Supplementary Material: On the Topological Protection of the Quantum Hall Effect in a Cavity

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I. PAULI-FIERZ HAMILTONIAN IN THE CENTER OF MASS FRAME

In this section we would like to give the details about the transformation of the Pauli-Fierz Hamiltonian in the center of mass (CM) and relative distances frame. The Hamiltonian of our system is [1]

$$H \;=\; \frac{1}{2m}\sum_{i=1}^{N}\left(\boldsymbol{\pi}_i + e\mathbf{A}\right)^2 + \sum_{i<l}^{N} W\left(|\mathbf{r}_i - \mathbf{r}_l|\right) + \hbar\omega\left(a^\dagger a + \frac{1}{2}\right). \tag{1}$$

where $\boldsymbol{\pi}_i = \mathrm{i}\hbar\nabla_i + e\mathbf{A}_{\mathrm{ext}}(\mathbf{r}_i)$ are the dynamical momenta, and $\mathbf{A}_{\mathrm{ext}}(\mathbf{r}) = -\mathbf{e}_x By$ describes the homogeneous magnetic field $\mathbf{B} = \nabla \times \mathbf{A}_{\mathrm{ext}}(\mathbf{r}) = B\mathbf{e}_z$. The cavity field $\mathbf{A} = \sqrt{\frac{\hbar}{2\epsilon_0\mathcal{V}\omega}}\mathbf{e}_x\left(a + a^\dagger\right)$ is characterized by the in-plane polarization vector $\mathbf{e}_x$ and the photon's bare frequency ω. Further, $W(\mathbf{r}_i - \mathbf{r}_j) = 1/4\pi\epsilon_0|\mathbf{r}_i - \mathbf{r}_j|$ is the Coulomb interaction between the electrons. For mathematical convenience we utilize a symmetric definition with respect to $\sqrt{N}$ for the coordinates in the CM frame as in Ref. [2]

$$\mathbf{R} = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\mathbf{r}_i \;\text{ and }\; \widetilde{\mathbf{r}}_j = \frac{\mathbf{r}_1 - \mathbf{r}_j}{\sqrt{N}} \;\text{ with }\; j > 1. \tag{2}$$

The original electronic coordinates in terms of the new ones $\{\mathbf{R}, \widetilde{\mathbf{r}}_j\}$ are

$$\mathbf{r}_1 = \frac{1}{\sqrt{N}}\left(\mathbf{R} + \sum_{j=2}^{N}\widetilde{\mathbf{r}}_j\right) \text{ and } \mathbf{r}_j = \frac{1}{\sqrt{N}}\left(\mathbf{R} + \sum_{j=2}^{N}\widetilde{\mathbf{r}}_j\right) - \sqrt{N}\widetilde{\mathbf{r}}_j \;\text{ with }\; j > 1.$$

The momenta of the electrons in the new coordinate system are $\nabla_1 = \left(\nabla_{\mathbf{R}} + \sum_{j=2}^{N}\widetilde{\nabla}_j\right)/\sqrt{N}$ and $\nabla_j = \left(\nabla_{\mathbf{R}} - \widetilde{\nabla}_j\right)/\sqrt{N}$ with $j > 1$. From these expression we can find the form of the electronic kinetic terms in the new frame

$$\sum_{i=1}^{N}\nabla_i^2 = \nabla_{\mathbf{R}}^2 + \frac{1}{N}\sum_{j=2}^{N}\widetilde{\nabla}_j^2 + \frac{1}{N}\sum_{j,k=2}^{N}\widetilde{\nabla}_j\cdot\widetilde{\nabla}_k \;\text{ and }\; \sum_{i=1}^{N}\nabla_i = \sqrt{N}\nabla_{\mathbf{R}}. \tag{3}$$

The interaction term between the cavity field and the electrons takes the form

$$\mathbf{A}\cdot\sum_{i=1}^{N}\mathrm{i}\hbar\nabla_i + e\mathbf{A}_{\mathrm{ext}}(\mathbf{r}_i) = \sqrt{N}\mathbf{A}\cdot\left(\mathrm{i}\hbar\nabla_{\mathbf{R}} + e\mathbf{A}_{\mathrm{ext}}(\mathbf{R})\right) \tag{4}$$

To complete our analysis we also give the expression for the purely electronic terms in the new frame. For the quadrature of the external magnetic field we have

$$\sum_{i=1}^{N}\mathbf{A}_{\mathrm{ext}}^2(\mathbf{r}_i) = \mathbf{A}_{\mathrm{ext}}^2(\mathbf{R}) + N\sum_{j=2}^{N}\mathbf{A}_{\mathrm{ext}}^2(\widetilde{\mathbf{r}}_j) - \left[\sum_{j=2}^{N}\mathbf{A}_{\mathrm{ext}}(\widetilde{\mathbf{r}}_j)\right]^2 \tag{5}$$

and for the bilinear term between the magnetic field and the momenta we have

$$\sum_{i=1}^{N}\mathbf{A}_{\text{ext}}(\mathbf{r}_i)\cdot\nabla_i=\mathbf{A}_{\text{ext}}(\mathbf{R})\cdot\nabla_{\mathbf{R}}+\sum_{j=2}^{N}\mathbf{A}_{\text{ext}}(\widetilde{\mathbf{r}}_j)\cdot\widetilde{\nabla}_j. \tag{6}$$

Finally, we give the expression for the interaction term W between the electrons.

$$\sum_{i<l}^{N}W\left(|\mathbf{r}_i-\mathbf{r}_l|\right)=\sum_{1<l}^{N}W(\sqrt{N}|\widetilde{\mathbf{r}}_l|)+\sum_{2\leq i<l}^{N}W\left(\sqrt{N}|\widetilde{\mathbf{r}}_i-\widetilde{\mathbf{r}}_l|\right)$$

Adding together all the different terms we find that the expression of the Hamiltonian in the new frame is the sum of two parts: (i) the center of mass part H_{com} which is coupled to the quantized field $\mathbf{A}$ and (ii) the relative distances H_{rel} which does not couple to the cavity field, $H=H_{\text{cm}}+H_{\text{rel}}$ where each part looks as

$$\begin{aligned}
H_{\text{cm}} &= \frac{1}{2m}\left(\mathrm{i}\hbar\nabla_{\mathbf{R}}+e\mathbf{A}_{\text{ext}}(\mathbf{R})+e\sqrt{N}\mathbf{A}\right)^2+\hbar\omega\left(a^\dagger a+\frac{1}{2}\right),\\
H_{\text{rel}} &= \frac{1}{2m}\sum_{j=2}^{N}\left(\frac{\mathrm{i}\hbar}{\sqrt{N}}\widetilde{\nabla}_j+e\sqrt{N}\mathbf{A}_{\text{ext}}(\widetilde{\mathbf{r}}_j)\right)^2-\frac{\hbar^2}{2mN}\sum_{j,l=2}^{N}\widetilde{\nabla}_j\cdot\widetilde{\nabla}_l-\frac{e^2}{2m}\left(\sum_{j=2}^{N}\mathbf{A}_{\text{ext}}(\widetilde{\mathbf{r}}_j)\right)^2\\
&+\sum_{1<l}^{N}W(\sqrt{N}|\widetilde{\mathbf{r}}_l|)+\sum_{2\leq i<l}^{N}W\left(\sqrt{N}|\widetilde{\mathbf{r}}_i-\widetilde{\mathbf{r}}_l|\right).
\end{aligned} \tag{7}$$

Lastly, it is important to demonstrate that the center of mass and relative distances degrees of freedom are independent by checking the commutation relations between their coordinates and momenta. Using the chain rule we have for the derivatives of the coordinates in the center of mass frame

$$\nabla_{\mathbf{R}}=\sum_{i=1}^{N}\frac{\partial\mathbf{r}_i}{\partial\mathbf{R}}\nabla_i=\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\nabla_i,\ \text{ and }\ \widetilde{\nabla}_j=\sum_{i=1}^{N}\frac{\partial\mathbf{r}_i}{\partial\widetilde{\mathbf{r}}_j}\nabla_i=\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\nabla_i-\sqrt{N}\nabla_j \text{ with } j>1.$$

From the above expressions it is clear that the momenta in the new coordinate frame commute

$$\left[\nabla_{\mathbf{R}},\widetilde{\nabla}_j\right]=0. \tag{8}$$

The next property to check is the commutation relations between the momenta and the coordinates.

$$[\nabla_{\mathbf{R}},\widetilde{\mathbf{r}}_j] = \frac{1}{N}\left[\sum_{i=1}^{N}\nabla_i,\mathbf{r}_1-\mathbf{r}_j\right]=0,\quad j>1, \tag{9}$$

$$\begin{aligned}
\left[\widetilde{\nabla}_j,\mathbf{R}\right] &= \left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\nabla_i-\sqrt{N}\nabla_j,\frac{1}{\sqrt{N}}\sum_{l=1}^{N}\mathbf{r}_l\right]=\\
&= \frac{1}{N}\sum_{i,l=1}^{N}[\nabla_i,\mathbf{r}_l]-\sum_{l=1}^{N}[\nabla_j,\mathbf{r}_l]=\frac{1}{N}\sum_{i,l=1}^{N}\delta_{il}-\sum_{l=1}^{N}\delta_{jl}=\frac{N}{N}-1=0.
\end{aligned} \tag{10}$$

We would also like to mention that the separation between the CM and the relative distances holds true also for an arbitrary amount of photon modes as long as the cavity field is considered to be homogeneous as it was shown in [3].

II. EXACT SOLUTION OF THE CM HAMILTONIAN AND LANDAU POLARITONS

Having demonstrated that only the CM of the electronic system couples to the cavity we will show that H_{cm} can be solved analytically. Let us see how this can be done. To proceed we expand the covariant kinetic term

$$H_{\text{cm}}=\frac{\mathbf{\Pi}^2}{2m}+\frac{e\sqrt{N}}{m}\mathbf{A}\cdot\mathbf{\Pi}+\underbrace{\frac{e^2N\mathbf{A}^2}{2m}+\hbar\omega\left(a^\dagger a+\frac{1}{2}\right)}_{H_p} \tag{11}$$

For the description of the photon operators we will introduce the displacement coordinate q and its conjugate momentum ∂_q as $a = \frac{1}{\sqrt{2}}(q + \partial/\partial q)$ and $a^\dagger$ defined by conjugation [1, 4]. The part H_p can be brought to diagonal form by the scaling transformation on the photonic displacement coordinate

$$u = q\sqrt{\frac{\widetilde{\omega}}{\omega}} \text{ where } \widetilde{\omega} = \sqrt{\omega^2 + \omega_d^2} \tag{12}$$

with $\omega_d = \sqrt{e^2 N/\epsilon_0 m \mathcal{V}}$ is the diamagnetic frequency depending on the electron density in the effective mode volume. After this transformation the CM Hamiltonian is

$$H_{\mathrm{cm}} = \frac{\mathbf{\Pi}^2}{2m} + \frac{e\sqrt{N}}{m}\mathbf{A}\cdot\mathbf{\Pi} + \frac{\hbar\widetilde{\omega}}{2}\left(-\frac{\partial^2}{\partial u^2} + u^2\right), \tag{13}$$

where the quantized field is now

$$\mathbf{A} = \sqrt{\frac{\hbar}{\epsilon_0 V \widetilde{\omega}}}\mathbf{e}_x u. \tag{14}$$

In the Landau gauge the Hamiltonian has translational invariance along the X coordinate which implies that the eigenfunctions in X are plane waves $e^{\mathrm{i}K_x X}$. We apply H_{cm} on the plane wave and we have

$$H_{\mathrm{cm}} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial Y^2} + \frac{m\omega_c^2}{2}\left(Y + \frac{\hbar K_x}{eB}\right)^2 - geBu\left(Y + \frac{\hbar K_x}{eB}\right) + \frac{\hbar\widetilde{\omega}}{2}\left(-\frac{\partial^2}{\partial u^2} + u^2\right)$$

where we also introduced the coupling constant $g = \omega_d\sqrt{\hbar/m\widetilde{\omega}}$. As a next step we define the coordinate

$$\bar{Y} = Y + \frac{\hbar K_x}{eB} \tag{15}$$

and the Hamiltonian simplifies further

$$H_{\mathrm{cm}} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial \bar{Y}^2} + \frac{m\omega_c^2}{2}\bar{Y}^2 - geBu\bar{Y} + \frac{\hbar\widetilde{\omega}}{2}\left[-\frac{\partial^2}{\partial u^2} + u^2\right]$$

The Hamiltonian consists of two coupled harmonic oscillators. It is convenient to perform another scaling transformation on $\bar{Y}$ and u

$$V_- = -u\sqrt{\frac{\hbar}{\widetilde{\omega}}} \text{ and } V_+ = \sqrt{m}\bar{Y}. \tag{16}$$

such that we have both harmonic oscillators in the form of having mass equal to 1. The Hamiltonian then becomes

$$H_{\mathrm{cm}} = -\frac{\hbar^2}{2}\sum_{l=\pm}\frac{\partial^2}{\partial V_l^2} + \frac{1}{2}\sum_{l,j=\pm} W_{lj}V_l V_j. \tag{17}$$

The matrix W

$$W = \begin{pmatrix} \omega_c^2 & \omega_d\omega_c \\ \omega_d\omega_c & \widetilde{\omega}^2 \end{pmatrix} \tag{18}$$

is real and symmetric, and as a consequence can be diagonalized by the orthogonal matrix O [5],

$$O = \begin{pmatrix} \frac{1}{\sqrt{1+\Lambda^2}} & \frac{\Lambda}{\sqrt{1+\Lambda^2}} \\ -\frac{\Lambda}{\sqrt{1+\Lambda^2}} & \frac{1}{\sqrt{1+\Lambda^2}} \end{pmatrix} \quad \text{where } \Lambda = \alpha - \sqrt{1+\alpha^2} \quad \text{and} \quad \alpha = \frac{\omega_c^2 - \widetilde{\omega}^2}{2\omega_d\omega_c}.$$

The eigenvalues of the matrix W give the new normal modes of the interacting light-matter system. We find them to be

$$\Omega_\pm^2 = \frac{1}{2}\left(\widetilde{\omega}^2 + \omega_c^2 \pm \sqrt{4\omega_d^2\omega_c^2 + (\widetilde{\omega}^2 - \omega_c^2)^2}\right). \tag{19}$$

The Hamiltonian after the orthogonal transformation takes the canonical form

$$H_{\text{cm}} = -\frac{\hbar^2}{2}\sum_{l=\pm}\frac{\partial^2}{\partial S_l^2} + \frac{1}{2}\sum_{l=\pm}\Omega_l^2 S_l^2. \tag{20}$$

The new coordinates S_l and conjugate momenta ∂_{S_l} are related to the old ones $\{V_l, \partial_{V_l}\}$ through the orthogonal matrix O,

$$S_l = \sum_{j=\pm} O_{jl} V_j \text{ and } \frac{\partial}{\partial S_l} = \sum_{j=\pm} O_{jl}\frac{\partial}{\partial V_j}. \tag{21}$$

Due to the fact that the matrix O is orthogonal the canonical commutation relations are satisfied which implies that we have two independent harmonic oscillators [5]. Thus, the eigenfunctions of the interacting system are Hermite functions Φ of the coordinates S_+ and S_-. The full set of eigenfunctions of the system is

$$\Psi_{K_x,n_+,n_-}(X,S_+,S_-) = e^{\mathrm{i}K_x X}\Phi_{n_+}(S_+)\Phi_{n_-}(S_-) \tag{22}$$

with eigenspectrum

$$E_{n_+,n_-} = \hbar\Omega_+\left(n_+ + \frac{1}{2}\right) + \hbar\Omega_-\left(n_- + \frac{1}{2}\right). \tag{23}$$

The frequencies Ω_+ (upper) and Ω_- (lower) are the two collective Landau polariton modes of the quantum Hall system in the cavity. For completeness, we note that the solution of the polaritons for the CM can be equivalently written in terms of annihilation $b_\pm$ and creation $b_\pm^\dagger$ operators for the polariton quasiparticles. In this representation H_{cm} is written as

$$H_{\text{cm}} = \hbar\Omega_+\left(b_+^\dagger b_+ + \frac{1}{2}\right) + \hbar\Omega_-\left(b_-^\dagger b_- + \frac{1}{2}\right) \tag{24}$$

with the polariton operators defined $b_\pm = S_\pm\sqrt{\frac{\Omega_\pm}{2}} + \sqrt{\frac{1}{2\Omega_\pm}}\partial_{S_\pm}$ [4]. It is worth to notice that in the limit $\omega \to 0$ the lower polariton frequency goes to zero, $\Omega_- \to 0$, which means that the system becomes gapless. In this limit the canonical transformation from the electron and photon basis $V_\pm$ to the polariton basis $S_\pm$ becomes singular.

III. FINITE TEMPERATURE TRANSPORT

In this section we present the general formalism employed for the finite temperature transport of the light-matter system. As we already showed the Hamiltonian of our system can be written as a sum of a CM and relative part $H = H_{\text{cm}} + H_{\text{rel}}$. To proceed we assume that the eigenstates of H_{cm} are $|\Phi_n\rangle$ and the eigenstates of H_{rel} are $|F_I\rangle$ such that it holds

$$H_{\text{cm}}|\Phi_n\rangle = E_n|\Phi_n\rangle \text{ and } H_{\text{rel}}|F_I\rangle = E_I|F_I\rangle \tag{25}$$

Then, the eigenstates of the full Hamiltonian H are

$$|\Psi_{nI}\rangle = |\Phi_n\rangle \otimes |F_I\rangle, \tag{26}$$

and the full eigenspectrum is $E_{nI} = E_n + E_I$. The Kubo formula for the optical conductivity of the system is [6, 7]

$$\sigma_{ab}(w) = \frac{\mathrm{i}}{w + \mathrm{i}\delta}\left(\frac{e^2 n_e}{m}\delta_{ab} + \frac{\chi_{ab}(w)}{A}\right) \quad \delta \to 0^+ \tag{27}$$

where $a, b = x, y, z$. The first term in the optical conductivity is the Drude term, while the second term is the current-current correlator in the frequency domain, which is defined as the Fourier transform of current-current correlator in the time domain

$$\chi_{ab}(t) = \frac{-\mathrm{i}\Theta(t)}{\hbar}\langle[J_a(t), J_b]\rangle, \tag{28}$$

with the current operators considered in the interaction picture $\mathbf{J}(t) = e^{\mathrm{i}Ht/\hbar}\mathbf{J}e^{-\mathrm{i}Ht/\hbar}$ [6]. In the canonical ensemble the expectation value of an operator $\mathcal{O}$ is defined as [7]

$$\langle \mathcal{O} \rangle = Tr\{\rho\mathcal{O}\} = \frac{1}{\mathcal{Z}}\sum_{n,I}\langle\Psi_{nI}|e^{-\beta H \mathcal{O}}|\Psi_{nI}\rangle \tag{29}$$

where the partition function is $\mathcal{Z} = \sum_{n,I} e^{-\beta E_n}e^{-\beta E_I}$. We will use these formulas now for the computation of the current correlation functions. The current response can be splitted into two parts

$$\chi_{ab}(t) = \frac{-\mathrm{i}\Theta(t)}{\hbar}\left(\langle J_a(t)J_b\rangle - \langle J_b J_a(t)\rangle\right). \tag{30}$$

Let us compute first the first term $\langle J_a(t)J_b\rangle$. We use the expression for the canonical ensemble and for the current operator in the interaction picture and we have

$$\begin{aligned}\langle J_a(t)J_b\rangle &= \frac{1}{\mathcal{Z}}\sum_{n,I} e^{-\beta E_{nI}}\langle\Psi_{nI}|e^{\mathrm{i}Ht/\hbar}J_a e^{-\mathrm{i}Ht/\hbar}J_b|\Psi_{nI}\rangle \\ &= \frac{1}{\mathcal{Z}}\sum_{n,I} e^{-\beta E_{nI}}e^{\mathrm{i}tE_{nI}/\hbar}\langle\Psi_{nI}|J_a e^{-\mathrm{i}Ht/\hbar}J_b|\Psi_{nI}\rangle.\end{aligned} \tag{31}$$

We introduce the identity $\mathbb{I} = \sum_{m,J}|\Psi_{mJ}\rangle\langle\Psi_{mJ}|$ in the above expression

$$\begin{aligned}\langle J_a(t)J_b\rangle &= \frac{1}{\mathcal{Z}}\sum_{n,m,J,I} e^{-\beta E_{nI}}e^{\mathrm{i}tE_{nI}/\hbar}\langle\Psi_{nI}|J_a e^{-\mathrm{i}Ht/\hbar}|\Psi_{mJ}\rangle\langle\Psi_{mJ}|J_b|\Psi_{nI}\rangle \\ &= \frac{1}{\mathcal{Z}}\sum_{n,m,J,I} e^{-\beta E_{nI}}e^{\mathrm{i}t(E_{nI}-E_{mJ})/\hbar}\langle\Psi_{nI}|J_a|\Psi_{mJ}\rangle\langle\Psi_{mJ}|J_b|\Psi_{nI}\rangle\end{aligned} \tag{32}$$

A. Current in the CM frame

Since we work in the CM frame in order to proceed we need examine how the current operator looks in the CM frame. The expression for the current operator can be obtained by computing the velocity operator of the electrons through the Heisenberg equation of motion [8]

$$\mathbf{v}_i = \frac{d\mathbf{r}_i}{dt} = \frac{\mathrm{i}}{\hbar}[H, \mathbf{r}_i] = \frac{1}{m}\left(-\mathrm{i}\hbar\nabla_i - e\mathbf{A}_{\mathrm{ext}}(\mathbf{r}_i) - e\mathbf{A}\right). \tag{33}$$

Then, the full gauge-invariant current operator is [8]

$$\mathbf{J} = e\sum_{i=1}^{N}\mathbf{v}_i = -\frac{\mathrm{i}e\hbar}{m_{\mathrm{e}}}\sum_{j=1}^{N}\nabla_j - \frac{e^2N}{m_{\mathrm{e}}}\mathbf{A} - \frac{e^2}{m_{\mathrm{e}}}\sum_{i=1}^{N}\mathbf{A}_{\mathrm{ext}}(\mathbf{r}_i). \tag{34}$$

We to go to the CM and relative distances frame and we utilize the expressions derived in Appendix I for all the relevant operators and we find for current operator

$$\mathbf{J} = \sqrt{N}\left[-\frac{\mathrm{i}e\hbar}{m}\nabla_{\mathbf{R}} - \frac{e^2}{m}\sqrt{N}\mathbf{A} - \frac{e^2}{m}\mathbf{A}_{\mathrm{ext}}(\mathbf{R})\right] \equiv \mathbf{J}_{\mathrm{cm}}. \tag{35}$$

The above result shows that the total current in the system is equal essentially to current of the CM and depends only on CM related operators. This property has the following important implication

$$\langle\Psi_{nI}|\mathbf{J}|\Psi_{mJ}\rangle = \delta_{IJ}\langle\Phi_n|\mathbf{J}|\Phi_m\rangle \tag{36}$$

using the above the expression for the current correlator simplifies

$$\langle J_a(t)J_b\rangle = \frac{1}{\mathcal{Z}}\sum_{n,m,I} e^{-\beta E_{nI}}e^{\mathrm{i}t(E_n-E_m)/\hbar}\langle\Phi_n|J_a|\Phi_m\rangle\langle\Phi_m|J_b|\Phi_n\rangle \tag{37}$$

We note to obtain the above we used that $E_{nI} - E_{mI} = E_n - E_m$. To complete the computation we need to multiply $\langle J_a(t)J_b\rangle$ with $\frac{-\mathrm{i}\Theta(t)}{\hbar}$ and Fourier transform into the frequency space

$$\frac{-\mathrm{i}\Theta(t)}{\hbar}\langle J_a(t)J_b\rangle \longrightarrow \frac{1}{\mathcal{Z}}\sum_{n,m,I} e^{-\beta E_{nI}}\frac{\langle\Phi_n|J_a|\Phi_m\rangle\langle\Phi_m|J_b|\Phi_n\rangle}{w+(E_n-E_m)/\hbar+\mathrm{i}\delta} = \frac{\sum_I e^{-\beta E_I}}{\sum_I e^{-\beta E_I}\sum_k e^{-\beta E_k}}\sum_{n,m,I} e^{-\beta E_n}\frac{\langle\Phi_n|J_a|\Phi_m\rangle\langle\Phi_m|J_b|\Phi_n\rangle}{w+(E_n-E_m)/\hbar+\mathrm{i}\delta}$$
$$= \frac{1}{\sum_k e^{-\beta E_k}}\sum_{n,m} e^{-\beta E_n}\frac{\langle\Phi_n|J_a|\Phi_m\rangle\langle\Phi_m|J_b|\Phi_n\rangle}{w+(E_n-E_m)/\hbar+\mathrm{i}\delta} \text{ with } \delta\to 0^+. \tag{38}$$

Following exactly the same procedure for the second term in Eq.(30) $\frac{\mathrm{i}\Theta(t)}{\hbar}\langle\mathbf{J}\mathbf{J}(t)\rangle$ we find the the expression for the current-current response function

$$\chi_{ab}(w) = \frac{1}{\sum_l e^{-\beta E_l}}\sum_{n,m}\left(e^{-\beta E_n} - e^{-\beta E_m}\right)\frac{\langle\Phi_n|J_a|\Phi_m\rangle\langle\Phi_m|J_b|\Phi_n\rangle}{w+(E_n-E_m)/\hbar+\mathrm{i}\delta} \text{ with } \delta\to 0^+. \tag{39}$$

From the above expression we see that current response function solely depends on the CM eigenstates and the CM eigenenergies. This is a consequence of homogeneity which implies the separability of the full Hamiltonian into CM and relative parts.

B. Application to Landau Polaritons

Having derived the general formula for the current response function of a homogeneous system, we will now apply to the Landau polaritons. For the polaritons we have the CM eigenstates $e^{\mathrm{i}K_xX}\phi_{n_+}(S_+)\phi_{n_-}(S_-) \equiv |K_xn_+n_-\rangle$ and the eigenergies $E_{n_+n_-} = \hbar\Omega_+\left(n_+ + \frac{1}{2}\right) + \hbar\Omega_-\left(n_- + \frac{1}{2}\right)$. Consequently the response functions take the form

$$\chi_{ab}(w) = \sum_{n_+,n_-,m_+,m_-,K_x,K'_x}\frac{e^{-\beta E_{n_+n_-}} - e^{-\beta E_{m_+m_-}}}{\mathcal{Z}_{\text{cm}}}\frac{\langle n_+n_-K'_x|J_a|K_xm_+m_-\rangle\langle K_xm_+m_-|J_b|K'_xn_+n_-\rangle}{w+(E_{n_+n_-}-E_{m_+m_-})/\hbar+\mathrm{i}\delta} \tag{40}$$

where $\mathcal{Z}_{\text{cm}} = \sum_{n_+,n_-,K_x} e^{-\beta E_{n_+n_-}}$ is the CM partition function. To proceed further we need the expressions for the current operators in the polaritonic basis. The x and y components of the current operator in terms of the polaritonic coordinates $S_\pm$ are

$$\begin{aligned} J_x &= \frac{e^2\sqrt{N}B}{m_e^{3/2}}\left[\frac{\sqrt{m}}{eB}\left(-\mathrm{i}\hbar\nabla_X - \hbar K_x\right) + \frac{S_+(1-\eta\Lambda)+S_-(\Lambda+\eta)}{\sqrt{1+\Lambda^2}}\right] \\ J_y &= -\frac{\mathrm{i}e\hbar}{m}\sum_{j=1}^{N}\partial_{y_j} = \frac{-\mathrm{i}e\hbar}{\sqrt{m}}\sqrt{\frac{N}{1+\Lambda^2}}\left[\partial_{S_+} + \Lambda\partial_{S_-}\right]. \end{aligned} \tag{41}$$

Moreover, the current operators can be written using the polaritonic annihilation and creation operators as follows

$$J_x = \frac{e^2\sqrt{N}B}{m_e^{3/2}}\sqrt{\frac{\hbar}{2(1+\Lambda^2)}}\left[\frac{\sqrt{m}}{eB}\left(-\mathrm{i}\hbar\nabla_X - \hbar K_x\right) + \frac{\Lambda+\eta}{\sqrt{\Omega_-}}\left(b_-^\dagger + b_-\right) + \frac{1-\eta\Lambda}{\sqrt{\Omega_+}}\left(b_+^\dagger + b_+\right)\right] \tag{42}$$

$$J_y = -\mathrm{i}e\sqrt{\frac{\hbar N}{2m(1+\Lambda^2)}}\left[\sqrt{\Omega_+}\left(b_+ - b_+^\dagger\right) + \Lambda\sqrt{\Omega_-}\left(b_- - b_-^\dagger\right)\right] \tag{43}$$

From the above we can obtain the matrix representation of the current operator on the polariton basis

$$\begin{aligned} \langle n_+n_-K'_x|J_x|K_xm_+m_-\rangle &= \frac{e^2\sqrt{N}B}{m_e^{3/2}}\sqrt{\frac{\hbar}{2(1+\Lambda^2)}}\Bigg[\frac{\Lambda+\eta}{\sqrt{\Omega_-}}\delta_{n_+m_+}\left(\sqrt{m_-+1}\,\delta_{n_-,m_-+1} + \sqrt{m_-}\,\delta_{n_-,m_--1}\right) \\ &+ \frac{1-\eta\Lambda}{\sqrt{\Omega_+}}\delta_{n_-m_-}\left(\sqrt{m_+}\,\delta_{n_+,m_+-1} + \sqrt{m_++1}\,\delta_{n_+,m_++1}\right)\Bigg]\delta_{K'_xK_x} \end{aligned} \tag{44}$$

$$\begin{aligned} \langle n_+n_-K'_x|J_y|K_xm_+m_-\rangle &= -\mathrm{i}e\sqrt{\frac{\hbar N}{2m(1+\Lambda^2)}}\Bigg[\sqrt{\Omega_+}\,\delta_{n_-m_-}\left(\sqrt{m_+}\,\delta_{n_+,m_+-1} - \sqrt{m_++1}\,\delta_{n_+,m_++1}\right) \\ &+ \Lambda\sqrt{\Omega_-}\,\delta_{n_+m_+}\left(\sqrt{m_-}\,\delta_{n_-,m_--1} - \sqrt{m_-+1}\,\delta_{n_-,m_-+1}\right)\Bigg]\delta_{K'_xK_x} \end{aligned} \tag{45}$$

The current operators are diagonal with respect to the plane-wave states $e^{\mathrm{i}K_x X}$ and consequently the current response functions simplifies to

$$\chi_{ab}(w) = \sum_{n_+,n_-,m_+,m_-} \frac{e^{-\beta E_{n_+n_-}} - e^{-\beta E_{m_+m_-}}}{\sum_{l_+,l_-} e^{-\beta E_{l_+l_-}}} \frac{\langle n_+n_-|J_a|m_+m_-\rangle\langle m_+m_-|J_b|n_+n_-\rangle}{w + (E_{n_+n_-} - E_{m_+m_-})/\hbar + \mathrm{i}\delta} \tag{46}$$

With the use of the above formula for current response functions the temperature dependent transport of the polariton system can be obtained. The corresponding results are shown in main text of the manuscript.

For completeness and as supporting computation, in Fig. 1 we provide the finite temperature transport of the polariton system for the experimentally reported parameters in Ref. [9], $\omega = 2\pi \times 0.14$THz and $n_{2\mathrm{d}} = 2 \times 10^{11}\mathrm{cm}^{-2}$, and broadening $\delta = 2\pi \times 5 \times 10^{-3}$THz which we used in Fig. 2 in the main text. In Fig. 1(a) we show the deviation of the Hall conductance from the topologically expected quantized values for two different plateaus $\nu = 8, 4$ ($B = 1, 2T$). In Fig. 1(b) we show the thermal behavior of the longitudinal conductance σ_{yy}. We observe the exponential thermal activation as expected. In the low temperature regime, $T < 0.4$K, we see that the modifications of quantum Hall transport are consistent with the $T = 0$ results show in Fig. 2 in the main text. Being more precise, the cavity-induced transport deviations at $T < 0.4K$ and for $B = 1T(\nu = 8)$ are $\sim 2 \times 10^{-4}$ and for $B = 2T(\nu = 4)$ are $\sim 5 \times 10^{-5}$. These values agree with the $T = 0$ transport for $\omega = 2\pi \times 0.14$THz shown in Fig. 2 in the main text.

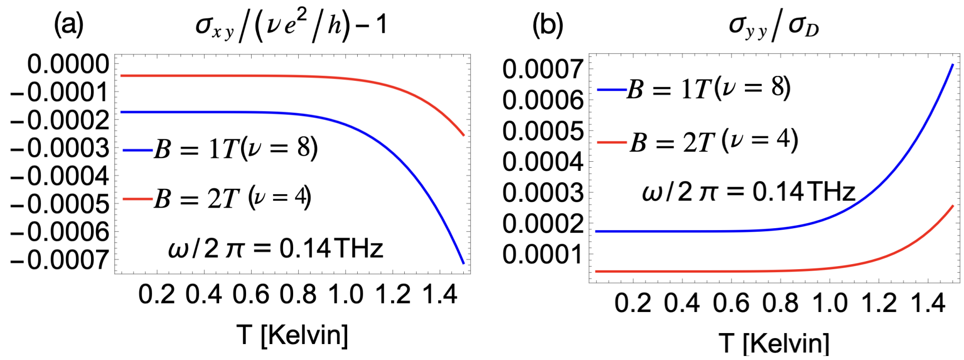

FIG. 1: Finite temperature quantum Hall transport in the cavity. The cavity frequency is $\omega = 2\pi \times 0.14$THz and 2d electron density $n_{2\mathrm{d}} = 2 \times 10^{11}\mathrm{cm}^{-2}$. The polariton broadening is $\delta = 2\pi \times 5 \times 10^{-3}$THz as in Fig. 2 in the main text. a) Deviations of the Hall conductance from the expected quantized values for two different plateaus $\nu = 8, 4$ corresponding to $B = 1, 2T$. b) Thermal activation of the longitudinal conductance σ_{yy} normalized by the Drude DC conductivity $\sigma_D = e^2 n_{2\mathrm{d}}/m\delta$. For low temperatures $T < 0.4K$ the finite temperature transport reproduces the $T = 0$ transport in Fig. 2 for $\omega = 2\pi \times 0.14$THz presented in the main text.

C. Zero Temperature Transport

Having derived the general formula for the current correlator $\chi_{ab}(w)$ at finite temperature, we will focus now at the transport properties at zero temperature, $T = 0$, where the topological protection and the quantization of the quantum Hall conductance are expected from the Thouless argument, as long as the system is gapped [10]. At $T = 0$ the ground state of the system is for $n_+ = n_- = 0$ and only the thermal prefactors corresponding to the ground state $e^{-\beta E_{00}}$ contribute to transport.

$$\chi_{ab}(w) = \sum_{m_+,m_-} \frac{\langle 00|J_a|m_+m_-\rangle\langle m_+m_-|J_b|00\rangle}{w + (E_{00} - E_{m_+m_-})/\hbar + \mathrm{i}\delta} - (00 \leftrightarrow m_+m_-) \tag{47}$$

Furthermore, the current operators are linear in the polaritonic annihilation and creation operators and thus allow only for single-polariton transitions to occur, which implies that inthe denominator of the response function only single polariton energies show up $\Omega_\pm$. Finally, using the formulas for the matrix representation of the components of the current operator we find the following analytically exact expressions for the transverse χ_{xy} and longitudinal χ_{yy} response functions

$$\begin{aligned}
\chi_{xy}(w) &= \frac{Ne^3B}{(1+\Lambda^2)m_e^2}\left[\Lambda(\Lambda+\eta)\frac{\mathrm{i}}{2}\left(\frac{1}{w+\Omega_-+\mathrm{i}\delta} + \frac{1}{w-\Omega_-+\mathrm{i}\delta}\right) + (1-\eta\Lambda)\frac{\mathrm{i}}{2}\left(\frac{1}{w+\Omega_++\mathrm{i}\delta} + \frac{1}{w-\Omega_++\mathrm{i}\delta}\right)\right], \\
\chi_{yy}(w) &= -\frac{Ne^2}{(1+\Lambda^2)m}\left[\frac{\Omega_+}{2}\left(\frac{1}{w+\Omega_++\mathrm{i}\delta} - \frac{1}{w-\Omega_++\mathrm{i}\delta}\right) + \frac{\Lambda^2\Omega_-}{2}\left(\frac{1}{w+\Omega_-+\mathrm{i}\delta} - \frac{1}{w-\Omega_-+\mathrm{i}\delta}\right)\right].
\end{aligned} \tag{48}$$

With the above results and using the Kubo formula, the expressions for the optical and the CD conductivities can be straightforwardly obtained.

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