Complex conformal transformations and zero-rest-mass fields

Bernardo Araneda*
Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut), Am Mühlenberg 1, D-14476 Potsdam, Germany

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We give a simple prescription for relating different solutions to the zero-rest-mass field equations in conformally flat space-time via complex conformal transformations and changes in reality conditions. We give several examples including linearized black holes. In particular, we show that the linearized Plebański-Demiański and Schwarzschild fields are related by a complex translation and a complex special conformal transformation. Similar results hold for the linearized Kerr and C-metric fields and for a peculiar toroidal singularity.

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I. INTRODUCTION

The Newman-Janis complex shift [1] is a method for obtaining the Kerr solution to the Einstein vacuum equations from the Schwarzschild solution via a complex coordinate transformation. The apparent arbitrariness in the way in which some of the metric functions must be complexified makes it difficult to establish whether it has a deep geometric origin [2]. The linear version of it, however, can be understood as a simple complex translation $z \to z - \mathrm{i}a$ [3]. The interest in this shift has been recently renewed in view of its applications to scattering amplitudes [4–6].

In this paper, we show that a complex translation followed by a complex special conformal transformation applied to the linearized Schwarzschild field produces the linearized Plebański-Demiański field (which is the linear limit of the most general type D vacuum space-time), and we furthermore show that this is just an example of a general and simple procedure in twistor space that applies to generic zero-rest-mass fields. This allows us to uncover complex coordinate transformations between, for example, the (linearized) Kerr and C-metric fields and a curious toroidal structure, as well as complex transformations applied to constant fields, hopfions/knotted fields, plane waves, etc. with arbitrary spin and algebraic type, as part of a unified framework.

It is interesting to note that the basic idea in the Newman-Janis shift can be traced back to at least 1887, when Appell [7] noticed that the point singularity $\{x = y = z = 0\}$ of the fundamental solution $(x^2 + y^2 + z^2)^{-1/2}$ to the Laplace equation in $\mathbb{R}^3$ is mapped to a ring singularity $\{x^2 + y^2 = a^2, z = 0\}$ under the complex translation $z \to z - \mathrm{i}a$. Synge [8] generalized this to remove the light-cone singularity of the fundamental solution $(t^2 - x^2 - y^2 - z^2)^{-1}$ to the relativistic wave equation in Minkowski space-time. Complex translations, and, more generally, complex Poincaré transformations, were then recognized by Trautman [9] as a powerful tool for the generation of new solutions to the scalar, Maxwell, and linearized gravity field equations in flat space-time; see also Ref. [10].

II. PRELIMINARIES

We will use twistor methods. For background on the aspects of twistor theory relevant to this work, we refer to Refs. [11–14]. We include the Appendix with some spinor conventions.

Let $\mathbb{CM}$ be complexified Minkowski space-time, with flat holomorphic metric $\eta = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$. The twistor space of $\mathbb{CM}$ is $\mathbb{PT} = \mathbb{CP}^3 \backslash \mathbb{CP}^1$, with homogeneous coordinates $Z^\alpha = (Z^0, Z^1, Z^2, Z^3)$. In 2-spinor notation, space-time coordinates are encoded in a 2×2 matrix $x^{AA'}$, and points of $\mathbb{PT}$ are represented by $Z^\alpha = (\omega^A, \pi_{A'})$, with $Z^0 = \omega^0$, $Z^1 = \omega^1$, $Z^2 = \pi_{0'}$, $Z^3 = \pi_{1'}$. The two spaces are related by the incidence relation

$$\omega^A = \mathrm{i}x^{AA'}\pi_{A'}. \quad (1)$$

The $\mathbb{CP}^1$ removed in the definition $\mathbb{PT} = \mathbb{CP}^3 \backslash \mathbb{CP}^1$ corresponds to the set $\{Z^2 = Z^3 = 0\}$ (i.e. $\{\pi_{A'} = 0\}$). This gives a fibration $\mathbb{PT} \to \mathbb{CP}^1$, where $\pi_{A'}$ are inhomogeneous coordinates on the base, and ω^A are coordinates on the fibers.

*bernardo.araneda@aei.mpg.de

From (1), one deduces that a point $x^{AA'} \in \mathbb{CM}$ corresponds to a holomorphic linear Riemann sphere $L_x = \mathbb{CP}^1 \subset \mathbb{PT}$ (a twistor line), while a point $Z^\alpha \in \mathbb{PT}$ corresponds to a totally null 2-surface in $\mathbb{CM}$ (an α-surface). The set $\{Z^2 = Z^3 = 0\}$ removed from $\mathbb{PT}$ is a twistor line $\mathbf{I}$ in the twistor space of conformally compactified Minkowski space-time $\mathbb{CM}^\sharp$. This twistor space is the compactification $\mathbb{CP}^3$; the line $\mathbf{I}$ corresponds to the vertex I of the null cone at infinity, and it can also be represented by the infinity twistor $I_{\alpha\beta}\mathrm{d}Z^\alpha \wedge \mathrm{d}Z^\beta = 2\,\mathrm{d}Z^2 \wedge \mathrm{d}Z^3$.

Two twistor lines L_x and L_y intersect if and only if the associated space-time points x and y are null separated. This means that the conformal structure of space-time is encoded in the intersection of twistor lines in $\mathbb{PT}$. More generally, twistor theory is conformally invariant, and twistors can be understood as the spinors of the (complexified) conformal group $\mathrm{SL}(4,\mathbb{C})$. In other words, twistor space carries a representation of $\mathrm{SL}(4,\mathbb{C})$: a complex *linear* transformation

$$Z^\alpha \mapsto T^\alpha{}_\beta Z^\beta, \qquad T^\alpha{}_\beta \in \mathrm{SL}(4,\mathbb{C}), \tag{2}$$

corresponds to a complex *conformal* transformation on space-time [15] (that is, to an element of the 15-complex-dimensional group of complex Poincaré transformations, complex dilations, and complex special conformal transformations). More precisely, the conformal group acts on the compactified space $\mathbb{CM}^\sharp$, since conformal inversions interchange the origin with I. The subgroup of $\mathrm{SL}(4,\mathbb{C})$ that leaves the line $\mathbf{I}$ in $\mathbb{CP}^3$ invariant is the Poincaré group.

We can express any $T^\alpha{}_\beta$ as a matrix:

$$T^\alpha{}_\beta = \begin{pmatrix} \theta^A{}_B & \tau^{AB'} \\ \nu_{A'B} & \tilde{\theta}_{A'}{}^{B'} \end{pmatrix}. \tag{3}$$

To describe the action on space-time coordinates, we distinguish three cases: (i) $\tau^{AB'} = 0 = \nu_{A'B}$, (ii) $\theta^A{}_B = \delta^A_B$, $\tilde{\theta}_{A'}{}^{B'} = \delta^{B'}_{A'}$, $\nu_{AB'} = 0$, and (iii) $\theta^A{}_B = \delta^A_B$, $\tilde{\theta}_{A'}{}^{B'} = \delta^{B'}_{A'}$, $\tau^{AB'} = 0$. Then, we find

$$\text{(i)}\ x'^{AA'} = \theta^A{}_B x^{BB'} (\tilde{\theta}^{-1})_{B'}{}^{A'}, \tag{4a}$$

$$\text{(ii)}\ x'^{AA'} = x^{AA'} + \xi^{AA'}, \qquad \xi^{AA'} = -\mathrm{i}\tau^{AA'}, \tag{4b}$$

$$\text{(iii)}\ x'^{AA'} = \frac{x^{AA'} - (x_b x^b) s^{AA'}}{(x_b x^b)(s_c s^c) - 2x_b s^b + 1}, \qquad s^{AA'} = -\frac{\mathrm{i}}{2}\nu^{AA'}. \tag{4c}$$

Thus, we see that $\theta^A{}_B$ and $\tilde{\theta}_{A'}{}^{B'}$ describe left and right Lorentz transformations and dilations, $\tau^{AB'}$ corresponds to translations, and $\nu_{AB'}$ corresponds to special conformal transformations.

Let $h > 0$ be a positive half-integer number. A zero-rest-mass field of helicity (or spin) h is a totally symmetric spinor $\varphi_{A'\ldots K'}$ with $2h$ indices such that

$$\nabla^{AA'} \varphi_{A'\ldots K'} = 0. \tag{5}$$

For $h = 0$, the corresponding equation is $\Box\varphi = 0$. The cases $h = \frac{1}{2}, 1, \frac{3}{2}, 2$ describe (massless) Dirac, Maxwell, Rarita-Schwinger, and linearized gravitational fields, respectively. The field equations (5) are conformally invariant, as long as $\varphi_{A'\ldots K'}$ has conformal weight -1. A classical result from twistor theory (see, e.g., Ref. [13]) establishes that the set of zero-rest-mass fields is isomorphic to the Čech cohomology group $\check{H}^1(\mathbb{PT}, \mathcal{O}(-2h-2))$, where $\mathcal{O}(k)$ is the sheaf of holomorphic functions on $\mathbb{PT}$ that are homogeneous of degree k. An explicit representation of this isomorphism is the Penrose transform: if $x^{AA'} \in \mathbb{CM}$ and $L_x = \mathbb{CP}^1$ is the associated twistor line, then any solution to (5) can be written as

$$\varphi_{A'\ldots K'}(x) = \frac{1}{2\pi\mathrm{i}} \oint_\Gamma f(Z)|_{L_x} \pi_{A'} \ldots \pi_{K'} \pi_{L'} \mathrm{d}\pi^{L'}, \tag{6}$$

where the twistor function f is homogeneous of degree $-2h-2$ and holomorphic except for a certain singularity region and the contour $\Gamma \subset L_x$ is such that it surrounds the singularities of f; see Ref. [12], Sec. 6.10. The relationship between $\varphi_{A'\ldots K'}$ and f is not unique, but f is just a representative of a cohomology class in $\check{H}^1(\mathbb{PT}, \mathcal{O}(-2h-2))$. For practical calculations, however, one can work with representatives, as long as the corresponding space-time result is cohomological invariant. This will be the case for the applications considered in this work.

A. Main idea

The basic observation in this work is the following. Consider a twistor function f, which generates a zero-rest-mass field $\varphi_{A'\ldots K'}$ via the Penrose transform. Consider also a linear transformation (2) in $\mathbb{PT}$, and put $Z'^\alpha = T^\alpha{}_\beta Z^\beta$. Define

$$f'(Z) := f(Z'). \tag{7}$$

The Penrose transform of f' will generate a new zero-rest-mass field $\varphi'_{A'\ldots K'}$ (of the same helicity). But we mentioned that (2) corresponds to a complex conformal transformation on space-time. Imposing different reality conditions on space-time coordinates before and after the transformation, this means that the fields $\varphi_{A'\ldots K'}$ and $\varphi'_{A'\ldots K'}$ will be two different solutions, which can be mapped to each other via a complex conformal transformation. We will illustrate this with several examples in the next section. We note that, even though we will apply the prescription (7) to twistor representatives and not to cohomology classes, this is sufficient for the purposes of this work, which are simply to use twistor theory as a tool to show how to relate different solutions via complex coordinate transformations.

III. COMPLEX TRANSFORMATIONS

For most of our examples of interest, we will need the following identity.

Proposition 1. Let $\alpha_{A'}, \beta_{A'}$ be two arbitrary (non-proportional) spinor fields, and let r, s be positive integers. Then,

$$\frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{\pi_{A'_1}\dots\pi_{A'_{2h}}\pi_{B'}\mathrm{d}\pi^{B'}}{(\alpha^{A'}\pi_{A'})^r(\beta^{A'}\pi_{A'})^s} = \frac{k}{(\alpha_{A'}\beta^{A'})^{2h+1}}\beta_{(A'_1}\dots\beta_{A'_{r-1}}\alpha_{A'_r}\dots\alpha_{A'_{2h})}, \tag{8}$$

where k is a constant, $2h = r + s - 2$, and the contour Γ separates the poles at $\pi_{A'} = \alpha_{A'}$ and $\pi_{A'} = \beta_{A'}$.

A simple way to show (8) is to contract the left-hand side with n factors of $\alpha^{A'}$ and $2h - n$ factors of $\beta^{A'}$, and then deduce that the integral will be equal to $(\alpha_{A'}\beta^{A'})^{-1}$ if $n = r - 1$ and zero otherwise; the right-hand side then follows straightforwardly.

A. Constant, elementary, and momentum states

An elementary state in twistor theory [11,12] is a zero-rest-mass field generated by a twistor function of the form

$$f(Z) = \frac{(C_\alpha Z^\alpha)^l (D_\alpha Z^\alpha)^m}{(A_\alpha Z^\alpha)^r (B_\alpha Z^\alpha)^s}, \tag{9}$$

for some $A_\alpha, \dots, D_\alpha$, where l, m, r, s are non-negative integers. The relevance of these functions comes from their utility as an alternative basis to momentum eigenstates (see Ref. [11], Sec. 4) and Refs. [13,16]. We will focus on the case $l = m = 0$, so that (9) takes the form

$$f(Z) = [\chi(Z)]^{-1}, \qquad \chi(Z) = (A_\alpha Z^\alpha)^r (B_\alpha Z^\alpha)^s. \tag{10}$$

Consider the simple case

$$\chi(Z) = (Z^2)^r (Z^3)^s. \tag{11}$$

The singular region of $f = \chi^{-1}$ is the algebraic set $\{\chi = 0\}$, which consists of two parallel planes $\mathbb{A} = \{Z^2 = 0\}$, $\mathbb{B} = \{Z^3 = 0\}$. These are in fact two fibers of the fibration $\mathbb{PT} \to \mathbb{CP}^1$ (as such, they do not intersect). The Penrose transform of $f = \chi^{-1}$ is a particular case of (8), with $\alpha^{A'} = o^{A'}, \beta^{A'} = \iota^{A'}$ (see the Appendix for our conventions). Thus, we immediately obtain the constant spinor field $\varphi_{A'_1\dots A'_{2h}} = k\iota_{(A'_1}\dots\iota_{A'_{r-1}}o_{A'_r}\dots o_{A'_{2h})}$, with $2h = r + s - 2$. For example, for $r = s = 2$, the corresponding (self-dual) Maxwell field is [see Eq. (23) below]

$$\mathcal{F} = \mathrm{d}t \wedge \mathrm{d}z + \mathrm{i}\mathrm{d}x \wedge \mathrm{d}y, \tag{12}$$

which (assuming t, x, y, z to be real) is a constant electric field in the z direction.

Now, writing (11) as in (10) with $A_\alpha = (0, 0, 1, 0)$, $B_\alpha = (0, 0, 0, 1)$, we apply an $\mathrm{SL}(4, \mathbb{C})$ transformation $Z^\alpha \mapsto Z'^\alpha = T^\alpha{}_\beta Z^\beta$ and put

$$\chi'(Z) \coloneqq \chi(Z') = (A'_\alpha Z^\alpha)^r (B'_\alpha Z^\alpha)^s, \tag{13}$$

where $A'_\beta = A_\alpha T^\alpha{}_\beta = (\nu_{0'B}, \tilde{\theta}_{0'}{}^{B'})$, $B'_\beta = B_\alpha T^\alpha{}_\beta = (\nu_{1'B}, \tilde{\theta}_{1'}{}^{B'})$ [$T^\alpha{}_\beta$ is given explicitly by (3)]. We see that A'_α and B'_α are only sensitive to the parts of $T^\alpha{}_\beta$ corresponding to right Lorentz rotations and dilations (contained in $\tilde{\theta}_{A'}{}^{B'}$) and special conformal transformations (contained in $\nu_{A'B}$). The singular set of $f' = \chi'^{-1}$ is again given by two planes in $\mathbb{PT}$, $\mathbb{A}' = \{A'_\alpha Z^\alpha = 0\}$, $\mathbb{B}' = \{B'_\alpha Z^\alpha = 0\}$, but now the planes intersect. This intersection is a twistor line, $\mathbb{A}' \cap \mathbb{B}' = L_q$, where, putting $A'_\alpha = (a_A, \tilde{a}^{A'})$, $B'_\alpha = (b_A, \tilde{b}^{A'})$, the point $q \in \mathbb{CM}$ is given by

$$q^{AA'} = \frac{\mathrm{i}}{(a_B b^B)}(b^A\tilde{a}^{A'} - a^A\tilde{b}^{A'}) = \frac{2\mathrm{i}}{(\nu_c\nu^c)}\nu^{AB'}\tilde{\theta}_{B'}{}^{A'} \tag{14}$$

(we assume ν^a to be non-null). The Penrose transform of $f' = \chi'^{-1}$ is again a particular case of (8), where now $\alpha^{A'} = \mathrm{i}x^{AA'}a_A + \tilde{a}^{A'}$, $\beta^{A'} = \mathrm{i}x^{AA'}b_A + \tilde{b}^{A'}$. The zero-rest-mass field is then given by the right-hand side of (8), with

$$\alpha_{A'}\beta^{A'} = k'(x_a - q_a)(x^a - q^a) \tag{15}$$

for some constant k'. The fields represented by the right-hand side of (8) with (15) are called “spin-h hopfions” or “knotted fields” [17,18]. The name comes from the fact that the principal spinors $\alpha_{A'}$ and $\beta_{A'}$ define in this case Robinson congruences, which are in turn related to the Hopf fibration (cf. Ref. [12], Sec. 6.2).

From the observation made around Eq. (7), we deduce that spin-h hopfions/knotted fields can be obtained via complex conformal transformations of constant fields. We notice that, for the case of null fields in electromagnetism, a complex transformation from a constant, null Maxwell field to a null electromagnetic hopfion was already given in Refs. [19,20], the null condition being essential. Our approach in this work shows that the complex transformation is valid for fields of arbitrary spin and arbitrary algebraic type, and it is just a particular example of the general framework given around Eq. (7).

1. *Intuitive interpretation*

Even though the above result that parallel spinors, which are simply constant fields on space-time, and spin-h hopfions/knotted fields, which have a quite complicated topological structure, are related by a complex conformal transformation may be difficult to anticipate if one only looks at their space-time description, from the twistor perspective we can get a fairly simple intuitive understanding of this phenomenon. Constant fields can be generated by two parallel planes in $\mathbb{PT}$ ($\{Z^2 = 0\}$ and $\{Z^3 = 0\}$), while spin-h hopfions can be generated by two intersecting planes.[1] In the former case, the planes do not intersect because we removed the line $\mathbf{I}$ from $\mathbb{CP}^3$ (which corresponds to infinity in space-time), while in the latter, the planes intersect in the twistor line L_q corresponding to a point q in space-time. If we think of the two parallel planes as "intersecting at infinity," then the above two twistor configurations are equivalent, so the corresponding physical configurations must be appropriately equivalent as well. It is also clear that the transformation must really be conformal, since Poincaré transformations preserve the line $\mathbf{I}$ (i.e., the infinity twistor $I_{\alpha\beta}$).

A little more formally, recall that, in order for conformal transformations to be well-defined everywhere in space-time, we must consider compactified Minkowski space $\mathbb{CM}^\sharp$. As mentioned in Sec. II, the twistor space of $\mathbb{CM}^\sharp$ (which is $\mathbb{CP}^3$) does contain $\mathbf{I}$, and the planes $\{Z^2 = 0\}$ and $\{Z^3 = 0\}$ intersect precisely in $\mathbf{I}$. The special conformal transformation relating parallel spinors and spin-h hopfions interchanges the line $\mathbf{I}$ (defined by $Z^2 = Z^3 = 0$) and the line L_q (defined by $Z'^2 = Z'^3 = 0$), or equivalently, it interchanges (via conformal inversion) the point $q \in \mathbb{CM}^\sharp$ with the vertex I of the light-cone at infinity (which is also a point in $\mathbb{CM}^\sharp$). In other words, if we interpret the point q as the location of the "source" of the spin-h hopfion, we see that the source of a constant field is a point at infinity.

2. *Momentum eigenstates*

Consider now twistor functions of the form

$$f(Z) = \frac{\exp(C_\alpha Z^\alpha / B_\alpha Z^\alpha)}{(A_\alpha Z^\alpha)(B_\alpha Z^\alpha)^{2h+1}}, \tag{16}$$

where $A_\alpha = (0, \tilde{a}^{A'})$, $B_\alpha = (0, \tilde{b}^{A'})$, $C_\alpha = (c_A, 0)$. The corresponding zero-rest-mass fields are momentum eigenstates (or plane waves; see Ref. [11], Sec. 4.4): choosing $\tilde{a}_{A'}\tilde{b}^{A'} = 1$, defining $k_a \coloneqq c_A \tilde{a}_{A'}$, and applying a slight variation of formula (8), we get the null fields

$$\varphi_{A'_1 \ldots A'_{2h}} = e^{\mathrm{i} x^a k_a} \tilde{a}_{A'_1} \ldots \tilde{a}_{A'_{2h}}. \tag{17}$$

Now, perform an arbitrary linear transformation $Z^\alpha \mapsto Z'^\alpha = T^\alpha{}_\beta Z^\beta$, and define $A'_\beta = A_\alpha T^\alpha{}_\beta = (a'_A, \tilde{a}'^{A'})$, $B'_\beta = B_\alpha T^\alpha{}_\beta$, $C'_\beta = C_\alpha T^\alpha{}_\beta$. Let $f'(Z) \coloneqq f(Z')$. Then, the Penrose transform of f' gives

$$\varphi'_{A'_1 \ldots A'_{2h}} = \frac{\exp\left[\frac{\alpha_{A'}\gamma^{A'}}{\alpha_{B'}\beta^{B'}}\right]}{(\alpha_{C'}\beta^{C'})^{2h+1}} \alpha_{A'_1} \ldots \alpha_{A'_{2h}}, \tag{18}$$

where $\alpha^{A'} = \mathrm{i} x^{AA'} a'_A + \tilde{a}'^{A'}$, etc. The prefactor can be written as

$$\frac{1}{(\alpha_{C'}\beta^{C'})^{2h+1}} \exp\left[\frac{\alpha_{A'}\gamma^{A'}}{\alpha_{B'}\beta^{B'}}\right] = \frac{c_1}{|x-q|^{2(2h+1)}} \exp\left[c_2 \frac{|x-p|^2}{|x-q|^2}\right]$$

for some constants c_1 and c_2 and some fixed points q^a and p^a defined analogously to (14). [We also put $|x-q|^2 \equiv (x_a - q_a)(x^a - q^a)$, etc.] Thus, a complex conformal transformation of a plane wave (17) produces a new null, zero-rest-mass field (18) with hopfion/knotted-like features. We notice that for the electromagnetic case ($h = 1$), similar results were obtained in Ref. [20] (see also Ref. [19]). The generalization (18) to arbitrary spin appears to be new. The $h = 2$ case might exhibit some interesting physical features, as it transforms a plane gravitational wave to a hopfion/knotted-like gravitational wave (which is different from the gravitational hopfion considered in Refs. [17,18]).

B. Linearized black holes

By "linearized black hole," we mean a spin-2 field [$h = 2$ in (5)] in $\mathbb{CM}$, which formally looks the same as the Weyl curvature spinor of a black hole (Petrov type D) solution [11]. As explained in Ref. [11], such fields are generated by twistor functions of the form

$$f(Z) = [\chi(Z)]^{-(h+1)}, \qquad \chi(Z) = Q_{\alpha\beta} Z^\alpha Z^\beta, \tag{19}$$

where $h = 2$, and $Q_{\alpha\beta}$ cannot be written as a product $A_{(\alpha}B_{\beta)}$ of only two twistors. The $h = 1$ case of (19) describes electromagnetic analogues such as the Coulomb field, the "magic" [21] (or "root-Kerr" [4]) field, or others (see examples below). We will, however, leave h in (19) arbitrary, so our construction also applies to higher-/lower-spin analogs.

The facts that $Q_{\alpha\beta} \neq A_{(\alpha}B_{\beta)}$ and that any conformal transformation of $A_{(\alpha}B_{\beta)}Z^\alpha Z^\beta$ gives $A'_{(\alpha}B'_{\beta)}Z^\alpha Z^\beta$ suggest that linearized black holes (and higher-/lower-spin analogs) cannot be obtained from complex conformal transformations of elementary states. However, we should be more careful since these are facts about representatives and not about cohomology classes. In other words, one would need to prove that the cohomology classes of (19) and (10)

[1]Indeed, this elementary observation was one of our basic motivations for this work.

cannot be connected by a linear transformation of the twistor variables. We will not attempt to do this and will simply study the case (19) separately.

For any (non-negative) integer h, the zero-rest-mass fields generated by (19) can be obtained as a particular case of formula (8), with $r = s = h + 1$. This is because on a generic twistor line L_x, using the incidence relation (1), we get $\chi|_{L_x} = K^{A'B'}\pi_{A'}\pi_{B'}$ for some symmetric $K^{A'B'}$, which can always be decomposed into principal spinors as $K^{A'B'} = \alpha^{(A'}\beta^{B')}$, and, assuming the generic case $K^{A'B'}K_{A'B'} \neq 0$, we have $\alpha_{A'}\beta^{A'} \neq 0$. The spinors $\alpha_{A'}$ and $\beta_{A'}$ contain the information of the roots of the second-order homogeneous polynomial $\chi|_{L_x} = K^{A'B'}\pi_{A'}\pi_{B'}$. More explicitly, in terms of a coordinate $\zeta = \frac{\pi_{1'}}{\pi_{0'}}$ on the Riemann sphere of x, we have $\chi|_{L_x} = (\pi_{0'})^2(A\zeta^2 + 2B\zeta + C)$, where $A = K^{1'1'}$, $B = K^{0'1'}$, $C = K^{0'0'}$. The roots are then

$$\zeta_\pm = \frac{1}{A}(-B \pm \Delta), \qquad \Delta := \sqrt{B^2 - AC}. \tag{20}$$

Putting $\alpha^{A'} = \sqrt{A}(o^{A'} + \zeta_+\iota^{A'})$, $\beta^{A'} = \sqrt{A}(o^{A'} + \zeta_-\iota^{A'})$, we get $K^{A'B'} = \alpha^{(A'}\beta^{B')}$ as required. We also see that $\alpha_{A'}\beta^{A'} = -2\Delta$. Finally, defining a spin frame $\hat{\alpha}_{A'} = \frac{1}{(\alpha_{B'}\beta^{B'})^{1/2}}\alpha_{A'}$, $\hat{\beta}_{A'} = \frac{1}{(\alpha_{B'}\beta^{B'})^{1/2}}\beta_{A'}$, we can express the field generated by (19) as

$$\varphi_{A'_1\ldots A'_{2h}} = \frac{k}{\Delta^{h+1}}\hat{\alpha}_{(A'_1}\ldots\hat{\alpha}_{A'_h}\hat{\beta}_{A'_{h+1}}\ldots\hat{\beta}_{A'_{2h})}, \tag{21}$$

where $\hat{\alpha}_{A'}\hat{\beta}^{A'} = 1$, and we redefined the constant k. From the invariant expression

$$\varphi^{A'_1\ldots A'_{2h}}\varphi_{A'_1\ldots A'_{2h}} \propto \frac{1}{\Delta^{2(h+1)}}, \tag{22}$$

we see that the field is not null and that it is singular at $\Delta = 0$. This will be useful for physical interpretation.

The case $h = 1$ (Maxwell fields) has a simple description in tensor terms: defining $\mathcal{F}_{ab} = \varphi_{A'B'}\epsilon_{AB}$, a calculation gives

$$\begin{aligned}\mathcal{F} = \frac{k}{\Delta^3}\bigg[&\frac{(A-C)}{2}(\mathrm{d}t\wedge\mathrm{d}x + \mathrm{i}\mathrm{d}y\wedge\mathrm{d}z)\\ &+ B(\mathrm{d}t\wedge\mathrm{d}z + \mathrm{i}\mathrm{d}x\wedge\mathrm{d}y)\\ &+ \frac{(A+C)}{2}(\mathrm{d}z\wedge\mathrm{d}x + \mathrm{i}\mathrm{d}y\wedge\mathrm{d}t)\bigg].\end{aligned} \tag{23}$$

1. Schwarzschild and Plebański-Demiański

Consider the twistor function (19) with

$$\chi(Z) = Z^0Z^3 - Z^1Z^2. \tag{24}$$

On a generic twistor line L_x, we get $A = \frac{x+\mathrm{i}y}{\sqrt{2}}$, $B = \frac{z}{\sqrt{2}}$, $C = -\frac{(x-\mathrm{i}y)}{\sqrt{2}}$ (we omit an overall factor of i). The function Δ in (20) is

$$\Delta = \frac{1}{\sqrt{2}}r, \qquad r := \sqrt{x^2 + y^2 + z^2}, \tag{25}$$

and the roots are $\zeta_\pm = \frac{-z\pm r}{x+\mathrm{i}y}$. The singularity region $\Delta = 0$ of the field (21) is then $x = y = z = 0$ (t arbitrary), which we can interpret as Coulomb/Schwarzschild-like behavior. For example, for $h = 1$ and $h = 2$,

$$\begin{aligned}\mathcal{F} &= \frac{k}{r^2}[\mathrm{d}t\wedge\mathrm{d}r - \mathrm{i}r^2\sin\theta\mathrm{d}\phi\wedge\mathrm{d}\theta],\\ \varphi_{A'B'C'D'} &= \frac{k}{r^3}\hat{\alpha}_{(A'}\hat{\alpha}_{B'}\hat{\beta}_{C'}\hat{\beta}_{D')},\end{aligned} \tag{26}$$

where (r, θ, ϕ) are standard spherical coordinates, defined by $x + \mathrm{i}y = r\sin\theta e^{\mathrm{i}\phi}$, $z = r\cos\theta$. The field $\mathcal{F}$ is precisely the (self-dual) Coulomb field, while the spin-2 field is the linearized Schwarzschild solution.

Now, consider a linear transformation $Z^\alpha \mapsto Z'^\alpha = T^\alpha{}_\beta Z^\beta$, with

$$T^\alpha{}_\beta = \begin{pmatrix} 1 & 0 & \frac{c}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & -\frac{c}{\sqrt{2}} \\ -\frac{1}{c\sqrt{2}} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{c\sqrt{2}} & 0 & \frac{1}{2} \end{pmatrix} \tag{27}$$

for some $c \in \mathbb{C}$, $c \neq 0$. One can check that $\det(T^\alpha{}_\beta) = 1$, so $T^\alpha{}_\beta \in \mathrm{SL}(4, \mathbb{C})$. Following the prescription (7) with (19) and (24), we get

$$\chi'(Z) := \chi(Z') = \frac{\sqrt{2}}{c}\left(Z^0Z^1 + \frac{c^2}{2}Z^2Z^3\right). \tag{28}$$

After some straightforward calculations, the new Δ (20), now denoted Δ', is

$$\Delta' = \frac{1}{4}\sqrt{(x_ax^a - c^2)^2 - 4c^2(x^2 + y^2)}. \tag{29}$$

To have an interpretation of the new field, we must analyze the set of points on space-time where $\Delta' = 0$. To this end, we separate c into real and imaginary parts as

$$c = a + \mathrm{i}b, \tag{30}$$

with a, b real. Writing also $(4\Delta')^2 = R + \mathrm{i}I$ (with R, I real), we get

$$\begin{aligned} R &= (x_a x^a)^2 + a^4 + b^4 - 6a^2 b^2 \\ &\quad + 2(a^2 - b^2)(z^2 - t^2 - x^2 - y^2), \\ I &= 4ab(a^2 - b^2 + z^2 - t^2 - x^2 - y^2), \end{aligned} \tag{31}$$

so $\Delta' = 0$ iff $R = I = 0$. The condition $I = 0$ gives $t^2 - z^2 + x^2 + y^2 = a^2 - b^2$. Replacing in $R = 0$, we get also $t^2 - z^2 - (x^2 + y^2) = \pm(a^2 + b^2)$. The $+$ sign leads to $t^2 - z^2 = a^2$ and $x^2 + y^2 = -b^2$, while the $-$ sign leads to $z^2 - t^2 = b^2$ and $x^2 + y^2 = a^2$. Assuming the generic case $a \neq 0$, $b \neq 0$ (see the next example for other cases), we find

$$\Delta' = 0 \Leftrightarrow z^2 - t^2 = b^2 \quad \text{and} \quad x^2 + y^2 = a^2. \tag{32}$$

This is exactly the singular structure of the Plebański-Demiański field [22]: two accelerating ring singularities $x^2 + y^2 = a^2$, each moving on one branch of the hyperbola $z^2 - t^2 = b^2$.

To interpret (27) in space-time terms, we express it as a composition of the basic transformations (3) and (4). We find

$$T^\alpha{}_\beta = S^\alpha{}_\gamma U^\gamma{}_\beta, \quad S^\alpha{}_\gamma = \begin{pmatrix} \delta^A_C & 0 \\ \frac{1}{c^2}\tau_{A'C} & \delta^{C'}_{A'} \end{pmatrix}, \quad U^\gamma{}_\beta = \begin{pmatrix} \delta^C_B & \tau^{CB'} \\ 0 & \delta^{B'}_{C'} \end{pmatrix}, \tag{33}$$

where the components of $\tau^{AB'}$ are $\tau^{00'} = \frac{c}{\sqrt{2}} = -\tau^{11'}$, $\tau^{01'} = \tau^{10'} = 0$ (see the Appendix for some useful identities). Using (4), we see that $U^\alpha{}_\beta$ corresponds to a translation along the vector field ξ^a, while $S^\alpha{}_\beta$ is a special conformal transformation along $s^a = \frac{1}{c^2}\xi^a$, where [recalling the definition (30)]

$$\xi^a = (0, 0, 0, -\mathrm{i}a + b). \tag{34}$$

Summarizing, we have the following proposition.

Proposition 2. The Plebański-Demiański field can be obtained from the Schwarzschild field (for any spin h, in particular for the linearized black hole solutions) by a complex translation along ξ^a followed by a complex special conformal transformation along $s^a = \frac{1}{(a+\mathrm{i}b)^2}\xi^a$, where ξ^a is given by (34).

This provides an interpretation for the transformation for Maxwell fields mentioned by Plebański and Demiański in Ref. [22], Eq. (4.65). To have some intuition about the appearance of two objects "out of one"; see the Conclusions.

2. Kerr and the C-metric

Let a be a real parameter, and consider

$$\chi(Z) = Z^0 Z^3 - Z^1 Z^2 + \sqrt{2} a\, Z^2 Z^3. \tag{35}$$

Note that this function can be obtained from (24) by a linear transformation (2) and (3) corresponding to a translation: this is the twistor version of the (linearized) Newman-Janis shift. We will, however, analyze this case independently of (24). On twistor lines, we find $A = \frac{x+\mathrm{i}y}{\sqrt{2}}$, $B = \frac{z-\mathrm{i}a}{\sqrt{2}}$, $C = -\frac{(x-\mathrm{i}y)}{\sqrt{2}}$, which gives

$$\Delta = \frac{1}{\sqrt{2}} r_{\mathrm{c}}, \qquad r_{\mathrm{c}} := \sqrt{x^2 + y^2 + (z - \mathrm{i}a)^2} = r - \mathrm{i}az/r, \tag{36}$$

where r is defined to be the real part of r_{c}. The singularity region $\Delta = 0$ of the fields (21) is now $x^2 + y^2 = a^2$, $z = 0$ (t arbitrary). This ring singularity allows us to associate (21) in this case to the (linearized) Kerr field and its higher-/lower-spin analogs. For example, introducing a spheroidal coordinate system (r, θ, ϕ) by $x + \mathrm{i}y = \sqrt{r^2 + a^2} \sin\theta e^{\mathrm{i}\phi}$, $z = r\cos\theta$, we find for the spin 1 and 2 cases

$$\begin{aligned} \mathcal{F} &= \frac{k}{(r - \mathrm{i}a\cos\theta)^2}[\mathrm{d}t \wedge (\mathrm{d}r + \mathrm{i}a\sin\theta \mathrm{d}\theta) \\ &\quad - \sin\theta \mathrm{d}\phi \wedge (a\sin\theta \mathrm{d}r + \mathrm{i}(r^2 + a^2)\mathrm{d}\theta)], \end{aligned} \tag{37}$$

$$\varphi_{A'B'C'D'} = \frac{k}{(r - \mathrm{i}a\cos\theta)^3} \hat{\alpha}_{(A'}\hat{\alpha}_{B'}\hat{\beta}_{C'}\hat{\beta}_{D')}. \tag{38}$$

The field (37) is the root-Kerr (or magic) solution [4], while (38) is the linearized Kerr solution. So, we can interpret a as an angular momentum parameter.

Consider now the transformation $Z^\alpha \mapsto Z'^\alpha = T^\alpha{}_\beta Z^\beta$ with

$$T^\alpha{}_\beta = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ -\frac{\lambda}{a\sqrt{2}} & 0 & \frac{1}{\lambda} & 0 \\ 0 & \frac{\lambda}{a\sqrt{2}} & 0 & \frac{1}{\lambda} \end{pmatrix}, \tag{39}$$

where $\lambda \neq 0$ is a complex parameter. Following the prescription (7) with (19) and (35), we find

$$\chi'(Z) := \chi(Z') = \frac{\sqrt{2}}{c}\left(Z^0 Z^1 + \frac{c^2}{2} Z^2 Z^3\right), \qquad c = \frac{2a}{\lambda^2}. \tag{40}$$

So, the new function χ' is formally the same as (28), and the new Δ' is again given by (29) [although the parameter c is now related to a different conformal transformation (39) and (40)]. To interpret the new field, we analyze the set of points where $\Delta' = 0$. This was already done below Eq. (32) when c is genuinely complex, in which case the new solution is the Plebański-Demiański field. So, it remains to analyze the cases c real or purely imaginary:

(i) Suppose first c purely imaginary, $c \equiv \mathrm{i}\alpha^{-1}$, $\alpha \in \mathbb{R}$. Then, $(4\Delta')^2 = (x_a x^a + \alpha^{-2})^2 + 4\alpha^{-2}(x^2 + y^2)$, so $\Delta' = 0$ iff $x^2 + y^2 = 0 = x_a x^a + \alpha^{-2}$. This gives $x = y = 0$, $z^2 - t^2 = \alpha^{-2}$. These are two points moving each on one branch of the hyperbola $z^2 - t^2 = \alpha^{-2}$: this is the *C-metric field*, with acceleration parameter α.

(ii) Suppose now c real (for concreteness assume $c > 0$). Then, we can write the equation $\Delta' = 0$ as $(x^2 + y^2 + z^2 + c^2 - t^2)^2 = 4c^2(x^2 + y^2)$. For $t = 0$, we have a ring singularity $x^2 + y^2 = c^2$, $z = 0$. For fixed $t \neq 0$, the equation describes a torus, where t is the radius of the tube and c is the distance from the center of the torus to the center of the tube. As time progresses, the torus evolves through the three possible tori[2]: the standard "ring torus" for $t < c$, a "horn torus" (no hole) at $t = c$, and a "spindle torus" (self-intersecting) for $t > c$. This toroidal singularity is quite peculiar, and we are not aware of a nonlinear solution in general relativity that can be associated to this field. We note, however, that this singularity has also been described in Ref. [23].

Finally, we need to interpret (39) in space-time terms. To this end, we note that

$$T^\alpha{}_\beta = D^\alpha{}_\gamma S^\gamma{}_\beta, \quad D^\alpha{}_\gamma = \begin{pmatrix} \lambda\delta^A_C & 0 \\ 0 & \lambda^{-1}\delta^{C'}_{A'} \end{pmatrix}, \quad S^\gamma{}_\beta = \begin{pmatrix} \delta^C_B & 0 \\ \nu_{C'B} & \delta^{B'}_{C'} \end{pmatrix}, \tag{41}$$

where the components of $\nu_{A'B}$ are $\nu_{0'0} = -\frac{\lambda^2}{a\sqrt{2}} = -\nu_{1'1}$, $\nu_{0'1} = \nu_{1'0} = 0$. Using (4), we see that $D^\alpha{}_\beta$ is a dilation with parameter λ, and $S^\alpha{}_\beta$ is a special conformal transformation along the vector field

$$s^a = \left(0, 0, 0, \frac{\lambda^2}{2\mathrm{i}a}\right). \tag{42}$$

Proposition 3 provides a summary.

Proposition 3. The conformal transformation (39), which consists of a complex special conformal transformation along (42) followed by a complex dilation with parameter λ, maps the Kerr field (for any spin h, in particular for the linearized black hole solutions) to (i) the Plebański-Demiański field if λ^2 is genuinely complex, (ii) the C-metric field if λ^2 is purely imaginary, and (iii) a toroidal singularity if λ^2 is real.

Note that the Schwarzschild field can also be transformed to the C-metric field and to the toroidal singularity [by assuming $a = 0$ or $b = 0$ in (30)]. On the other hand, given that the Plebański-Demiański field can be obtained from the Schwarzschild field by a translation followed by a special conformal transformation, and that the Kerr field is itself obtained from a translation of Schwarzschild, one might think that to go from Kerr to Plebański-Demiański one only needs a special conformal transformation. But the above result shows that a complex dilation is also needed. In view of the definition of c in (40), we see that the action of the dilation is to effectively complexify the angular momentum parameter (so that it becomes c).

C. Spherical scalar waves

As a final example of complex conformal transformations, we can try to combine the twistor functions of linearized black holes (19) with the ones of plane waves (16). That is, consider

$$f(Z) = \frac{\exp(C_\alpha Z^\alpha / B_\alpha Z^\alpha)}{(Q_{\alpha\beta} Z^\alpha Z^\beta)^{h+1}} \tag{43}$$

where $B_\alpha = (0, b^{A'})$, $C_\alpha = (c_A, 0)$, and $Q_{\alpha\beta}$ cannot be expressed as a product of only two twistors. For spin other than zero (i.e., $h > 0$), the calculation of the contour integrals involved in the zero-rest-mass fields associated to (43) is quite involved. We will, for simplicity, restrict ourselves to the scalar case $h = 0$. A calculation shows that the Penrose transform of the twistor function (43) with $h = 0$ is

$$\varphi(x^a) = \frac{1}{\alpha_{A'}\beta^{A'}} \exp\left[\mathrm{i}\frac{x^{AA'} c_A \alpha_{A'}}{\alpha_{B'} b^{B'}}\right] \tag{44}$$

(in this and the following expressions, we will omit irrelevant overall numerical constants), where $\alpha_{A'}$ and $\beta_{A'}$ are the spinor fields defined by $(Q_{\alpha\beta} Z^\alpha Z^\beta)|_{L_x} = \alpha^{A'}\beta^{B'}\pi_{A'}\pi_{B'}$. Choosing $c_A = o_A$, $b^{A'} = \frac{1}{\sqrt{2}}\iota^{A'}$ and recalling the definitions (20) of Δ and $\zeta_\pm$, we get

$$\varphi(x^a) = \frac{e^{\mathrm{i}(t+z+\zeta_+(x+\mathrm{i}y))}}{\Delta}. \tag{45}$$

For example, choosing $Q_{\alpha\beta}$, $Q'_{\alpha\beta}$ to be given by (24) and (28), respectively, we get the scalar waves

$$\varphi(x^a) = \frac{1}{r}\exp\left(\mathrm{i}(t + r)\right), \tag{46a}$$

$$\varphi'(x^a) = \frac{1}{\Delta'}\exp\left[\mathrm{i}(t + z) + \frac{\mathrm{i}}{2(t - z)} \times (-t^2 + z^2 - x^2 - y^2 + c^2 + \Delta')\right], \tag{46b}$$

[2] See, e.g., https://en.wikipedia.org/wiki/Torus.

where $r = \sqrt{x^2+y^2+z^2}$ and Δ' is defined in (29). We see that (46a) represents a spherical scalar wave, and (46b) is a much more complicated configuration, but the two solutions (46a) and (46b) are related by the complex conformal transformation mentioned in Proposition 2.

IV. COMMENTS ON NONLINEAR FIELDS

As emphasized by Newman [24], Flaherty [2], and others, the fact that complex coordinate transformations produce new solutions to real field equations can be understood (assuming real analyticity) via consideration of holomorphic extensions, holomorphic coordinate transformations, and the imposition of new reality conditions. The coordinate transformations should also preserve some additional structure, e.g., the Minkowski metric, or the conformal structure in the current manuscript.[3]

While the above works well for linear theories in (conformally) flat space-time and it has been the topic of this manuscript, the understanding of the fully nonlinear Newman-Janis shift [1] in general relativity is much less satisfactory. In general relativity, one would wish to preserve the Einstein equations, i.e., to map a solution to another solution. This is guaranteed to be the case if one considers holomorphic extensions. The nonlinear Newman-Janis shift, however, does not correspond to a holomorphic extension, and the fact that it produces a new solution does not seem to be an automatic consequence of the procedure. What is more, the transformation between Schwarzschild and Kerr *cannot* be holomorphic (at least in the above sense of analytic continuation), since, as explained by Newman [24], the solutions have different numbers of holomorphic Killing vectors.

As is well known, the main ambiguity in the nonlinear Newman-Janis shift is in the way in which some functions in the metric must be complexified. In particular, the function $\frac{2m}{r}$ must be replaced by $\frac{m}{r}+\frac{m}{\bar{r}}$ in order for the trick to work. More generally (see Ref. [25]), the idea is that a function $f(r)$ must be replaced by a function $F(r,\bar{r})$ that reduces to $f(r)$ on the real slice. We can actually use this to give a very simple version of the trick, as follows. (We are not aware that this form of the trick has been given before.) Consider a complex manifold with local complex coordinates (T,X,Y,Z) and a complex nonholomorphic metric

$$g = \mathrm{d}T^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 - \mathrm{d}Z^2 + \Phi(R,\bar{R})(L_a \mathrm{d}x^a)^2, \tag{47}$$

where $R = \sqrt{X^2+Y^2+Z^2}$ and

$$\Phi(R,\bar{R}) = \frac{m}{R} + \frac{m}{\bar{R}} \tag{48}$$

$$L_a \mathrm{d}x^a = \frac{1}{\sqrt{2}}\left[\mathrm{d}T + \left(\frac{1-|\zeta_+|^2}{1+|\zeta_+|^2}\right)\mathrm{d}Z + \left(\frac{\zeta_+ + \bar{\zeta}_+}{1+|\zeta_+|^2}\right)\mathrm{d}X + \mathrm{i}\left(\frac{\zeta_+ - \bar{\zeta}_+}{1+|\zeta_+|^2}\right)\mathrm{d}Y\right], \tag{49}$$

$$\zeta_+ = \frac{-Z+R}{X+\mathrm{i}Y}, \tag{50}$$

with m a real parameter. Define two real slices $S_1 = \{T=t, X=x, Y=y, Z=z\}$ and $S_2 = \{T=t, X=x, Y=y, Z=z-\mathrm{i}a\}$, where t, x, y, and z are real and a is a real parameter. Then, a calculation shows that g restricted to S_1 is the Schwarzschild metric, and g restricted to S_2 is the Kerr metric.

The above version of the trick singles out Schwarzschild and Kerr from a complex space by simply selecting two different real slices related in the usual Newman-Janis way $z \to z - \mathrm{i}a$ (in particular, it does not involve a change from spherical to spheroidal coordinates—see Ref. [26]—and it also shows that the null vectors L_a of Schwarzschild and Kerr correspond to two points of the Riemann sphere related by $z \to z - \mathrm{i}a$). However, the procedure is arbitrary in that, in going from Schwarzschild to Kerr, one still has to make the arbitrary replacement $\frac{2m}{r} \to \frac{m}{r} + \frac{m}{\bar{r}}$ in (48). Moreover, replacements of this sort, without any justification, can be used to argue that any two metrics are "related" by a complex coordinate transformation. Let us illustrate this with a transformation from Schwarzschild to the C-metric.

Proposition 4. Consider a complex manifold with local complex coordinates $(U,V,W,\tilde{W})$ and a nonholomorphic metric

$$g = \Omega^{-2}[2(\mathrm{d}U\mathrm{d}V + \mathrm{d}W\mathrm{d}\tilde{W}) + \mathcal{F}(U,\bar{U})\mathrm{d}V^2 + \mathcal{G}(W,\bar{W})\mathrm{d}\tilde{W}^2], \tag{51}$$

where, separating $(U,V,W,\tilde{W})$ into real and imaginary parts according to $U = u + \mathrm{i}u'$, $V = v - \mathrm{i}v'$, $W = w + \mathrm{i}w'$, $\tilde{W} = \tilde{w} - \mathrm{i}\tilde{w}'$:

$$\Omega(U,\bar{U},W,\bar{W}) = u + \alpha(w' - u'), \tag{52a}$$

[3]Otherwise, all (say, non-null) Maxwell fields could be deemed as "equivalent," since, given any two of them, one can be mapped to the other by the transformation that takes the Darboux coordinates of the first to the Darboux coordinates of the other. Mathematically, this is the statement that all symplectic forms are locally equivalent.

$$\mathcal{F}(U,\bar{U}) = u^2(1-2mu) - (u'^2-1)(1+2m\alpha u'), \quad (52b)$$

$$\mathcal{G}(W,\bar{W}) = 1 - w^2 - (1-w'^2)(1+2\alpha m w'), \quad (52c)$$

and m and α are two real parameters. Define two real slices by $S_1 = \{U = u + \mathrm{i}, V = v, W = w + \mathrm{i}, \tilde{W} = \tilde{w}\}$ and $S_2 = \{U = \mathrm{i}u', V = -\mathrm{i}v', W = 1 + \mathrm{i}w', \tilde{W} = -\mathrm{i}\tilde{w}'\}$. Then, Eq. (51) restricted to S_1 is the Schwarzschild metric, and Eq. (51) restricted to S_2 is the C-metric.

To show this, notice first that

$$g|_{S_1} = \frac{1}{u^2}[2(\mathrm{d}u\mathrm{d}v + \mathrm{d}w\mathrm{d}\tilde{w}) + u^2(1-2mu)\mathrm{d}v^2 + (1-w^2)\mathrm{d}\tilde{w}^2], \quad (53)$$

$$g|_{S_2} = \frac{1}{\alpha^2(w'-u')^2}[2(\mathrm{d}u'\mathrm{d}v' + \mathrm{d}w'\mathrm{d}\tilde{w}') + (u'^2-1) \times (1+\alpha m u')\mathrm{d}v'^2 + (1-w'^2)(1+2\alpha m w')\mathrm{d}\tilde{w}'^2], \quad (54)$$

and define new coordinates $(t_{\mathrm{s}}, r, \theta, \phi)$ and (τ, x, y, φ) by

$$u = \frac{1}{r}, \quad v = t_{\mathrm{s}} + \int \frac{\mathrm{d}r}{1-\frac{2m}{r}}, \quad w = -\cos\theta, \quad \tilde{w} = \mathrm{i}\phi - \int \frac{\mathrm{d}\theta}{\sin\theta}, \quad (55)$$

$$u' = -y, \quad v' = \tau + \int \frac{\mathrm{d}y}{F(y)}, \quad w' = x, \quad \tilde{w}' = \mathrm{i}\varphi - \int \frac{\mathrm{d}x}{G(x)}, \quad (56)$$

where $F(y) = (y^2-1)(1-2\alpha m y)$, $G(x) = (1-x^2)(1+2\alpha m x)$. Then, a short calculation gives the standard forms

$$g|_{S_1} = \left(1 - \frac{2m}{r}\right)\mathrm{d}t_{\mathrm{s}}^2 - \frac{\mathrm{d}r^2}{(1-\frac{2m}{r})} - r^2(\mathrm{d}\theta^2 + \sin\theta^2\mathrm{d}\phi^2), \quad (57)$$

$$g|_{S_2} = \frac{1}{\alpha^2(x+y)^2}\left[F(y)\mathrm{d}\tau^2 - \frac{\mathrm{d}y^2}{F(y)} - \frac{\mathrm{d}x^2}{G(x)} - G(x)\mathrm{d}\varphi^2\right]. \quad (58)$$

The above shows that one can start from the Schwarzschild metric written in the form (53), "complexify" the functions $u^2(1-2mu)$, $(1-w^2)$ in such a way so as to obtain (52b) and (52c), and then make a complex coordinate change (or choose a new real slice) and obtain the C-metric (54) and (58). But this procedure is obviously completely arbitrary, since there are many ways of complexifying the functions in (53).

We note that the (double-Kerr-Schild) form of the Schwarzschild and C metrics (53) and (54) was found by using the facts that the space-times are conformally (Lorentzian) Kähler [2] and that any Kähler metric has a double Kerr-Schild structure (as is not hard to show). The metrics inside the square brackets in (53) and (54) are actually the Kähler metrics associated to these space-times [27]. (The fact that these solutions are double-Kerr-Schild is known from the work of Plebański and Demiański [22], but it is perhaps not straightforward to deduce (53) and (54) from the expressions given in Ref. [22].)

V. CONCLUSIONS

We gave a simple procedure for relating different solutions to the zero-rest-mass field equations via complex coordinate transformations, by exploiting the fact that the conformal group acts linearly on twistor space. In particular, we showed that a complex translation followed by a complex special conformal transformation of the (linearized) Schwarzschild field produces the (linearized) Plebański-Demiański field. We also gave numerous other examples (constant fields, hopfions, waves, etc.) and (hopefully) provided a twistor intuition of why some of these transformations can be anticipated without calculation.

The fact that a complex translation of a pointlike source produces a rotating source can already be intuitively anticipated from the Appell trick [7] at the Newtonian level, while the Newman-Janis shift generalizes this to the relativistic level. Interestingly, a complex special conformal transformation has the effect of producing either *two* accelerating ring singularities, two accelerating pointlike singularities, or a curious toroidal singularity (depending on the values of the parameters involved in the transformation). The apparent transformation of "one object into two" has to do with the fact that a special conformal transformation must be more properly applied in conformally compactified Minkowski space, and from the point of view of this space, a Coulomb field is actually double-valued (as it changes sign when crossing conformal infinity); see Ref. [12], Sec. 9.4.

While our procedure for linear fields is unambiguous and essentially algorithmic, we argued that an analogous construction for nonlinear fields is not so clear, at least not in the way in which the usual nonlinear Newman-Janis shift is performed (that is, at the metric level). We illustrated this with a "complex transformation" that relates the (nonlinear) Schwarzschild and C metrics, but we noticed that the procedure is completely artificial and nonunique. Part of the difficulty has to do with the fact that one is attempting to perform the complex transformation at the metric level, whereas in field theory the transformation is done in the curvature tensor (Maxwell, Weyl, and higher spin), which is a holomorphic object as it has definite chirality. In any case, since the recent applications of the Newman-Janis shift to amplitudes [4,5] make use of the field strength version of the trick, it is possible that the approach in this paper can be applied to examine other kinds of scattering processes.

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APPENDIX: CONVENTIONS

The complexified spin group $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})$ has two independent basic representations, $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$; we say that they have "opposite" chirality, and we indicate this with the two different kinds of indices $A,B,\ldots$ and $A',B',\ldots$, which take values 0,1 in both cases. The vector representation is $(\frac{1}{2},\frac{1}{2})\cong(\frac{1}{2},0)\otimes(0,\frac{1}{2})$, so it has indices $a\cong AA'$, $b\cong BB'$, etc. Accordingly, the (complexified) Minkowskian coordinates (t,x,y,z) of a point x^a are encoded in the 2×2 matrix $x^{AA'}$ as

$$x^{00'}=\frac{1}{\sqrt{2}}(t+z),\qquad x^{01'}=\frac{1}{\sqrt{2}}(x+\mathrm{i}y),$$
$$x^{10'}=\frac{1}{\sqrt{2}}(x-\mathrm{i}y),\qquad x^{11'}=\frac{1}{\sqrt{2}}(t-z).\tag{A1}$$

These components can be thought of as taken with respect to two constant spin dyads (o_A,ι_A), $(o_{A'},\iota_{A'})$ (which are in general not complex conjugates): $x^{00'}=x^{AA'}o_Ao_{A'}$, $x^{01'}=x^{AA'}o_A\iota_{A'}$, $x^{10'}=x^{AA'}\iota_Ao_{A'}$, $x^{11'}=x^{AA'}\iota_A\iota_{A'}$. We raise and lower spinor indices with the (skew-symmetric) spin metrics ϵ_{AB} and $\epsilon_{A'B'}$ and their inverses, according to $\omega^A=\epsilon^{AB}\omega_B$, $\varphi_A=\varphi^B\epsilon_{BA}$, etc.

Similarly, the Minkowskian components of a vector field $V=V^t\partial_t+V^x\partial_x+V^y\partial_y+V^z\partial_z\cong(V^t,V^x,V^y,V^z)$ are equivalently encoded in the spinor components

$$V^{00'}=\frac{1}{\sqrt{2}}(V^t+V^z),\qquad V^{01'}=\frac{1}{\sqrt{2}}(V^x+\mathrm{i}V^y),$$
$$V^{10'}=\frac{1}{\sqrt{2}}(V^x-\mathrm{i}V^y),\qquad V^{11'}=\frac{1}{\sqrt{2}}(V^t-V^z).\tag{A2}$$

The inverse transformation is

$$V^t=\frac{(V^{00'}+V^{11'})}{\sqrt{2}},\qquad V^x=\frac{(V^{01'}+V^{10'})}{\sqrt{2}},$$
$$V^y=\frac{(V^{01'}-V^{10'})}{\sqrt{2}\mathrm{i}},\qquad V^z=\frac{(V^{00'}-V^{11'})}{\sqrt{2}}.\tag{A3}$$

Putting $V_{AA'}=\epsilon_{AB}\epsilon_{A'B'}V^{BB'}$, we also have

$$V_{00'}=V^{11'},\qquad V_{10'}=-V^{01'},$$
$$V_{01'}=-V^{10'},\qquad V_{11'}=V^{00'}.\tag{A4}$$

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