Eccentricity evolution of spinning binaries

Sayak Datta*

Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut), D-30167 Hannover, Germany and Leibniz Universität Hannover, D-30167 Hannover, Germany

(Dated: June 23, 2023)

We study the evolution of the eccentricity of an eccentric orbit with spinning components. We develop a prescription to express the evolving eccentricity in terms of initial eccentricity and frequency. For that purpose we considered the spins to be perpendicular to the orbital plane. Using this we found an analytical result for the contribution of spin in eccentricity evolution. As a result, we expressed orbital eccentricity in a series of initial eccentricity and gravitational wave frequency. The prescription developed here can easily be used to find arbitrarily higher-order contributions of initial eccentricity. With the eccentricity evolution at hand, we computed the evolving energy and angular momentum fluxes for eccentric orbit with spinning components. This result can be used to construct the waveforms of spinning compact objects in an eccentric orbit.

I. INTRODUCTION

In recent times, the detection of gravitational waves (GWs) from the coalescence of compact binaries with ground-based detectors [1, 2] has opened up a new era of astronomy [3, 4]. Most of the sources are believed to be black hole (BH) binaries. The merger of two neutron stars (NSs) was also observed in the event GW170817 [5], and possibly also GW190425 [6]. More recently, detections of GW200105 and GW200115 [7] were also made where it is believed that it is BH-NS binary. Currently, the existing detectors are continuously being upgraded. Alongside, there are proposals for several next-generation ground-based detectors such as the Einstein telescope [8] and cosmic explorer [9]. These detectors will be significantly more sensitive compared to the current detectors. As a result, it will be possible to measure very small features in the signals. Similarly, space-based detectors such as Laser Interferometer Space Antenna (LISA) [10] are also being built. LISA will observe binaries comprising supermassive bodies. These sources will be either very loud or will last very long in the detector for the detector to measure very small features in the signal. Therefore modeling the signals as accurately as possible has become a necessity.

In the context of GW astronomy, primarily the focus has been on the circular orbits. This is reasonable as we expect the stellar mass binaries to have low eccentricities [11, 12]. However, compact binaries that formed via the dynamical interactions in dense stellar environments or through the Kozai-Lidov processes [13, 14], are expected to retain residual eccentricities $e_0 \gtrsim .1$ during observation with ground-based detectors [15]. Therefore, the measurement of nonzero eccentricities of the binaries may shed light on our understanding of the formation channels. On the other hand extreme mass ratio inspirals are also expected to have large eccentricities in the observable band [16]. Hence, it is crucial to understand and model the effects of eccentricity on a binary.

Interestingly, some of the recent analyses [17–21] support the presence of eccentricity in the observed binary black hole (BBH) events. Alongside, it was argued in Ref. [22], that the presence of even smaller eccentricities $e_0 \sim .01 - .05$ may induce systematic biases in parameter estimation analyses. Since the next generation ground-based detectors will have improved low-frequency sensitivity compared to the current generation detectors, these detectors can make confident observations of eccentric systems [23].

There have been efforts in the past to model the inspiral waveforms from compact binary mergers [24–32]. Although the effect of spins has been modeled within the post-Newtonian formalism [33–43], a combined treatment including spins and eccentricity is largely absent. There have been efforts such as those of Ref. [44–53] that attempt to address this concern to some extent. However, we still lack waveforms as per the requirement.

Although there are some works on eccentric waveforms, mostly they do not consider the effect of the interaction between spin and eccentricity. Keeping this in mind, we will try to find the eccentricity evolution and fluxes due to such interaction. In Ref. [47] the equations governing the time evolution of the orbital parameters, and in particular the eccentricity, including the spin-orbit and the spin-spin couplings were computed. These are needed to accurately compute the post-Newtonian approximation up to 2PN accuracy. However, these results are not sufficient to compute GW waveforms. The computed results depend on the "instantaneous" eccentricity of the orbit. However, in the inspiral time scale, the eccentricity of an orbit evolves. Hence, it is important to express the instantaneous eccentricity in terms of some "initial eccentricity" and GW frequency. In the leading order of initial eccentricity for nonspinning orbit this was explored in a few works [25, 30, 54]. Some, higher order terms of initial eccentricity were computed in ref. [55], albeit for non-spinning binaries. However, in ref. [32] a more comprehensive prescription was developed for non-spinning binaries, that can be used to iteratively find the arbitrarily higher order terms in initial eccentricities and

*Electronic address: sayak.datta@aei.mpg.de

GW frequency.

In the current work we extend the results of Ref. [32] to spinning binaries. Therefore, it will be for the first time that the eccentricity evolution of spinning binaries will be analytically expressed in terms of initial eccentricity and GW frequency. This as a result provides eccentricity-spin coupling terms in fluxes. We develop the prescription for spins perpendicular to the orbital plane. The prescription discussed here can be extended to arbitrarily high order in eccentricity, in principle. With the newly found expressions, we also compute the leading order shifts in the fluxes due to the eccentricity-spin coupling terms. Therefore these results can be directly used to model the GW waveforms for spinning bodies in an eccentric orbit.

II. EQUATIONS OF ECCENTRIC ORBITS

Our purpose in the current work is to formulate a method that can be used to analytically compute the eccentricity of an orbit with spinning components with respect to the emitted GW frequency. As a result, this can be used to compute frequency-dependent fluxes and consequently the GW waveform. For this purpose we will primarily follow the notations and discussions in Ref. [47]. The spin-orbit couplings appear at 1.5PN order and spin-spin couplings appear at 2PN order. For this reason Ref. [47] considered only the Newtonian and spin-coupling terms in the equations of motion. For simplicity, we will use a system of units where $G = c = M = 1$, where M is the total mass of the system. The boldface represents a 3−vector and a hat above represents a unit vector. The Lagrangian of the system in the center of mass frame used in Ref. [44, 56]:

$$\begin{aligned}\mathcal{L} =&\frac{\nu}{2}v^2 + \frac{\nu}{r} + \frac{\nu}{2}(\mathbf{v}\times\mathbf{a}).\xi - \frac{2\nu}{r^3}(\mathbf{x}\times\mathbf{v}).(\zeta+\xi)\\ &+ \frac{1}{r^3}\mathbf{S_1}.\mathbf{S_2} - \frac{3}{r^5}(\mathbf{x}.\mathbf{S_1})(\mathbf{x}.\mathbf{S_2}),\end{aligned} \tag{1}$$

where $\nu = m_1 m_2$ and

$$\zeta = \mathbf{S_1} + \mathbf{S_2} \tag{2}$$

$$\xi = \tfrac{m_2}{m_1}\mathbf{S_1} + \tfrac{m_1}{m_2}\mathbf{S_2}. \tag{3}$$

A quasi-Keplerian solution to the equation of motion derived from the Lagrangian in Eq. (1) was found in Ref. [47]

$$r = a(1 - e_r\cos u) + f_r\cos 2(v-\psi) \tag{4}$$

$$\phi = (1+k)v + f_{\phi,1}\sin(v-2\psi) + f_{\phi,2}\sin 2(v-\psi) \tag{5}$$

$$v = 2\arctan\left(\sqrt{\tfrac{1+e_\phi}{1-e_\phi}}\tan\tfrac{u}{2}\right) \tag{6}$$

$$l = n(t-t_0) = u - e_t\sin u, \tag{7}$$

where (r, ϕ) is a polar coordinate system in the plane of motion, n is the mean motion, u, v, and l are the eccentric, true, and mean anomalies, a is the semi-major axis, e_t, e_r, and e_ϕ are eccentricities, k is the perihelion precession, and the f_i are constants [47].

In the current work, we will compute only leading order shifts in energy and angular momentum fluxes due to spin-eccentricity couplings. Therefore, we only require the following leading order results of reduced energy (E) and angular averaged reduced angular momentum (L)[47],

$$E = -\tfrac{x}{2} \tag{8}$$

$$L = \tfrac{\sqrt{1-e_t^2}}{x^{1/2}}\left[1 - \tfrac{x^{3/2}\beta(4,3-e_t^2)}{2(1-e_t^2)^{3/2}} - \tfrac{x^2\gamma_1}{2(1-e_t^2)^2}\right], \tag{9}$$

where,

$$\beta(a,b) = a\beta_a + b\beta_b \tag{10}$$

$$\beta_a = \hat{\mathbf{J}}.\zeta \tag{11}$$

$$\beta_b = \hat{\mathbf{J}}.\xi \tag{12}$$

$$\gamma_1 = \tfrac{1}{\nu}\left[\mathbf{S_1}.\mathbf{S_2} - 3(\hat{\mathbf{J}}.\mathbf{S_1})(\hat{\mathbf{J}}.\mathbf{S_2})\right] \tag{13}$$

$$\gamma_2 = \tfrac{1}{\nu}|\hat{\mathbf{J}}\times\mathbf{S}_1||\hat{\mathbf{J}}\times\mathbf{S}_2|, \tag{14}$$

where $\mathbf{S}_i$ are the individual spins and $\mathbf{J}$ is the reduced orbital angular momentum discussed in Ref. [47]. $\mathbf{J}$ helps encapsulate the effects of spin precession. It can be checked from Eq. (10) in Ref. [47], when the spins of the bodies are either aligned or antialigned with the direction perpendicular to the orbital plane, say $\hat{\mathbf{L}}$, $\mathbf{J}$ is also aligned or antialigned with $\hat{\mathbf{L}}$. In such a case therefore, $\hat{\mathbf{J}}||\hat{\mathbf{L}}$.

Spin and eccentricity dependence of several orbital quantities were found in Ref. [47]. These can also be expressed in a similar manner as follows,

$$k = -\tfrac{x^{3/2}\beta(4,3)}{(1-e_t^2)^{3/2}} - \tfrac{3x^2\gamma_1}{2(1-e_t^2)^2} \tag{15}$$

$$e_r = e_t\left[1 - \tfrac{x^{3/2}\beta(2,1)}{\sqrt{1-e_t^2}} - \tfrac{x^2\gamma_1}{2(1-e_t^2)}\right] \tag{16}$$

$$e_\phi = e_t\left[1 - \tfrac{x^{3/2}\beta(2,2)}{\sqrt{1-e_t^2}} - \tfrac{x^2\gamma_1}{(1-e_t^2)}\right] \tag{17}$$

$$f_r = -\tfrac{x}{2(1-e_t^2)}\gamma_2 \tag{18}$$

$$f_{\phi,1} = -\tfrac{e_t x^2}{(1-e_t^2)^2}\gamma_2 \tag{19}$$

$$f_{\phi,2} = -\tfrac{x^2}{4(1-e_t^2)^2}\gamma_2. \tag{20}$$

In the quasi-Keplerian formalism, PN expansions of elliptical orbit quantities are performed most naturally in terms of the radial orbit angular frequency $\omega_r \equiv n$. This frequency is the mean motion or periastron-to-periastron angular frequency. However, the limitation of this frequency is that correspondence with the circular orbit limit is not straightforward. Unlike the eccentric case, the circular orbit quantities are more naturally expanded in terms of the azimuthal or ϕ−angular frequency

$\omega_\phi \equiv \xi_\phi/M$. For this purpose we use the variable x, defined as $\xi_\phi = x^{3/2}$ [25]. The relationship between these two frequencies is as follows,

$$\frac{n}{\xi_\phi} = 1 - \frac{3x}{1-e_t^2} - \left[18 - 28\nu + \left(51 - 26\nu\right)e_t^2\right]\frac{x^2}{4(1-e_t^2)^2} \equiv \mathcal{F}(x, e_t(x)). \tag{21}$$

III. ECCENTRICITY EVOLUTION

In this section, our focus will be on computing the eccentricity of a binary in terms of binary frequency. Currently, these results are available for non-spinning binaries [25, 32]. In this section, we will construct a prescription that can be used to find the eccentricity in the power of initial eccentricity and frequency to an arbitrary power for a spinning binary.

In Sec. II we discussed the parametric solution in the quasi-Keplerian approach. This is done using the constants of the motions considering only the conservative part. However, during orbital motion, the system emits GWs that carry away energy and angular momentum. As a result, the conserved quantities start to change in the inspiral time scale. This corresponds to the dissipative part of the equation of motion. Conventionally it is tackled by solving the evolution equation of the conserved quantities in the inspiral time scale. For the current work, we only require the evolution equation of the e_t and n. Their evolution equation can be expressed as [47, 54],

$$\begin{aligned}\frac{dn}{dt} &= \frac{\nu x^{11/2}}{(1-e_t^2)^{7/2}}\Big(\mathcal{N}_n + x\mathcal{N}_1 + x^{3/2}\mathcal{N}_{hered} - x^{3/2}\mathcal{N}_\beta - x^2\mathcal{N}_\sigma + x^2\mathcal{N}_\tau\Big)\\ 2e_t\frac{de_t}{dt} &= \frac{-\nu x^4}{(1-e_t^2)^{5/2}}\Big(2e_t^2\mathcal{E}_n + 2e_t^2 x\mathcal{E}_1 + 2e_t^2 x^{3/2}\mathcal{E}_{hered} - x^{3/2}\mathcal{E}_\beta - x^2\mathcal{E}_\sigma + x^2\mathcal{E}_\tau\Big).\end{aligned} \tag{22}$$

To find an expression of de_t/dx it is required to find dx/dt. With the result at hand, we can compute using, $de_t/dx = (de_t/dt)/(dx/dt)$. To find dx/dt we used Eq. (22) along with Eq. (21). We find,

$$\frac{dn}{dt} = \frac{dx}{dt}\left(\frac{3}{2}x^{1/2}\mathcal{F}(x,e_t) + x^{3/2}\Big(\frac{\partial\mathcal{F}(x,e_t)}{\partial x} + \frac{\partial\mathcal{F}(x,e_t)}{\partial e_t}\frac{de_t}{dx}\Big)\right). \tag{23}$$

Once an expression of de_t/dx is found, the right-hand side of the equation can be expressed in a series expansion of e_t where the coefficients of each term depend on x. This equation then is solved to find the expression of e_t in terms of an initial eccentricity e_0 and x. In this section, we will develop a prescription that can be used to find very high-order powers in e_0.

To demonstrate the prescription, we will keep up to power e_t^3. Then the eccentricity evolution can be expressed as follows,

$$\frac{de_t}{dx} = e_t^{-1}f_{-1}(x) + e_t f_1(x) + e_t^3 f_3(x) + \mathcal{O}(e_t^4). \tag{24}$$

It will be demonstrated later that for aligned or antialigned spins $f_{-1}(x) = 0$. Hence, we will fix $f_{-1}(x) = 0$ from here on. In such case, the equation simplifies to,

$$\frac{de_t}{dx} = e_t f_1(x) + e_t^3 f_3(x) + \mathcal{O}(e_t^4). \tag{25}$$

Note, we have not explicitly specified the functional expression of $f_1(x)$ and $f_3(x)$. Once the prescription is described, we will substitute them with the required PN accuracy.

From Eq. (25) the resulting solution can be found by integrating the equation on both sides. During the process, we identify that $e_t \to e_0$, i.e. the initial eccentricity, when $x \to x_0$, i.e. the reference frequency. we find,

$$\begin{aligned}\int_{e_0}^{e_t}\frac{de_t}{e_t} &= \int_{x_0}^{x} dx\Big(f_1(x)+e_t^2 f_3(x)+\mathcal{O}(e_t^3)\Big)\\ \ln(\frac{e_t}{e_0}) &= \int_{x_0}^{x} dx f_1(x)+\int_{x_0}^{x} dx e_t^2 f_3(x)+\mathcal{O}(e_t^3)\\ e_t =& e_0 e^{\int_{x_0}^{x} dx f_1(x)} e^{\int_{x_0}^{x} dx e_t^2 f_3(x)+\mathcal{O}(e_t^3)}.\end{aligned} \tag{26}$$

Note, that the first integral on the right-hand side is independent of the eccentricity while the second one depends on it. Therefore although the second integral can not be computed without the explicit knowledge of e_t, the first integral can be computed. Hence the expression can be rearranged as,

$$e_t = e_0\frac{e^{F_1(x)}}{e^{F_1(x_0)}} e^{\int_{x_0}^{x} dx e_t^2 f_3(x)+\mathcal{O}(e_t^3)}. \tag{27}$$

The second integral has e_t inside the integral. Therefore, without the exact knowledge of e_t in terms of x, this integral can not be computed exactly. However, the eccentricity e_t inside the second integral can be replaced with the above equation and as a result, a leading order term of the integral can be computed. Therefore, although an exact integral is not computable, an approximate result can be found which is exact to a particular order. This as a result can be used to find the next order term. This can be continued for arbitrary powers of e_0. In this work, we will only compute up to e_0^3 term as Eq. (24) keeps only up to e_t^3. But in principle, this can be continued iteratively by considering higher order terms in Eq. (24). After rearranging the expressions they can be expressed as,

$$\begin{aligned}\frac{e_t e^{F_1(x_0)}}{e_0 e^{F_1(x)}} =& e^{e_0^2\int_{x_0}^{x} dx\frac{e^{2F_1(x)}}{e^{2F_1(x_0)}}\left(e^{2\int_{x_0}^{x} d\bar{x} e_t^2 f_3(\bar{x})+\mathcal{O}(e_t^3)}\right) f_3(x)+\mathcal{O}(e_t^3)}\\ \frac{e_t e^{F_1(x_0)}}{e_0 e^{F_1(x)}} =& \Big(1+e_0^2\int_{x_0}^{x} dx\frac{e^{2F_1(x)}}{e^{2F_1(x_0)}}\Big(e^{2\int_{x_0}^{x} d\bar{x} e_t^2 f_3(\bar{x})+\mathcal{O}(e_t^3)}\Big) f_3(x)+\mathcal{O}(e_0^3)\Big)\\ \frac{e_t e^{F_1(x_0)}}{e_0 e^{F_1(x)}} =& \Big(1+e_0^2\int_{x_0}^{x} dx\frac{e^{2F_1(x)}}{e^{2F_1(x_0)}}\Big(1+\mathcal{O}(e_0^2)\Big) f_3(x)+\mathcal{O}(e_0^3)\Big).\end{aligned} \tag{28}$$

This as a result boils down to a series of e_0, where the coefficients of the expansions are integrals in x. This approach can be used consecutively after deriving individual coefficients of a particular order. Interestingly, to find the schematic structure of the coefficients in integral form of an arbitrary power of e_0 it is not required to know $f_i(x)$. The expression can be derived in terms of the integrals of $f_i(x)$s.

The above results can be expressed in a further simplified and compact form as,

$$\begin{aligned}e_t =& \frac{e^{F_1(x)}}{e^{F_1(x_0)}}(e_0+e_0^3\int_{x_0}^{x} dx\frac{e^{2F_1(x)}}{e^{2F_1(x_0)}} f_3(x)+\mathcal{O}(e_0^4))\\ e_t =& e_0\frac{\mathcal{A}(x)}{\mathcal{A}(x_0)}+e_0^3\Big(\frac{\mathcal{A}(x)\mathcal{B}(x)}{\mathcal{A}(x_0)\mathcal{A}_2(x_0)}-\frac{\mathcal{A}(x)\mathcal{B}(x_0)}{\mathcal{A}(x_0)\mathcal{A}_2(x_0)}\Big)\\ \equiv& \epsilon_1 e_0+\epsilon_3 e_0^3,\end{aligned} \tag{29}$$

where,

$$\begin{aligned}\mathcal{A}(x) =& \quad e^{F_1(x)}\\ \mathcal{A}_2(x) =& \quad e^{2F_1(x)}\\ \mathcal{B}(x)-\mathcal{B}(x_0) =& \int_{x_0}^{x} dx e^{2F_1(x)} f_3(x).\end{aligned} \tag{30}$$

These expressions provide us with the eccentricity evolution with respect to the frequency. These can be used for further computations for fluxes and waveforms. In the later sections, we will derive equations for the f_is. Using the derived expression finally, we will find the explicit expressions for the evolving eccentricity. Once the eccentricity in terms of x is known we will use it to compute the shifts in the fluxes.

IV. ALIGNED AND ANTIALIGNED ORBITS

In the last section, we constructed a prescription that can be used to compute eccentricity evolution in terms of initial eccentricity and x. Using this we have found the general expression for e_t up to e_0^3. Although, we limited ourselves to order e_0^3, in principle this can be easily extended further.

In this work, we want to compute the leading order shifts due to spin-eccentricity coupling. Hence we will use 2PN order expressions of f_1 and f_3. However, in 2PN order we will keep only the spin-dependent terms. The required expressions are as follows [47, 54]:

$$\mathcal{N}_n = \frac{1}{5}(96+292e_t^2+37e_t^4) \tag{31}$$
$$\mathcal{N}_1 = \frac{1}{(1-e_t^2)}\left(-\frac{4846}{35}-\frac{264\nu}{5}+e_t^2\left(\frac{5001}{35}-570\nu\right)+e_t^4\left(\frac{2489}{4}-\frac{5061\nu}{10}\right)+e_t^6\left(\frac{11717}{280}-\frac{148\nu}{5}\right)\right) \tag{32}$$
$$\mathcal{N}_{hered} = 4\pi\frac{96(1-e_t^2)^{7/2}}{5}\left(1+\frac{2335}{192}e_t^2+\frac{42955}{768}e_t^4\right) \tag{33}$$
$$\mathcal{N}_\beta = \frac{1}{10(1-e_t^2)^{3/2}}\beta(3088+15528e_t^2+7026e_t^4+195e_t^6, 2160+11720e_t^2+5982e_t^4+207e_t^6) \tag{34}$$
$$\mathcal{N}_\sigma = \frac{1}{160(1-e_t^2)^2}\sigma(21952+128544e_t^2+73752e_t^4+3084e_t^6, 64576+373472e_t^2+210216e_t^4 +8532e_t^6, 131344e_t^2+127888e_t^4+7593e_t^6) \tag{35}$$
$$\mathcal{N}_\tau = \frac{1}{320(1-e_t^2)^2}\tau(448+4256e_t^2+3864e_t^4+252e_t^6, 64+608e_t^2+552e_t^4+36e_t^6, 16e_t^2+80e_t^4+9e_t^6). \tag{36}$$

$$\mathcal{E}_n = \left(\frac{304}{15}+\frac{121}{15}e_t^2\right) \tag{37}$$
$$\mathcal{E}_1 = \frac{1}{(1-e_t^2)}\left(-\frac{939}{35}-\frac{4084\nu}{45}+e_t^2\left(\frac{29917}{105}-\frac{7753}{30}\nu\right)+e_t^4\left(\frac{13929}{280}-\frac{1664\nu}{45}\right)\right) \tag{38}$$
$$\mathcal{E}_{hered} = \frac{32}{5}\frac{985(1-e_t^2)^{5/2}}{48}\pi\left(1+\frac{21729}{3940}e_t^2-\frac{7007}{788}e_t^4\right) \tag{39}$$
$$\mathcal{E}_\beta = \frac{e_t^2}{15(1-e_t^2)^{3/2}}\beta(13048+12000e_t^2+789e_t^4, 9208+10026e_t^2+835e_t^4) \tag{40}$$
$$\mathcal{E}_\sigma = \frac{1}{240(1-e_t^2)^2}\sigma(-320+101664e_t^2+116568e_t^4+9420e_t^6, -320+296672e_t^2+333624e_t^4 +26820e_t^6, 88432e_t^2+161872e_t^4+16521e_t^6) \tag{41}$$
$$\mathcal{E}_\tau = \frac{1}{480(1-e_t^2)^2}\tau(-320+2720e_t^2+5880e_t^4+540e_t^6, -320-160e_t^2+1560e_t^4+180e_t^6, 16e_t^2+80e_t^4+9e_t^6). \tag{42}$$

$\sigma(a,b,c)$ and $\tau(a,b,c)$ are spin dependent functions defined as follows,

$$\sigma(a,b,c) = a\sigma_a+b\sigma_b+c\sigma_c, \tag{43}$$
$$\tau(a,b,c) = a\tau_a+b\tau_b+c\tau_c. \tag{44}$$

The functions defined above σ_i and τ_i, where $i=a,b,c$, depend on the spin magnitude of both of the bodies in the binary and their directions. The expressions of these functions in terms of individual spins and their components can be expressed as follows,

$$\sigma_a = \frac{1}{\nu}\mathbf{S_1}.\mathbf{S_2} \tag{45}$$
$$\sigma_b = -\frac{1}{\nu}(\hat{\mathbf{J}}.\mathbf{S_1})(\hat{\mathbf{J}}.\mathbf{S_2}) \tag{46}$$
$$\sigma_c = \frac{1}{\nu}|\hat{\mathbf{J}}\times\mathbf{S_1}||\hat{\mathbf{J}}\times\mathbf{S_2}|\cos 2\psi \tag{47}$$
$$\tau_a = \sum_{i=1}^2\frac{1}{m_i^2}\mathbf{S_i^2} \tag{48}$$
$$\tau_b = -\sum_{i=1}^2\frac{1}{m_i^2}(\hat{\mathbf{J}}.\mathbf{S_i})^2 \tag{49}$$
$$\tau_c = \sum_{i=1}^2\frac{1}{m_i^2}|\hat{\mathbf{J}}\times\mathbf{S_i}|^2\cos 2\psi_i, \tag{50}$$

where, ψ_i is the angle subtended of $\mathbf{S}_i$ in the plane of motion and the periastron line. When the spins of the bodies are aligned or antialigned with $\hat{\mathbf{L}}$, $\hat{\mathbf{J}}||\hat{\mathbf{L}}$. As a result, $\sigma_c=\tau_c=0$, $\sigma_a=-\sigma_b=S_1S_2/\nu$, $\tau_a=-\tau_b=\sum_{i=1}^2\frac{S_i^2}{m_i^2}$, and $\gamma_2=0$, where $S_i=\mathbf{S}_i.\hat{\mathbf{L}}$. Using the above expressions it is straightforward to compute $f_1(x)$ and $f_3(x)$. Then using Eq. (30) further computation can be done. We find that f_{-1} contributes at 2-PN order for the spinning case. For non-spinning cases, it vanishes. The expression is as follows,

$$f_{-1}(x) = -\frac{5}{192}x(2\sigma_a+2\sigma_b-\tau_a-\tau_b). \tag{51}$$

This behaviour of f_{-1} arises because e_t^{-1} comes primarily due to $\mathcal{E}_\sigma$ and $\mathcal{E}_\tau$. Note in the above expression $f_{-1}(x)$ vanishes if $\sigma_a=-\sigma_b$ and $\tau_a=-\tau_b$. Therefore, this term does not contribute when the spins of the bodies are aligned or antialigned with $\hat{\mathbf{L}}$. In such a case we can apply the prescription constructed in the last section, and eccentricity evolution can be found. We find,

$$f_1(x) = -\frac{19}{12x}\left(1+\frac{(2833-5516\nu)x}{3192}+\frac{1}{456}x^{3/2}(-2452\beta_a-1776\beta_b+1131\pi)+\frac{x^2\left(6618528\sigma_a+91728\tau_a\right)}{1072512}\right)$$
$$f_3(x) = \frac{3323}{576x}\left(1+\frac{(472943-653228\nu)x}{159504}+\frac{x^{3/2}(-353345\beta_a-270423\beta_b+159321\pi)}{39876}+\frac{x^2\left(3432250080\sigma_a+39238416\tau_a\right)}{375153408}\right). \tag{52}$$

With this, we can compute the eccentricity evolution as follows,

$$x^{19/12}\mathcal{A}(x) =1+\frac{(5516\nu-2833)x}{2016}+\frac{x^{3/2}}{432}(2452\beta_a+1776\beta_b-1131\pi)+\frac{x^2(-39711168\sigma_a-550368\tau_a)}{8128512}. \tag{53}$$

$$x^{19/6}\mathcal{A}_2(x) =1+\frac{(5516\nu-2833)x}{1008}+x^{3/2}\Big(\frac{613\beta_a}{54}+\frac{74\beta_b}{9}-\frac{377\pi}{72}\Big)+\frac{x^2(-19855584\sigma_a-275184\tau_a)}{2032128}. \tag{54}$$

$$\begin{aligned}\mathcal{B}(x) =&-\frac{3323}{1824x^{19/6}}\Big(1+\frac{19(576485\nu+64718)x}{5443074}-\frac{19x^{3/2}(890535\pi-2(893893\beta_a+517017\beta_b))}{7177680}\\ &+\frac{19x^2(-1049901048\sigma_a-52036236\tau_a)}{11817332352}\Big).\end{aligned} \tag{55}$$

$$\begin{aligned}\mathcal{A}(x)\mathcal{B}(x) =&-\frac{3323}{1824x^{19/4}}\Big(1+\frac{(137845708\nu-34236165)x}{29029728}+\frac{x^{3/2}(12451319\beta_a+8192481\beta_b-5951955\pi)}{1196280}\\ &+\frac{x^2(-3(598429191360\sigma_a+13780542048\tau_a))}{273111681024}\Big).\end{aligned} \tag{56}$$

V. ORBITAL QUANTITIES

In the previous section, we formulated a prescription to compute eccentricity evolution. We expressed eccentricity in a series expansion of e_0. This result considers the spins of the components to be (anti)aligned with $\hat{\mathbf{L}}$. As discussed in Sec. II, several conservative quantities of an eccentric orbit can be expressed in terms of the PN parameter x and eccentricity e_t. These quantities also evolve as the orbit evolves in the inspiral time scale. With the knowledge of e_t in a PN expansion, it is possible to find expressions for these quantities also. Since, f_r, $f_{\phi,1}$, $f_{\phi,2}$ are proportional to γ_2, they all vanish. By using Eq. (29) in Eq. (15) and simplifying we find the eccentricity and spin-dependent shifts as follows,

$$\delta_{\chi,e_0}k = -\tfrac{3}{2}e_0^2\Big\{\epsilon_1^2x^{3/2}\big(4\beta_a+3\beta_b\big)+2\epsilon_1^2x^2\gamma_1\Big\}+e_0^4\Big\{-\tfrac{3}{8}x^{3/2}(4\beta_a+3\beta_b)\big(5\epsilon_1^4+8\epsilon_1\epsilon_3\big)-\tfrac{3}{2}\gamma_1x^2\big(3\epsilon_1^4+4\epsilon_1\epsilon_3\big)\Big\} \tag{57}$$

$$\begin{aligned}\delta_{\chi,e_0}e_r =& -\tfrac{e_0\epsilon_1}{2}\big(-2+4\beta_ax^{3/2}+2\beta_bx^{3/2}+\gamma_1x^2\big)\\ &-\tfrac{e_0^3}{2}\big(-2\epsilon_3+2\beta_a\epsilon_1^3x^{3/2}+\beta_b\epsilon_1^3x^{3/2}+4\beta_a\epsilon_3x^{3/2}+2\beta_b\epsilon_3x^{3/2}+\epsilon_1^3\gamma_1x^2+\epsilon_3\gamma_1x^2\big)\end{aligned} \tag{58}$$

$$\begin{aligned}\delta_{\chi,e_0}e_\phi =& -e_0\epsilon_1\big(-1+2\beta_ax^{3/2}+2\beta_bx^{3/2}+\gamma_1x^2\big)\\ &-e_0^3\big(-\epsilon_3+\beta_a\epsilon_1^3x^{3/2}+\beta_b\epsilon_1^3x^{3/2}+2\beta_a\epsilon_3x^{3/2}+2\beta_b\epsilon_3x^{3/2}+\epsilon_1^3\gamma_1x^2+\epsilon_3\gamma_1x^2\big).\end{aligned} \tag{59}$$

VI. ENERGY AND ANGULAR MOMENTUM FLUXES

In the context of GW astronomy, the most crucial thing is to model the GW waveform emitted by a system. This boils down to modeling both the amplitude and the phase of the waveform as these quantities evolve. These results are usually computed from the knowledge of the energy flux ($\dot{E}$) and the angular momentum flux ($\dot{L}$) from the GW sources. These fluxes can be separated into explicit spin-independent and spin-dependent parts.

$$\dot{E} = \nu(\dot{E}_N+\dot{E}_{SO}+\dot{E}_{SS}) \tag{60}$$
$$\dot{L} = \nu(\dot{L}_N+\dot{L}_{SO}+\dot{L}_{SS}), \tag{61}$$

where SO and SS represent the spin-orbit and spin-spin coupling respectively. These expressions depend on orbital energy and angular momentum as follows,

$$\dot{E}_N = -\frac{(-2E)^{3/2}}{15L^7}(96+292A^2+37A^4) \tag{62}$$

$$\dot{E}_{SO} = \frac{(-2E)^{3/2}}{10L^{10}}\beta(2704+7320A^2+2490A^4+65A^6, 1976+5096A^2+1569A^4+32A^6) \tag{63}$$

$$\dot{E}_{SS} = \frac{(-2E)^{3/2}}{960L^{11}}\Big[2\sigma(42048+154272A^2+75528A^4+3084A^6, 124864+450656A^2+215544A^4+8532A^6, 131344A^2+127888A^4+7593A^6)-\tau(448+4256A^2+3864A^4+252A^6, 64+608A^2+552A^4+36A^6, 16A^2+80A^4+9A^6)\Big], \tag{64}$$

$$\dot{L}_N = -\frac{4(-2E)^{3/2}}{5L^4}(8+7A^2) \tag{65}$$

$$\dot{L}_{SO} = \frac{(-2E)^{3/2}}{15L^7}\beta(2264+2784A^2+297A^4, 1620+1852A^2+193A^4) \tag{66}$$

$$\dot{L}_{SS} = \frac{(-2E)^{3/2}}{20L^8}\Big[2\sigma(552+996A^2+132A^4, 1616+2868A^2+381A^4, 894A^2+186A^4) -(8+24A^2+3A^4)\tau(2,1,0)\Big], \tag{67}$$

where $A=\sqrt{1+2EL^2}$.

Using Eq. (8) and the expression of A it is possible to find a PN expression for the fluxes. We use δ_{χ,e_0} to represent the sift of $\dot{E}$ and $\dot{L}$ due to both the nonzero spin and eccentricity. We compute the leading order eccentricity-spin coupling terms arising from spin-orbit and spin-spin coupling.

$$\delta_{\chi,e_0}\dot{E}_N = \mathcal{E}_{N,\chi,2}e_0^2+\mathcal{E}_{N,\chi,4}e_0^4 \tag{68}$$

$$\delta_{\chi,e_0}\dot{L}_N = \mathcal{L}_{N,\chi,2}e_0^2+\mathcal{L}_{N,\chi,4}e_0^4 \tag{69}$$

$$\delta_{\chi,e_0}\dot{E}_{SO} = \mathcal{E}_{SO,2}e_0^2+\mathcal{E}_{SO,4}e_0^4 \tag{70}$$

$$\delta_{\chi,e_0}\dot{E}_{SS} = \mathcal{E}_{SS,2}e_0^2+\mathcal{E}_{SS,4}e_0^4 \tag{71}$$

$$\delta_{\chi,e_0}\dot{L}_{SO} = \mathcal{L}_{SO,2}e_0^2+\mathcal{L}_{SO,4}e_0^4 \tag{72}$$

$$\delta_{\chi,e_0}\dot{L}_{SS} = \mathcal{L}_{SS,2}e_0^2+\mathcal{L}_{SS,4}e_0^4. \tag{73}$$

In the above expression, we have decomposed the energy and angular momentum fluxes in a series expansion with respect to initial eccentricity. We have kept only up to e_0^4. With more computation of eccentricity evolution, this can be extended further. The coefficients of these expansions are themselves series expansion in PN parameter x. The corresponding expressions are computed and demonstrated in the following,

$$\mathcal{E}_{SO,2} = \frac{4x^{10/3}(2605\beta_a+1872\beta_b)}{5\mathcal{A}_0^2} \tag{74}$$

$$\mathcal{L}_{SO,2} = \frac{2x^{11/6}(5354\beta_a+3761\beta_b)}{15\mathcal{A}_0^2} \tag{75}$$

$$\mathcal{E}_{SS,2} = -\frac{2x^{23/6}(3916\sigma_a+15\tau_a)}{5\mathcal{A}_0^2} \tag{76}$$

$$\mathcal{L}_{SS,2} = -\frac{2x^{7/3}(1532\sigma_a+7\tau_a)}{5\mathcal{A}_0^2}. \tag{77}$$

$$\mathcal{E}_{N,\chi,2} = -\frac{x^{11/6}}{1905120\mathcal{A}_0^2}\Big(-2352x^{3/2}(-1236868\beta_a-883848\beta_b)+x^2(540800064\gamma_1+157(-4963896\sigma_a-68796\tau_a))\Big). \tag{78}$$

$$\mathcal{L}_{N,\chi,2} = -\frac{x^{1/3}}{635040\mathcal{A}_0^2}\Big(-2352x^{3/2}(-131996\beta_a-92580\beta_b)+x^2(50295168\gamma_1-23\,(4963896\sigma_a+68796\tau_a))\Big). \tag{79}$$

$$\mathcal{E}_{SO,4} = \frac{x^{1/6}}{1140\mathcal{A}_0^4\mathcal{A}_2(x_0)}\Big(-8656415\mathcal{A}_0^2\beta_a-6220656\mathcal{A}_0^2\beta_b+9080100\mathcal{A}_2(x_0)\beta_a+6462546\mathcal{A}_2(x_0)\beta_b\Big) \tag{80}$$

$$\mathcal{L}_{SO,4} = \frac{1}{6840\mathcal{A}_0^4\mathcal{A}_2(x_0)x^{4/3}}\Big(-17791342\mathcal{A}_0^2\beta_a-12497803\mathcal{A}_0^2\beta_b+12708720\mathcal{A}_2(x_0)\beta_a+8861220\mathcal{A}_2(x_0)\beta_b\Big) \tag{81}$$

$$\mathcal{E}_{SS,4} = \frac{x^{2/3}}{2280\mathcal{A}_0^4\mathcal{A}_2(x_0)}\Big(13012868\mathcal{A}_0^2\sigma_a+49845\mathcal{A}_0^2\tau_a-15439704\mathcal{A}_2(x_0)\sigma_a-71820\mathcal{A}_2(x_0)\tau_a\Big) \tag{82}$$

$$\mathcal{L}_{SS,4} = \frac{1}{2280\mathcal{A}_0^4\mathcal{A}_2(x_0)x^{5/6}}\Big(5090836\mathcal{A}_0^2\sigma_a+23261\mathcal{A}_0^2\tau_a-4189956\mathcal{A}_2(x_0)\sigma_a-20406\mathcal{A}_2(x_0)\tau_a\Big). \tag{83}$$

$$x^{4/3}\mathcal{E}_{N,\chi,4} = \frac{1}{3162194380800\mathcal{A}_0^4\mathcal{A}_2(x_0)}\Bigg[\mathcal{A}_0^2\Big[428064x^{3/2}(46434089278\beta_a + 32454792462\beta_b) + 40x^2\Big\{81767076876576\gamma_1 - 157\Big(880186480992\sigma_a + 16828189560\tau_a\Big)\Big\}\Big] + 4357080\mathcal{A}_2(x_0)\Big[1344x^{3/2}(-3426812\beta_a - 2390010\beta_b) - x^2\Big\{759773952\gamma_1 - 605\Big(2836512\sigma_a + 39312\tau_a\Big)\Big\}\Big]\Bigg]. \tag{84}$$

$$x^{17/6}\mathcal{L}_{N,\chi,4} = \frac{336x^{3/2}(-364012\beta_a - 249321\beta_b) + x^2\Big\{19\Big(2836512\sigma_a + 39312\tau_a\Big) - 18724608\gamma_1\Big\}}{90720\mathcal{A}_0^4} + \frac{1}{131758099200\mathcal{A}_0^2\mathcal{A}_2(x_0)}\Bigg[53508x^{3/2}(5167489562\beta_a + 3528306258\beta_b) + 5x^2\Big\{7604453368512\gamma_1 - 23\Big(880186480992\sigma_a + 16828189560\tau_a\Big)\Big\}\Bigg], \tag{85}$$

where $\mathcal{A}_0 = \mathcal{A}(x = x_0)$. Here we have computed the shift in fluxes only due to the leading order spin-orbit and spin-spin interaction. These fluxes can further be used to compute the shift in the emitted GWs. Therefore these expressions will be necessary to construct GW waveforms for spinning bodies in an eccentric orbit.

VII. DISCUSSION

The main result of the paper is the formulation of the prescription that can be used to compute the eccentricity evolution of a spinning binary to very high order. To our knowledge, this is for the first time where spin and e_0 couplings have been computed in the expression of eccentricity evolution. It provides us with a frequency-dependent evolution of eccentricity in terms of initial eccentricity. We also discussed how it can be extended to higher orders iteratively. We considered the spins to be (anti)aligned to simplify the calculations. As a result in the current case $f_{-1}(x) = 0$. With the computed PN expansion of eccentricity, we added several orbital quantities and the fluxes of energy and angular momentum. These fluxes are crucial for the computation of the GW waveform. Hence, these results can be used to model GW waveforms considering eccentricity-spin coupling.

In general $f_{-1}(x) \neq 0$. Considering such cases, need more studies which we will pursue in the future. In such a case different approaches combining analytical and numerical fitting may be required. In the presence of nonvanishing $f_{-1}(x)$ integrating the equation analytically becomes challenging. To address this a semi-analytical approach can be constructed. By multiplying both side with $2e_t$ in Eq. (24) it can be found,

$$\begin{aligned}\frac{de_t^2}{dx} =& 2f_{-1}(x) + 2e_t^2 f_1(x) + 2e_t^4 f_3(x) + \mathcal{O}(e_t^5) \\ \equiv& f_0(x) + e_t^2 f_2(x) + e_t^4 f_4(x) + \mathcal{O}(e_t^5)\end{aligned} \tag{86}$$

The semi-analytic approach can be constructed by defining $\tilde{e}_t^2 = \bar{e}_t^2 + \Delta e_t^2$. Then rather than solving the exact differential equation satisfied by e_t, we can solve for two approximated differential equations such as

$$\frac{d\Delta e_t^2}{dx} = f_0(x) \tag{87}$$

$$\frac{d\bar{e}_t^2}{dx} = \sum_{n=2}^{\infty} \bar{e}_t^n f_n(x). \tag{88}$$

Eq. (87) can be solved analytically. Similarly, Eq. (88) can be solved analytically in an iterative manner as discussed in the current paper. This will provide us with an analytical expression for $\tilde{e}_t$. Note, $\tilde{e}_t \neq e_t$ in general. However, as long as $\frac{\tilde{e}_t - e_t}{e_t}$ is lesser than the observable statistical error, this systematic error can be ignored. In such a case by comparing a numerically found solution for e_t from the differential equation and comparing it with $\tilde{e}_t$ we can estimate the systematic error and its significance compared to the statistical error. This needs to be explored in detail.

Acknowledgements

We want to thank K. G. Arun, Antoine Klein, Chandra Kant Mishra, and Khun Sang Phukon for the very useful discussions. We also like to thank Chandra Kant Mishra, and Khun Sang Phukon for reading an earlier draft and providing us with valuable comments.

[1] J. Aasi et al. (LIGO Scientific), Class. Quant. Grav. **32**, 074001 (2015), 1411.4547.
[2] F. Acernese et al. (VIRGO), Class. Quant. Grav. **32**, 024001 (2015), 1408.3978.
[3] B. P. Abbott et al. (LIGO Scientific, Virgo), Phys. Rev. **X9**, 031040 (2019), 1811.12907.
[4] R. Abbott et al. (LIGO Scientific, Virgo), Phys. Rev. X **11**, 021053 (2021), 2010.14527.
[5] B. P. Abbott et al. (Virgo, LIGO Scientific), Phys. Rev. Lett. **119**, 161101 (2017), 1710.05832.
[6] B. Abbott et al. (LIGO Scientific, Virgo), Astrophys. J. Lett. **892**, L3 (2020), 2001.01761.
[7] R. Abbott et al. (LIGO Scientific, KAGRA, VIRGO), Astrophys. J. Lett. **915**, L5 (2021), 2106.15163.
[8] M. Maggiore, C. Van Den Broeck, N. Bartolo, E. Belgacem, D. Bertacca, M. A. Bizouard, M. Branchesi, S. Clesse, S. Foffa, J. García-Bellido, et al., Journal of Cosmology and Astroparticle Physics **2020**, 050 (2020).
[9] D. Reitze et al., Bull. Am. Astron. Soc. **51**, 035 (2019), 1907.04833.
[10] P. Amaro-Seoane et al. (LISA) (2017), 1702.00786.
[11] P. C. Peters, Phys. Rev. **136**, B1224 (1964).
[12] P. C. Peters and J. Mathews, Phys. Rev. **131**, 435 (1963).
[13] Y. Kozai, Astron. J. **67**, 591 (1962).
[14] M. L. Lidov, Planetary and Space Science **9**, 719 (1962).
[15] B. P. Abbott et al. (LIGO Scientific, Virgo), Astrophys. J. **883**, 149 (2019), 1907.09384.
[16] P. Amaro-Seoane, Living Rev. Rel. **21**, 4 (2018), 1205.5240.
[17] R. Abbott et al. (LIGO Scientific, Virgo), Astrophys. J. Lett. **900**, L13 (2020), 2009.01190.
[18] C. Kimball et al., Astrophys. J. Lett. **915**, L35 (2021), 2011.05332.
[19] I. M. Romero-Shaw, P. D. Lasky, and E. Thrane, Astrophys. J. Lett. **921**, L31 (2021), 2108.01284.
[20] I. M. Romero-Shaw, P. D. Lasky, and E. Thrane, Astrophys. J. **940**, 171 (2022), 2206.14695.
[21] E. O'Shea and P. Kumar (2021), 2107.07981.
[22] M. Favata, C. Kim, K. G. Arun, J. Kim, and H. W. Lee, Phys. Rev. D **105**, 023003 (2022), 2108.05861.
[23] M. E. Lower, E. Thrane, P. D. Lasky, and R. Smith, Phys. Rev. D **98**, 083028 (2018), 1806.05350.
[24] C. K. Mishra, K. G. Arun, and B. R. Iyer, Phys. Rev. D **91**, 084040 (2015), 1501.07096.
[25] B. Moore, M. Favata, K. G. Arun, and C. K. Mishra, Phys. Rev. D **93**, 124061 (2016), 1605.00304.
[26] S. Tanay, M. Haney, and A. Gopakumar, Phys. Rev. D **93**, 064031 (2016), 1602.03081.
[27] Y. Boetzel, C. K. Mishra, G. Faye, A. Gopakumar, and B. R. Iyer, Phys. Rev. D **100**, 044018 (2019), 1904.11814.
[28] M. Ebersold, Y. Boetzel, G. Faye, C. K. Mishra, B. R. Iyer, and P. Jetzer, Phys. Rev. D **100**, 084043 (2019), 1906.06263.
[29] C. Konigsdorffer and A. Gopakumar, Phys. Rev. D **73**, 124012 (2006), gr-qc/0603056.
[30] B. Moore and N. Yunes, Class. Quant. Grav. **36**, 185003 (2019), 1903.05203.
[31] K. Paul and C. K. Mishra (2022), 2211.04155.
[32] S. Datta (2023), 2305.03771.
[33] S. Marsat, A. Bohe, G. Faye, and L. Blanchet, Class. Quant. Grav. **30**, 055007 (2013), 1210.4143.
[34] J. Hartung and J. Steinhoff, Annalen Phys. **523**, 783 (2011), 1104.3079.
[35] M. Levi and J. Steinhoff, JHEP **09**, 219 (2015), 1501.04956.
[36] A. Bohe, S. Marsat, G. Faye, and L. Blanchet, Class. Quant. Grav. **30**, 075017 (2013), 1212.5520.
[37] A. Bohé, S. Marsat, and L. Blanchet, Class. Quant. Grav. **30**, 135009 (2013), 1303.7412.
[38] S. Marsat, A. Bohé, L. Blanchet, and A. Buonanno, Class. Quant. Grav. **31**, 025023 (2014), 1307.6793.
[39] A. Bohé, G. Faye, S. Marsat, and E. K. Porter, Class. Quant. Grav. **32**, 195010 (2015), 1501.01529.
[40] L. Blanchet, A. Buonanno, and G. Faye, Phys. Rev. D **84**, 064041 (2011), 1104.5659.
[41] K. Chatziioannou, A. Klein, N. Yunes, and N. Cornish, Phys. Rev. D **88**, 063011 (2013), 1307.4418.
[42] Q. Henry, S. Marsat, and M. Khalil, Phys. Rev. D **106**, 124018 (2022), 2209.00374.
[43] K. G. Arun, A. Buonanno, G. Faye, and E. Ochsner, Phys. Rev. **D79**, 104023 (2009), [Erratum: Phys. Rev.D84,049901(2011)], 0810.5336.
[44] L. E. Kidder, Phys. Rev. D **52**, 821 (1995), gr-qc/9506022.
[45] A. Buonanno, G. Faye, and T. Hinderer, Phys. Rev. D **87**, 044009 (2013), 1209.6349.
[46] J. Majar and M. Vasuth, Phys. Rev. D **77**, 104005 (2008), 0806.2273.
[47] A. Klein and P. Jetzer, Phys. Rev. D **81**, 124001 (2010), 1005.2046.
[48] A. Klein, Y. Boetzel, A. Gopakumar, P. Jetzer, and L. de Vittori, Phys. Rev. D **98**, 104043 (2018), 1801.08542.
[49] A. Klein (2021), 2106.10291.
[50] L. Blanchet, G. Faye, B. R. Iyer, and S. Sinha, Class. Quant. Grav. **25**, 165003 (2008), [Erratum: Class.Quant.Grav. 29, 239501 (2012)], 0802.1249.
[51] L. E. Kidder, Phys. Rev. D **77**, 044016 (2008), 0710.0614.
[52] L. Blanchet, Living Rev. Rel. **17**, 2 (2014), 1310.1528.
[53] K. S. Phukon, A. Gupta, S. Bose, and P. Jain, Phys. Rev. D **100**, 124008 (2019), 1904.03985.
[54] K. G. Arun, L. Blanchet, B. R. Iyer, and S. Sinha, Phys. Rev. D **80**, 124018 (2009), 0908.3854.
[55] N. Yunes, K. G. Arun, E. Berti, and C. M. Will, Phys. Rev. D **80**, 084001 (2009), [Erratum: Phys.Rev.D 89, 109901 (2014)], 0906.0313.
[56] L. E. Kidder, C. M. Will, and A. G. Wiseman, Phys. Rev. D **47**, R4183 (1993), gr-qc/9211025.