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Conservative scattering of spinning black holes at fourth post-Minkowskian order

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Using the $\mathcal{N}=1$ supersymmetric, spinning worldline quantum field theory formalism we compute the conservative spin-orbit part of the momentum impulse Δp_i^μ, spin kick ΔS_i^μ and scattering angle θ from the scattering of two spinning massive bodies (black holes or neutron stars) up to fourth post-Minkowskian (PM) order. These three-loop results extend the state-of-the-art for generically spinning binaries from 3PM to 4PM. They are obtained by employing recursion relations for the integrand construction and advanced multi-loop Feynman integral technology in the causal (in-in) worldline quantum field theory framework to directly produce classical observables. We focus on the conservative contribution (including tail effects) and outline the computations for the dissipative contributions as well. Our spin-orbit results agree with N^{3}LO post-Newtonian and test-body data in the respective limits. We also re-confirm the conservative 4PM non-spinning results.

High-precision predictions for the gravitational waves emitted from the interaction of compact binaries are essential for data analysis of gravitational wave detectors [1–6]. They are the prerequisites to address fundamental questions in astro-, gravitational, particle and nuclear physics through observations of gravitational waves. The third generation of detectors — LISA, Einstein Telescope and Cosmic Explorer [7], scheduled to go online in the 2030s — will reach an experimental accuracy that goes well beyond the present state-of-the-art in analytical and numerical gravitational wave physics [8, 9]. This situation has sparked a renewed effort to extend and innovate traditional approaches to the classical relativistic two-body problem.

The early inspiral phase of a bound two-body system, or a small-deflection scattering scenario, is characterized by a scale separation between the relative distance of the compact bodies and their sizes. Here, the weakness of the gravitational field enables an analytic, perturbative treatment: one models black holes (BHs) or neutron stars (NSs) as two massive, spinning point particles that interact gravitationally, controlled by a perturbative expansion in Newton's coupling G [10–12]. In addition, finite-size and tidal effects may be included in the point-particle model by coupling higher-dimensional operators to the particle's worldline theory in the logic of effective field theory (EFT) [13–15]. Typically for bound orbits this is done in a post-Newtonian (PN) expansion in *both* G and the relative velocity v of the bodies [16–27]; however, quantum field theory (QFT) based methods using the post-Minkowskian (PM) expansion solely in G are rapidly developing — see [28–30] for reviews.

These PM-based QFT methods derive from the well-studied perturbative quantization of Einstein's theory of gravity about flat space-time backgrounds. Here, state-of-the-art Feynman integral technology may be fruitfully ported to the realm of classical general relativity: integration by parts methods (IBP) for a reduction to master integrals [31–33], their computation via differential equations [33, 34] and the method of regions to determine boundary values [35]. Clearly, the natural habitat for the PM expansion is the scattering scenario. Still, the scattering data in the conservative sector informs also the (conservative) bound case through a matching to a real [36–42] or effective-one-body [43–47] Hamiltonian.

At present two complementary QFT-based approaches are being pursued: based on scattering amplitudes [37, 48–60] and worldlines [61–77]. In the scattering scenario there are three key observables: the deflection of momentum (known as impulse), the change of spin vectors (known as spin kick) and the Bremsstrahlung (waveform) in the far-field limit. Ignoring finite-size, tidal and spin effects the impulse is known at 3PM (two-loop) [51, 52, 54, 62, 63, 77] and 4PM (three-loop) [64, 65, 78–82] order. Preliminary work has also begun on the 5PM scattering angle, using electrodynamics as a toy model [83]. In the case of spinning binaries the impulse is known up to quintic spin interactions [84, 85] at 2PM order, and up to quadratic spin interactions (including the spin kick) at 3PM order [42, 74][41] together with partial results at all spin orders [85–88]. Leading tidal effects have been computed at 2PM [89, 90] and 3PM order [63, 75, 91]. For the Bremsstrahlung waveform the leading-order result without [66, 71, 92–95] and with spin [69, 72] or tidal effects [68, 75] was recently updated to next-to-leading order for the non-spinning case [96–99].

In this Letter we provide the conservative, spin-orbit contributions to the impulse and spin kick at 4PM accuracy, together with the total scattering angle. These results provide the basis to refine effective one-body Hamiltonians and resummed scattering prescriptions for high-precision gravitational wave physics. Our worldline quantum field theory (WQFT) hinges on three innovations to the EFT approach for gravitational scattering: (i) quantizing *both* the worldline degrees of freedom and the gravitational field allows for a diagrammatic formulation of the classical perturbation theory yielding the observables as one-point functions of the worldline or gravitational fields [70], (ii) capturing the spin of the compact objects through a supersymmetric worldline theory [73],

(iii) the Schwinger-Keldysh (in-in) initial value formulation of WQFT that induces the use of retarded propagators and a causality flow in the diagrammatic expansion [75].

Supersymmetric in-in WQFT formalism. — The effective worldline theory of spinning bodies (Kerr BHs or NSs) with masses m_i and space-time coordinates $x_i^\mu(\tau)$ on a general D-dimensional space-time with metric $g_{\mu\nu}$ is described up to quadratic order in spin by an $\mathcal{N}=2$ supersymmetric worldline theory [73]. As we are focusing on the spin-orbit (linear-in-spin) dynamics here, the $\mathcal{N}=1$ incarnation of this theory will suffice:

$$S=-\sum_{i=1}^{2} m_i\int \mathrm{d}\tau\left[\tfrac{1}{2}g_{\mu\nu}\dot{x}_i^\mu\dot{x}_i^\nu+i\psi_{i,a}\frac{\mathrm{D}\psi_i^a}{\mathrm{D}\tau}\right]+S_{\mathrm{EH}}\,. \quad (1)$$

The real anti-commuting vectors $\psi_i^a(\tau)$ are defined in a flat tangent space using the vierbein e_a^μ and $\frac{\mathrm{D}\psi_i^a}{\mathrm{D}\tau}=\dot{\psi}_i^a+\dot{x}^\mu\omega_\mu{}^{ab}\psi_{i,b}$ with the spin-connection $\omega_\mu{}^{ab}$ (our metric is mostly minus). We work in $D=4-2\epsilon$ dimensions with S_{EH} the bulk Einstein-Hilbert action including a gauge-fixing term; the process of dimensional regularisation, wherein we ultimately send $\epsilon\to 0$, is aided by only this part of the full action needing to be lifted to D dimensions. The $\psi_i^a(\tau)$ carry the spin degrees of freedom with the spin tensors $S_i^{\mu\nu}=-im_i\psi_i^\mu\psi_i^\nu$ and the Pauli-Lubanski vectors $S_i^\mu=m_ia_i^\mu=\frac{1}{2}\epsilon^\mu_{\ \nu\rho\sigma}v_i^\nu S_i^{\rho\sigma}$.

We expand the fields around their respective backgrounds: the metric $g_{\mu\nu}=\eta_{\mu\nu}+\kappa h_{\mu\nu}$, with $\kappa=\sqrt{32\pi G}$, and the worldlines

$$x_i^\mu(\tau)=b_i^\mu+v_i^\mu\tau+z_i^\mu(\tau)\,,\quad \psi_i^\mu(\tau)=\Psi_i^\mu+\psi'^\mu_i(\tau)\,, \quad (2)$$

where $\{b_i^\mu, v_i^\mu, \Psi_i^\mu\}$ are the initial (background) parameters of the two bodies. Using background symmetries we set $b\cdot v_i=\Psi_i\cdot v_i=0$ where $b^\mu=|b|\,\hat{b}^\mu=b_2^\mu-b_1^\mu$ is the covariant impact parameter. We also introduce the Lorentz factor $\gamma=v_1\cdot v_2$ and the relative velocity $v=\sqrt{\gamma^2-1}/\gamma$.

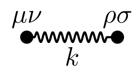
Causal observables including radiative effects arise from the Schwinger-Keldysh (in-in) formalism applied to WQFT [75] where one doubles the fields: $h_{\mu\nu}\to(h_{\mu\nu}^{(1)},h_{\mu\nu}^{(2)})$ and $Z_i^\mu\to(Z_i^{(1)\mu},Z_i^{(2)\mu})$ introducing the worldline "super-fields" $Z_i=\{z_i,\psi_i'\}$. Causal one-point functions follow from the in-in path integral

$$\langle\mathcal{O}\rangle:=\int\mathcal{D}[h_{\mu\nu}^{(1,2)},Z_i^{(1,2)\mu}]e^{i(S[\{\,\}^{(1)}]-S[\{\,\}^{(2)}]^*)}\mathcal{O}\,, \quad (3)$$

normalized such that $\langle 1\rangle=1$ and with $\{\,\}^{(n)}$ denoting the (n)'th copy of the doubled fields. The key property we exploit is that the WQFT *tree-level one-point functions* $\langle Z_i^{(n)}\rangle$ solve the classical equations of motion. Moreover, the computation of one-point functions of in-in WQFT reduces to the use of *retarded* propagators combined with the standard in-out WQFT Feynman rules [75]. This formalism yields an efficient QFT-based scheme to solve the classical equations perturbatively.

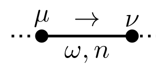
Conservative observable can in turn be defined by neglecting all interactions between $h_{\mu\nu}^{(1)}$ and $h_{\mu\nu}^{(2)}$. This may be achieved by using the in-in formalism only for the worldlines while keeping the in-out formalism for the gravitons and projecting on the real part of observables [100, 101]. This separation of conservative effects at 4PM has proven its efficiency for the non-spinning results [65, 79].

WQFT Feynman rules. — The graviton propagator in de Donder gauge with Feynman prescription reads



$$=\frac{iP_{\mu\nu;\rho\sigma}}{k^2+i0^+}\,, \quad (4)$$

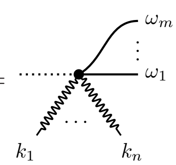
with $P_{\mu\nu;\rho\sigma}:=\eta_{\mu(\rho}\eta_{\sigma)\nu}-\frac{1}{D-2}\eta_{\mu\nu}\eta_{\rho\sigma}$ while the worldline propagators associated with z_i^μ and $\psi_i'^\mu$ read, respectively



$$=\frac{-i\eta^{\mu\nu}}{m_i(\omega+i0^+)^n}\begin{cases}n=2 \text{ for } z_i^\mu\,,\\ n=1 \text{ for } \psi_i'^\mu\,.\end{cases} \quad (5)$$

The arrow on the propagators indicates the momentum or energy flow on the *retarded* propagators. Importantly, the Feynman graviton propagators reflect our focus on conservative observables. Full dissipative results may be obtained by using retarded propagators instead. The Feynman vertices of the spinning WQFT to lower multiplicities have been exposed in [73]. The generic worldline vertex couples n gravitons to m worldline fields and reads

$V_{n|m}=$



$$\sim\; m\kappa^n\, e^{ik\cdot b}\delta\Big(k\cdot v+\sum_{i=1}^n\omega_i\Big)\times \times\begin{pmatrix}\text{polynomial in } \omega_i,k_j\\ \text{of degree } 2n+m\end{pmatrix} \quad (6)$$

where $k^\mu=\sum_{i=1}^n k_i^\mu$ is the total outflowing four-momentum and the dotted outgoing line symbolizes the background parameters $\{b^\mu,v^\mu,\Psi^\mu\}$ of Eq. (2). We see that only energy is conserved on the worldline. The bulk graviton vertices are generic. At 4PM order we need the worldline vertices $V_{n|m}$ above for $\{n=1,\ldots,4; m=0,\ldots,5-n\}$, and the 3-,4-,5-graviton vertices.

Momentum impulse and spin kick. — The momentum impulse $\Delta p_i^\mu:=[p_i^\mu]_{\tau=-\infty}^{\tau=+\infty}$ and spin kick $\Delta S_i^\mu:=[S_i^\mu]_{\tau=-\infty}^{\tau=+\infty}$ follow from the one-point functions

$$\Delta p_i^\mu=m_i\int_{-\infty}^{\infty}\mathrm{d}\tau\left\langle\frac{\mathrm{d}^2x_i^\mu(\tau)}{\mathrm{d}\tau^2}\right\rangle=-m_i\omega^2\left\langle z_i^\mu(\omega)\right\rangle\big|_{\omega=0}\,,$$
$$\Delta\psi_i^\mu=\int_{-\infty}^{\infty}\mathrm{d}\tau\left\langle\frac{\mathrm{d}\psi_i^\mu(\tau)}{\mathrm{d}\tau}\right\rangle=-i\omega\left\langle\psi_i'^\mu(\omega)\right\rangle\big|_{\omega=0}\,, \quad (7)$$

where we have Fourier transformed to momentum space. Both observables are given as the sum of all diagrams at a given PM order with one outgoing Z_i^μ line with vanishing energy. The spin kick is subsequently derived from the kick of the Grassmann variable as in Ref. [42]. The workflow of our computation proceeds as follows:

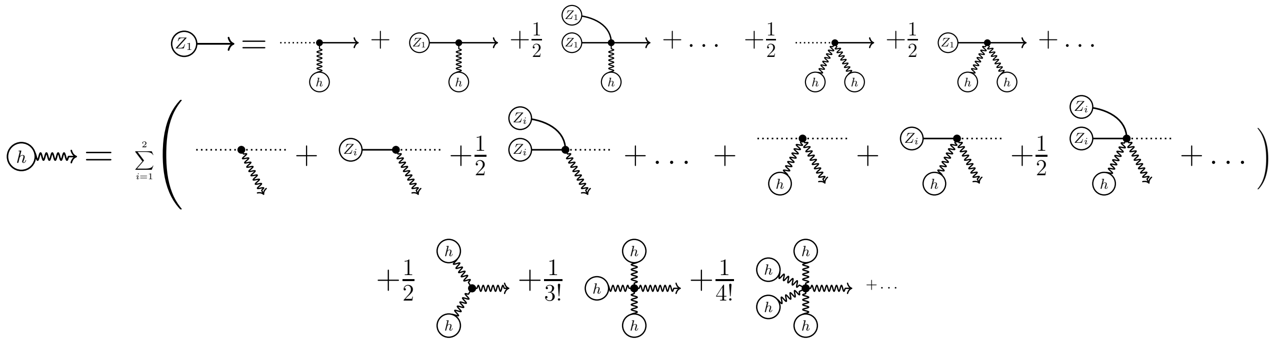


FIG. 1: Berends-Giele type recursion relation to construct $\langle Z_i^\mu(\omega)\rangle$ and $\langle h_{\mu\nu}(k)\rangle$ perturbatively. The causality flow is always from the Z_i and h blobs to the outgoing line. They are equivalent to the PM-expanded geodesic and Einstein equations.

Integrand generation. — The 4PM impulse and spin-kick integrands are generated recursively via Berends-Giele type relations. The one-point functions for the worldline "super-fields" $Z_i = \{z_i, \psi_i'\}$ and for the graviton are represented as

$$\langle Z_i(\omega)\rangle = \textcircled{Z_i}\!\xrightarrow[\omega]{\rightarrow}, \quad \langle h_{\mu\nu}(k)\rangle = \textcircled{h}\!\underset{k}{\overset{\rightarrow}{\rightsquigarrow}}. \qquad (8)$$

Their recursive definitions follow from the Schwinger-Dyson equations and are depicted in Fig. 1. Spelling this out systematically to order G^4 allows for an algorithmic construction of the integrand: in our case, we efficiently inserted Feynman rules into the generated trees using `FORM` [102]. There are 201 graphs contributing to the 4PM impulse in the non-spinning case, 529 with spin and 253 contributing to the 4PM spin kick.

Reduction to scalar integrals. — A generic 4PM diagram after performing the worldline energy integrals via the δ-functions in Eq. (6) takes the form

$$\begin{aligned}&\int_q e^{-iq\cdot b}\delta(q\cdot v_1)\delta(q\cdot v_2)\times \qquad (9)\\ &\quad\int_{\ell_1,\ell_2,\ell_3}\frac{\mathrm{num}[\ell_i]}{D_1\cdots D_{12}}\delta(\ell_1\cdot v_{i_1})\delta(\ell_2\cdot v_{i_2})\delta(\ell_3\cdot v_{i_3}),\end{aligned}$$

where the D_i are either linear or massless propagators depending on the loop momenta ℓ_i, velocities v_i and momentum transfer q. The numerators $\mathrm{num}[\ell_i]$ are polynomial in loop momenta. Tensor reduction of $\mathrm{num}[\ell_i]$ to scalar integrals is performed by expanding the loop momenta on a basis dual to v_i^μ and q^μ, as demonstrated in the 3PM case [74]. The only dimensionful quantity in the 3-loop ℓ_i integral is the momentum transfer q^μ. Hence, $|q| = \sqrt{-q^2}$ may be scaled out, and the remaining 3-loop integrals depend only on the Lorentz factor γ.

The specific choice of three $\delta(\ell_k \cdot v_{i_k})$ functions in Eq. (9) follows the mass dependence of a given diagram, which scales as $m_1 m_2 m_{i_1} m_{i_2} m_{i_3}$. Diagrams are thereby grouped into two categories: test-body contributions with mass dependence $m_1^4 m_2$ or $m_1 m_2^4$ and comparable-mass contributions $m_1^3 m_2^2$, $m_1^2 m_2^3$ — see Fig. 2. For the conservative impulse we can easily reconstruct the $m_1 m_2^4$ and $m_1^2 m_2^3$ components using $\Delta p^\mu_{1,\mathrm{cons}} = -\Delta p^\mu_{2,\mathrm{cons}}$, the impulse on the second body being given simply by relabeling the two worldlines. When computing $\Delta\psi^\mu_{1,\mathrm{cons}}$ no similar relation exists; however, the integrals in opposing mass sectors are also related by a trivial relabeling.

Integral families and reduction to masters. — There are three integral families that need to be reduced to master integrals. The first 4PM family is ($i = 1, 2$)

$$I^{[i](\sigma_1,\sigma_2,\sigma_3)}_{n_1,n_2,\dots,n_{12}} = \int_{\ell_1,\ell_2,\ell_3}\frac{\delta(\ell_1\cdot v_i)\delta(\ell_2\cdot v_1)\delta(\ell_3\cdot v_1)}{D_1^{n_1}D_2^{n_2}\dots D_{12}^{n_{12}}} \qquad (10a)$$

with the propagators ($j = 1, 2, 3$ and $k = 1, 2$):

$$\begin{aligned}&D_1 = \ell_1\cdot v_{\bar{\imath}} + \sigma_1 i0^+,\ D_{1+k} = \ell_{1+k}\cdot v_2 + \sigma_{1+k} i0^+,\\ &D_4 = (\ell_1+\ell_2+\ell_3+q)^2,\ D_5 = (\ell_1+\ell_2+q)^2, \qquad (10b)\\ &D_{5+k} = (\ell_k+\ell_3)^2,\ D_{7+j} = \ell_j^2,\ D_{10+k} = (\ell_k+q)^2,\end{aligned}$$

and $\bar{1} = 2$, $\bar{2} = 1$. The $I^{[1]}$ and $I^{[2]}$ families contribute to the test-body and comparable-mass regimes respectively. The other 4PM family is given by

$$J^{(\sigma_1,\sigma_2,\sigma_3)}_{n_1,n_2,\dots,n_{12}} := \int_{\ell_1,\ell_2,\ell_3}\frac{\delta(\ell_1\cdot v_1)\delta(\ell_2\cdot v_1)\delta(\ell_3\cdot v_2)}{D_1^{n_1}D_2^{n_2}\dots D_{12}^{n_{12}}} \qquad (11a)$$

with ($j = 1, 2, 3$, $k = 1, 2$)

$$\begin{aligned}&D_k = \ell_k\cdot v_2 + \sigma_k i0^+,\ D_3 = \ell_3\cdot v_1 + \sigma_3 i0^+,\\ &D_{3+k} = (\ell_k-\ell_3)^2,\ D_6 = (\ell_1-\ell_2)^2, \qquad (11b)\\ &D_{6+j} = \ell_j^2,\ D_{9+j} = (\ell_j+q)^2.\end{aligned}$$

Each family splits into two branches: even (b-type) or odd (v-type) in the number of worldline propagators. In the non-spinning impulse, these integrals multiply terms proportional to b^μ, v_i^μ respectively (22). Using integration-by-parts (IBP) relations [103–105] we reduce the families to 23 master integrals for the I-b and I-v types each as well as 64 of J-b type and 66 of J-v type. The complete spinning impulse computation (including dissipation) results in approximately 10^5 integrals for reduction to scalar masters.

Differential equations. — To solve for the master integrals we employ the method of canonical differential equations (DEs) [34]. Each master integral family

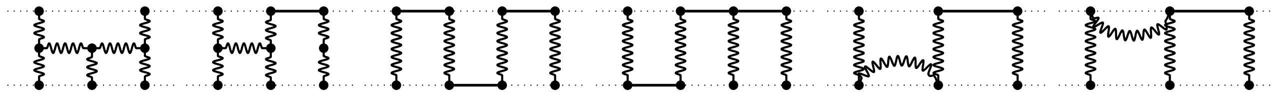


FIG. 2: Examples of comparable-mass graph topologies with mass dependence $m_1^2 m_2^3$ contributing to the 4PM calculation. One should attach an outgoing worldline to *any* worldline node and apply the resulting causality flow. The corresponding scalar integrals feature as top sectors in the differential equations.

is grouped into a vector $\vec{I}$ ordered according to the number of active propagators. The DE in $x = \gamma - \sqrt{\gamma^2-1}$ reads $\mathrm{d}\vec{I}/\mathrm{d}x = M(\epsilon, x)\,\vec{I}$ with a lower-block triangular matrix $M(\epsilon, x)$. Finding a transformation matrix T that brings us to a canonical basis with an ϵ factorized DE $\mathrm{d}\vec{\tilde{I}}/\mathrm{d}x = \epsilon A(x)\,\vec{\tilde{I}}$ is a highly involved procedure in which we employ the packages [106, 107]. The resulting symbol alphabet is $\{x, 1+x, 1-x, 1+x^2\}$, and we encounter elliptic integrals in the J-b family [81, 106].

Fixing boundary conditions . — Boundary conditions on the master integrals are determined in the static limit ($\gamma \to 1$, $v \to 0$) using the *method of regions* [60, 79, 108, 109] to expand the integrand in v. Regions in the static limit are characterized by different scalings of the bulk graviton loop momenta with potential (P) and radiative (R) modes defined by relative scalings of their spacial and timelike components:

$$\ell_i^{\mathrm{P}} = (\ell_i^0, \boldsymbol{\ell}_i) \sim (v, 1)\,, \qquad \ell_i^{\mathrm{R}} = (\ell_i^0, \boldsymbol{\ell}_i) \sim (v, v)\,. \quad (12)$$

Only gravitons which may go on-shell can be radiative and there are at most two of these defining the three regions: (PP), (RR) and (PR+RP). The regions (PP) and (PR+RP) are purely conservative and dissipative respectively, while the (RR) region carries both kinds of effects. In the (PP) region the integrals reduce to test-body integrals described by the $I^{[1]}$ family (10); in the (RR) region they reduce to double-mushroom integrals like the first graph of Fig. 3. Either way, the boundary integrals are independent of γ and thus functions only of $D = 4-2\epsilon$. In our conservative observables we include only the PP and RR regions as the (PR+RP) region generates terms odd in v.

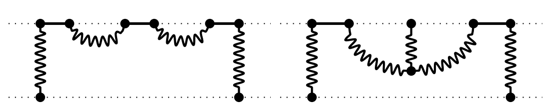


FIG. 3: Two examples of graphs contributing in the RR region but not the PP region.

Reaching 4PM order introduces the physical phenomenon of tail effects [65, 79]. In an observable X (impulse, spin kick or scattering angle) poles in $\epsilon = \frac{4-D}{2}$ appear in the (PP) and conservative (RR) contributions:

$$X^{(\mathrm{PP})} = \frac{P(\gamma)}{2\epsilon} + \dots, \; X^{(\mathrm{RR})}_{\mathrm{cons}} = -v^{-4\epsilon}\frac{P(\gamma)}{2\epsilon} + \dots, \quad (13)$$

higher-order terms being finite as $\epsilon \to 0$. Non-analytic dependence on $v^{-4\epsilon}$ in the (RR) region is a direct consequence of the velocity scaling of the two radiative gravitons. The cancelation of these poles when assembling X_{cons} introduces logarithmic velocity dependence:

$$X_{\mathrm{cons}} = X^{\mathrm{PP}} + X^{\mathrm{RR}}_{\mathrm{cons}} = P(\gamma)\log\frac{\gamma-1}{2} + \dots\,, \quad (14)$$

the dots indicating terms that are rational in $\sqrt{\gamma^2-1}$ in the static limit.

Results. — We begin with $\Delta p^{(4)\mu}_{i,\mathrm{cons}}$, the G^4 component of the impulse $\Delta p^{\mu}_{i,\mathrm{cons}}$. It may be decomposed as

$$\Delta p^{(4)\mu}_{\mathrm{cons},1} = \frac{m_1^2 m_2^2}{|b|^4}\sum_{l,\sigma=b,v}\rho_l^{(\sigma)\mu}\Bigg[\Big(\frac{m_2^2}{m_1}c_l^{(\sigma)}(\gamma) + \frac{m_1^2}{m_2}\bar{c}_l^{(\sigma)}(\gamma)\Big) + \sum_\alpha F_\alpha^{(\sigma)}(\gamma)\Big(m_2 d^{(\sigma)}_{\alpha,l}(\gamma) + m_1\bar{d}^{(\sigma)}_{\alpha,l}(\gamma)\Big)\Bigg]\,, \quad (15)$$

where the basis vectors and spin structures $\rho_l^{(b,v)\mu}$ are

$$\rho_l^{(b)\mu} = \left\{\hat{b}^\mu, \frac{a_i\cdot\hat{L}}{|b|}\,\hat{b}^\mu, \frac{a_i\cdot\hat{b}}{|b|}\,\hat{L}^\mu\right\}\,, \quad \rho_l^{(v)\mu} = \left\{v_j^\mu, \frac{a_i\cdot\hat{L}}{|b|}\,v_j^\mu, \frac{a_i\cdot v_{\bar{\imath}}}{|b|}\,\hat{L}^\mu\right\}\,. \quad (16)$$

There are five and eight elements in $\rho_l^{(b)\mu}$ and $\rho_l^{(v)\mu}$ respectively, and the normalized angular momentum $\hat{L}^\mu = \epsilon_{\nu\rho\sigma}{}^{\mu}v_1^\nu v_2^\rho \hat{b}^\sigma/\gamma v$. The $c_l^{(\sigma)}(\gamma)$ and $d^{(\sigma)}_{\alpha,l}(\gamma)$ and their barred counterparts are rational functions (up to integer powers of $\sqrt{\gamma^2-1}$). All non-trivial dependence on γ is contained in the 16 functions $F_\alpha^{(b)}(\gamma)$ with $\gamma_\pm = \gamma \pm 1$:

$$\begin{aligned} F_\alpha^{(b)}(\gamma) = \{&1, \mathrm{arccosh}[\gamma], \log[\gamma], \log\left[\frac{\gamma_\pm}{2}\right], \\ &\mathrm{arccosh}^2[\gamma], \mathrm{arccosh}[\gamma]\log\left[\frac{\gamma_\pm}{2}\right], \log\left[\frac{\gamma_+}{2}\right]\log\left[\frac{\gamma_-}{2}\right], \\ &\log^2\left[\frac{\gamma_+}{2}\right], \mathrm{Li}_2\left[\pm\frac{\gamma_-}{\gamma_+}\right], \mathrm{Li}_2\left[\sqrt{\frac{\gamma_-}{\gamma_+}}\right], \\ &\mathrm{K}^2\left[\frac{\gamma_-}{\gamma_+}\right], \mathrm{E}^2\left[\frac{\gamma_-}{\gamma_+}\right], \mathrm{K}\left[\frac{\gamma_-}{\gamma_+}\right]\mathrm{E}\left[\frac{\gamma_-}{\gamma_+}\right]\}\,, \end{aligned} \quad (17)$$

and the much simpler set $F_\alpha^{(v)} = \{1, \mathrm{arccosh}[\gamma]\}$. The first line of Eq. (17) includes transcendental weight-1 functions, the second and third lines weight-2 functions and the final line quadratic combinations of elliptic functions of the first and second kind.

The G^4 component of the spin kick $\Delta S^{(4)\mu}_{i,\rm cons}$ admits a similar decomposition, involving the same functions $F^{(b,v)}_\alpha$ but a different set of basis vectors and spin structures — see the supplementary material for details. As checks on these two observables we have confirmed: (i) the cancellation of all $1/\epsilon$ poles occurring between the (PP) and (RR) regions; (ii) conservation of p_i^2, S_i^2 and the $\mathcal{N}=1$ global supercharge $Q_i=p_i\cdot\psi_i$. While the first two only check the simpler terms carrying $F^{(v)}_\alpha$, the latter also compares $F^{(b)}_\alpha$ terms between $\Delta p^{(4)\mu}_{i,\rm cons}$ and $\Delta\psi^{(4)\mu}_{i,\rm cons}$, and thus $\Delta S^{(4)\mu}_{i,\rm cons}$.

We also define the total scattering angle θ for generic spin configurations as

$$\sin\frac{\theta}{2}=\frac{|\Delta p^\mu_{i,\rm cons}|}{2p_\infty},\qquad \theta=\frac{E}{M}\sum_{n,m}\left(\frac{GM}{|b|}\right)^n\frac{\theta^{(n,m)}}{|b|^m}\,, \tag{18}$$

with $p_\infty=m_1m_2\sqrt{\gamma^2-1}/E$, total energy $E=|p_1^\mu+p_2^\mu|$ and total mass $M=m_1+m_2$, n and m counting PM and spin orders respectively. The 4PM spin-orbit contribution is

$$\begin{aligned}\theta^{(4,1)}_{\rm cons}=&\sum_{\alpha=1}^{16}\pi\nu\left(s_+h^{(+)}_\alpha(\gamma)+\delta\, s_-h^{(-)}_\alpha(\gamma)\right)F^{(b)}_\alpha(\gamma)\\&-\frac{21\pi\gamma\left(33\gamma^4-30\gamma^2+5\right)\left(13s_+-3\delta s_-\right)}{32\left(\gamma^2-1\right)^{5/2}}\,,\end{aligned} \tag{19}$$

where the test-body contributions (second line) agree with the geodesic motion in a Kerr background [110]. Here we use $\nu=m_1m_2/M^2$, $\delta=(m_2-m_1)/M$ and $s_\pm=(a_1\pm a_2)\cdot\hat{L}$; the 32 polynomial functions $h^{(\pm)}_\alpha(\gamma)$ are given in the supplementary material (20). We have checked this result against the corresponding N^{3}LO PN [111, 112] literature, and found agreement by taking the PN expansion. All of our results are included in an ancillary file attached to the `arXiv` submission of this Letter.

Outlook. — Having produced a complete set of 4PM linear-in-spin conservative scattering observables — and successfully compared them with N^{3}LO spin-orbit PN [111, 112]— our next step will be upgrading them to include dissipative effects, as has already been done in the non-spinning case [80, 81]. This will require two changes to our setup: retarded graviton propagators in place of time-symmetric Feynman (see Ref. [75]) and incorporation of the (PR+RP) regions when fixing boundary conditions on master integrals. Notwithstanding the added complexity, quadratic-in-spin order is also an achievable target — corresponding N^{3}LO quadratic-in-spin PN results are already available [113, 114]. In the near future we also seek to use these results to describe bound orbits, the main obstacle being the aforementioned tail effect [65, 79]. Recent Numerical Relativity simulations of spinning black holes on hyperbolic-like orbits [115] also offer us future numerical comparisons of the scattering angle θ.

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SUPPLEMENTARY MATERIAL

Scattering angle. — The 32 rational functions $h_\alpha^{\pm}(\gamma)$ (up to integer powers of $\sqrt{\gamma^2-1}$) appearing in the spin-orbit contribution to the scattering angle (19) take the explicit form

$$\begin{aligned}
h_1^{(+)} &= \frac{3\pi^2(\gamma+1)^2(1225\gamma^8+1225\gamma^7-1875\gamma^6-1875\gamma^5+795\gamma^4+3035\gamma^3-3601\gamma^2+1775\gamma-448)}{192(\gamma+1)^3\sqrt{\gamma^2-1}} - \frac{1}{192\gamma^8(\gamma+1)(\gamma^2-1)^{5/2}}\Big(22050\gamma^{19} \\
&\quad + 33075\gamma^{18} - 71725\gamma^{17} - 123397\gamma^{16} + 186555\gamma^{15} + 67503\gamma^{14} - 89885\gamma^{13} - 190167\gamma^{12} + 181103\gamma^{11} + 137042\gamma^{10} - 506830\gamma^9 \\
&\quad + 407004\gamma^8 - 33671\gamma^7 - 33671\gamma^6 + 8501\gamma^5 + 8501\gamma^4 - 1885\gamma^3 - 1885\gamma^2 + 315\gamma + 315\Big) \\
h_1^{(-)} &= \frac{3\pi^2(-1225\gamma^8-1225\gamma^7+1875\gamma^6+1875\gamma^5-795\gamma^4-1115\gamma^3+401\gamma^2-111\gamma+64)(\gamma+1)^2}{192(\gamma+1)^3\sqrt{\gamma^2-1}} + \frac{1}{192\gamma^8(\gamma+1)(\gamma^2-1)^{5/2}}\Big(22050\gamma^{19} \\
&\quad + 33075\gamma^{18} - 71725\gamma^{17} - 115333\gamma^{16} + 96699\gamma^{15} + 140871\gamma^{14} - 56261\gamma^{13} - 73191\gamma^{12} - 6593\gamma^{11} + 27498\gamma^{10} - 3718\gamma^9 \\
&\quad + 9004\gamma^8 - 1491\gamma^7 - 1491\gamma^6 + 313\gamma^5 + 313\gamma^4 + 95\gamma^3 + 95\gamma^2 - 105\gamma - 105\Big) \\
h_2^{(+)} &= -\frac{\gamma\left(2\gamma^2-3\right)\left(420\gamma^6-1180\gamma^5-875\gamma^4-2587\gamma^3+10515\gamma^2-8829\gamma+2752\right)}{8(\gamma-1)^3(\gamma+1)^4} \\
h_2^{(-)} &= -\frac{\gamma\left(2\gamma^2-3\right)\left(160\gamma^5-25\gamma^4+199\gamma^3-739\gamma^2+493\gamma-112\right)}{8(\gamma-1)^3(\gamma+1)^4} \\
h_3^{(+)} &= \frac{-3675\gamma^{10}+11750\gamma^8-13720\gamma^6-10342\gamma^4-74445\gamma^2-11200}{16\left(\gamma^2-1\right)^{5/2}} \qquad h_3^{(-)} = \frac{3675\gamma^{10}-11750\gamma^8+13720\gamma^6-5018\gamma^4+5837\gamma^2+448}{16\left(\gamma^2-1\right)^{5/2}} \\
h_4^{(+)} &= \frac{3675\gamma^{10}-11750\gamma^8+1680\gamma^7+17940\gamma^6+27396\gamma^5+20930\gamma^4+85992\gamma^3+150881\gamma^2+71748\gamma+21588}{32\left(\gamma^2-1\right)^{5/2}} \\
h_4^{(-)} &= \frac{-3675\gamma^{10}+11750\gamma^8-14028\gamma^6-2724\gamma^5+3478\gamma^4-6872\gamma^3-10473\gamma^2-2884\gamma-876}{32\left(\gamma^2-1\right)^{5/2}} \\
h_5^{(+)} &= \frac{420\gamma^6-635\gamma^5-330\gamma^4-2977\gamma^3+10125\gamma^2-8984\gamma+2597}{8(\gamma-1)(\gamma+1)^2\sqrt{\gamma^2-1}} \qquad h_5^{(-)} = \frac{77\gamma^5-108\gamma^4+277\gamma^3-661\gamma^2+498\gamma-107}{8(\gamma-1)(\gamma+1)^2\sqrt{\gamma^2-1}} \\
h_6^{(+)} &= -9h_6^{(-)} = -\frac{27\gamma^3\left(3-2\gamma^2\right)^2\left(7\gamma^2-3\right)}{16\left(\gamma^2-1\right)^{7/2}} \qquad h_7^{(+)} = -9h_7^{(-)} = \frac{27\gamma^2\left(14\gamma^6-201\gamma^4+212\gamma^2+87\right)}{8\left(\gamma^2-1\right)^3} \\
h_8^{(+)} &= -9h_8^{(-)} = \frac{27\gamma^2\left(14\gamma^4-27\gamma^2+9\right)}{8\left(\gamma^2-1\right)^2} \qquad h_9^{(+)} = -\frac{3\left(63\gamma^4+203\gamma^3-247\gamma^2+89\gamma-28\right)}{4(\gamma+1)\sqrt{\gamma^2-1}} \qquad h_9^{(-)} = -\frac{3\left(-7\gamma^4-27\gamma^3+23\gamma^2-9\gamma+4\right)}{4(\gamma+1)\sqrt{\gamma^2-1}} \\
h_{10}^{(+)} &= \frac{3\left(35\gamma^4+180\gamma^3+84\gamma^2+72\gamma+7\right)}{\left(\gamma^2-1\right)^{3/2}} \qquad h_{10}^{(-)} = \frac{3\left(-5\gamma^4-20\gamma^3-8\gamma^2-8\gamma-1\right)}{\left(\gamma^2-1\right)^{3/2}} \\
h_{11}^{(+)} &= \frac{3\left(2240\gamma^6-3520\gamma^5-1504\gamma^4+4832\gamma^3-2784\gamma^2+672\gamma\right)}{64(\gamma-1)^2(\gamma+1)^3} \\
&\quad + \frac{3\left(-1225\gamma^{12}-1225\gamma^{11}+4325\gamma^{10}+4325\gamma^9-5770\gamma^8-3530\gamma^7+11546\gamma^6+12442\gamma^5-1277\gamma^4-9341\gamma^3-7151\gamma^2-2671\gamma-448\right)}{64(\gamma-1)^2(\gamma+1)^3\sqrt{\gamma^2-1}} \\
h_{11}^{(-)} &= \frac{3\left(-320\gamma^6+320\gamma^5+288\gamma^4-416\gamma^3+288\gamma^2-96\gamma\right)}{64(\gamma-1)^2(\gamma+1)^3} \\
&\quad + \frac{3\left(1225\gamma^{12}+1225\gamma^{11}-4325\gamma^{10}-4325\gamma^9+5770\gamma^8+5450\gamma^7-4506\gamma^6-4378\gamma^5+1149\gamma^4+1789\gamma^3+623\gamma^2+239\gamma+64\right)}{64(\gamma-1)^2(\gamma+1)^3\sqrt{\gamma^2-1}} \\
h_{12}^{(+)} &= \frac{3\gamma\left(1225\gamma^8-3100\gamma^6+2670\gamma^4+4884\gamma^2+2385\right)}{32\left(\gamma^2-1\right)^{3/2}} \qquad h_{12}^{(-)} = \frac{3\gamma\left(-1225\gamma^8+3100\gamma^6-2670\gamma^4+236\gamma^2-337\right)}{32\left(\gamma^2-1\right)^{3/2}} \\
h_{13}^{(+)} &= -\frac{6\gamma\left(2\gamma^2-3\right)\left(35\gamma^3-55\gamma^2+29\gamma-7\right)}{(\gamma-1)^2(\gamma+1)^3} \qquad h_{13}^{(-)} = -\frac{6\gamma\left(2\gamma^2-3\right)\left(-5\gamma^3+5\gamma^2-3\gamma+1\right)}{(\gamma-1)^2(\gamma+1)^3} \\
h_{14}^{(+)} &= \frac{120\gamma^4+13740\gamma^3+31705\gamma^2+22164\gamma+5167}{8\left(\gamma^2-1\right)^{5/2}} \qquad h_{14}^{(-)} = \frac{120\gamma^4-660\gamma^3-2515\gamma^2-2468\gamma-733}{8\left(\gamma^2-1\right)^{5/2}} \\
h_{15}^{(+)} &= \frac{5\gamma\left(24\gamma^4+3509\gamma^2+3539\right)}{8(\gamma-1)^2(\gamma+1)\sqrt{\gamma^2-1}} \qquad h_{15}^{(-)} = \frac{\gamma\left(120\gamma^4-1135\gamma^2-2113\right)}{8(\gamma-1)^2(\gamma+1)\sqrt{\gamma^2-1}} \\
h_{16}^{(+)} &= \frac{-240\gamma^5-13860\gamma^4-35090\gamma^3-53869\gamma^2-35390\gamma-5167}{8\left(\gamma^2-1\right)^{5/2}} \qquad h_{16}^{(-)} = \frac{-240\gamma^5+540\gamma^4+2270\gamma^3+4983\gamma^2+4226\gamma+733}{8\left(\gamma^2-1\right)^{5/2}}
\end{aligned} \tag{20}$$

Here the $(\pm)$ upper indices label the coupling to the spin-orbit components $s_\pm = (a_1 \pm a_2)\cdot \hat{L}$ via s_+ to $h_\alpha^{(+)}$ and $\delta\, s_-$ to $h_\alpha^{(-)}$. The indices $\alpha = 1,\ldots,16$ correspond to the entries of the function space vector $F_\alpha^{(b)}$, i.e.

$$F_\alpha^{(b)}(\gamma) = \{1, \text{arccosh}[\gamma], \log[\gamma], \log\left[\frac{\gamma_+}{2}\right], \log\left[\frac{\gamma_-}{2}\right], \text{arccosh}^2[\gamma], \text{arccosh}[\gamma]\log\left[\frac{\gamma_+}{2}\right], \text{arccosh}[\gamma]\log\left[\frac{\gamma_-}{2}\right], \quad (21)$$
$$\log\left[\frac{\gamma_+}{2}\right]\log\left[\frac{\gamma_-}{2}\right], \log^2\left[\frac{\gamma_+}{2}\right], \text{Li}_2\left[\frac{\gamma_-}{\gamma_+}\right], \text{Li}_2\left[-\frac{\gamma_-}{\gamma_+}\right], \text{Li}_2\left[\sqrt{\frac{\gamma_-}{\gamma_+}}\right], \text{K}^2\left[\frac{\gamma_-}{\gamma_+}\right], \text{E}^2\left[\frac{\gamma_-}{\gamma_+}\right], \text{K}\left[\frac{\gamma_-}{\gamma_+}\right]\text{E}\left[\frac{\gamma_-}{\gamma_+}\right]\},$$

which provides a precise ordering for the list in Eq. (17).

Spin kick. — The spin kick may be expanded in terms of the basis vectors

$$\tilde{\rho}^{(b)\mu} = \left\{\hat{b}^\mu a_1\cdot v_2, \hat{b}\cdot a_1\, v_1^\mu, \hat{b}\cdot a_1\, v_2^\mu\right\}, \qquad \tilde{\rho}^{(v)\mu} = \left\{\hat{b}\cdot a_1\,\hat{b}^\mu, \hat{b}\cdot a_2\,\hat{b}^\mu, a_1\cdot v_2\, v_2^\mu\right\}, \quad (22)$$

and takes the schematic form

$$\Delta S^{(4)\mu}_{\text{cons},1} = \frac{m_1^2 m_2^2}{|b|^4}\sum_{l,\sigma}\tilde{\rho}_l^{(\sigma)\mu}\left[\left(\frac{m_2^2}{m_1}e_l^{(\sigma)}(\gamma) + \frac{m_1^2}{m_2}\bar{e}_l^{(\sigma)}(\gamma)\right) + \sum_\alpha F_\alpha^{(\sigma)}(\gamma)\Big(m_2 f_{\alpha,l}^{(\sigma)}(\gamma) + m_1 \bar{f}_{\alpha,l}^{(\sigma)}(\gamma)\Big)\right]. \quad (23)$$

Here $e_l^{(\sigma)}(\gamma)$ and $f_{\alpha,l}^{(\sigma)}(\gamma)$ and their barred counterparts are rational functions (again, up to integer powers of $\sqrt{\gamma^2-1}$). For the full expression we refer the reader to the ancillary file.

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