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On the Topological Protection of the Quantum Hall Effect in a Cavity

Vasil Rokaj,[1, 2, *] Jie Wang,[2, 3, †] John Sous,[4, 5] Markus Penz,[6] Michael Ruggenthaler,[7] and Angel Rubio[7, 8, ‡]

[1] *ITAMP, Center for Astrophysics | Harvard & Smithsonian, Cambridge, USA*
[2] *Department of Physics, Harvard University, Cambridge, USA*
[3] *Center of Mathematical Sciences and Applications, Harvard University, Cambridge, USA*
[4] *Department of Physics, Stanford University, Stanford, USA*
[5] *Stanford Institute for Theoretical Physics, Stanford University, Stanford, USA*
[6] *Basic Research Community for Physics, Innsbruck, Austria*
[7] *Max Planck Institute for the Structure and Dynamics of Matter, Hamburg, Germany*
[8] *Center for Computational Quantum Physics, Flatiron Institute, New York, USA*

We study the quantum Hall effect in a two-dimensional homogeneous electron gas coupled to a quantum cavity field. As initially pointed out by Kohn, Galilean invariance for a homogeneous quantum Hall system implies that the electronic center of mass (CM) decouples from the electron-electron interaction, and the energy of the CM mode, also known as Kohn mode, is equal to the single particle cyclotron transition. In this work, we point out that strong light-matter hybridization between the Kohn mode and the cavity photons gives rise to collective hybrid modes between the Landau levels and the photons. We provide the exact solution for the collective Landau polaritons and we demonstrate the weakening of topological protection at zero temperature due to the existence of the lower polariton mode which is softer than the Kohn mode. This provides an intrinsic mechanism for the recently observed topological breakdown of the quantum Hall effect in a cavity [Appugliese et al., Science 375, 1030-1034 (2022)]. Importantly, our theory predicts the cavity suppression of the thermal activation gap in the quantum Hall transport. Our work paves the way for future developments in the cavity control of quantum materials.

Interaction and topology give rise to rich exotic phases of matter, among which the integer quantum Hall (IQH) effect and the fractional quantum Hall (FQH) effect stand out [1–4]. On the other side, over the last decade, great progress has been achieved in the manipulation of quantum materials with the use of quantum fields originating from a cavity [5–15]. Specifically, for two-dimensional (2d) materials in magnetic fields, ultrastrong coupling of the Landau levels to the cavity field and the observation of Landau polariton quasiparticles have been achieved [16–20]. Recently, modifications of the magnetotransport properties inside a cavity due to Landau polaritons were reported [21, 22] and most significantly cavity modifications of the IQH transport was demonstrated [23, 24]. The experimental phenomena was argued to originate from a disorder-assisted cavity-mediated long-range hopping [25].

In this work, given that in experiments the GaAs samples have low disorder and that the cavity field is homogeneous in the bulk of the cavity [23], we study the quantum Hall system in the homogeneous limit with vanishing disorder and we propose an alternative theory for the observed cavity modified IQH transport [23]. Our theory highlights the importance of the hybridization between cavity photons and the collective Kohn mode in the quantum Hall system, and provides the exact solution for the polariton modes of the light-matter system. In connection to the experimental findings [23], our theory draws the picture that the transport in the hybrid system is strongly influenced by the polariton states, in contrast to the standard quantum Hall transport which is purely electronic. Crucially, the low energy physics is dictated by the lower polariton mode which is softer than the cyclotron mode. The softening of the cyclotron mode signals the weakened topological protection and provides an intrinsic mechanism for the recently observed topological breakdown [23]. Importantly, our theory predicts that the cavity suppresses the thermal activation gap which can be studied experimentally in the temperature dependence of the quantum Hall transport in the cavity.

Model Hamiltonian.—The model considers a two-dimensional electron gas coupled to a strong magnetic field and a single-mode homogeneous cavity field, as schematically depicted in Fig. 1(a). The system is described by the Pauli-Fierz Hamiltonian [26–28]

$$\hat{H}=\sum_{i=1}^{N}\frac{\left(\boldsymbol{\pi}_i+e\hat{\mathbf{A}}\right)^2}{2m_e}+\hbar\omega\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)+\sum_{i<j}W(\mathbf{r}_i-\mathbf{r}_j), \tag{1}$$

where $\boldsymbol{\pi}_i=\mathrm{i}\hbar\boldsymbol{\nabla}_i+e\mathbf{A}_{\mathrm{ext}}(\mathbf{r}_i)$ are the dynamical momenta of the electrons and $\mathbf{A}_{\mathrm{ext}}(\mathbf{r})=-\mathbf{e}_xBy$ describes the homogeneous magnetic field $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}_{\mathrm{ext}}(\mathbf{r})=B\mathbf{e}_z$. The cavity field $\hat{\mathbf{A}}=\sqrt{\frac{\hbar}{2\epsilon_0\mathcal{V}\omega}}\mathbf{e}_x\left(\hat{a}+\hat{a}^\dagger\right)$ is characterized by the in-plane polarization vector $\mathbf{e}_x$ and the photon's bare frequency ω. The $\mathcal{V}$ and ϵ_0 are the effective mode volume and the dielectric constant, respectively, and the ladder operators $\hat{a}$ and $\hat{a}^\dagger$ represent the bare photon fields. We have parameterized the bare electron dispersion by an effective mass term and assumed Galilean invariance. With Galilean invariance in a purely homogeneous system, the

* vasil.rokaj@cfa.harvard.edu
† jiewang.phy@gmail.com
‡ angel.rubio@mpsd.mpg.de

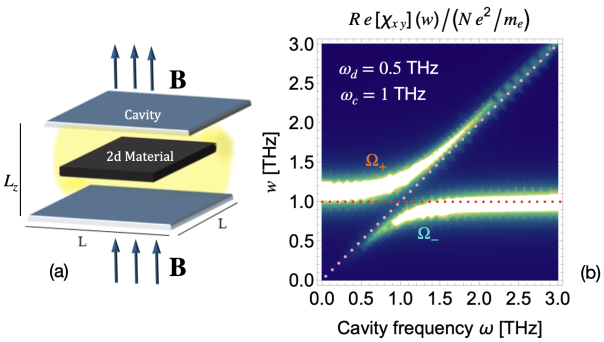

FIG. 1. (a) Two-dimensional material confined inside a cavity. The distance between the cavity mirrors is L_z while the whole system is placed perpendicular to a homogeneous magnetic field $\mathbf{B}$. (b) Real part of the linear current response function $\chi_{xy}(w)$ for a quantum Hall system coupled to a terahertz cavity. The light-matter coupling is controlled by the diamagnetic frequency ω_d which is the strong coupling regime $\omega_d = 0.5$THz, while the broadening parameter is chosen $\delta = 0.05$THz. We see the two hybrid modes, the upper Ω_+ and lower Ω_- polariton. Compared to the cyclotron mode $\omega_c = 1$THz, Ω_- is softer. This signals the weakened topological protection of the hybrid system.

CM is decoupled from the relative motion of the electrons, regardless of the interaction strength [29]. Note that besides Galilean invariance, our theory assumes a homogeneous cavity field. The kinetics of the CM and its coupling to light is best described in terms of the CM coordinate $\mathbf{R} = (X, Y) = \sum_{i=1}^{N} \mathbf{r}_i/\sqrt{N}$ where N is the total particle number. The Hamiltonian describing the coupling of the CM to light reads

$$\hat{H}_{\rm cm} = \frac{1}{2m_e}\left(\mathbf{\Pi} + e\sqrt{N}\hat{\mathbf{A}}\right)^2 + \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \qquad (2)$$

where $\mathbf{\Pi} = \mathrm{i}\hbar\nabla_{\mathbf{R}} + e\mathbf{A}_{\rm ext}(\mathbf{R})$ is the canonical momentum of the CM. It is important to mention that if we break either Galilean invariance or consider a spatially inhomogeneous cavity field, the relative degrees of freedom will couple to quantum light. The CM Hamiltonian has the form of two coupled harmonic oscillators, one for the Landau level transition and one for the photons. In many cases such a Hamiltonian is known as the Hopfield Hamiltonian which can be solved by the Hopfield transformation [30]. The Hopfield model has been employed in previous works for the description of single-particle Landau level transitions coupled to cavity photons [19, 22]. Here, it shows up for the collective coupling of the electrons which emerges naturally through the CM. After the Hopfield transformation we find

$$\hat{H}_{\rm cm} = \hbar\Omega_+\left(\hat{b}^\dagger_+\hat{b}_+ + \frac{1}{2}\right) + \hbar\Omega_-\left(\hat{b}^\dagger_-\hat{b}_- + \frac{1}{2}\right) \qquad (3)$$

where $\{\hat{b}^\dagger_\pm, \hat{b}_\pm\}$ are the creation and annihilation operators of the Landau polariton quasiparticles. We provide the details about the CM Hamiltoninan and its diagonalization in the Supplementary Material. The $\Omega_\pm$ are the upper and lower Landau polariton modes respectively,

$$\Omega^2_\pm = \frac{\omega^2 + \omega_d^2 + \omega_c^2}{2} \pm \sqrt{\omega_d^2\omega_c^2 + \left(\frac{\omega^2 + \omega_d^2 - \omega_c^2}{2}\right)^2} \qquad (4)$$

where $\omega_d = \sqrt{e^2N/m_e\epsilon_0\mathcal{V}}$ is the diamagnetic frequency originating from the $\hat{\mathbf{A}}^2$ which depends on the number of electrons N and the effective mode volume $\mathcal{V}$. To define the polariton operators we represent the photon annihilation operator in terms of a displacement coordinate q and its conjugate momentum as $\hat{a} = (q + \partial_q)/\sqrt{2}$, with $\hat{a}^\dagger$ obtained via conjugation [26, 27]. In this representation the polariton operators $\{\hat{b}_\pm, \hat{b}^\dagger_\pm\}$ can be written in terms of mixed, polaritonic coordinates as $S_\pm = \sqrt{\hbar/2\Omega_\pm}\left(\hat{b}_\pm + \hat{b}^\dagger_\pm\right)$ with

$$S_+ = \frac{\sqrt{m_e}\bar{Y} + q\Lambda\sqrt{\hbar/\omega}}{\sqrt{1+\Lambda^2}} \text{ and } S_- = \frac{-q\sqrt{\hbar/\omega} + \sqrt{m_e}\Lambda\bar{Y}}{\sqrt{1+\Lambda^2}}$$

where $\bar{Y} = Y + \frac{\hbar K_x}{eB}$ is the guiding center and K_x is the electronic wave number in the x-direction. Also we introduced the parameter $\Lambda = \alpha - \sqrt{1+\alpha^2}$ with $\alpha = \left(\omega_c^2 - \omega^2 - \omega_d^2\right)/2\omega_d\omega_c$ which quantifies the mixing between electronic and photonic degrees of freedom.

Behavior of polaritons.—The Landau polariton modes $\Omega_\pm$ depend on the cavity frequency ω and the number of electrons through the diamagnetic frequency ω_d. The behavior of the polariton modes $\Omega_\pm$ as a function of the cavity frequency can be understood from their exact expressions Eq. (4) also shown in Fig. 1(b). Before the avoided crossing the upper polariton Ω_+ follows the cyclotron frequency ω_c while the lower polariton the bare cavity mode ω. After the avoided crossing the situation is inverted. At the resonance point $\omega = \omega_c$ the two modes are separated by the Rabi splitting $\Omega_R = \Omega_+ - \Omega_-$ which is approximately proportional to the diamagnetic frequency $\Omega_R \approx \omega_d$. The lower polariton is the most important one for the low energy physics of the system and we will show that its behavior controls the transport properties of the 2d quantum Hall system. Approaching the limit $\omega \to 0$, the lower polariton mode becomes gapless reproducing the result in Refs. [15, 20]. In addition, Ω_- decreases as a function of the light-matter coupling strength, controlled via the diamagnetic frequency ω_d, i.e., $\Omega_- < \omega_c$ when $\omega_d > 0$. In what follows, we discuss the implications of the polariton states for the quantum Hall transport at zero and finite temperature.

Fragility of topological protection against polariton lifetimes and ultrastrong light-matter coupling.—A clean or weakly disordered quantum Hall system at zero temperature, $T = 0$, as long as it is gapped, is expected to be topologically protected [31]. However, the softening of the cyclotron mode, due to the emergence of the lower polariton, indicates that the topological protection of the hybrid system is weakened by the light-matter coupling.

Due to the gap reduction of the lower polariton the transport of the system can be more easily affected by disorder, which leads to a finite lifetime for the polariton quasiparticles. The polariton lifetimes will be included phenomenologically and we will see that their effect combined with ultrastrong coupling enables the breakdown of topological protection [23, 24].

The gauge-invariant current operator in the case of homogeneous fields solely depends on the CM canonical momentum and the cavity field [15, 32] $\hat{\mathbf{J}} = -\frac{e\sqrt{N}}{m_e}\left(\mathbf{\Pi} + e\sqrt{N}\hat{\mathbf{A}}\right)$. Due to this property and the separability of $\hat{H}_{\text{cm}}$ from the electronic correlations we can compute the transport of the system by focusing only on the states of $\hat{H}_{\text{cm}}$. At $T = 0$ the system is in the polariton vacuum $|\Psi_{\text{gs}}\rangle = |0_+\rangle|0_-\rangle$ which is annihilated by both polariton operators $\hat{b}_\pm$. Given this state, we employ the standard Kubo formalism [33] for the computation of the current correlators $\chi_{ab}(t) = -\mathrm{i}\Theta(t)\langle\Psi_{\text{gs}}|[\hat{J}_a(t), \hat{J}_b]|\Psi_{\text{gs}}\rangle/\hbar$ in the time domain which we can transform to the frequency domain in order to obtain the optical conductivities [33] $\sigma_{ab}(w) = \frac{\mathrm{i}}{w+\mathrm{i}\delta}\left(\frac{e^2 n_{2d}}{m_e}\delta_{ab} + \frac{\chi_{ab}(w)}{A}\right)$ where A is the area of the 2d material, δ is the broadening parameter, and δ_{ab} the Kronecker delta with $a, b \in \{x, y\}$. The full details for the transport computations are provided in the Supplementary Material. The poles of the linear response functions $\chi_{ab}(w)$ identify the optical responses of the system and its excitations which are shown in Fig. 1(b). In addition, using the Kubo formula we find the Hall and longitudinal DC ($w = 0$) conductivities

$$\sigma_{xy} = \frac{e^2\nu}{h(1+\Lambda^2)}\left[\frac{\Lambda(\Lambda+\eta)}{\Omega_-^2/\omega_c^2 + \delta^2/\omega_c^2} + \frac{1-\eta\Lambda}{\Omega_+^2/\omega_c^2 + \delta^2/\omega_c^2}\right]$$
$$\sigma_{yy} = \sigma_D\left[1 - \frac{1}{1+\Lambda^2}\left(\frac{\Omega_+^2}{\Omega_+^2+\delta^2} + \frac{\Lambda^2\Omega_-^2}{\Omega_-^2+\delta^2}\right)\right] \qquad (5)$$

where $\eta = \omega_d/\omega_c$. We note that $\sigma_D = e^2 n_{2d}/m_e\delta$ is the Drude DC conductivity, and that in the expression for the Hall conductance we introduced the Landau level filling factor $\nu = n_{2d}h/eB$ [34, 35]. Taking the value of the broadening parameter to zero $\delta \to 0$ we find that the Hall conductance is quantized $\sigma_{xy} = e^2\nu/h$, consistent with the Thouless flux insertion argument [31]. In the last step we used two properties of the mixing parameter $1 - \eta\Lambda = \Omega_+^2/\omega_c^2$ and $\Lambda(\Omega_-^2/\omega_c^2 - 1) = \eta$ which can be exactly deduced from the definition of Λ.

The polariton lifetimes are responsible for the broadening in the transmission spectra observed experimentally in quantum Hall systems coupled to cavities [17, 18, 23]. The total lifetime is a result of several mechanisms, scattering by impurities in the material, radiative decay due to interaction with the electromagnetic vacuum [27], coupling to phonons, as well as to the substrate. Here, we phenomenologically model the polariton lifetime as $\tau = 1/\delta$ by keeping a finite broadening δ which enables to model the experimental optical spectra as for example in Fig. 1(b).

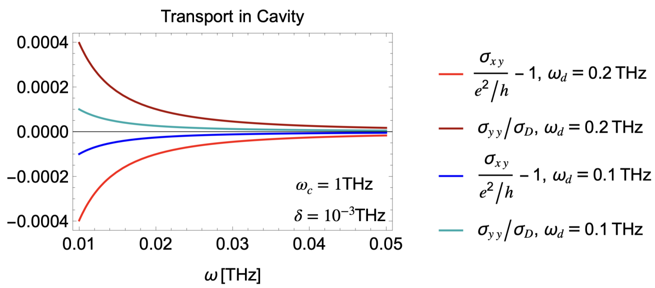

FIG. 2. Quantum Hall transport in a cavity at $T = 0$ with a finite broadening $\delta = 10^{-3}$THz which models the finite lifetime of the polariton excitations. For ultrastrong light-matter coupling $\omega_d = 0.1 - 0.2$THz and for a small cavity frequency, ω, when compared to the cyclotron mode $\omega_c = 1$THz, we see that the Hall and the longitudinal conductivities deviate from the topologically expected values, 1 and 0, respectively.

Motivated by the experimental measurements in Refs. [18, 21, 23] we choose $\delta = 10^{-3}$THz and in Fig.2 we plot σ_{xy} and σ_{yy} under ultrastrong light-matter coupling, which is quantified by the diamagnetic frequency ω_d. In the regime where the cavity frequency is much smaller than the cyclotron frequency, we see that $\sigma_{xy}/(e^2/h)$ deviates from unity and at the same time σ_{yy} deviates from zero. Both phenomena signal the breakdown of topological protection. The deviations from the expected values occur off-resonance, for a small cavity frequency, because in this regime the lower polariton gap Ω_- is significantly reduced (see Fig. 1(b)). Importantly, in Fig. 2 we see that the effects on the transport get significantly enhanced as we increase ω_d from 0.1THz to 0.2THz. This demonstrates that it is the interplay between the ultrastrong light-matter coupling and the finite polariton lifetime that causes the effects on transport. This intuitive physical picture is in agreement with the observed breakdown of topological protection recently reported in Ref [23], and the disorder-assisted cavity-mediated hopping mechanism [25].

It is worth to mention that the above analysis is consistent with the result in the long-wavelength limit (also known as optical limit) $\omega \to 0$ and $\delta = 0$ [36]. The Hall conductivity for $\delta = 0$ is perfectly quantized for all $\omega > 0$ (finite energy gap) but drops to $e^2\nu/h/(1+\eta^2)$ for $\omega \to 0$ (gapless) as it was shown in Ref. [36]. This is the point where the canonical transformation to the polariton basis becomes singular. In this sense, the phenomenological broadening parameter δ is a physical way to regularize the optical limit result.

Cavity suppression of the thermal activation gap.— In addition, finite temperature transport properties are strongly influenced by coupling the electrons to the cavity field. This can be understood from the formula for the thermal behavior of the longitudinal transport $\sigma_{yy}(T)/\sigma_{yy}(T=0) \approx \exp(-\beta\Delta)$ where Δ is the activation gap of the system and $\beta = kT$. For the light-matter coupled system, $\Delta = \Omega_-$. Thereby for the IQH effect, the coupling to the cavity generally speaking reduces the activation gap from the bare Landau level gap ω_c to Ω_- and makes the Hall transport easier to be modified by temperature.

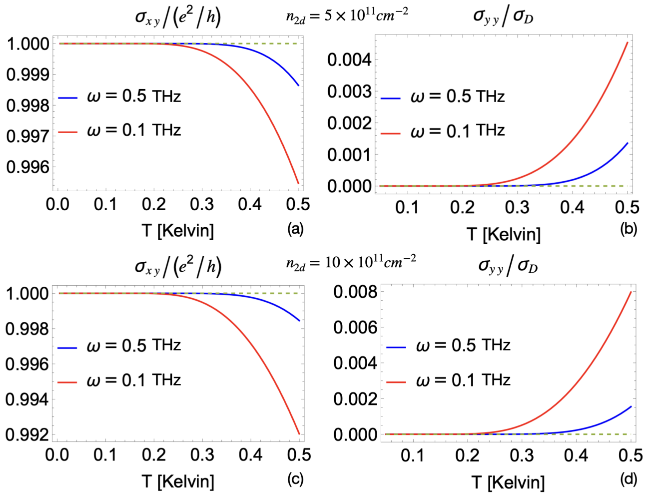

FIG. 3. Low temperature transport of the quantum Hall system at magnetic field strength $B = 1T$ coupled to a cavity, for different values of the light-matter coupling. In (a) and (b) $n_{2d} = 5\times10^{11}\text{cm}^{-2}$ while in (c) and (d) $n_{2d} = 10\times10^{11}\text{cm}^{-2}$. We see that the light-matter coupling strongly affects the quantum Hall transport. For the smaller cavity frequency $\omega = 0.1$THz and the larger electron density the deviation from the topologically protected values maximizes. This relates to the behavior of lower polariton mode Ω_- which controls the thermal activation in the system. The broadening parameter is chosen very small $\delta = 10^{-4}$THz in order to guarantee numerical convergence without influencing transport.

The quantitative description of the thermal activation gap can be obtained from the finite temperature linear-response Kubo formula [33], through $\chi_{\alpha\beta}(\omega)$ which is the retarded current-current correlation function,

$$\chi_{ab}(w) = \sum_{M,Q} \frac{e^{-\beta E_M} - e^{-\beta E_Q}}{\mathcal{Z}} \frac{\langle\Psi_M|\hat{J}_a|\Psi_Q\rangle\langle\Psi_Q|\hat{J}_b|\Psi_M\rangle}{w + (E_M - E_Q)/\hbar + \mathrm{i}\delta} \tag{6}$$

where $|\Psi_M\rangle, |\Psi_Q\rangle$ are the many-body states with eigenenergies E_M, E_Q respectively, and $\mathcal{Z}$ is the partition function $\mathcal{Z} = \sum_M \exp(-\beta E_M)$. The dominant contribution for the transport is from the ground-state, and the next leading order contribution is from the first excited state. The activation gap is accounted by the exponential suppression of the contribution to conductivity from the excited states. Since the lowest excitation in the system is Ω_- (see Fig. 1) we expect the lower polariton to control the low temperature transport of the system. The details of the temperature dependent transport formalism are given in the Supplementary Material.

For the temperature dependent computations presented in Fig. 3 we use parameters in the same regime as the ones reported experimentally in Ref. [23]. The magnetic field strength is chosen $B = 1T$ where a quantum Hall plateau is reported in Ref. [23], and the cavity is in the terahertz regime, as in the experiments. For the geometry we consider in Fig.1, the diamagnetic frequency ω_d, which controls the light-matter hybridization, can be estimated through the electron density n_{2d} as $\omega_d = \sqrt{e^2N/m_e\epsilon_0\mathcal{V}} = \sqrt{e^2n_{2d}\omega/\pi c m_e\epsilon_0}$ where we used also the expression for the fundamental cavity frequency $\omega = \pi c/L_z$ [20]. Here, the electron density is in the range $n_{2d} = 5 - 10\times10^{11}\text{cm}^{-2}$ in accordance with experimentally reported values [23].

Figure 3 demonstrates that indeed the transport properties of the quantum Hall system can be modified by coupling strongly to the cavity field. From the behavior of both conductivities it is evident that the dependence of transport on temperature is enhanced for the lower cavity frequency $\omega = 0.1$THz. This is can be directly connected to gap reduction in the system as the lower polariton Ω_- takes a smaller value for a smaller cavity frequency. In addition to this important finding we see that the temperature effect is also enhanced by the electron density n_{2d} by comparing Figs.3 (a) and (b) to Figs.3 (c) and (d). This is to be expected since the electron density is crucial for the light-matter coupling strength as it controls the diamagnetic frequency ω_d.

Connections to Experiments and Future Directions.— The above analysis suggests that the activation gap of the hybrid system is strongly suppressed by coupling to cavity modes. Importantly, our model enables the theoretical estimate of the activation gap and direct comparison to experiment. It is an interesting prospect to test experimentally our prediction that the activation gap should follow the lower polariton excitation.

Further, we comment on the FQH effect which is stabilized by electron-electron interactions. In samples with low disorder, the activation gap of the FQH effect is given by the many-body gap, typically determined by the magneto-roton energy [37], which we assume it to be smaller than Ω_- and therefore protected from the cavity induced phenomena. This picture is consistent with the experimental observations that FQH plateaus are much less modified by the cavity in comparison to the integer ones [23]. From this analysis we anticipate that the FQH effect starts to be modified at low temperature when the lower polariton becomes softer than the many-body gap.

To summarize, using a Galilean invariant quantum Hall model coupled to a homogeneous single-mode cavity field, we are able to provide the exact solution for the hybrid, polariton states and discuss their experimental implications for quantum Hall transport in cavities. We find that the lower polariton Ω_- is significantly softer than the bare cyclotron mode and leads to the weakening of the topological protection of the hybrid system. Thus, our theory provides an intrinsic mechanism for the recently observed breakdown of the topological protection of the IQH effect due to cavity vacuum fluctuations [23]. In addition our theory predicts that the modification of transport by temperature is enhanced by the cavity, as the thermal activation gap is suppressed due to the lower polariton mode. Having understood analytically the homogeneous setting, our work paves the way for future investigations going beyond the homogeneous approximation for the cavity field, such that the interaction between the CM polariton modes with the electron-electron correlations comes into play. This could potentially lead to polariton-induced topological order and novel correlated phases between light and matter. Another important direction to be taken is the inclusion of disorder and impurity scattering in our model which will be crucial for a more precise understanding of transport in materials under strong coupling. Finally, incorporating leakage and the multimode structure of the cavity will enable a more realistic description of transport phenomena in complex electromagnetic environments [38].

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SUPPLEMENTARY MATERIAL

I. PAULI-FIERZ HAMILTONIAN IN THE CENTER OF MASS FRAME

In this section we would like to give the details about the transformation of the Pauli-Fierz Hamiltonian in the center of mass (CM) and relative distances frame. The Hamiltonian of our system is

$$\hat{H} = \frac{1}{2m_{\mathrm{e}}}\sum_{i=1}^{N}\left(\boldsymbol{\pi}_i + e\hat{\mathbf{A}}\right)^2 + \sum_{i<l}^{N} W\left(|\mathbf{r}_i - \mathbf{r}_l|\right) + \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \tag{7}$$

where $\boldsymbol{\pi}_i = \mathrm{i}\hbar\nabla_i + e\mathbf{A}_{\mathrm{ext}}(\mathbf{r}_i)$ are the canonical momenta, and $\mathbf{A}_{\mathrm{ext}}(\mathbf{r}) = -\mathbf{e}_x By$ describes the homogeneous magnetic field $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}_{\mathrm{ext}}(\mathbf{r}) = B\mathbf{e}_z$. The cavity field $\hat{\mathbf{A}} = \sqrt{\frac{\hbar}{2\epsilon_0\mathcal{V}\omega}}\mathbf{e}_x\left(\hat{a}+\hat{a}^\dagger\right)$ is characterized by the in-plane polarization vector $\mathbf{e}_x$ and the photon's bare frequency ω. For mathematical convenience we utilize a symmetric definition with respect to $\sqrt{N}$ for the coordinates in the CM frame as in Ref. [39]

$$\mathbf{R} = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\mathbf{r}_i \ \text{ and } \ \widetilde{\mathbf{r}}_j = \frac{\mathbf{r}_1 - \mathbf{r}_j}{\sqrt{N}} \ \text{ with } \ j > 1. \tag{8}$$

The original electronic coordinates in terms of the new ones $\{\mathbf{R}, \widetilde{\mathbf{r}}_j\}$ are

$$\mathbf{r}_1 = \frac{1}{\sqrt{N}}\left(\mathbf{R} + \sum_{j=2}^{N}\widetilde{\mathbf{r}}_j\right) \ \text{ and } \mathbf{r}_j = \frac{1}{\sqrt{N}}\left(\mathbf{R} + \sum_{j=2}^{N}\widetilde{\mathbf{r}}_j\right) - \sqrt{N}\widetilde{\mathbf{r}}_j \ \text{ with } \ j > 1.$$

The momenta of the electrons in the new coordinate system are $\nabla_1 = \left(\nabla_{\mathbf{R}} + \sum_{j=2}^{N}\widetilde{\nabla}_j\right)/\sqrt{N}$ and $\nabla_j = \left(\nabla_{\mathbf{R}} - \widetilde{\nabla}_j\right)/\sqrt{N}$ with $j > 1$. From these expression we can find the form of the electronic kinetic terms in the new frame

$$\sum_{i=1}^{N}\nabla_i^2 = \nabla_{\mathbf{R}}^2 + \frac{1}{N}\sum_{j=2}^{N}\widetilde{\nabla}_j^2 + \frac{1}{N}\sum_{j,k=2}^{N}\widetilde{\nabla}_j\cdot\widetilde{\nabla}_k \ \text{ and } \ \sum_{i=1}^{N}\nabla_i = \sqrt{N}\nabla_{\mathbf{R}}. \tag{9}$$

The interaction term between the cavity field and the electrons takes the form

$$\hat{\mathbf{A}}\cdot\sum_{i=1}^{N}\mathrm{i}\hbar\nabla_i + e\mathbf{A}_{\mathrm{ext}}(\mathbf{r}_i) = \sqrt{N}\hat{\mathbf{A}}\cdot\left(\mathrm{i}\hbar\nabla_{\mathbf{R}} + e\mathbf{A}_{\mathrm{ext}}(\mathbf{R})\right) \tag{10}$$

To complete our analysis we also give the expression for the purely electronic terms in the new frame. For the quadrature of the external magnetic field we have

$$\sum_{i=1}^{N}\mathbf{A}_{\mathrm{ext}}^2(\mathbf{r}_i) = \mathbf{A}_{\mathrm{ext}}^2(\mathbf{R}) + N\sum_{j=2}^{N}\mathbf{A}_{\mathrm{ext}}^2(\widetilde{\mathbf{r}}_j) - \left[\sum_{j=2}^{N}\mathbf{A}_{\mathrm{ext}}(\widetilde{\mathbf{r}}_j)\right]^2 \tag{11}$$

and for the bilinear term between the magnetic field and the momenta we have

$$\sum_{i=1}^{N}\mathbf{A}_{\mathrm{ext}}(\mathbf{r}_i)\cdot\nabla_i = \mathbf{A}_{\mathrm{ext}}(\mathbf{R})\cdot\nabla_{\mathbf{R}} + \sum_{j=2}^{N}\mathbf{A}_{\mathrm{ext}}(\widetilde{\mathbf{r}}_j)\cdot\widetilde{\nabla}_j. \tag{12}$$

Finally, we give the expression for the interaction term $\hat{W}$ between the electrons.

$$\sum_{i<l}^{N} W\left(|\mathbf{r}_i - \mathbf{r}_l|\right) = \sum_{1<l}^{N} W(\sqrt{N}|\widetilde{\mathbf{r}}_l|) + \sum_{2\le i<l}^{N} W\left(\sqrt{N}|\widetilde{\mathbf{r}}_i - \widetilde{\mathbf{r}}_l|\right)$$

Adding together all the different terms we find that the expression of the Hamiltonian in the new frame is the sum of two parts: (i) the center of mass part $\hat{H}_{\text{com}}$ which is coupled to the quantized field $\hat{\mathbf{A}}$ and (ii) the relative distances $\hat{H}_{\text{rel}}$ which does not couple to the cavity field, $\hat{H} = \hat{H}_{\text{cm}} + \hat{H}_{\text{rel}}$ where each part looks as

$$\hat{H}_{\text{cm}} = \frac{1}{2m_e}\left(\mathrm{i}\hbar\nabla_{\mathbf{R}} + e\mathbf{A}_{\text{ext}}(\mathbf{R}) + e\sqrt{N}\hat{\mathbf{A}}\right)^2 + \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right),$$

$$\hat{H}_{\text{rel}} = \frac{1}{2m_e}\sum_{j=2}^{N}\left(\frac{\mathrm{i}\hbar}{\sqrt{N}}\widetilde{\nabla}_j + e\sqrt{N}\mathbf{A}_{\text{ext}}(\widetilde{\mathbf{r}}_j)\right)^2 - \frac{\hbar^2}{2m_eN}\sum_{j,l=2}^{N}\widetilde{\nabla}_j\cdot\widetilde{\nabla}_l - \frac{e^2}{2m_e}\left(\sum_{j=2}^{N}\mathbf{A}_{\text{ext}}(\widetilde{\mathbf{r}}_j)\right)^2 + \hat{W}. \tag{13}$$

Lastly, it is important to demonstrate that the center of mass and relative distances degrees of freedom are independent by checking the commutation relations between their coordinates and momenta. Using the chain rule we have for the derivatives of the coordinates in the center of mass frame

$$\nabla_{\mathbf{R}} = \sum_{i=1}^{N}\frac{\partial\mathbf{r}_i}{\partial\mathbf{R}}\nabla_i = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\nabla_i, \quad \text{and} \quad \widetilde{\nabla}_j = \sum_{i=1}^{N}\frac{\partial\mathbf{r}_i}{\partial\widetilde{\mathbf{r}}_j}\nabla_i = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\nabla_i - \sqrt{N}\nabla_j \text{ with } j>1.$$

From the above expressions it is clear that the momenta in the new coordinate frame commute

$$\left[\nabla_{\mathbf{R}}, \widetilde{\nabla}_j\right] = 0. \tag{14}$$

The next property to check is the commutation relations between the momenta and the coordinates.

$$[\nabla_{\mathbf{R}}, \widetilde{\mathbf{r}}_j] = \frac{1}{N}\left[\sum_{i=1}^{N}\nabla_i, \mathbf{r}_1 - \mathbf{r}_j\right] = 0, \quad j>1, \tag{15}$$

$$\begin{aligned}\left[\widetilde{\nabla}_j, \mathbf{R}\right] &= \left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\nabla_i - \sqrt{N}\nabla_j, \frac{1}{\sqrt{N}}\sum_{l=1}^{N}\mathbf{r}_l\right] = \\ &= \frac{1}{N}\sum_{i,l=1}^{N}[\nabla_i, \mathbf{r}_l] - \sum_{l=1}^{N}[\nabla_j, \mathbf{r}_l] = \frac{1}{N}\sum_{i,l=1}^{N}\delta_{il} - \sum_{l=1}^{N}\delta_{jl} = \frac{N}{N} - 1 = 0.\end{aligned} \tag{16}$$

We would also like to mention that the separation between the CM and the relative distances holds true also for an arbitrary amount of photon modes as long as the cavity field is considered to be homogeneous as it was shown in [40].

II. EXACT SOLUTION OF THE CM HAMILTONIAN AND LANDAU POLARITONS

Having demonstrated that only the CM of the electronic system couples to the cavity we will show that $\hat{H}_{\text{cm}}$ can be solved analytically. Let us see how this can be done. To proceed we expand the covariant kinetic term

$$\hat{H}_{\text{cm}} = \frac{\mathbf{\Pi}^2}{2m_e} + \frac{e\sqrt{N}}{m_e}\hat{\mathbf{A}}\cdot\mathbf{\Pi} + \underbrace{\frac{e^2N\hat{\mathbf{A}}^2}{2m_e} + \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)}_{\hat{H}_p} \tag{17}$$

For the description of the photon operators we will introduce the displacement coordinate q and its conjugate momentum ∂_q as $\hat{a} = \frac{1}{\sqrt{2}}(q + \partial/\partial q)$ and $\hat{a}^\dagger$ defined by conjugation [26, 41]. The part $\hat{H}_p$ can be brought to diagonal form by the scaling transformation on the photonic displacement coordinate

$$u = q\sqrt{\frac{\widetilde{\omega}}{\omega}} \text{ where } \widetilde{\omega} = \sqrt{\omega^2 + \omega_d^2} \tag{18}$$

with $\omega_d = \sqrt{e^2N/\epsilon_0 m_e\mathcal{V}}$ is the diamagnetic frequency depending on the electron density in the effective mode volume. After this transformation the CM Hamiltonian is

$$\hat{H}_{\text{cm}} = \frac{\mathbf{\Pi}^2}{2m_e} + \frac{e\sqrt{N}}{m_e}\hat{\mathbf{A}}\cdot\mathbf{\Pi} + \frac{\hbar\widetilde{\omega}}{2}\left(-\frac{\partial^2}{\partial u^2} + u^2\right), \tag{19}$$

where the quantized field is now

$$\hat{\mathbf{A}} = \sqrt{\frac{\hbar}{\epsilon_0 V \widetilde{\omega}}}\mathbf{e}_x u. \tag{20}$$

In the Landau gauge the Hamiltonian has translational invariance along the X coordinate which implies that the eigenfunctions in X are plane waves $e^{\mathrm{i}K_x X}$. We apply $\hat{H}_{\mathrm{cm}}$ on the plane wave and we have

$$\hat{H}_{\mathrm{cm}} = -\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial Y^2} + \frac{m_e\omega_c^2}{2}\left(Y + \frac{\hbar K_x}{eB}\right)^2 - geBu\left(Y + \frac{\hbar K_x}{eB}\right) + \frac{\hbar\widetilde{\omega}}{2}\left(-\frac{\partial^2}{\partial u^2} + u^2\right)$$

where we also introduced the coupling constant $g = \omega_d\sqrt{\hbar/m_e\widetilde{\omega}}$. As a next step we define the coordinate

$$\bar{Y} = Y + \frac{\hbar K_x}{eB} \tag{21}$$

and the Hamiltonian simplifies further

$$\hat{H}_{\mathrm{cm}} = -\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial \bar{Y}^2} + \frac{m_e\omega_c^2}{2}\bar{Y}^2 - geBu\bar{Y} + \frac{\hbar\widetilde{\omega}}{2}\left[-\frac{\partial^2}{\partial u^2} + u^2\right]$$

The Hamiltonian consists of two coupled harmonic oscillators. It is convenient to perform another scaling transformation on $\bar{Y}$ and u

$$V_- = -u\sqrt{\frac{\hbar}{\widetilde{\omega}}} \text{ and } V_+ = \sqrt{m_e}\bar{Y}. \tag{22}$$

such that we have both harmonic oscillators in the form of having mass equal to 1. The Hamiltonian then becomes

$$\hat{H}_{\mathrm{cm}} = -\frac{\hbar^2}{2}\sum_{l=\pm}\frac{\partial^2}{\partial V_l^2} + \frac{1}{2}\sum_{l,j=\pm} W_{lj}V_lV_j. \tag{23}$$

The matrix W

$$W = \begin{pmatrix} \omega_c^2 & \omega_d\omega_c \\ \omega_d\omega_c & \widetilde{\omega}^2 \end{pmatrix} \tag{24}$$

is real and symmetric, and as a consequence can be diagonalized by the orthogonal matrix O [42],

$$O = \begin{pmatrix} \frac{1}{\sqrt{1+\Lambda^2}} & \frac{\Lambda}{\sqrt{1+\Lambda^2}} \\ -\frac{\Lambda}{\sqrt{1+\Lambda^2}} & \frac{1}{\sqrt{1+\Lambda^2}} \end{pmatrix} \quad \text{where } \Lambda = \alpha - \sqrt{1+\alpha^2} \quad \text{and} \quad \alpha = \frac{\omega_c^2 - \widetilde{\omega}^2}{2\omega_d\omega_c}.$$

The eigenvalues of the matrix W give the new normal modes of the interacting light-matter system. We find them to be

$$\Omega_\pm^2 = \frac{1}{2}\left(\widetilde{\omega}^2 + \omega_c^2 \pm \sqrt{4\omega_d^2\omega_c^2 + (\widetilde{\omega}^2 - \omega_c^2)^2}\right). \tag{25}$$

The Hamiltonian after the orthogonal transformation takes the canonical form

$$\hat{H}_{\mathrm{cm}} = -\frac{\hbar^2}{2}\sum_{l=\pm}\frac{\partial^2}{\partial S_l^2} + \frac{1}{2}\sum_{l=\pm}\Omega_l^2 S_l^2. \tag{26}$$

The new coordinates S_l and conjugate momenta ∂_{S_l} are related to the old ones $\{V_l, \partial_{V_l}\}$ through the orthogonal matrix O,

$$S_l = \sum_{j=\pm} O_{jl}V_j \text{ and } \frac{\partial}{\partial S_l} = \sum_{j=\pm} O_{jl}\frac{\partial}{\partial V_j}. \tag{27}$$

Due to the fact that the matrix O is orthogonal the canonical commutation relations are satisfied which implies that we have two independent harmonic oscillators [42]. Thus, the eigenfunctions of the interacting system are Hermite functions Φ of the coordinates S_+ and S_-. The full set of eigenfunctions of the system is

$$\Psi_{K_x,n_+,n_-}(X, S_+, S_-) = e^{\mathrm{i}K_x X}\Phi_{n_+}(S_+)\Phi_{n_-}(S_-) \tag{28}$$

with eigenspectrum

$$E_{n_+,n_-} = \hbar\Omega_+ \left(n_+ + \frac{1}{2}\right) + \hbar\Omega_- \left(n_- + \frac{1}{2}\right). \tag{29}$$

The frequencies Ω_+ (upper) and Ω_- (lower) are the two collective Landau polariton modes of the quantum Hall system in the cavity. For completeness, we note that the solution of the polaritons for the CM can be equivalently written in terms of annihilation $\hat{b}_\pm$ and creation $\hat{b}^\dagger_\pm$ operators for the polariton quasiparticles. In this representation $\hat{H}_{\rm cm}$ is written as

$$\hat{H}_{\rm cm} = \hbar\Omega_+ \left(\hat{b}^\dagger_+\hat{b}_+ + \frac{1}{2}\right) + \hbar\Omega_- \left(\hat{b}^\dagger_-\hat{b}_- + \frac{1}{2}\right) \tag{30}$$

with the polariton operators defined $\hat{b}_\pm = S_\pm\sqrt{\frac{\Omega_\pm}{2}} + \sqrt{\frac{1}{2\Omega_\pm}}\partial_{S_\pm}$ [41]. It is worth to notice that in the optical limit $\omega \to 0$ the lower polariton frequency goes to zero, $\Omega_- \to 0$, and the canonical transformation from the electron and photon basis $V_\pm$ to the polariton basis $S_\pm$ becomes singular.

III. FINITE TEMPERATURE TRANSPORT

In this section we present the general formalism employed for the finite temperature transport of the light-matter system. As we already showed the Hamiltonian of our system can be written as a sum of a CM and relative part $\hat{H} = \hat{H}_{\rm cm} + \hat{H}_{\rm rel}$. To proceed we assume that the eigenstates of $\hat{H}_{\rm cm}$ are $|\Phi_n\rangle$ and the eigenstates of $\hat{H}_{\rm rel}$ are $|F_I\rangle$ such that it holds

$$\hat{H}_{\rm cm}|\Phi_n\rangle = E_n|\Phi_n\rangle \;\; \text{and} \;\; \hat{H}_{\rm rel}|F_I\rangle = E_I|F_I\rangle \tag{31}$$

Then, the eigenstates of the full Hamiltonian $\hat{H}$ are

$$|\Psi_{nI}\rangle = |\Phi_n\rangle \otimes |F_I\rangle, \tag{32}$$

and the full eigenspectrum is $E_{nI} = E_n + E_I$. The Kubo formula for the optical conductivity of the system is [33, 43]

$$\sigma_{ab}(w) = \frac{\mathrm{i}}{w + \mathrm{i}\delta}\left(\frac{e^2 n_e}{m_e}\delta_{ab} + \frac{\chi_{ab}(w)}{A}\right) \quad \delta \to 0^+ \tag{33}$$

where $a, b = x, y, z$. The first term in the optical conductivity is the Drude term, while the second term is the current-current correlator in the frequency domain, which is defined as the Fourier transform of current-current correlator in the time domain

$$\chi_{ab}(t) = \frac{-\mathrm{i}\Theta(t)}{\hbar}\langle[\hat{J}_a(t), \hat{J}_b]\rangle, \tag{34}$$

with the current operators considered in the interaction picture $\mathbf{\hat{J}}(t) = e^{\mathrm{i}Ht/\hbar}\mathbf{\hat{J}}e^{-\mathrm{i}\hat{H}t/\hbar}$ [33]. In the canonical ensemble the expectation value of an operator $\hat{\mathcal{O}}$ is defined as [43]

$$\langle\hat{\mathcal{O}}\rangle = Tr\{\hat{\rho}\hat{\mathcal{O}}\} = \frac{1}{\mathcal{Z}}\sum_{n,I}\langle\Psi_{nI}|e^{-\beta\hat{H}}\hat{\mathcal{O}}|\Psi_{nI}\rangle \tag{35}$$

where the partition function is $\mathcal{Z} = \sum_{n,I} e^{-\beta E_n}e^{-\beta E_I}$. We will use these formulas now for the computation of the current correlation functions. The current response can be splitted into two parts

$$\chi_{ab}(t) = \frac{-\mathrm{i}\Theta(t)}{\hbar}\left(\langle\hat{J}_a(t)\hat{J}_b\rangle - \langle\hat{J}_b\hat{J}_a(t)\rangle\right). \tag{36}$$

Let us compute first the first term $\langle\hat{J}_a(t)\hat{J}_b\rangle$. We use the expression for the canonical ensemble and for the current operator in the interaction picture and we have

$$\begin{aligned}\langle\hat{J}_a(t)\hat{J}_b\rangle &= \frac{1}{\mathcal{Z}}\sum_{n,I} e^{-\beta E_{nI}}\langle\Psi_{nI}|e^{\mathrm{i}Ht/\hbar}\hat{J}_a e^{-\mathrm{i}\hat{H}t/\hbar}\hat{J}_b|\Psi_{nI}\rangle \\ &= \frac{1}{\mathcal{Z}}\sum_{n,I} e^{-\beta E_{nI}}e^{\mathrm{i}tE_{nI}/\hbar}\langle\Psi_{nI}|\hat{J}_a e^{-\mathrm{i}\hat{H}t/\hbar}\hat{J}_b|\Psi_{nI}\rangle.\end{aligned} \tag{37}$$

We introduce the identity $\mathbb{I} = \sum_{m,J} |\Psi_{mJ}\rangle\langle\Psi_{mJ}|$ in the above expression

$$\begin{aligned}\langle \hat{J}_a(t)\hat{J}_b\rangle &= \frac{1}{\mathcal{Z}} \sum_{n,m,J,I} e^{-\beta E_{nI}} e^{\mathrm{i}tE_{nI}/\hbar} \langle\Psi_{nI}|\hat{J}_a e^{-\mathrm{i}\hat{H}t/\hbar}|\Psi_{mJ}\rangle\langle\Psi_{mJ}|\hat{J}_b|\Psi_{nI}\rangle \\ &= \frac{1}{\mathcal{Z}} \sum_{n,m,J,I} e^{-\beta E_{nI}} e^{\mathrm{i}t(E_{nI}-E_{mJ})/\hbar} \langle\Psi_{nI}|\hat{J}_a|\Psi_{mJ}\rangle\langle\Psi_{mJ}|\hat{J}_b|\Psi_{nI}\rangle \end{aligned} \tag{38}$$

A. Current in the CM frame

Since we work in the CM frame in order to proceed we need examine how the current operator looks in the CM frame. The expression for the current operator can be obtained by computing the velocity operator of the electrons through the Heisenberg equation of motion [32]

$$\mathbf{v}_i = \frac{d\mathbf{r}_i}{dt} = \frac{\mathrm{i}}{\hbar}[\hat{H}, \mathbf{r}_i] = \frac{1}{m_e}\left(-\mathrm{i}\hbar\nabla_i - e\mathbf{A}_{\text{ext}}(\mathbf{r}_i) - e\hat{\mathbf{A}}\right). \tag{39}$$

Then, the full gauge-invariant current operator is [32]

$$\hat{\mathbf{J}} = e\sum_{i=1}^{N}\mathbf{v}_i = -\frac{\mathrm{i}e\hbar}{m_\mathrm{e}}\sum_{j=1}^{N}\nabla_j - \frac{e^2N}{m_\mathrm{e}}\hat{\mathbf{A}} - \frac{e^2}{m_\mathrm{e}}\sum_{i=1}^{N}\mathbf{A}_{\text{ext}}(\mathbf{r}_i). \tag{40}$$

We to go to the CM and relative distances frame and we utilize the expressions derived in Appendix I for all the relevant operators and we find for current operator

$$\hat{\mathbf{J}} = \sqrt{N}\left[-\frac{\mathrm{i}e\hbar}{m_e}\nabla_{\mathbf{R}} - \frac{e^2}{m_e}\sqrt{N}\hat{\mathbf{A}} - \frac{e^2}{m_e}\mathbf{A}_{\text{ext}}(\mathbf{R})\right] \equiv \hat{\mathbf{J}}_{\text{cm}}. \tag{41}$$

The above result shows that the total current in the system is equal essentially to current of the CM and depends only on CM related operators. This property has the following important implication

$$\langle\Psi_{nI}|\hat{\mathbf{J}}|\Psi_{mJ}\rangle = \delta_{IJ}\langle\Phi_n|\hat{\mathbf{J}}|\Phi_m\rangle \tag{42}$$

using the above the expression for the current correlator simplifies

$$\langle \hat{J}_a(t)\hat{J}_b\rangle = \frac{1}{\mathcal{Z}} \sum_{n,m,I} e^{-\beta E_{nI}} e^{\mathrm{i}t(E_n-E_m)/\hbar} \langle\Phi_n|\hat{J}_a|\Phi_m\rangle\langle\Phi_m|\hat{J}_b|\Phi_n\rangle \tag{43}$$

We note to obtain the above we used that $E_{nI} - E_{mI} = E_n - E_m$. To complete the computation we need to multiply $\langle \hat{J}_a(t)\hat{J}_b\rangle$ with $\frac{-\mathrm{i}\Theta(t)}{\hbar}$ and Fourier transform into the frequency space

$$\begin{aligned}\frac{-\mathrm{i}\Theta(t)}{\hbar}\langle \hat{J}_a(t)\hat{J}_b\rangle \longrightarrow \frac{1}{\mathcal{Z}}\sum_{n,m,I} e^{-\beta E_{nI}} \frac{\langle\Phi_n|\hat{J}_a|\Phi_m\rangle\langle\Phi_m|\hat{J}_b|\Phi_n\rangle}{w + (E_n - E_m)/\hbar + \mathrm{i}\delta} &= \frac{\sum_I e^{-\beta E_I}}{\sum_I e^{-\beta E_I}\sum_k e^{-\beta E_k}} \sum_{n,m,I} e^{-\beta E_n} \frac{\langle\Phi_n|\hat{J}_a|\Phi_m\rangle\langle\Phi_m|\hat{J}_b|\Phi_n\rangle}{w + (E_n - E_m)/\hbar + \mathrm{i}\delta} \\ = \frac{1}{\sum_k e^{-\beta E_k}} \sum_{n,m} e^{-\beta E_n} \frac{\langle\Phi_n|\hat{J}_a|\Phi_m\rangle\langle\Phi_m|\hat{J}_b|\Phi_n\rangle}{w + (E_n - E_m)/\hbar + \mathrm{i}\delta} \quad \text{with} \quad \delta \to 0^+. \end{aligned} \tag{44}$$

Following exactly the same procedure for the second term in Eq.(36) $\frac{\mathrm{i}\Theta(t)}{\hbar}\langle\hat{\mathbf{J}}\hat{\mathbf{J}}(t)\rangle$ we find the the expression for the current-current response function

$$\chi_{ab}(w) = \frac{1}{\sum_l e^{-\beta E_l}} \sum_{n,m}\left(e^{-\beta E_n} - e^{-\beta E_m}\right) \frac{\langle\Phi_n|\hat{J}_a|\Phi_m\rangle\langle\Phi_m|\hat{J}_b|\Phi_n\rangle}{w + (E_n - E_m)/\hbar + \mathrm{i}\delta} \quad \text{with} \quad \delta \to 0^+. \tag{45}$$

From the above expression we see that current response function solely depends on the CM eigenstates and the CM eigenenergies. This is a consequence of homogeneity which implies the separability of the full Hamiltonian into CM and relative parts.

B. Application to Landau Polaritons

Having derived the general formula for the current response function of a homogeneous system, we will now apply to the Landau polaritons. For the polaritons we have the CM eigenstates $e^{\mathrm{i}K_x X}\phi_{n_+}(S_+)\phi_{n_-}(S_-) \equiv |K_x n_+ n_-\rangle$ and the eigenergies $E_{n_+n_-} = \hbar\Omega_+\left(n_+ + \frac{1}{2}\right) + \hbar\Omega_-\left(n_- + \frac{1}{2}\right)$. Consequently the response functions take the form

$$\chi_{ab}(w) = \sum_{n_+,n_-,m_+,m_-,K_x,K'_x} \frac{e^{-\beta E_{n_+n_-}} - e^{-\beta E_{m_+m_-}}}{\mathcal{Z}_{\mathrm{cm}}} \frac{\langle n_+n_-K'_x|\hat{J}_a|K_x m_+m_-\rangle\langle K_x m_+m_-|\hat{J}_b|K'_x n_+n_-\rangle}{w + (E_{n_+n_-} - E_{m_+m_-})/\hbar + \mathrm{i}\delta} \tag{46}$$

where $\mathcal{Z}_{\mathrm{cm}} = \sum_{n_+,n_-,K_x} e^{-\beta E_{n_+n_-}}$ is the CM partition function. To proceed further we need the expressions for the current operators in the polaritonic basis. The x and y components of the current operator in terms of the polaritonic coordinates $S_\pm$ are

$$\begin{aligned}\hat{J}_x &= \frac{e^2\sqrt{N}B}{m_e^{3/2}}\left[\frac{\sqrt{m_e}}{eB}\left(-\mathrm{i}\hbar\nabla_X - \hbar K_x\right) + \frac{S_+(1-\eta\Lambda) + S_-(\Lambda+\eta)}{\sqrt{1+\Lambda^2}}\right]\\ \hat{J}_y &= -\frac{\mathrm{i}e\hbar}{m_e}\sum_{j=1}^{N}\partial_{y_j} = \frac{-\mathrm{i}e\hbar}{\sqrt{m_e}}\sqrt{\frac{N}{1+\Lambda^2}}\left[\partial_{S_+} + \Lambda\partial_{S_-}\right].\end{aligned} \tag{47}$$

Moreover, the current operators can be written using the polaritonic annihilation and creation operators as follows

$$\hat{J}_x = \frac{e^2\sqrt{N}B}{m_e^{3/2}}\sqrt{\frac{\hbar}{2(1+\Lambda^2)}}\left[\frac{\sqrt{m_e}}{eB}\left(-\mathrm{i}\hbar\nabla_X - \hbar K_x\right) + \frac{\Lambda+\eta}{\sqrt{\Omega_-}}\left(\hat{b}_-^\dagger + \hat{b}_-\right) + \frac{1-\eta\Lambda}{\sqrt{\Omega_+}}\left(\hat{b}_+^\dagger + \hat{b}_+\right)\right] \tag{48}$$

$$\hat{J}_y = -\mathrm{i}e\sqrt{\frac{\hbar N}{2m_e(1+\Lambda^2)}}\left[\sqrt{\Omega_+}\left(\hat{b}_+ - \hat{b}_+^\dagger\right) + \Lambda\sqrt{\Omega_-}\left(\hat{b}_- - \hat{b}_-^\dagger\right)\right] \tag{49}$$

From the above we can obtain the matrix representation of the current operator on the polariton basis

$$\begin{aligned}\langle n_+n_-K'_x|\hat{J}_x|K_x m_+m_-\rangle = \frac{e^2\sqrt{N}B}{m_e^{3/2}}\sqrt{\frac{\hbar}{2(1+\Lambda^2)}}\Bigg[&\frac{\Lambda+\eta}{\sqrt{\Omega_-}}\delta_{n_+m_+}\left(\sqrt{m_-+1}\,\delta_{n_-,m_-+1} + \sqrt{m_-}\,\delta_{n_-,m_--1}\right)\\ &+ \frac{1-\eta\Lambda}{\sqrt{\Omega_+}}\delta_{n_-m_-}\left(\sqrt{m_+}\,\delta_{n_+,m_+-1} + \sqrt{m_++1}\,\delta_{n_+,m_++1}\right)\Bigg]\delta_{K'_xK_x}\end{aligned} \tag{50}$$

$$\begin{aligned}\langle n_+n_-K'_x|\hat{J}_y|K_x m_+m_-\rangle = -\mathrm{i}e\sqrt{\frac{\hbar N}{2m_e(1+\Lambda^2)}}\Bigg[&\sqrt{\Omega_+}\,\delta_{n_-m_-}\left(\sqrt{m_+}\,\delta_{n_+,m_+-1} - \sqrt{m_++1}\,\delta_{n_+,m_++1}\right)\\ &+ \Lambda\sqrt{\Omega_-}\,\delta_{n_+m_+}\left(\sqrt{m_-}\,\delta_{n_-,m_--1} - \sqrt{m_-+1}\,\delta_{n_-,m_-+1}\right)\Bigg]\delta_{K'_xK_x}\end{aligned} \tag{51}$$

The current operators are diagonal with respect to the plane-wave states $e^{\mathrm{i}K_xX}$ and consequently the current response functions simplifies to

$$\chi_{ab}(w) = \sum_{n_+,n_-,m_+,m_-} \frac{e^{-\beta E_{n_+n_-}} - e^{-\beta E_{m_+m_-}}}{\sum_{l_+,l_-} e^{-\beta E_{l_+l_-}}} \frac{\langle n_+n_-|\hat{J}_a|m_+m_-\rangle\langle m_+m_-|\hat{J}_b|n_+n_-\rangle}{w + (E_{n_+n_-} - E_{m_+m_-})/\hbar + \mathrm{i}\delta} \tag{52}$$

With the use of the above formula for current response functions the temperature dependent transport of the polariton system can be obtained. The corresponding results are shown in main text of the manuscript.

C. Zero Temperature Transport

Having derived the general formula for the current correlator $\chi_{ab}(w)$ at finite temperature, we will focus now at the transport properties at zero temperature, $T = 0$, where the topological protection and the quantization of the quantum Hall conductance are expected from the Thouless argument, as long as the system is gapped [31]. At $T = 0$

the ground state of the system is for $n_+ = n_- = 0$ and only the thermal prefactors corresponding to the ground state $e^{-\beta E_{00}}$ contribute to transport.

$$\chi_{ab}(w) = \sum_{m_+,m_-} \frac{\langle 00|\hat{J}_a|m_+m_-\rangle\langle m_+m_-|\hat{J}_b|00\rangle}{w + (E_{00} - E_{m_+m_-})/\hbar + \mathrm{i}\delta} - (00 \leftrightarrow m_+m_-) \tag{53}$$

Furthermore, the current operators are linear in the polaritonic annihilation and creation operators and thus allow only for single-polariton transitions to occur, which implies that inthe denominator of the response function only single polariton energies show up $\Omega_\pm$. Finally, using the formulas for the matrix representation of the components of the current operator we find the following analytically exact expressions for the transverse χ_{xy} and longitudinal χ_{yy} response functions

$$\begin{aligned}
\chi_{xy}(w) &= \frac{Ne^3B}{(1+\Lambda^2)m_e^2}\Big[\Lambda(\Lambda+\eta)\frac{\mathrm{i}}{2}\left(\frac{1}{w+\Omega_-+\mathrm{i}\delta} + \frac{1}{w-\Omega_-+\mathrm{i}\delta}\right) + (1-\eta\Lambda)\frac{\mathrm{i}}{2}\left(\frac{1}{w+\Omega_++\mathrm{i}\delta} + \frac{1}{w-\Omega_++\mathrm{i}\delta}\right)\Big], \\
\chi_{yy}(w) &= -\frac{Ne^2}{(1+\Lambda^2)m_e}\left[\frac{\Omega_+}{2}\left(\frac{1}{w+\Omega_++\mathrm{i}\delta} - \frac{1}{w-\Omega_++\mathrm{i}\delta}\right) + \frac{\Lambda^2\Omega_-}{2}\left(\frac{1}{w+\Omega_-+\mathrm{i}\delta} - \frac{1}{w-\Omega_-+\mathrm{i}\delta}\right)\right].
\end{aligned} \tag{54}$$

With the above results and using the Kubo formula, the expressions for the optical and the CD conductivities can be straightforwardly obtained.