# Conservative Scattering of Spinning Black Holes at Fourth Post-Minkowskian Order

Gustav Uhre Jakobsen,[1,2,*] Gustav Mogull,[1,2,†] Jan Plefka,[1,‡] Benjamin Sauer,[1,§] and Yingxuan Xu[3,∥]

[1]*Institut für Physik und IRIS Adlershof, Humboldt Universität zu Berlin, Zum Großen Windkanal 2, 12489 Berlin, Germany*
[2]*Max Planck Institute for Gravitational Physics (Albert Einstein Institute), Am Mühlenberg 1, 14476 Potsdam, Germany*
[3]*Institut für Physik, Humboldt Universität zu Berlin, Newtonstraße 15, 12489 Berlin, Germany*

(Received 26 June 2023; accepted 18 September 2023; published 11 October 2023)

Using the $\mathcal{N}=1$ supersymmetric, spinning worldline quantum field theory formalism, we compute the conservative spin-orbit part of the momentum impulse $\Delta p_i^\mu$, spin kick $\Delta S_i^\mu$, and scattering angle $\theta$ from the scattering of two spinning massive bodies (black holes or neutron stars) up to fourth post-Minkowskian (PM) order. These three-loop results extend the state of the art for generically spinning binaries from 3PM to 4PM. They are obtained by employing recursion relations for the integrand construction and advanced multiloop Feynman integral technology in the causal (in-in) worldline quantum field theory framework to directly produce classical observables. We focus on the conservative contribution (including tail effects) and outline the computations for the dissipative contributions as well. Our spin-orbit results agree with next-to-next-to-next-to-leading-order post-Newtonian and test-body data in the respective limits. We also reconfirm the conservative 4PM nonspinning results.

DOI: 10.1103/PhysRevLett.131.151401

High-precision predictions for the gravitational waves emitted from the interaction of compact binaries are essential for data analysis of gravitational wave detectors [1–6]. They are the prerequisites to address fundamental questions in astro-, gravitational, particle, and nuclear physics through observations of gravitational waves. The third generation of detectors—LISA, Einstein Telescope, and Cosmic Explorer [7] scheduled to go on-line in the 2030s—will reach an experimental accuracy that goes well beyond the present state of the art in analytical and numerical gravitational wave physics [8,9]. This situation has sparked a renewed effort to extend and innovate traditional approaches to the classical relativistic two-body problem.

The early inspiral phase of a bound two-body system, or a small-deflection scattering scenario, is characterized by a scale separation between the relative distance of the compact bodies and their sizes. Here, the weakness of the gravitational field enables an analytic, perturbative treatment: one models black holes (BHs) or neutron stars (NSs) as two massive, spinning point particles that interact gravitationally, controlled by a perturbative expansion in Newton's coupling $G$ [10–12]. In addition, finite-size and tidal effects may be included in the point-particle model by coupling higher-dimensional operators to the particle's worldline theory in the logic of effective field theory (EFT) [13–15]. Typically, for bound orbits this is done in a post-Newtonian (PN) expansion in both $G$ and the relative velocity $v$ of the bodies, both in the traditional [11,16–19] and effective-field-theory-based [13–15,20–30] approaches. The present state of the art is approaching the 5PN level [29–36] including next-to-next-to-next-to-leading-order (N$^3$LO) spin effects [37–43].

However, quantum field theory (QFT)-based methods using the PM expansion [44–47], i.e., perturbative in $G$ but exact in $v$, are rapidly developing (see Refs. [48–50] for reviews): They derive from the well-studied perturbative quantization of Einstein's theory of gravity about flat space-time backgrounds. Here, state-of-the-art Feynman integral technology may be fruitfully ported to the realm of classical general relativity: integration by parts (IBP) methods for a reduction to master integrals [51–53], their computation via differential equations [53,54], and the method of regions to determine boundary values [55]. Clearly, the natural habitat for the PM expansion is the scattering scenario. Still, the scattering data in the conservative sector inform also the (conservative) bound case through a matching to a real [56–62] or effective-one-body [63–67] Hamiltonian.

At present, two complementary QFT-based approaches to the post-Minkowskian (PM) expansion are being pursued: based on scattering amplitudes [57,68–84] and worldlines [85–99]. In the scattering scenario, there are three key observables: the deflection of momentum (known as impulse), the change of spin vectors (known as spin kick), and the bremsstrahlung (waveform) in the far-field limit. Ignoring finite-size, tidal, and spin effects, the impulse is known at 3PM (two-loop) [73,74,76,84,86,87] and 4PM (three-loop) [88,89,100–104] order. Preliminary work has also begun on the 5PM scattering angle using electrodynamics as a toy model [105]. In the case of spinning binaries, the impulse is known up to quintic spin interactions [106,107] at 2PM order, and up to quadratic spin interactions (including the spin kick) at 3PM order [61,62,98] together with partial results at all spin orders [107–111]. Leading tidal effects have been computed at 2PM [112,113] and 3PM order [87,99,114]. For the bremsstrahlung waveform, the leading-order result without [90,95,115–118] and with spin [93,96] or tidal effects [92,99] was recently updated to next-to-leading order for the nonspinning case [119–122].

In this Letter, we provide the conservative, spin-orbit contributions to the impulse and spin kick at 4PM accuracy, together with the total scattering angle. These results provide the basis to refine effective-one-body Hamiltonians and resummed scattering prescriptions for high-precision gravitational wave physics. Our worldline quantum field theory (WQFT) hinges on three innovations to the EFT approach for gravitational scattering: (i) quantizing both the worldline degrees of freedom and the gravitational field allows for a diagrammatic formulation of the classical perturbation theory yielding the observables as one-point functions of the worldline or gravitational fields [94], (ii) capturing the spin of the compact objects through a supersymmetric worldline theory [97], and (iii) the Schwinger-Keldysh (in-in) initial value formulation of WQFT that induces the use of retarded propagators and a causality flow in the diagrammatic expansion [99].

*Supersymmetric in-in WQFT formalism.*—The effective worldline theory of spinning bodies (Kerr BHs or NSs) with masses $m_i$ and space-time coordinates $x_i^\mu(\tau)$ on a general $D$-dimensional space-time with metric $g_{\mu\nu}$ is described up to quadratic order in spin by an $\mathcal{N}=2$ supersymmetric worldline theory [97]. As we are focusing on the spin-orbit (linear-in-spin) dynamics here, the $\mathcal{N}=1$ incarnation of this theory will suffice:

$$S = -\sum_{i=1}^{2} m_i \int d\tau \left[\frac{1}{2} g_{\mu\nu}\dot{x}_i^\mu \dot{x}_i^\nu + i\psi_{i,a}\frac{D\psi_i^a}{D\tau}\right] + S_{\mathrm{EH}}. \qquad (1)$$

The real anticommuting vectors $\psi_i^a(\tau)$ are defined in a flat tangent space using the vierbein $e_a^\mu$ and $(D\psi_i^a/D\tau) = \dot{\psi}_i^a + \dot{x}_i^\mu \omega_\mu^{ab}\psi_{i,b}$ with the spin connection $\omega_\mu^{ab}$ (our metric is mostly minus). We work in $D = 4-2\epsilon$ dimensions with $S_{\mathrm{EH}}$ the bulk Einstein-Hilbert action including a gauge-fixing term; the process of dimensional regularization, wherein we ultimately send $\epsilon \to 0$, is aided by only this part of the full action needing to be lifted to $D$ dimensions. The $\psi_i^a(\tau)$ carry the spin degrees of freedom with the spin tensors $S_i^{\mu\nu} = -im_i\psi_i^\mu\psi_i^\nu$ and the Pauli-Lubanski vectors $S_i^\mu = m_i a_i^\mu = \frac{1}{2}\epsilon^\mu_{\nu\rho\sigma}v_i^\nu S_i^{\rho\sigma}$.

We expand the fields around their respective backgrounds: the metric $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, with $\kappa = \sqrt{32\pi G}$, and the worldlines

$$x_i^\mu(\tau) = b_i^\mu + v_i^\mu\tau + z_i^\mu(\tau), \qquad \psi_i^\mu(\tau) = \Psi_i^\mu + \psi_i'^\mu(\tau), \qquad (2)$$

where $\{b_i^\mu, v_i^\mu, \Psi_i^\mu\}$ are the initial (background) parameters of the two bodies. Using background symmetries, we set $b\cdot v_i = \Psi_i \cdot v_i = 0$ where $b^\mu = |b|\hat{b}^\mu = b_2^\mu - b_1^\mu$ is the covariant impact parameter. We also introduce the Lorentz factor $\gamma = v_1\cdot v_2$ and the relative velocity $v = \sqrt{\gamma^2-1}/\gamma$.

Causal observables including radiative effects arise from the Schwinger-Keldysh (in-in) formalism applied to WQFT [99] where one doubles the fields: $h_{\mu\nu} \to (h_{\mu\nu}^{(1)}, h_{\mu\nu}^{(2)})$ and $Z_i^\mu \to (Z_i^{(1)\mu}, Z_i^{(2)\mu})$ introducing the worldline "superfields" $Z_i = \{z_i, \psi_i'\}$. Causal one-point functions follow from the in-in path integral

$$\langle \mathcal{O} \rangle \coloneqq \int \mathcal{D}[h_{\mu\nu}^{(1,2)}, Z_i^{(1,2)\mu}] e^{i(S[\{\}^{(1)}] - S[\{\}^{(2)}]^*)}\mathcal{O} \qquad (3)$$

normalized such that $\langle 1\rangle = 1$ and with $\{\}^{(n)}$ denoting the $(n)$ th copy of the doubled fields. The key property we exploit is that the WQFT tree-level one-point functions $\langle Z_i^{(n)}\rangle$ solve the classical equations of motion. Moreover, the computation of one-point functions of in-in WQFT reduces to the use of retarded propagators combined with the standard in-out WQFT Feynman rules [99]. This formalism yields an efficient QFT-based scheme to solve the classical equations perturbatively.

The conservative observable can in turn be defined by neglecting all interactions between $h_{\mu\nu}^{(1)}$ and $h_{\mu\nu}^{(2)}$. This may be achieved by using the in-in formalism only for the worldlines while keeping the in-out formalism for the gravitons and projecting on the real part of observables [123,124]. This separation of conservative effects at 4PM has proven its efficiency for the nonspinning results [89,101].

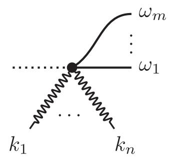
*WQFT Feynman rules.*—The graviton propagator in de Donder gauge with Feynman prescription reads

$$\overset{\mu\nu}{\bullet}\!\!\underset{k}{\rule{2em}{0.4pt}}\!\!\overset{\rho\sigma}{\bullet} = \frac{iP_{\mu\nu;\rho\sigma}}{k^2 + i0^+}, \qquad (4)$$

with $P_{\mu\nu;\rho\sigma} \coloneqq \eta_{\mu(\rho}\eta_{\sigma)\nu} - [1/(D-2)]\eta_{\mu\nu}\eta_{\rho\sigma}$, while the worldline propagators associated with $z_i^\mu$ and $\psi_i'^\mu$ read, respectively,

$$\cdots\overset{\mu}{\bullet}\!\!\underset{\omega,n}{\overset{\rightarrow}{\rule{2em}{0.4pt}}}\!\!\overset{\nu}{\bullet}\cdots = \frac{-i\eta^{\mu\nu}}{m_i(\omega+i0^+)^n}\begin{cases} n=2 \text{ for } z_i^\mu, \\ n=1 \text{ for } \psi_i'^\mu. \end{cases} \qquad (5)$$

The arrow on the propagators indicates the momentum or energy flow on the retarded propagators. Importantly, the Feynman graviton propagators reflect our focus on conservative observables. Full dissipative results may be obtained by using retarded propagators instead. The Feynman vertices of the spinning WQFT to lower multiplicities have been exposed in [97]. The generic worldline vertex couples $n$ gravitons to $m$ worldline fields and reads



$$V_{n|m} = \ \sim \ m\kappa^n\, e^{ik\cdot b}\delta\left(k\cdot v+\sum_{i=1}^{n}\omega_i\right)\times \times\begin{pmatrix}\text{polynomial in } \omega_i, k_j\\ \text{of degree } 2n+m\end{pmatrix}, \tag{6}$$

where $k^\mu = \sum_{i=1}^{n} k_i^\mu$ is the total outflowing four-momentum, and the dotted outgoing line symbolizes the background parameters $\{b^\mu, v^\mu, \Psi^\mu\}$ of Eq. (2). We see that only energy is conserved on the worldline. The bulk graviton vertices are generic. At 4PM order, we need the worldline vertices $V_{n|m}$ above for $\{n=1,\ldots,4; m=0,\ldots,5-n\}$, and the three-, four-, and five-graviton vertices.

*Momentum impulse and spin kick.*—The momentum impulse $\Delta p_i^\mu \coloneqq [p_i^\mu]_{\tau=-\infty}^{\tau=+\infty}$ and spin kick $\Delta S_i^\mu \coloneqq [S_i^\mu]_{\tau=-\infty}^{\tau=+\infty}$ follow from the one-point functions

$$\Delta p_i^\mu = m_i \int_{-\infty}^{\infty} d\tau \left\langle \frac{d^2 x_i^\mu(\tau)}{d\tau^2}\right\rangle = -m_i\omega^2\langle z_i^\mu(\omega)\rangle|_{\omega=0},$$
$$\Delta\psi_i^\mu = \int_{-\infty}^{\infty} d\tau \left\langle \frac{d\psi_i^\mu(\tau)}{d\tau}\right\rangle = -i\omega\langle \psi_i'^\mu(\omega)\rangle|_{\omega=0}, \tag{7}$$

where we have Fourier transformed to momentum space. Both observables are given as the sum of all diagrams at a given PM order with one outgoing $Z_i^\mu$ line with vanishing energy. The spin kick is subsequently derived from the kick of the Grassmann variable as in Ref. [62].

*Integrand generation.*—The 4PM impulse and spin-kick integrands are generated recursively via Berends-Giele-type relations. The one-point functions for the worldline "superfields" $Z_i = \{z_i, \psi_i'\}$ and for the graviton are represented as

$$\langle Z_i(\omega)\rangle = (Z_i)\xrightarrow[\omega]{\to}, \quad \langle h_{\mu\nu}(k)\rangle = (h)\overset{\to}{\underset{k}{\rightsquigarrow}}. \tag{8}$$

Their recursive definitions follow from the Schwinger-Dyson equations and are depicted in Fig. 1. Spelling this out systematically to order $G^4$ allows for an algorithmic construction of the integrand: In our case, we efficiently inserted Feynman rules into the generated trees using FORM [125]. There are 201 graphs contributing to the 4PM impulse in the nonspinning case, 529 with spin, and 253 contributing to the 4PM spin kick.

*Reduction to scalar integrals.*—A generic 4PM diagram after performing the worldline energy integrals via the $\delta$ functions in Eq. (6) takes the form

$$\int_q e^{-iqb}\delta(q\cdot v_1)\delta(q\cdot v_2)$$
$$\times \int_{\ell_1,\ell_2,\ell_3} \frac{\text{num}[\ell_i]}{D_1\cdots D_{12}}\delta(\ell_1\cdot v_{i_1})\delta(\ell_2\cdot v_{i_2})\delta(\ell_3\cdot v_{i_3}), \tag{9}$$

where the $D_i$ are either linear or massless propagators depending on the loop momenta $\ell_i$, velocities $v_i$, and momentum transfer $q$. The numerators num$[\ell_i]$ are polynomial in loop momenta. Tensor reduction of num$[\ell_i]$ to scalar integrals is performed by expanding the loop momenta on a basis dual to $v_i^\mu$ and $q^\mu$, as demonstrated in the 3PM case [98]. The only dimensionful quantity in the three-loop $\ell_i$ integral is the momentum transfer $q^\mu$. Hence, $|q| = \sqrt{-q^2}$ may be scaled out, and the remaining three-loop integrals depend only on the Lorentz factor $\gamma$.

The specific choice of three $\delta(\ell_k v_{i_k})$ functions in Eq. (9) follows the mass dependence of a given diagram, which scales as $m_1 m_2 m_{i_1} m_{i_2} m_{i_3}$. Diagrams are thereby grouped into two categories: test-body contributions with mass dependence $m_1^4 m_2$ or $m_1 m_2^4$ and comparable-mass contributions $m_1^3 m_2^2$, $m_1^2 m_2^3$; see Fig. 2. For the conservative impulse, we can easily reconstruct the $m_1 m_2^4$ and $m_1^2 m_2^3$ components using $\Delta p_{1,\text{cons}}^\mu = -\Delta p_{2,\text{cons}}^\mu$, the impulse on the second body being given simply by relabeling the two worldlines. When computing $\Delta\psi_{1,\text{cons}}^\mu$, no similar relation exists; however, the integrals in opposing mass sectors are also related by a trivial relabeling.

*Integral families and reduction to masters.*—There are three integral families that need to be reduced to master integrals. The first 4PM family is ($i = 1, 2$),

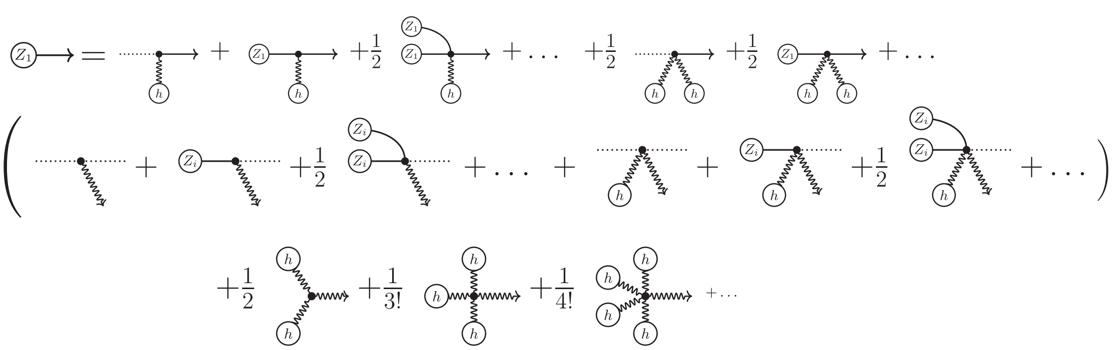



FIG. 1. Berends-Giele-type recursion relation to construct $\langle Z_i^\mu(\omega)\rangle$ and $\langle h_{\mu\nu}(k)\rangle$ perturbatively. The causality flow is always from the $Z_i$ and $h$ blobs to the outgoing line. They are equivalent to the PM-expanded geodesic and Einstein equations.

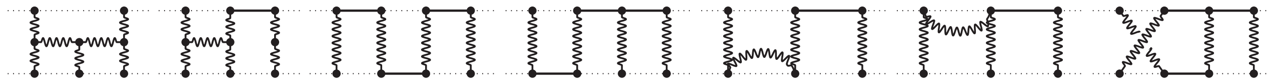


FIG. 2. Examples of comparable-mass graphs with mass dependence $m_1^2 m_2^3$ contributing to the 4PM calculation. One should attach an outgoing worldline to any worldline node and apply the resulting causality flow. The corresponding scalar integrals feature as top sectors in the differential equations: All graphs can be described by the $J$ family (11), except for the last graph belonging to the $I$ family (10). The first two graphs give rise to the elliptic functions in the final result. The second-to-last graph is nonzero only in the (PR + RP) region and therefore does not contribute to the conservative results in this Letter.

$$I^{[i](\sigma_1,\sigma_2,\sigma_3)}_{n_1,n_2,\ldots,n_{12}} = \int_{\ell_1,\ell_2,\ell_3} \frac{\delta(\ell_1\cdot v_i)\delta(\ell_2\cdot v_1)\delta(\ell_3\cdot v_1)}{D_1^{n_1}D_2^{n_2}\cdots D_{12}^{n_{12}}} \tag{10a}$$

with the propagators ($j = 1, 2, 3$ and $k = 1, 2$):

$$\begin{aligned} D_1 &= \ell_1\cdot v_{\bar{i}} + \sigma_1 i0^+, \quad D_{1+k} = \ell_{1+k}\cdot v_2 + \sigma_{1+k} i0^+, \\ D_4 &= (\ell_1+\ell_2+\ell_3+q)^2, \quad D_5 = (\ell_1+\ell_2+q)^2, \\ D_{5+k} &= (\ell_k+\ell_3)^2, \quad D_{7+j} = \ell_j^2, \quad D_{10+k} = (\ell_k+q)^2, \end{aligned} \tag{10b}$$

and $\bar{1} = 2, \bar{2} = 1$. The $I^{[1]}$ and $I^{[2]}$ families contribute to the test-body and comparable-mass regimes, respectively. The other 4PM family is given by

$$J^{(\sigma_1,\sigma_2,\sigma_3)}_{n_1,n_2,\ldots,n_{12}} := \int_{\ell_1,\ell_2,\ell_3} \frac{\delta(\ell_1\cdot v_1)\delta(\ell_2\cdot v_1)\delta(\ell_3\cdot v_2)}{D_1^{n_1}D_2^{n_2}\cdots D_{12}^{n_{12}}} \tag{11a}$$

with ($j = 1, 2, 3$, $k = 1, 2$),

$$\begin{aligned} D_k &= \ell_k\cdot v_2 + \sigma_k i0^+, \qquad D_3 = \ell_3\cdot v_1 + \sigma_3 i0^+, \\ D_{3+k} &= (\ell_k - \ell_3)^2, \qquad D_6 = (\ell_1-\ell_2)^2, \\ D_{6+j} &= \ell_j^2, \qquad D_{9+j} = (\ell_j+q)^2. \end{aligned} \tag{11b}$$

Each family splits into two branches: even ($b$ type) or odd ($v$ type) in the number of worldline propagators. In the nonspinning impulse, these integrals multiply terms proportional to $b^\mu$, $v_i^\mu$, respectively (16). Using integration by parts (IBP) relations [126–129], we reduce the families to 23 master integrals for the $I$-$b$ and $I$-$v$ types each, as well as 64 of $J$-$b$ type and 66 of $J$-$v$ type. The complete spinning impulse computation (including dissipation) results in approximately $10^5$ integrals for reduction to scalar masters.

*Differential equations.*—To solve for the master integrals, we employ the method of canonical differential equations (DEs) [54]. Each master integral family is grouped into a vector $\vec{I}$ ordered according to the number of active propagators. The DE in $x = \gamma - \sqrt{\gamma^2-1}$ reads $d\vec{I}/dx = M(\epsilon, x)\vec{I}$ with a lower-block triangular matrix $M(\epsilon, x)$. Finding a transformation matrix $T$ that brings us to a canonical basis with an $\epsilon$ factorized DE $d\vec{\tilde{I}}/dx = \epsilon A(x)\vec{\tilde{I}}$ is a highly involved procedure in which we employ the packages [130–133]. The resulting symbol alphabet is $\{x, 1+x, 1-x, 1+x^2\}$, and we encounter elliptic integrals in the $J$-$b$ family [103,130].

*Fixing boundary conditions.*—Boundary conditions on the master integrals are determined in the static limit ($\gamma \to 1$, $v \to 0$) using the method of regions [82,101,134,135] to expand the integrand in $v$. Regions in the static limit are characterized by different scalings of the bulk graviton loop momenta with potential (P) and radiative (R) modes defined by relative scalings of their spacial and timelike components:

$$\ell_i^{\mathrm{P}} = (\ell_i^0, \boldsymbol{\ell}_i) \sim (v, 1), \qquad \ell_i^{\mathrm{R}} = (\ell_i^0, \boldsymbol{\ell}_i) \sim (v, v). \tag{12}$$

Only gravitons which may go on shell can be radiative, and there are at most two of these defining the three regions: (PP), (RR), and (PR + RP). The regions (PP) and (PR + RP) are purely conservative and dissipative, respectively, while the (RR) region carries both kinds of effects. In the (PP) region, the integrals reduce to test-body integrals described by the $I^{[1]}$ family (10); in the (RR) region, they reduce to double-mushroom integrals like the first graph of Fig. 3. Either way, the boundary integrals are independent of $\gamma$ and thus functions only of $D = 4 - 2\epsilon$. In our conservative observables, we include only the (PP) and (RR) regions as the (PR + RP) region generates terms odd in $v$.

Reaching 4PM order introduces the physical phenomenon of tail effects [89,101]. In the 4PM contributions to an observable $X$ (impulse, spin kick, or scattering angle), poles in $\epsilon = [(4-D)/2]$ appear in the (PP) and conservative (RR) contributions:

$$X^{(\mathrm{PP})} = \frac{P(\gamma)}{2\epsilon} + \ldots, \qquad X^{(\mathrm{RR})}_{\mathrm{cons}} = -v^{-4\epsilon}\frac{P(\gamma)}{2\epsilon} + \ldots, \tag{13}$$

higher-order terms being finite as $\epsilon \to 0$. Nonanalytic dependence on $v^{-4\epsilon}$ in the (RR) region is a direct consequence of the velocity scaling of the two radiative gravitons. The cancellation of these poles when assembling $X_{\mathrm{cons}}$ introduces logarithmic velocity dependence:

$$X_{\mathrm{cons}} = X^{(\mathrm{PP})} + X^{(\mathrm{RR})}_{\mathrm{cons}} = P(\gamma)\log\frac{\gamma-1}{2} + \ldots, \tag{14}$$

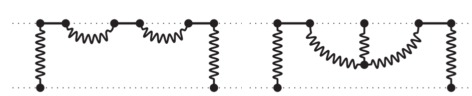


FIG. 3. Two examples of graphs contributing in the (RR) region but not the (PP) region.

the dots indicating terms that are rational in $\sqrt{\gamma^2-1}$ in the static limit.

*Results.*—We begin with $\Delta p^{(4)\mu}_{i,\text{cons}}$, the $G^4$ component of the impulse $\Delta p^\mu_{i,\text{cons}}$. It may be decomposed as

$$\Delta p^{(4)\mu}_{\text{cons},1} = \frac{m_1^2 m_2^2}{|b|^4} \sum_{l,\sigma=b,v} \rho_l^{(\sigma)\mu} \left[ \left( \frac{m_2^2}{m_1} c_l^{(\sigma)}(\gamma) + \frac{m_1^2}{m_2} \bar{c}_l^{(\sigma)}(\gamma) \right) + \sum_\alpha F_\alpha^{(\sigma)}(\gamma) \left( m_2 d^{(\sigma)}_{\alpha,l}(\gamma) + m_1 \bar{d}^{(\sigma)}_{\alpha,l}(\gamma) \right) \right], \quad (15)$$

where the basis vectors and spin structures $\rho_l^{(b,v)\mu}$ are

$$\rho_l^{(b)\mu} = \left\{ \hat{b}^\mu, \frac{a_i \cdot \hat{L}}{|b|} \hat{b}^\mu, \frac{a_i \cdot \hat{b}}{|b|} \hat{L}^\mu \right\},$$
$$\rho_l^{(v)\mu} = \left\{ v_j^\mu, \frac{a_i \cdot \hat{L}}{|b|} v_j^\mu, \frac{a_i \cdot v_{\bar{\imath}}}{|b|} \hat{L}^\mu \right\}. \quad (16)$$

There are five and eight elements in $\rho_l^{(b)\mu}$ and $\rho_l^{(v)\mu}$, respectively, and the normalized angular momentum $\hat{L}^\mu = \epsilon^\mu_{\nu\rho\sigma} v_1^\nu v_2^\rho \hat{b}^\sigma/\gamma v$. The $c_l^{(\sigma)}(\gamma)$ and $d_{\alpha,l}^{(\sigma)}(\gamma)$ and their barred counterparts are rational functions (up to integer powers of $\sqrt{\gamma^2-1}$). All nontrivial dependence on $\gamma$ is contained in the 16 functions $F_\alpha^{(b)}(\gamma)$ with $\gamma_\pm = \gamma \pm 1$:

$$F_\alpha^{(b)}(\gamma) = \Bigg\{ 1, \operatorname{arccosh}[\gamma], \log[\gamma], \log\left[\frac{\gamma_\pm}{2}\right],$$
$$\operatorname{arccosh}^2[\gamma], \operatorname{arccosh}[\gamma] \log\left[\frac{\gamma_\pm}{2}\right], \log\left[\frac{\gamma_+}{2}\right] \log\left[\frac{\gamma_-}{2}\right],$$
$$\log^2\left[\frac{\gamma_+}{2}\right], \operatorname{Li}_2\left[\pm\frac{\gamma_-}{\gamma_+}\right], \operatorname{Li}_2\left[\sqrt{\frac{\gamma_-}{\gamma_+}}\right],$$
$$K^2\left[\frac{\gamma_-}{\gamma_+}\right], E^2\left[\frac{\gamma_-}{\gamma_+}\right], K\left[\frac{\gamma_-}{\gamma_+}\right] E\left[\frac{\gamma_-}{\gamma_+}\right] \Bigg\}, \quad (17)$$

and the much simpler set $F_\alpha^{(v)} = \{1, \operatorname{arccosh}[\gamma]\}$. The first line of Eq. (17) includes transcendental weight-1 functions, the second and third lines weight-2 functions, and the final line quadratic combinations of elliptic functions of the first and second kind. The barred coefficients $\bar{c}_l^{(\sigma)}$ and $\bar{d}_l^\sigma$ may be obtained from the unbarred ones by relabeling using $\Delta p^{(4)\mu}_{\text{cons},1} = -\Delta p^{(4)\mu}_{\text{cons},2}$.

The $G^4$ component of the spin kick $\Delta S^{(4)\mu}_{i,\text{cons}}$ admits a similar decomposition involving the same functions $F_\alpha^{(b,v)}$ but a different set of basis vectors and spin structures:

$$\tilde{\rho}^{(b)\mu} = \left\{ \frac{a_1 \cdot v_2}{|b|} \hat{b}^\mu, \frac{a_1 \cdot \hat{b}}{|b|} v_j^\mu \right\},$$
$$\tilde{\rho}^{(v)\mu} = \left\{ \frac{a_1 \cdot \hat{b}}{|b|} \hat{b}^\mu, \frac{a_1 \cdot v_2}{|b|} v_j^\mu \right\}, \quad (18)$$

and takes the schematic form

$$\Delta S^{(4)\mu}_{\text{cons},1} = \frac{m_1^2 m_2^2}{|b|^3} \sum_{l,\sigma} \tilde{\rho}_l^{(\sigma)\mu} \left[ \left( \frac{m_2^2}{m_1} e_l^{(\sigma)}(\gamma) + \frac{m_1^2}{m_2} \bar{e}_l^{(\sigma)}(\gamma) \right) + \sum_\alpha F_\alpha^{(\sigma)}(\gamma) \left( m_2 f^{(\sigma)}_{\alpha,l}(\gamma) + m_1 \bar{f}^{(\sigma)}_{\alpha,l}(\gamma) \right) \right]. \quad (19)$$

Here, $e_l^{(\sigma)}(\gamma)$ and $f_{\alpha,l}^{(\sigma)}(\gamma)$ and their barred counterparts are rational functions (again, up to integer powers of $\sqrt{\gamma^2-1}$). For the full expression, we refer the reader to the Supplemental Material [136].

As checks on these two observables, we have confirmed (i) the cancellation of all $1/\epsilon$ poles occurring between the (PP) and (RR) regions and the (ii) conservation of $p_i^2$, $S_i^2$, and the $\mathcal{N}=1$ global supercharge $Q_i = p_i \cdot \psi_i$. While the first two only check the simpler terms carrying $F_\alpha^{(v)}$, the latter also compares $F_\alpha^{(b)}$ terms between $\Delta p^{(4)\mu}_{i,\text{cons}}$ and $\Delta \psi^{(4)\mu}_{i,\text{cons}}$, and thus, $\Delta S^{(4)\mu}_{i,\text{cons}}$.

We also define the total scattering angle $\theta$ for generic spin configurations as

$$\sin\frac{\theta}{2} = \frac{|\Delta p^\mu_{i,\text{cons}}|}{2 p_\infty}, \qquad \theta = \frac{E}{M} \sum_{n,m} \left( \frac{GM}{|b|} \right)^n \frac{\theta^{(n,m)}}{|b|^m}, \quad (20)$$

with $p_\infty = m_1 m_2 \sqrt{\gamma^2-1}/E$, total energy $E = |p_1^\mu + p_2^\mu|$, and total mass $M = m_1 + m_2$, $n$ and $m$ counting PM and spin orders, respectively. The 4PM spin-orbit contribution is

$$\theta^{(4,1)}_{\text{cons}} = \sum_{\alpha=1}^{16} \pi\nu \left( s_+ h_\alpha^{(+)}(\gamma) + \delta s_- h_\alpha^{(-)}(\gamma) \right) F_\alpha^{(b)}(\gamma) - \frac{21\pi\gamma(33\gamma^4 - 30\gamma^2 + 5)(13 s_+ - 3\delta s_-)}{32(\gamma^2-1)^{5/2}}, \quad (21)$$

where the test-body contributions (second line) agree with the geodesic motion in a Kerr background [137]. Here we use the mass parameters $\nu = m_1 m_2/M^2$ and $\delta = (m_2 - m_1)/M$, and we have defined $s_\pm = -(a_1 \pm a_2) \cdot \hat{L}$. The 32 polynomial functions $h_\alpha^{(\pm)}(\gamma)$ are given in the Supplemental Material Eq. (23) [136]. We have checked this result against the corresponding N$^3$LO PN [39,41] literature and found agreement by taking the PN expansion. The tail term $P_\theta^{(4)}(\gamma)$ of the scattering angle is simply related to the 3PM radiated energy $E^{(3)}_{\text{rad}}$ as follows:

$$P_\theta^{(4)}(\gamma) = E \frac{\partial E^{(3)}_{\text{rad}}}{\partial J}, \quad (22)$$

where $J = p_\infty |b|$ is the initial angular momentum. This equation follows the pattern derived in Ref. [138] and constitutes another nontrivial check of our results. All of our results are included in the Supplemental Material [136].

*Outlook.*—Having produced a complete set of 4PM linear-in-spin conservative scattering observables—and successfully compared them with N$^3$LO spin-orbit PN [39,41]—our next step will be upgrading them to include dissipative effects, as has already been done in the nonspinning case [102,103]. This will require two changes to our setup: retarded graviton propagators in place of time-symmetric Feynman (see Ref. [99]) and incorporation of the (PR + RP) regions when fixing boundary conditions on master integrals. Notwithstanding the added complexity, quadratic-in-spin order is also an achievable target—corresponding N$^3$LO quadratic-in-spin PN results are already available [38,40]. In the near future, we also seek to use these results to describe bound orbits, the main obstacle being the aforementioned tail effect [89,101]. Recent numerical relativity simulations of spinning black holes on hyperboliclike orbits [139] also offer us future numerical comparisons of the scattering angle $\theta$.

We thank Alessandra Buonanno, Christoph Dlapa, Gregor Kälin, Jung-Wook Kim, Zhengwen Liu, Raj Patil, Chia-Hsien Shen, and Jan Steinhoff for enlightening discussions and Peter Uwer for help with high-performance computing. This work is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) Project No. 417533893/GRK2575 "Rethinking Quantum Field Theory".

*gustav.uhre.jakobsen@physik.hu-berlin.de
†gustav.mogull@aei.mpg.de
‡jan.plefka@hu-berlin.de
§benjamin.sauer@hu-berlin.de
‖yingxu@physik.hu-berlin.de

[1] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Observation of Gravitational Waves from a Binary Black Hole Merger, Phys. Rev. Lett. **116**, 061102 (2016).

[2] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral, Phys. Rev. Lett. **119**, 161101 (2017).

[3] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GWTC-1: A Gravitational-Wave Transient Catalog of Compact Binary Mergers Observed by LIGO and Virgo during the First and Second Observing Runs, Phys. Rev. X **9**, 031040 (2019).

[4] R. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GWTC-2: Compact Binary Coalescences Observed by LIGO and Virgo during the First Half of the Third Observing Run, Phys. Rev. X **11**, 021053 (2021).

[5] R. Abbott *et al.* (LIGO Scientific and VIRGO Collaborations), GWTC-2.1: Deep extended catalog of compact binary coalescences observed by LIGO and Virgo during the first half of the third observing run, arXiv:2108.01045.

[6] R. Abbott *et al.* (LIGO Scientific, VIRGO, and KAGRA Collaborations), GWTC-3: Compact binary coalescences observed by LIGO and Virgo during the second part of the third observing run, arXiv:2111.03606.

[7] S. W. Ballmer *et al.*, Snowmass2021 cosmic frontier white paper: Future gravitational-wave detector facilities, arXiv:2203.08228.

[8] M. Pürrer and C.-J. Haster, Gravitational waveform accuracy requirements for future ground-based detectors, Phys. Rev. Res. **2**, 023151 (2020).

[9] V. Kalogera *et al.*, The next generation global gravitational wave observatory: The science book, arXiv:2111.06990.

[10] K. Westpfahl and M. Goller, Gravitational scattering of two relativistic particles in postlinear approximation, Lett. Nuovo Cimento **26**, 573 (1979).

[11] L. Bel, T. Damour, N. Deruelle, J. Ibanez, and J. Martin, Poincaré-invariant gravitational field and equations of motion of two pointlike objects: The postlinear approximation of general relativity, Gen. Relativ. Gravit. **13**, 963 (1981).

[12] T. Damour, High-energy gravitational scattering and the general relativistic two-body problem, Phys. Rev. D **97**, 044038 (2018).

[13] W. D. Goldberger and I. Z. Rothstein, An effective field theory of gravity for extended objects, Phys. Rev. D **73**, 104029 (2006).

[14] R. A. Porto, The effective field theorist's approach to gravitational dynamics, Phys. Rep. **633**, 1 (2016).

[15] M. Levi, Effective field theories of post-Newtonian gravity: A comprehensive review, Rep. Prog. Phys. **83**, 075901 (2020).

[16] L. Blanchet, Gravitational radiation from post-Newtonian sources and inspiralling compact binaries, Living Rev. Relativity **17**, 2 (2014).

[17] G. Schäfer and P. Jaranowski, Hamiltonian formulation of general relativity and post-Newtonian dynamics of compact binaries, Living Rev. Relativity **21**, 7 (2018).

[18] T. Futamase and Y. Itoh, The post-Newtonian approximation for relativistic compact binaries, Living Rev. Relativity **10**, 2 (2007).

[19] M. E. Pati and C. M. Will, Post-Newtonian gravitational radiation and equations of motion via direct integration of the relaxed Einstein equations. 1. Foundations, Phys. Rev. D **62**, 124015 (2000).

[20] W. D. Goldberger and I. Z. Rothstein, Towers of gravitational theories, Gen. Relativ. Gravit. **38**, 1537 (2006).

[21] W. D. Goldberger and A. Ross, Gravitational radiative corrections from effective field theory, Phys. Rev. D **81**, 124015 (2010).

[22] B. Kol and M. Smolkin, Non-relativistic gravitation: From Newton to Einstein and back, Classical Quantum Gravity **25**, 145011 (2008).

[23] W. D. Goldberger, Les Houches Lectures on Effective Field Theories and Gravitational Radiation, in *Proceedings of the Les Houches Summer School, Session LXXXVI*; arXiv:hep-ph/0701129.

[24] S. Foffa and R. Sturani, Effective field theory methods to model compact binaries, Classical Quantum Gravity **31**, 043001 (2014).

[25] I. Z. Rothstein, Progress in effective field theory approach to the binary inspiral problem, Gen. Relativ. Gravit. **46**, 1726 (2014).

[26] C. R. Galley and M. Tiglio, Radiation reaction and gravitational waves in the effective field theory approach, Phys. Rev. D **79**, 124027 (2009).

[27] S. Foffa, P. Mastrolia, R. Sturani, C. Sturm, and W. J. Torres Bobadilla, Static Two-Body Potential at Fifth Post-Newtonian Order, Phys. Rev. Lett. **122**, 241605 (2019).

[28] J. Blümlein, A. Maier, P. Marquard, and G. Schäfer, The fifth-order post-Newtonian Hamiltonian dynamics of two-body systems from an effective field theory approach: Potential contributions, Nucl. Phys. **B965**, 115352 (2021).

[29] D. Bini, T. Damour, and A. Geralico, Binary dynamics at the fifth and fifth-and-a-half post-Newtonian orders, Phys. Rev. D **102**, 024062 (2020).

[30] D. Bini, T. Damour, and A. Geralico, Sixth post-Newtonian nonlocal-in-time dynamics of binary systems, Phys. Rev. D **102**, 084047 (2020).

[31] J. Blümlein, A. Maier, P. Marquard, and G. Schäfer, Testing binary dynamics in gravity at the sixth post-Newtonian level, Phys. Lett. B **807**, 135496 (2020).

[32] D. Bini, T. Damour, and A. Geralico, Sixth post-Newtonian local-in-time dynamics of binary systems, Phys. Rev. D **102**, 024061 (2020).

[33] D. Bini, T. Damour, A. Geralico, S. Laporta, and P. Mastrolia, Gravitational dynamics at $O(G^6)$: Perturbative gravitational scattering meets experimental mathematics, arXiv:2008.09389.

[34] J. Blümlein, A. Maier, P. Marquard, and G. Schäfer, The fifth-order post-Newtonian Hamiltonian dynamics of two-body systems from an effective field theory approach, Nucl. Phys. **B983**, 115900 (2022).

[35] S. Foffa and R. Sturani, Hereditary terms at next-to-leading order in two-body gravitational dynamics, Phys. Rev. D **101**, 064033 (2020).

[36] G. L. Almeida, A. Müller, S. Foffa, and R. Sturani, Conservative binary dynamics from gravitational tail emission processes, arXiv:2307.05327.

[37] M. Levi and Z. Yin, Completing the fifth PN precision frontier via the EFT of spinning gravitating objects, J. High Energy Phys. 04 (2023) 079.

[38] J.-W. Kim, M. Levi, and Z. Yin, $N^3LO$ quadratic-in-spin interactions for generic compact binaries, J. High Energy Phys. 03 (2023) 098.

[39] J.-W. Kim, M. Levi, and Z. Yin, $N^3LO$ spin-orbit interaction via the EFT of spinning gravitating objects, J. High Energy Phys. 05 (2023) 184.

[40] M. K. Mandal, P. Mastrolia, R. Patil, and J. Steinhoff, Gravitational quadratic-in-spin Hamiltonian at NNNLO in the post-Newtonian framework, J. High Energy Phys. 07 (2023) 128.

[41] M. K. Mandal, P. Mastrolia, R. Patil, and J. Steinhoff, Gravitational spin-orbit Hamiltonian at NNNLO in the post-Newtonian framework, J. High Energy Phys. 03 (2023) 130.

[42] M. Levi, A. J. Mcleod, and M. Von Hippel, $N^3LO$ gravitational quadratic-in-spin interactions at $G^4$, J. High Energy Phys. 07 (2021) 116.

[43] M. Levi, A. J. Mcleod, and M. Von Hippel, $N^3LO$ gravitational spin-orbit coupling at order $G^4$, J. High Energy Phys. 07 (2021) 115.

[44] B. Bertotti, On gravitational motion, Nuovo Cimento **4**, 898 (1956).

[45] R. P. Kerr, The Lorentz-covariant approximation method in general relativity I, Nuovo Cimento **13**, 469 (1959).

[46] M. Portilla, Scattering of two gravitating particles: Classical approach, J. Phys. A **13**, 3677 (1980).

[47] K. Westpfahl, High-speed scattering of charged and uncharged particles in general relativity, Fortschr. Phys. **33**, 417 (1985).

[48] D. A. Kosower, R. Monteiro, and D. O'Connell, The SAGEX review on scattering amplitudes Chapter 14: Classical gravity from scattering amplitudes, J. Phys. A **55**, 443015 (2022).

[49] N. E. J. Bjerrum-Bohr, P. H. Damgaard, L. Plante, and P. Vanhove, The SAGEX review on scattering amplitudes Chapter 13: Post-Minkowskian expansion from scattering amplitudes, J. Phys. A **55**, 443014 (2022).

[50] A. Buonanno, M. Khalil, D. O'Connell, R. Roiban, M. P. Solon, and M. Zeng, Snowmass white paper: Gravitational waves and scattering amplitudes, arXiv:2204.05194.

[51] S. Laporta, High precision calculation of multiloop Feynman integrals by difference equations, Int. J. Mod. Phys. A **15**, 5087 (2000).

[52] A. V. Smirnov, Algorithm FIRE—Feynman integral reduction, J. High Energy Phys. 10 (2008) 107.

[53] T. Gehrmann and E. Remiddi, Differential equations for two loop four point functions, Nucl. Phys. **B580**, 485 (2000).

[54] J. M. Henn, Multiloop Integrals in Dimensional Regularization Made Simple, Phys. Rev. Lett. **110**, 251601 (2013).

[55] M. Beneke and V. A. Smirnov, Asymptotic expansion of Feynman integrals near threshold, Nucl. Phys. **B522**, 321 (1998).

[56] C. Cheung, I. Z. Rothstein, and M. P. Solon, From Scattering Amplitudes to Classical Potentials in the Post-Minkowskian Expansion, Phys. Rev. Lett. **121**, 251101 (2018).

[57] N. E. J. Bjerrum-Bohr, P. H. Damgaard, G. Festuccia, L. Planté, and P. Vanhove, General Relativity from Scattering Amplitudes, Phys. Rev. Lett. **121**, 171601 (2018).

[58] A. Cristofoli, N. E. J. Bjerrum-Bohr, P. H. Damgaard, and P. Vanhove, Post-Minkowskian Hamiltonians in general relativity, Phys. Rev. D **100**, 084040 (2019).

[59] Z. Bern, A. Luna, R. Roiban, C.-H. Shen, and M. Zeng, Spinning black hole binary dynamics, scattering amplitudes, and effective field theory, Phys. Rev. D **104**, 065014 (2021).

[60] D. Kosmopoulos and A. Luna, Quadratic-in-spin Hamiltonian at $\mathcal{O}(G^2)$ from scattering amplitudes, J. High Energy Phys. 07 (2021) 037.

[61] F. Febres Cordero, M. Kraus, G. Lin, M. S. Ruf, and M. Zeng, Conservative Binary Dynamics with a Spinning Black Hole at $O(G^3)$ from Scattering Amplitudes, Phys. Rev. Lett. **130**, 021601 (2023).

[62] G. U. Jakobsen and G. Mogull, Linear response, Hamiltonian, and radiative spinning two-body dynamics, Phys. Rev. D **107**, 044033 (2023).

[63] A. Buonanno and T. Damour, Effective one-body approach to general relativistic two-body dynamics, Phys. Rev. D **59**, 084006 (1999).

[64] A. Buonanno and T. Damour, Transition from inspiral to plunge in binary black hole coalescences, Phys. Rev. D **62**, 064015 (2000).

[65] A. Antonelli, A. Buonanno, J. Steinhoff, M. van de Meent, and J. Vines, Energetics of two-body Hamiltonians in post-Minkowskian gravity, Phys. Rev. D **99**, 104004 (2019).

[66] P. H. Damgaard and P. Vanhove, Remodeling the effective one-body formalism in post-Minkowskian gravity, Phys. Rev. D **104**, 104029 (2021).
[67] M. Khalil, A. Buonanno, J. Steinhoff, and J. Vines, Energetics and scattering of gravitational two-body systems at fourth post-Minkowskian order, Phys. Rev. D **106**, 024042 (2022).
[68] Y. Iwasaki, Quantum theory of gravitation vs. classical theory: Fourth-order potential, Prog. Theor. Phys. **46**, 1587 (1971).
[69] B. R. Holstein and J. F. Donoghue, Classical Physics and Quantum Loops, Phys. Rev. Lett. **93**, 201602 (2004).
[70] D. Neill and I. Z. Rothstein, Classical space-times from the $S$ matrix, Nucl. Phys. **B877**, 177 (2013).
[71] A. Luna, I. Nicholson, D. O'Connell, and C. D. White, Inelastic black hole scattering from charged scalar amplitudes, J. High Energy Phys. 03 (2018) 044.
[72] N. E. J. Bjerrum-Bohr, J. F. Donoghue, and P. Vanhove, On-shell techniques and universal results in quantum gravity, J. High Energy Phys. 02 (2014) 111.
[73] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon, and M. Zeng, Scattering Amplitudes and the Conservative Hamiltonian for Binary Systems at Third Post-Minkowskian Order, Phys. Rev. Lett. **122**, 201603 (2019).
[74] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon, and M. Zeng, Black hole binary dynamics from the double copy and effective theory, J. High Energy Phys. 10 (2019) 206.
[75] N. E. J. Bjerrum-Bohr, L. Planté, and P. Vanhove, Post-Minkowskian radial action from soft limits and velocity cuts, J. High Energy Phys. 03 (2022) 071.
[76] C. Cheung and M. P. Solon, Classical gravitational scattering at $\mathcal{O}(G^3)$ from Feynman diagrams, J. High Energy Phys. 06 (2020) 144.
[77] N. E. J. Bjerrum-Bohr, P. H. Damgaard, L. Planté, and P. Vanhove, The amplitude for classical gravitational scattering at third Post-Minkowskian order, J. High Energy Phys. 08 (2021) 172.
[78] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, Universality of ultra-relativistic gravitational scattering, Phys. Lett. B **811**, 135924 (2020).
[79] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, The eikonal approach to gravitational scattering and radiation at $\mathcal{O}(G^3)$, J. High Energy Phys. 07 (2021) 169.
[80] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, Classical gravitational observables from the eikonal operator, Phys. Lett. B **843**, 138049 (2023).
[81] T. Damour, Radiative contribution to classical gravitational scattering at the third order in $G$, Phys. Rev. D **102**, 124008 (2020).
[82] E. Herrmann, J. Parra-Martinez, M. S. Ruf, and M. Zeng, Radiative classical gravitational observables at $\mathcal{O}(G^3)$ from scattering amplitudes, J. High Energy Phys. 10 (2021) 148.
[83] P. H. Damgaard, K. Haddad, and A. Helset, Heavy black hole effective theory, J. High Energy Phys. 11 (2019) 070.
[84] A. Brandhuber, G. Chen, G. Travaglini, and C. Wen, Classical gravitational scattering from a gauge-invariant double copy, J. High Energy Phys. 10 (2021) 118.
[85] G. Kälin and R. A. Porto, Post-Minkowskian effective field theory for conservative binary dynamics, J. High Energy Phys. 11 (2020) 106.
[86] G. Kälin, Z. Liu, and R. A. Porto, Conservative Dynamics of Binary Systems to Third Post-Minkowskian Order from the Effective Field Theory Approach, Phys. Rev. Lett. **125**, 261103 (2020).
[87] G. Kälin, Z. Liu, and R. A. Porto, Conservative tidal effects in compact binary systems to next-to-leading post-Minkowskian order, Phys. Rev. D **102**, 124025 (2020).
[88] C. Dlapa, G. Kälin, Z. Liu, and R. A. Porto, Dynamics of binary systems to fourth Post-Minkowskian order from the effective field theory approach, Phys. Lett. B **831**, 137203 (2022).
[89] C. Dlapa, G. Kälin, Z. Liu, and R. A. Porto, Conservative Dynamics of Binary Systems at Fourth Post-Minkowskian Order in the Large-Eccentricity Expansion, Phys. Rev. Lett. **128**, 161104 (2022).
[90] S. Mougiakakos, M. M. Riva, and F. Vernizzi, Gravitational bremsstrahlung in the post-Minkowskian effective field theory, Phys. Rev. D **104**, 024041 (2021).
[91] M. M. Riva and F. Vernizzi, Radiated momentum in the post-Minkowskian worldline approach via reverse unitarity, J. High Energy Phys. 11 (2021) 228.
[92] S. Mougiakakos, M. M. Riva, and F. Vernizzi, Gravitational Bremsstrahlung with Tidal Effects in the Post-Minkowskian Expansion, Phys. Rev. Lett. **129**, 121101 (2022).
[93] M. M. Riva, F. Vernizzi, and L. K. Wong, Gravitational bremsstrahlung from spinning binaries in the post-Minkowskian expansion, Phys. Rev. D **106**, 044013 (2022).
[94] G. Mogull, J. Plefka, and J. Steinhoff, Classical black hole scattering from a worldline quantum field theory, J. High Energy Phys. 02 (2021) 048.
[95] G. U. Jakobsen, G. Mogull, J. Plefka, and J. Steinhoff, Classical Gravitational Bremsstrahlung from a Worldline Quantum Field Theory, Phys. Rev. Lett. **126**, 201103 (2021).
[96] G. U. Jakobsen, G. Mogull, J. Plefka, and J. Steinhoff, Gravitational Bremsstrahlung and Hidden Supersymmetry of Spinning Bodies, Phys. Rev. Lett. **128**, 011101 (2022).
[97] G. U. Jakobsen, G. Mogull, J. Plefka, and J. Steinhoff, SUSY in the sky with gravitons, J. High Energy Phys. 01 (2022) 027.
[98] G. U. Jakobsen and G. Mogull, Conservative and Radiative Dynamics of Spinning Bodies at Third Post-Minkowskian Order Using Worldline Quantum Field Theory, Phys. Rev. Lett. **128**, 141102 (2022).
[99] G. U. Jakobsen, G. Mogull, J. Plefka, and B. Sauer, All things retarded: Radiation-reaction in worldline quantum field theory, J. High Energy Phys. 10 (2022) 128.
[100] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C.-H. Shen, M. P. Solon, and M. Zeng, Scattering Amplitudes and Conservative Binary Dynamics at $\mathcal{O}(G^4)$, Phys. Rev. Lett. **126**, 171601 (2021).
[101] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C.-H. Shen, M. P. Solon, and M. Zeng, Scattering Amplitudes, the Tail Effect, and Conservative Binary Dynamics at $O(G^4)$, Phys. Rev. Lett. **128**, 161103 (2022).

[102] C. Dlapa, G. Kälin, Z. Liu, J. Neef, and R. A. Porto, Radiation Reaction and Gravitational Waves at Fourth Post-Minkowskian Order, Phys. Rev. Lett. **130**, 101401 (2023).
[103] C. Dlapa, G. Kälin, Z. Liu, and R. A. Porto, Bootstrapping the relativistic two-body problem, J. High Energy Phys. 08 (2023) 109.
[104] N. E. J. Bjerrum-Bohr, L. Planté, and P. Vanhove, Effective field theory and applications: Weak field observables from scattering amplitudes in quantum field theory, arXiv:2212.08957.
[105] Z. Bern, E. Herrmann, R. Roiban, M. S. Ruf, A. V. Smirnov, V. A. Smirnov *et al.*, Conservative binary dynamics at order $O(\alpha^5)$ in electrodynamics, arXiv:2305.08981.
[106] Z. Bern, D. Kosmopoulos, A. Luna, R. Roiban, and F. Teng, Binary Dynamics through the Fifth Power of Spin at $O(G^2)$, Phys. Rev. Lett. **130**, 201402 (2023).
[107] R. Aoude, K. Haddad, and A. Helset, Classical Gravitational Spinning-Spinless Scattering at $O(G^2 S\infty)$, Phys. Rev. Lett. **129**, 141102 (2022).
[108] J. Vines, Scattering of two spinning black holes in post-Minkowskian gravity, to all orders in spin, and effective-one-body mappings, Classical Quantum Gravity **35**, 084002 (2018).
[109] F. Alessio and P. Di Vecchia, Radiation reaction for spinning black-hole scattering, Phys. Lett. B **832**, 137258 (2022).
[110] R. Aoude, K. Haddad, and A. Helset, Classical gravitational scattering amplitude at $O(G^2 S_1\infty S_2\infty)$, Phys. Rev. D **108**, 024050 (2023).
[111] F. Alessio, Kerr binary dynamics from minimal coupling and double copy, arXiv:2303.12784.
[112] D. Bini, T. Damour, and A. Geralico, Scattering of tidally interacting bodies in post-Minkowskian gravity, Phys. Rev. D **101**, 044039 (2020).
[113] K. Haddad and A. Helset, Tidal effects in quantum field theory, J. High Energy Phys. 12 (2020) 024.
[114] C. Cheung and M. P. Solon, Tidal Effects in the Post-Minkowskian Expansion, Phys. Rev. Lett. **125**, 191601 (2020).
[115] K. S. Thorne and S. J. Kovacs, The generation of gravitational waves. I. Weak-field sources., Astrophys. J. **200**, 245 (1975).
[116] R. J. Crowley and K. S. Thorne, The generation of gravitational waves. 2. The postlinear formalism revisited, Astrophys. J. **215**, 624 (1977).
[117] S. J. Kovacs and K. S. Thorne, The generation of gravitational waves. 3. Derivation of bremsstrahlung formulas, Astrophys. J. **217**, 252 (1977).
[118] S. J. Kovacs and K. S. Thorne, The generation of gravitational waves. 4. bremsstrahlung, Astrophys. J. **224**, 62 (1978).
[119] A. Brandhuber, G. R. Brown, G. Chen, S. De Angelis, J. Gowdy, and G. Travaglini, One-loop gravitational bremsstrahlung and waveforms from a heavy-mass effective field theory, J. High Energy Phys. 06 (2023) 048.
[120] A. Herderschee, R. Roiban, and F. Teng, The sub-leading scattering waveform from amplitudes, J. High Energy Phys. 06 (2023) 004.
[121] A. Georgoudis, C. Heissenberg, and I. Vazquez-Holm, Inelastic exponentiation and classical gravitational scattering at one loop, J. High Energy Phys. 06 (2023) 126.
[122] A. Elkhidir, D. O'Connell, M. Sergola, and I. A. Vazquez-Holm, Radiation and reaction at one loop, arXiv:2303.06211.
[123] C. R. Galley, Classical Mechanics of Nonconservative Systems, Phys. Rev. Lett. **110**, 174301 (2013).
[124] G. Kälin, J. Neef, and R. A. Porto, Radiation-reaction in the effective field theory approach to post-Minkowskian dynamics, J. High Energy Phys. 01 (2023) 140.
[125] B. Ruijl, T. Ueda, and J. Vermaseren, FORM version 4.2, arXiv:1707.06453.
[126] R. N. Lee, LITERED 1.4: A powerful tool for reduction of multiloop integrals, J. Phys. Conf. Ser. **523**, 012059 (2014).
[127] A. V. Smirnov and F. S. Chuharev, FIRE6: Feynman integral reduction with modular arithmetic, Comput. Phys. Commun. **247**, 106877 (2020).
[128] P. Maierhöfer, J. Usovitsch, and P. Uwer, KIRA—A Feynman integral reduction program, Comput. Phys. Commun. **230**, 99 (2018).
[129] J. Klappert, F. Lange, P. Maierhöfer, and J. Usovitsch, Integral reduction with KIRA 2.0 and finite field methods, Comput. Phys. Commun. **266**, 108024 (2021).
[130] C. Dlapa, J. M. Henn, and F. J. Wagner, An algorithmic approach to finding canonical differential equations for elliptic Feynman integrals, J. High Energy Phys. 08 (2023) 120.
[131] C. Meyer, Algorithmic transformation of multi-loop master integrals to a canonical basis with CANONICA, Comput. Phys. Commun. **222**, 295 (2018).
[132] M. Prausa, EPSILON: A tool to find a canonical basis of master integrals, Comput. Phys. Commun. **219**, 361 (2017).
[133] R. N. Lee, LIBRA: A package for transformation of differential systems for multiloop integrals, Comput. Phys. Commun. **267**, 108058 (2021).
[134] V. A. Smirnov, *Analytic Tools for Feynman Integrals* (Springer-Verlag, Berlin, 2012), Vol. 250, 10.1007/978-3-642-34886-0.
[135] T. Becher, A. Broggio, and A. Ferroglia, Introduction to soft-collinear effective theory, Lect. Notes Phys. **896**, 1 (2015).
[136] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.131.151401 for full analytic expression of the 4PM spin-orbit scattering angle.
[137] P. H. Damgaard, J. Hoogeveen, A. Luna, and J. Vines, Scattering angles in Kerr metrics, Phys. Rev. D **106**, 124030 (2022).
[138] D. Bini and T. Damour, Gravitational scattering of two black holes at the fourth post-Newtonian approximation, Phys. Rev. D **96**, 064021 (2017).
[139] S. Hopper, A. Nagar, and P. Rettegno, Strong-field scattering of two spinning black holes: Numerics versus analytics, Phys. Rev. D **107**, 124034 (2023).