $\dim E_d$ is a polynomial in d. If λ is regular, then the degree (in d) of $\dim E_d$ is $\frac{1}{2}\dim \mathfrak{g}/\mathfrak{h}$.

For determining the leading term of $\mathcal{PI}_X(F_d)$, as mentioned in the Introduction, we can specialize the result of [Bismut et al. 2017, Theorem 0.1] as in Section 8 of that work for the symmetric space X. Here to emphasize $\mathcal{PI}_X(F_d)$ being a polynomial in d, we state a result of [Müller and Pfaff 2013a, Proposition 1.3] as follows.

Proposition 7.5.2. *Suppose that λ is nondegenerate and that $\lambda_0 = 0$. Then there exists a constant $C_{X,\lambda} \neq 0$ such that*

$$\mathcal{PI}_X(F_d) = C_{X,\lambda}\, d \dim E_d + R(d), \tag{7.5.5}$$

where $R(d)$ is a polynomial whose degree is no greater than the degree of $\dim E_d$.

Remark 7.5.3. Note that Müller and Pfaff [2013a, Proposition 1.3] proved Proposition 7.5.2 by reducing the problems to the cases $G = \mathrm{SL}_3(\mathbb{R})$ and $\mathrm{SO}^0(p,q)$ ($pq > 1$ odd). In particular, for certain examples of λ, they also worked out explicitly the constant $C_{X,\lambda}$ [Müller and Pfaff 2013a, Corollaries 1.4, 1.5].

Similarly, if we take a nonzero λ_0, we can repeat their computations for $G = \mathrm{SL}_3(\mathbb{R})$ and $\mathrm{SO}^0(p,q)$ ($pq > 1$ odd) in order to get more explicit information on the leading terms of $\mathcal{PI}_X(F_d)$.

An important step in Müller and Pfaff's proof to Proposition 7.5.2 is reducing the computation of $\mathcal{PI}_X(F_d)$ to the cases where $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{R})$ or $\mathfrak{so}(p,q)$ with $pq > 1$ odd. Such reduction is already explained in Section 4.2. More precisely, we have

$$X = X_1 \times X_2, \tag{7.5.6}$$

where X_1 is one case listed in (4.2.1), and X_2 is a symmetric space with rank 0.

We use the notation in Section 4.2 and assume G to be semisimple. Let λ_i, $\lambda_{0,i}$ be dominant weights of U_i, $i = 1, 2$, such that

$$\lambda = \lambda_1 + \lambda_2, \quad \lambda_0 = \lambda_{0,1} + \lambda_{0,2}. \tag{7.5.7}$$

Now we consider the sequence $d\lambda + \lambda_0$, $d \in \mathbb{N}$. Then

$$E_d = E_{d\lambda_1+\lambda_{0,1}} \otimes E_{d\lambda_2+\lambda_{0,2}}. \tag{7.5.8}$$

Since G_2 is equal rank, the nondegeneracy of λ with respect to θ is equivalent to the nondegeneracy of λ_1 with respect to θ_1. Then by Proposition 4.2.2, after taking the Mellin transform, we have

$$\mathcal{MI}_X(F_d, s) = [e(TX_2, \nabla^{TX_2})]^{\max 2} \dim E_{d\lambda_2+\lambda_{0,2}} \mathcal{MI}_{X_1}(F_{d\lambda_1+\lambda_{0,1}}, s). \tag{7.5.9}$$

Then

$$\mathcal{PI}_X(F_d) = [e(TX_2, \nabla^{TX_2})]^{\max 2} \dim E_{d\lambda_2+\lambda_{0,2}} \mathcal{PI}_{X_1}(F_{d\lambda_1+\lambda_{0,1}}). \tag{7.5.10}$$

Then we only need to evaluate $\mathcal{PI}_{X_1}(F_{d\lambda_1+\lambda_{0,1}})$ explicitly, which has been dealt with in [Müller and Pfaff 2013a, Section 6].

7.6. *Asymptotic elliptic orbital integrals.*

Definition 7.6.1. A function $f(d)$ in d is called an exponential polynomial in d if it is a finite sum of the term $c_{j,s}e^{2\pi\sqrt{-1}sd}d^j$ with $j \in \mathbb{N}$, $s \in \mathbb{R}$, $c_{j,s} \in \mathbb{C}$. The largest $j \geq 0$ such that $c_{j,s} \neq 0$ in $f(d)$ is called the degree of $f(d)$.

We say that the oscillating term $e^{2\pi\sqrt{-1}sd}$ is nice if $s \in \mathbb{Q}$. We say that an exponential polynomial $f(d)$ in d is nice if all its oscillating terms are nice.

Remark 7.6.2. If $f(d)$ is a nice exponential polynomial in d, then there exists an $N_0 \in \mathbb{N}_{>0}$ such that the function $f(dN_0)$ is a polynomial in d.

Note that by (5.4.18), $\varphi_\gamma(\sigma, \eta_\omega(d\lambda + \lambda_0))$ is an oscillating term in d, which is nice when $\gamma \in T$ is of finite order. The following theorem is a direct consequence of Theorem 7.2.10.

Theorem 7.6.3. *Suppose that λ is nondegenerate, and that $\gamma = k \in T$. Then, for sufficiently large d, $\mathcal{PE}_{X,\gamma}(F_d)$ is an exponential polynomial in d. Moreover, we have*

$$\mathcal{PE}_{X,\gamma}(F_d) = 2\pi c_G(\gamma) \cdot \sum_{\substack{\omega \in W_u \\ \sigma \in W^1(\gamma)}} \varepsilon(\omega)\varphi_\gamma(\sigma, \eta_\omega(d\lambda + \lambda_0)) \int_0^{|da_{\lambda,\omega} + b_{\lambda_0,\omega}|} P^\gamma_{\omega,\sigma,d\lambda+\lambda_0}(t)\, dt. \tag{7.6.1}$$

If we consider $G = \mathrm{Spin}(1, 2n+1)$, $n \geq 1$, as in [Fedosova 2015], then up to a constant, the exponential polynomial $\sum_{\sigma\in W^1(\gamma)} \varphi_\gamma(\sigma, \eta_\omega(d\lambda+\lambda_0)) P^\gamma_{\omega,\sigma,d\lambda+\lambda_0}(t)$ is just the one defined in [Fedosova 2015, Proposition 5.1]. This way, our results are compatible with her results in [Fedosova 2015, Theorem 1,1] for hyperbolic orbifolds.

Remark 7.6.4. Let $\mathrm{Char}(A)$ denote the character ring of the complex representations of a compact Lie group A. One key ingredient in (7.2.66) is an explicit decomposition of characters of U into characters of $U_M(\gamma)^0$. In the diagram below, we give two different ways of getting to this decomposition:

$$\begin{tikzcd} & \mathrm{Char}(U(\gamma)^0) \arrow[dr, "\otimes\Lambda^\bullet\mathfrak{n}(\gamma)^*_{\mathbb{C}}"] & \\ \mathrm{Char}(U) \arrow[ur, "\text{Kirillov for } \gamma\in U"] \arrow[dr, "\otimes\Lambda^\bullet\mathfrak{n}^*_{\mathbb{C}}"'] & & \mathrm{Char}(U_M(\gamma)^0) \\ & \mathrm{Char}(U_M) \arrow[ur, "\text{Kirillov for } \gamma\in U_M"] & \end{tikzcd} \tag{7.6.2}$$

The formula in (7.2.66) is obtained by the computations along the lower path in (7.6.2). We also have the upper path, which is essentially the geometric localization formula obtained in Theorem 6.0.1.

We will use the same notation as in Section 6. The following theorem is a consequence of the geometric localization formula obtained in Theorem 6.0.1.

For $k\in T$, let $W^1_U(k)\subset W(U,T_U)$ be defined as in (5.4.14) with respect to $R^+(\mathfrak{u},\mathfrak{t}_U)$. For $\sigma\in W^1_U(k)$, the term $\varphi^U_k(\sigma, d\lambda+\lambda_0)$ defined as in (6.0.8) is an oscillating term, which is nice if k is of finite order.

Theorem 7.6.5. *Suppose that $\gamma = k \in T$ is elliptic and that λ is nondegenerate with respect to θ. Then, for $\sigma\in W^1_U(k)$, $\sigma\lambda\in P_{++}(\widetilde{U}(k))$ is nondegenerate with respect to the Cartan involution θ on $\mathfrak{z}(k)$. For $d\in\mathbb{N}$, let $E^k_{\sigma,d}$ be the irreducible unitary representation of $\widetilde{U}(k)$ with highest weight $d\sigma\lambda+\sigma(\lambda_0+\rho_{\mathfrak{u}})-\rho_{\mathfrak{u}(k)}$. This way we get a sequence of flat vector bundles $\{F^k_{\sigma,d}\}_{d\in\mathbb{N}}$ on $X(k)$. Then, for sufficiently large d, we have*

$$\mathcal{P}\mathcal{E}_{X,\gamma}(F_d) = \sum_{\sigma\in W^1_U(k)} \varphi^U_k(\sigma, d\lambda+\lambda_0)\mathcal{P}\mathcal{I}_{X(k)}(F^k_{\sigma,d}). \tag{7.6.3}$$

Proof. The nondegeneracy of $\sigma\lambda$ ($\sigma\in W^1_U(k)$) follows easily from the nondegeneracy of λ and the definition of $W^1_U(k)$. For proving this theorem, we only need to prove (7.6.3). Actually, by Theorem 6.0.1, for $t>0$, we get

$$\mathcal{E}_{X,\gamma}(F_d,t) = \sum_{\sigma\in W^1_U(k)} \varphi^U_k(\sigma, d\lambda+\lambda_0)\mathcal{I}_{X(k)}(F^k_{\sigma,d},t), \tag{7.6.4}$$

Then (7.6.3) follows from the linearity of Mellin transform. ☐

8. A proof of Theorem 1.0.2

In this section, we complete the proof of Theorem 1.0.2; then Theorem 1.0.1 (and Theorem 1.0.1′) follows as a consequence. We assume that G is a connected linear real reductive Lie group with $\delta(G)=1$ and compact center, so that U is a compact Lie group.

8.1. ***A lower bound for the Hodge Laplacian on X.*** We use the notation from Section 4. Recall that $e_1, \dots, e_m$ is an orthogonal basis of TX or $\mathfrak{p}$. Put

$$C^{\mathfrak{g},H} = -\sum_{j=1}^{m} e_j^2 \in U\mathfrak{g}. \tag{8.1.1}$$

Let $C^{\mathfrak{g},H,E}$ be its action on E via ρ^E. Then

$$C^{\mathfrak{g},E} = C^{\mathfrak{g},H,E} + C^{\mathfrak{k},E}. \tag{8.1.2}$$

Let $\Delta^{H,X}$ be the Bochner–Laplace operator on the bundle $\Lambda^\bullet(T^*X) \otimes F$ associated with the unitary connection $\nabla^{\Lambda^\bullet(T^*X)\otimes F,u}$. Put

$$\begin{aligned}\Theta(F) = \tfrac{1}{4}S^X - \tfrac{1}{8}\langle R^{TX}(e_i, e_j)e_k, e_\ell\rangle c(e_i)c(e_j)\hat{c}(e_k)\hat{c}(e_\ell)\\ - C^{\mathfrak{g},H,E} + \tfrac{1}{2}\big(c(e_i)c(e_j) - \hat{c}(e_i)\hat{c}(e_j)\big)R^F(e_i, e_j),\end{aligned} \tag{8.1.3}$$

where R^F is the curvature of the unitary connection ∇^F on F.

Then $\Theta(F)$ is a self-adjoint section of $\operatorname{End}(\Lambda^\bullet(T^*X) \otimes F)$, which is parallel with respect to $\nabla^{\Lambda^\bullet(T^*X)\otimes F,u}$. Equivalently, $\Theta(F)$ is an element in $\operatorname{End}(\Lambda^\bullet(\mathfrak{p}^*) \otimes E)$ which commutes with the K-action. By [Bismut et al. 2017, equation (8.39)], we have

$$\boldsymbol{D}^{X,F,2} = -\Delta^{H,X} + \Theta(F). \tag{8.1.4}$$

Then, for $s \in \Omega^\bullet_c(X, F)$, we have

$$\langle \boldsymbol{D}^{X,F,2}s, s\rangle_{L_2} \geq \langle \Theta(F)s, s\rangle_{L_2}. \tag{8.1.5}$$

Let $\Delta^{H,X,i}$ denote the Bochner–Laplace operator acting on $\Omega^i(X, F)$, and let $p^{H,i}_t(x, x')$ be the kernel of $\exp(t\Delta^{H,X,i}/2)$ on X with respect to dx'. We will denote by $p^{H,i}_t(g) \in \operatorname{End}(\Lambda^i(\mathfrak{p}^*) \otimes E)$ its lift to G explained in Section 3.2. Let Δ^X_0 be the scalar Laplacian on X with the heat kernel $p^{X,0}_t$.

Let $\|p^{H,i}_t(g)\|$ be the operator norm of $p^{H,i}_t(g)$ in $\operatorname{End}(\Lambda^i(\mathfrak{p}^*) \otimes E)$. By [Müller and Pfaff 2013b, Proposition 3.1], if $g \in G$; then

$$\|p^{H,i}_t(g)\| \leq p^{X,0}_t(g). \tag{8.1.6}$$

Let p^H_t be the kernel of $\exp(t\Delta^{H,X}/2)$, then

$$p^H_t = \bigoplus_{i=1}^{p} p^{H,i}_t. \tag{8.1.7}$$

Let $q^{X,F}_t$ be the heat kernel associated with $\boldsymbol{D}^{X,F,2}/2$, by (8.1.4), for $g \in X$,

$$q^{X,F}_t(g) = \exp\Big(-\frac{t\Theta(F)}{2}\Big)p^H_t(g). \tag{8.1.8}$$

Recall that $P_{++}(U)$ is the set of dominant weights of U with respect to $R^+(\mathfrak{u}, \mathfrak{t}_U)$ defined in Section 5.3. As in Section 7.3, we fix $\lambda, \lambda_0 \in P_{++}(U)$ such that λ is nondegenerate with respect to θ. Recall that, for $d \in \mathbb{N}$, (E_d, ρ^{E_d}) is the irreducible unitary representation of U with highest weight $d\lambda + \lambda_0$, which extends uniquely to a representation of G. By [Bismut et al. 2011, Théorème 3.2; 2017, Theorem 4.4, Remark 4.5; Müller and Pfaff 2013a, Proposition 7.5], there exist $c > 0$, $C > 0$ such that, for $d \in \mathbb{N}$,

$$\Theta(F_d) \geq cd^2 - C, \tag{8.1.9}$$

where the estimate d^2 comes from the positive operator $C^{\mathfrak{g},H,E_d}$. By (8.1.4), (8.1.5), (8.1.9), we get

$$\boldsymbol{D}^{X,F_d,2} \geq cd^2 - C. \tag{8.1.10}$$

Lemma 8.1.1. *There exists $d_0 \in \mathbb{N}$ and $c_0 > 0$ such that if $d \geq d_0$, $g \in G$,*

$$\|q_t^{X,F_d}(g)\| \leq e^{-c_0 d^2 t} p_t^{X,0}(g). \tag{8.1.11}$$

Proof. By (8.1.9), there exist $d_0 \in \mathbb{N}$, $c' > 0$ such that if $d \geq d_0$,

$$\Theta(F_d) \geq c' d^2. \tag{8.1.12}$$

Then if $t > 0$,

$$\left\| \exp\left(-\frac{t\Theta(F_d)}{2} \right) \right\| \leq e^{-\frac{1}{2}c' d^2 t}. \tag{8.1.13}$$

By (8.1.6), (8.1.7), (8.1.8), (8.1.13), we get (8.1.11). □

The locally symmetric orbifold Z is defined as $\Gamma \backslash X$, where Γ is a cocompact discrete subgroup of G. For $\gamma \in \Gamma$, the number $m_\gamma \geq 0$ is given by (3.3.3), which only depends on the conjugacy class of γ (in G or Γ). Recall that $E[\Gamma]$ is the finite set of elliptic conjugacy classes in Γ.

For $t > 0$, $x \in X$, $\gamma \in \Gamma$, set

$$v_t(F_d, \gamma, x) = \operatorname{Tr}_s^{\Lambda^\bullet(T^*X)\otimes F_d}\left[\left(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2} \right) q_t^{X,F_d}(x, \gamma(x))\gamma \right]. \tag{8.1.14}$$

Then by Lemma 8.1.1, we have the following result.

Lemma 8.1.2. *There exist $C_0 > 0$, $c_0 > 0$ such that if d is large enough, for $t > 0$, $x \in X$, $\gamma \in \Gamma$,*

$$|v_t(F_d, \gamma, x)| \leq C_0 (\dim E_d) e^{-c_0 d^2 t} p_t^{X,0}(x, \gamma(x)). \tag{8.1.15}$$

Set

$$m_\Gamma = \inf_{[\gamma] \in [\Gamma] - E[\Gamma]} m_\gamma. \tag{8.1.16}$$

By [Liu 2018, Proposition 1.8.5], $m_\Gamma > 0$.

Proposition 8.1.3. *There exist constants $C > 0$, $c > 0$ such that if $x \in X$, $t \in]0, 1]$, then*

$$\sum_{\gamma \in \Gamma, \gamma \text{ nonelliptic}} p_t^{X,0}(x, \gamma(x)) \leq C \exp\left(-\frac{c}{t} \right). \tag{8.1.17}$$

Proof. By [Donnelly 1979, Theorem 3.3], there exists $C_0 > 0$ such that when $0 < t \leq 1$,

$$p_t^{X,0}(x, x') \leq C_0 t^{-\frac{m}{2}} \exp\left(-\frac{d^2(x, x')}{4t} \right). \tag{8.1.18}$$

By [Liu 2018, Lemma 1.8.6], there exist $c>0,\ C>0$ such that for $R>0,\ x\in X$,

$$\#\{\gamma\in\Gamma \mid \gamma \text{ nonelliptic}, d_\gamma(x)\leq R\}\leq C\exp(cR). \tag{8.1.19}$$

By (8.1.16), (8.1.18), (8.1.19), and using the same arguments as in the proof of [Müller and Pfaff 2013b, Proposition 3.2], we get (8.1.17). □

8.2. ***A proof of Theorem 1.0.2.*** In this subsection, we complete our proof of Theorem 1.0.2. Note that every elliptic element $\gamma\in\Gamma$ is of finite order, then part (2) of Theorem 1.0.2 is an easy consequence of Theorem 7.6.5. We only need to prove part (1). We restate it as follows.

Proposition 8.2.1. *Let $\Gamma\subset G$ be a cocompact discrete subgroup and set $Z=\Gamma\backslash X$. There exists $c>0$ such that, for d large enough,*

$$\mathcal{T}(Z,F_d)=\frac{\mathrm{Vol}(Z)}{|S|}\mathcal{PI}_X(F_d)+\sum_{[\gamma]\in E^+[\Gamma]}\frac{\mathrm{Vol}(\Gamma\cap Z(\gamma)\backslash X(\gamma))}{|S(\gamma)|}\mathcal{PE}_{X,\gamma}(F_d)+\mathcal{O}(e^{-cd}), \tag{8.2.1}$$

where $E^+[\Gamma]=E^+[\Gamma]\backslash\{[1]\}$ is the finite set of nontrivial elliptic classes in $[\Gamma]$.

Proof. By (8.1.10), we have

$$\boldsymbol{D}^{Z,F_d,2}\geq cd^2-C. \tag{8.2.2}$$

Then if d is large enough, we have

$$H^\bullet(Z,F_d)=0. \tag{8.2.3}$$

Then $\mathcal{T}(Z,F_d)$ can be computed using (2.2.15).

As in (2.2.12), for $t>0$, set

$$b(F_d,t)=\Big(1+2t\frac{\partial}{\partial t}\Big)\mathrm{Tr}_s\Big[\Big(N^{\Lambda^\bullet(T^*Z)}-\frac{m}{2}\Big)\exp\Big(-\frac{t\boldsymbol{D}^{Z,F_d,2}}{2}\Big)\Big]. \tag{8.2.4}$$

As in [Bismut et al. 2017, Section 7.2], by (8.2.2), there exist constants $\tilde{c}>0,\ \widetilde{C}>0$ such that, for d large enough and for $t>\frac{1}{d}$,

$$|b(F_d,t)|\leq\widetilde{C}\exp(-\tilde{c}d-\tilde{c}t). \tag{8.2.5}$$

By (2.2.15), we have

$$\mathcal{T}(Z,F_d)=-\int_0^{+\infty}b(F_d,t)\,\frac{dt}{t}. \tag{8.2.6}$$

We rewrite it as

$$\mathcal{T}(Z,F_d)=-\int_{1/d}^{+\infty}b(F_d,t)\,\frac{dt}{t}-\int_0^d b\Big(\frac{F_d,t}{d^2}\Big)\,\frac{dt}{t}. \tag{8.2.7}$$

By (8.2.5), there exists $c>0$ such that, for d large enough,

$$\int_{1/d}^{+\infty}b(F_d,t)\,\frac{dt}{t}=\mathcal{O}(e^{-cd}). \tag{8.2.8}$$

By (3.5.1), (8.1.14), (8.2.4), we get

$$b(F_d,t)=\Big(1+2t\frac{\partial}{\partial t}\Big)\int_Z \frac{1}{|S|}\sum_{\gamma\in\Gamma} v_t(F_d,\gamma,z)\,dz. \tag{8.2.9}$$

We split the sum in (8.2.9) into two parts,

$$\sum_{\gamma\in\Gamma,\gamma\text{ elliptic}} + \sum_{\gamma\in\Gamma,\gamma\text{ nonelliptic}}, \tag{8.2.10}$$

so that we write

$$b(F_d,t)=b_{\text{elliptic}}(F_d,t)+b_{\text{nonelliptic}}(F_d,t). \tag{8.2.11}$$

Similar to Selberg's trace formula in Section 3.5, we get

$$b_{\text{elliptic}}(F_d,t)=\sum_{[\gamma]\in E[\Gamma]}\frac{\mathrm{Vol}(\Gamma\cap Z(\gamma)\backslash X(\gamma))}{|S(\gamma)|}\Big(1+2t\frac{\partial}{\partial t}\Big)\mathcal{E}_{X,\gamma}(F_d,t). \tag{8.2.12}$$

By (7.4.2) and (7.6.4), the terms in $\mathcal{E}_{X,\gamma}(F_d,t)$ are of the form

$$t^{-j+\frac{1}{2}}\exp(-2\pi^2 t(da'+b')^2)Q(d), \tag{8.2.13}$$

where $Q(d)$ is a nice exponential polynomial in d, and $a',b'\in\mathbb{R}$ with $a'\neq 0$ due to the nondegeneracy of λ. By (8.2.13), there exists $c>0$ such that, for d large enough,

$$\int_0^d b_{\text{elliptic}}\Big(F_d,\frac{t}{d^2}\Big)\frac{dt}{t}=\int_0^{+\infty} b_{\text{elliptic}}(F_d,t)\frac{dt}{t}+\mathcal{O}(e^{-cd}). \tag{8.2.14}$$

Using Proposition 7.1.1 and by (8.2.13), we get

$$\mathcal{P}\mathcal{E}_{X,\gamma}(F_d)=-\int_0^{+\infty}\Big(1+2t\frac{\partial}{\partial t}\Big)\mathcal{E}_{X,\gamma}(F_d,t)\frac{dt}{t}. \tag{8.2.15}$$

Now we consider the contribution from the nonelliptic elements. If $x\in X$, put

$$h_t(F_d,x)=\frac{1}{|S|}\sum_{\gamma\in\Gamma,\gamma\text{ nonelliptic}} v_t(F_d,\gamma,x). \tag{8.2.16}$$

Then

$$b_{\text{nonelliptic}}(F_d,t)=\Big(1+2t\frac{\partial}{\partial t}\Big)\int_Z h_t(F_d,z)\,dz. \tag{8.2.17}$$

Now we prove the following uniform estimates for $x\in X$:

$$\int_0^d\Big(1+2t\frac{\partial}{\partial t}\Big)h_{t/d^2}(F_d,x)\frac{dt}{t}=\mathcal{O}(e^{-cd}). \tag{8.2.18}$$

Indeed, using Lemma 8.1.2 and Proposition 8.1.3, there exist $C>0,\ c'>0,\ c''>0$ such that if d is large enough, $0<t\leq d$, then

$$|h_{t/d^2}(F_d,x)|\leq C\dim(E_d)e^{-c't}\exp\Big(-\frac{c''d^2}{t}\Big). \tag{8.2.19}$$

Recall that $\dim E_d$ is a polynomial in d. Then by (8.2.19), we have

$$\begin{aligned} \left|\int_0^1 h_{t/d^2}(F_d,x)\,\frac{dt}{t}\right| &\le Ce^{-c''d^2/2}\dim(E_d)\int_0^1 e^{-c''d^2/2t}\,\frac{dt}{t} = \mathcal{O}(e^{-cd}),\\ \left|\int_1^d h_{t/d^2}(F_d,x)\frac{dt}{t}\right| &\le Ce^{-c''d}\,\dim(E_d)\int_1^d e^{-c't}\,\frac{dt}{t} = \mathcal{O}(e^{-cd}). \end{aligned} \tag{8.2.20}$$

By (8.2.19)–(8.2.20), we get (8.2.18).

At last, we assembly together (8.2.7), (8.2.8), (8.2.11), (8.2.14)–(8.2.18), we get exactly (8.2.1). □

Note that since $\mathcal{T}(Z,F_d)$ is always a real number, (8.2.1) still holds if we take the real part of $\mathcal{P}\mathcal{E}_{X,\gamma}(F_d)$ instead.

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