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Leonid Makar-Limanov

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Department of Mathematics
Wayne State University
Detroit, MI 48202
USA

Faculty of Mathematics and Computer Science
Weizmann Institute of Science
Rehovot 7610001
Israel

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On locally nilpotent derivations of Danielewski domains.

L. Makar-Limanov *†

Abstract

Let $p(Z) \in \mathbb{C}[Z]$ be a polynomial of degree d. In this note I'll show that if positive natural numbers n, m, and d are relatively prime then up to an automorphism there is at most one nonzero irreducible locally nilpotent derivation on the domain $\mathbb{C}[X, Y, Z]/(X^nY^m - p(Z))$.

Introduction.

In this note we take the field $\mathbb{C}$ of complex numbers as the ground field. In fact it is essential only that the ground field has characteristic zero. Also all appearing rings are domains.

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Let $R = \mathbb{C}[x, y]$. It is well known (see [Re]) that the kernel of a nonzero locally nilpotent derivation of R is $\mathbb{C}[u]$ where u is an image of x under an automorphism. More recently a similar result was proved for domains $\mathbb{C}[X, Y, Z]/(X^nY - p(Z))$, $n > 1$, $\deg(p(Z)) > 1$ (see [ML1] and [FMJ]) and $\mathbb{C}[X, Y, Z]/(XY - p(Z))$ where $\deg(p(Z)) > 0$ (see [Da] and [ML2]). Here we will look from this point of view on the domains R given by $\mathbb{C}[X, Y, Z]/(X^nY^m - p(Z))$ where $p(Z)$ is a monic polynomial and n, m, and $d = \deg(p(Z))$ are relatively prime positive natural numbers. If $d = 1$ then the corresponding domains are actually isomorphic to $\mathbb{C}[x, y]$. It turns out that a nonzero locally nilpotent derivation (lnd for short) exists on R only if $(d-1)(n-1)(m-1) = 0$. So as far as the description of lnds is concerned we have only previously described cases. On the other hand we have here more direct proofs, which are substantially shorter.

Definitions, notations and technical lemmas.

Here we recall briefly some necessary notions and facts.

Let A be a $\mathbb{C}$-algebra. A $\mathbb{C}$-homomorphism ∂ of A is called a *derivation* of A if it satisfies the Leibniz rule: $\partial(ab) = \partial(a)b + a\partial(b)$.

A derivation is *irreducible* if $\partial(A)$ does not belong to a proper principal ideal. (So, since (0) is not a proper ideal, according to this definition zero derivation is irreducible!)

We will be using in the next section so called Jacobian derivations on $\mathbb{C}[X, Y, Z]$. Let us take any two $p, q \in C[X, Y, Z]$. Then $\partial(r) = \mathrm{J}(p, q, r)$ where $\mathrm{J}(p, q, r)$ denotes the Jacobian, i. e. the determinant of the corresponding Jacobi matrix, is a derivation. Let us recall that $\mathrm{J}(p, q, r)$ is also skew symmetric.

Any derivation ∂ determines two subalgebras of A. One is the kernel of ∂ which is usually denoted by A^∂ and is called the *ring of ∂-constants*.

The other is $\mathrm{Nil}_A(\partial)$, the *ring of nilpotency of* ∂ :
$\mathrm{Nil}_A(\partial) = \{a \in A | \partial^n(a) = 0, n >> 1\}$.
In other words $a \in \mathrm{Nil}_A(\partial)$ if for a sufficiently large natural number n we have $\partial^n(a) = 0$.

Both A^∂ and $\mathrm{Nil}_A(\partial)$ are subalgebras of A because of the Leibniz rule.

We will call a derivation *locally nilpotent* if $\mathrm{Nil}_A(\partial) = A$.

The best examples of lnds (locally nilpotent derivations) are the partial derivatives on the rings of polynomials $\mathbb{C}[x_1, ..., x_n]$.

With the help of a locally nilpotent derivation acting on A, we can define a function $\deg_\partial$ by $\deg_\partial(a) = \max(n|\partial^n(a) \neq 0)$ if $a \in A^* = A \setminus 0$ and $\deg_\partial(0) = -\infty$.

Then the function $\deg_\partial$ is a degree function, i.e.,
$\deg_\partial(a + b) \leq \max(\deg_\partial(a), \deg_\partial(b))$ and
$\deg_\partial(ab) = \deg_\partial(a) + \deg_\partial(b)$.

Two locally nilpotent derivations are *equivalent* if the corresponding degree function are the same.

By definition $\deg_\partial$ has only nonnegative values on A^* and $a \in A^\partial \setminus 0$ if $\deg_\partial(a) = 0$. So it is clear that the ring A^∂ is "factorially closed"; i. e., if $a, b \in A^*$ and $ab \in A^\partial$, then $a, b \in A^\partial$.

Let F be the field of fractions of A. Any derivation ∂ can be extended to a derivation on F by the "calculus" formula $\partial(ab^{-1}) = (\partial(a)b - a\partial(b))b^{-2}$. We will denote this extended derivation also by ∂.

Lemma 1. Let ∂ be a locally nilpotent nonzero derivation of A. Then there exists an element $t \in F$ for which $\partial(t) = 1$ and $\mathrm{Nil}_F(\partial) = F^{\partial}[t]$.

Proof. ∂ is a nonzero derivation so $A \neq A^{\partial}$ and there exists an $a \in A \setminus A^{\partial}$. Put $r = \partial^n(a)$ and $s = \partial(r)$ where $n = \deg_{\partial}(a) - 1$. Then $r \notin A^{\partial},\ s \in A^{\partial}$ and $\partial(t) = 1$ for $t = rs^{-1}$. Observe that $s \in A^{\partial}$, we will use this fact later.

It is clear that $F^{\partial}[t] \subset \mathrm{Nil}_F(\partial)$. Let $a \in \mathrm{Nil}_F(\partial)$. We will use induction on $\deg_{\partial}(a) = n$ to show the opposite inclusion. If $a \in F$ and $\deg_{\partial}(a) = 0$ then $a \in F^{\partial}$ by definition. Let us make the step from $\deg_{\partial}(a) = n - 1$ to $\deg_{\partial}(a) = n$. If $\deg_{\partial}(a) = n$ then $\deg_{\partial}(\partial(a)) = n - 1$ and by induction $\partial(a) = \sum_{i=0}^{n-1} a_i t^{n-1-i}$ for some $a_i \in F^{\partial}$. Let $f = \sum_{i=0}^{n-1} (n-i)^{-1} a_i t^{n-i}$. Then $\partial(f) = \partial(a)$. So $\partial(a - f) = 0$ which means that $a = f + a_n$ where $a_n \in F^{\partial}$. $\square$

Remark 1. It is clear that $\deg_{\partial}$ and $\deg_t$ are the same functions. This, of course, gives a proof of the properties of $\deg_{\partial}$ mentioned above. See also [FLN].

Remark 2. A^{∂} is algebraically closed in A. Indeed, if $a \notin A^{\partial}$ then it is represented by a polynomial of positive degree and $p(a)$ also has a positive degree for any nonzero polynomial p.

Lemma 2. Let ∂ be a nonzero lnd of A. If $\partial = a\epsilon$ where $a \in A$ and ϵ is a derivation of A then $\partial(a) = 0$ and ϵ is an lnd.

Proof. We want to show that $\deg_{\partial}(a) = 0$. It is clear that $A^{\epsilon} = A^{\partial}$. If $\deg_{\partial}(a) > 0$ then $\deg_{\partial}(\partial(b)) = \deg_{\partial}(a\epsilon(b)) = \deg_{\partial}(a) + \deg_{\partial}(\epsilon(b)) > 0$ for any $b \notin A^{\partial}$. So if $b \notin A^{\partial}$ then $\partial(b) \notin A^{\partial}$ which means that ∂ is not an lnd if

$A \neq A^\partial$ i. e. $\partial \neq 0$. So $\partial(a) = 0$. Therefore $\deg_\partial(\epsilon(b)) = \deg_\partial(b) - 1$ for any $b \notin A^\partial$. Hence ϵ is an lnd. Even more, $\deg_\partial = \deg_\epsilon$. $\square$

Remark 3. We see that any nonzero lnd is equivalent to an irreducible lnd.

Lemma 3. F^∂ is the field of fractions of A^∂.

Proof. This proof was suggested by Ofer Hadas. Let $a, b \in A$ and $r = ab^{-1} \in F^\partial$. Assume also that $\deg_\partial(a)$ is minimal possible for all presentations of r as a fraction. Now, $\partial(r) = (\partial(a)b - a\partial(b))b^{-2} = 0$. So $ab^{-1} = \partial(a)\partial(b)^{-1}$ and $\deg_\partial(\partial(a)) < \deg_\partial(a)$. To avoid a contradiction we have to assume that $\deg_\partial(a) = 0$, so a and b are in A^∂. $\square$

Remark 4. Since $F = F^\partial(t)$ the transcendence degree $\mathrm{trdeg}(F^\partial) = \mathrm{trdeg}(F) - 1$. Furthermore, $\mathrm{trdeg}(F) = \mathrm{trdeg}(A)$, $\mathrm{trdeg}(F^\partial) = \mathrm{trdeg}(A^\partial)$ and $\mathrm{trdeg}(A^\partial) = \mathrm{trdeg}(A) - 1$.

Lemma 4. Let $Q \in \mathbb{C}[X, Y, Z]$ be an irreducible polynomial, $S = \mathbb{C}[X, Y, Z]/(Q)$ be the corresponding factor ring, and π the projection of $\mathbb{C}[X, Y, Z]$ on S. Assume that there is a nonzero lnd ∂ on S. Let $H \in \mathbb{C}[X, Y, Z]$ be such that $\pi(H) \in S^\partial \setminus \mathbb{C}$. Then $\epsilon(\pi(G)) = \pi(\mathrm{J}(Q, H, G))$ defines an lnd on S which is equivalent to ∂.

Proof. Expression $\pi(\mathrm{J}(Q, H, G))$ defines a derivation on S. To check this we should first verify that if $\pi(G_1) = \pi(G_2)$ then $\epsilon(\pi(G_1)) = \epsilon(\pi(G_2))$. In this case $G_2 = G_1 + PQ$ and

$\mathrm{J}(Q, H, G_2) = \mathrm{J}(Q, H, G_1 + PQ) = \mathrm{J}(Q, H, G_1) + \mathrm{J}(Q, H, PQ) =$
$\mathrm{J}(Q, H, G_1) + \mathrm{J}(Q, H, P)Q.$

Since $\mathrm{J}(Q, H, P)Q \in (Q)$ we see that $\pi(\mathrm{J}(Q, H, G_2)) = \pi(\mathrm{J}(Q, H, G_1))$.

The linear homomorphism ϵ is a derivation because

$$\mathrm{J}(Q,H,G_1G_2) = \mathrm{J}(Q,H,G_1)G_2 + G_1\mathrm{J}(Q,H,G_2)$$

and π is a linear homomorphism.

Lnd ∂ defines a degree function on S and we can lift $\deg_\partial$ on $\mathbb{C}[X,Y,Z]$ to obtain a function deg on $\mathbb{C}[X,Y,Z]:\ \deg(G) = \deg_\partial(\pi(G))$. This function is nearly an ordinary degree function with the only difference being that there are many polynomials in $\mathbb{C}[X,Y,Z]$ with $\deg = -\infty$: if $G \in (Q)$ then (and only then) $\deg(G) = -\infty$.

Consider the subring of the field of fractions of S consisting of fractions with denominators in $S^\partial \setminus 0$ and denote the result by $\mathcal{B}$. This is a subring since S^∂ is closed under multiplication. As we know ∂ can be extended on $\mathcal{B}$ and by the proof of Lemma 1 $\mathcal{B}$ contains an element t for which $\partial(t) = 1$. (The derivation ∂ is an lnd on $\mathcal{B}$.)

Denote by K the set of all polynomials in $\mathbb{C}[X,Y,Z]$ with degree zero, i.e. the preimage of $S^\partial \setminus 0$. Let $\mathcal{A} = \mathbb{C}[X,Y,Z]_K$ be the subring of the field of rational functions $\mathbb{C}(X,Y,Z)$ consisting of fractions with denominators in K. Since K is closed under multiplication $\mathcal{A}$ is a ring. The projection π can be extended to $\mathcal{A}$ with image $\mathcal{B}$. Take any preimage T of $t:\ \pi(T) = t$.

By Lemma 1 any element $b \in \mathcal{B}$ can be written as $b = \sum_{i=0}^n b_i t^{n-i}$ where $b_i \in \mathcal{B}^\partial$. Hence any element a of $\mathcal{A}$ can be written as $a = \sum_{i=0}^n a_i T^{n-i}$ where $\pi(a_i) \in \mathcal{B}^\partial$, i.e. $a_i \in L$, the field of fractions of K. So

$$1 = \mathrm{J}(X,Y,Z) = \sum \mathrm{J}(X_iT^i, Y_jT^j, Z_kT^k)$$

where $\pi(X_i),\ \pi(Y_j),\ \pi(Z_k) \in \mathcal{B}^\partial$.

Using that the Jacobian is skew-symmetric and is a derivation in every argument we can rewrite each of these summands as a linear combination with coefficients in $\mathcal{A}$ of the Jacobians of the following two types: $\mathrm{J}(U_1, U_2, U_3)$ and $\mathrm{J}(U_1, U_2, T)$ where $\pi(U_i) \in \mathcal{B}^{\partial}$.

We are going to show that $\mathrm{J}(U_1, U_2, U_3) \in (Q)$ and that $\mathrm{J}(U_1, U_2, T)$ is congruent modulo (Q) to $\mathrm{J}(Q, H, T)$ multiplied by an element of $\mathcal{A}$.

Since $\pi(U_i) \in \mathcal{B}^{\partial}$ and $\mathrm{trdeg}(\mathcal{B}^{\partial}) = 1$ (Remark 4) elements $\pi(U_i)$ and $\pi(H)$ are algebraically dependent. Therefore for any pair U_i, H there is a polynomial f_i such that $f_i(H, U_i) = P_i Q$. We can assume that all f_i are irreducible.

Now, some boring computations.

$\mathrm{J}(f_1(H, U_1), U_2, U_3) = \mathrm{J}(H, U_2, U_3)\frac{\partial f_1}{\partial H} + \mathrm{J}(U_1, U_2, U_3)\frac{\partial f_1}{\partial U_1} =$

$\mathrm{J}(P_1 Q, U_2, U_3) \equiv P_1 \mathrm{J}(Q, U_2, U_3) \pmod{(Q)}$.

Since f_1 is irreducible and $H,\ U_1 \in K$ both $\frac{\partial f_1}{\partial H}$ and $\frac{\partial f_1}{\partial U_1}$ are in $K \setminus (Q)$ and it remains to show that $\mathrm{J}(H, U_2, U_3) \in (Q)$ and $\mathrm{J}(Q, U_2, U_3) \in (Q)$.

Next, $\mathrm{J}(H, f_2, U_3) = \mathrm{J}(H, H, U_3)\frac{\partial f_2}{\partial H} + \mathrm{J}(H, U_2, U_3)\frac{\partial f_2}{\partial U_2} = \mathrm{J}(H, P_2 Q, U_3) \equiv P_2 \mathrm{J}(H, Q, U_3) \pmod{(Q)}$ and $\mathrm{J}(H, U_2, U_3)\frac{\partial f_2}{\partial U_2} \equiv P_2 \mathrm{J}(H, Q, U_3) \pmod{(Q)}$;

$J(Q, f_2, U_3) = J(Q, H, U_3)\frac{\partial f_2}{\partial H} + J(Q, U_2, U_3)\frac{\partial f_2}{\partial U_2} = J(Q, P_2 Q, U_3) \equiv 0 \pmod{(Q)}$.

It remains to show that $\mathrm{J}(Q, H, U_3) \equiv 0 \pmod{(Q)}$.

$\mathrm{J}(Q, H, f_3) = \mathrm{J}(Q, H, H)\frac{\partial f_3}{\partial H} + \mathrm{J}(Q, H, U_3)\frac{\partial f_3}{\partial U_3} = \mathrm{J}(Q, H, P_3 Q) \equiv 0 \pmod{(Q)}$.

Hence $\mathrm{J}(Q, H, U_3) \equiv 0 \pmod{(Q)}$ and $\mathrm{J}(U_1, U_2, U_3) \equiv 0 \pmod{(Q)}$.

Finally we will check that Jacobians $\mathrm{J}(U_1, U_2, T)$ are congruent modulo (Q) to $\mathrm{J}(Q, H, T)$ multiplied by an element of $\mathcal{A}$.

$\mathrm{J}(f_1(H, U_1), U_2, T) = \mathrm{J}(H, U_2, T)\frac{\partial f_1}{\partial H} + \mathrm{J}(U_1, U_2, T)\frac{\partial f_1}{\partial U_1} =$

$\mathrm{J}(P_1 Q, U_2, T) \equiv P_1 \mathrm{J}(Q, U_2, T) \pmod{(Q)}$.

$\mathrm{J}(H, f_2, T) = \mathrm{J}(H, H, T)\frac{\partial f_2}{\partial H} + \mathrm{J}(H, U_2, T)\frac{\partial f_2}{\partial U_2} = \mathrm{J}(H, P_2Q, T) \equiv$
$P_2\mathrm{J}(H, Q, T) \pmod{(Q)}$; $\mathrm{J}(H, U_2, T)\frac{\partial f_2}{\partial U_2} \equiv P_2\mathrm{J}(H, Q, T) \pmod{(Q)}$.
$\mathrm{J}(Q, f_2, T) = \mathrm{J}(Q, H, T)\frac{\partial f_2}{\partial H} + \mathrm{J}(Q, U_2, T)\frac{\partial f_2}{\partial U_2} = \mathrm{J}(Q, P_2Q, T) \equiv 0 \pmod{(Q)}$.

The derivative $\frac{\partial f_2}{\partial U_2}$ is a polynomial in H and U_2 which are preimages of elements from $\mathcal{B}^\partial$. The projection $\pi(\frac{\partial f_2}{\partial U_2}) \in S^\partial \setminus 0$ because we assumed that f_2 is an irreducible polynomial. Hence $\mathrm{J}(H, U_2, T)$ and $\mathrm{J}(Q, U_2, T)$ are proportional to $\mathrm{J}(Q, H, T)$ with coefficients from $\mathcal{A}$ and thus $\mathrm{J}(U_1, U_2, T)$ is congruent modulo (Q) to $\mathrm{J}(Q, H, T)$ multiplied by an element of $\mathcal{A}$.

Therefore $1 = \mathrm{J}(X, Y, Z) \equiv a\mathrm{J}(Q, H, T) \pmod{(Q)}$ for some $a \in \mathcal{A}$, i. e. $1 = \pi(a\mathrm{J}(Q, H, T)) = \pi(a)\pi(\mathrm{J}(Q, H, T))$. Since $\pi(a) \in \mathcal{B}$ its ∂-degree is nonnegative. Hence $\deg_\partial(a) = \deg_\partial(\mathrm{J}(Q, H, T)) = 0$.

To finish the proof observe that we showed that
(a) $\mathrm{J}(Q, H, U) \in (Q)$ if $\deg_\partial(U) = 0$, so $\epsilon(u) = 0$ if $u \in S^\partial$;
(b) $\deg_\partial(\mathrm{J}(Q, H, T)) = 0$, so $\epsilon(t) \in S^\partial \setminus \mathbb{C}$.
So ϵ is an lnd on S and $\ker(\epsilon) = \ker(\partial)$ since $\ker(\epsilon) \supset \ker(\partial)$ and $\ker(\epsilon)$ and $\ker(\partial)$ are algebraically closed in S (see Remark 2). Then (b) shows that ∂ and ϵ give the same degree function and therefore are equivalent. □

Remark 5. We will be using the following description of $\epsilon(g)$:
$\epsilon(g) \equiv \mathrm{J}(Q, H, G)$ where G is a preimage of g. To make it a derivation on S we will consider the right side modulo the ideal (Q).

Remark 6. It turns out that a similar description of lnds is possible for any finitely generated domain (see [ML3]).

Let us also recall the following construction for $\mathbb{C}[X, Y, Z]$. We can take some real valued weights $w(X)$, $w(Y)$, and $w(Z)$, define $w(X^iY^jZ^k) =$

$iw(X) + jw(Y) + kw(Z)$, and extend w to polynomials by defining $w(p)$ be the maximal weight among the weights of all monomials which are present in p with nonzero coefficients. Then any $p \in \mathbb{C}[X, Y, Z]$ can be written as $p = \sum_{i=u}^{v} p_i$ where each p_i is homogeneous, i. e. consists only of monomials with the same weight, and $w(p_i) < w(p_{i+1})$. We will call $\bar{p} = p_v$ the leading form of p.

Results and proofs.

Theorem 1. If $R = \mathbb{C}[X, Y, Z]/(Q)$, where $Q = X^n Y^m - p(Z)$ and p is a polynomial of degree d, has a nonzero lnd and m, n, and d are relatively prime then $(d-1)(n-1)(m-1) = 0$.

Proof. Let ∂ be a nonzero lnd on R and let $h \in \ker(\partial) \setminus \mathbb{C}$. We may assume that $\deg_z(h) < d$ since $z^d = x^n y^m + (z^d - p(z))$ in R. Let us replace this derivation by ϵ described in Remark 5: $\epsilon(g) \equiv \mathrm{J}(Q, H, G)$ where H is a preimage of h such that $\deg_Z(H) < d$.

Let us take weights $w(X) = m + dN$, $w(Y) = -n$, and $w(Z) = nN$ where N is a natural number. The leading form $\bar{Q}$ of Q is $X^n Y^m - Z^d$ for any N. The leading form $\bar{H}$ of H may depend on N. Let us check that by taking N sufficiently large we can make $\bar{H} = X^i Y^j Z^k$. Indeed, if monomials $X^{i_1} Y^{j_1} Z^{k_1}$ and $X^{i_2} Y^{j_2} Z^{k_2}$ are in $\bar{H}$ then $N(di_1 + nk_1) + mi_1 - nj_1 = N(di_2 + nk_2) + mi_2 - nj_2$. If $N > m \deg_X(H) + n \deg_Y(H)$ then $di_1 + nk_1 = di_2 + nk_2$ and therefore $mi_1 - nj_1 = mi_2 - nj_2$. Hence

$$d(i_1 - i_2) + n(k_1 - k_2) = 0$$

and

$$m(i_1 - i_2) - n(j_1 - j_2) = 0$$

We assumed that $(n, m, d) = 1$. Therefore $i_1 - i_2 = ns$, $k_1 - k_2 = -ds$, and $j_1 - j_2 = ms$ where s is an integer. But then $s = 0$ since $|k_1 - k_2| < d$.

Let us fix such a sufficiently large N for which $\bar{H}$ is a monomial $X^i Y^j Z^k$.

Consider now a derivation $\bar{\epsilon}(G) = \mathrm{J}(\bar{Q}, \bar{H}, G)$. We can observe that the projection of this derivation on $\bar{R} = \mathbb{C}[X, Y, Z]/(\bar{Q})$ is locally nilpotent on $\bar{R}$. Indeed, it is easy to see that $\mathrm{J}(\bar{Q}, \bar{H}, \bar{G})$ is either $\overline{\mathrm{J}(Q, H, G)}$ or zero. Since ϵ is lnd on R we know that after several applications of a derivation $D(-) = \mathrm{J}(Q, H, -)$ to G we obtain a polynomial which is divisible by Q. It implies, of course, that the leading form of this polynomial is divisible by the leading form of Q. So if we apply at most the same number of times $\bar{\epsilon}$ to $\bar{G}$ we get a polynomial which is divisible by $\bar{Q}$. It may happen that we'll get zero or a polynomial which is divisible by $\bar{Q}$ on one of the previous steps.

Condition $(n, m, d) = 1$ makes $\bar{Q} = X^n Y^m - Z^d$ irreducible. Hence $\bar{R}$ is a domain. As we saw, in this setting the product of two nonzero elements is an $\bar{\epsilon}$-constant only if both factors are constants. Since $\bar{\epsilon}(\bar{H}) = \bar{\epsilon}(X^i Y^j Z^k) = 0$ we can conclude that either x, or y, or z is a constant of $\pi(\bar{\epsilon})$. (Here x, y, and z are the images of X, Y, and Z in $\bar{R}$.) So according to Lemma 4 one of the derivations $\epsilon_x(-) = J(X^n Y^m - Z^d, X, -)$, $\epsilon_y(-) = J(X^n Y^m - Z^d, Y, -)$, $\epsilon_z(-) = J(X^n Y^m - Z^d, Z, -)$ induces a locally nilpotent derivation on $\bar{R}$.

Now, $\epsilon_x(X) = 0$, $\epsilon_x(Y) = -dZ^{d-1}$, $\epsilon_x(Z) = -mX^n Y^{m-1}$. To see when the induced derivation is an lnd let us use the degree function defined by this derivation on $\bar{R}$. Denote by d_x, d_y, and d_z the degrees of x, y, and z

correspondingly. Then $d_x = 0$, $d_y - 1 = (d-1)d_z$, and $d_z - 1 = (m-1)d_y$. Thus $-2 = (m-2)d_y + (d-2)d_z$. Since d_y and d_z are natural numbers this equality is possible only if either $m = 1$ or $d = 1$. In both these cases $\pi(\epsilon_x)$ is an lnd.

For ϵ_y we have $\epsilon_y(X) = dZ^{d-1}$, $\epsilon_y(Y) = 0$, $\epsilon_y(Z) = nX^{n-1}Y^m$. This case is similar to the previous one and $\pi(\epsilon_y)$ is an lnd if and only if either $n = 1$ or $d = 1$.

Finally $\epsilon_z(X) = mX^nY^{m-1}$, $\epsilon_z(Y) = -nX^{n-1}Y^m$, $\epsilon_z(Z) = 0$. Using the degree function which would be defined by $\pi(\epsilon_z)$ we can see that $\pi(\epsilon_z)$ is never an lnd.

This finishes the proof of Theorem 1. □

We have now the following cases in which there is a nonzero lnd on R: $d = 1$ which corresponds to the polynomial ring in two variables independently of values of n and m; $n = 1$; $m = 1$.

If $d > 1$ and R has a nonzero lnd then either $n = 1$ of $m = 1$ and we may assume without loss of generality that $m = 1$: if $m \neq 1$, $n = 1$ we will switch x and y.

From now on $d > 1$ and R is given by a relation $x^n y = p(z)$.

Theorem 2. Let ∂ be a nonzero lnd of $R = \mathbb{C}[X,Y,Z]/(Q)$ where $Q = X^nY - p(Z)$ and let $h \in \ker(\partial) \setminus \mathbb{C}$. Then there exists an automorphism α of R such that $\alpha(h) = q(x)$.

Proof. We will be choosing different weights for X, Y, and Z in the course of the proof of this Theorem. Since for all these choices the weight of Z will be positive and $nw(X) + w(Y) = dw(Z)$, the leading form of Q for these weights will be $X^nY - Z^d$.

As above, we can take a preimage H of h for which $\deg_Z(H) < d$. Let us use again the weights $w_1(X) = 1 + dN$, $w_1(Y) = -n$, and $w_1(Z) = nN$. As we saw in the proof of Theorem 1 we can conclude that if N is very large then the leading form $\bar{H}$ of H is either X^i or Y^j. (It cannot be a product X^iY^j since then $\ker(\pi(\bar{\epsilon})) \ni x, y$ which is possible only if $\pi(\bar{\epsilon}) = 0$.) We can also observe that if $\bar{H} = Y^j$ with our choice of N then $h \in \mathbb{C}[y]$ since then $w_1(H) < 0$ while the weight of any monomial which contains X or Z is positive if N is large enough. (This, of course, imply that $\ker(\partial) = \mathbb{C}[y]$ and $(n-1)(d-1) = 0$; so $n = 1$ and there exists an automorphism of R sending y to x.)

Let us use now different weights: $w_2(X) = -1$, $w_2(Y) = n + dN$, and $w_2(Z) = N$. Again, if N is sufficiently large the leading form of H is a monomial. We already know that this monomial is either X^i or Y^j.

If it is X^i then $h \in \mathbb{C}[x]$ and $\pi(\epsilon_x(g)) \equiv \mathrm{J}(X^nY - p(Z), X, G)$ is indeed an lnd, and if it is Y^j then $n = 1$.

So we see that if $(n-1)(d-1) \neq 0$ then $h \in \mathbb{C}[x]$. It remains to consider the case $n = 1$ with an additional assumption that $h \notin (\mathbb{C}[x] \cup \mathbb{C}[y])$. Then the leading form of H relative to w_1 is and X^a and the leading form of H relative to w_2 is Y^b.

Since $x \to y$, $y \to x$, $z \to z$ is an automorphism of R when $n = 1$ we may also assume that $a \geq b$.

Let us now chose natural positive weights $w_3(X) = \rho$, $w_3(Y) = \sigma$, $w_3(Z) = \tau$ so that $a\rho = b\sigma$, $\rho + \sigma = d\tau$, and ρ and τ are relatively prime. (If

k divides ρ and τ then k divides σ and we can cancel it.)

Denote by $\bar{H}_3$ the leading form of H relative to w_3. Then $\bar{H}_3$ contains both X^a and Y^b. Indeed if $w_3(H) = a\rho = b\sigma$, then both X^a and Y^b are in $\bar{H}_3$. Otherwise, since $\rho > 0$, $w_3(H) > a\rho$ and $\bar{H}_3$ contains a monomial $X^iY^jZ^k$ for which $i\rho + j\sigma + k\tau > a\rho$.

To bring this to a contradiction let us consider the weights
$w_4(X) = \rho - d\delta_1$, $w_4(Y) = \sigma - d\delta_2$, $w_4(Z) = \tau - \delta_1 - \delta_2$ where δ_1 and δ_2 satisfy the following conditions:

1) $da\delta_1 + db\delta_2 + \deg_Z(\bar{H}_3)(\delta_1 + \delta_2) < w_3(\bar{H}_3) - a\rho$.

2) δ_1 and δ_2 are positive irrational numbers which are linearly independent over the field of rational numbers.

3) $w_4(X) > 0, \quad w_4(Y) > 0, \quad w_4(Z) > 0$.

Then $\bar{H}_4$ for w_4 is a monomial in force of condition 2) and this monomial cannot be neither X^a nor Y^b since in force of condition 1) $w_4(X^iY^jZ^k) = w_3(H) - id\delta_1 - jd\delta_2 - k(\delta_1 + \delta_2) > w_3(X^a) = w_3(Y^b)$ while $w_4(X^a) = a\rho - ad\delta_1 < w_3(X^a)$ and $w_4(Y^b) = b\sigma - bd\delta_2 < w_3(Y^b)$. As we already know it is impossible and hence $\bar{H}_3 = \mu X^a + \ldots + \nu Y^b$.

Consider now $X^b\bar{H}_3$. This polynomial can be rewritten as a polynomial $\psi \in \mathbb{C}[X, Z]$ since $XY = Z^d$ in $\bar{R}$.

The polynomial ψ is ρ, τ homogeneous, so $\psi = c\prod_i(X^\tau - c_iZ^\rho)$ and $\bar{H}_3 = c\prod_i(X^\tau - c_iZ^\rho)X^{-b}$. Let us replace $\bar{H}_3$ by $\bar{H}_3^d$.

Lemma 5. $(x^\tau - c_iz^\rho)^d x^{-\rho} \in \bar{R}$.

Proof. It is sufficient to show that any monomial $x^{i\tau-\rho}z^{(d-i)\rho} \in \bar{R}$. Of course, any monomial of this kind with $i\tau - \rho \geq 0$ is in $\bar{R}$. If $i\tau - \rho < 0$ then

$(d-i)\rho > d(\rho - i\tau)$ since $d\tau = \rho + \sigma$ and the corresponding monomial is equal to $y^{\rho-i\tau}z^{(d-i)\rho-d(\rho-i\tau)} \in \bar{R}$. $\square$

The form $\bar{H}_3^d$ can be written as $c\prod_i[(X^\tau - c_iZ^\rho)^d X^{-\rho}]$ and each of the factors $(x^\tau - c_iz^\rho)^d x^{-\rho}$ belongs to $\bar{R}$.

As we know the derivation which is induced on $\bar{R}$ by $\bar{\epsilon}(-) = \mathrm{J}(XY - Z^d, \bar{H}_3, -)$ is an lnd. Since $\bar{\epsilon}(\bar{H}_3^d) = 0$ each of these factors is in the kernel of the $\pi(\bar{\epsilon})$ and if there are two different factors then $\ker(\pi(\bar{\epsilon}))$ has the transcendence degree 2 and $\pi(\bar{\epsilon}) = 0$. Since it is not the case, there is just one factor. Furthermore, since $x \to \lambda x$, $y \to \lambda^{-1}y$, $z \to z$ is an automorphism of $\bar{R}$ it remains to find out for which ρ, τ, and d the derivation of $\bar{R}$ given by $\pi(\bar{\epsilon})(g) \equiv \mathrm{J}(XY - Z^d, (X^\tau - Z^\rho)^d X^{-\rho}, G)$ is an lnd.

Let us compute $\pi(\bar{\epsilon})(z)$:

$\pi(\bar{\epsilon})(z) \equiv \mathrm{J}(XY - Z^d, (X^\tau - Z^\rho)^d X^{-\rho}, Z) = \mathrm{J}_{X,Y}(XY, (X^\tau - Z^\rho)^d X^{-\rho}) =$
$-X[d(X^\tau - Z^\rho)^{d-1}\tau X^{\tau-1-\rho} - \rho(X^\tau - Z^\rho)^d X^{-\rho-1}] \equiv$
$[\rho - \tau d x^\tau (x^\tau - z^\rho)^{-1}](x^\tau - z^\rho)^d x^{-\rho}$.

Now, $\pi(\bar{\epsilon})((x^\tau - z^\rho)^d x^{-\rho}) = 0$, so $\pi(\bar{\epsilon})(x^\tau(x^\tau - z^\rho)^{-1}) \neq 0$ since $\rho \neq d\tau$.

Let us denote by deg the degree induced by $\bar{\epsilon}$. Then $\deg((x^\tau - z^\rho)^d x^{-\rho}) = 0$ and $\deg(z) - 1 = \deg(\rho - \tau d x^\tau (x^\tau - z^\rho)^{-1}) \neq 0$.

We see that $\deg(x^\tau(x^\tau - z^\rho)^{-1}) = \deg(z) - 1 > 0$. This is possible only if $\deg(x^\tau) = \deg(z^\rho) > \deg(x^\tau - z^\rho)$. So
$\tau\deg(x) - \rho\deg(z) = 0$,
$\tau\deg(x) - \deg(z) - \deg(x^\tau - z^\rho) = -1$ and
$\rho\deg(x) - d\deg(x^\tau - z^\rho) = 0$.

Solving this system we obtain $\deg(x^\tau - z^\rho) = \rho^2[\rho^2 - \tau d(\rho - 1)]^{-1}$. Now,

$\rho^2 - \tau d(\rho-1) = \rho^2 - (\rho+\sigma)(\rho-1) = \rho+\sigma-\rho\sigma = 1-(\rho-1)(\sigma-1)$ since $\tau d = \rho+\sigma$. Since $\deg(x^\tau - z^\rho) > 0$ we should have $1-(\rho-1)(\sigma-1) > 0$ which is possible only if $(\rho-1)(\sigma-1) = 0$. Since $\rho a = \sigma b$ and $a \geq b$ we have $\sigma \geq \rho$ and so $\rho = 1$ if $\bar{\partial} = \pi(\bar{\epsilon})$ is an lnd on $\bar{R}$.

If $\rho = 1$ then $\deg(x^\tau - z) = 1,\ \deg(x) = d,\ \deg(z) = d\tau$. Hence if $\bar{\partial}$ is an lnd then $\bar{\partial}(x^\tau - z) = \lambda_1 \in \bar{R}^{\bar{\partial}}$.

Since $\bar{\partial}(x^\tau - z) \equiv \mathrm{J}(XY - Z^d, (X^\tau - Z)^d X^{-1}, X^\tau - Z) = -(X^\tau - Z)^d X^{-1} \equiv -(x^\tau - z)^d x^{-1} = \lambda_1 \in \bar{R}^{\bar{\partial}}$ we can put $x^\tau - z = \lambda_1 t$ where $\bar{\partial}(t) = 1$. Then $x = -\lambda_1^{d-1} t^d \in \mathrm{Nil}_{\bar{R}}(\bar{\partial}),\ z = x^\tau - \lambda_1 t \in \mathrm{Nil}_{\bar{R}}(\bar{\partial})$ and $y = z^d x^{-1} = -\lambda_1^{1-d}((-\lambda_1)^{(d-1)\tau} t^{d\tau-1} - \lambda_1)^d \in \mathrm{Nil}_{\bar{R}}(\bar{\partial})$, i.e. $\bar{\partial}$ is an lnd on $\bar{R}$.

We checked that if $a \geq b$ then $\bar{H}_3 = c(X^\tau - c_1 Z)^k X^{-b}$. Therefore the leading form of h relative to the weight given by $w_3(x) = \rho,\ w_3(y) = \sigma,\ w_3(z) = \tau$ is $c(x^\tau - c_1 z)^k x^{-b}$.

Observe that a homomorphism β given by $x \to x$, $y \to (p(z + c_1^{-1} x^\tau)) x^{-1}$, $z \to z + c_1^{-1} x^\tau$ is an automorphism of R. If we apply this automorphism to h then the leading form of h, as an element of $\bar{R}$ becomes $c[x^\tau - c_1(z + c_1^{-1} x^\tau)]^k x^{-b} = c(-c_1 z)^k x^{-b} = \nu y^b$. (Hence $k = bd$.)

Therefore $\deg_y(\beta(h)) = \deg_y(h)$ while $\deg_x(\beta(h)) < \deg_x(h)$. If $\beta(h) \in \mathbb{C}[y]$ we can finish the proof since $x \to y$, $y \to x$, $z \to z$ is an automorphism of R. If $\beta(h) \notin \mathbb{C}[y]$ we can find an automorphism which will decrease either $\deg_x$ or $\deg_y$ of $\beta(h)$. Since these degrees cannot decrease indefinitely, a composition of several automorphisms of this type and, possibly, an automorphism exchanging x and y gives an automorphism α such that $\alpha(h) = q(x)$. $\square$

Conclusion.

We proved that there is only the zero lnd on $R = \mathbb{C}[X,Y,Z]/(X^nY^m - p(z))$, $\deg(p) = d$ when $(d-1)(m-1)(n-1) \neq 0$ and $(d,m,n) = 1$; when $(d-1)(m-1) \neq 0$ and $n = 1$ or when $(d-1)(n-1) \neq 0$ and $m = 1$ all nonzero lnds have the same kernel; when $d = 1$ or when $n = m = 1$ there are lnds with different kernels but each kernel can be mapped on a "standard" one by an automorphism.

Lemma 6. Locally nilpotent derivations of a domain A with the same kernel are equivalent to each other.

Proof. Assume that nonzero lnds ∂_1 and ∂_2 of A have the same kernel K. We know that $\mathrm{Nil}_F(\partial_1) = F^{\partial_1}[t_1]$ and $\mathrm{Nil}_F(\partial_2) = F^{\partial_2}[t_2]$ where F is the field of fractions of A (Lemma 1) and that $F^{\partial_1} = F^{\partial_2} = L = \mathrm{Frac}(K)$ (Lemma 3). We may assume that $at_1 \in A$ for some $a \in K \setminus 0$ (see the proof of Lemma 1). Then $\partial_2^i(at_1) = a\partial_2^i(t_1)$ for any i. Hence $t_1 \in \mathrm{Nil}_F(\partial_2)$ and $t_1 = \sum_i f_i t_2^i$ where $f_i \in L$. Similarly, $t_2 = \sum_j f_j t_1^j$ where $f_j \in L$. Hence $\deg_{t_2}(t_1) = \deg_{t_1}(t_2) = 1$ and Lemma is proved. □

Remark 7. All these derivations are proportional to each other over F^∂ and any linear combination of these derivations with coefficients in K is again an lnd with the kernel K. By Lemma 2 at least one of these derivations is irreducible. If A is not a unique factorization domain then there may be several irreducible derivations among these derivations. (It would be interesting to find an example.)

Theorem 3. If R is a ring satisfying conditions of Theorem 1 then, up to an automorphism (and multiplication by $c \in \mathbb{C}$), there is just one nonzero irreducible lnd of R. It is defined by $\partial(x) = 0,\ \partial(y) = p'(z),\ \partial(z) = x^n$.

Proof. If ϵ is an lnd of R with $R^\epsilon = \mathbb{C}[x]$ then $\epsilon = \frac{q_1(x)}{q_2(x)}\partial$ and we can assume that polynomials $q_1,\ q_2$ are relatively prime. We can find two polynomials $p_1,\ p_2 \in \mathbb{C}[x]$ such that the lnd $\epsilon_1 = p_1\epsilon + p_2\partial = \frac{1}{q_2(x)}\partial$. Therefore $\epsilon_1(y) = \frac{p'(z)}{q_2(x)} \in R$ and $\epsilon_1(z) = \frac{x^n}{q_2(x)} \in R$. If $q_2(x) \notin \mathbb{C}$ then $\frac{p'(z)}{x} \in R = \mathbb{C}[x, \frac{p(z)}{x^n}, z]$. Assume that $\frac{p'(z)}{x} = r(x, \frac{p(z)}{x^n}, z)$ where $r(x, y, z) \in \mathbb{C}[x, y, z]$. Let us take $w(x) = 1,\ w(z) = \lambda$ where λ is a positive irrational number, such that all monomials of $r(x, y, z)$ have different weights. Then $w(\frac{p'(z)}{x}) = i + j(d\lambda - n) + k\lambda$ for some nonnegative integers $i,\ j,\ k$, i.e. $(d-1)\lambda - 1 = i + j(d\lambda - n) + k\lambda$. Since λ is irrational, $i - jn + 1 = 0$ and $jd + k - d + 1 = 0$. Hence $j = 0$. But then $i = -1$, which is impossible. Hence $q_2 \in \mathbb{C}$. $\square$

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Address: Department of Mathematics, Wayne State University, Detroit, MI 48202, USA

Faculty of Mathematics and Computer Science, Weizmann Institute of Science, Rehovot, 7610001 Israel.

E-mail address: lml@wayne.edu