

RECEIVED: May 29, 2023

REVISED: August 12, 2023

ACCEPTED: August 17, 2023

PUBLISHED: September 5, 2023

Pseudo entropy of primary operators in $T\bar{T}/J\bar{T}$ -deformed CFTs

Song He,^{a,c} Jie Yang,^{a,b} Yu-Xuan Zhang^a and Zi-Xuan Zhao^a

^aCenter for Theoretical Physics and College of Physics, Jilin University, Changchun 130012, People's Republic of China

^bSchool of Mathematical Sciences, Capital Normal University, Beijing 100048, People's Republic of China

^cMax Planck Institute for Gravitational Physics (Albert Einstein Institute), Am Mühlenberg 1, 14476 Golm, Germany

E-mail: hesong@jlu.edu.cn, yangjie@cnu.edu.cn,
yuxuanz20@mails.jlu.edu.cn, zzx23@mails.jlu.edu.cn

ABSTRACT: In this work, we investigate the time evolution of the pseudo-(Rényi) entropy after local primary operator quenches in 2D CFTs with $T\bar{T}/J\bar{T}$ -deformation. Using perturbation theory, we analyze the corrections to the second pseudo-Rényi entropy at the late time, which exhibit a universal form, while its early-time behavior is model-dependent. Moreover, we uncover nontrivial time-dependent effects arising from the first-order deformation of the k^{th} pseudo-Rényi entropy at the late time. Additionally, drawing inspiration from the gravitational side, specifically the gluing of two cutoff AdS geometries, we investigate the k^{th} pseudo-Rényi entropy for vacuum states characterized by distinct $T\bar{T}$ -deformation parameters, as well as for primary states acting on different deformed vacuum states. Our findings reveal additional corrections compared to the results of pseudo-Rényi entropy for globally deformed vacuum states.

KEYWORDS: AdS-CFT Correspondence, Conformal and W Symmetry, Field Theories in Lower Dimensions, Integrable Field Theories

ARXIV EPRINT: [2305.10984](https://arxiv.org/abs/2305.10984)

Contents

1	Introduction	1
2	Setup	3
2.1	The $T\bar{T}$ -deformation of the correlation functions	3
2.2	The pseudo entropy and the replica method	5
3	Second pseudo-Rényi entropy in $T\bar{T}$-deformed CFTs	6
4	Pseudo entropy in $T\bar{T}$-deformed CFTs	10
4.1	k^{th} Rényi entropy in $T\bar{T}$ -deformed CFTs	10
4.2	k^{th} pseudo-Rényi entropy in $T\bar{T}$ -deformed CFTs	12
5	Pseudo entropy in $T\bar{T}$-deformed CFTs with different deformed states	13
6	Pseudo-Rényi entropy in $J\bar{T}$-deformed CFTs	19
7	Conclusion and prospect	20
A	Details of integrals in $T\bar{T}$-deformed pseudo-Rényi entropy	21
B	The $T\bar{T}$-deformation of the k^{th} pseudo-Rényi entropy by the replica method	23
C	Details of integrals in pseudo-Rényi entropy with different deformed states	25

1 Introduction

The AdS/CFT correspondence [1–3] provides a holographic description of a quantum gravity theory in an asymptotically anti-de Sitter (AdS) spacetime with a conformal field theory (CFT) at its asymptotic boundary. Its discovery has inspired significant exploration regarding quantum information theory in the high-energy physics community in recent years, covering a range of topics such as quantum entanglement [4–10], the emergence of geometry [11–13], and the black hole information paradox [14–18].

Recently, there has been a surge of interest in 2D CFTs deformed by $T\bar{T}/J\bar{T}$ operator [19–21]. Though the deformations are irrelevant in the sense of the renormalization group, owing to the integrability [19, 21] and holographic duality [22–26], they have been extensively investigated in recent years [27–56]. The $T\bar{T}/J\bar{T}$ -deformation triggers a flow of the original

theory along a trajectory in the space of field theory. For the $T\bar{T}$ -deformation, the trajectory of the deformed action satisfies the following condition:

$$\frac{dS(\lambda)}{d\lambda} = \int d^2z \sqrt{g}(T\bar{T})_\lambda, \tag{1.1}$$

where λ represents the coupling constant of the $T\bar{T}$ operator. $S(\lambda = 0)$ corresponds to the action of the undeformed theory. In the flat space, the $T\bar{T}$ operator can be expressed as

$$T\bar{T} = -\frac{1}{4}\det(T_{\mu\nu}) = T_{zz}T_{\bar{z}\bar{z}} - T_{z\bar{z}}T_{\bar{z}z}, \tag{1.2}$$

with $T = T_{zz}$ and $\bar{T} = T_{\bar{z}\bar{z}}$. In this paper, we mainly focus on the perturbation theory of $S(\lambda)$. The $T\bar{T}$ -deformed action, up to the first order of λ , is given by

$$S(\lambda) = S(\lambda = 0) + \lambda \int d^2z \sqrt{g}(T\bar{T})_{\lambda=0} + \mathcal{O}(\lambda^2), \tag{1.3}$$

and we refer to $(T\bar{T})_{\lambda=0}$ as simply $T\bar{T}$. We treat the $J\bar{T}$ -deformation in the same manner.

Pseudo entropy is a generalization of the entanglement entropy proposed via the AdS/CFT correspondence and post-selection [57]. Specifically, the pseudo entropy is a two-state vector version of the entanglement entropy, defined as follows. Let $|\psi\rangle$ and $|\varphi\rangle$ be two non-orthogonal states in the Hilbert space \mathcal{H}_S of a composite quantum system $S = A \cup B$. We initially construct an operator acting on \mathcal{H}_S which is called the *transition matrix* [57, 58]

$$\mathcal{T}^{\psi|\varphi} \equiv \frac{|\psi\rangle\langle\varphi|}{\langle\varphi|\psi\rangle} = \frac{\rho_\psi\rho_\varphi}{\text{tr}[\rho_\psi\rho_\varphi]}. \tag{1.4}$$

We then define the so-called *reduced transition matrix* $\mathcal{T}_A^{\psi|\varphi}$ for subsystem A by tracing out the degrees of freedom in the subsystem B from the transition matrix (1.4), that is, $\mathcal{T}_A^{\psi|\varphi} = \text{Tr}_B[\mathcal{T}^{\psi|\varphi}]$. The pseudo entropy of A is then defined by the von Neumann entropy of $\mathcal{T}_A^{\psi|\varphi}$, i.e.,

$$S_A = -\text{tr}[\mathcal{T}_A^{\psi|\varphi} \log \mathcal{T}_A^{\psi|\varphi}]. \tag{1.5}$$

Furthermore, we can introduce the so-called *pseudo-Rényi entropy* $S_A^{(k)}$, a generalization of the Rényi entropy

$$S_A^{(k)} = \frac{1}{1-k} \log \text{tr}[(\rho_A)^k], \tag{1.6}$$

utilizing the transition matrix. The pseudo-Rényi entropy is defined by replacing the reduced density matrix in the Rényi entropy (1.6) with the reduced transition matrix $\mathcal{T}_A^{\psi|\varphi}$,

$$S_A^{(k)} = \frac{1}{1-k} \log \text{tr}[(\mathcal{T}_A^{\psi|\varphi})^k]. \tag{1.7}$$

Note that the pseudo-Rényi entropy (1.7) gives the pseudo entropy (1.5) in the limit of $k \rightarrow 1$.

It has been shown that pseudo entropy, like entanglement entropy, can be used to quantify the topological contributions of entanglement of excited states in two-dimensional rational CFTs [59–61].¹ The behavior of the entanglement entropy in the deformed CFTs, especially in the context of the $T\bar{T}$ - and $J\bar{T}$ -deformations, has been extensively studied in recent years [81–85]. However, little is known about the pseudo-Rényi entropy in $T\bar{T}/J\bar{T}$ -deformed CFTs. Another motivation for investigating pseudo-Rényi entropy in $T\bar{T}/J\bar{T}$ -deformed CFTs comes from the holography. It is thought that pseudo entropy has its holographic dual, and the holographic dual of the positive sign $T\bar{T}$ -deformed two-dimensional holographic CFT was proposed to be AdS₃ gravity with a finite radius cut-off. So the properties of the pseudo-Rényi entropy in the deformed CFTs would provide insight into the structure of the deformed theories and their relationship to the undeformed CFTs. Therefore, investigating the pseudo-Rényi entropy in the deformed CFTs is an interesting and important research direction.

This paper aims to investigate the pseudo-Rényi entropy in the $T\bar{T}/J\bar{T}$ -deformed CFTs. Our inquiry can be traced back to the investigations on the Rényi entropy of vacuum states [81] and locally excited states [86] in the $T\bar{T}/J\bar{T}$ -deformed theories. It has been observed that, with the first-order perturbation of the deformation, the second Rényi entanglement entropy of locally excited states acquires a nontrivial time dependence. It would be captivating to investigate the correction of the pseudo-Rényi entropy in the deformed CFTs. Besides, inspired by gluing together two cutoff AdS geometries and the cutoff AdS/ $T\bar{T}$ -deformed-CFT(cAdS/dCFT) correspondence, we are interested in exploring the pseudo-Rényi entropy of two states with two different deformation parameters.

The structure of this paper is as follows. In section 2, we briefly overview the $T\bar{T}$ -deformation of the correlation functions and the replica method for locally excited states in 2D CFTs. In section 3, we concentrate on the second pseudo-Rényi entropy of the $T\bar{T}$ -deformed theories. In section 4, we investigate the excess of the k^{th} Rényi entropy of locally excited states and the k^{th} pseudo-Rényi entropy of locally excited states in $T\bar{T}$ -deformed CFTs. In section 5, we compute the k^{th} pseudo-Rényi entropy of two vacuum states and two primary states with different deformation parameters in $T\bar{T}$ -deformed CFTs. In section 6, we extend the analysis from sections 3 and 4 to the $J\bar{T}$ -deformed theory. Finally, in section 7, we provide concluding remarks and discuss possible future research directions. Some technical details of the calculations are presented in the appendices.

2 Setup

2.1 The $T\bar{T}$ -deformation of the correlation functions

In this subsection, we briefly overview the $T\bar{T}$ -deformation in 2D CFTs and its effect on the correlation functions in [86].

By using the Ward identity, the first-order correction to the k -point correlation function of primary operators in $T\bar{T}$ -deformed CFTs can be expressed as $\langle \mathcal{O}_1(z_1, \bar{z}_1) \cdots \mathcal{O}_k(z_k, \bar{z}_k) \rangle_\lambda$, where λ is the coupling constant of the $T\bar{T}$ -deformation. The expression for the first-order

¹Please refer to [62–80] for other recent development of pseudo entropy.

correction is as follows:

$$\begin{aligned}
 & \langle \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \dots \mathcal{O}_k(z_k, \bar{z}_k) \rangle_\lambda \\
 &= \lambda \int d^2z \langle T\bar{T}(z, \bar{z}) \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \dots \mathcal{O}_k(z_k, \bar{z}_k) \rangle \\
 &= \lambda \int d^2z \left(\sum_{i=1}^k \left(\frac{h_i}{(z - z_i)^2} + \frac{\partial_{z_i}}{z - z_i} \right) \right) \left(\sum_{i=1}^k \left(\frac{\bar{h}_i}{(\bar{z} - \bar{z}_i)^2} + \frac{\partial_{\bar{z}_i}}{\bar{z} - \bar{z}_i} \right) \right) \\
 & \quad \times \langle \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \dots \mathcal{O}_k(z_k, \bar{z}_k) \rangle.
 \end{aligned} \tag{2.1}$$

To obtain the result, we take advantage of the fact that any correlation function involving $T_{z\bar{z}}$ vanishes in a 2D CFT. Specifically, we begin with the two-point correlation function of primary operators \mathcal{O}_1 and \mathcal{O}_2 with conformal weights (h, \bar{h}) . The two-point function of \mathcal{O}_1 and \mathcal{O}_2 can be written as

$$\langle \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \rangle = \frac{C_{12}}{z_{12}^{2h} \bar{z}_{12}^{2\bar{h}}}, \tag{2.2}$$

where $z_{ij} = z_i - z_j$, $\bar{z}_{ij} = \bar{z}_i - \bar{z}_j$. The Ward identity of the two-point function gives us the following relation:

$$\begin{aligned}
 & \langle T(z) \bar{T}(\bar{z}) \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \rangle \\
 &= \left(\frac{h\bar{h}z_{12}^2 \bar{z}_{12}^2}{(z - z_1)^2 (z - z_2)^2 (\bar{z} - \bar{z}_1)^2 (\bar{z} - \bar{z}_2)^2} - \frac{4\pi\bar{h}}{(z - z_1)(\bar{z}_1 - \bar{z}_2)} \delta^{(2)}(z - z_1) \right. \\
 & \quad \left. - \frac{2\pi\bar{h}}{(z - z_2)(\bar{z}_1 - \bar{z}_2)} \delta^{(2)}(z - z_2) \right) \langle \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \rangle.
 \end{aligned} \tag{2.3}$$

The appearance of the delta function arises from the action of ∂_{z_i} on terms such as $(\bar{z} - \bar{z}_i)^{-1}$, which leads to a non-dynamical contribution. After the integration over spacetime, this contribution gives rise to a UV divergence. To simplify the following analysis, we omit these terms. We can obtain the first-order correction of the two-point function resulting from the $T\bar{T}$ -deformation using dimension regularization

$$\langle \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \rangle_\lambda = \lambda h \bar{h} \frac{4\pi}{|z_{12}|^2} \left(\frac{4}{\epsilon} + 2 \log(|z_{12}|^2) + 2 \log \pi + 2\gamma - 5 \right) \langle \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \rangle. \tag{2.4}$$

Next, we compute the four-point function within the framework of $T\bar{T}$ -deformed CFTs. Specifically, the four-point function of primary operators can be written as follows:

$$\langle \mathcal{O}_1^\dagger(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \mathcal{O}_1^\dagger(z_3, \bar{z}_3) \mathcal{O}_2(z_4, \bar{z}_4) \rangle = \frac{G(\eta, \bar{\eta})}{z_{13}^{2h} z_{23}^{2h} \bar{z}_{13}^{2\bar{h}} \bar{z}_{24}^{2\bar{h}}}, \tag{2.5}$$

with the cross ratios

$$\eta := \frac{z_{12} z_{34}}{z_{13} z_{24}}, \quad \bar{\eta} := \frac{\bar{z}_{12} \bar{z}_{34}}{\bar{z}_{13} \bar{z}_{24}}, \tag{2.6}$$

and $G(\eta, \bar{\eta})$ is the conformal block which is a function of cross ratios.

The first-order correction to this four-point function due to the $T\bar{T}$ -deformation is given by:

$$\begin{aligned}
 & \langle \mathcal{O}_1^\dagger(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \mathcal{O}_1^\dagger(z_3, \bar{z}_3) \mathcal{O}_2(z_4, \bar{z}_4) \rangle_\lambda \\
 &= \lambda \int d^2z \left\{ \left(\frac{hz_{13}^2}{(z-z_1)^2(z-z_3)^2} + \frac{hz_{24}^2}{(z-z_2)^2(z-z_4)^2} + \frac{z_{23}z_{14}}{\prod_{i=1}^4(z-z_i)} \frac{\eta G(\eta, \bar{\eta})}{G(\eta, \bar{\eta})} \right) \right. \\
 & \quad \times \left(\frac{\bar{h}\bar{z}_{13}^2}{(\bar{z}-\bar{z}_1)^2(\bar{z}-\bar{z}_3)^2} + \frac{\bar{h}\bar{z}_{24}^2}{(\bar{z}-\bar{z}_2)^2(\bar{z}-\bar{z}_4)^2} + \frac{\bar{z}_{23}\bar{z}_{14}}{\prod_{i=1}^4(\bar{z}-\bar{z}_i)} \frac{\bar{\eta} G(\eta, \bar{\eta})}{G(\eta, \bar{\eta})} \right) \\
 & \quad - \eta \bar{\eta} \frac{z_{23}z_{14}\bar{z}_{23}\bar{z}_{14}}{\prod_{i=1}^4(z-z_i)(\bar{z}-\bar{z}_i)} \frac{\partial_\eta G(\eta, \bar{\eta}) \partial_{\bar{\eta}} G(\eta, \bar{\eta})}{G(\eta, \bar{\eta})^2} + \eta \bar{\eta} \frac{z_{23}z_{14}\bar{z}_{23}\bar{z}_{14}}{\prod_{i=1}^4(z-z_i)(\bar{z}-\bar{z}_i)} \frac{\partial_\eta \partial_{\bar{\eta}} G(\eta, \bar{\eta})}{G(\eta, \bar{\eta})} \\
 & \quad - \frac{4\pi \bar{h} \bar{z}_{13}^2}{(z-z_1)(\bar{z}-\bar{z}_1)(\bar{z}-\bar{z}_3)^2} \delta^{(2)}(z-z_1) - \frac{4\pi \bar{h} \bar{z}_{13}^2}{(z-z_3)(\bar{z}-\bar{z}_3)(\bar{z}-\bar{z}_1)^2} \delta^{(2)}(z-z_3) \\
 & \quad - \frac{4\pi \bar{h} \bar{z}_{24}^2}{(z-z_2)(\bar{z}-\bar{z}_2)(\bar{z}-\bar{z}_4)^2} \delta^{(2)}(z-z_2) - \frac{4\pi \bar{h} \bar{z}_{24}^2}{(z-z_4)(\bar{z}-\bar{z}_4)(\bar{z}-\bar{z}_2)^2} \delta^{(2)}(z-z_4) \\
 & \quad - \frac{\bar{z}_{12}\bar{z}_{34} \partial_{\bar{\eta}} G(\eta, \bar{\eta})}{G(\eta, \bar{\eta})} \left[\frac{2\pi \delta^{(2)}(z-z_1)}{(z-z_1)(\bar{z}-\bar{z}_2)(\bar{z}-\bar{z}_3)(\bar{z}-\bar{z}_4)} + \frac{2\pi \delta^{(2)}(z-z_2)}{(z-z_2)(\bar{z}-\bar{z}_1)(\bar{z}-\bar{z}_3)(\bar{z}-\bar{z}_4)} \right. \\
 & \quad \left. + \frac{2\pi \delta^{(2)}(z-z_3)}{(z-z_3)(\bar{z}-\bar{z}_1)(\bar{z}-\bar{z}_2)(\bar{z}-\bar{z}_4)} + \frac{2\pi \delta^{(2)}(z-z_4)}{(z-z_4)(\bar{z}-\bar{z}_1)(\bar{z}-\bar{z}_2)(\bar{z}-\bar{z}_3)} \right] \Big\} \\
 & \times \langle \mathcal{O}_1^\dagger(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \mathcal{O}_1^\dagger(z_3, \bar{z}_3) \mathcal{O}_2(z_4, \bar{z}_4) \rangle. \tag{2.7}
 \end{aligned}$$

2.2 The pseudo entropy and the replica method

This subsection presents a concise overview of the replica method for constructing the pseudo-Rényi entropy with locally excited states in rational conformal field theories (RCFTs). We start by considering a RCFT defined on a plane, where the vacuum state is denoted as $|\Omega\rangle$. By applying the primary operators $\mathcal{O}(x_1)$ and $\mathcal{O}^\dagger(x_2)$ to the vacuum state, we create two locally excited states

$$|\psi_1\rangle \equiv e^{-\epsilon H} \mathcal{O}(x_1) |\Omega\rangle, \quad |\psi_2\rangle \equiv e^{-\epsilon H} \mathcal{O}^\dagger(x_2) |\Omega\rangle, \tag{2.8}$$

where the infinitesimal parameter ϵ is introduced to suppress the high energy modes [87]. Subsequently, we utilize these states to construct a real-time evolved transition matrix denoted as $\mathcal{T}^{1|2}(t)$:

$$\mathcal{T}^{1|2}(t) \equiv e^{-iHt} \frac{|\psi_1\rangle \langle \psi_2|}{\langle \psi_2 | \psi_1 \rangle} e^{iHt}. \tag{2.9}$$

To obtain the reduced transition matrix of subsystem A at time t , we trace out the degrees of freedom of A^c (the complement of A) from the transition matrix $\mathcal{T}^{1|2}(t)$, giving us $\mathcal{T}_A^{1|2}(t) = \text{tr}_{A^c} [\mathcal{T}^{1|2}(t)]$.

It turns out that the excess of the k^{th} pseudo-Rényi entropy of A with respect to the ground state, defined as $\Delta S^{(k)}(\mathcal{T}_A^{1|2}(t)) := S^{(k)}(\mathcal{T}_A^{1|2}(t)) - S^{(k)}(\text{tr}_{A^c} [|\Omega\rangle \langle \Omega|])$, is of the

form [60]

$$\Delta S^{(k)}(\mathcal{T}_A^{1|2}(t)) = \frac{1}{1-k} \left[\log \left\langle \prod_{j=1}^k \mathcal{O}(w_{2j-1}, \bar{w}_{2j-1}) \mathcal{O}^\dagger(w_{2j}, \bar{w}_{2j}) \right\rangle_{\Sigma_k} - k \log \langle \mathcal{O}(w_1, \bar{w}_1) \mathcal{O}^\dagger(w_2, \bar{w}_2) \rangle_{\Sigma_1} \right], \quad (2.10)$$

according to the replica method. In eq. (2.10), Σ_k denotes an k -sheet Riemann surface with cuts on each copy corresponding to A , and $(w_{2j-1}, \bar{w}_{2j-1})$ and (w_{2j}, \bar{w}_{2j}) are coordinates on the j^{th} -sheet surface. The term in the first line of eq. (2.10) is given by a $2k$ -point correlation function on Σ_k , and the one in the second line is calculated from a two-point function on Σ_1 . The coordinates are chosen as

$$\begin{aligned} w_{2j-1} &= x_1 + t - i\epsilon, & w_{2j} &= x_2 + t + i\epsilon, \\ \bar{w}_{2j-1} &= x_1 - t + i\epsilon, & \bar{w}_{2j} &= x_2 - t - i\epsilon. \end{aligned} \quad (j = 1, 2, \dots, k) \quad (2.11)$$

The $2k$ -point correlation function on Σ_k in eq. (2.10) can be evaluated with the help of a conformal mapping of Σ_k to the complex plane Σ_1 . For convenience, the subsystem is chosen such that $A = [0, \infty)$ hereafter. We can then map Σ_k to Σ_1 using the simple conformal mapping

$$w = z^k. \quad (2.12)$$

We subsequently utilize the mapping given by eq. (2.12) to express the $2k$ -point function on Σ_k ,

$$\begin{aligned} &\langle \mathcal{O}(w_1, \bar{w}_1) \mathcal{O}^\dagger(w_2, \bar{w}_2) \dots \mathcal{O}(w_{2k-1}, \bar{w}_{2k-1}) \mathcal{O}^\dagger(w_{2k}, \bar{w}_{2k}) \rangle_{\Sigma_k} \\ &= \left(\prod_{j=1}^{2k} \left| \frac{dw_j}{dz_j} \right|^{-2h} \right) \langle \mathcal{O}(w_1, \bar{w}_1) \mathcal{O}^\dagger(w_2, \bar{w}_2) \dots \mathcal{O}(w_{2k-1}, \bar{w}_{2k-1}) \mathcal{O}^\dagger(w_{2k}, \bar{w}_{2k}) \rangle_{\Sigma_1}. \end{aligned} \quad (2.13)$$

The $2k$ points z_1, z_2, \dots, z_{2k} in the z -coordinates are given by

$$\begin{aligned} z_{2j+1} &= e^{2\pi i \frac{j+1/2}{k}} (-x_1 - t + i\epsilon)^{\frac{1}{k}}, & \bar{z}_{2j+1} &= e^{-2\pi i \frac{j+1/2}{k}} (-x_1 + t - i\epsilon)^{\frac{1}{k}}, \\ z_{2j+2} &= e^{2\pi i \frac{j+1/2}{k}} (-x_2 - t - i\epsilon)^{\frac{1}{k}}, & \bar{z}_{2j+2} &= e^{-2\pi i \frac{j+1/2}{k}} (-x_2 + t + i\epsilon)^{\frac{1}{k}}, \end{aligned} \quad (j = 0, \dots, k-1). \quad (2.14)$$

3 Second pseudo-Rényi entropy in $T\bar{T}$ -deformed CFTs

The late-time excess of pseudo-Rényi entropy of locally excited operators has a universal behavior: the logarithmic quantum dimension of the operator, while the early-time one does not. Investigating the late-time behavior of the excess of pseudo-Rényi entropy and its behavior in the early time in $T\bar{T}$ -deformed CFTs is intriguing.

In [81], the authors discussed Rényi entropy in the $T\bar{T}$ -deformed CFTs for vacuum theory. It is shown that up to the first order of the deformation parameter λ ,

$$\Delta S_{A,\lambda}^{(k)} = \frac{-k\lambda}{1-k} \int_{\Sigma_1} [\langle T\bar{T} \rangle_{\Sigma_k} - \langle T\bar{T} \rangle_{\Sigma_1}] + \mathcal{O}(\lambda^2), \quad (3.1)$$

where we denote $\Delta S_{A,\lambda}^{(k)}$ as the correction to the undeformed Rényi entropy.

Now let us consider second pseudo-Rényi entropy in $T\bar{T}$ -deformed CFTs

$$\begin{aligned}
& - \left(\Delta S_{A,0}^{(2)} + \Delta S_{A,\lambda}^{(2)} \right) \\
& = \log \left(\langle \mathcal{O}(w_1, \bar{w}_1) \mathcal{O}^\dagger(w_2, \bar{w}_2) \mathcal{O}(w_3, \bar{w}_3) \mathcal{O}^\dagger(w_4, \bar{w}_4) \rangle_{\Sigma_2} \right. \\
& \quad \left. + 2\lambda \int d^2 w \langle T\bar{T}(w, \bar{w}) \mathcal{O}(w_1, \bar{w}_1) \mathcal{O}^\dagger(w_2, \bar{w}_2) \mathcal{O}(w_3, \bar{w}_3) \mathcal{O}^\dagger(w_4, \bar{w}_4) \rangle_{\Sigma_2} \right) \\
& \quad - 2 \log \left(\langle \mathcal{O}(w_1, \bar{w}_1) \mathcal{O}^\dagger(w_2, \bar{w}_2) \rangle_{\Sigma_1} + \lambda \int d^2 w \langle T\bar{T}(w, \bar{w}) \mathcal{O}(w_1, \bar{w}_1) \mathcal{O}^\dagger(w_2, \bar{w}_2) \rangle_{\Sigma_1} \right).
\end{aligned} \tag{3.2}$$

The derivation of (3.2) is a simplified version of appendix B where we set $\lambda_1 = \lambda_2$ and $k = 2$.²

Under the conformal mapping (2.12) for $k = 2$, the stress tensors transform as

$$T(w) = \left(\frac{dz}{dw} \right)^2 T(z) + \frac{c}{12} \{z, w\}, \quad \bar{T}(\bar{w}) = \left(\frac{d\bar{z}}{d\bar{w}} \right)^2 \bar{T}(\bar{z}) + \frac{c}{12} \{\bar{z}, \bar{w}\}, \tag{3.3}$$

where

$$\{z, w\} = \frac{z'''}{z'} - \frac{3}{2} \frac{z''^2}{z'^2}, \quad \left(z' = \frac{dz}{dw} \right)$$

is the Schwarzian derivative, and c is the central charge. The $T\bar{T}$ operator thus transforms as

$$T\bar{T}(w, \bar{w}) = \frac{1}{4z^2} \frac{1}{4\bar{z}^2} \left(T(z) + \frac{c}{8z^2} \right) \left(\bar{T}(\bar{z}) + \frac{c}{8\bar{z}^2} \right). \tag{3.4}$$

By utilizing this transformation formula and performing an expansion around $\lambda = 0$, we obtain expressions for the *zero*th and first-order terms in λ :

$$\begin{aligned}
& - \left(\Delta S_{A,0}^{(2)} + \Delta S_{A,\lambda}^{(2)} \right) \\
& = \log \left(\prod_{i=1}^4 \left| \frac{dw_i}{dz_i} \right|^{-2h} \frac{\langle \mathcal{O}(w_1, \bar{w}_1) \mathcal{O}^\dagger(w_2, \bar{w}_2) \mathcal{O}(w_3, \bar{w}_3) \mathcal{O}^\dagger(w_4, \bar{w}_4) \rangle_{\Sigma_1}}{\left(\langle \mathcal{O}(w_1, \bar{w}_1) \mathcal{O}^\dagger(w_2, \bar{w}_2) \rangle_{\Sigma_1} \right)^2} \right) \\
& \quad + 2\lambda \int d^2 z \frac{1}{4|z|^2} \frac{\langle \left(T(z) + \frac{c}{8z^2} \right) \left(\bar{T}(\bar{z}) + \frac{c}{8\bar{z}^2} \right) \mathcal{O}(w_1, \bar{w}_1) \mathcal{O}^\dagger(w_2, \bar{w}_2) \mathcal{O}(w_3, \bar{w}_3) \mathcal{O}^\dagger(w_4, \bar{w}_4) \rangle_{\Sigma_1}}{\langle \mathcal{O}(w_1, \bar{w}_1) \mathcal{O}^\dagger(w_2, \bar{w}_2) \mathcal{O}(w_3, \bar{w}_3) \mathcal{O}^\dagger(w_4, \bar{w}_4) \rangle_{\Sigma_1}} \\
& \quad - 2\lambda \int d^2 w \frac{\langle T\bar{T}(w, \bar{w}) \mathcal{O}(w_1, \bar{w}_1) \mathcal{O}^\dagger(w_2, \bar{w}_2) \rangle_{\Sigma_1}}{\langle \mathcal{O}(w_1, \bar{w}_1) \mathcal{O}^\dagger(w_2, \bar{w}_2) \rangle_{\Sigma_1}} + \mathcal{O}(\lambda^2).
\end{aligned} \tag{3.5}$$

Let us focus on the large c case. At the leading order, that is the $\mathcal{O}(c^2)$ order, eq. (3.5) is evaluated as

$$2\lambda \frac{c^2}{64} \int d^2 z \frac{1}{4|z|^6} = 6\lambda \frac{2\pi c^2}{64} \int_0^\infty d\rho \frac{1}{4\rho^5}. \tag{3.6}$$

²Notice that there are some typos in [86], the exact coupling in its section 3 and section 5.1 should be $\frac{\lambda}{3}$ instead of λ .

To regularize this formula, we introduce the IR cutoff and UV cutoff by replacing $(0, \infty)$ with $(\frac{1}{\Lambda}, \Lambda)$ and we obtain

$$2\lambda \frac{c^2}{64} \int d^2z \frac{1}{4|z|^6} \stackrel{\text{cutoff}}{\rightarrow} 6\lambda \frac{2\pi c^2}{64} \frac{\tilde{\Lambda}^4}{16}. \quad (3.7)$$

At the next order, which is of order $\mathcal{O}(c)$, eq. (3.5) can be evaluated as

$$\begin{aligned} & \frac{c}{8} 2\lambda \int d^2z \frac{1}{4|z|^2} \frac{\langle (\frac{1}{z^2} T(z) + \frac{1}{\bar{z}^2} \bar{T}(\bar{z})) \mathcal{O}(w_1, \bar{w}_1) \mathcal{O}^\dagger(w_2, \bar{w}_2) \mathcal{O}(w_3, \bar{w}_3) \mathcal{O}^\dagger(w_4, \bar{w}_4) \rangle_{\Sigma_1}}{\langle \mathcal{O}(w_1, \bar{w}_1) \mathcal{O}^\dagger(w_2, \bar{w}_2) \mathcal{O}(w_3, \bar{w}_3) \mathcal{O}^\dagger(w_4, \bar{w}_4) \rangle_{\Sigma_1}} \\ &= 2\lambda \frac{c\pi}{64} \left(\frac{h}{|z_1|^4} + \frac{h}{|z_2|^4} + \frac{h}{|z_3|^4} + \frac{h}{|z_4|^4} - \frac{2hz_1}{z_{13}|z_1|^4} + \frac{2hz_3}{z_{13}|z_3|^4} - \frac{2hz_2}{z_{24}|z_2|^4} + \frac{2hz_4}{z_{24}|z_4|^4} \right. \\ & \quad \left. + \left[\frac{z_1}{|z_1|^4} \left(-\frac{1}{z_{13}} + \frac{1}{z_{12}} \right) + \frac{z_2}{|z_2|^4} \left(-\frac{1}{z_{24}} - \frac{1}{z_{12}} \right) + \frac{z_3}{|z_3|^4} \left(\frac{1}{z_{34}} + \frac{1}{z_{13}} \right) + \frac{z_4}{|z_4|^4} \left(-\frac{1}{z_{34}} + \frac{1}{z_{24}} \right) \right] \frac{\eta \partial_\eta G(\eta, \bar{\eta})}{G(\eta, \bar{\eta})} \right) \\ & \quad + 2\lambda \frac{c\pi}{64} \left(\frac{h}{|\bar{z}_1|^4} + \frac{h}{|\bar{z}_2|^4} + \frac{h}{|\bar{z}_3|^4} + \frac{h}{|\bar{z}_4|^4} - \frac{2h\bar{z}_1}{\bar{z}_{13}|\bar{z}_1|^4} + \frac{2h\bar{z}_3}{\bar{z}_{13}|\bar{z}_3|^4} - \frac{2h\bar{z}_2}{\bar{z}_{24}|\bar{z}_2|^4} + \frac{2h\bar{z}_4}{\bar{z}_{24}|\bar{z}_4|^4} \right. \\ & \quad \left. + \left[\frac{\bar{z}_1}{|\bar{z}_1|^4} \left(-\frac{1}{\bar{z}_{13}} + \frac{1}{\bar{z}_{12}} \right) + \frac{\bar{z}_2}{|\bar{z}_2|^4} \left(-\frac{1}{\bar{z}_{24}} - \frac{1}{\bar{z}_{12}} \right) + \frac{\bar{z}_3}{|\bar{z}_3|^4} \left(\frac{1}{\bar{z}_{34}} + \frac{1}{\bar{z}_{13}} \right) + \frac{\bar{z}_4}{|\bar{z}_4|^4} \left(-\frac{1}{\bar{z}_{34}} + \frac{1}{\bar{z}_{24}} \right) \right] \frac{\bar{\eta} \partial_{\bar{\eta}} G(\eta, \bar{\eta})}{G(\eta, \bar{\eta})} \right). \end{aligned} \quad (3.8)$$

The derivation of this equation is shown in appendix A. When we take the late-time limit $t \rightarrow \infty$, the conformal block in the four-point correlation function is

$$G(\eta, \bar{\eta}) \sim \mathcal{F}_{00}(1 - \eta)^{-2h} \bar{\eta}^{-2h}, \quad (3.9)$$

where $(\mathcal{F}_{00})^{-1} = d_{\mathcal{O}}$ is the quantum dimension of \mathcal{O} .

Substitute (3.9) into the (3.8) and we find the correction to the second pseudo-Rényi entropy in large c limit at the late time:

$$\begin{aligned} & -\Delta S_{A,\lambda}^{(2)} = \\ & 6\lambda \frac{2\pi c^2}{64} \frac{\tilde{\Lambda}^4}{16} + 2\lambda \frac{4(x_1^2 - x_2^2)hc\pi}{64} \\ & \left(\frac{-t^2 (\sqrt{-t-x_1}\sqrt{t-x_1} + \sqrt{-t-x_2}\sqrt{t-x_2})}{(-t-x_1)^{3/2}(t-x_1)^{3/2}(-t-x_2)^{3/2}(t-x_2)^{3/2} (\sqrt{-t-x_1} + \sqrt{-t-x_2}) (\sqrt{t-x_1} - \sqrt{t-x_2})} \right. \\ & \quad \left. + \frac{x_1^2 \sqrt{-t-x_2}\sqrt{t-x_2} + x_2^2 \sqrt{-t-x_1}\sqrt{t-x_1}}{(-t-x_1)^{3/2}(t-x_1)^{3/2}(-t-x_2)^{3/2}(t-x_2)^{3/2} (\sqrt{-t-x_1} + \sqrt{-t-x_2}) (\sqrt{t-x_1} - \sqrt{t-x_2})} \right) \\ & + \mathcal{O}(\epsilon, c^0, \lambda^2). \end{aligned} \quad (3.10)$$

This result is universal in all RCFTs. However, if the two insertion coordinates coincide, eq. (3.10) does not yield the same result as the second Rényi entropy derived in [86]. This discrepancy arises primarily because in [86], we initially set $x_1 \rightarrow x_2$ and perform an expansion in ϵ , whereas in the case of the second pseudo-Rényi entropy, we first expand in ϵ and then take the limit $x_1 \rightarrow x_2$. Notably, the two limits, $\epsilon \rightarrow 0$ and $x_1 \rightarrow x_2$, do not commute, as demonstrated in [61].

Since the two insertion coordinates are different, the conformal block $G(\eta, \bar{\eta})$ may diverge for some models at the early time. The rest of this subsection will discuss the early-time correction to the second pseudo-Rényi entropy in $T\bar{T}$ -deformed CFTs for the free scalar and Ising model as examples.

Example 1: The free scalar. The free scalar can be described by the operator $\mathcal{O} = \frac{1}{\sqrt{2}}(e^{\frac{i}{2}\phi} + e^{-\frac{i}{2}\phi})$, whose conformal dimension is $h = \bar{h} = \frac{1}{8}$. The conformal block of free scalar can be written as

$$G(\eta, \bar{\eta}) = \frac{1 + |\eta| + |1 - \eta|}{2\sqrt{|\eta||1 - \eta|}}. \quad (3.11)$$

Substituting (3.11) into (3.8), we find that the corrections to the second pseudo-Rényi entropy of free scalar at the early time in large c limit is

$$\begin{aligned} -\Delta S_{A,\lambda}^{(2)} = & 6\lambda \frac{2\pi c^2 \tilde{\Lambda}^4}{64 \cdot 16} + \frac{1}{x_1^2 x_2^2 (x_1 - x_2)^4} \left(4x_1^6 - 16x_2 x_1^5 + 2x_2^2 \left(\left(\frac{(x_1 - x_2)^4}{x_1^2 x_2^2} \right)^{3/4} + 14 \right) x_1^4 \right. \\ & + \left(\sqrt{-x_1} \sqrt{-x_2} \left(\sqrt{\frac{x_1^2 + 4\sqrt{-x_1} \sqrt{-x_2} x_1 + 6x_2 x_1 + x_2^2 + 4\sqrt{-x_1} \sqrt{-x_2} x_2}{x_1 x_2}} \right. \right. \\ & \left. \left. - \sqrt{\frac{4\sqrt{-x_2} (-x_1)^{3/2} + 4\sqrt{-x_1} (-x_2)^{3/2} + x_1^2 + x_2^2 + 6x_1 x_2}{x_1 x_2}} \right) \left(\frac{(x_1 - x_2)^4}{x_1^2 x_2^2} \right)^{3/4} \right. \\ & \left. + 4x_2 \left(\left(\frac{(x_1 - x_2)^4}{x_1^2 x_2^2} \right)^{3/4} - 8 \right) \right) x_2^2 x_1^3 \\ & + \left(\sqrt{-x_1} \sqrt{-x_2} \left(\sqrt{\frac{x_1^2 + 4\sqrt{-x_1} \sqrt{-x_2} x_1 + 6x_2 x_1 + x_2^2 + 4\sqrt{-x_1} \sqrt{-x_2} x_2}{x_1 x_2}} \right. \right. \\ & \left. \left. - \sqrt{\frac{4\sqrt{-x_2} (-x_1)^{3/2} + 4\sqrt{-x_1} (-x_2)^{3/2} + x_1^2 + x_2^2 + 6x_1 x_2}{x_1 x_2}} \right) \left(\frac{(x_1 - x_2)^4}{x_1^2 x_2^2} \right)^{3/4} \right. \\ & \left. + 2x_2 \left(\left(\frac{(x_1 - x_2)^4}{x_1^2 x_2^2} \right)^{3/4} + 14 \right) \right) x_2^3 x_1^2 - 16x_2^5 x_1 + 4x_2^6 \Big). \end{aligned} \quad (3.12)$$

Notice that the results (3.10) and (3.12) are highly different from that in [86] because the insertion coordinates of two states generating the transition matrix are different.

Example 2: Ising model. The $2n$ -point correlation function of the spin operator in the Ising model can be written as

$$\langle \sigma(z_1, \bar{z}_1) \dots \sigma(z_{2n}, \bar{z}_{2n}) \rangle^2 = \frac{1}{2^n} \sum_{\substack{\epsilon_i = \pm 1 \\ i=1, \dots, 2n \\ \sum \epsilon_i = 0}} \prod_{i < j} |z_i - z_j|^{\epsilon_i \epsilon_j / 2} \quad (3.13)$$

The first-order corrections to the second pseudo-Rényi entropy of the spin operator at the early time is

$$\begin{aligned} -\Delta S_{A,\lambda}^{(2)} = & 6\lambda \frac{2\pi c^2 \tilde{\Lambda}^4}{64 \cdot 16} + 2\lambda \frac{c\pi}{64} \left(\frac{-(x_1^2)^{3/2} + 3\sqrt{x_2^2 x_1^2} + 3\sqrt{x_1^2 x_2^2} + 4x_1 x_2 \left(\sqrt{x_1^2} + \sqrt{x_2^2} \right)^2}{4(x_1^2)^{9/8} \sqrt{-2\sqrt{-x_1} \sqrt{-x_2} - x_1 - x_2} \sqrt{2\sqrt{-x_1} \sqrt{-x_2} - x_1 - x_2} (x_2^2)^{9/8}} \right. \\ & \left. + \frac{2x_2^2 \sqrt{x_1^2} + 2(x_1^2)^{5/4} - (x_2^2)^{5/4}}{4(x_1^2)^{9/8} \sqrt{-2\sqrt{-x_1} \sqrt{-x_2} - x_1 - x_2} \sqrt{2\sqrt{-x_1} \sqrt{-x_2} - x_1 - x_2} (x_2^2)^{7/8}} + \frac{4}{x_1^2} + \frac{4}{x_2^2} \right) + \mathcal{O}(\epsilon, c^0, \lambda^2). \end{aligned} \quad (3.14)$$

Since all correlation functions in the Ising model are known, one can easily calculate the correction to the pseudo-Rényi entropy in $T\bar{T}$ -deformed CFTs at the early time. As eq. (3.14) shows, the earlier time behavior of the pseudo-Rényi entropy depends on the initial setup and details of the theories.

4 Pseudo entropy in $T\bar{T}$ -deformed CFTs

In the previous section, we perform a perturbative calculation of the second pseudo-Rényi entropy in $T\bar{T}$ -deformed CFT at the late time and corrections to the second pseudo-Rényi entropy of free scalar and Ising model at the early time. However, to obtain the correction to the pseudo entropy, it is reasonable to examine the k^{th} pseudo-Rényi entropy and perform an analytic continuation for k goes to 1 in $T\bar{T}$ -deformed CFT. As demonstrated earlier, the conformal block for the second pseudo-Rényi entropy did not have a universal structure at the early stage. Therefore, in this section, firstly, we will investigate the k^{th} Rényi entropy at both the early and the late time in $T\bar{T}$ -deformed CFTs. Then we only consider the $T\bar{T}$ -deformation of the k^{th} pseudo-Rényi entropy at the late time.

4.1 k^{th} Rényi entropy in $T\bar{T}$ -deformed CFTs

Under the conformal map (2.12), the $T\bar{T}$ operators transform as

$$T(w) = \frac{1}{k^2 z^{2k-2}} \left(T(z) + \frac{c(k^2-1)}{24} \frac{1}{z^2} \right), \quad \bar{T}(\bar{w}) = \frac{1}{k^2 \bar{z}^{2k-2}} \left(\bar{T}(\bar{z}) + \frac{c(k^2-1)}{24} \frac{1}{\bar{z}^2} \right). \quad (4.1)$$

The correction to the undeformed Rényi entropy seen in appendix B is

$$\Delta S_{A,\lambda}^{(k)} = \frac{k\lambda}{k-1} \int_{\Sigma_1} d^2 w \left(\frac{\langle T\bar{T}(w, \bar{w}) \mathcal{O}_1 \dots \mathcal{O}_{2k}^\dagger \rangle_{\Sigma_k}}{\langle \mathcal{O}_1 \dots \mathcal{O}_{2k}^\dagger \rangle_{\Sigma_k}} - \frac{\langle T\bar{T}(w, \bar{w}) \mathcal{O}_1 \mathcal{O}_2^\dagger \rangle_{\Sigma_1}}{\langle \mathcal{O}_1 \mathcal{O}_2^\dagger \rangle_{\Sigma_1}} \right), \quad (4.2)$$

where $\mathcal{O}(z_i, \bar{z}_i)$ is denoted by \mathcal{O}_i . Notice that if we take all primary operators as identity operators, (4.2) becomes the Rényi entropy for vacuum states in $T\bar{T}$ -deformed CFTs, and it matches the result in [81] precisely.

Using the conformal transformation (2.12) between Σ_k and Σ_1 , we can rewrite this formula as

$$\Delta S_{A,\lambda}^{(k)} = \frac{k\lambda}{k-1} \int_{\Sigma_1} d^2 z \frac{1}{k^2 |z|^{2k-2}} \left(\frac{\left\langle \left(T(z) + \frac{c(k^2-1)}{24} \frac{1}{z^2} \right) \left(\bar{T}(\bar{z}) + \frac{c(k^2-1)}{24} \frac{1}{\bar{z}^2} \right) \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k}) \right\rangle_{\Sigma_1}}{\langle \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k}) \rangle_{\Sigma_1}} - \frac{\langle T\bar{T}(w, \bar{w}) \mathcal{O}_1 \mathcal{O}_2^\dagger \rangle_{\Sigma_1}}{\langle \mathcal{O}_1 \mathcal{O}_2^\dagger \rangle_{\Sigma_1}} \right). \quad (4.3)$$

We still focus on the large c case. At the leading order $\mathcal{O}(c^2)$, by introducing the IR cutoff $\frac{1}{\Lambda}$, the correction to the k^{th} Rényi entropy is

$$\frac{k}{k-1} \lambda \frac{c^2(k^2-1)^2}{576} \int d^2 z \frac{1}{k^2 |z|^{2k+2}} \xrightarrow{\text{cutoff}} 3k\lambda \frac{c^2(k^2-1)^2 \pi \tilde{\Lambda}^{2k}}{576k^3}. \quad (4.4)$$

The derivation of this formula is shown in appendix A. At the next-to-leading order, the integral in the first line of (4.3) can be divided into two parts at the late time. The first part, due to $T(z)$ acting on the $2k$ -point correlation function, can be written as

$$\begin{aligned}
 \Delta S_{A,\lambda,part1}^{(k)} &\equiv \frac{k\lambda}{k-1} \int_{\Sigma_1} d^2z \frac{1}{k^2 |z|^{2k-2}} \frac{c(k^2-1)}{24} \frac{1}{\bar{z}^2} \frac{\langle (T(z)\mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k})) \rangle_{\Sigma_1}}{\langle \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k}) \rangle_{\Sigma_1}} \\
 &= \frac{k}{k-1} \lambda \frac{1}{k^2} \frac{c(k^2-1)}{24} \int d^2z \frac{1}{|z|^{2k-2} \bar{z}^2} \frac{1}{\langle \mathcal{O}(z_1, \bar{z}_1) \mathcal{O}^\dagger(z_2, \bar{z}_2) \dots \mathcal{O}(z_{2k-1}, \bar{z}_{2k-1}) \mathcal{O}^\dagger(z_{2k}, \bar{z}_{2k}) \rangle_{\Sigma_1}} \\
 &\quad \left(\sum_{i=1}^{2k} \frac{h}{(z-z_i)^2} + \sum_{j=0}^{k-2} \left(\frac{\partial_{2j+1}}{z-z_{2j+1}} + \frac{\partial_{2j+4}}{z-z_{2j+4}} \right) + \frac{\partial_2}{z-z_2} + \frac{\partial_{2k-1}}{z-z_{2k-1}} \right) \\
 &\quad \times \langle \mathcal{O}(z_1, \bar{z}_1) \mathcal{O}^\dagger(z_2, \bar{z}_2) \dots \mathcal{O}(z_{2k-1}, \bar{z}_{2k-1}) \mathcal{O}^\dagger(z_{2k}, \bar{z}_{2k}) \rangle_{\Sigma_1} \\
 &= \frac{k}{k-1} \lambda \frac{1}{k^2} \frac{c(k^2-1)}{24} \left(\sum_{i=1}^{2k} \frac{h\pi}{k} \frac{1}{|z_i|^{2k}} + \sum_{j=0}^{k-2} \frac{\pi}{k} \left(\frac{-2h}{z_{2j+1}-z_{2j+4}} \frac{z_{2j+1}}{|z_{2j+1}|^{2k}} + \frac{2h}{z_{2j+1}-z_{2j+4}} \frac{z_{2j+4}}{|z_{2j+4}|^{2k}} \right) \right. \\
 &\quad \left. + \frac{\pi}{k} \left(\frac{-2h}{z_2-z_{2k-1}} \frac{z_2}{|z_2|^2} + \frac{2h}{z_2-z_{2k-1}} \frac{z_{2k-1}}{|z_{2k-1}|^2} \right) \right) \tag{4.5}
 \end{aligned}$$

where in the last equality, we use the fact

$$\lim_{t \rightarrow \infty} (z_{2i+4} - z_{2i+1}) \simeq \frac{w_2 - w_1}{kt} e^{2\pi i \frac{i+1}{k}} t^{\frac{1}{k}} \simeq 0, \tag{4.6}$$

and the fact that the holomorphic part of the $2k$ -point correlation function can be written as

$$\begin{aligned}
 &\langle \mathcal{O}(z_1) \mathcal{O}^\dagger(z_2) \dots \mathcal{O}(z_{2k-1}) \mathcal{O}^\dagger(z_{2k}) \rangle_{\Sigma_1} \\
 &\sim (F_{00}[\mathcal{O}])^{k-1} \langle \mathcal{O}(z_1) \mathcal{O}^\dagger(z_4) \rangle_{\Sigma_1} \dots \langle \mathcal{O}(z_{2k+1}) \mathcal{O}^\dagger(z_{2k+4}) \rangle_{\Sigma_1}. \tag{4.7}
 \end{aligned}$$

The second part due to $\bar{T}(\bar{z})$ acting on the $2k$ -point correlation function can be written as

$$\begin{aligned}
 \Delta S_{A,\lambda,part2}^{(k)} &\equiv \frac{k\lambda}{k-1} \int_{\Sigma_1} d^2z \frac{1}{k^2 |z|^{2k-2}} \frac{c(k^2-1)}{24} \frac{1}{z^2} \frac{\langle (\bar{T}(z)\mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k})) \rangle_{\Sigma_1}}{\langle \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k}) \rangle_{\Sigma_1}} \\
 &= \frac{k}{k-1} \lambda \frac{1}{k^2} \frac{c(k^2-1)}{24} \int d^2z \frac{1}{|z|^{2k-2} z^2} \frac{1}{\langle \mathcal{O}(z_1, \bar{z}_1) \mathcal{O}^\dagger(z_2, \bar{z}_2) \dots \mathcal{O}(z_{2k-1}, \bar{z}_{2k-1}) \mathcal{O}^\dagger(z_{2k}, \bar{z}_{2k}) \rangle_{\Sigma_1}} \\
 &\quad \left(\sum_{i=1}^{2k} \frac{\bar{h}}{(\bar{z}-\bar{z}_i)^2} + \sum_{j=0}^{k-1} \left(\frac{\bar{\partial}_{2j+1}}{\bar{z}-\bar{z}_{2j+1}} + \frac{\bar{\partial}_{2j+2}}{\bar{z}-\bar{z}_{2j+2}} \right) \right) \\
 &\quad \times \langle \mathcal{O}(w_1, \bar{w}_1) \mathcal{O}^\dagger(w_2, \bar{w}_2) \dots \mathcal{O}(w_{2k-1}, \bar{w}_{2k-1}) \mathcal{O}^\dagger(w_{2k}, \bar{w}_{2k}) \rangle_{\Sigma_1} \\
 &= \frac{k}{k-1} \lambda \frac{1}{k^2} \frac{c(k^2-1)}{24} \left(\sum_{i=1}^{2k} \frac{\bar{h}\pi}{k} \frac{1}{|\bar{z}_i|^{2k}} + \sum_{j=0}^{k-1} \frac{\pi}{k} \left(\frac{-2\bar{h}}{\bar{z}_{2j+1}-\bar{z}_{2j+2}} \frac{\bar{z}_{2j+1}}{|\bar{z}_{2j+1}|^{2k}} + \frac{2\bar{h}}{\bar{z}_{2j+1}-\bar{z}_{2j+2}} \frac{\bar{z}_{2j+2}}{|\bar{z}_{2j+2}|^{2k}} \right) \right), \tag{4.8}
 \end{aligned}$$

where in the last equality, we use the fact

$$\lim_{t \rightarrow \infty} (\bar{z}_{2i+2} - \bar{z}_{2i+1}) \simeq \frac{w_1 - w_2}{kt} e^{-2\pi i \frac{i+\frac{1}{2}}{k}} t^{\frac{1}{k}} \simeq 0, \tag{4.9}$$

and the fact that the anti-holomorphic part of the $2k$ -point correlation function can be written as

$$\begin{aligned} & \langle \mathcal{O}(\bar{z}_1) \mathcal{O}^\dagger(\bar{z}_2) \dots \mathcal{O}(\bar{z}_{2k-1}) \mathcal{O}^\dagger(\bar{z}_{2k}) \rangle_{\Sigma_1} \\ & \sim (F_{00}[\mathcal{O}])^{k-1} \langle \mathcal{O}(\bar{z}_1) \mathcal{O}^\dagger(\bar{z}_2) \rangle_{\Sigma_1} \dots \langle \mathcal{O}(\bar{z}_{2k-1}) \mathcal{O}^\dagger(\bar{z}_{2k}) \rangle_{\Sigma_1}. \end{aligned} \quad (4.10)$$

Now we replace all coordinates in (4.5) and (4.8) with

$$\begin{aligned} z_{2k+1} &= e^{2\pi i \frac{k+1/2}{n}} (-x-t+i\epsilon)^{\frac{1}{n}}, & \bar{z}_{2k+1} &= e^{-2\pi i \frac{k+1/2}{n}} (-x+t-i\epsilon)^{\frac{1}{n}}, \\ z_{2k+2} &= e^{2\pi i \frac{k+1/2}{n}} (-x-t-i\epsilon)^{\frac{1}{n}}, & \bar{z}_{2k+2} &= e^{-2\pi i \frac{k+1/2}{n}} (-x+t+i\epsilon)^{\frac{1}{n}}, \quad (k=0, \dots, n-1). \end{aligned} \quad (4.11)$$

We get the correction to the k^{th} Rényi entropy in the $T\bar{T}$ -deformed CFT at the late time

$$\Delta S_{A,\lambda}^{(k)} = 3k\lambda \frac{c^2(k^2-1)^2 \pi \tilde{\Lambda}^{2k}}{576k^3} - \frac{k}{k-1} \lambda \frac{h\pi c(k^2-1)}{24k^3} \frac{4k^2 t}{(t-x)(t+x)^2} + \mathcal{O}(\epsilon, c^0, \lambda^2). \quad (4.12)$$

We next consider the integral (4.3) at the early time. The coordinates and the $2k$ -point correlation function have the following property:

$$\begin{aligned} & \lim_{t \rightarrow 0} (z_{2i+2} - z_{2i+1}) \simeq 0, \quad \lim_{t \rightarrow 0} (\bar{z}_{2i+2} - \bar{z}_{2i+1}) \simeq 0, \\ & \langle \mathcal{O}(z_1, \bar{z}_1) \mathcal{O}^\dagger(z_2, \bar{z}_2) \dots \mathcal{O}(z_{2k-1}, \bar{z}_{2k-1}) \mathcal{O}^\dagger(z_{2k}, \bar{z}_{2k}) \rangle_{\Sigma_1} \\ & \sim \langle \mathcal{O}(z_1, \bar{z}_1) \mathcal{O}^\dagger(z_2, \bar{z}_2) \rangle_{\Sigma_1} \dots \langle \mathcal{O}(z_{2k-1}, \bar{z}_{2k-1}) \mathcal{O}^\dagger(z_{2k}, \bar{z}_{2k}) \rangle_{\Sigma_1}. \end{aligned} \quad (4.13)$$

We can get the correction to the k^{th} Rényi entropy at the early time.

$$\Delta S_{A,\lambda}^{(k)} = 3k\lambda \frac{c^2(k^2-1)^2 \pi \tilde{\Lambda}^{2k}}{576k^3} - \frac{k}{k-1} \lambda \frac{h\pi c(k^2-1)}{24k^3} \frac{8k^2 t^2}{(t^2-x^2)^2} + \mathcal{O}(\epsilon, c^0, \lambda^2). \quad (4.14)$$

Results (4.12) and (4.14) are precisely the results we find in [86] if we take the interval of A in [86] to infinity and set $k = 2$. Taking the limit of $k \rightarrow 1$ for (4.12) and (4.14), we obtain the early-time and late-time behavior of the entanglement entropy under the $T\bar{T}$ -deformation up to the first-order, respectively. Namely,

$$\Delta S_{A,\lambda} = \begin{cases} -\lambda \frac{2\pi hc}{3} \frac{t^2}{(t^2-x^2)^2}, & t < |x|, \\ -\lambda \frac{\pi hc}{3} \frac{t}{(t-x)(t+x)^2}, & t > |x|. \end{cases} \quad (4.15)$$

In this scenario, the $T\bar{T}$ deformation does not alter the late-time dynamics of $\Delta S_{A,\lambda}$, which exhibits a temporal evolution following a $\frac{1}{t^2}$ pattern. It is important to note that a singularity emerges in (4.15) at $t = x$, and this singularity arises from the derivative acting on the $2k$ -point correlation function, which discontinuously changes at that point.

4.2 k^{th} pseudo-Rényi entropy in $T\bar{T}$ -deformed CFTs

Thanks to the property of the $2k$ -point correlation function (4.13) at the early time, we can derive the correction to the k^{th} Rényi entropy in $T\bar{T}$ -deformed CFTs. However, this

property is no longer valid for the pseudo entropy since the two insertion coordinates x_1 and x_2 are different. Therefore we only consider the $T\bar{T}$ -deformation of the k^{th} pseudo-Rényi entropy at the late time in this subsection.

Combining (4.5) and (4.8), and replacing all the coordinates with (2.14), we get the correction to the k^{th} pseudo-Rényi entropy in the $T\bar{T}$ -deformed CFT,

$$\begin{aligned} \Delta S_{A,\lambda}^{(k)} = & 3k\lambda \frac{c^2(k^2-1)^2\pi\tilde{\Lambda}^{2k}}{576k^3} \\ & + \frac{k}{k-1}\lambda \frac{h\pi c(k^2-1)}{24k^3} \left(\frac{2k \left((-t+x_1)^{\frac{1}{k}}(t-x_1)^{\frac{1}{k}} - e^{\frac{2\pi i}{k}}(-t-x_2)^{\frac{1}{k}}(t-x_2)^{\frac{1}{k}} \right) (x_1^2-x_2^2)}{(t^2-x_1^2) \left((-t-x_1)^{\frac{1}{k}} - e^{\frac{2\pi i}{k}}(-t-x_2)^{\frac{1}{k}} \right) \left((t-x_1)^{\frac{1}{k}} - (t-x_2)^{\frac{1}{k}} \right) (t^2-x_2^2)} \right) \\ & + \mathcal{O}(\epsilon, c^0, \lambda^2). \end{aligned} \tag{4.16}$$

The result (4.16) precisely matches the result we find in the previous section 3 if we take $k = 2$.

Taking the limit of $k \rightarrow 1$ for (4.16), we obtain the late-time correction of the pseudo entropy under the $T\bar{T}$ -deformation up to the first-order which is

$$\Delta S_{A,\lambda} = \frac{\lambda ch\pi}{6} \frac{((-t-x_1)(t-x_1) - (-t-x_2)(t-x_2))(x_1^2-x_2^2)}{(t^2-x_1^2)(-x_1^2+x_2^2)(t^2-x_2^2)}. \tag{4.17}$$

In the limit of late time, it is evident that $\Delta S_{A,\lambda} \sim \frac{1}{t^2}$. Therefore, the first-order $T\bar{T}$ deformation does not alter the late-time behavior of the pseudo-entropy, which is consistent with the late-time behavior of the excess entanglement entropy derived in (4.15). Moreover, there exist singularities at $t = x_1$ or $t = x_2$, which emerge due to the non-smooth behavior of the correlation function at those specific points. These singularities exhibit similarities to the finding discussed in the previous subsection.

5 Pseudo entropy in $T\bar{T}$ -deformed CFTs with different deformed states

In the previous section, we have explicitly shown the correction of the k^{th} pseudo-Rényi entropy after the $T\bar{T}$ -deformation, up to the first order of the deformation coefficient. Recent research about the $T\bar{T}$ -deformed CFTs has focused on how to glue two different AdS spacetimes [88, 89] and its holographic dual. Inspired by the cAdS/dCFT correspondence, in this section, we will consider the pseudo-Rényi entropy for states with different $T\bar{T}$ -deformation on the CFT side.

We first consider the transition matrix for the vacuum state taking the form

$$\mathcal{T}^{1|2} \equiv \frac{|\Omega_{\lambda_1}\rangle\langle\Omega_{\lambda_2}|}{\langle\Omega_{\lambda_2}|\Omega_{\lambda_1}\rangle}, \tag{5.1}$$

where the vacuum states are deformed by $T\bar{T}$ operators with different coupling constants λ_1 and λ_2 , and this transition matrix is defined in the boundary theory of a glued AdS spacetime with two cutoff AdS spacetimes gluing together along a space-like hypersurface,

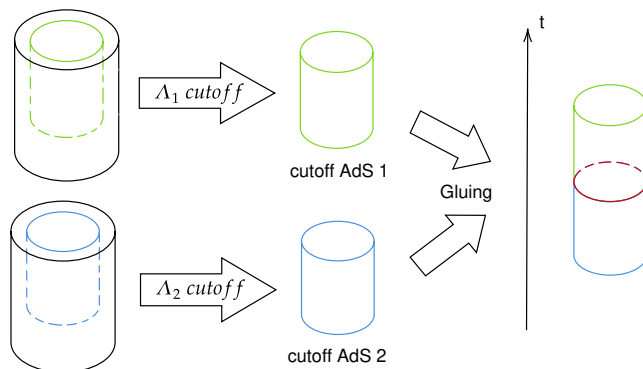


Figure 1. Gluing of two cutoff AdS spacetimes along their space-like boundaries.

as shown in figure 1. In the rest of this section, we mainly consider the case that the deformation parameters, λ_1 and λ_2 , are infinitesimally small. For $\lambda_1 \sim 0$ and $\lambda_2 \sim 0$, the deformed vacuum states, $|\Omega_{\lambda_1}\rangle$ and $|\Omega_{\lambda_2}\rangle$, can be expanded according to the complete basis of the undeformed Hilbert space \mathcal{H} . Therefore, in the sense of perturbation theory, the deformation of the k^{th} pseudo-Rényi entropy can be derived through the replica method shown in appendix B, which is

$$\Delta S_{A,\lambda_1,\lambda_2}^{(k)} = \frac{k}{k-1} \left[\lambda_1 \int_{\text{LHP}} d^2w \left(\langle T\bar{T}(w, \bar{w}) \rangle_{\Sigma_k} - \langle T\bar{T}(w, \bar{w}) \rangle_{\Sigma_1} \right) + \lambda_2 \int_{\text{UHP}} d^2w \left(\langle T\bar{T}(w, \bar{w}) \rangle_{\Sigma_k} - \langle T\bar{T}(w, \bar{w}) \rangle_{\Sigma_1} \right) \right]. \quad (5.2)$$

In eq. (5.2), LHP denotes the lower half-plane of Σ_1 , while UHP denotes the upper half-plane of Σ_1 .

Using the transformation of the $T\bar{T}$ operator between Σ_k and Σ_1 , the integral becomes

$$\begin{aligned} \Delta S_{A,\lambda_1,\lambda_2}^{(k)} &= \frac{k}{k-1} \left[\lambda_1 \int_{\text{LHP}} d^2z \frac{1}{k^2 |z|^{2k-2}} \left\langle \left(T(z) + \frac{c(k^2-1)}{24z^2} \right) \left(\bar{T}(\bar{z}) + \frac{c(k^2-1)}{24\bar{z}^2} \right) \right\rangle_{\Sigma_1} \right. \\ &\quad \left. + \lambda_2 \int_{\text{UHP}} d^2z \frac{1}{k^2 |z|^{2k-2}} \left\langle \left(T(z) + \frac{c(k^2-1)}{24z^2} \right) \left(\bar{T}(\bar{z}) + \frac{c(k^2-1)}{24\bar{z}^2} \right) \right\rangle_{\Sigma_1} \right] \\ &= \frac{k}{k-1} \left[\lambda_1 \int_{\text{LHP}} d^2z \frac{1}{k^2 |z|^{2k-2}} \frac{c(k^2-1)}{24z^2} \frac{c(k^2-1)}{24\bar{z}^2} \right. \\ &\quad \left. + \lambda_2 \int_{\text{UHP}} d^2z \frac{1}{k^2 |z|^{2k-2}} \frac{c(k^2-1)}{24z^2} \frac{c(k^2-1)}{24\bar{z}^2} \right] \\ &= \lambda_1 \frac{c^2(k-1)(k+1)^2}{1152k^2} \tilde{\Lambda}_1^{2k} + \lambda_2 \frac{c^2(k-1)(k+1)^2}{1152k^2} \tilde{\Lambda}_2^{2k}, \end{aligned} \quad (5.3)$$

where $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ are two different IR cutoffs. Since we are dealing with two distinct deformation couplings, there is no requirement for the two cutoffs to be identical. However, in the case of vacuum states, if we exchange the indices ($1 \leftrightarrow 2$), the path integral constructed through the replica method should remain invariant. This symmetry is also satisfied by the result in eq. (5.3). In the scenario where the two deformation parameters are equal, we must set $\tilde{\Lambda}_1 = \tilde{\Lambda}_2$. Under this condition, eq. (5.3) reverts to eq. (4.4).

In the subsequent part of this section, we shall examine the pseudo-Rényi entropy derived from primary operators acting on distinct vacuum states with two different infinitesimally small constants in $T\bar{T}$ -deformed CFT. The states used to construct the transition matrix are now expressed as follows:

$$|\psi_1\rangle = \mathcal{O}(w_1)|\Omega_{\lambda_1}\rangle, \quad |\psi_2\rangle = \mathcal{O}(w_2)|\Omega_{\lambda_2}\rangle. \quad (5.4)$$

According to appendix B, the deformed k^{th} pseudo-Rényi entropy is now taking the form

$$\begin{aligned} \Delta S_{A,\lambda_1,\lambda_2}^{(k)} &= \frac{k}{k-1} \lambda_1 \int_{\text{LHP of } \Sigma_1} d^2w \left(\frac{\langle T\bar{T}(w, \bar{w}) \mathcal{O}_1 \dots \mathcal{O}_{2k}^\dagger \rangle_{\Sigma_k}}{\langle \mathcal{O}_1 \dots \mathcal{O}_{2k}^\dagger \rangle_{\Sigma_k}} - \frac{\langle T\bar{T}(w, \bar{w}) \mathcal{O}_1 \mathcal{O}_2^\dagger \rangle_{\Sigma_1}}{\langle \mathcal{O}_1 \mathcal{O}_2^\dagger \rangle_{\Sigma_1}} \right) \\ &+ \frac{k}{k-1} \lambda_2 \int_{\text{UHP of } \Sigma_1} d^2w \left(\frac{\langle T\bar{T}(w, \bar{w}) \mathcal{O}_1 \dots \mathcal{O}_{2k}^\dagger \rangle_{\Sigma_k}}{\langle \mathcal{O}_1 \dots \mathcal{O}_{2k}^\dagger \rangle_{\Sigma_k}} - \frac{\langle T\bar{T}(w, \bar{w}) \mathcal{O}_1 \mathcal{O}_2^\dagger \rangle_{\Sigma_1}}{\langle \mathcal{O}_1 \mathcal{O}_2^\dagger \rangle_{\Sigma_1}} \right). \end{aligned} \quad (5.5)$$

Through the calculation in appendix C, we have the correction to the k^{th} pseudo-Rényi entropy with different deformed parameters of $\mathcal{O}(c)$ under $T\bar{T}$ deformation at the late time

$$\begin{aligned} \Delta S_{A,\lambda_1,\lambda_2}^{(k)} &= \\ &\frac{c(k+1)}{24k^2} \lambda_1 \left\{ \frac{\left((-t+x_1)^{\frac{1}{k}} (t-x_1)^{\frac{1}{k}} - e^{\frac{2\pi i}{k}} (-t-x_2)^{\frac{1}{k}} (t-x_2)^{\frac{1}{k}} \right) (x_1^2 - x_2^2)}{(t^2 - x_1^2) \left((-t-x_1)^{\frac{1}{k}} - e^{\frac{2\pi i}{k}} (-t-x_2)^{\frac{1}{k}} \right) \left((t-x_1)^{\frac{1}{k}} - (t-x_2)^{\frac{1}{k}} \right) (t^2 - x_2^2)} \right. \\ &+ \frac{1}{2} \pi \left[\frac{(x_1 - x_2) (2t - x_1 - x_2) \left((t-x_1)^{1/k} + (t-x_2)^{1/k} \right)}{(t-x_1)^2 (t-x_2)^2 \left((t-x_1)^{1/k} - (t-x_2)^{1/k} \right)} + \frac{2 \left((-t-x_1)^{\frac{1}{k}-2} - e^{\frac{2i\pi}{k}} (-t-x_2)^{\frac{1}{k}-2} \right)}{-(-t-x_1)^{1/k} + e^{\frac{2i\pi}{k}} (-t-x_2)^{1/k}} \right. \\ &\left. \left. + 2t^2 + 2t(x_1 + x_2) + 2 \left(\frac{1}{t+x_2} + \frac{1}{t+x_1} \right) + x_1^2 + x_2^2 \right] \right\} + \mathcal{O}(\epsilon, c^0, \lambda_1^2) \\ &+ \frac{c(k+1)}{24k^2} \lambda_2 \left\{ \frac{\left((-t+x_1)^{\frac{1}{k}} (t-x_1)^{\frac{1}{k}} - e^{\frac{2\pi i}{k}} (-t-x_2)^{\frac{1}{k}} (t-x_2)^{\frac{1}{k}} \right) (x_1^2 - x_2^2)}{(t^2 - x_1^2) \left((-t-x_1)^{\frac{1}{k}} - e^{\frac{2\pi i}{k}} (-t-x_2)^{\frac{1}{k}} \right) \left((t-x_1)^{\frac{1}{k}} - (t-x_2)^{\frac{1}{k}} \right) (t^2 - x_2^2)} \right. \\ &- \frac{1}{2} \pi \left[\frac{(x_1 - x_2) (2t - x_1 - x_2) \left((t-x_1)^{1/k} + (t-x_2)^{1/k} \right)}{(t-x_1)^2 (t-x_2)^2 \left((t-x_1)^{1/k} - (t-x_2)^{1/k} \right)} + \frac{2 \left((-t-x_1)^{\frac{1}{k}-2} - e^{\frac{2i\pi}{k}} (-t-x_2)^{\frac{1}{k}-2} \right)}{-(-t-x_1)^{1/k} + e^{\frac{2i\pi}{k}} (-t-x_2)^{1/k}} \right. \\ &\left. \left. + 2t^2 + 2t(x_1 + x_2) + 2 \left(\frac{1}{t+x_2} + \frac{1}{t+x_1} \right) + x_1^2 + x_2^2 \right] \right\} + \mathcal{O}(\epsilon, c^0, \lambda_2^2). \end{aligned} \quad (5.6)$$

Notice that there are also some possible cutoff terms with ϵ_1 and ϵ_2 corresponding to the LHP and UHP separately and we have regularized them through minimal subtraction seen in (C.4)-(C.11). When the two deformation parameters are identical, we require that both cutoffs, ϵ_1 and ϵ_2 , are set to the same value ϵ . In this case, formula (5.6), including the ignored cutoff terms, will precisely give rise to the result in eq. (4.16). However, when $\lambda_1 \neq \lambda_2$, the correction to the k^{th} pseudo-Rényi entropy may involve additional nontrivial terms compared to (4.16).

When the deformation parameters are large enough, the two deformed theories denoted as CFT^1 and CFT^2 are far from the undeformed theory, so states in CFT^1 and CFT^2 belong

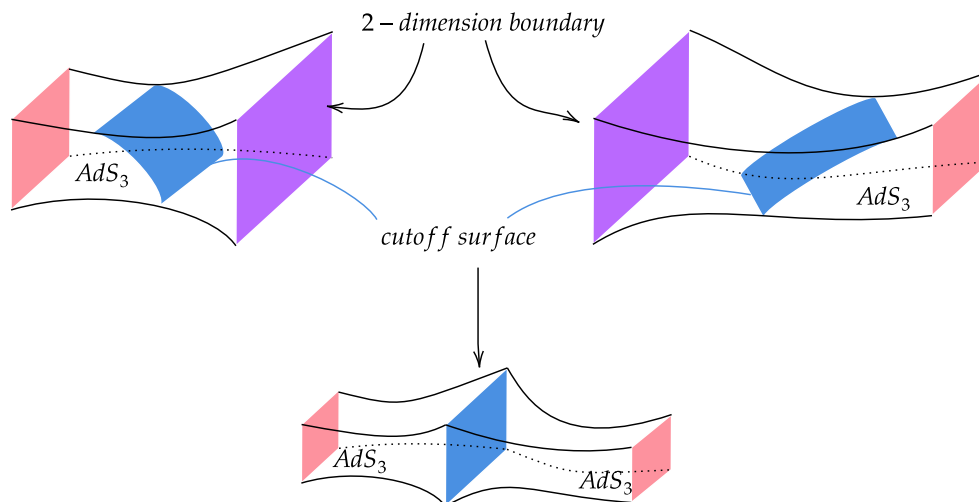


Figure 2. Gluing of two cutoff AdS_3 along their time-like boundaries.

to two different Hilbert space \mathcal{H}_1 and \mathcal{H}_2 . The gluing procedure in figure 1 gives rise to an invalid transition matrix, since ordinarily the transition matrix is defined with states in the same Hilbert space [57].³ In order to establish a well-defined transition matrix, we could consider the tensor product theory $CFT^1 \otimes CFT^2$ which has been extensively investigated in [90]. From the holographic view, such tensor product theory may relate to the gluing of two cutoff spacetimes along their time-like cutoff boundaries (or branes) as shown in figure 2. In the rest of this section, we will concentrate on pseudo entropy constructed from states of the tensor product $CFT^1 \otimes CFT^2$.

Let us first consider a sufficiently simple tensor product theory as a warm-up. The Hamiltonian of the theory is given by

$$H = H_1 \otimes I_2 + I_1 \otimes H_2, \tag{5.7}$$

where H_i ($i=1,2$) is the Hamiltonian of the corresponding deformed theory CFT^i and I_i ($i=1,2$) is the identity operator for CFT^i . The inner product of $|\varphi\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle$ is defined as

$$\langle \varphi | \varphi \rangle = \langle \varphi_1 | \varphi_1 \rangle \otimes \langle \varphi_2 | \varphi_2 \rangle = \langle \varphi_1 | \varphi_1 \rangle \cdot \langle \varphi_2 | \varphi_2 \rangle. \tag{5.8}$$

We then construct two states of the tensor product theory by acting tensor product operators on the deformed vacuum state $|\Omega_{\lambda_1}\rangle \otimes |\Omega_{\lambda_2}\rangle$

$$\begin{aligned} |\psi_1\rangle &= \mathcal{O}(w_1) \otimes \mathcal{O}'(w_2) |\Omega_{\lambda_1}\rangle \otimes |\Omega_{\lambda_2}\rangle = \mathcal{O}(w_1) |\Omega_{\lambda_1}\rangle \otimes \mathcal{O}'(w_2) |\Omega_{\lambda_2}\rangle, \\ |\psi_2\rangle &= \mathcal{O}(w_2) \otimes \mathcal{O}'(w_1) |\Omega_{\lambda_1}\rangle \otimes |\Omega_{\lambda_2}\rangle = \mathcal{O}(w_2) |\Omega_{\lambda_1}\rangle \otimes \mathcal{O}'(w_1) |\Omega_{\lambda_2}\rangle. \end{aligned} \tag{5.9}$$

Before calculating the time evolution of pseudo-Rényi entropy of $|\psi_1\rangle$ and $|\psi_2\rangle$ using the replica method, we would like to show specifically the path integral in the tensor product

³We thank the anonymous referee for pointing this out to us.

theory. As an explicit example, we first consider the correlation function of $\mathcal{O} \otimes \mathcal{O}'$ in the tensor product theory, which is

$$\begin{aligned} \langle \mathcal{O}(w_1) \otimes \mathcal{O}'(w_2) \rangle_{\text{CFT}^1 \otimes \text{CFT}^2} &= \frac{1}{Z} \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \mathcal{O}(\phi_1) \otimes \mathcal{O}'(\phi_2) \exp\{-\mathcal{L}_1 - \mathcal{L}_2\} \\ &= \frac{1}{Z_1} \int \mathcal{D}\phi_1 \mathcal{O}(\phi_1) \exp\{-\mathcal{L}_1\} \cdot \frac{1}{Z_2} \int \mathcal{D}\phi_2 \mathcal{O}'(\phi_2) \exp\{-\mathcal{L}_2\} \\ &= \langle \mathcal{O}(w_1) \rangle_{\text{CFT}^1} \cdot \langle \mathcal{O}'(w_2) \rangle_{\text{CFT}^2}. \end{aligned} \quad (5.10)$$

The transition matrix of $|\psi_1\rangle$ and $|\psi_2\rangle$ is

$$\begin{aligned} \mathcal{T}^{\psi_1|\psi_2} &= \frac{\mathcal{O}(w_1) \otimes \mathcal{O}'(w_2) |\Omega_{\lambda_1}\rangle \otimes |\Omega_{\lambda_2}\rangle \langle \Omega_1| \otimes \langle \Omega_2| \mathcal{O}^\dagger(w_2) \otimes \mathcal{O}'^\dagger(w_1)}{\langle \Omega_1| \mathcal{O}^\dagger(w_2) \mathcal{O}(w_1) |\Omega_1\rangle \cdot \langle \Omega_2| \mathcal{O}'^\dagger(w_1) \mathcal{O}'(w_2) |\Omega_2\rangle} \\ &= \frac{\mathcal{O}(w_1) |\Omega_{\lambda_1}\rangle \langle \Omega_1| \mathcal{O}^\dagger(w_2)}{\langle \Omega_1| \mathcal{O}^\dagger(w_2) \mathcal{O}(w_1) |\Omega_1\rangle} \otimes \frac{\mathcal{O}'(w_2) |\Omega_{\lambda_2}\rangle \langle \Omega_2| \mathcal{O}'^\dagger(w_1)}{\langle \Omega_2| \mathcal{O}'^\dagger(w_1) \mathcal{O}'(w_2) |\Omega_2\rangle}, \end{aligned} \quad (5.11)$$

and after using replica method $(\mathcal{T}_A^{\psi_1|\psi_2})^k$ becomes

$$(\mathcal{T}_A^{\psi_1|\psi_2})^k = \left(\text{tr}_{\bar{A}} \frac{\mathcal{O}(w_1) |\Omega_{\lambda_1}\rangle \langle \Omega_1| \mathcal{O}^\dagger(w_2)}{\langle \Omega_1| \mathcal{O}^\dagger(w_2) \mathcal{O}(w_1) |\Omega_1\rangle} \right)^k \otimes \left(\text{tr}_{\bar{A}} \frac{\mathcal{O}'(w_2) |\Omega_{\lambda_2}\rangle \langle \Omega_2| \mathcal{O}'^\dagger(w_1)}{\langle \Omega_2| \mathcal{O}'^\dagger(w_1) \mathcal{O}'(w_2) |\Omega_2\rangle} \right)^k. \quad (5.12)$$

Therefore, the correction to the k^{th} pseudo-Rényi entropy for tensor product theory can be written as

$$\begin{aligned} \Delta S_{A, \lambda_1, \lambda_2}^{(k)} &= \frac{k}{k-1} (\lambda_1 + \lambda_2) \int_{\Sigma_1} d^2w \left(\frac{\langle T\bar{T}(w, \bar{w}) \mathcal{O}_1 \otimes \mathcal{O}'_1 \dots \mathcal{O}_{2k}^\dagger \otimes \mathcal{O}'_{2k}^\dagger \rangle_{\Sigma_k}}{\langle \mathcal{O}_1 \otimes \mathcal{O}'_1 \dots \mathcal{O}_{2k}^\dagger \otimes \mathcal{O}'_{2k}^\dagger \rangle_{\Sigma_k}} \right. \\ &\quad \left. - \frac{\langle T\bar{T}(w, \bar{w}) \mathcal{O}_1 \otimes \mathcal{O}'_1 \mathcal{O}_2^\dagger \otimes \mathcal{O}'_2^\dagger \rangle_{\Sigma_1}}{\langle \mathcal{O}_1 \otimes \mathcal{O}'_1 \mathcal{O}_2^\dagger \otimes \mathcal{O}'_2^\dagger \rangle_{\Sigma_1}} \right). \end{aligned} \quad (5.13)$$

In the above discussion, our focus is on the condition that the two deformed theories remain independent. However, as demonstrated in [88], if we consider the gluing of two AdS spacetimes, we should add an interaction term H_{int} to the Hamiltonian of tensor product theory (5.7) which probably can be understood in terms of an irrelevant deformation that couples the two deformed CFTs together, where the author in [90] call it sequential flow, and this term arises due to the exchange of gravitons between two AdS spacetimes. Under such flow, the Lagrangian of the tensor product theory with interactions up to the first order of deformation parameters can be expressed as [90]

$$L = L_1 + L_2 + \lambda_3 [(T_1 + T_2)(\bar{T}_1 + \bar{T}_2)] + \lambda_1 T_1 \bar{T}_1 + \lambda_2 T_2 \bar{T}_2. \quad (5.14)$$

Due to the interaction term, the transition matrix of interest cannot be decomposed as (5.11) in this case

$$\mathcal{T}^{\psi_1|\psi_2} = \frac{\mathcal{O}(w_1) \otimes \mathcal{O}'(w_2) |\Omega\rangle_{\text{SF}} \langle \Omega| \mathcal{O}^\dagger(w_2) \otimes \mathcal{O}'^\dagger(w_1)}{\langle \Omega| \mathcal{O}^\dagger(w_2) \otimes \mathcal{O}'^\dagger(w_1) \mathcal{O}(w_1) \otimes \mathcal{O}'(w_2) |\Omega\rangle_{\text{SF}}}, \quad (5.15)$$

where $|\Omega\rangle_{\text{SF}}$ denotes the vacuum of the sequential-flowed theory (5.14). We still can expand the correlation function under sequential flow perturbatively, and the result is

$$\begin{aligned}
 & \langle \mathcal{O} \otimes \mathcal{O}' \rangle_{\text{SF}} \\
 &= \frac{1}{\sqrt{Z_{\text{SF}}}} \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \mathcal{O} \otimes \mathcal{O}' \exp\{-\mathcal{L}_1 - \mathcal{L}_2 - \lambda_3[(T_1 + T_2)(\bar{T}_1 + \bar{T}_2)] - \lambda_1 T_1 \bar{T}_1 - \lambda_2 T_2 \bar{T}_2\} \\
 &= \frac{1}{\sqrt{Z_0}} \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \mathcal{O} \otimes \mathcal{O}' \exp\{-\mathcal{L}_1 - \mathcal{L}_2\} (1 - \lambda_3[(T_1 + T_2)(\bar{T}_1 + \bar{T}_2)] - \lambda_1 T_1 \bar{T}_1 - \lambda_2 T_2 \bar{T}_2) \\
 &= \langle \mathcal{O} \otimes \mathcal{O}' \rangle - \lambda_3 \int d^2z \langle (T_1(z) + T_2(z))(\bar{T}_1(\bar{z}) + \bar{T}_2(\bar{z})) \mathcal{O} \otimes \mathcal{O}' \rangle \\
 &\quad - \lambda_1 \int d^2z \langle T_1(z) \bar{T}_1(\bar{z}) \mathcal{O} \otimes \mathcal{O}' \rangle - \lambda_2 \int d^2z \langle T_2(z) \bar{T}_2(\bar{z}) \mathcal{O} \otimes \mathcal{O}' \rangle
 \end{aligned} \tag{5.16}$$

where Z_{SF} is the partition function under sequential flow which can be expressed as

$$\begin{aligned}
 Z_{\text{SF}} &= \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \exp\{-\mathcal{L}_1 - \mathcal{L}_2 - \lambda_3[(T_1 + T_2)(\bar{T}_1 + \bar{T}_2)] - \lambda_1 T_1 \bar{T}_1 - \lambda_2 T_2 \bar{T}_2\} \\
 &= Z_0 - (\lambda_3 + \lambda_1) \int d^2z \langle T_1(z) \bar{T}_1(\bar{z}) \rangle - (\lambda_3 + \lambda_2) \int d^2z \langle T_2(z) \bar{T}_2(\bar{z}) \rangle \\
 &\quad - \lambda_3 \int d^2z (\langle T_1(z) \rangle \langle \bar{T}_2(\bar{z}) \rangle + \langle \bar{T}_1(\bar{z}) \rangle \langle T_2(z) \rangle).
 \end{aligned} \tag{5.17}$$

Using the method in section 4 we can derive the correction to the k^{th} pseudo-Rényi entropy for tensor product theory under sequential flow

$$\begin{aligned}
 \Delta S_{A, \lambda_1, \lambda_2, \lambda_3}^{(k)} &= \\
 & \frac{k}{k-1} (\lambda_1 + \lambda_3) \int_{\Sigma_1} d^2w \left(\frac{\langle T_1 \bar{T}_1(w, \bar{w}) \mathcal{O}_1 \otimes \mathcal{O}'_1 \dots \mathcal{O}_{2k}^\dagger \otimes \mathcal{O}'_{2k} \rangle_{\Sigma_k}}{\langle \mathcal{O}_1 \otimes \mathcal{O}'_1 \dots \mathcal{O}_{2k}^\dagger \otimes \mathcal{O}'_{2k} \rangle_{\Sigma_k}} - \frac{\langle T_1 \bar{T}_1(w, \bar{w}) \mathcal{O}_1 \otimes \mathcal{O}'_1 \mathcal{O}_2^\dagger \otimes \mathcal{O}'_2 \rangle_{\Sigma_1}}{\langle \mathcal{O}_1 \otimes \mathcal{O}'_1 \mathcal{O}_2^\dagger \otimes \mathcal{O}'_2 \rangle_{\Sigma_1}} \right) \\
 & + \frac{k}{k-1} (\lambda_2 + \lambda_3) \int_{\Sigma_1} d^2w \left(\frac{\langle T_2 \bar{T}_2(w, \bar{w}) \mathcal{O}_1 \otimes \mathcal{O}'_1 \dots \mathcal{O}_{2k}^\dagger \otimes \mathcal{O}'_{2k} \rangle_{\Sigma_k}}{\langle \mathcal{O}_1 \otimes \mathcal{O}'_1 \dots \mathcal{O}_{2k}^\dagger \otimes \mathcal{O}'_{2k} \rangle_{\Sigma_k}} - \frac{\langle T_2 \bar{T}_2(w, \bar{w}) \mathcal{O}_1 \otimes \mathcal{O}'_1 \mathcal{O}_2^\dagger \otimes \mathcal{O}'_2 \rangle_{\Sigma_1}}{\langle \mathcal{O}_1 \otimes \mathcal{O}'_1 \mathcal{O}_2^\dagger \otimes \mathcal{O}'_2 \rangle_{\Sigma_1}} \right) \\
 & + \frac{k}{k-1} \lambda_3 \int_{\Sigma_1} d^2w \left(\frac{\langle T_1 \bar{T}_2(w, \bar{w}) \mathcal{O}_1 \otimes \mathcal{O}'_1 \dots \mathcal{O}_{2k}^\dagger \otimes \mathcal{O}'_{2k} \rangle_{\Sigma_k}}{\langle \mathcal{O}_1 \otimes \mathcal{O}'_1 \dots \mathcal{O}_{2k}^\dagger \otimes \mathcal{O}'_{2k} \rangle_{\Sigma_k}} - \frac{\langle T_1 \bar{T}_2(w, \bar{w}) \mathcal{O}_1 \otimes \mathcal{O}'_1 \mathcal{O}_2^\dagger \otimes \mathcal{O}'_2 \rangle_{\Sigma_1}}{\langle \mathcal{O}_1 \otimes \mathcal{O}'_1 \mathcal{O}_2^\dagger \otimes \mathcal{O}'_2 \rangle_{\Sigma_1}} \right) \\
 & + \frac{k}{k-1} \lambda_3 \int_{\Sigma_1} d^2w \left(\frac{\langle \bar{T}_1 T_2(w, \bar{w}) \mathcal{O}_1 \otimes \mathcal{O}'_1 \dots \mathcal{O}_{2k}^\dagger \otimes \mathcal{O}'_{2k} \rangle_{\Sigma_k}}{\langle \mathcal{O}_1 \otimes \mathcal{O}'_1 \dots \mathcal{O}_{2k}^\dagger \otimes \mathcal{O}'_{2k} \rangle_{\Sigma_k}} - \frac{\langle \bar{T}_1 T_2(w, \bar{w}) \mathcal{O}_1 \otimes \mathcal{O}'_1 \mathcal{O}_2^\dagger \otimes \mathcal{O}'_2 \rangle_{\Sigma_1}}{\langle \mathcal{O}_1 \otimes \mathcal{O}'_1 \mathcal{O}_2^\dagger \otimes \mathcal{O}'_2 \rangle_{\Sigma_1}} \right),
 \end{aligned} \tag{5.18}$$

where the correlation function containing the term such as $T_1 \bar{T}_2$ can be expressed as

$$\langle T_1 \bar{T}_2(w, \bar{w}) \mathcal{O}_1 \otimes \mathcal{O}'_1 \dots \mathcal{O}_{2k}^\dagger \otimes \mathcal{O}'_{2k} \rangle = \langle T_1(w) \mathcal{O}_1 \dots \mathcal{O}_{2k}^\dagger \rangle \cdot \langle \bar{T}_2(\bar{w}) \mathcal{O}'_1 \dots \mathcal{O}'_{2k} \rangle, \tag{5.19}$$

and the term such as $T_1 \bar{T}_1$ can be expressed as

$$\langle T_1 \bar{T}_1(w, \bar{w}) \mathcal{O}_1 \otimes \mathcal{O}'_1 \dots \mathcal{O}_{2k}^\dagger \otimes \mathcal{O}'_{2k} \rangle = \langle T_1 \bar{T}_1(w, \bar{w}) \mathcal{O}_1 \dots \mathcal{O}_{2k}^\dagger \rangle \cdot \langle \bar{\mathcal{O}}'_1 \dots \mathcal{O}'_{2k} \rangle. \tag{5.20}$$

All terms in (5.18) can be computed similarly using the results in (4.4), (4.5) and (4.8). However, comparing (5.18) and (5.13), we find the result in (5.18) has additional corrections shown in the last two lines of (5.18), which can not be derived through the renaming of λ_1 and λ_2 in (5.13), and these corrections are totally due to the interaction between the two CFTs of the tensor product theory.

The above computations are an initial attempt in RCFTs under the sequential flow [90]. In order to gain insight into the connection between the gluing procedure and the sequential flow, we expect to find the holographic pseudo-Rényi entropy of transition matrix (5.15) in the concept of “holography without boundaries” initiated in [88]. The holographic calculations in glued AdS and the field-theoretic calculations in holographic CFTs under sequential flow are necessary and will be served as our future work.

6 Pseudo-Rényi entropy in $J\bar{T}$ -deformed CFTs

So far, we have discussed the $T\bar{T}$ -deformation of the pseudo-Rényi entropy. In this section, we study the $J\bar{T}$ -deformation of the pseudo entropy, with $J\bar{T}$ operator constructed from a chiral U(1) current J and stress tensor \bar{T} .

The $J\bar{T}$ -deformed action is the trajectory on the space of field theory satisfying

$$\frac{dS}{d\lambda} = \int d^2z (J\bar{T})_\lambda. \tag{6.1}$$

Similar to the $T\bar{T}$ -deformation, we regard the deformation as a perturbative theory, in which case the action can be written as

$$S(\lambda) = S(\lambda = 0) + \lambda \int d^2z \sqrt{g} J\bar{T} + \mathcal{O}(\lambda^2), \tag{6.2}$$

where $(J\bar{T})_{\lambda=0}$ is denoted by $J\bar{T}$ later on. The first-order correction to the correlation function by using the Ward identity is

$$\begin{aligned} & \langle \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \dots \mathcal{O}_k(z_k, \bar{z}_k) \rangle_\lambda \\ &= \lambda \int d^2z \langle J\bar{T}(z, \bar{z}) \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \dots \mathcal{O}_k(z_k, \bar{z}_k) \rangle \end{aligned} \tag{6.3}$$

$$\begin{aligned} &= \lambda \int d^2z \left(\sum_{i=1}^k \frac{q_i}{z - z_i} \right) \left(\sum_{i=1}^k \left(\frac{\bar{h}_i}{(\bar{z} - \bar{z}_i)^2} + \frac{\partial_{\bar{z}_i}}{\bar{z} - \bar{z}_i} \right) \right) \\ & \times \langle \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \dots \mathcal{O}_k(z_k, \bar{z}_k) \rangle. \end{aligned} \tag{6.4}$$

We then calculate the correction to the k^{th} pseudo-Rényi entropy in the $J\bar{T}$ -deformed CFT.

Under the conformal map (2.12), the U(1) current and the stress tensor transform as

$$J\bar{T}(w, \bar{w}) = \frac{dz}{dw} J(z) \frac{1}{4\bar{z}^2} \left(\bar{T}(\bar{z}) + \frac{c}{8\bar{z}^2} \right). \tag{6.5}$$

The correction to the undeformed k^{th} pseudo-Rényi entropy can be derived similarly from that in the previous section. Thus we have

$$\Delta S_{A,\lambda}^{(k)} = \frac{k\lambda}{k-1} \int_{\Sigma_1} d^2w \left(\frac{\langle J\bar{T}(w, \bar{w}) \mathcal{O}_1 \dots \mathcal{O}_{2k}^\dagger \rangle_{\Sigma_k}}{\langle \mathcal{O}_1 \dots \mathcal{O}_{2k}^\dagger \rangle_{\Sigma_k}} - \frac{\langle J\bar{T}(w, \bar{w}) \mathcal{O}_1 \mathcal{O}_2^\dagger \rangle_{\Sigma_1}}{\langle \mathcal{O}_1 \mathcal{O}_2^\dagger \rangle_{\Sigma_1}} \right). \tag{6.6}$$

We still focus on the large c case. At $\mathcal{O}(c)$, $\Delta S_{A,\lambda}^{(k)}$ at the late time can be written as

$$\begin{aligned} \Delta S_{A,\lambda}^{(k)} &= \frac{k\lambda}{k-1} \int d^2z \frac{c}{8\bar{z}^2} \frac{1}{k\bar{z}^{k-1}} \frac{\langle J(z)\mathcal{O}_1(z_1, \bar{z}_1)\mathcal{O}_2^\dagger(z_2, \bar{z}_2)\dots\mathcal{O}_{2k-1}(z_{2k-1}, \bar{z}_{2k-1})\mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k})\rangle_{\Sigma_1}}{\langle \mathcal{O}_1(z_1, \bar{z}_1)\mathcal{O}_2^\dagger(z_2, \bar{z}_2)\dots\mathcal{O}_{2k-1}(z_{2k-1}, \bar{z}_{2k-1})\mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k})\rangle_{\Sigma_1}} \\ &= \frac{k\lambda}{k-1} \int d^2z \frac{c}{8\bar{z}^2} \frac{1}{k\bar{z}^{k-1}} \left(\sum_{i=1}^{2k} \frac{q_i}{z-z_i} \right) \\ &= \frac{2\lambda\pi c}{8(k-1)} \sum_{i=1}^{2k} \int_{|z_i|}^{\infty} \frac{q_i z_i^k}{\rho^{2k+1}} = \frac{2\lambda\pi c}{16k(1-k)} \sum_{i=1}^{2k} \frac{q_i z_i^k}{|z_i|^{2k}}. \end{aligned} \tag{6.7}$$

Substituting (2.14) into (6.7), we get the first-order corrections to the k^{th} pseudo-Rényi entropy in $J\bar{T}$ -deformed CFT,

$$\Delta S_{A,\lambda}^{(k)} = \frac{2\lambda\pi c}{8(k-1)} \sum_{i=0}^{k-1} \left(\frac{q_{2i+1}}{-t+x_1} + \frac{q_{2i+2}}{-t+x_2} \right) + \mathcal{O}(c^0, \lambda^2). \tag{6.8}$$

Specifically, we consider the $k=2$ case,

$$\Delta S_{A,\lambda}^{(2)} = 2\lambda \frac{c\pi}{32} \left(\frac{q_1+q_3}{-t+x_1} + \frac{q_2+q_4}{-t+x_2} \right) + \mathcal{O}(c^0, \lambda^2). \tag{6.9}$$

If we take the limit $x_1 \rightarrow x_2$, (6.9) returns to the result we found in [86]. Using the method mentioned in subsection 4.1, one can also calculate the late-time and early-time corrections to the entanglement entropy in $J\bar{T}$ -deformed CFTs.

7 Conclusion and prospect

This paper employs perturbation theory to investigate the pseudo-Rényi entropy associated with locally excited states in the two-dimensional $T\bar{T}/J\bar{T}$ deformed CFTs. Our primary objective is to derive nontrivial corrections to the pseudo-Rényi entropy of the undeformed theory. Initially, we provide a comprehensive review of the calculation of two-point and four-point correlation functions using the Ward identity in deformed CFTs and the pseudo-Rényi entropy obtained through the replica method in undeformed CFTs. We then derive the leading order corrections to the second pseudo-Rényi entropy and find that it may depend on the introduced infrared (IR) cutoff and may also depend on the ultraviolet (UV) cutoff, depending on whether the subsystem A is finite. Furthermore, at order $\mathcal{O}(c)$, we discover that the second pseudo-Rényi entropy may acquire nontrivial corrections in the deformed CFTs at the late time, as demonstrated in eqs. (3.10) and (6.9). However, the early-time correction becomes more intricate, and we illustrate its behavior using the free scalar and Ising models as specific examples.

In addition, we extend our investigation to the k^{th} Rényi entropy, examining its behavior both at the early and late times as expressed in eqs. (4.12) and (4.14), respectively. Subsequently, we calculate the k -th pseudo-Rényi entropy of locally excited states at the late time, using the fact that the $2k$ -point correlation function can be deformed as the product of k two-point functions. We observe nontrivial corrections to the k^{th} pseudo-Rényi

entropy at the late time, as presented in eq. (4.16). Notably, this result aligns precisely with the findings from section 3.

Finally, drawing inspiration from gluing two AdS spacetimes with distinct cutoffs and the correspondence between cutoff AdS (cAdS) and $T\bar{T}$ -deformed CFT (dCFT), we separately investigate the pseudo-Rényi entropy for vacuum states characterized by distinct $T\bar{T}$ -deformation parameters, and for primary states acting on different deformed vacuum states. At the leading order, we find that the corrections to the pseudo-Rényi entropy may depend on two different cutoffs, as indicated in eq. (5.3). For the pseudo-Rényi entropy associated with two primary operators acting on different deformed vacuum states, we obtain additional nontrivial terms in eq. (5.6) compared to eq. (4.16).

Our primary focus centers on the computation of corrections induced by $T\bar{T}/J\bar{T}$ -deformation to the pseudo-Rényi entropy in RCFTs. Specifically, by employing the $T\bar{T}/J\bar{T}$ deformation, it is possible to directly compute corrections to the pseudo-Rényi entropy for different local operators in various theoretical frameworks, including Liouville CFT, holographic CFT, and other relevant approaches. Moreover, exploring the definition of an operator within the framework of $T\bar{T}$ -deformation, constructing pseudo-Rényi entropy for locally excited states in the AdS context, connecting distinct cutoff AdS geometries, and calculating the pseudo-Rényi entropy for locally excited states with different deformation parameters on the cAdS side could provide valuable insights. However, these captivating avenues remain unexplored and await future research endeavors.

Acknowledgments

We thank Yang Liu, Hongfei Shu, Yuan Sun, Hongan Zeng, and Long Zhao for the valuable discussions. S.H. would appreciate the financial support from the Fundamental Research Funds for the Central Universities, the Max Planck Partner group, and the Natural Science Foundation of China Grants (No. 12075101, No. 12235016). J.Y. is partially supported by the Natural Science Foundation of China Grants (No. 11971322).

A Details of integrals in $T\bar{T}$ -deformed pseudo-Rényi entropy

In this appendix, we evaluate the integral of (3.8) in detail. It mainly takes the following forms

$$\int d^2z \frac{1}{|z|^2} \frac{1}{z^2} \frac{1}{(\bar{z} - \bar{z}_i)} \quad \text{and} \quad \int d^2z \frac{1}{|z|^2} \frac{1}{z^2} \frac{1}{(\bar{z} - \bar{z}_i)^2} \tag{A.1}$$

as well as their complex conjugate. Now we evaluate them accordingly. Firstly

$$\int d^2z \frac{1}{|z|^2} \frac{1}{z^2} \frac{1}{(\bar{z} - \bar{z}_i)^2} = \int_0^\infty d\rho \int_0^{2\pi} d\theta \frac{1}{\rho^3 e^{2i\theta}} \frac{1}{(\rho e^{-i\theta} - \bar{z}_i)^2}. \tag{A.2}$$

Integrating out θ , we find

$$\int d^2z \frac{1}{|z|^2} \frac{1}{z^2} \frac{1}{(\bar{z} - \bar{z}_i)^2} = \int d\rho \frac{e^{i\rho} \frac{i\rho}{e^{i\theta} \bar{z}_i - \rho} + i \log(\bar{z}_i - \rho e^{-i\theta})}{\rho^5} \Big|_{\theta=0}^{\theta=2\pi}. \tag{A.3}$$

Let us consider the process where θ runs from 0 to 2π . When $|\frac{\rho}{\bar{z}_i}| > 1$, $\rho e^{-i\theta}$ circles counterclockwise around \bar{z}_i . Hence $\log(\bar{z}_i - \rho e^{-i\theta})$ contributes a factor $-2\pi i$. Therefore we have

$$\log(\bar{z}_i - \rho e^{-i\theta}) = \begin{cases} 0 & |\frac{\rho}{\bar{z}_i}| < 1 \\ -2\pi i & |\frac{\rho}{\bar{z}_i}| > 1 \end{cases}. \quad (\text{A.4})$$

The integral becomes

$$\int d^2 z \frac{1}{|z|^2} \frac{1}{z^2} \frac{1}{(\bar{z} - \bar{z}_i)^2} = \int_{|\bar{z}_i|}^{\infty} \frac{2\pi}{\rho^5} d\rho = \frac{2\pi}{4|\bar{z}_i|^4}. \quad (\text{A.5})$$

Secondly the other integral in (A.1) can be evaluated as

$$\begin{aligned} & \int d^2 z \frac{1}{|z|^2} \frac{1}{z^2} \frac{1}{(\bar{z} - \bar{z}_i)} = \int_0^{\infty} d\rho \int_0^{2\pi} d\theta \frac{1}{\rho^3 e^{2i\theta}} \frac{1}{(\rho e^{-i\theta} - \bar{z}_i)} \\ & = \int d\rho \frac{i e^{-i\theta} \rho + i \bar{z}_i \log(\bar{z}_i - \rho e^{-i\theta})}{\rho^5} \Big|_{\theta=0}^{\theta=2\pi} \int_{|\bar{z}_i|}^{\infty} \frac{2\pi \bar{z}_i}{\rho^5} d\rho = \frac{2\pi \bar{z}_i}{4|\bar{z}_i|^4}. \end{aligned} \quad (\text{A.6})$$

Other integrals in (3.8) can be evaluated in a similar way. We find that those terms coupled with \bar{T} become

$$\begin{aligned} & \frac{c}{8} 2\lambda \int d^2 z \frac{1}{4|z|^2} \frac{\langle (\frac{1}{z^2} \bar{T}(\bar{z})) \mathcal{O}(w_1, \bar{w}_1) \mathcal{O}^\dagger(w_2, \bar{w}_2) \mathcal{O}(w_3, \bar{w}_3) \mathcal{O}^\dagger(w_4, \bar{w}_4) \rangle_{\Sigma_1}}{\langle \mathcal{O}(w_1, \bar{w}_1) \mathcal{O}^\dagger(w_2, \bar{w}_2) \mathcal{O}(w_3, \bar{w}_3) \mathcal{O}^\dagger(w_4, \bar{w}_4) \rangle_{\Sigma_1}} \\ & = 2\lambda \frac{c}{32} \int d^2 z \frac{1}{|z|^2} \frac{1}{z^2} \left(\sum_{i=1}^4 \frac{h}{(\bar{z} - \bar{z}_i)^2} - \frac{2h}{(\bar{z} - \bar{z}_1)(\bar{z}_1 - \bar{z}_3)} + \frac{2h}{(\bar{z} - \bar{z}_3)(\bar{z}_1 - \bar{z}_3)} \right. \\ & \quad \left. - \frac{2h}{(\bar{z} - \bar{z}_2)(\bar{z}_2 - \bar{z}_4)} + \frac{2h}{(\bar{z} - \bar{z}_4)(\bar{z}_2 - \bar{z}_4)} \right. \\ & \quad \left. + \left[\frac{1}{\bar{z} - \bar{z}_1} \left(-\frac{1}{\bar{z}_{13}} + \frac{1}{\bar{z}_{12}} \right) + \frac{1}{\bar{z} - \bar{z}_2} \left(-\frac{1}{\bar{z}_{24}} - \frac{1}{\bar{z}_{12}} \right) + \frac{1}{\bar{z} - \bar{z}_3} \left(\frac{1}{\bar{z}_{34}} + \frac{1}{\bar{z}_{13}} \right) + \frac{1}{\bar{z} - \bar{z}_4} \left(-\frac{1}{\bar{z}_{34}} + \frac{1}{\bar{z}_{24}} \right) \right] \frac{\bar{\eta} \partial_{\bar{\eta}} G(\eta, \bar{\eta})}{G(\eta, \bar{\eta})} \right) \\ & = 2\lambda \frac{c\pi}{64} \left(\frac{h}{|\bar{z}_1|^4} + \frac{h}{|\bar{z}_2|^4} + \frac{h}{|\bar{z}_3|^4} + \frac{h}{|\bar{z}_4|^4} - \frac{2h\bar{z}_1}{\bar{z}_{13}|\bar{z}_1|^4} + \frac{2h\bar{z}_3}{\bar{z}_{13}|\bar{z}_3|^4} - \frac{2h\bar{z}_2}{\bar{z}_{24}|\bar{z}_2|^4} + \frac{2h\bar{z}_4}{\bar{z}_{24}|\bar{z}_4|^4} \right. \\ & \quad \left. + \left[\frac{\bar{z}_1}{|\bar{z}_1|^4} \left(-\frac{1}{\bar{z}_{13}} + \frac{1}{\bar{z}_{12}} \right) + \frac{\bar{z}_2}{|\bar{z}_2|^4} \left(-\frac{1}{\bar{z}_{24}} - \frac{1}{\bar{z}_{12}} \right) + \frac{\bar{z}_3}{|\bar{z}_3|^4} \left(\frac{1}{\bar{z}_{34}} + \frac{1}{\bar{z}_{13}} \right) + \frac{\bar{z}_4}{|\bar{z}_4|^4} \left(-\frac{1}{\bar{z}_{34}} + \frac{1}{\bar{z}_{24}} \right) \right] \frac{\bar{\eta} \partial_{\bar{\eta}} G(\eta, \bar{\eta})}{G(\eta, \bar{\eta})} \right). \end{aligned} \quad (\text{A.7})$$

Those terms coupled with T are

$$\begin{aligned} & \frac{c}{8} 2\lambda \int d^2 z \frac{1}{4|z|^2} \frac{\langle (\frac{1}{z^2} T(z)) \mathcal{O}(w_1, \bar{w}_1) \mathcal{O}^\dagger(w_2, \bar{w}_2) \mathcal{O}(w_3, \bar{w}_3) \mathcal{O}^\dagger(w_4, \bar{w}_4) \rangle_{\Sigma_1}}{\langle \mathcal{O}(w_1, \bar{w}_1) \mathcal{O}^\dagger(w_2, \bar{w}_2) \mathcal{O}(w_3, \bar{w}_3) \mathcal{O}^\dagger(w_4, \bar{w}_4) \rangle_{\Sigma_1}} \\ & = 2\lambda \frac{c}{32} \int d^2 z \frac{1}{|z|^2} \frac{1}{z^2} \left(\sum_{i=1}^4 \frac{h}{(z - z_i)^2} - \frac{2h}{(z - z_1)(z_1 - z_3)} + \frac{2h}{(z - z_3)(z_1 - z_3)} \right. \\ & \quad \left. - \frac{2h}{(z - z_2)(z_2 - z_4)} + \frac{2h}{(z - z_4)(z_2 - z_4)} \right. \\ & \quad \left. + \left[\frac{1}{z - z_1} \left(-\frac{1}{z_{13}} + \frac{1}{z_{12}} \right) + \frac{1}{z - z_2} \left(-\frac{1}{z_{24}} - \frac{1}{z_{12}} \right) + \frac{1}{z - z_3} \left(\frac{1}{z_{34}} + \frac{1}{z_{13}} \right) + \frac{1}{z - z_4} \left(-\frac{1}{z_{34}} + \frac{1}{z_{24}} \right) \right] \frac{\eta \partial_\eta G(\eta, \bar{\eta})}{G(\eta, \bar{\eta})} \right) \\ & = 2\lambda \frac{c\pi}{64} \left(\frac{h}{|z_1|^4} + \frac{h}{|z_2|^4} + \frac{h}{|z_3|^4} + \frac{h}{|z_4|^4} - \frac{2hz_1}{z_{13}|z_1|^4} + \frac{2hz_3}{z_{13}|z_3|^4} - \frac{2hz_2}{z_{24}|z_2|^4} + \frac{2hz_4}{z_{24}|z_4|^4} \right. \\ & \quad \left. + \left[\frac{z_1}{|z_1|^4} \left(-\frac{1}{z_{13}} + \frac{1}{z_{12}} \right) + \frac{z_2}{|z_2|^4} \left(-\frac{1}{z_{24}} - \frac{1}{z_{12}} \right) + \frac{z_3}{|z_3|^4} \left(\frac{1}{z_{34}} + \frac{1}{z_{13}} \right) + \frac{z_4}{|z_4|^4} \left(-\frac{1}{z_{34}} + \frac{1}{z_{24}} \right) \right] \frac{\eta \partial_\eta G(\eta, \bar{\eta})}{G(\eta, \bar{\eta})} \right). \end{aligned} \quad (\text{A.8})$$

We then list the results of those integrals in (4.5) and (4.8) using the same method mentioned above.

$$\begin{aligned}
 \int d^2z \frac{1}{|z|^{2k-2}} \frac{1}{\bar{z}^2 (z - z_i)^2} &= \frac{\pi}{k |z_i|^{2k}}. \\
 \int d^2z \frac{1}{|z|^{2k-2}} \frac{1}{\bar{z}^2 (z - z_i)} &= \frac{\pi z_i}{k |z_i|^{2k}}. \\
 \int d^2z \frac{1}{|\bar{z}|^{2k-2}} \frac{1}{z^2 (\bar{z} - \bar{z}_i)^2} &= \frac{\pi}{k |\bar{z}_i|^{2k}}. \\
 \int d^2z \frac{1}{|\bar{z}|^{2k-2}} \frac{1}{z^2 (\bar{z} - \bar{z}_i)} &= \frac{\pi \bar{z}_i}{k |\bar{z}_i|^{2k}}.
 \end{aligned} \tag{A.9}$$

B The $T\bar{T}$ -deformation of the k^{th} pseudo-Rényi entropy by the replica method

This appendix provides a derivation of the leading-order corrections of the k^{th} pseudo-Rényi entropy in 2D CFTs under the $T\bar{T}$ -deformation, using the replica trick.

We first generate the states of interest by applying a local operator to the vacuum of the $T\bar{T}$ -deformed theory with varying parameters, that is

$$|\psi_i\rangle = \mathcal{O}(x_i, \tau_i) |\Omega_{\lambda_i}\rangle, \quad (i = 1, 2) \tag{B.1}$$

with $\mathcal{O}(x_i, \tau_i) \equiv \mathcal{O}(w_i, \bar{w}_i) = e^{H_{\lambda_i} \tau_i} \mathcal{O}(x_i, \tau = 0) e^{-H_{\lambda_i} \tau_i}$. $|\Omega_{\lambda_i}\rangle$ and H_{λ_i} denote the vacuum and the Hamiltonian of the deformed theory with the deformation parameter λ_i , respectively. By the standard replica trick, we can express the essential part of the k^{th} pseudo-Rényi entropy, namely the trace of the k^{th} power of the reduced transition matrix, as the following Euclidean path integral.

$$\text{tr}[(\mathcal{T}_A^{\psi_1|\psi_2})^k] = \left(\begin{array}{c} \mathcal{L}_{\lambda_2} \\ \mathcal{O}^\dagger(w_2) \\ \hline \mathcal{O}(w_1) \\ \mathcal{L}_{\lambda_1} \end{array} \right)_{\tau=+\infty}^{-k} \times \left(\begin{array}{c} \mathcal{L}_{\lambda_2} \\ \mathcal{O}(w_1) \\ \mathcal{L}_{\lambda_1} \\ \mathcal{L}_{\lambda_2} \\ \mathcal{O}^\dagger(w_2) \\ \mathcal{L}_{\lambda_1} \\ \mathcal{O}(w_3) \\ \mathcal{L}_{\lambda_2} \\ \mathcal{O}^\dagger(w_4) \\ \mathcal{L}_{\lambda_1} \\ \vdots \\ \mathcal{L}_{\lambda_2} \\ \mathcal{O}^\dagger(w_{2k}) \\ \mathcal{L}_{\lambda_1} \\ \mathcal{O}(w_{2k-1}) \end{array} \right)_{\tau=-\infty} \tag{B.2}$$

The first factor of the right-hand side of eq. (B.2) corresponds to a two-point function on the complex plane Σ_1 , while the second factor corresponds to a $2k$ -point function on the k -sheeted Riemann surface Σ_k . Notably, the $T\bar{T}$ -deformed theory with parameter λ_1 lives on the lower half-plane (LHP) of each sheeted surface, whereas the deformed theory with parameter λ_2 resides on the upper half-plane (UHP) of each sheeted surface.

Rewriting the above result as the correlation functions and expanding it according to the order of $\lambda_{1(2)}$, we obtain⁴

$$\begin{aligned}
 & \text{tr}[(\mathcal{T}_A^{\psi_1|\psi_2})^k] = \\
 & \frac{Z_k}{Z_1^k} \times \frac{\langle \mathcal{O}(w_1)\mathcal{O}^\dagger(w_2)\dots\mathcal{O}(w_{2k-1})\mathcal{O}^\dagger(w_{2k}) \rangle_{\Sigma_k}}{\langle \mathcal{O}(w_1)\mathcal{O}^\dagger(w_2) \rangle_{\Sigma_1}^k} \\
 & \times \left(1 - \frac{k\lambda_1 \int_{\text{LHP}} d^2w \langle T\bar{T}(w)\mathcal{O}(w_1)\dots\mathcal{O}^\dagger(w_{2k}) \rangle_{\Sigma_k} + k\lambda_2 \int_{\text{UHP}} d^2w \langle T\bar{T}(w)\mathcal{O}(w_1)\dots\mathcal{O}^\dagger(w_{2k}) \rangle_{\Sigma_k}}{\langle \mathcal{O}(w_1)\mathcal{O}^\dagger(w_2)\dots\mathcal{O}(w_{2k-1})\mathcal{O}^\dagger(w_{2k}) \rangle_{\Sigma_k}} \right. \\
 & \left. + \frac{k\lambda_1 \int_{\text{LHP}} d^2w \langle T\bar{T}(w)\mathcal{O}(w_1)\mathcal{O}^\dagger(w_2) \rangle_{\Sigma_1} + k\lambda_2 \int_{\text{UHP}} d^2w \langle T\bar{T}(w)\mathcal{O}(w_1)\mathcal{O}^\dagger(w_2) \rangle_{\Sigma_1}}{\langle \mathcal{O}(w_1)\mathcal{O}^\dagger(w_2) \rangle_{\Sigma_1}} \right) \quad (\text{B.3})
 \end{aligned}$$

+sub-leading order terms.

The two factors in the first line expression come from the vacuum contribution of the k^{th} Rényi entropy of subsystem A (denoted as $S_{A,\text{vacuum}}^{(k)}$) and the operator \mathcal{O} 's contribution of the k^{th} pseudo-Rényi entropy (denoted as $\Delta S_{A,0}^{(k)}$), respectively. Substituting (B.3) into (1.7), we obtain the k^{th} pseudo-Rényi entropy of subsystem A under the $T\bar{T}$ -deformation

$$\begin{aligned}
 & \Delta S_{A,\lambda_1,\lambda_2}^{(k)} \\
 & = S_{A,\lambda_1,\lambda_2}^{(k)} - S_{A,\text{vacuum}}^{(k)} - \Delta S_{A,0}^{(k)} \\
 & = \frac{1}{1-k} \log \left(1 - \frac{k\lambda_1 \int_{\text{LHP}} d^2w \langle T\bar{T}(w)\mathcal{O}(w_1)\dots\mathcal{O}^\dagger(w_{2k}) \rangle_{\Sigma_k} + k\lambda_2 \int_{\text{UHP}} d^2w \langle T\bar{T}(w)\mathcal{O}(w_1)\dots\mathcal{O}^\dagger(w_{2k}) \rangle_{\Sigma_k}}{\langle \mathcal{O}(w_1)\mathcal{O}^\dagger(w_2)\dots\mathcal{O}(w_{2k-1})\mathcal{O}^\dagger(w_{2k}) \rangle_{\Sigma_k}} \right. \\
 & \quad \left. + \frac{k\lambda_1 \int_{\text{LHP}} d^2w \langle T\bar{T}(w)\mathcal{O}(w_1)\mathcal{O}^\dagger(w_2) \rangle_{\Sigma_1} + k\lambda_2 \int_{\text{UHP}} d^2w \langle T\bar{T}(w)\mathcal{O}(w_1)\mathcal{O}^\dagger(w_2) \rangle_{\Sigma_1}}{\langle \mathcal{O}(w_1)\mathcal{O}^\dagger(w_2) \rangle_{\Sigma_1}} \right) \\
 & \quad + \text{sub-leading order terms} \\
 & = \frac{k}{k-1} \left(\frac{\lambda_1 \int_{\text{LHP}} d^2w \langle T\bar{T}(w)\mathcal{O}(w_1)\dots\mathcal{O}^\dagger(w_{2k}) \rangle_{\Sigma_k} + \lambda_2 \int_{\text{UHP}} d^2w \langle T\bar{T}(w)\mathcal{O}(w_1)\dots\mathcal{O}^\dagger(w_{2k}) \rangle_{\Sigma_k}}{\langle \mathcal{O}(w_1)\mathcal{O}^\dagger(w_2)\dots\mathcal{O}(w_{2k-1})\mathcal{O}^\dagger(w_{2k}) \rangle_{\Sigma_k}} \right. \\
 & \quad \left. - \frac{\lambda_1 \int_{\text{LHP}} d^2w \langle T\bar{T}(w)\mathcal{O}(w_1)\mathcal{O}^\dagger(w_2) \rangle_{\Sigma_1} + \lambda_2 \int_{\text{UHP}} d^2w \langle T\bar{T}(w)\mathcal{O}(w_1)\mathcal{O}^\dagger(w_2) \rangle_{\Sigma_1}}{\langle \mathcal{O}(w_1)\mathcal{O}^\dagger(w_2) \rangle_{\Sigma_1}} \right) \quad (\text{B.4}) \\
 & \quad + \text{sub-leading order terms.}
 \end{aligned}$$

Note that eq. (B.4) reduces to (4.2) when $\lambda_1 = \lambda_2 = \lambda$, and it reduces to (5.2) when $\mathcal{O} = \mathbb{I}$, i.e., the identity operator.

⁴We denote (w, \bar{w}) by (w) in \mathcal{O} and $T\bar{T}$ for simplicity.

C Details of integrals in pseudo-Rényi entropy with different deformed states

In this appendix, we evaluate the integral of (5.5) in detail. The deformed pseudo-Rényi entropy according to appendix B can be evaluated as

$$\begin{aligned}
 & \Delta S_{A,\lambda_1,\lambda_2}^{(k)} \\
 &= \frac{k\lambda_1}{k-1} \int_{\text{LHP of } \Sigma_1} d^2z \frac{1}{k^2 |z|^{2k-2}} \\
 & \quad \left(\frac{\left\langle \left(T(z) + \frac{c(k^2-1)}{24} \frac{1}{z^2} \right) (\bar{T}(\bar{z}) + \frac{c(k^2-1)}{24} \frac{1}{\bar{z}^2}) \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k}) \right\rangle_{\Sigma_1}}{\langle \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k}) \rangle_{\Sigma_1}} - \frac{\langle T\bar{T}(w, \bar{w}) \mathcal{O}_1 \mathcal{O}_2^\dagger \rangle_{\Sigma_1}}{\langle \mathcal{O}_1 \mathcal{O}_2^\dagger \rangle_{\Sigma_1}} \right) \\
 &+ \frac{k\lambda_2}{k-1} \int_{\text{UHP of } \Sigma_1} d^2z \frac{1}{k^2 |z|^{2k-2}} \\
 & \quad \left(\frac{\left\langle \left(T(z) + \frac{c(k^2-1)}{24} \frac{1}{z^2} \right) (\bar{T}(\bar{z}) + \frac{c(k^2-1)}{24} \frac{1}{\bar{z}^2}) \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k}) \right\rangle_{\Sigma_1}}{\langle \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k}) \rangle_{\Sigma_1}} - \frac{\langle T\bar{T}(w, \bar{w}) \mathcal{O}_1 \mathcal{O}_2^\dagger \rangle_{\Sigma_1}}{\langle \mathcal{O}_1 \mathcal{O}_2^\dagger \rangle_{\Sigma_1}} \right). \tag{C.1}
 \end{aligned}$$

Similar to the discussion above, we still focus on the large c case. The leading order of the integral (C.1) is simple, so we do not evaluate it in detail. We will calculate the integral (C.1) of $\mathcal{O}(c)$ in the rest of this appendix. At $\mathcal{O}(c)$, using the conformal transformation (2.12) and Ward Identity, the integral (C.1) can be rewritten as

$$\begin{aligned}
 & \Delta S_{A,\lambda_1,\lambda_2}^{(k)} \\
 &= \frac{k\lambda_1}{k-1} \int_{\text{LHP of } \Sigma_1} d^2z \frac{1}{k^2 |z|^{2k-2}} \\
 & \quad \left(\frac{c(k^2-1)}{24z^2} \frac{\langle T(z) \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k}) \rangle_{\Sigma_1}}{\langle \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k}) \rangle_{\Sigma_1}} + \frac{c(k^2-1)}{24z^2} \frac{\langle \bar{T}(\bar{z}) \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k}) \rangle_{\Sigma_1}}{\langle \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k}) \rangle_{\Sigma_1}} \right) \\
 &+ \frac{k\lambda_2}{k-1} \int_{\text{UHP of } \Sigma_1} d^2z \frac{1}{k^2 |z|^{2k-2}} \\
 & \quad \left(\frac{c(k^2-1)}{24\bar{z}^2} \frac{\langle T(z) \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k}) \rangle_{\Sigma_1}}{\langle \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k}) \rangle_{\Sigma_1}} + \frac{c(k^2-1)}{24\bar{z}^2} \frac{\langle \bar{T}(\bar{z}) \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k}) \rangle_{\Sigma_1}}{\langle \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k}) \rangle_{\Sigma_1}} \right) \\
 &= \frac{k\lambda_1}{k-1} \int_{\text{LHP of } \Sigma_1} d^2z \frac{1}{k^2 |z|^{2k-2}} \frac{c(k^2-1)}{24 \langle \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k}) \rangle_{\Sigma_1}} \\
 & \quad \left(\frac{1}{z^2} \left(\sum_{i=1}^{2k} \frac{h}{(z-z_i)^2} + \sum_{j=0}^{k-2} \left(\frac{\partial_{2j+1}}{z-z_{2j+1}} + \frac{\partial_{2j+4}}{z-z_{2j+4}} \right) + \frac{\partial_2}{z-z_2} + \frac{\partial_{2k-1}}{z-z_{2k-1}} \right) \right. \\
 & \quad \times \langle \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k}) \rangle_{\Sigma_1} \\
 & \quad \left. + \frac{1}{z^2} \left(\sum_{i=1}^{2k} \frac{\bar{h}}{(\bar{z}-\bar{z}_i)^2} + \sum_{j=0}^{k-1} \left(\frac{\bar{\partial}_{2j+1}}{\bar{z}-\bar{z}_{2j+1}} + \frac{\bar{\partial}_{2j+2}}{\bar{z}-\bar{z}_{2j+2}} \right) \right) \langle \bar{T}(\bar{z}) \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k}) \rangle_{\Sigma_1} \right)
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{k\lambda_2}{k-1} \int_{\text{UHP of } \Sigma_1} d^2z \frac{1}{k^2|z|^{2k-2}} \frac{c(k^2-1)}{24 \langle \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k}) \rangle_{\Sigma_1}} \\
 & \left(\frac{1}{\bar{z}^2} \left(\sum_{i=1}^{2k} \frac{h}{(z-z_i)^2} + \sum_{j=0}^{k-2} \left(\frac{\partial_{2j+1}}{z-z_{2j+1}} + \frac{\partial_{2j+4}}{z-z_{2j+4}} \right) + \frac{\partial_2}{z-z_2} + \frac{\partial_{2k-1}}{z-z_{2k-1}} \right) \right. \\
 & \times \left. \langle \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k}) \rangle_{\Sigma_1} \right. \\
 & \left. + \frac{1}{z^2} \left(\sum_{i=1}^{2k} \frac{\bar{h}}{(\bar{z}-\bar{z}_i)^2} + \sum_{j=0}^{k-1} \left(\frac{\bar{\partial}_{2j+1}}{\bar{z}-\bar{z}_{2j+1}} + \frac{\bar{\partial}_{2j+2}}{\bar{z}-\bar{z}_{2j+2}} \right) \right) \langle \bar{T}(\bar{z}) \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_{2k}^\dagger(z_{2k}, \bar{z}_{2k}) \rangle_{\Sigma_1} \right). \tag{C.2}
 \end{aligned}$$

All the integrals can be fully evaluated after using the polar coordinate and taking proper regularization, and we will show the following integral as an example:

$$\begin{aligned}
 & \int_{\text{LHP of } \Sigma_1} d^2z \frac{z^2}{|z|^{2k+2} (z-z_i)^2} = \int_0^\infty \int_{-\pi}^0 d\rho d\theta \frac{e^{2i\theta}}{\rho^{2k-1} (\rho e^{i\theta} - z_i)^2} \\
 & = \begin{cases} \int_\epsilon^{|z_i|} \rho^{-1-2k} \left(-\pi + \frac{2i\rho z_i}{\rho^2 - z_i^2} + 2i \text{Arctanh} \left(\frac{z_i}{\rho} \right) \right) & z_i \in \text{LHP and } \rho < |z_i| \\ \int_{|z_i|}^\infty \rho^{-1-2k} \left(\pi + \frac{2i\rho z_i}{\rho^2 - z_i^2} + 2i \text{Arctanh} \left(\frac{z_i}{\rho} \right) \right) & z_i \in \text{LHP and } \rho > |z_i| \\ \int_\epsilon^\infty \rho^{-1-2k} \left(\pi + \frac{2i\rho z_i}{\rho^2 - z_i^2} + 2i \text{Arctanh} \left(\frac{z_i}{\rho} \right) \right) & z_i \in \text{UHP} \end{cases} \tag{C.3}
 \end{aligned}$$

where we have introduced ϵ as the cutoff. Integrate out ρ in (C.3) and regularize it through minimal subtraction, then we have

$$\int_{\text{LHP of } \Sigma_1} d^2z \frac{z^2}{|z|^{2k+2} (z-z_i)^2} = \begin{cases} \frac{\pi}{k|z_i|^{2k}} + \frac{\pi}{2kz_i^{2k}} + \frac{\pi}{z_i} & z_i \in \text{LHP} \\ \frac{\pi}{2kz_i^{2k}} + \frac{\pi}{z_i} & z_i \in \text{UHP} \end{cases}. \tag{C.4}$$

Other integrals in (C.2) can be evaluated similarly, so we only list their results below.

$$\int_{\text{LHP of } \Sigma_1} d^2z \frac{z^2}{|z|^{2k+2} (z-z_i)} = \begin{cases} \frac{z_i \pi}{k|z_i|^{2k}} - \frac{\pi}{2kz_i^{2k-1}} & z_i \in \text{LHP} \\ -\frac{\pi}{2kz_i^{2k-1}} & z_i \in \text{UHP} \end{cases}. \tag{C.5}$$

$$\int_{\text{UHP of } \Sigma_1} d^2z \frac{z^2}{|z|^{2k+2} (z-z_i)^2} = \begin{cases} \frac{\pi}{k|z_i|^{2k}} - \frac{\pi}{2kz_i^{2k}} - \frac{\pi}{z_i} & z_i \in \text{UHP} \\ -\frac{\pi}{2kz_i^{2k}} - \frac{\pi}{z_i} & z_i \in \text{LHP} \end{cases} \tag{C.6}$$

$$\int_{\text{UHP of } \Sigma_1} d^2z \frac{z^2}{|z|^{2k+2} (z-z_i)} = \begin{cases} \frac{z_i \pi}{k|z_i|^{2k}} + \frac{\pi}{2kz_i^{2k-1}} & z_i \in \text{UHP} \\ \frac{\pi}{2kz_i^{2k-1}} & z_i \in \text{LHP} \end{cases}. \tag{C.7}$$

$$\int_{\text{UHP of } \Sigma_1} d^2z \frac{\bar{z}^2}{|z|^{2k+2} (\bar{z}-\bar{z}_i)^2} = \begin{cases} \frac{\pi}{k|\bar{z}_i|^{2k}} + \frac{\pi}{2k\bar{z}_i^{2k}} + \frac{\pi}{\bar{z}_i} & \bar{z}_i \in \text{LHP} \\ \frac{\pi}{2k\bar{z}_i^{2k}} + \frac{\pi}{\bar{z}_i} & \bar{z}_i \in \text{UHP} \end{cases} \tag{C.8}$$

$$\int_{\text{LHP of } \Sigma_1} d^2 z \frac{\bar{z}^2}{|z|^{2k+2} (\bar{z} - \bar{z}_i)^2} = \begin{cases} \frac{\pi}{k|\bar{z}_i|^{2k}} - \frac{\pi}{2k\bar{z}_i^{2k}} - \frac{\pi}{\bar{z}_i} & \bar{z}_i \in \text{UHP} \\ -\frac{\pi}{2k\bar{z}_i^{2k}} - \frac{\pi}{\bar{z}_i} & \bar{z}_i \in \text{LHP} \end{cases} \quad (\text{C.9})$$

$$\int_{\text{UHP of } \Sigma_1} d^2 z \frac{z^2}{|z|^{2k+2} (z - z_i)^2} = \begin{cases} \frac{\bar{z}_i \pi}{k|\bar{z}_i|^{2k}} + \frac{\pi}{2k\bar{z}_i^{2k-1}} & \bar{z}_i \in \text{LHP} \\ \frac{\pi}{2k\bar{z}_i^{2k-1}} & \bar{z}_i \in \text{UHP} \end{cases}. \quad (\text{C.10})$$

$$\int_{\text{LHP of } \Sigma_1} d^2 z \frac{z^2}{|z|^{2k+2} (z - z_i)^2} = \begin{cases} \frac{\bar{z}_i \pi}{k|\bar{z}_i|^{2k}} - \frac{\pi}{2k\bar{z}_i^{2k-1}} & \bar{z}_i \in \text{UHP} \\ -\frac{\pi}{2k\bar{z}_i^{2k-1}} & \bar{z}_i \in \text{LHP} \end{cases}. \quad (\text{C.11})$$

Using results (C.4)–(C.11) and (C.2), we get the correction to pseudo-Rényi entropy of primary operators with different $T\bar{T}$ -deformation parameters

$$\begin{aligned} & \Delta S_{A, \lambda_1, \lambda_2}^{(k)} \\ &= \frac{c(k+1)}{24k} \lambda_1 \left[\sum_{z_i \in \text{LHP}} \frac{\pi h}{k|z_i|^{2k}} - \sum_{z_{2i+1} \in \text{LHP}} \frac{2\pi h}{z_{2i+1} - z_{2i+4}} \frac{z_{2i+1}}{|z_{2i+1}|^{2k}} + \sum_{z_{2i+4} \in \text{LHP}} \frac{2\pi h}{z_{2i+1} - z_{2i+4}} \frac{z_{2i+4}}{|z_{2i+4}|^{2k}} \right. \\ & \quad \sum_{\bar{z}_i \in \text{UHP}} \frac{\pi h}{k|\bar{z}_i|^{2k}} - \sum_{\bar{z}_{2i+1} \in \text{UHP}} \frac{2\pi \bar{h}}{\bar{z}_{2i+1} - \bar{z}_{2i+2}} \frac{\bar{z}_{2i+1}}{|\bar{z}_{2i+1}|^{2k}} + \sum_{\bar{z}_{2i+2} \in \text{UHP}} \frac{2\pi \bar{h}}{\bar{z}_{2i+1} - \bar{z}_{2i+2}} \frac{\bar{z}_{2i+2}}{|\bar{z}_{2i+2}|^{2k}} \\ & \quad + \sum_{j=1}^{2k} \left(\frac{\pi}{2kz_i^{2k}} + \frac{\pi}{z_i} \right) + \sum_{j=0}^{k-2} \frac{\pi}{k} \left(\frac{-h}{z_{2j+1} - z_{2j+4}} \frac{z_{2j+1}}{z_{2j+1}^{2k}} + \frac{h}{z_{2j+1} - z_{2j+4}} \frac{z_{2j+4}}{z_{2j+4}^{2k}} \right) \\ & \quad + \frac{\pi}{k} \left(\frac{-h}{z_2 - z_{2k-1}} \frac{z_2}{z_2^{2k}} + \frac{h}{z_2 - z_{2k-1}} \frac{z_{2k-1}}{z_{2k-1}^{2k}} \right) - \sum_{j=1}^{2k} \left(\frac{\pi}{2k\bar{z}_i^{2k}} + \frac{\pi}{\bar{z}_i} \right) \\ & \quad \left. - \sum_{j=0}^{k-1} \frac{\pi}{k} \left(\frac{-\bar{h}}{\bar{z}_{2j+1} - \bar{z}_{2j+2}} \frac{\bar{z}_{2j+1}}{\bar{z}_{2j+1}^{2k}} + \frac{\bar{h}}{\bar{z}_{2j+1} - \bar{z}_{2j+2}} \frac{\bar{z}_{2j+2}}{\bar{z}_{2j+2}^{2k}} \right) \right] \\ & + \frac{c(k+1)}{24k} \lambda_2 \left[\sum_{z_i \in \text{UHP}} \frac{\pi h}{k|z_i|^{2k}} - \sum_{z_{2i+1} \in \text{UHP}} \frac{2\pi h}{z_{2i+1} - z_{2i+4}} \frac{z_{2i+1}}{|z_{2i+1}|^{2k}} + \sum_{z_{2i+4} \in \text{UHP}} \frac{2\pi h}{z_{2i+1} - z_{2i+4}} \frac{z_{2i+4}}{|z_{2i+4}|^{2k}} \right. \\ & \quad \sum_{\bar{z}_i \in \text{LHP}} \frac{\pi h}{k|\bar{z}_i|^{2k}} - \sum_{\bar{z}_{2i+1} \in \text{LHP}} \frac{2\pi \bar{h}}{\bar{z}_{2i+1} - \bar{z}_{2i+2}} \frac{\bar{z}_{2i+1}}{|\bar{z}_{2i+1}|^{2k}} + \sum_{\bar{z}_{2i+2} \in \text{LHP}} \frac{2\pi \bar{h}}{\bar{z}_{2i+1} - \bar{z}_{2i+2}} \frac{\bar{z}_{2i+2}}{|\bar{z}_{2i+2}|^{2k}} \\ & \quad - \sum_{j=1}^{2k} \left(\frac{\pi}{2kz_i^{2k}} + \frac{\pi}{z_i} \right) - \sum_{j=0}^{k-2} \frac{\pi}{k} \left(\frac{-h}{z_{2j+1} - z_{2j+4}} \frac{z_{2j+1}}{z_{2j+1}^{2k}} + \frac{h}{z_{2j+1} - z_{2j+4}} \frac{z_{2j+4}}{z_{2j+4}^{2k}} \right) \\ & \quad - \frac{\pi}{k} \left(\frac{-h}{z_2 - z_{2k-1}} \frac{z_2}{z_2^{2k}} + \frac{h}{z_2 - z_{2k-1}} \frac{z_{2k-1}}{z_{2k-1}^{2k}} \right) + \sum_{j=1}^{2k} \left(\frac{\pi}{2k\bar{z}_i^{2k}} + \frac{\pi}{\bar{z}_i} \right) \\ & \quad \left. + \sum_{j=0}^{k-1} \frac{\pi}{k} \left(\frac{-\bar{h}}{\bar{z}_{2j+1} - \bar{z}_{2j+2}} \frac{\bar{z}_{2j+1}}{\bar{z}_{2j+1}^{2k}} + \frac{\bar{h}}{\bar{z}_{2j+1} - \bar{z}_{2j+2}} \frac{\bar{z}_{2j+2}}{\bar{z}_{2j+2}^{2k}} \right) \right] \quad (\text{C.12}) \end{aligned}$$

Replacing all the coordinates in (C.12) with (2.14), we get the correction to the k^{th} pseudo-Rényi entropy with different deformed parameters under $T\bar{T}$ deformation at the late time

$$\begin{aligned}
 \Delta S_{A,\lambda_1,\lambda_2}^{(k)} = & \frac{c(k+1)}{24k^2} \lambda_1 \left\{ \frac{\left((-t+x_1)^{\frac{1}{k}} (t-x_1)^{\frac{1}{k}} - e^{\frac{2\pi i}{k}} (-t-x_2)^{\frac{1}{k}} (t-x_2)^{\frac{1}{k}} \right) (x_1^2 - x_2^2)}{\left(t^2 - x_1^2 \right) \left((-t-x_1)^{\frac{1}{k}} - e^{\frac{2\pi i}{k}} (-t-x_2)^{\frac{1}{k}} \right) \left((t-x_1)^{\frac{1}{k}} - (t-x_2)^{\frac{1}{k}} \right) (t^2 - x_2^2)} \right. \\
 & + \frac{1}{2} \pi \left[\frac{(x_1-x_2)(2t-x_1-x_2) \left((t-x_1)^{1/k} + (t-x_2)^{1/k} \right)}{(t-x_1)^2 (t-x_2)^2 \left((t-x_1)^{1/k} - (t-x_2)^{1/k} \right)} + \frac{2 \left((-t-x_1)^{\frac{1}{k}-2} - e^{\frac{2i\pi}{k}} (-t-x_2)^{\frac{1}{k}-2} \right)}{-(-t-x_1)^{1/k} + e^{\frac{2i\pi}{k}} (-t-x_2)^{1/k}} \right] \\
 & \left. + 2t^2 + 2t(x_1+x_2) + 2 \left(\frac{1}{t+x_2} + \frac{1}{t+x_1} \right) + x_1^2 + x_2^2 \right\} + \mathcal{O}(\epsilon, c^0, \lambda_1^2) \\
 & + \frac{c(k+1)}{24k^2} \lambda_2 \left\{ \frac{\left((-t+x_1)^{\frac{1}{k}} (t-x_1)^{\frac{1}{k}} - e^{\frac{2\pi i}{k}} (-t-x_2)^{\frac{1}{k}} (t-x_2)^{\frac{1}{k}} \right) (x_1^2 - x_2^2)}{\left(t^2 - x_1^2 \right) \left((-t-x_1)^{\frac{1}{k}} - e^{\frac{2\pi i}{k}} (-t-x_2)^{\frac{1}{k}} \right) \left((t-x_1)^{\frac{1}{k}} - (t-x_2)^{\frac{1}{k}} \right) (t^2 - x_2^2)} \right. \\
 & - \frac{1}{2} \pi \left[\frac{(x_1-x_2)(2t-x_1-x_2) \left((t-x_1)^{1/k} + (t-x_2)^{1/k} \right)}{(t-x_1)^2 (t-x_2)^2 \left((t-x_1)^{1/k} - (t-x_2)^{1/k} \right)} + \frac{2 \left((-t-x_1)^{\frac{1}{k}-2} - e^{\frac{2i\pi}{k}} (-t-x_2)^{\frac{1}{k}-2} \right)}{-(-t-x_1)^{1/k} + e^{\frac{2i\pi}{k}} (-t-x_2)^{1/k}} \right] \\
 & \left. + 2t^2 + 2t(x_1+x_2) + 2 \left(\frac{1}{t+x_2} + \frac{1}{t+x_1} \right) + x_1^2 + x_2^2 \right\} + \mathcal{O}(\epsilon, c^0, \lambda_2^2). \tag{C.13}
 \end{aligned}$$

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] J.M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [[hep-th/9711200](#)] [[INSPIRE](#)].
- [2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett. B* **428** (1998) 105 [[hep-th/9802109](#)] [[INSPIRE](#)].
- [3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)] [[INSPIRE](#)].
- [4] H. Casini and M. Huerta, *A Finite entanglement entropy and the c-theorem*, *Phys. Lett. B* **600** (2004) 142 [[hep-th/0405111](#)] [[INSPIRE](#)].
- [5] P. Calabrese and J.L. Cardy, *Entanglement entropy and quantum field theory*, *J. Stat. Mech.* **0406** (2004) P06002 [[hep-th/0405152](#)] [[INSPIRE](#)].
- [6] A. Kitaev and J. Preskill, *Topological entanglement entropy*, *Phys. Rev. Lett.* **96** (2006) 110404 [[hep-th/0510092](#)] [[INSPIRE](#)].
- [7] H. Casini, I. Salazar Landea and G. Torroba, *The g-theorem and quantum information theory*, *JHEP* **10** (2016) 140 [[arXiv:1607.00390](#)] [[INSPIRE](#)].
- [8] T. Nishioka, *Entanglement entropy: holography and renormalization group*, *Rev. Mod. Phys.* **90** (2018) 035007 [[arXiv:1801.10352](#)] [[INSPIRE](#)].

- [9] E. Witten, *APS Medal for Exceptional Achievement in Research: Invited article on entanglement properties of quantum field theory*, *Rev. Mod. Phys.* **90** (2018) 045003 [[arXiv:1803.04993](#)] [[INSPIRE](#)].
- [10] H. Casini and M. Huerta, *Lectures on entanglement in quantum field theory*, *PoS TASI2021* (2023) 002 [[arXiv:2201.13310](#)] [[INSPIRE](#)].
- [11] M. Van Raamsdonk, *Building up spacetime with quantum entanglement*, *Gen. Rel. Grav.* **42** (2010) 2323 [[arXiv:1005.3035](#)] [[INSPIRE](#)].
- [12] J. Maldacena and L. Susskind, *Cool horizons for entangled black holes*, *Fortsch. Phys.* **61** (2013) 781 [[arXiv:1306.0533](#)] [[INSPIRE](#)].
- [13] M. Rangamani and T. Takayanagi, *Holographic Entanglement Entropy*, Springer (2017) [[DOI:10.1007/978-3-319-52573-0](#)] [[INSPIRE](#)].
- [14] S.W. Hawking, *Breakdown of Predictability in Gravitational Collapse*, *Phys. Rev. D* **14** (1976) 2460 [[INSPIRE](#)].
- [15] S.D. Mathur, *The Information paradox: A Pedagogical introduction*, *Class. Quant. Grav.* **26** (2009) 224001 [[arXiv:0909.1038](#)] [[INSPIRE](#)].
- [16] A. Almheiri, D. Marolf, J. Polchinski and J. Sully, *Black Holes: Complementarity or Firewalls?*, *JHEP* **02** (2013) 062 [[arXiv:1207.3123](#)] [[INSPIRE](#)].
- [17] G. Penington, *Entanglement Wedge Reconstruction and the Information Paradox*, *JHEP* **09** (2020) 002 [[arXiv:1905.08255](#)] [[INSPIRE](#)].
- [18] A. Almheiri, N. Engelhardt, D. Marolf and H. Maxfield, *The entropy of bulk quantum fields and the entanglement wedge of an evaporating black hole*, *JHEP* **12** (2019) 063 [[arXiv:1905.08762](#)] [[INSPIRE](#)].
- [19] F.A. Smirnov and A.B. Zamolodchikov, *On space of integrable quantum field theories*, *Nucl. Phys. B* **915** (2017) 363 [[arXiv:1608.05499](#)] [[INSPIRE](#)].
- [20] A. Cavaglià, S. Negro, I.M. Szécsényi and R. Tateo, *$T\bar{T}$ -deformed 2D Quantum Field Theories*, *JHEP* **10** (2016) 112 [[arXiv:1608.05534](#)] [[INSPIRE](#)].
- [21] M. Guica, *An integrable Lorentz-breaking deformation of two-dimensional CFTs*, *SciPost Phys.* **5** (2018) 048 [[arXiv:1710.08415](#)] [[INSPIRE](#)].
- [22] L. McGough, M. Mezei and H. Verlinde, *Moving the CFT into the bulk with $T\bar{T}$* , *JHEP* **04** (2018) 010 [[arXiv:1611.03470](#)] [[INSPIRE](#)].
- [23] M. Guica and R. Monten, *$T\bar{T}$ and the mirage of a bulk cutoff*, *SciPost Phys.* **10** (2021) 024 [[arXiv:1906.11251](#)] [[INSPIRE](#)].
- [24] P. Kraus, J. Liu and D. Marolf, *Cutoff AdS_3 versus the $T\bar{T}$ deformation*, *JHEP* **07** (2018) 027 [[arXiv:1801.02714](#)] [[INSPIRE](#)].
- [25] S. Chakraborty, A. Giveon and D. Kutasov, *$J\bar{T}$ deformed CFT_2 and string theory*, *JHEP* **10** (2018) 057 [[arXiv:1806.09667](#)] [[INSPIRE](#)].
- [26] A. Bzowski and M. Guica, *The holographic interpretation of $J\bar{T}$ -deformed CFTs*, *JHEP* **01** (2019) 198 [[arXiv:1803.09753](#)] [[INSPIRE](#)].
- [27] S. Dubovsky, V. Gorbenko and M. Mirbabayi, *Asymptotic fragility, near AdS_2 holography and $T\bar{T}$* , *JHEP* **09** (2017) 136 [[arXiv:1706.06604](#)] [[INSPIRE](#)].

- [28] J. Cardy, *The $T\bar{T}$ deformation of quantum field theory as random geometry*, *JHEP* **10** (2018) 186 [[arXiv:1801.06895](#)] [[INSPIRE](#)].
- [29] O. Aharony and T. Vaknin, *The TT^* deformation at large central charge*, *JHEP* **05** (2018) 166 [[arXiv:1803.00100](#)] [[INSPIRE](#)].
- [30] G. Bonelli, N. Doroud and M. Zhu, *$T\bar{T}$ -deformations in closed form*, *JHEP* **06** (2018) 149 [[arXiv:1804.10967](#)] [[INSPIRE](#)].
- [31] S. Datta and Y. Jiang, *$T\bar{T}$ deformed partition functions*, *JHEP* **08** (2018) 106 [[arXiv:1806.07426](#)] [[INSPIRE](#)].
- [32] O. Aharony et al., *Modular invariance and uniqueness of $T\bar{T}$ deformed CFT*, *JHEP* **01** (2019) 086 [[arXiv:1808.02492](#)] [[INSPIRE](#)].
- [33] R. Conti, S. Negro and R. Tateo, *The $T\bar{T}$ perturbation and its geometric interpretation*, *JHEP* **02** (2019) 085 [[arXiv:1809.09593](#)] [[INSPIRE](#)].
- [34] C.-K. Chang, C. Ferko and S. Sethi, *Supersymmetry and $T\bar{T}$ deformations*, *JHEP* **04** (2019) 131 [[arXiv:1811.01895](#)] [[INSPIRE](#)].
- [35] R. Conti, L. Iannella, S. Negro and R. Tateo, *Generalised Born-Infeld models, Lax operators and the $T\bar{T}$ perturbation*, *JHEP* **11** (2018) 007 [[arXiv:1806.11515](#)] [[INSPIRE](#)].
- [36] X. Jiang, P. Wang, H. Wu and H. Yang, *Timelike entanglement entropy and $T\bar{T}$ deformation*, *Phys. Rev. D* **108** (2023) 046004 [[arXiv:2302.13872](#)] [[INSPIRE](#)].
- [37] H.-S. Jeong, W.-B. Pan, Y.-W. Sun and Y.-T. Wang, *Holographic study of $T\bar{T}$ like deformed HV QFTs: holographic entanglement entropy*, *JHEP* **02** (2023) 018 [[arXiv:2211.00518](#)] [[INSPIRE](#)].
- [38] D.J. Gross, J. Kruthoff, A. Rolph and E. Shaghoulian, *Hamiltonian deformations in quantum mechanics, $T\bar{T}$, and the SYK model*, *Phys. Rev. D* **102** (2020) 046019 [[arXiv:1912.06132](#)] [[INSPIRE](#)].
- [39] H. Geng, *$T\bar{T}$ Deformation and the Complexity = Volume Conjecture*, *Fortsch. Phys.* **68** (2020) 2000036 [[arXiv:1910.08082](#)] [[INSPIRE](#)].
- [40] S. He, J.-R. Sun and Y. Sun, *The correlation function of (1,1) and (2,2) supersymmetric theories with $T\bar{T}$ deformation*, *JHEP* **04** (2020) 100 [[arXiv:1912.11461](#)] [[INSPIRE](#)].
- [41] M. Asrat and J. Kudler-Flam, *$T\bar{T}$, the entanglement wedge cross section, and the breakdown of the split property*, *Phys. Rev. D* **102** (2020) 045009 [[arXiv:2005.08972](#)] [[INSPIRE](#)].
- [42] H. Ouyang and H. Shu, *$T\bar{T}$ deformation of chiral bosons and Chern-Simons AdS_3 gravity*, *Eur. Phys. J. C* **80** (2020) 1155 [[arXiv:2006.10514](#)] [[INSPIRE](#)].
- [43] Y. Li and Y. Zhou, *Cutoff AdS_3 versus $T\bar{T}$ CFT_2 in the large central charge sector: correlators of energy-momentum tensor*, *JHEP* **12** (2020) 168 [[arXiv:2005.01693](#)] [[INSPIRE](#)].
- [44] S. Ebert, H.-Y. Sun and Z. Sun, *$T\bar{T}$ deformation in SCFTs and integrable supersymmetric theories*, *JHEP* **09** (2021) 082 [[arXiv:2011.07618](#)] [[INSPIRE](#)].
- [45] M. Medenjak, G. Policastro and T. Yoshimura, *$T\bar{T}$ -Deformed Conformal Field Theories out of Equilibrium*, *Phys. Rev. Lett.* **126** (2021) 121601 [[arXiv:2011.05827](#)] [[INSPIRE](#)].
- [46] S. Ebert, C. Ferko, H.-Y. Sun and Z. Sun, *$T\bar{T}$ deformations of supersymmetric quantum mechanics*, *JHEP* **08** (2022) 121 [[arXiv:2204.05897](#)] [[INSPIRE](#)].

- [47] Y. Li, *Comments on large central charge $T\bar{T}$ deformed conformal field theory and cutoff AdS holography*, [arXiv:2012.14414](#) [[INSPIRE](#)].
- [48] S. He, *Note on higher-point correlation functions of the $T\bar{T}$ or $J\bar{T}$ deformed CFTs*, *Sci. China Phys. Mech. Astron.* **64** (2021) 291011 [[arXiv:2012.06202](#)] [[INSPIRE](#)].
- [49] H. Jiang and G. Tartaglino-Mazzucchelli, *Supersymmetric $J\bar{T}$ and $T\bar{J}$ deformations*, *JHEP* **05** (2020) 140 [[arXiv:1911.05631](#)] [[INSPIRE](#)].
- [50] S. Chakraborty and A. Hashimoto, *Entanglement entropy for $T\bar{T}$, $J\bar{T}$, $T\bar{J}$ deformed holographic CFT*, *JHEP* **02** (2021) 096 [[arXiv:2010.15759](#)] [[INSPIRE](#)].
- [51] M. He and Y.-H. Gao, *$T\bar{T}/J\bar{T}$ -deformed WZW models from Chern-Simons AdS_3 gravity with mixed boundary conditions*, *Phys. Rev. D* **103** (2021) 126019 [[arXiv:2012.05726](#)] [[INSPIRE](#)].
- [52] O. Aharony and N. Barel, *Correlation functions in $T\bar{T}$ -deformed Conformal Field Theories*, *JHEP* **08** (2023) 035 [[arXiv:2304.14091](#)] [[INSPIRE](#)].
- [53] S. He, Y. Li, Y.-Z. Li and Y. Zhang, *Holographic torus correlators of stress tensor in AdS_3/CFT_2* , *JHEP* **06** (2023) 116 [[arXiv:2303.13280](#)] [[INSPIRE](#)].
- [54] D. Basu, Lavish and B. Paul, *Entanglement negativity in $T\bar{T}$ -deformed CFTs*, *Phys. Rev. D* **107** (2023) 126026 [[arXiv:2302.11435](#)] [[INSPIRE](#)].
- [55] H.-S. Jeong, K.-Y. Kim and M. Nishida, *Entanglement and Rényi entropy of multiple intervals in $T\bar{T}$ -deformed CFT and holography*, *Phys. Rev. D* **100** (2019) 106015 [[arXiv:1906.03894](#)] [[INSPIRE](#)].
- [56] S. He, P.H.C. Lau, Z.-Y. Xian and L. Zhao, *Quantum chaos, scrambling and operator growth in $T\bar{T}$ deformed SYK models*, *JHEP* **12** (2022) 070 [[arXiv:2209.14936](#)] [[INSPIRE](#)].
- [57] Y. Nakata et al., *New holographic generalization of entanglement entropy*, *Phys. Rev. D* **103** (2021) 026005 [[arXiv:2005.13801](#)] [[INSPIRE](#)].
- [58] W.-Z. Guo, S. He and Y.-X. Zhang, *Constructible reality condition of pseudo entropy via pseudo-Hermiticity*, *JHEP* **05** (2023) 021 [[arXiv:2209.07308](#)] [[INSPIRE](#)].
- [59] T. Nishioka, T. Takayanagi and Y. Taki, *Topological pseudo entropy*, *JHEP* **09** (2021) 015 [[arXiv:2107.01797](#)] [[INSPIRE](#)].
- [60] W.-Z. Guo, S. He and Y.-X. Zhang, *On the real-time evolution of pseudo-entropy in 2d CFTs*, *JHEP* **09** (2022) 094 [[arXiv:2206.11818](#)] [[INSPIRE](#)].
- [61] S. He, J. Yang, Y.-X. Zhang and Z.-X. Zhao, *Pseudo-entropy for descendant operators in two-dimensional conformal field theories*, [arXiv:2301.04891](#) [[INSPIRE](#)].
- [62] A. Mollabashi et al., *Pseudo Entropy in Free Quantum Field Theories*, *Phys. Rev. Lett.* **126** (2021) 081601 [[arXiv:2011.09648](#)] [[INSPIRE](#)].
- [63] G. Camilo and A. Prudenziati, *Twist operators and pseudo entropies in two-dimensional momentum space*, [arXiv:2101.02093](#) [[INSPIRE](#)].
- [64] A. Mollabashi et al., *Aspects of pseudoentropy in field theories*, *Phys. Rev. Res.* **3** (2021) 033254 [[arXiv:2106.03118](#)] [[INSPIRE](#)].
- [65] K. Goto, M. Nozaki and K. Tamaoka, *Subregion spectrum form factor via pseudoentropy*, *Phys. Rev. D* **104** (2021) L121902 [[arXiv:2109.00372](#)] [[INSPIRE](#)].
- [66] M. Miyaji, *Island for gravitationally prepared state and pseudo entanglement wedge*, *JHEP* **12** (2021) 013 [[arXiv:2109.03830](#)] [[INSPIRE](#)].

- [67] I. Akal et al., *Page curve under final state projection*, *Phys. Rev. D* **105** (2022) 126026 [[arXiv:2112.08433](#)] [[INSPIRE](#)].
- [68] J. Mukherjee, *Pseudo Entropy in U(1) gauge theory*, *JHEP* **10** (2022) 016 [[arXiv:2205.08179](#)] [[INSPIRE](#)].
- [69] Y. Ishiyama, R. Kojima, S. Matsui and K. Tamaoka, *Notes on pseudo entropy amplification*, *PTEP* **2022** (2022) 093B10 [[arXiv:2206.14551](#)] [[INSPIRE](#)].
- [70] K. Doi et al., *Pseudoentropy in dS/CFT and Timelike Entanglement Entropy*, *Phys. Rev. Lett.* **130** (2023) 031601 [[arXiv:2210.09457](#)] [[INSPIRE](#)].
- [71] Z. Li, Z.-Q. Xiao and R.-Q. Yang, *On holographic time-like entanglement entropy*, *JHEP* **04** (2023) 004 [[arXiv:2211.14883](#)] [[INSPIRE](#)].
- [72] K. Doi et al., *Timelike entanglement entropy*, *JHEP* **05** (2023) 052 [[arXiv:2302.11695](#)] [[INSPIRE](#)].
- [73] Z. Chen, *Complex-valued Holographic Pseudo Entropy via Real-time AdS/CFT Correspondence*, [arXiv:2302.14303](#) [[INSPIRE](#)].
- [74] K. Narayan and H.K. Saini, *Notes on time entanglement and pseudo-entropy*, [arXiv:2303.01307](#) [[INSPIRE](#)].
- [75] X. Jiang, P. Wang, H. Wu and H. Yang, *Timelike entanglement entropy in dS₃/CFT₂*, [arXiv:2304.10376](#) [[INSPIRE](#)].
- [76] C.-S. Chu and H. Parihar, *Time-like entanglement entropy in AdS/BCFT*, *JHEP* **06** (2023) 173 [[arXiv:2304.10907](#)] [[INSPIRE](#)].
- [77] P.-Z. He and H.-Q. Zhang, *Timelike Entanglement Entropy from Rindler Method*, [arXiv:2307.09803](#) [[INSPIRE](#)].
- [78] A.J. Parzygnat, T. Takayanagi, Y. Taki and Z. Wei, *SVD Entanglement Entropy*, [arXiv:2307.06531](#) [[INSPIRE](#)].
- [79] D. Chen, X. Jiang and H. Yang, *Holographic $T\bar{T}$ deformed entanglement entropy in dS₃/CFT₂*, [arXiv:2307.04673](#) [[INSPIRE](#)].
- [80] W.-Z. Guo and J. Zhang, *Sum rule for pseudo Rényi entropy*, [arXiv:2308.05261](#) [[INSPIRE](#)].
- [81] B. Chen, L. Chen and P.-X. Hao, *Entanglement entropy in $T\bar{T}$ -deformed CFT*, *Phys. Rev. D* **98** (2018) 086025 [[arXiv:1807.08293](#)] [[INSPIRE](#)].
- [82] M. He and Y. Sun, *Holographic entanglement entropy in $T\bar{T}$ -deformed AdS₃*, *Nucl. Phys. B* **990** (2023) 116190 [[arXiv:2301.04435](#)] [[INSPIRE](#)].
- [83] S. He, Z.-C. Liu and Y. Sun, *Entanglement entropy and modular Hamiltonian of free fermion with deformations on a torus*, *JHEP* **09** (2022) 247 [[arXiv:2207.06308](#)] [[INSPIRE](#)].
- [84] W. Donnelly and V. Shyam, *Entanglement entropy and $T\bar{T}$ deformation*, *Phys. Rev. Lett.* **121** (2018) 131602 [[arXiv:1806.07444](#)] [[INSPIRE](#)].
- [85] S. Ashkenazi, S. Chakraborty, Z. Ma and T. Shachar, *Linear response of entanglement entropy to $T\bar{T}$ in massive QFTs*, *JHEP* **04** (2023) 077 [[arXiv:2302.06688](#)] [[INSPIRE](#)].
- [86] S. He and H. Shu, *Correlation functions, entanglement and chaos in the $T\bar{T}$ / $J\bar{T}$ -deformed CFTs*, *JHEP* **02** (2020) 088 [[arXiv:1907.12603](#)] [[INSPIRE](#)].
- [87] P. Calabrese and J.L. Cardy, *Evolution of entanglement entropy in one-dimensional systems*, *J. Stat. Mech.* **0504** (2005) P04010 [[cond-mat/0503393](#)] [[INSPIRE](#)].

- [88] T. Kawamoto, S.-M. Ruan and T. Takayanagia, *Gluing AdS/CFT*, *JHEP* **07** (2023) 080 [[arXiv:2303.01247](#)] [[INSPIRE](#)].
- [89] L. Apolo, P.-X. Hao, W.-X. Lai and W. Song, *Glue-on AdS holography for $T\bar{T}$ -deformed CFTs*, *JHEP* **06** (2023) 117 [[arXiv:2303.04836](#)] [[INSPIRE](#)].
- [90] C. Ferko and S. Sethi, *Sequential flows by irrelevant operators*, *SciPost Phys.* **14** (2023) 098 [[arXiv:2206.04787](#)] [[INSPIRE](#)].