Pseudo entropy of primary operators in $T\bar{T}/J\bar{T}$-deformed CFTs

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Abstract

In this work, we investigate the time evolution of the pseudo-(Rényi) entropy after local primary operator quenches in 2D CFTs with $T\bar{T}/J\bar{T}$-deformation. Using perturbation theory, we analyze the corrections to the second pseudo-Rényi entropy at the late time, which exhibit a universal form, while its early-time behavior is model-dependent. Moreover, we uncover nontrivial time-dependent effects arising from the first-order deformation of the kth pseudo-Rényi entropy at the late time. Additionally, drawing inspiration from the gravitational side, specifically the gluing of two cutoff AdS geometries, we investigate the kth pseudo-Rényi entropy for vacuum states characterized by distinct $T\bar{T}$-deformation parameters, as well as for primary states acting on different deformed vacuum states. Our findings reveal additional corrections compared to the results of pseudo-Rényi entropy for globally deformed vacuum states.

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Contents

1 Introduction

The AdS/CFT correspondence [1–3] provides a holographic description of a quantum gravity theory in an asymptotically anti-de Sitter (AdS) spacetime with a conformal field theory (CFT) at its asymptotic boundary. Its discovery has inspired significant exploration regarding quantum information theory in the high-energy physics community in recent years, covering a range of topics such as quantum entanglement [4–10], the emergence of geometry [11–13], and the black hole information paradox [14–18].

Recently, there has been a surge of interest in 2D CFTs deformed by $T\bar{T}/J\bar{T}$ operator [19–21]. Though the deformations are irrelevant in the sense of the renormalization group, owing to the integrability [19, 21] and holographic duality [22–26], they have been extensively investigated in recent years [27–53]. The $T\bar{T}/J\bar{T}$-deformation triggers a flow of the original theory along a trajectory in the space of field theory. For the $T\bar{T}$-deformation, the trajectory

of the deformed action satisfies the following condition:

$$\frac{dS(\lambda)}{d\lambda} = \int d^2 z \sqrt{g}(T\bar{T})_\lambda, \tag{1}$$

where λ represents the coupling constant of the $T\bar{T}$ operator. $S(\lambda = 0)$ corresponds to the action of the undeformed theory. In the flat space, the $T\bar{T}$ operator can be expressed as

$$T\bar{T} = -\frac{1}{4}\det(T_{\mu\nu}) = T_{zz}T_{\bar{z}\bar{z}} - T_{z\bar{z}}T_{z\bar{z}}, \tag{2}$$

with $T = T_{zz}$ and $\bar{T} = T_{\bar{z}\bar{z}}$. In this paper, we mainly focus on the perturbation theory of $S(\lambda)$. The $T\bar{T}$-deformed action, up to the first order of λ, is given by

$$S(\lambda) = S(\lambda = 0) + \lambda \int d^2 z \sqrt{g}(T\bar{T})_{\lambda=0} + \mathcal{O}(\lambda^2), \tag{3}$$

and we refer to $(T\bar{T})_{\lambda=0}$ as simply $T\bar{T}$. We treat the $J\bar{T}$-deformation in the same manner.

Pseudo entropy is a generalization of the entanglement entropy proposed via the AdS/CFT correspondence and post-selection [54]. Specifically, the pseudo entropy is a two-state vector version of the entanglement entropy, defined as follows. Let $|\psi\rangle$ and $|\varphi\rangle$ be two non-orthogonal states in the Hilbert space $\mathcal{H}_\mathcal{S}$ of a composite quantum system $\mathcal{S} = A \cup B$. We initially construct an operator acting on $\mathcal{H}_\mathcal{S}$ which is called the *transition matrix* [54,55]

$$\mathcal{T}^{\psi|\varphi} \equiv \frac{|\psi\rangle\langle\varphi|}{\langle\varphi|\psi\rangle} = \frac{\rho_\psi \rho_\varphi}{\mathrm{tr}[\rho_\psi \rho_\varphi]}. \tag{4}$$

We then define the so-called *reduced transition matrix* $\mathcal{T}_A^{\psi|\varphi}$ for subsystem A by tracing out the degrees of freedom in the subsystem B from the transition matrix (4), that is, $\mathcal{T}_A^{\psi|\varphi} = \mathrm{Tr}_B[\mathcal{T}^{\psi|\varphi}]$. The pseudo entropy of A is then defined by the von Neumann entropy of $\mathcal{T}_A^{\psi|\varphi}$, i.e.,

$$S_A = -\mathrm{tr}\big[\mathcal{T}_A^{\psi|\varphi} \log \mathcal{T}_A^{\psi|\varphi}\big]. \tag{5}$$

Furthermore, we can introduce the so-called *pseudo-Rényi entropy* $S_A^{(k)}$, a generalization of the Rényi entropy

$$S_A^{(k)} = \frac{1}{1-k} \log \mathrm{tr}[(\rho_A)^k], \tag{6}$$

utilizing the transition matrix. The pseudo-Rényi entropy is defined by replacing the reduced density matrix in the Rényi entropy (6) with the reduced transition matrix $\mathcal{T}_A^{\psi|\varphi}$,

$$S_A^{(k)} = \frac{1}{1-k} \log \mathrm{tr}\big[(\mathcal{T}_A^{\psi|\varphi})^k\big]. \tag{7}$$

Note that the pseudo-Rényi entropy (7) gives the pseudo entropy (5) in the limit of $k \to 1$.

It has been shown that pseudo entropy, like entanglement entropy, can be used to quantify the topological contributions of entanglement of excited states in two-dimensional rational CFTs

[56–58][5]. The behavior of the entanglement entropy in the deformed CFTs, especially in the context of the $T\bar{T}$- and $J\bar{T}$-deformations, has been extensively studied in recent years [74–78]. However, little is known about the pseudo-Rényi entropy in $T\bar{T}/J\bar{T}$-deformed CFTs. Another motivation for investigating pseudo-Rényi entropy in $T\bar{T}/J\bar{T}$-deformed CFTs comes from the holography. It is thought that pseudo entropy has its holographic dual, and the holographic dual of the positive sign $T\bar{T}$-deformed two-dimensional holographic CFT was proposed to be AdS$_3$ gravity with a finite radius cut-off. So the properties of the pseudo-Rényi entropy in the deformed CFTs would provide insight into the structure of the deformed theories and their relationship to the undeformed CFTs. Therefore, investigating the pseudo-Rényi entropy in the deformed CFTs is an interesting and important research direction.

This paper aims to investigate the pseudo-Rényi entropy in the $T\bar{T}/J\bar{T}$-deformed CFTs. Our inquiry can be traced back to the investigations on the Rényi entropy of vacuum states [74] and locally excited states [79] in the $T\bar{T}/J\bar{T}$-deformed theories. It has been observed that, with the first-order perturbation of the deformation, the second Rényi entanglement entropy of locally excited states acquires a nontrivial time dependence. It would be captivating to investigate the correction of the pseudo-Rényi entropy in the deformed CFTs. Besides, inspired by gluing together two cutoff AdS geometries and the cutoff AdS/$T\bar{T}$-deformed-CFT(cAdS/dCFT) correspondence, we are interested in exploring the pseudo-Rényi entropy of two states with two different deformation parameters.

The structure of this paper is as follows. In section 2, we briefly overview the $T\bar{T}$-deformation of the correlation functions and the replica method for locally excited states in 2D CFTs. In section 3, we concentrate on the second pseudo-Rényi entropy of the $T\bar{T}$-deformed theories. In section 4, we investigate the excess of the kth Rényi entropy of locally excited states and the kth pseudo-Rényi entropy of locally excited states in $T\bar{T}$-deformed CFTs. In section 5, we compute the kth pseudo-Rényi entropy of two vacuum states and two primary states with different deformation parameters in $T\bar{T}$-deformed CFTs. In section 6, we extend the analysis from sections 3 and 4 to the $J\bar{T}$-deformed theory. Finally, in section 7, we provide concluding remarks and discuss possible future research directions. Some technical details of the calculations are presented in the appendices.

2 Setup

2.1 The $T\bar{T}$-deformation of the correlation functions

In this subsection, we briefly overview the $T\bar{T}$-deformation in 2D CFTs and its effect on the correlation functions in [79].

By using the Ward identity, the first-order correction to the k-point correlation function of

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[5]Please refer to [59–73] for other recent development of pseudo entropy.

primary operators in $T\bar{T}$-deformed CFTs can be expressed as $\langle \mathcal{O}_1(z_1, \bar{z}_1) \cdots \mathcal{O}_k(z_k, \bar{z}_k) \rangle_\lambda$, where λ is the coupling constant of the $T\bar{T}$-deformation. The expression for the first-order correction is as follows:

$$
\begin{aligned}
&\langle \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \ldots \mathcal{O}_k(z_k, \bar{z}_k) \rangle_\lambda \\
=& \lambda \int d^2 z \, \langle T\bar{T}(z, \bar{z}) \mathcal{O}_1(z_1, \bar{z}_1) \, \mathcal{O}_2(z_2, \bar{z}_2) \cdots \mathcal{O}_k(z_k, \bar{z}_k) \rangle \\
=& \lambda \int d^2 z \left(\sum_{i=1}^{k} \left(\frac{h_i}{(z - z_i)^2} + \frac{\partial_{z_i}}{z - z_i} \right) \right) \left(\sum_{i=1}^{k} \left(\frac{\bar{h}_i}{(\bar{z} - \bar{z}_i)^2} + \frac{\partial_{\bar{z}_i}}{\bar{z} - \bar{z}_i} \right) \right) \\
& \times \langle \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \ldots \mathcal{O}_k(z_k, \bar{z}_k) \rangle.
\end{aligned}
\tag{8}
$$

To obtain the result, we take advantage of the fact that any correlation function involving $T_{z\bar{z}}$ vanishes in a 2D CFT. Specifically, we begin with the two-point correlation function of primary operators $\mathcal{O}_1$ and $\mathcal{O}_2$ with conformal weights $(h, \bar{h})$. The two-point function of $\mathcal{O}_1$ and $\mathcal{O}_2$ can be written as

$$
\langle \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \rangle = \frac{C_{12}}{z_{12}^{2h} \bar{z}_{12}^{2\bar{h}}},
\tag{9}
$$

where $z_{ij} = z_i - z_j$, $\bar{z}_{ij} = \bar{z}_i - \bar{z}_j$. The Ward identity of the two-point function gives us the following relation:

$$
\begin{aligned}
&\langle T(z) \bar{T}(\bar{z}) \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \rangle \\
=& \Bigg(\frac{h\bar{h} z_{12}^2 \bar{z}_{12}^2}{(z - z_1)^2 (z - z_2)^2 (\bar{z} - \bar{z}_1)^2 (\bar{z} - \bar{z}_2)^2} - \frac{4\pi \bar{h}}{(z - z_1)(\bar{z}_1 - \bar{z}_2)} \delta^{(2)}(z - z_1) \\
& - \frac{2\pi \bar{h}}{(z - z_2)(\bar{z}_1 - \bar{z}_2)} \delta^{(2)}(z - z_2) \Bigg) \langle \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \rangle.
\end{aligned}
\tag{10}
$$

The appearance of the delta function arises from the action of ∂_{z_i} on terms such as $(\bar{z} - \bar{z}_i)^{-1}$, which leads to a non-dynamical contribution. After the integration over spacetime, this contribution gives rise to a UV divergence. To simplify the following analysis, we omit these terms. We can obtain the first-order correction of the two-point function resulting from the $T\bar{T}$-deformation using dimension regularization

$$
\langle \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \rangle_\lambda = \lambda h \bar{h} \frac{4\pi}{|z_{12}|^2} \left(\frac{4}{\epsilon} + 2\log(|z_{12}|^2) + 2\log \pi + 2\gamma - 5 \right) \langle \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \rangle.
\tag{11}
$$

Next, we compute the four-point function within the framework of $T\bar{T}$-deformed CFTs. Specifically, the four-point function of primary operators can be written as follows:

$$
\langle \mathcal{O}_1^\dagger(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \mathcal{O}_1^\dagger(z_3, \bar{z}_3) \mathcal{O}_2(z_4, \bar{z}_4) \rangle = \frac{G(\eta, \bar{\eta})}{z_{13}^{2h} z_{23}^{2h} \bar{z}_{13}^{2\bar{h}} \bar{z}_{24}^{2\bar{h}}},
\tag{12}
$$

with the cross ratios

$$
\eta := \frac{z_{12} z_{34}}{z_{13} z_{24}}, \quad \bar{\eta} := \frac{\bar{z}_{12} \bar{z}_{34}}{\bar{z}_{13} \bar{z}_{24}},
\tag{13}
$$

4

and $G(\eta, \bar{\eta})$ is the conformal block which is a function of cross ratios.

The first-order correction to this four-point function due to the $T\bar{T}$-deformation is given by:

$$
\langle \mathcal{O}_1^\dagger(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \mathcal{O}_1^\dagger(z_3, \bar{z}_3) \mathcal{O}_2(z_4, \bar{z}_4) \rangle_\lambda
$$

$$
= \lambda \int d^2 z \left\{ \left(\frac{h z_{13}^2}{(z - z_1)^2 (z - z_3)^2} + \frac{h z_{24}^2}{(z - z_2)^2 (z - z_4)^2} + \frac{z_{23} z_{14}}{\prod_{i=1}^4 (z - z_i)} \frac{\eta G(\eta, \bar{\eta})}{G(\eta, \bar{\eta})} \right) \right.
$$

$$
\times \left(\frac{\bar{h} \bar{z}_{13}^2}{(\bar{z} - \bar{z}_1)^2 (\bar{z} - \bar{z}_3)^2} + \frac{\bar{h} \bar{z}_{24}^2}{(\bar{z} - \bar{z}_2)^2 (\bar{z} - \bar{z}_4)^2} + \frac{\bar{z}_{23} \bar{z}_{14}}{\prod_{i=1}^4 (\bar{z} - \bar{z}_i)} \frac{\bar{\eta} G(\eta, \bar{\eta})}{G(\eta, \bar{\eta})} \right)
$$

$$
- \eta \bar{\eta} \frac{z_{23} z_{14} \bar{z}_{23} \bar{z}_{14}}{\prod_{i=1}^4 (z - z_i)(\bar{z} - \bar{z}_i)} \frac{\partial_\eta G(\eta, \bar{\eta}) \partial_{\bar{\eta}} G(\eta, \bar{\eta})}{G(\eta, \bar{\eta})^2} + \eta \bar{\eta} \frac{z_{23} z_{14} \bar{z}_{23} \bar{z}_{14}}{\prod_{i=1}^4 (z - z_i)(\bar{z} - \bar{z}_i)} \frac{\partial_\eta \partial_{\bar{\eta}} G(\eta, \bar{\eta})}{G(\eta, \bar{\eta})}
$$

$$
- \frac{4\pi \bar{h} \bar{z}_{13}^2}{(z - z_1)(\bar{z} - \bar{z}_1)(\bar{z} - \bar{z}_3)^2} \delta^{(2)}(z - z_1) - \frac{4\pi \bar{h} \bar{z}_{13}^2}{(z - z_3)(\bar{z} - \bar{z}_3)(\bar{z} - \bar{z}_1)^2} \delta^{(2)}(z - z_3)
$$

$$
- \frac{4\pi \bar{h} \bar{z}_{24}^2}{(z - z_2)(\bar{z} - \bar{z}_2)(\bar{z} - \bar{z}_4)^2} \delta^{(2)}(z - z_2) - \frac{4\pi \bar{h} \bar{z}_{24}^2}{(z - z_4)(\bar{z} - \bar{z}_4)(\bar{z} - \bar{z}_2)^2} \delta^{(2)}(z - z_2)
$$

$$
- \frac{\bar{z}_{12} \bar{z}_{34} \partial_{\bar{\eta}} G(\eta, \bar{\eta})}{G(\eta, \bar{\eta})} \left[\frac{2\pi \delta^{(2)}(z - z_1)}{(z - z_1)(\bar{z} - \bar{z}_2)(\bar{z} - \bar{z}_3)(\bar{z} - \bar{z}_4)} + \frac{2\pi \delta^{(2)}(z - z_2)}{(z - z_2)(\bar{z} - \bar{z}_1)(\bar{z} - \bar{z}_3)(\bar{z} - \bar{z}_4)} \right.
$$

$$
\left. + \frac{2\pi \delta^{(2)}(z - z_3)}{(z - z_3)(\bar{z} - \bar{z}_1)(\bar{z} - \bar{z}_2)(\bar{z} - \bar{z}_4)} + \frac{2\pi \delta^{(2)}(z - z_4)}{(z - z_4)(\bar{z} - \bar{z}_1)(\bar{z} - \bar{z}_2)(\bar{z} - \bar{z}_3)} \right] \right\}
$$

$$
\times \langle \mathcal{O}_1^\dagger(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \mathcal{O}_1^\dagger(z_3, \bar{z}_3) \mathcal{O}_2(z_4, \bar{z}_4) \rangle. \tag{14}
$$

2.2 The pseudo entropy and the replica method

This subsection presents a concise overview of the replica method for constructing the pseudo-Rényi entropy with locally excited states in rational conformal field theories (RCFTs). We start by considering a RCFT defined on a plane, where the vacuum state is denoted as $|\Omega\rangle$. By applying the primary operators $\mathcal{O}(x_1)$ and $\mathcal{O}^\dagger(x_2)$ to the vacuum state, we create two locally excited states

$$
|\psi_1\rangle \equiv e^{-\epsilon H} \mathcal{O}(x_1)|\Omega\rangle, \quad |\psi_2\rangle \equiv e^{-\epsilon H} \mathcal{O}^\dagger(x_2)|\Omega\rangle, \tag{15}
$$

where the infinitesimal parameter ϵ is introduced to suppress the high energy modes [80]. Subsequently, we utilize these states to construct a real-time evolved transition matrix denoted as $\mathcal{T}^{1|2}(t)$:

$$
\mathcal{T}^{1|2}(t) \equiv e^{-iHt} \frac{|\psi_1\rangle\langle\psi_2|}{\langle\psi_2|\psi_1\rangle} e^{iHt}. \tag{16}
$$

To obtain the reduced transition matrix of subsystem A at time t, we trace out the degrees of freedom of A^c (the complement of A) from the transition matrix $\mathcal{T}^{1|2}(t)$, giving us $\mathcal{T}_A^{1|2}(t) = \text{tr}A^c[\mathcal{T}^{1|2}(t)]$.

It turns out that the excess of the kth pseudo-Rényi entropy of A with respect to the ground

state, defined as $\Delta S^{(k)}(\mathcal{T}_A^{1|2}(t)) := S^{(k)}(\mathcal{T}_A^{1|2}(t)) - S^{(k)}\big(\mathrm{tr}_{A^c}\big[|\Omega\rangle\langle\Omega|\big]\big)$, is of the form [57]

$$\Delta S^{(k)}(\mathcal{T}_A^{1|2}(t)) = \frac{1}{1-k}\Big[\log\langle\prod_{j=1}^{k}\mathcal{O}(w_{2j-1},\bar{w}_{2j-1})\mathcal{O}^\dagger(w_{2j},\bar{w}_{2j})\rangle_{\Sigma_k}$$
$$-k\log\langle\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\rangle_{\Sigma_1}\Big], \tag{17}$$

according to the replica method. In Eq.(17), Σ_k denotes an k-sheet Riemann surface with cuts on each copy corresponding to A, and $(w_{2j-1},\bar{w}_{2j-1})$ and $(w_{2j},\bar{w}_{2j})$ are coordinates on the jth-sheet surface. The term in the first line of Eq.(17) is given by a $2k$-point correlation function on Σ_k, and the one in the second line is calculated from a two-point function on Σ_1. The coordinates are chosen as

$$w_{2j-1} = x_1 + t - i\epsilon, \quad w_{2j} = x_2 + t + i\epsilon,$$
$$\bar{w}_{2j-1} = x_1 - t + i\epsilon, \quad \bar{w}_{2j} = x_2 - t - i\epsilon. \quad (j=1,2,...,k) \tag{18}$$

The $2k$-point correlation function on Σ_k in Eq.(17) can be evaluated with the help of a conformal mapping of Σ_k to the complex plane Σ_1. For convenience, the subsystem is chosen such that $A = [0,\infty)$ hereafter. We can then map Σ_k to Σ_1 using the simple conformal mapping

$$w = z^k. \tag{19}$$

We subsequently utilize the mapping given by Eq. (19) to express the $2k$-point function on Σ_k,

$$\langle\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\ldots\mathcal{O}(w_{2k-1},\bar{w}_{2k-1})\mathcal{O}^\dagger(w_{2k},\bar{w}_{2k})\rangle_{\Sigma_k}$$
$$= \left(\prod_{j=1}^{2k}\left|\frac{dw_j}{dz_j}\right|^{-2h}\right)\langle\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\ldots\mathcal{O}(w_{2k-1},\bar{w}_{2k-1})\mathcal{O}^\dagger(w_{2k},\bar{w}_{2k})\rangle_{\Sigma_1}. \tag{20}$$

The $2k$ points $z_1, z_2, ..., z_{2k}$ in the z-coordinates are given by

$$z_{2j+1} = e^{2\pi i\frac{j+1/2}{k}}(-x_1 - t + i\epsilon)^{\frac{1}{k}}, \quad \bar{z}_{2j+1} = e^{-2\pi i\frac{j+1/2}{k}}(-x_1 + t - i\epsilon)^{\frac{1}{k}},$$
$$z_{2j+2} = e^{2\pi i\frac{j+1/2}{k}}(-x_2 - t - i\epsilon)^{\frac{1}{k}}, \quad \bar{z}_{2j+2} = e^{-2\pi i\frac{j+1/2}{k}}(-x_2 + t + i\epsilon)^{\frac{1}{k}}, \quad (j=0,...,k-1). \tag{21}$$

3 Second pseudo-Rényi entropy in $T\bar{T}$-deformed CFTs

The late-time excess of pseudo-Rényi entropy of locally excited operators has a universal behavior: the logarithmic quantum dimension of the operator, while the early-time one does not. Investigating the late-time behavior of the excess of pseudo-Rényi entropy and its behavior in the early time in $T\bar{T}$-deformed CFTs is intriguing.

In [74], the authors discussed Rényi entropy in the $T\bar{T}$-deformed CFTs for vacuum theory. It is shown that up to the first order of the deformation parameter λ,

$$\Delta S_{A,\lambda}^{(k)} = \frac{-k\lambda}{1-k}\int_{\Sigma_1}[\langle T\bar{T}\rangle_{\Sigma_k} - \langle T\bar{T}\rangle_{\Sigma_1}] + \mathcal{O}(\lambda^2), \tag{22}$$

where we denote $\Delta S_{A,\lambda}^{(k)}$ as the correction to the undeformed Rényi entropy.

Now let us consider second pseudo-Rényi entropy in $T\bar{T}$-deformed CFTs

$$
\begin{aligned}
& -\left(\Delta S_{A,0}^{(2)} + \Delta S_{A,\lambda}^{(2)}\right) \\
&= \log\left(\langle\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\mathcal{O}(w_3,\bar{w}_3)\mathcal{O}^\dagger(w_4,\bar{w}_4)\rangle_{\Sigma_2}\right. \\
&\left. + 2\lambda\int d^2w\langle T\bar{T}(w,\bar{w})\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\mathcal{O}(w_3,\bar{w}_3)\mathcal{O}^\dagger(w_4,\bar{w}_4)\rangle_{\Sigma_2}\right) \\
& - 2\log\left(\langle\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\rangle_{\Sigma_1} + \lambda\int d^2w\langle T\bar{T}(w,\bar{w})\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\rangle_{\Sigma_1}\right). \quad (23)
\end{aligned}
$$

The derivation of (23) is a simplified version of Appendix B where we set $\lambda_1 = \lambda_2$ and $k = 2$ [6].

Under the conformal mapping (19) for $k = 2$, the stress tensors transform as

$$
T(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}\{z,w\}, \quad \bar{T}(\bar{w}) = \left(\frac{d\bar{z}}{d\bar{w}}\right)^2 \bar{T}(\bar{z}) + \frac{c}{12}\{\bar{z},\bar{w}\}, \quad (24)
$$

where

$$
\{z,w\} = \frac{z'''}{z'} - \frac{3}{2}\frac{z''^2}{z'^2}, \quad \left(z' = \frac{dz}{dw}\right)
$$

is the Schwarzian derivative, and c is the central charge. The $T\bar{T}$ operator thus transforms as

$$
T\bar{T}(w,\bar{w}) = \frac{1}{4z^2}\frac{1}{4\bar{z}^2}\left(T(z) + \frac{c}{8z^2}\right)\left(\bar{T}(\bar{z}) + \frac{c}{8\bar{z}^2}\right). \quad (25)
$$

By utilizing this transformation formula and performing an expansion around $\lambda = 0$, we obtain expressions for the $zero^{th}$ and first-order terms in λ:

$$
\begin{aligned}
& -\left(\Delta S_{A,0}^{(2)} + \Delta S_{A,\lambda}^{(2)}\right) \\
&= \log\left(\prod_{i=1}^{4}\left|\frac{dw_i}{dz_i}\right|^{-2h}\frac{\langle\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\mathcal{O}(w_3,\bar{w}_3)\mathcal{O}^\dagger(w_4,\bar{w}_4)\rangle_{\Sigma_1}}{\left(\langle\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\rangle_{\Sigma_1}\right)^2}\right) \\
& + 2\lambda\int d^2z\frac{1}{4|z|^2}\frac{\langle\left(T(z) + \frac{c}{8z^2}\right)\left(\bar{T}(\bar{z}) + \frac{c}{8\bar{z}^2}\right)\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\mathcal{O}(w_3,\bar{w}_3)\mathcal{O}^\dagger(w_4,\bar{w}_4)\rangle_{\Sigma_1}}{\langle\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\mathcal{O}(w_3,\bar{w}_3)\mathcal{O}^\dagger(w_4,\bar{w}_4)\rangle_{\Sigma_1}} \\
& - 2\lambda\int d^2w\frac{\langle T\bar{T}(w,\bar{w})\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\rangle_{\Sigma_1}}{\langle\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\rangle_{\Sigma_1}} + \mathcal{O}(\lambda^2). \quad (26)
\end{aligned}
$$

Let us focus on the large c case. At the leading order, that is the $\mathcal{O}(c^2)$ order, Eq. (26) is evaluated as

$$
2\lambda\frac{c^2}{64}\int d^2z\frac{1}{4|z|^6} = 6\lambda\frac{2\pi c^2}{64}\int_0^\infty d\rho\frac{1}{4\rho^5}. \quad (27)
$$

To regularize this formula, we introduce the IR cutoff and UV cutoff by replacing $(0,\infty)$ with $(\frac{1}{\tilde{\Lambda}},\Lambda)$ and we obtain

$$
2\lambda\frac{c^2}{64}\int d^2z\frac{1}{4|z|^6} \xrightarrow{\text{cutoff}} 6\lambda\frac{2\pi c^2}{64}\frac{\tilde{\Lambda}^4}{16}. \quad (28)
$$

---

[6]Notice that there are some typos in [79], the exact coupling in its section 3 and section 5.1 should be $\frac{\lambda}{3}$ instead of λ.

At the next order, which is of order $\mathcal{O}(c)$, Eq. (26) can be evaluated as

$$\frac{c}{8}2\lambda \int d^2z \frac{1}{4\left|z\right|^2} \frac{\left\langle \left(\frac{1}{\bar{z}^2}T(z) + \frac{1}{z^2}\bar{T}(\bar{z})\right)\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\mathcal{O}(w_3,\bar{w}_3)\mathcal{O}^\dagger(w_4,\bar{w}_4)\right\rangle_{\Sigma_1}}{\left\langle \mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\mathcal{O}(w_3,\bar{w}_3)\mathcal{O}^\dagger(w_4,\bar{w}_4)\right\rangle_{\Sigma_1}}$$

$$=2\lambda\frac{c\pi}{64}\left(\frac{h}{\left|z_1\right|^4} + \frac{h}{\left|z_2\right|^4} + \frac{h}{\left|z_3\right|^4} + \frac{h}{\left|z_4\right|^4} - \frac{2hz_1}{z_{13}\left|z_1\right|^4} + \frac{2hz_3}{z_{13}\left|z_3\right|^4} - \frac{2hz_2}{z_{24}\left|z_2\right|^4} + \frac{2hz_4}{z_{24}\left|z_4\right|^4}\right.$$

$$\left.+\left[\frac{z_1}{\left|z_1\right|^4}\left(-\frac{1}{z_{13}} + \frac{1}{z_{12}}\right) + \frac{z_2}{\left|z_2\right|^4}\left(-\frac{1}{z_{24}} - \frac{1}{z_{12}}\right) + \frac{z_3}{\left|z_3\right|^4}\left(\frac{1}{z_{34}} + \frac{1}{z_{13}}\right) + \frac{z_4}{\left|z_4\right|^4}\left(-\frac{1}{z_{34}} + \frac{1}{z_{24}}\right)\right]\frac{\eta\partial_\eta G(\eta,\bar{\eta})}{G(\eta,\bar{\eta})}\right)$$

$$+2\lambda\frac{c\pi}{64}\left(\frac{h}{\left|\bar{z}_1\right|^4} + \frac{h}{\left|\bar{z}_2\right|^4} + \frac{h}{\left|\bar{z}_3\right|^4} + \frac{h}{\left|\bar{z}_4\right|^4} - \frac{2h\bar{z}_1}{\bar{z}_{13}\left|\bar{z}_1\right|^4} + \frac{2h\bar{z}_3}{\bar{z}_{13}\left|\bar{z}_3\right|^4} - \frac{2h\bar{z}_2}{\bar{z}_{24}\left|\bar{z}_2\right|^4} + \frac{2h\bar{z}_4}{\bar{z}_{24}\left|\bar{z}_4\right|^4}\right.$$

$$\left.+\left[\frac{\bar{z}_1}{\left|\bar{z}_1\right|^4}\left(-\frac{1}{\bar{z}_{13}} + \frac{1}{\bar{z}_{12}}\right) + \frac{\bar{z}_2}{\left|\bar{z}_2\right|^4}\left(-\frac{1}{\bar{z}_{24}} - \frac{1}{\bar{z}_{12}}\right) + \frac{\bar{z}_3}{\left|\bar{z}_3\right|^4}\left(\frac{1}{\bar{z}_{34}} + \frac{1}{\bar{z}_{13}}\right) + \frac{\bar{z}_4}{\left|\bar{z}_4\right|^4}\left(-\frac{1}{\bar{z}_{34}} + \frac{1}{\bar{z}_{24}}\right)\right]\frac{\bar{\eta}\partial_{\bar{\eta}} G(\eta,\bar{\eta})}{G(\eta,\bar{\eta})}\right).$$

$$\tag{29}$$

The derivation of this equation is shown in Appendix A. When we take the late-time limit $t \to \infty$, the conformal block in the four-point correlation function is

$$G(\eta,\bar{\eta}) \sim \mathcal{F}_{00}(1-\eta)^{-2h}\bar{\eta}^{-2h}, \tag{30}$$

where $(\mathcal{F}_{00})^{-1} = d_{\mathcal{O}}$ is the quantum dimension of $\mathcal{O}$.

Substitute (30) into the (29) and we find the correction to the second pseudo-Rényi entropy in large c limit at the late time:

$$-\Delta S_{A,\lambda}^{(2)} = 6\lambda\frac{2\pi c^2}{64}\frac{\tilde{\Lambda}^4}{16} + 2\lambda\frac{4(x_1^2 - x_2^2)hc\pi}{64}$$

$$\left(\frac{-t^2\left(\sqrt{-t-x_1}\sqrt{t-x_1} + \sqrt{-t-x_2}\sqrt{t-x_2}\right)}{(-t-x_1)^{3/2}(t-x_1)^{3/2}(-t-x_2)^{3/2}(t-x_2)^{3/2}\left(\sqrt{-t-x_1} + \sqrt{-t-x_2}\right)\left(\sqrt{t-x_1} - \sqrt{t-x_2}\right)}\right.$$

$$+\left.\frac{x_1^2\sqrt{-t-x_2}\sqrt{t-x_2} + x_2^2\sqrt{-t-x_1}\sqrt{t-x_1}}{(-t-x_1)^{3/2}(t-x_1)^{3/2}(-t-x_2)^{3/2}(t-x_2)^{3/2}\left(\sqrt{-t-x_1} + \sqrt{-t-x_2}\right)\left(\sqrt{t-x_1} - \sqrt{t-x_2}\right)}\right)$$

$$+\mathcal{O}(\epsilon, c^0, \lambda^2). \tag{31}$$

This result is universal in all RCFTs. However, if the two insertion coordinates coincide, Eq. (31) does not yield the same result as the second R'enyi entropy derived in [79]. This discrepancy arises primarily because in [79], we initially set $x_1 \to x_2$ and perform an expansion in ϵ, whereas in the case of the second pseudo-R'enyi entropy, we first expand in ϵ and then take the limit $x_1 \to x_2$. Notably, the two limits, $\epsilon \to 0$ and $x_1 \to x_2$, do not commute, as demonstrated in [58].

Since the two insertion coordinates are different, the conformal block $G(\eta,\bar{\eta})$ may diverge for some models at the early time. The rest of this subsection will discuss the early-time correction to the second pseudo-Rényi entropy in $T\bar{T}$-deformed CFTs for the free scalar and Ising model as examples.

Example 1. The free scalar

The free scalar can be described by the operator $\mathcal{O} = \frac{1}{\sqrt{2}}(e^{\frac{i}{2}\phi} + e^{-\frac{i}{2}\phi})$, whose conformal dimension is $h = \bar{h} = \frac{1}{8}$. The conformal block of free scalar can be written as

$$G(\eta, \bar{\eta}) = \frac{1 + |\eta| + |1 - \eta|}{2\sqrt{|\eta|\,|1-\eta|}}.$$ (32)

Substituting (32) into (29), we find that the corrections to the second pseudo-Rényi entropy of free scalar at the early time in large c limit is

$$
\begin{aligned}
-\Delta S_{A,\lambda}^{(2)} =& 6\lambda \frac{2\pi c^2}{64} \frac{\tilde{\Lambda}^4}{16} + \frac{1}{x_1^2 x_2^2 (x_1 - x_2)^4} \left(4x_1^6 - 16x_2 x_1^5 + 2x_2^2 \left(\left(\frac{(x_1 - x_2)^4}{x_1^2 x_2^2} \right)^{3/4} + 14 \right) x_1^4 \right.\\
& + \left(\sqrt{-x_1}\sqrt{-x_2} \left(\sqrt{\frac{x_1^2 + 4\sqrt{-x_1}\sqrt{-x_2}x_1 + 6x_2 x_1 + x_2^2 + 4\sqrt{-x_1}\sqrt{-x_2}x_2}{x_1 x_2}} \right.\right.\\
& \left.\left. - \sqrt{\frac{4\sqrt{-x_2}(-x_1)^{3/2} + 4\sqrt{-x_1}(-x_2)^{3/2} + x_1^2 + x_2^2 + 6x_1 x_2}{x_1 x_2}} \right) \left(\frac{(x_1 - x_2)^4}{x_1^2 x_2^2} \right)^{3/4} \right.\\
& + 4x_2 \left(\left(\frac{(x_1 - x_2)^4}{x_1^2 x_2^2} \right)^{3/4} - 8 \right) \right) x_2^2 x_1^3\\
& + \left(\sqrt{-x_1}\sqrt{-x_2} \left(\sqrt{\frac{x_1^2 + 4\sqrt{-x_1}\sqrt{-x_2}x_1 + 6x_2 x_1 + x_2^2 + 4\sqrt{-x_1}\sqrt{-x_2}x_2}{x_1 x_2}} \right.\right.\\
& \left.\left. - \sqrt{\frac{4\sqrt{-x_2}(-x_1)^{3/2} + 4\sqrt{-x_1}(-x_2)^{3/2} + x_1^2 + x_2^2 + 6x_1 x_2}{x_1 x_2}} \right) \left(\frac{(x_1 - x_2)^4}{x_1^2 x_2^2} \right)^{3/4} \right.\\
& \left. + 2x_2 \left(\left(\frac{(x_1 - x_2)^4}{x_1^2 x_2^2} \right)^{3/4} + 14 \right) \right) x_2^3 x_1^2 - 16x_2^5 x_1 + 4x_2^6 \right).
\end{aligned}
$$ (33)

Notice that the results (31) and (33) are highly different from that in [79] because the insertion coordinates of two states generating the transition matrix are different.

Example 2. Ising model

The $2n$-point correlation function of the spin operator in the Ising model can be written as

$$\langle \sigma(z_1, \bar{z}_1) \ldots \sigma(z_{2n}, \bar{z}_{2n}) \rangle^2 = \frac{1}{2^n} \sum_{\substack{\epsilon_i = \pm 1 \\ i=1,\ldots,2n \\ \sum \epsilon_i = 0}} \prod_{i<j} |z_i - z_j|^{\epsilon_i \epsilon_j / 2}$$ (34)

The first-order corrections to the second pseudo-Rényi entropy of the spin operator at the early time is

$$
\begin{aligned}
-\Delta S_{A,\lambda}^{(2)} =& 6\lambda \frac{2\pi c^2}{64} \frac{\tilde{\Lambda}^4}{16} + 2\lambda \frac{c\pi}{64} \left(\frac{-(x_1^2)^{3/2} + 3\sqrt{x_2^2}x_1^2 + 3\sqrt{x_1^2}x_2^2 + 4x_1 x_2 \left(\sqrt[4]{x_1^2} + \sqrt[4]{x_2^2} \right)^2}{4(x_1^2)^{9/8}\sqrt{-2\sqrt{-x_1}\sqrt{-x_2} - x_1 - x_2}\sqrt{2\sqrt{-x_1}\sqrt{-x_2} - x_1 - x_2}(x_2^2)^{9/8}} \right.\\
& \left. + \frac{2x_2^2 \sqrt[4]{x_1^2} + 2(x_1^2)^{5/4} - (x_2^2)^{5/4}}{4(x_1^2)^{9/8}\sqrt{-2\sqrt{-x_1}\sqrt{-x_2} - x_1 - x_2}\sqrt{2\sqrt{-x_1}\sqrt{-x_2} - x_1 - x_2}(x_2^2)^{7/8}} + \frac{4}{x_1^2} + \frac{4}{x_2^2} \right) + \mathcal{O}(\epsilon, c^0, \lambda^2).
\end{aligned}
$$ (35)

Since all correlation functions in the Ising model are known, one can easily calculate the correction to the pesudo-Rényi entropy in $T\bar{T}$-deformed CFTs at the early time. As Eq. (35) shows, the earlier time behavior of the pesudo-Rényi entropy depends on the initial setup and details of the theories.

4 Pseudo entropy in $T\bar{T}$-deformed CFTs

In the previous section, we perform a perturbative calculation of the second pseudo-Rényi entropy in $T\bar{T}$-deformed CFT at the late time and corrections to the second pseudo-Rényi entropy of free scalar and Ising model at the early time. However, to obtain the correction to the pseudo entropy, it is reasonable to examine the kth pseudo-Rényi entropy and perform an analytic continuation for k goes to 1 in $T\bar{T}$-deformed CFT. As demonstrated earlier, the conformal block for the second pseudo-Rényi entropy did not have a universal structure at the early stage. Therefore, in this section, firstly, we will investigate the kth Rényi entropy at both the early and the late time in $T\bar{T}$-deformed CFTs. Then we only consider the $T\bar{T}$-deformation of the kth pseudo-Rényi entropy at the late time.

4.1 k^{th} Rényi entropy in $T\bar{T}$-deformed CFTs

Under the conformal map (19), the $T\bar{T}$ operators transform as

$$T(w) = \frac{1}{k^2 z^{2k-2}}\left(T(z) + \frac{c(k^2-1)}{24}\frac{1}{z^2}\right), \quad \bar{T}(\bar{w}) = \frac{1}{k^2 \bar{z}^{2k-2}}\left(\bar{T}(\bar{z}) + \frac{c(k^2-1)}{24}\frac{1}{\bar{z}^2}\right). \quad (36)$$

The correction to the undeformed Rényi entropy seen in appendix B is

$$\Delta S_{A,\lambda}^{(k)} = \frac{k\lambda}{k-1}\int_{\Sigma_1} d^2w \left(\frac{\langle T\bar{T}(w,\bar{w})\mathcal{O}_1...\mathcal{O}_{2k}^\dagger\rangle_{\Sigma_k}}{\langle \mathcal{O}_1...\mathcal{O}_{2k}^\dagger\rangle_{\Sigma_k}} - \frac{\langle T\bar{T}(w,\bar{w})\mathcal{O}_1\mathcal{O}_2^\dagger\rangle_{\Sigma_1}}{\langle \mathcal{O}_1\mathcal{O}_2^\dagger\rangle_{\Sigma_1}}\right), \quad (37)$$

where $\mathcal{O}(z_i,\bar{z}_i)$ is denoted by $\mathcal{O}_i$. Notice that if we take all primary operators as identity operators, (37) becomes the Rényi entropy for vacuum states in $T\bar{T}$-deformed CFTs, and it matches the result in [74] precisely.

Using the conformal transformation (19) between Σ_k and Σ_1, we can rewrite this formula as

$$\Delta S_{A,\lambda}^{(k)} = \frac{k\lambda}{k-1}\int_{\Sigma_1} d^2z \frac{1}{k^2 |z|^{2k-2}}$$
$$\left(\frac{\left\langle (T(z)+\frac{c(k^2-1)}{24}\frac{1}{z^2})(\bar{T}(\bar{z})+\frac{c(k^2-1)}{24}\frac{1}{\bar{z}^2})\mathcal{O}_1(z_1,\bar{z}_1)...\mathcal{O}_{2k}^\dagger(z_{2k},\bar{z}_{2k})\right\rangle_{\Sigma_1}}{\langle \mathcal{O}_1(z_1,\bar{z}_1)...\mathcal{O}_{2k}^\dagger(z_{2k},\bar{z}_{2k})\rangle_{\Sigma_1}} - \frac{\langle T\bar{T}(w,\bar{w})\mathcal{O}_1\mathcal{O}_2^\dagger\rangle_{\Sigma_1}}{\langle \mathcal{O}_1\mathcal{O}_2^\dagger\rangle_{\Sigma_1}}\right).$$
$$(38)$$

We still focus on the large c case. At the leading order $\mathcal{O}(c^2)$, by introducing the IR cutoff $\frac{1}{\tilde{\Lambda}}$, the correction to the kth Rényi entropy is

$$\frac{k}{k-1}\lambda\frac{c^2(k^2-1)^2}{576}\int d^2z\frac{1}{k^2\,|z|^{2k+2}}\xrightarrow{\text{cutoff}}3k\lambda\frac{c^2(k^2-1)^2\pi\tilde{\Lambda}^{2k}}{576k^3}. \tag{39}$$

The derivation of this formula is shown in Appendix A. At the next-to-leading order, the integral in the first line of (38) can be divided into two parts at the late time. The first part, due to $T(z)$ acting on the $2k$-point correlation function, can be written as

$$\Delta S^{(k)}_{A,\lambda,part1} \equiv \frac{k\lambda}{k-1}\int_{\Sigma_1} d^2z\frac{1}{k^2\,|z|^{2k-2}}\frac{c(k^2-1)}{24}\frac{1}{\bar{z}^2}\frac{\langle(T(z)\mathcal{O}_1(z_1,\bar{z}_1)...\mathcal{O}^\dagger_{2k}(z_{2k},\bar{z}_{2k})\rangle_{\Sigma_1}}{\langle\mathcal{O}_1(z_1,\bar{z}_1)...\mathcal{O}^\dagger_{2k}(z_{2k},\bar{z}_{2k})\rangle_{\Sigma_1}}$$

$$=\frac{k}{k-1}\lambda\frac{1}{k^2}\frac{c(k^2-1)}{24}\int d^2z\frac{1}{|z|^{2k-2}\,\bar{z}^2}\frac{1}{\langle\mathcal{O}(z_1,\bar{z}_1)\mathcal{O}^\dagger(z_2,\bar{z}_2)\ldots\mathcal{O}(z_{2k-1},\bar{z}_{2k-1})\mathcal{O}^\dagger(z_{2k},\bar{z}_{2k})\rangle_{\Sigma_1}}$$

$$\left(\sum_{i=1}^{2k}\frac{h}{(z-z_i)^2}+\sum_{j=0}^{k-2}\left(\frac{\partial_{2j+1}}{z-z_{2j+1}}+\frac{\partial_{2j+4}}{z-z_{2j+4}}\right)+\frac{\partial_2}{z-z_2}+\frac{\partial_{2k-1}}{z-z_{2k-1}}\right)$$

$$\times\langle\mathcal{O}(z_1,\bar{z}_1)\mathcal{O}^\dagger(z_2,\bar{z}_2)\ldots\mathcal{O}(z_{2k-1},\bar{z}_{2k-1})\mathcal{O}^\dagger(z_{2k},\bar{z}_{2k})\rangle_{\Sigma_1}$$

$$=\frac{k}{k-1}\lambda\frac{1}{k^2}\frac{c(k^2-1)}{24}\left(\sum_{i=1}^{2k}\frac{h\pi}{k}\frac{1}{|z_i|^{2k}}+\sum_{j=0}^{k-2}\frac{\pi}{k}\left(\frac{-2h}{z_{2j+1}-z_{2j+4}}\frac{z_{2j+1}}{|z_{2j+1}|^{2k}}+\frac{2h}{z_{2j+1}-z_{2j+4}}\frac{z_{2j+4}}{|z_{2j+4}|^{2k}}\right)\right.$$

$$\left.+\frac{\pi}{k}\left(\frac{-2h}{z_2-z_{2k-1}}\frac{z_2}{|z_2|^2}+\frac{2h}{z_2-z_{2k-1}}\frac{z_{2k-1}}{|z_{2k-1}|^2}\right)\right) \tag{40}$$

where in the last equality, we use the fact

$$\lim_{t\to\infty}(z_{2i+4}-z_{2i+1})\simeq\frac{w_2-w_1}{kt}e^{2\pi i\frac{i+1}{k}}t^{\frac{1}{k}}\simeq0, \tag{41}$$

and the fact that the holomorphic part of the $2k$-point correlation function can be written as

$$\langle\mathcal{O}(z_1)\mathcal{O}^\dagger(z_2)\ldots\mathcal{O}(z_{2k-1})\mathcal{O}^\dagger(z_{2k})\rangle_{\Sigma_1}$$

$$\sim(F_{00}[\mathcal{O}])^{k-1}\langle\mathcal{O}(z_1)\mathcal{O}^\dagger(z_4)\rangle_{\Sigma_1}\ldots\langle\mathcal{O}(z_{2k+1})\mathcal{O}^\dagger(z_{2k+4})\rangle_{\Sigma_1}. \tag{42}$$

The second part due to $\bar{T}(\bar{z})$ acting on the $2k$-point correlation function can be written as

$$\Delta S^{(k)}_{A,\lambda,part2} \equiv \frac{k\lambda}{k-1}\int_{\Sigma_1} d^2z\frac{1}{k^2\,|z|^{2k-2}}\frac{c(k^2-1)}{24}\frac{1}{z^2}\frac{\langle(\bar{T}(z)\mathcal{O}_1(z_1,\bar{z}_1)...\mathcal{O}^\dagger_{2k}(z_{2k},\bar{z}_{2k})\rangle_{\Sigma_1}}{\langle\mathcal{O}_1(z_1,\bar{z}_1)...\mathcal{O}^\dagger_{2k}(z_{2k},\bar{z}_{2k})\rangle_{\Sigma_1}}$$

$$=\frac{k}{k-1}\lambda\frac{1}{k^2}\frac{c(k^2-1)}{24}\int d^2z\frac{1}{|z|^{2k-2}\,z^2}\frac{1}{\langle\mathcal{O}(z_1,\bar{z}_1)\mathcal{O}^\dagger(z_2,\bar{z}_2)\ldots\mathcal{O}(z_{2k-1},\bar{z}_{2k-1})\mathcal{O}^\dagger(z_{2k},\bar{z}_{2k})\rangle_{\Sigma_1}}$$

$$\left(\sum_{i=1}^{2k}\frac{\bar{h}}{(\bar{z}-\bar{z}_i)^2}+\sum_{j=0}^{k-1}\left(\frac{\bar{\partial}_{2j+1}}{\bar{z}-\bar{z}_{2j+1}}+\frac{\bar{\partial}_{2j+2}}{\bar{z}-\bar{z}_{2j+2}}\right)\right)$$

$$\times\langle\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\ldots\mathcal{O}(w_{2k-1},\bar{w}_{2k-1})\mathcal{O}^\dagger(w_{2k},\bar{w}_{2k})\rangle_{\Sigma_1}$$

$$=\frac{k}{k-1}\lambda\frac{1}{k^2}\frac{c(k^2-1)}{24}\left(\sum_{i=1}^{2k}\frac{\bar{h}\pi}{k}\frac{1}{|\bar{z}_i|^{2k}}+\sum_{j=0}^{k-1}\frac{\pi}{k}\left(\frac{-2\bar{h}}{\bar{z}_{2j+1}-\bar{z}_{2j+2}}\frac{\bar{z}_{2j+1}}{|\bar{z}_{2j+1}|^{2k}}+\frac{2\bar{h}}{\bar{z}_{2j+1}-\bar{z}_{2j+2}}\frac{\bar{z}_{2j+2}}{|\bar{z}_{2j+2}|^{2k}}\right)\right), \tag{43}$$

where in the last equality, we use the fact

$$\lim_{t\to\infty}(\bar{z}_{2i+2}-\bar{z}_{2i+1}) \simeq \frac{w_1-w_2}{kt}e^{-2\pi i\frac{i+\frac{1}{2}}{k}}t^{\frac{1}{k}} \simeq 0, \tag{44}$$

and the fact that the anti-holomorphic part of the $2k$-point correlation function can be written as

$$\langle\mathcal{O}(\bar{z}_1)\mathcal{O}^\dagger(\bar{z}_2)\dots\mathcal{O}(\bar{z}_{2k-1})\mathcal{O}^\dagger(\bar{z}_{2k})\rangle_{\Sigma_1}$$
$$\sim(F_{00}[\mathcal{O}])^{k-1}\langle\mathcal{O}(\bar{z}_1)\mathcal{O}^\dagger(\bar{z}_2)\rangle_{\Sigma_1}\dots\langle\mathcal{O}(\bar{z}_{2k-1})\mathcal{O}^\dagger(\bar{z}_{2k})\rangle_{\Sigma_1}. \tag{45}$$

Now we replace all coordinates in (40) and (43) with

$$z_{2k+1} = e^{2\pi i\frac{k+1/2}{n}}(-x-t+i\epsilon)^{\frac{1}{n}}, \quad \bar{z}_{2k+1} = e^{-2\pi i\frac{k+1/2}{n}}(-x+t-i\epsilon)^{\frac{1}{n}},$$
$$z_{2k+2} = e^{2\pi i\frac{k+1/2}{n}}(-x-t-i\epsilon)^{\frac{1}{n}}, \quad \bar{z}_{2k+2} = e^{-2\pi i\frac{k+1/2}{n}}(-x+t+i\epsilon)^{\frac{1}{n}}, \quad (k=0,...,n-1). \tag{46}$$

We get the correction to the kth Rényi entropy in the $T\bar{T}$-deformed CFT at the late time

$$\Delta S_{A,\lambda}^{(k)} = 3k\lambda\frac{c^2(k^2-1)^2\pi\tilde{\Lambda}^{2k}}{576k^3} - \frac{k}{k-1}\lambda\frac{h\pi c(k^2-1)}{24k^3}\frac{4k^2t}{(t-x)(t+x)^2} + \mathcal{O}(\epsilon,c^0,\lambda^2). \tag{47}$$

We next consider the integral (38) at the early time. The coordinates and the $2k$-point correlation function have the following property:

$$\lim_{t\to 0}(z_{2i+2}-z_{2i+1}) \simeq 0, \quad \lim_{t\to 0}(\bar{z}_{2i+2}-\bar{z}_{2i+1}) \simeq 0,$$
$$\langle\mathcal{O}(z_1,\bar{z}_1)\mathcal{O}^\dagger(z_2,\bar{z}_2)\dots\mathcal{O}(z_{2k-1},\bar{z}_{2k-1})\mathcal{O}^\dagger(z_{2k},\bar{z}_{2k})\rangle_{\Sigma_1}$$
$$\sim\langle\mathcal{O}(z_1,\bar{z}_1)\mathcal{O}^\dagger(z_2,\bar{z}_2)\rangle_{\Sigma_1}\dots\langle\mathcal{O}(z_{2k-1},\bar{z}_{2k-1})\mathcal{O}^\dagger(z_{2k},\bar{z}_{2k})\rangle_{\Sigma_1}. \tag{48}$$

We can get the correction to the kth Rényi entropy at the early time.

$$\Delta S_{A,\lambda}^{(k)} = 3k\lambda\frac{c^2(k^2-1)^2\pi\tilde{\Lambda}^{2k}}{576k^3} - \frac{k}{k-1}\lambda\frac{h\pi c(k^2-1)}{24k^3}\frac{8k^2t^2}{(t^2-x^2)^2} + \mathcal{O}(\epsilon,c^0,\lambda^2). \tag{49}$$

Results (47) and (49) are precisely the results we find in [79] if we take the interval of A in [79] to infinity and set $k=2$. Taking the limit of $k\to 1$ for (47) and (49), we obtain the early-time and late-time behavior of the entanglement entropy under the $T\bar{T}$-deformation up to the first-order, respectively. Namely,

$$\Delta S_{A,\lambda} = \begin{cases} -\lambda\frac{2\pi hc}{3}\frac{t^2}{(t^2-x^2)^2}, & t < |x|, \\ -\lambda\frac{\pi hc}{3}\frac{t}{(t-x)(t+x)^2}, & t > |x|. \end{cases} \tag{50}$$

In this scenario, the $T\bar{T}$ deformation does not alter the late-time dynamics of $\Delta S_{A,\lambda}$, which exhibits a temporal evolution following a $\frac{1}{t^2}$ pattern. It is important to note that a singularity emerges in (50) at $t=x$, and this singularity arises from the derivative acting on the $2k$-point correlation function, which discontinuously changes at that point.

4.2 k^{th} pseudo-Rényi entropy in $T\bar{T}$-deformed CFTs

Thanks to the property of the $2k$-point correlation function (48) at the early time, we can derive the correction to the kth Rényi entropy in $T\bar{T}$-deformed CFTs. However, this property is no longer valid for the pseudo entropy since the two insertion coordinates x_1 and x_2 are different. Therefore we only consider the $T\bar{T}$-deformation of the kth pseudo-Rényi entropy at the late time in this subsection.

Combining (40) and (43), and replacing all the coordinates with (21), we get the correction to the kth pseudo-Rényi entropy in the $T\bar{T}$-deformed CFT,

$$
\Delta S_{A,\lambda}^{(k)} = 3k\lambda \frac{c^2(k^2-1)^2 \pi \tilde{\Lambda}^{2k}}{576k^3}
$$
$$
+ \frac{k}{k-1}\lambda \frac{h\pi c(k^2-1)}{24k^3} \left(\frac{2k\left((-t+x_1)^{\frac{1}{k}}(t-x_1)^{\frac{1}{k}} - e^{\frac{2\pi i}{k}}(-t-x_2)^{\frac{1}{k}}(t-x_2)^{\frac{1}{k}}\right)(x_1^2-x_2^2)}{(t^2-x_1^2)\left((-t-x_1)^{\frac{1}{k}} - e^{\frac{2\pi i}{k}}(-t-x_2)^{\frac{1}{k}}\right)\left((t-x_1)^{\frac{1}{k}} - (t-x_2)^{\frac{1}{k}}\right)(t^2-x_2^2)} \right)
$$
$$
+ \mathcal{O}(\epsilon, c^0, \lambda^2). \tag{51}
$$

The result (51) precisely matches the result we find in the previous section 3 if we take $k = 2$.

Taking the limit of $k \to 1$ for (51), we obtain the late-time correction of the pseudo entropy under the $T\bar{T}$-deformation up to the first-order which is

$$
\Delta S_{A,\lambda} = \frac{\lambda ch\pi}{6} \frac{((-t-x_1)(t-x_1) - (-t-x_2)(t-x_2))(x_1^2-x_2^2)}{(t^2-x_1^2)(-x_1^2+x_2^2)(t^2-x_2^2)}. \tag{52}
$$

In the limit of late time, it is evident that $\Delta S_{A,\lambda} \sim \frac{1}{t^2}$. Therefore, the first-order $T\bar{T}$ deformation does not alter the late-time behavior of the pseudo-entropy, which is consistent with the late-time behavior of the excess entanglement entropy derived in (50). Moreover, there exist singularities at $t = x_1$ or $t = x_2$, which emerge due to the non-smooth behavior of the correlation function at those specific points. These singularities exhibit similarities to the finding discussed in the previous subsection.

5 Pseudo entropy in $T\bar{T}$-deformed CFTs with different deformed states

In the previous section, we have explicitly shown the correction of the kth pseudo-Rényi entropy after the $T\bar{T}$-deformation, up to the first order of the deformation coefficient. Recent research about the $T\bar{T}$-deformed CFTs has focused on how to glue two different AdS spacetimes [81,82] and its holographic dual. Inspired by the cAdS/dCFT correspondence, in this section, we will consider the pseudo-Rényi entropy for states with different $T\bar{T}$-deformation on the CFT side.

We first consider the transition matrix for the vacuum state taking the form

$$
\mathcal{T}^{1|2} \equiv \frac{|\Omega_{\lambda_1}\rangle\langle\Omega_{\lambda_2}|}{\langle\Omega_{\lambda_2}|\Omega_{\lambda_1}\rangle}, \tag{53}
$$

where the vacuum states are deformed by $T\bar{T}$ operators with different coupling constants λ_1 and λ_2. The deformation of the kth pseudo-Rényi entropy, in this case, can be derived through the replica method shown in appendix B, which is

$$\Delta S_{A,\lambda_1,\lambda_2}^{(k)} = \frac{k}{k-1}\left[\lambda_1 \int_{\text{LHP}} d^2w \left(\langle T\bar{T}(w,\bar{w})\rangle_{\Sigma_k} - \langle T\bar{T}(w,\bar{w})\rangle_{\Sigma_1}\right)\right.$$
$$\left. + \lambda_2 \int_{\text{UHP}} d^2w \left(\langle T\bar{T}(w,\bar{w})\rangle_{\Sigma_k} - \langle T\bar{T}(w,\bar{w})\rangle_{\Sigma_1}\right)\right]. \tag{54}$$

In Eq. (54), LHP denotes the lower half-plane of Σ_1, while UHP denotes the upper half-plane of Σ_1.

Using the transformation of the $T\bar{T}$ operator between Σ_k and Σ_1, the integral becomes

$$\Delta S_{A,\lambda_1,\lambda_2}^{(k)} = \frac{k}{k-1}\left[\lambda_1 \int_{\text{LHP}} d^2z \frac{1}{k^2 |z|^{2k-2}} \left\langle \left(T(z) + \frac{c(k^2-1)}{24z^2}\right)\left(\bar{T}(\bar{z}) + \frac{c(k^2-1)}{24\bar{z}^2}\right)\right\rangle_{\Sigma_1}\right.$$
$$\left. + \lambda_2 \int_{\text{UHP}} d^2z \frac{1}{k^2 |z|^{2k-2}} \left\langle \left(T(z) + \frac{c(k^2-1)}{24z^2}\right)\left(\bar{T}(\bar{z}) + \frac{c(k^2-1)}{24\bar{z}^2}\right)\right\rangle_{\Sigma_1}\right]$$
$$= \frac{k}{k-1}\left[\lambda_1 \int_{\text{LHP}} d^2z \frac{1}{k^2 |z|^{2k-2}} \frac{c(k^2-1)}{24z^2}\frac{c(k^2-1)}{24\bar{z}^2}\right.$$
$$\left. + \lambda_2 \int_{\text{UHP}} d^2z \frac{1}{k^2 |z|^{2k-2}} \frac{c(k^2-1)}{24z^2}\frac{c(k^2-1)}{24\bar{z}^2}\right]$$
$$= \lambda_1 \frac{c^2(k-1)(k+1)^2}{1152k^2}\tilde{\Lambda}_1^{2k} + \lambda_2 \frac{c^2(k-1)(k+1)^2}{1152k^2}\tilde{\Lambda}_2^{2k}, \tag{55}$$

where $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ are two different IR cutoffs. Since we are dealing with two distinct deformation couplings, there is no requirement for the two cutoffs to be identical. However, in the case of vacuum states, if we exchange the indices $(1 \leftrightarrow 2)$, the path integral constructed through the replica method should remain invariant. This symmetry is also satisfied by the result in Eq. (55). In the scenario where the two deformation parameters are equal, we must set $\tilde{\Lambda}_1 = \tilde{\Lambda}_2$. Under this condition, Eq. (55) reverts to Eq. (39).

In the subsequent part of this section, we shall examine the pseudo-Rényi entropy derived from primary operators acting on distinct vacuum states with two different deformation constants in $T\bar{T}$-deformed CFT. The states used to construct the transition matrix are now expressed as follows:

$$|\psi_1\rangle = \mathcal{O}(w_1)|\Omega_{\lambda_1}\rangle, \quad |\psi_2\rangle = \mathcal{O}(w_2)|\Omega_{\lambda_2}\rangle. \tag{56}$$

According to appendix B, the deformed k^{th} pseudo-Rényi entropy is now taking the form

$$\Delta S_{A,\lambda_1,\lambda_2}^{(k)} = \frac{k}{k-1}\lambda_1 \int_{\text{LHP of } \Sigma_1} d^2w \left(\frac{\langle T\bar{T}(w,\bar{w})\mathcal{O}_1...\mathcal{O}_{2k}^\dagger\rangle_{\Sigma_k}}{\langle \mathcal{O}_1...\mathcal{O}_{2k}^\dagger\rangle_{\Sigma_k}} - \frac{\langle T\bar{T}(w,\bar{w})\mathcal{O}_1\mathcal{O}_2^\dagger\rangle_{\Sigma_1}}{\langle \mathcal{O}_1\mathcal{O}_2^\dagger\rangle_{\Sigma_1}}\right)$$
$$+ \frac{k}{k-1}\lambda_2 \int_{\text{UHP of } \Sigma_1} d^2w \left(\frac{\langle T\bar{T}(w,\bar{w})\mathcal{O}_1...\mathcal{O}_{2k}^\dagger\rangle_{\Sigma_k}}{\langle \mathcal{O}_1...\mathcal{O}_{2k}^\dagger\rangle_{\Sigma_k}} - \frac{\langle T\bar{T}(w,\bar{w})\mathcal{O}_1\mathcal{O}_2^\dagger\rangle_{\Sigma_1}}{\langle \mathcal{O}_1\mathcal{O}_2^\dagger\rangle_{\Sigma_1}}\right). \tag{57}$$

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Through the calculation in appendix C, we have the correction to the kth pseudo-Rényi entropy with different deformed parameters of $O(c)$ under $T\bar{T}$ deformation at the late time

$$
\begin{aligned}
&\Delta S^{(k)}_{A,\lambda_1,\lambda_2} = \\
&\frac{c(k+1)}{24k^2}\lambda_1\Bigg\{\frac{\left((-t+x_1)^{\frac{1}{k}}(t-x_1)^{\frac{1}{k}}-e^{\frac{2\pi i}{k}}(-t-x_2)^{\frac{1}{k}}(t-x_2)^{\frac{1}{k}}\right)(x_1^2-x_2^2)}{(t^2-x_1^2)\left((-t-x_1)^{\frac{1}{k}}-e^{\frac{2\pi i}{k}}(-t-x_2)^{\frac{1}{k}}\right)\left((t-x_1)^{\frac{1}{k}}-(t-x_2)^{\frac{1}{k}}\right)(t^2-x_2^2)} \\
&+\frac{1}{2}\pi\Bigg[\frac{(x_1-x_2)(2t-x_1-x_2)\left((t-x_1)^{1/k}+(t-x_2)^{1/k}\right)}{(t-x_1)^2(t-x_2)^2\left((t-x_1)^{1/k}-(t-x_2)^{1/k}\right)}+\frac{2\left((-t-x_1)^{\frac{1}{k}-2}-e^{\frac{2i\pi}{k}}(-t-x_2)^{\frac{1}{k}-2}\right)}{-(-t-x_1)^{1/k}+e^{\frac{2i\pi}{k}}(-t-x_2)^{1/k}} \\
&+2t^2+2t(x_1+x_2)+2\left(\frac{1}{t+x_2}+\frac{1}{t+x_1}\right)+x_1^2+x_2^2\Bigg]\Bigg\}+\mathcal{O}(\epsilon,c^0,\lambda_1^2) \\
&+\frac{c(k+1)}{24k^2}\lambda_2\Bigg\{\frac{\left((-t+x_1)^{\frac{1}{k}}(t-x_1)^{\frac{1}{k}}-e^{\frac{2\pi i}{k}}(-t-x_2)^{\frac{1}{k}}(t-x_2)^{\frac{1}{k}}\right)(x_1^2-x_2^2)}{(t^2-x_1^2)\left((-t-x_1)^{\frac{1}{k}}-e^{\frac{2\pi i}{k}}(-t-x_2)^{\frac{1}{k}}\right)\left((t-x_1)^{\frac{1}{k}}-(t-x_2)^{\frac{1}{k}}\right)(t^2-x_2^2)} \\
&-\frac{1}{2}\pi\Bigg[\frac{(x_1-x_2)(2t-x_1-x_2)\left((t-x_1)^{1/k}+(t-x_2)^{1/k}\right)}{(t-x_1)^2(t-x_2)^2\left((t-x_1)^{1/k}-(t-x_2)^{1/k}\right)}+\frac{2\left((-t-x_1)^{\frac{1}{k}-2}-e^{\frac{2i\pi}{k}}(-t-x_2)^{\frac{1}{k}-2}\right)}{-(-t-x_1)^{1/k}+e^{\frac{2i\pi}{k}}(-t-x_2)^{1/k}} \\
&+2t^2+2t(x_1+x_2)+2\left(\frac{1}{t+x_2}+\frac{1}{t+x_1}\right)+x_1^2+x_2^2\Bigg]\Bigg\}+\mathcal{O}(\epsilon,c^0,\lambda_2^2). \tag{58}
\end{aligned}
$$

Notice that there are also some possible cutoff terms with ϵ_1 and ϵ_2 corresponding to the LHP and UHP separately and we have regularized them through minimal subtraction seen in (84)-(91). When the two deformation parameters are identical, we require that both cutoffs, ϵ_1 and ϵ_2, are set to the same value ϵ. In this case, formula (93), including the ignored cutoff terms, will precisely give rise to the result in Eq. (51). However, when $\lambda_1 \neq \lambda_2$, the correction to the kth pseudo-Rényi entropy may involve additional nontrivial terms compared to (51). We can further extend our research to holographic CFT, where we expect that the extended investigation in holographic CFT may offer valuable insights, particularly in relation to the boundary conditions during the gluing process [81, 82].

6 Pseudo-Rényi entropy in $J\bar{T}$-deformed CFTs

So far, we have discussed the $T\bar{T}$-deformation of the pseudo-Rényi entropy. In this section, we study the $J\bar{T}$-deformation of the pseudo entropy, with $J\bar{T}$ operator constructed from a chiral $U(1)$ current J and stress tensor $\bar{T}$.

The $J\bar{T}$-deformed action is the trajectory on the space of field theory satisfying

$$
\frac{dS}{d\lambda} = \int d^2z\,(J\bar{T})_\lambda. \tag{59}
$$

Similar to the $T\bar{T}$-deformation, we regard the deformation as a perturbative theory, in which

case the action can be written as

$$S(\lambda) = S(\lambda = 0) + \lambda \int d^2 z \sqrt{g} J\bar{T} + \mathcal{O}(\lambda^2), \tag{60}$$

where $(J\bar{T})_{\lambda=0}$ is denoted by $J\bar{T}$ later on. The first-order correction to the correlation function by using the Ward identity is

$$\langle \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \dots \mathcal{O}_k(z_k, \bar{z}_k) \rangle_\lambda$$
$$= \lambda \int d^2 z \left\langle J\bar{T}(z, \bar{z}) \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \cdots \mathcal{O}_k(z_k, \bar{z}_k) \right\rangle \tag{61}$$
$$= \lambda \int d^2 z \left(\sum_{i=1}^k \frac{q_i}{z - z_i} \right) \left(\sum_{i=1}^k \left(\frac{\bar{h}_i}{(\bar{z} - \bar{z}_i)^2} + \frac{\partial_{\bar{z}_i}}{\bar{z} - \bar{z}_i} \right) \right)$$
$$\times \langle \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \dots \mathcal{O}_k(z_k, \bar{z}_k) \rangle. \tag{62}$$

We then calculate the correction to the kth pseudo-Rényi entropy in the $J\bar{T}$-deformed CFT.

Under the conformal map (19), the $U(1)$ current and the stress tensor transform as

$$J\bar{T}(w, \bar{w}) = \frac{dz}{dw} J(z) \frac{1}{4\bar{z}^2} \left(\bar{T}(\bar{z}) + \frac{c}{8\bar{z}^2} \right). \tag{63}$$

The correction to the undeformed k^{th} pseudo-Rényi entropy can be derived similarly from that in the previous section. Thus we have

$$\Delta S_{A,\lambda}^{(k)} = \frac{k\lambda}{k-1} \int_{\Sigma_1} d^2 w \left(\frac{\langle J\bar{T}(w, \bar{w}) \mathcal{O}_1 \dots \mathcal{O}_{2k}^\dagger \rangle_{\Sigma_k}}{\langle \mathcal{O}_1 \dots \mathcal{O}_{2k}^\dagger \rangle_{\Sigma_k}} - \frac{\langle J\bar{T}(w, \bar{w}) \mathcal{O}_1 \mathcal{O}_2^\dagger \rangle_{\Sigma_1}}{\langle \mathcal{O}_1 \mathcal{O}_2^\dagger \rangle_{\Sigma_1}} \right). \tag{64}$$

We still focus on the large c case. At $\mathcal{O}(c)$, $\Delta S_{A,\lambda}^{(k)}$ at the late time can be written as

$$\Delta S_{A,\lambda}^{(k)} = \frac{k\lambda}{k-1} \int d^2 z \frac{c}{8\bar{z}^2} \frac{1}{k\bar{z}^{k-1}} \frac{\langle J(z) \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_2^\dagger(z_2, \bar{z}_2) \dots \mathcal{O}_{2k-1}(z_{2k-1}, \bar{z}_{2k-1}) \mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k}) \rangle_{\Sigma_1}}{\langle \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_2^\dagger(z_2, \bar{z}_2) \dots \mathcal{O}_{2k-1}(z_{2k-1}, \bar{z}_{2k-1}) \mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k}) \rangle_{\Sigma_1}}$$
$$= \frac{k\lambda}{k-1} \int d^2 z \frac{c}{8\bar{z}^2} \frac{1}{k\bar{z}^{k-1}} \left(\sum_{i=1}^{2k} \frac{q_i}{z - z_i} \right)$$
$$= \frac{2\lambda\pi c}{8(k-1)} \sum_{i=1}^{2k} \int_{|z_i|}^\infty \frac{q_i z_i^k}{\rho^{2k+1}} = \frac{2\lambda\pi c}{16k(1-k)} \sum_{i=1}^{2k} \frac{q_i z_i^k}{|z_i|^{2k}}. \tag{65}$$

Substituting (21) into (65), we get the first-order corrections to the kth pseudo-Rényi entropy in $J\bar{T}$-deformed CFT,

$$\Delta S_{A,\lambda}^{(k)} = \frac{2\lambda\pi c}{8(k-1)} \sum_{i=0}^{k-1} \left(\frac{q_{2i+1}}{-t + x_1} + \frac{q_{2i+2}}{-t + x_2} \right) + \mathcal{O}(c^0, \lambda^2). \tag{66}$$

Specifically, we consider the $k = 2$ case,

$$\Delta S_{A,\lambda}^{(2)} = 2\lambda \frac{c\pi}{32} \left(\frac{q_1 + q_3}{-t + x_1} + \frac{q_2 + q_4}{-t + x_2} \right) + \mathcal{O}(c^0, \lambda^2). \tag{67}$$

If we take the limit $x_1 \to x_2$, (67) returns to the result we found in [79]. Using the method mentioned in subsection 4.1, one can also calculate the late-time and early-time corrections to the entanglement entropy in $J\bar{T}$-deformed CFTs.

7   Conclusion and prospect

This paper employs perturbation theory to investigate the pseudo-Rényi entropy associated with locally excited states in the two-dimensional $T\bar{T}/J\bar{T}$ deformed CFTs. Our primary objective is to derive nontrivial corrections to the pseudo-Rényi entropy of the undeformed theory. Initially, we provide a comprehensive review of the calculation of two-point and four-point correlation functions using the Ward identity in deformed CFTs and the pseudo-Rényi entropy obtained through the replica method in undeformed CFTs. We then derive the leading order corrections to the second pseudo-Rényi entropy and find that it may depend on the introduced infrared (IR) cutoff and may also depend on the ultraviolet (UV) cutoff, depending on whether the subsystem A is finite. Furthermore, at order $\mathcal{O}(c)$, we discover that the second pseudo-Rényi entropy may acquire nontrivial corrections in the deformed CFTs at the late time, as demonstrated in Eqs. (31) and (67). However, the early-time correction becomes more intricate, and we illustrate its behavior using the free scalar and Ising models as specific examples.

In addition, we extend our investigation to the kth Rényi entropy, examining its behavior both at the early and late times as expressed in Eqs. (47) and (49), respectively. Subsequently, we calculate the k-th pseudo-Rényi entropy of locally excited states at the late time, using the fact that the $2k$-point correlation function can be deformed as the product of k two-point functions. We observe nontrivial corrections to the kth pseudo-Rényi entropy at the late time, as presented in Eq. (51). Notably, this result aligns precisely with the findings from section 3.

Finally, drawing inspiration from gluing two AdS spacetimes with distinct cutoffs and the correspondence between cutoff AdS (cAdS) and $T\bar{T}$-deformed CFT (dCFT), we separately investigate the pseudo-Rényi entropy for vacuum states characterized by distinct $T\bar{T}$-deformation parameters, and for primary states acting on different deformed vacuum states. At the leading order, we find that the corrections to the pseudo-Rényi entropy may depend on two different cutoffs, as indicated in Eq. (55). For the pseudo-Rényi entropy associated with two primary operators acting on different deformed vacuum states, we obtain additional nontrivial terms in Eq. (93) compared to Eq. (51).

Our primary focus centers on the computation of corrections induced by $T\bar{T}/J\bar{T}$-deformation to the pseudo-Rényi entropy in RCFTs. Specifically, by employing the $T\bar{T}/J\bar{T}$ deformation, it is possible to directly compute corrections to the pseudo-Rényi entropy for different local operators in various theoretical frameworks, including Liouville CFT, holographic CFT, and other relevant approaches. Moreover, exploring the definition of an operator within the framework of $T\bar{T}$-deformation, constructing pseudo-Rényi entropy for locally excited states in the AdS context, connecting distinct cutoff AdS geometries, and calculating the pseudo-Rényi entropy for locally excited states with different deformation parameters on the cAdS side could provide valuable insights. However, these captivating avenues remain unexplored and await future

research endeavors.

Acknowledgements

A Details of integrals in $T\bar{T}$-deformed pseudo-Rényi entropy

In this appendix, we evaluate the integral of (29) in detail. It mainly takes the following forms

$$\int d^2 z \frac{1}{|z|^2} \frac{1}{z^2} \frac{1}{(\bar{z} - \bar{z}_i)} \quad \text{and} \quad \int d^2 z \frac{1}{|z|^2} \frac{1}{z^2} \frac{1}{(\bar{z} - \bar{z}_i)^2} \tag{68}$$

as well as their complex conjugate. Now we evaluate them accordingly. Firstly

$$\int d^2 z \frac{1}{|z|^2} \frac{1}{z^2} \frac{1}{(\bar{z} - \bar{z}_i)^2} = \int_0^\infty d\rho \int_0^{2\pi} d\theta \frac{1}{\rho^3 e^{2i\theta}} \frac{1}{(\rho e^{-i\theta} - \bar{z}_i)^2}. \tag{69}$$

Integrating out θ, we find

$$\int d^2 z \frac{1}{|z|^2} \frac{1}{z^2} \frac{1}{(\bar{z} - \bar{z}_i)^2} = \int d\rho \frac{\frac{i\rho}{e^{i\theta}\bar{z}_i - \rho} + i \log(\bar{z}_i - \rho e^{-i\theta})}{\rho^5} \Big|_{\theta=0}^{\theta=2\pi}. \tag{70}$$

Let us consider the process where θ runs from 0 to 2π. When $\left|\frac{\rho}{\bar{z}_i}\right| > 1$, $\rho e^{-i\theta}$ circles counterclockwise around $\bar{z}_i$. Hence $\log(\bar{z}_i - \rho e^{-i\theta})$ contributes a factor $-2\pi i$. Therefore we have

$$\log(\bar{z}_i - \rho e^{-i\theta}) = \begin{cases} 0 & \left|\frac{\rho}{\bar{z}_i}\right| < 1 \\ -2\pi i & \left|\frac{\rho}{\bar{z}_i}\right| > 1 \end{cases}. \tag{71}$$

The integral becomes

$$\int d^2 z \frac{1}{|z|^2} \frac{1}{z^2} \frac{1}{(\bar{z} - \bar{z}_i)^2} = \int_{|\bar{z}_i|}^\infty \frac{2\pi}{\rho^5} d\rho = \frac{2\pi}{4 |\bar{z}_i|^4}. \tag{72}$$

Secondly the other integral in (68) can be evaluated as

$$\int d^2 z \frac{1}{|z|^2} \frac{1}{z^2} \frac{1}{(\bar{z} - \bar{z}_i)} = \int_0^\infty d\rho \int_0^{2\pi} d\theta \frac{1}{\rho^3 e^{2i\theta}} \frac{1}{(\rho e^{-i\theta} - \bar{z}_i)}$$

$$= \int d\rho \frac{ie^{-i\theta}\rho + i\bar{z}_i \log(\bar{z}_i - \rho e^{-i\theta})}{\rho^5} \Big|_{\theta=0}^{\theta=2\pi} \int_{|\bar{z}_i|}^\infty \frac{2\pi \bar{z}_i}{\rho^5} d\rho = \frac{2\pi \bar{z}_i}{4 |\bar{z}_i|^4}. \tag{73}$$

Other integrals in (29) can be evaluated in a similar way. We find that those terms coupled with $\bar{T}$ become

$$\frac{c}{8}2\lambda\int d^2z \frac{1}{4\,|z|^2}\frac{\langle\left(\frac{1}{\bar{z}^2}\bar{T}(\bar{z})\right)\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\mathcal{O}(w_3,\bar{w}_3)\mathcal{O}^\dagger(w_4,\bar{w}_4)\rangle_{\Sigma_1}}{\langle\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\mathcal{O}(w_3,\bar{w}_3)\mathcal{O}^\dagger(w_4,\bar{w}_4)\rangle_{\Sigma_1}}$$

$$=2\lambda\frac{c}{32}\int d^2z\frac{1}{|z|^2}\frac{1}{\bar{z}^2}\left(\sum_{i=1}^4\frac{h}{(\bar{z}-\bar{z}_i)^2}-\frac{2h}{(\bar{z}-\bar{z}_1)(\bar{z}_1-\bar{z}_3)}+\frac{2h}{(\bar{z}-\bar{z}_3)(\bar{z}_1-\bar{z}_3)}\right.$$

$$-\frac{2h}{(\bar{z}-\bar{z}_2)(\bar{z}_2-\bar{z}_4)}+\frac{2h}{(\bar{z}-\bar{z}_4)(\bar{z}_2-\bar{z}_4)}$$

$$\left.+[\frac{1}{\bar{z}-\bar{z}_1}(-\frac{1}{\bar{z}_{13}}+\frac{1}{\bar{z}_{12}})+\frac{1}{\bar{z}-\bar{z}_2}(-\frac{1}{\bar{z}_{24}}-\frac{1}{\bar{z}_{12}})+\frac{1}{\bar{z}-\bar{z}_3}(\frac{1}{\bar{z}_{34}}+\frac{1}{\bar{z}_{13}})+\frac{1}{\bar{z}-\bar{z}_4}(-\frac{1}{\bar{z}_{34}}+\frac{1}{\bar{z}_{24}})]\frac{\bar{\eta}\partial_{\bar{\eta}}G(\eta,\bar{\eta})}{G(\eta,\bar{\eta})}\right)$$

$$=2\lambda\frac{c\pi}{64}\left(\frac{h}{|\bar{z}_1|^4}+\frac{h}{|\bar{z}_2|^4}+\frac{h}{|\bar{z}_3|^4}+\frac{h}{|\bar{z}_4|^4}-\frac{2h\bar{z}_1}{\bar{z}_{13}\,|\bar{z}_1|^4}+\frac{2h\bar{z}_3}{\bar{z}_{13}\,|\bar{z}_3|^4}-\frac{2h\bar{z}_2}{\bar{z}_{24}\,|\bar{z}_2|^4}+\frac{2h\bar{z}_4}{\bar{z}_{24}\,|\bar{z}_4|^4}\right.$$

$$\left.+[\frac{\bar{z}_1}{|\bar{z}_1|^4}(-\frac{1}{\bar{z}_{13}}+\frac{1}{\bar{z}_{12}})+\frac{\bar{z}_2}{|\bar{z}_2|^4}(-\frac{1}{\bar{z}_{24}}-\frac{1}{\bar{z}_{12}})+\frac{\bar{z}_3}{|\bar{z}_3|^4}(\frac{1}{\bar{z}_{34}}+\frac{1}{\bar{z}_{13}})+\frac{\bar{z}_4}{|\bar{z}_4|^4}(-\frac{1}{\bar{z}_{34}}+\frac{1}{\bar{z}_{24}})]\frac{\bar{\eta}\partial_{\bar{\eta}}G(\eta,\bar{\eta})}{G(\eta,\bar{\eta})}\right).$$

$$(74)$$

Those terms coupled with T are

$$\frac{c}{8}2\lambda\int d^2z\frac{1}{4\,|z|^2}\frac{\langle\left(\frac{1}{z^2}T(z)\right)\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\mathcal{O}(w_3,\bar{w}_3)\mathcal{O}^\dagger(w_4,\bar{w}_4)\rangle_{\Sigma_1}}{\langle\mathcal{O}(w_1,\bar{w}_1)\mathcal{O}^\dagger(w_2,\bar{w}_2)\mathcal{O}(w_3,\bar{w}_3)\mathcal{O}^\dagger(w_4,\bar{w}_4)\rangle_{\Sigma_1}}$$

$$=2\lambda\frac{c}{32}\int d^2z\frac{1}{|z|^2}\frac{1}{z^2}\left(\sum_{i=1}^4\frac{h}{(z-z_i)^2}-\frac{2h}{(z-z_1)(z_1-z_3)}+\frac{2h}{(z-z_3)(z_1-z_3)}\right.$$

$$-\frac{2h}{(z-z_2)(z_2-z_4)}+\frac{2h}{(z-z_4)(z_2-z_4)}$$

$$\left.+[\frac{1}{z-z_1}(-\frac{1}{z_{13}}+\frac{1}{z_{12}})+\frac{1}{z-z_2}(-\frac{1}{z_{24}}-\frac{1}{z_{12}})+\frac{1}{z-z_3}(\frac{1}{z_{34}}+\frac{1}{z_{13}})+\frac{1}{z-z_4}(-\frac{1}{z_{34}}+\frac{1}{z_{24}})]\frac{\eta\partial_\eta G(\eta,\bar{\eta})}{G(\eta,\bar{\eta})}\right)$$

$$=2\lambda\frac{c\pi}{64}\left(\frac{h}{|z_1|^4}+\frac{h}{|z_2|^4}+\frac{h}{|z_3|^4}+\frac{h}{|z_4|^4}-\frac{2hz_1}{z_{13}\,|z_1|^4}+\frac{2hz_3}{z_{13}\,|z_3|^4}-\frac{2hz_2}{z_{24}\,|z_2|^4}+\frac{2hz_4}{z_{24}\,|z_4|^4}\right.$$

$$\left.+[\frac{z_1}{|z_1|^4}(-\frac{1}{z_{13}}+\frac{1}{z_{12}})+\frac{z_2}{|z_2|^4}(-\frac{1}{z_{24}}-\frac{1}{z_{12}})+\frac{z_3}{|z_3|^4}(\frac{1}{z_{34}}+\frac{1}{z_{13}})+\frac{z_4}{|z_4|^4}(-\frac{1}{z_{34}}+\frac{1}{z_{24}})]\frac{\eta\partial_\eta G(\eta,\bar{\eta})}{G(\eta,\bar{\eta})}\right).$$

$$(75)$$

We then list the results of those integrals in (40) and (43) using the same method mentioned above.

$$\int d^2z\frac{1}{|z|^{2k-2}\,\bar{z}^2}\frac{1}{(z-z_i)^2}=\frac{\pi}{k\,|z_i|^{2k}}.$$

$$\int d^2z\frac{1}{|z|^{2k-2}\,\bar{z}^2}\frac{1}{(z-z_i)}=\frac{\pi z_i}{k\,|z_i|^{2k}}.$$

$$\int d^2z\frac{1}{|\bar{z}|^{2k-2}\,z^2}\frac{1}{(\bar{z}-\bar{z}_i)^2}=\frac{\pi}{k\,|\bar{z}_i|^{2k}}.$$

$$\int d^2z\frac{1}{|\bar{z}|^{2k-2}\,z^2}\frac{1}{(\bar{z}-\bar{z}_i)}=\frac{\pi\bar{z}_i}{k\,|\bar{z}_i|^{2k}}.$$

$$(76)$$

B The $T\bar{T}$-deformation of the kth pseudo-Rényi entropy by the replica method

This appendix provides a derivation of the leading-order corrections of the kth pseudo-Rényi entropy in 2D CFTs under the $T\bar{T}$-deformation, using the replica trick.

We first generate the states of interest by applying a local operator to the vacuum of the $T\bar{T}$-deformed theory with varying parameters, that is

$$|\psi_i\rangle = \mathcal{O}(x_i, \tau_i)|\Omega_{\lambda_i}\rangle, \ (i = 1, 2) \tag{77}$$

with $\mathcal{O}(x_i, \tau_i) \equiv \mathcal{O}(w_i, \bar{w}_i) = e^{H_{\lambda_i}\tau_i}\mathcal{O}(x_i, \tau = 0)e^{-H_{\lambda_i}\tau_i}$. $|\Omega_{\lambda_i}\rangle$ and H_{λ_i} denote the vacuum and the Hamiltonian of the deformed theory with the deformation parameter λ_i, respectively. By the standard replica trick, we can express the essential part of the kth pseudo-Rényi entropy, namely the trace of the kth power of the reduced transition matrix, as the following Euclidean path integral.

$$\text{tr}[(\mathcal{T}_A^{\psi_1|\psi_2})^k] = \left(\begin{array}{c} \end{array} \right)^{-k} \times \qquad . \tag{78}$$

The first factor of the right-hand side of Eq. (78) corresponds to a two-point function on the complex plane Σ_1, while the second factor corresponds to a $2k$-point function on the k-sheeted Riemann surface Σ_k. Notably, the $T\bar{T}$-deformed theory with parameter λ_1 lives on the lower half-plane (LHP) of each sheeted surface, whereas the deformed theory with parameter λ_2 resides on the upper half-plane (UHP) of each sheeted surface.

Rewriting the above result as the correlation functions and expanding it according to the

order of $\lambda_{1(2)}$, we obtain[7]

$$
\begin{aligned}
&\text{tr}[(\mathcal{T}_A^{\psi_1|\psi_2})^k]\\
&=\frac{Z_k}{Z_1^k}\times\frac{\langle\mathcal{O}(w_1)\mathcal{O}^\dagger(w_2)...\mathcal{O}(w_{2k-1})\mathcal{O}^\dagger(w_{2k})\rangle_{\Sigma_k}}{\langle\mathcal{O}(w_1)\mathcal{O}^\dagger(w_2)\rangle_{\Sigma_1}^k}\\
&\quad\times\left(1-\frac{k\lambda_1\int_{\text{LHP}}\mathrm{d}^2w\langle T\bar{T}(w)\mathcal{O}(w_1)...\mathcal{O}^\dagger(w_{2k})\rangle_{\Sigma_k}+k\lambda_2\int_{\text{UHP}}\mathrm{d}^2w\langle T\bar{T}(w)\mathcal{O}(w_1)...\mathcal{O}^\dagger(w_{2k})\rangle_{\Sigma_k}}{\langle\mathcal{O}(w_1)\mathcal{O}^\dagger(w_2)...\mathcal{O}(w_{2k-1})\mathcal{O}^\dagger(w_{2k})\rangle_{\Sigma_k}}\right.\\
&\quad\left.+\frac{k\lambda_1\int_{\text{LHP}}\mathrm{d}^2w\langle T\bar{T}(w)\mathcal{O}(w_1)\mathcal{O}^\dagger(w_2)\rangle_{\Sigma_1}+k\lambda_2\int_{\text{UHP}}\mathrm{d}^2w\langle T\bar{T}(w)\mathcal{O}(w_1)\mathcal{O}^\dagger(w_2)\rangle_{\Sigma_1}}{\langle\mathcal{O}(w_1)\mathcal{O}^\dagger(w_2)\rangle_{\Sigma_1}}\right)\quad(79)
\end{aligned}
$$

+ sub-leading order terms.

The two factors in the first line expression come from the vacuum contribution of the kth Rényi entropy of subsystem A (denoted as $S_{A,\text{vacuum}}^{(k)}$) and the operator $\mathcal{O}$'s contribution of the kth pseudo-Rényi entropy (denoted as $\Delta S_{A,0}^{(k)}$), respectively. Substituting (79) into (7), we obtain the kth pseudo-Rényi entropy of subsystem A under the $T\bar{T}$-deformation

$$
\begin{aligned}
&\Delta S_{A,\lambda_1,\lambda_2}^{(k)}=S_{A,\lambda_1,\lambda_2}^{(k)}-S_{A,\text{vacuum}}^{(k)}-\Delta S_{A,0}^{(k)}\\
&=\frac{1}{1-k}\log\left(1-\frac{k\lambda_1\int_{\text{LHP}}\mathrm{d}^2w\langle T\bar{T}(w)\mathcal{O}(w_1)...\mathcal{O}^\dagger(w_{2k})\rangle_{\Sigma_k}+k\lambda_2\int_{\text{UHP}}\mathrm{d}^2w\langle T\bar{T}(w)\mathcal{O}(w_1)...\mathcal{O}^\dagger(w_{2k})\rangle_{\Sigma_k}}{\langle\mathcal{O}(w_1)\mathcal{O}^\dagger(w_2)...\mathcal{O}(w_{2k-1})\mathcal{O}^\dagger(w_{2k})\rangle_{\Sigma_k}}\right.\\
&\quad\left.+\frac{k\lambda_1\int_{\text{LHP}}\mathrm{d}^2w\langle T\bar{T}(w)\mathcal{O}(w_1)\mathcal{O}^\dagger(w_2)\rangle_{\Sigma_1}+k\lambda_2\int_{\text{UHP}}\mathrm{d}^2w\langle T\bar{T}(w)\mathcal{O}(w_1)\mathcal{O}^\dagger(w_2)\rangle_{\Sigma_1}}{\langle\mathcal{O}(w_1)\mathcal{O}^\dagger(w_2)\rangle_{\Sigma_1}}\right)\\
&\quad+\text{sub-leading order terms}\\
&=\frac{k}{k-1}\left(\frac{\lambda_1\int_{\text{LHP}}\mathrm{d}^2w\langle T\bar{T}(w)\mathcal{O}(w_1)...\mathcal{O}^\dagger(w_{2k})\rangle_{\Sigma_k}+\lambda_2\int_{\text{UHP}}\mathrm{d}^2w\langle T\bar{T}(w)\mathcal{O}(w_1)...\mathcal{O}^\dagger(w_{2k})\rangle_{\Sigma_k}}{\langle\mathcal{O}(w_1)\mathcal{O}^\dagger(w_2)...\mathcal{O}(w_{2k-1})\mathcal{O}^\dagger(w_{2k})\rangle_{\Sigma_k}}\right.\\
&\quad\left.-\frac{\lambda_1\int_{\text{LHP}}\mathrm{d}^2w\langle T\bar{T}(w)\mathcal{O}(w_1)\mathcal{O}^\dagger(w_2)\rangle_{\Sigma_1}+\lambda_2\int_{\text{UHP}}\mathrm{d}^2w\langle T\bar{T}(w)\mathcal{O}(w_1)\mathcal{O}^\dagger(w_2)\rangle_{\Sigma_1}}{\langle\mathcal{O}(w_1)\mathcal{O}^\dagger(w_2)\rangle_{\Sigma_1}}\right)\quad(80)
\end{aligned}
$$

+ sub-leading order terms.

Note that Eq. (80) reduces to (37) when $\lambda_1=\lambda_2=\lambda$, and it reduces to (54) when $\mathcal{O}=\mathbb{I}$, i.e., the identity operator.

----

[7]We denote $(w,\bar{w})$ by (w) in $\mathcal{O}$ and $T\bar{T}$ for simplicity.

C Details of integrals in pseudo-Rényi entropy with diverge deformed states

In this appendix, we evaluate the integral of (57) in detail. The deformed pseudo-Rényi entropy according to appendix B can be evaluated as

$$
\begin{aligned}
&\Delta S^{(k)}_{A,\lambda_1,\lambda_2} \\
&= \frac{k\lambda_1}{k-1} \int_{\text{LHP of } \Sigma_1} d^2z \frac{1}{k^2 |z|^{2k-2}} \\
&\left(\frac{\left\langle (T(z) + \frac{c(k^2-1)}{24}\frac{1}{z^2})(\bar{T}(\bar{z}) + \frac{c(k^2-1)}{24}\frac{1}{\bar{z}^2})\mathcal{O}_1(z_1,\bar{z}_1)...\mathcal{O}^\dagger_{2k}(z_{2k},\bar{z}_{2k}) \right\rangle_{\Sigma_1}}{\langle \mathcal{O}_1(z_1,\bar{z}_1)...\mathcal{O}^\dagger_{2k}(z_{2k},\bar{z}_{2k})\rangle_{\Sigma_1}} - \frac{\langle T\bar{T}(w,\bar{w})\mathcal{O}_1\mathcal{O}^\dagger_2\rangle_{\Sigma_1}}{\langle \mathcal{O}_1\mathcal{O}^\dagger_2\rangle_{\Sigma_1}} \right) \\
&+ \frac{k\lambda_2}{k-1} \int_{\text{UHP of } \Sigma_1} d^2z \frac{1}{k^2 |z|^{2k-2}} \\
&\left(\frac{\left\langle (T(z) + \frac{c(k^2-1)}{24}\frac{1}{z^2})(\bar{T}(\bar{z}) + \frac{c(k^2-1)}{24}\frac{1}{\bar{z}^2})\mathcal{O}_1(z_1,\bar{z}_1)...\mathcal{O}^\dagger_{2k}(z_{2k},\bar{z}_{2k}) \right\rangle_{\Sigma_1}}{\langle \mathcal{O}_1(z_1,\bar{z}_1)...\mathcal{O}^\dagger_{2k}(z_{2k},\bar{z}_{2k})\rangle_{\Sigma_1}} - \frac{\langle T\bar{T}(w,\bar{w})\mathcal{O}_1\mathcal{O}^\dagger_2\rangle_{\Sigma_1}}{\langle \mathcal{O}_1\mathcal{O}^\dagger_2\rangle_{\Sigma_1}} \right).
\end{aligned}
\tag{81}
$$

Similar to the discussion above, we still focus on the large c case. The leading order of the integral (81) is simple, so we do not evaluate it in detail. We will calculate the integral (81) of $\mathcal{O}(c)$ in the rest of this appendix. At $\mathcal{O}(c)$, using the conformal transformation (19) and Ward

Identity, the integral (81) can be rewritten as

$$
\Delta S_{A,\lambda_1,\lambda_2}^{(k)}
$$
$$
=\frac{k\lambda_1}{k-1}\int_{\text{LHP of }\Sigma_1}d^2z\frac{1}{k^2\left|z\right|^{2k-2}}
$$
$$
\left(\frac{c(k^2-1)}{24\bar{z}^2}\frac{\left\langle T(z)\mathcal{O}_1(z_1,\bar{z}_1)...\mathcal{O}_{2k}^\dagger(z_{2k},\bar{z}_{2k})\right\rangle_{\Sigma_1}}{\langle\mathcal{O}_1(z_1,\bar{z}_1)...\mathcal{O}_{2k}^\dagger(z_{2k},\bar{z}_{2k})\rangle_{\Sigma_1}}+\frac{c(k^2-1)}{24z^2}\frac{\left\langle \bar{T}(\bar{z})\mathcal{O}_1(z_1,\bar{z}_1)...\mathcal{O}_{2k}^\dagger(z_{2k},\bar{z}_{2k})\right\rangle_{\Sigma_1}}{\langle\mathcal{O}_1(z_1,\bar{z}_1)...\mathcal{O}_{2k}^\dagger(z_{2k},\bar{z}_{2k})\rangle_{\Sigma_1}}\right)
$$
$$
+\frac{k\lambda_2}{k-1}\int_{\text{UHP of }\Sigma_1}d^2z\frac{1}{k^2\left|z\right|^{2k-2}}
$$
$$
\left(\frac{c(k^2-1)}{24\bar{z}^2}\frac{\left\langle T(z)\mathcal{O}_1(z_1,\bar{z}_1)...\mathcal{O}_{2k}^\dagger(z_{2k},\bar{z}_{2k})\right\rangle_{\Sigma_1}}{\langle\mathcal{O}_1(z_1,\bar{z}_1)...\mathcal{O}_{2k}^\dagger(z_{2k},\bar{z}_{2k})\rangle_{\Sigma_1}}+\frac{c(k^2-1)}{24z^2}\frac{\left\langle \bar{T}(\bar{z})\mathcal{O}_1(z_1,\bar{z}_1)...\mathcal{O}_{2k}^\dagger(z_{2k},\bar{z}_{2k})\right\rangle_{\Sigma_1}}{\langle\mathcal{O}_1(z_1,\bar{z}_1)...\mathcal{O}_{2k}^\dagger(z_{2k},\bar{z}_{2k})\rangle_{\Sigma_1}}\right)
$$
$$
=\frac{k\lambda_1}{k-1}\int_{\text{LHP of }\Sigma_1}d^2z\frac{1}{k^2\left|z\right|^{2k-2}}\frac{c(k^2-1)}{24\langle\mathcal{O}_1(z_1,\bar{z}_1)...\mathcal{O}_{2k}^\dagger(z_{2k},\bar{z}_{2k})\rangle_{\Sigma_1}}
$$
$$
\left(\frac{1}{\bar{z}^2}\left(\sum_{i=1}^{2k}\frac{h}{(z-z_i)^2}+\sum_{j=0}^{k-2}\left(\frac{\partial_{2j+1}}{z-z_{2j+1}}+\frac{\partial_{2j+4}}{z-z_{2j+4}}\right)+\frac{\partial_2}{z-z_2}+\frac{\partial_{2k-1}}{z-z_{2k-1}}\right)\right.
$$
$$
\times\left\langle\mathcal{O}_1(z_1,\bar{z}_1)...\mathcal{O}_{2k}^\dagger(z_{2k},\bar{z}_{2k})\right\rangle_{\Sigma_1}
$$
$$
\left.+\frac{1}{z^2}\left(\sum_{i=1}^{2k}\frac{\bar{h}}{(\bar{z}-\bar{z}_i)^2}+\sum_{j=0}^{k-1}\left(\frac{\bar{\partial}_{2j+1}}{\bar{z}-\bar{z}_{2j+1}}+\frac{\bar{\partial}_{2j+2}}{\bar{z}-\bar{z}_{2j+2}}\right)\right)\left\langle\bar{T}(\bar{z})\mathcal{O}_1(z_1,\bar{z}_1)...\mathcal{O}_{2k}^\dagger(z_{2k},\bar{z}_{2k})\right\rangle_{\Sigma_1}\right)
$$
$$
+\frac{k\lambda_2}{k-1}\int_{\text{UHP of }\Sigma_1}d^2z\frac{1}{k^2\left|z\right|^{2k-2}}\frac{c(k^2-1)}{24\langle\mathcal{O}_1(z_1,\bar{z}_1)...\mathcal{O}_{2k}^\dagger(z_{2k},\bar{z}_{2k})\rangle_{\Sigma_1}}
$$
$$
\left(\frac{1}{\bar{z}^2}\left(\sum_{i=1}^{2k}\frac{h}{(z-z_i)^2}+\sum_{j=0}^{k-2}\left(\frac{\partial_{2j+1}}{z-z_{2j+1}}+\frac{\partial_{2j+4}}{z-z_{2j+4}}\right)+\frac{\partial_2}{z-z_2}+\frac{\partial_{2k-1}}{z-z_{2k-1}}\right)\right.
$$
$$
\times\left\langle\mathcal{O}_1(z_1,\bar{z}_1)...\mathcal{O}_{2k}^\dagger(z_{2k},\bar{z}_{2k})\right\rangle_{\Sigma_1}
$$
$$
\left.+\frac{1}{z^2}\left(\sum_{i=1}^{2k}\frac{\bar{h}}{(\bar{z}-\bar{z}_i)^2}+\sum_{j=0}^{k-1}\left(\frac{\bar{\partial}_{2j+1}}{\bar{z}-\bar{z}_{2j+1}}+\frac{\bar{\partial}_{2j+2}}{\bar{z}-\bar{z}_{2j+2}}\right)\right)\left\langle\bar{T}(\bar{z})\mathcal{O}_1(z_1,\bar{z}_1)...\mathcal{O}_{2k}^\dagger(z_{2k},\bar{z}_{2k})\right\rangle_{\Sigma_1}\right).
$$
$$
(82)
$$

All the integrals can be fully evaluated after using the polar coordinate and taking proper regularization, and we will show the following integral as an example:

$$
\int_{\text{LHP of }\Sigma_1}d^2z\frac{z^2}{\left|z\right|^{2k+2}\left(z-z_i\right)^2}=\int_0^\infty\int_{-\pi}^0d\rho d\theta\frac{e^{2i\theta}}{\rho^{2k-1}(\rho e^{i\theta}-z_i)^2}
$$
$$
=\begin{cases}\int_\epsilon^{\left|z_i\right|}\rho^{-1-2k}\left(-\pi+\frac{2i\rho z_i}{\rho^2-z_i^2}+2i\text{Arctanh}\left(\frac{z_i}{\rho}\right)\right) & z_i\in\text{LHP }and\ \rho<\left|z_i\right|\\\int_{\left|z_i\right|}^\infty\rho^{-1-2k}\left(\pi+\frac{2i\rho z_i}{\rho^2-z_i^2}+2i\text{Arctanh}\left(\frac{z_i}{\rho}\right)\right) & z_i\in\text{LHP }and\ \rho>\left|z_i\right|\\\int_\epsilon^\infty\rho^{-1-2k}\left(\pi+\frac{2i\rho z_i}{\rho^2-z_i^2}+2i\text{Arctanh}\left(\frac{z_i}{\rho}\right)\right) & z_i\in\text{UHP}\end{cases}\quad(83)
$$

where we have introduced ϵ as the cutoff. Integrate out ρ in (83) and regularize it through minimal subtraction, then we have

$$\int_{\text{LHP of } \Sigma_1} d^2 z \frac{z^2}{|z|^{2k+2}(z-z_i)^2} = \begin{cases} \frac{\pi}{k|z_i|^{2k}} + \frac{\pi}{2kz_i^{2k}} + \frac{\pi}{z_i} & z_i \in \text{LHP} \\ \frac{\pi}{2kz_i^{2k}} + \frac{\pi}{z_i} & z_i \in \text{UHP} \end{cases}. \tag{84}$$

Other integrals in (82) can be evaluated similarly, so we only list their results below.

$$\int_{\text{LHP of } \Sigma_1} d^2 z \frac{z^2}{|z|^{2k+2}(z-z_i)} = \begin{cases} \frac{z_i \pi}{k|z_i|^{2k}} - \frac{\pi}{2kz_i^{2k-1}} & z_i \in \text{LHP} \\ -\frac{\pi}{2kz_i^{2k-1}} & z_i \in \text{UHP} \end{cases}. \tag{85}$$

$$\int_{\text{UHP of } \Sigma_1} d^2 z \frac{z^2}{|z|^{2k+2}(z-z_i)^2} = \begin{cases} \frac{\pi}{k|z_i|^{2k}} - \frac{\pi}{2kz_i^{2k}} - \frac{\pi}{z_i} & z_i \in \text{UHP} \\ -\frac{\pi}{2kz_i^{2k}} - \frac{\pi}{z_i} & z_i \in \text{LHP} \end{cases} \tag{86}$$

$$\int_{\text{UHP of } \Sigma_1} d^2 z \frac{z^2}{|z|^{2k+2}(z-z_i)} = \begin{cases} \frac{z_i \pi}{k|z_i|^{2k}} + \frac{\pi}{2kz_i^{2k-1}} & z_i \in \text{UHP} \\ \frac{\pi}{2kz_i^{2k-1}} & z_i \in \text{LHP} \end{cases}. \tag{87}$$

$$\int_{\text{UHP of } \Sigma_1} d^2 z \frac{\bar{z}^2}{|z|^{2k+2}(\bar{z}-\bar{z}_i)^2} = \begin{cases} \frac{\pi}{k|\bar{z}_i|^{2k}} + \frac{\pi}{2k\bar{z}_i^{2k}} + \frac{\pi}{\bar{z}_i} & \bar{z}_i \in \text{LHP} \\ \frac{\pi}{2k\bar{z}_i^{2k}} + \frac{\pi}{\bar{z}_i} & \bar{z}_i \in \text{UHP} \end{cases} \tag{88}$$

$$\int_{\text{LHP of } \Sigma_1} d^2 z \frac{\bar{z}^2}{|z|^{2k+2}(\bar{z}-\bar{z}_i)^2} = \begin{cases} \frac{\pi}{k|\bar{z}_i|^{2k}} - \frac{\pi}{2k\bar{z}_i^{2k}} - \frac{\pi}{\bar{z}_i} & \bar{z}_i \in \text{UHP} \\ -\frac{\pi}{2k\bar{z}_i^{2k}} - \frac{\pi}{\bar{z}_i} & \bar{z}_i \in \text{LHP} \end{cases} \tag{89}$$

$$\int_{\text{UHP of } \Sigma_1} d^2 z \frac{z^2}{|z|^{2k+2}(z-z_i)} = \begin{cases} \frac{\bar{z}_i \pi}{k|\bar{z}_i|^{2k}} + \frac{\pi}{2k\bar{z}_i^{2k-1}} & \bar{z}_i \in \text{LHP} \\ \frac{\pi}{2k\bar{z}_i^{2k-1}} & \bar{z}_i \in \text{UHP} \end{cases}. \tag{90}$$

$$\int_{\text{LHP of } \Sigma_1} d^2 z \frac{z^2}{|z|^{2k+2}(z-z_i)} = \begin{cases} \frac{\bar{z}_i \pi}{k|\bar{z}_i|^{2k}} - \frac{\pi}{2k\bar{z}_i^{2k-1}} & \bar{z}_i \in \text{UHP} \\ -\frac{\pi}{2k\bar{z}_i^{2k-1}} & \bar{z}_i \in \text{LHP} \end{cases}. \tag{91}$$

Using results (84)-(91) and (82), we get the correction to pseudo-Rényi entropy of primary operators with different $T\bar{T}$-deformation parameters

$$
\Delta S^{(k)}_{A,\lambda_1,\lambda_2}
$$

$$
=\frac{c(k+1)}{24k}\lambda_1\left[\sum_{z_i\in\text{LHP}}\frac{\pi h}{k\,|z_i|^{2k}}-\sum_{z_{2i+1}\in\text{LHP}}\frac{2\pi h}{z_{2i+1}-z_{2i+4}}\frac{z_{2i+1}}{|z_{2i+1}|^{2k}}+\sum_{z_{2i+4}\in\text{LHP}}\frac{2\pi h}{z_{2i+1}-z_{2i+4}}\frac{z_{2i+4}}{|z_{2i+4}|^{2k}}\right.
$$

$$
\sum_{\bar{z}_i\in\text{UHP}}\frac{\pi h}{k\,|\bar{z}_i|^{2k}}-\sum_{\bar{z}_{2i+1}\in\text{UHP}}\frac{2\pi\bar{h}}{\bar{z}_{2i+1}-\bar{z}_{2i+2}}\frac{\bar{z}_{2i+1}}{|\bar{z}_{2i+1}|^{2k}}+\sum_{\bar{z}_{2i+2}\in\text{UHP}}\frac{2\pi\bar{h}}{\bar{z}_{2i+1}-\bar{z}_{2i+2}}\frac{\bar{z}_{2i+2}}{|\bar{z}_{2i+2}|^{2k}}
$$

$$
+\sum_{j=1}^{2k}\left(\frac{\pi}{2kz_i^{2k}}+\frac{\pi}{z_i}\right)+\sum_{j=0}^{k-2}\frac{\pi}{k}\left(\frac{-h}{z_{2j+1}-z_{2j+4}}\frac{z_{2j+1}}{z_{2j+1}^{2k}}+\frac{h}{z_{2j+1}-z_{2j+4}}\frac{z_{2j+4}}{z_{2j+4}^{2k}}\right)
$$

$$
+\frac{\pi}{k}\left(\frac{-h}{z_2-z_{2k-1}}\frac{z_2}{z_2^2}+\frac{h}{z_2-z_{2k-1}}\frac{z_{2k-1}}{z_{2k-1}^2}\right)-\sum_{j=1}^{2k}\left(\frac{\pi}{2k\bar{z}_i^{2k}}+\frac{\pi}{\bar{z}_i}\right)
$$

$$
\left.-\sum_{j=0}^{k-1}\frac{\pi}{k}\left(\frac{-\bar{h}}{\bar{z}_{2j+1}-\bar{z}_{2j+2}}\frac{\bar{z}_{2j+1}}{\bar{z}_{2j+1}^{2k}}+\frac{\bar{h}}{\bar{z}_{2j+1}-\bar{z}_{2j+2}}\frac{\bar{z}_{2j+2}}{\bar{z}_{2j+2}^{2k}}\right)\right]
$$

$$
+\frac{c(k+1)}{24k}\lambda_2\left[\sum_{z_i\in\text{UHP}}\frac{\pi h}{k\,|z_i|^{2k}}-\sum_{z_{2i+1}\in\text{UHP}}\frac{2\pi h}{z_{2i+1}-z_{2i+4}}\frac{z_{2i+1}}{|z_{2i+1}|^{2k}}+\sum_{z_{2i+4}\in\text{UHP}}\frac{2\pi h}{z_{2i+1}-z_{2i+4}}\frac{z_{2i+4}}{|z_{2i+4}|^{2k}}\right.
$$

$$
\sum_{\bar{z}_i\in\text{LHP}}\frac{\pi h}{k\,|\bar{z}_i|^{2k}}-\sum_{\bar{z}_{2i+1}\in\text{LHP}}\frac{2\pi\bar{h}}{\bar{z}_{2i+1}-\bar{z}_{2i+2}}\frac{\bar{z}_{2i+1}}{|\bar{z}_{2i+1}|^{2k}}+\sum_{\bar{z}_{2i+2}\in\text{LHP}}\frac{2\pi\bar{h}}{\bar{z}_{2i+1}-\bar{z}_{2i+2}}\frac{\bar{z}_{2i+2}}{|\bar{z}_{2i+2}|^{2k}}
$$

$$
-\sum_{j=1}^{2k}\left(\frac{\pi}{2kz_i^{2k}}+\frac{\pi}{z_i}\right)-\sum_{j=0}^{k-2}\frac{\pi}{k}\left(\frac{-h}{z_{2j+1}-z_{2j+4}}\frac{z_{2j+1}}{z_{2j+1}^{2k}}+\frac{h}{z_{2j+1}-z_{2j+4}}\frac{z_{2j+4}}{z_{2j+4}^{2k}}\right)
$$

$$
-\frac{\pi}{k}\left(\frac{-h}{z_2-z_{2k-1}}\frac{z_2}{z_2^2}+\frac{h}{z_2-z_{2k-1}}\frac{z_{2k-1}}{z_{2k-1}^2}\right)+\sum_{j=1}^{2k}\left(\frac{\pi}{2k\bar{z}_i^{2k}}+\frac{\pi}{\bar{z}_i}\right)
$$

$$
\left.+\sum_{j=0}^{k-1}\frac{\pi}{k}\left(\frac{-\bar{h}}{\bar{z}_{2j+1}-\bar{z}_{2j+2}}\frac{\bar{z}_{2j+1}}{\bar{z}_{2j+1}^{2k}}+\frac{\bar{h}}{\bar{z}_{2j+1}-\bar{z}_{2j+2}}\frac{\bar{z}_{2j+2}}{\bar{z}_{2j+2}^{2k}}\right)\right]
\tag{92}
$$

Replacing all the coordinates in (92) with (21), we get the correction to the kth pseudo-Rényi entropy with different deformed parameters under $T\bar{T}$ deformation at the late time

$$
\Delta S^{(k)}_{A,\lambda_1,\lambda_2} =
$$

$$
\frac{c(k+1)}{24k^2}\lambda_1 \Bigg\{ \frac{\left((-t+x_1)^{\frac{1}{k}}(t-x_1)^{\frac{1}{k}} - e^{\frac{2\pi i}{k}}(-t-x_2)^{\frac{1}{k}}(t-x_2)^{\frac{1}{k}}\right)(x_1^2-x_2^2)}{(t^2-x_1^2)\left((-t-x_1)^{\frac{1}{k}} - e^{\frac{2\pi i}{k}}(-t-x_2)^{\frac{1}{k}}\right)\left((t-x_1)^{\frac{1}{k}} - (t-x_2)^{\frac{1}{k}}\right)(t^2-x_2^2)}
$$

$$
+\frac{1}{2}\pi\Bigg[\frac{(x_1-x_2)(2t-x_1-x_2)\left((t-x_1)^{1/k}+(t-x_2)^{1/k}\right)}{(t-x_1)^2(t-x_2)^2\left((t-x_1)^{1/k}-(t-x_2)^{1/k}\right)} + \frac{2\left((-t-x_1)^{\frac{1}{k}-2} - e^{\frac{2i\pi}{k}}(-t-x_2)^{\frac{1}{k}-2}\right)}{-(-t-x_1)^{1/k} + e^{\frac{2i\pi}{k}}(-t-x_2)^{1/k}}
$$

$$
+2t^2 + 2t(x_1+x_2) + 2\left(\frac{1}{t+x_2}+\frac{1}{t+x_1}\right) + x_1^2 + x_2^2\Bigg]\Bigg\} + \mathcal{O}(\epsilon,c^0,\lambda_1^2)
$$

$$
+\frac{c(k+1)}{24k^2}\lambda_2 \Bigg\{ \frac{\left((-t+x_1)^{\frac{1}{k}}(t-x_1)^{\frac{1}{k}} - e^{\frac{2\pi i}{k}}(-t-x_2)^{\frac{1}{k}}(t-x_2)^{\frac{1}{k}}\right)(x_1^2-x_2^2)}{(t^2-x_1^2)\left((-t-x_1)^{\frac{1}{k}} - e^{\frac{2\pi i}{k}}(-t-x_2)^{\frac{1}{k}}\right)\left((t-x_1)^{\frac{1}{k}} - (t-x_2)^{\frac{1}{k}}\right)(t^2-x_2^2)}
$$

$$
-\frac{1}{2}\pi\Bigg[\frac{(x_1-x_2)(2t-x_1-x_2)\left((t-x_1)^{1/k}+(t-x_2)^{1/k}\right)}{(t-x_1)^2(t-x_2)^2\left((t-x_1)^{1/k}-(t-x_2)^{1/k}\right)} + \frac{2\left((-t-x_1)^{\frac{1}{k}-2} - e^{\frac{2i\pi}{k}}(-t-x_2)^{\frac{1}{k}-2}\right)}{-(-t-x_1)^{1/k} + e^{\frac{2i\pi}{k}}(-t-x_2)^{1/k}}
$$

$$
+2t^2 + 2t(x_1+x_2) + 2\left(\frac{1}{t+x_2}+\frac{1}{t+x_1}\right) + x_1^2 + x_2^2\Bigg]\Bigg\} + \mathcal{O}(\epsilon,c^0,\lambda_2^2). \tag{93}
$$

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