Discussion Papers of the
Max Planck Institute for
Research on Collective Goods
2023/5

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May 2023

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May 15, 2023

Abstract

This expository note uses Hilbert's "infinite hotel", a hotel where one can always find place for another guest even if the hotel is already full, to illustrate the failure of the First Welfare Theorem in "large-square" economies that have infinitely many participants as well as infinitely many goods. Hilbert's hotel with infinitely many guests has a similar mathematical structure as the overlapping-generations model of Allais (1947) and Samuelson (1958). The phenomenon of "dynamic inefficiency" in such models represents a failure of the First Welfare Theorem in "large-square" economies, rather than frictions from the sequential nature of markets.

Key Words: Hilbert's hotel, overlapping-generations models, dynamic inefficiency, First Welfare Theorem.

JEL: D15, D61, E62.

Hilberts' Hotel. The mathematician David Hilbert introduced the allegory of an infinite hotel to explain the notion of "infinity". An infinite hotel is one in which one can always accommodate another guest, even when the hotel is already full: Put the new guest into room 1, the guest from room 1 into room 2, and so on. In this hotel, scarcity is different from what we are used to. This has implications for welfare analysis. In particular, the First Welfare Theorem need not hold. It is instructive to see why.

Forgetting about new guests, consider the allocation of rooms when participants care about which rooms they are in. Think of the hotel as an exchange economy in which, for $n = 1, 2, ...$, agent n is endowed with room n and there is a complete market system with one market for each room. Suppose that agent n gets utility $u_n > 0$ from staying in room n, utility $u_{n+1} > 0$ from staying in room $n+1$ and no utility at all from staying in any other room. If $u_{n+1} = 2u_n$, one easily sees that a price system satisfying $p_1 = 1$ and $p_n = 3p_{n-1}$ for $n = 2, 3, ...$

supports the initial allocation as a competitive equilibrium allocation.[1] However, a Pareto improvement is obtained by moving the guest from room 1 into room 2, the guest from room 2 into room 3, and so on.[2]

Why doesn't the First Welfare Theorem hold for this hotel? The usual proof of the theorem begins by observing that, if an alternative allocation provides each participant with greater utility than the competitive equilibrium allocation, then for each participant the consumption plan under the new allocation must be unaffordable at the equilibrium prices. Upon adding this inequality over all consumers, one finds that the value at equilibrium prices of aggregate consumption under the alternative allocation must exceed the value of aggregate consumption under the competitive equilibrium allocation and therefore the value of the aggregate available resources. This leads to the conclusion that the alternative allocation cannot be feasible: For at least one good, the alternative allocation must stipulate consumption in excess of the resources available for providing this good.

With infinitely many participants and infinitely many goods, this argument can break down because taking sums over all consumers may not be admissible. In the case of Hilbert's hotel, with equilibrium prices satisfying $p_1 = 1$ and $p_n = 3p_{n-1}$ for $n > 1$, the value of aggregate consumption of the first N consumers under the alternative allocation - consumer n in room $n+1$ - is $\sum_{n=1}^{N} 3^n$. This exceeds the value $\sum_{n=1}^{N} 3^{n-1}$ of aggregate consumption of the first N consumers under the competitive allocation, but, as N goes out of bounds, both sums go out of bounds, and one cannot say that $\sum_{n=1}^{\infty} 3^n$ exceeds $\sum_{n=1}^{\infty} 3^{n-1}$. The value of aggregate resources at competitive equilibrium prices is also unbounded. In the infinite hotel, the notion of scarcity can be fundamentally different from what we are used to.

In this discussion, the fact that, at competitive equilibrium prices values of aggregate consumption vectors and aggregate available resources are unbounded hinges on the fact that these prices grow with n. Such growth is mandated by the specification of consumer preferences. If preferences satisfied $u_{n+1} = \frac{1}{2}u_n$ for all n, a price system satisfying $p_n = \frac{2}{3}p_{n-1}$ for all n would support the initial allocation as a competitive equilibrium. At these prices, the common value of aggregate consumption and aggregate resources would be equal to $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1} = 3 < \infty$, and the proof of the First Welfare Theorem would go through. The equilibrium allocation would be Pareto efficient.

In this analysis, Hilbert's hotel appears as an ingredient of a large-square economy, an economy that has an infinity of agents as well as an infinity of goods. In such an economy, the equilibrium values of aggregate consumption and aggregate endowments involve double sums with infinitely many terms.

[1] The conclusion would hold even if rooms were divisible and the utility of agent n was given as $xu_n + yu_{n+1}$, where x and y are the shares of rooms n and $n+1$ that the agent "enjoys".

[2] This argument involves the infinite number of participants as well as the infinite number of rooms. If the hotel was occupied by a king with an infinite retinue, and only the preferences of the king mattered, there would be no scope for Pareto inefficiency. Representative agent models involve such kings.

These sums are not well defined unless equilibrium prices decline sufficiently quickly as one goes down the list of goods. Whether they do so, depends on the underlying preferences of the infinitely many participants.[3]

OLG Economies as Large-Square Economies. Hilbert's hotel with an infinite number of rooms and an infinite number of guests has a similar mathematical structure as the overlapping-generations model of Allais (1947, Appendix 2) and Samuelson (1958) without capital and without production. In the simplest version of this model, there is a single perishable good in each period, people live for two periods and have an endowment E of the good in the period in which they are born and no endowment in the next period. If each person gets utility $c_1 + 2c_2$ out of consuming c_1 in the first period and c_2 in the second period of life, the autarky allocation, in which every participant consumes the endowment in the first period of life and consumes nothing in the second period of life, is an equilibrium allocation in a complete market system *ex ante*, with equilibrium prices for the consumption good in periods $t = 1, 2, , ...$ given as $p_1 = 1$ and $p_t = 3p_{t-1}$ for $t > 1$. If all generations have the same size, this allocation is Pareto-dominated by one in which all participants give their initial endowment to a member of the preceding generation and in the next period receives the initial endowment of a member of the next generation. Again the proof of the First Welfare Theorem does not work because the value of aggregate consumption and aggregate resources at the equilibrium prices is unbounded.

The similarity of this overlapping-generations model to Hilbert's hotel indicates that this inefficency has little to do with the time structure of the overlapping-generations model, with the interpretation of goods' indices as "periods", or with the lack of a complete market system *ex ante.* To be sure, such a market system is ruled out if generation t cannot do anything before the period when it born, but this fact is irrelevant if the market system is sequentially complete in the sense that the sequence of markets with goods t and $t+1$ traded in period t yields the same allocations as a complete market system *ex ante.*

Because the overlapping-generations model here does not involve capital, the inefficiency of competitive equilibrium allocations cannot be due to overaccumulation of capital. It concerns the allocation of different consumption goods (rooms) over different people. The inefficiency hinges on the specification of consumers' preferences and on the equilibrium price system that these preferences induce. If utility from first-period consumption c_1 and second-period consumption c_2 was given as $c_1 + \frac{1}{2}c_2$, the autarky allocation would be Pareto efficient, and a price system with $p_1 = 1$ and $p_t = \frac{2}{3}p_{t-1}$ for $t = 2, 3, ...$ would support the autarky allocation as a competitive equilibrium allocation. At these prices, the values of aggregate consumption and aggregate resources at these prices would be finite, and the proof of the First Welfare Theorem would go through. The equilibrium allocation would be Pareto efficient.

[3] For a general formulation, see Balasko and Shell (1980).

In each case, the question is whether or not there is enough discounting at the margin so that, in a complete market system *ex ante*, the (present-value) prices of goods to be delivered in period t decline sufficiently quickly with t so that the sums involved in the proof of the First Welfare Theorem are well defined. In a model with a representative agent maximizing a discounted sum of utilities, this condition would always be satisfied. It is also satisfied in a economy with finitely many consumers who maximize discounted sums of utilities.[4]

With an infinite number of consumers, discounting at the level of individuals is not sufficient to have present-value prices of goods to be delivered in period t decline sufficiently quickly with t. In the example above, there is some discounting because a person born in period t cares nothing about consumption in periods $t' > t + 1$. However, this discounting is not uniform across agents. For each t, no matter how large, there is some agent who cares a lot about consumption in period t. Moreover, the relative price for consumption in period $t + 1$ versus period t depends on the preferences of the people who are born in period t and who care about consumption in periods t and $t + 1$. The validity of the First Welfare Theorem then depends on whether these conditions on relative prices for consumption in $t+1$ versus consumption in t imply relative prices for consumption in $t + 1$ versus consumption in period one that decline sufficiently quickly with t.

In an overlapping-generations economy with production, investment and capital come in because, with nonlinear utility functions and high rates of return on assets, agents may devote large parts of their resources to later consumption so that marginal utilities of later consumption and therefore also the equilibrium relative prices of later consumption over earlier consumption are relatively small. If asset returns are certain, the condition for efficiency depends on whether rates of return on assets that are held exceed the growth rate of the economy. If asset returns are uncertain, portfolio choice considerations imply that the certainty equivalent of the uncertain marginal rates of return on assets that are held must be the same for all assets; the condition for efficiency then depends on whether this certainty equivalent exceeds the population growth rate of the economy. If population growth rates themselves are given by a sequence of independent and identically distributed random variables, the condition for efficiency depends on whether the common certainty equivalent of the uncertain marginal rates of return on assets that are held exceeds the certainty equivalent of a fictitious asset with a rate of return from one period to the next that is equal to the population growth rate between those two periods.[5]

[4] As discussed in Bewley (1972), with infinitely many commodities, some form of discounting for commodities with "high index values", e.g., some form of discounting of "late" consumption, is implied by the requirement that the price system belong to the set of continuous linear functionals on the commodity space so that "values" of commodity vectors are well-defined and well-behaved.

[5] For proofs of these statement, see Hellwig (2021, 2023).

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