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Energy conditions for non-timelike massive thin shells

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Abstract

We study energy conditions for non-timelike massive thin shells in arbitrary $n(\geq 3)$ dimensions. It is shown that the induced energy-momentum tensor $t_{\mu\nu}$ on a shell Σ is of the Hawking-Ellis type I if Σ is spacelike and either of type I, II, or III if Σ is null. Then, we derive simple equivalent representations of the standard energy conditions for $t_{\mu\nu}$. In particular, on a spacelike shell or on a null shell with non-vanishing surface current, $t_{\mu\nu}$ inevitably violates the dominant energy condition. Those fully general results are obtained without imposing a spacetime symmetry and can be used in any theory of gravity. Lastly, several applications of the main results are presented in general relativity in four dimensions.

Contents

1 Introduction

In gravitation physics, a massive thin shell Σ is a junction hypersurface of two spacetimes on which there is a non-vanishing induced energy-momentum tensor $t_{\mu\nu}$. Possible embeddings of Σ in a given set of bulk spacetimes and the resulting $t_{\mu\nu}$ are determined by the *junction conditions*. The first junction conditions require that the induced metrics on both sides of Σ are the same. Then, the second junction conditions relate $t_{\mu\nu}$ and the jump of the extrinsic curvature (transverse curvature) of Σ if Σ is non-null (null). In general relativity, the second junction conditions are derived from the Einstein equations and referred to as the Israel junction conditions for non-null Σ [1] and the Barrabès-Israel junction conditions for null Σ [2]. The Barrabès-Israel junction conditions have been reformulated by Poisson to provide a simple characterization of the thin-shell energy-momentum tensor $t_{\mu\nu}$ [3]. (The junction conditions in general relativity are summarized in Sec. 3 in the textbook [4].)

In fact, in order to grasp the essence of general relativistic gravitational phenomena, massive thin shells have been used to construct simple models in a very wide variety of contexts. It is not possible to list all the papers, but examples of such phenomena are gravitational collapse [5–8], growth of cosmic voids [9–11] and bubbles [12–18], and brane cosmology [19–23]. An exact model for the mass-inflation instability of the inner horizon of a charged black hole has also been constructed using a null shell [24]. The junction conditions have also been established in a large class of scalar-tensor theories of gravity for non-null Σ [25–27] as well as for general Σ [28] and in Einstein-Gauss-Bonnet higher curvature gravity for non-null Σ [29, 30]. (See [31] for recent developments in the research of junction conditions.)

For such a shell model to be physically reasonable, $t_{\mu\nu}$ induced on the shell should satisfy at least some of the standard energy conditions [32, 33]. (See [34] for a nice review of the energy conditions.) In arbitrary $n(\geq 3)$ dimensions, an energy-momentum tensor $T_{\mu\nu}$ is classified into the four Hawking-Ellis types (three types for $n = 2$) [33, 35–38] and equivalent representations of the standard energy conditions in terms of the orthonormal components of $T_{\mu\nu}$ in the canonical frame are available for each type [33, 39]. Even so, finding the canonical orthonormal frame by local Lorentz transformations is often not easy. Accordingly, we derived equivalent representations of the standard energy conditions in the case where the orthonormal components of $T_{\mu\nu}$ admit only a single off-diagonal "space-time" component [40]. Relations between the first junction conditions and the energy conditions were discussed in [41] for a timelike Σ. If a shell Σ is timelike, due to the Lorentzian signature on Σ, the energy conditions for $t_{\mu\nu}$ induced on the shell have to be examined in each case individually using those results. In contrast, if a shell is non-timelike, the situation is drastically simplified. In this paper, we will present simple representations of the standard energy conditions for $t_{\mu\nu}$ on a non-timelike Σ embedding in an $n(\geq 2)$-dimensional spacetime.

The present article is organized as follows. In the next section, after reviewing the standard energy conditions, the Hawking-Ellis classification of an energy-momentum tensor, and the junction conditions, we will present our main results. In Sec. 3, we will apply the main results in several physical situations in general relativity in four dimensions. Throughout this article, the signature of the Minkowski spacetime is $(-, +, \ldots, +)$, and Greek indices run over all spacetime indices. Other types of indices will be specified in the main text. We adopt the units such that $c = 1$ and use $\kappa_n := 8\pi G_n$ instead of the n-dimensional gravitational constant G_n. The conventions of curvature tensors such as $[\nabla_\rho, \nabla_\sigma]V^\mu = R^\mu{}_{\nu\rho\sigma}V^\nu$ and $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$.

2　Energy conditions for massive thin-shells

We follow the definitions and notations adopted in [33] for the energy conditions and the Hawking-Ellis types. In $n(\geq 2)$ dimensions, the standard energy conditions for an energy-momentum tensor $T_{\mu\nu}$ are stated as follows:

- *Null* energy condition (NEC): $T_{\mu\nu}k^\mu k^\nu \geq 0$ for any null vector k^μ.

- *Weak* energy condition (WEC): $T_{\mu\nu}v^\mu v^\nu \geq 0$ for any timelike vector v^μ.

- *Dominant* energy condition (DEC): $T_{\mu\nu}v^\mu v^\nu \geq 0$ and $J_\mu J^\mu \leq 0$ hold for any timelike vector v^μ, where $J^\mu := -T^\mu{}_\nu v^\nu$ is an energy-flux vector for an observer with its tangent vector v^μ.

- *Strong* energy condition (SEC): $\left(T_{\mu\nu} - \frac{1}{n-2}Tg_{\mu\nu}\right)v^\mu v^\nu \geq 0$ for any timelike vector v^μ.

The SEC is defined only for $n \geq 3$ and it is equivalent to the timelike convergence condition $R_{\mu\nu}v^\mu v^\nu \geq 0$ for any timelike vector in general relativity without a cosmological constant. We note that, although $n \geq 3$ is assumed in [33], the results there for the NEC, WEC, and DEC are also valid for $n = 2$.

2.1　Hawking-Ellis types

An orthonormal frame is defined by a set of n orthonormal basis vectors $\{E^\mu_{(\alpha)}\}$ $(\alpha = 0, 1, \cdots, n-1)$ that satisfy

$$E^\mu_{(\alpha)}E_{(\beta)\mu} = \eta_{(\alpha)(\beta)} = \mathrm{diag}(-1, 1, \cdots, 1), \tag{2.1}$$

which is equivalent to $g_{\mu\nu} = \eta_{(\alpha)(\beta)}E^{(\alpha)}_\mu E^{(\beta)}_\nu$. The Minkowski metric $\eta_{(\alpha)(\beta)}$ in the orthonormal frame and its inverse $\eta^{(\alpha)(\beta)}$ are respectively used to lower and raise the indices

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(α). Components of $T_{\mu\nu}$ in the orthonormal frame are given by $T_{(\alpha)(\beta)} = T_{\mu\nu} E^{\mu}_{(\alpha)} E^{\nu}_{(\beta)}$. Orthonormal frame has a degree of freedom provided by local Lorentz transformations $E^{\mu}_{(\alpha)} \to \tilde{E}^{\mu}_{(\alpha)} := L_{(\alpha)}{}^{(\beta)} E^{\mu}_{(\beta)}$, where $L_{(\alpha)}{}^{(\beta)}$ satisfies $L_{(\alpha)}{}^{(\gamma)} L_{(\beta)}{}^{(\delta)} \eta_{(\gamma)(\delta)} = \eta_{(\alpha)(\beta)}$. $T_{(\alpha)(\beta)}$ behaves as a scalar under a diffeomorphism and as a two-tensor under a local Lorentz transformation. We refer to such a mathematical object as a Lorentz-covariant tensor. According to this terminology, a basis vector $E^{\mu}_{(\alpha)}$ is a Lorentz-covariant vector.

The Hawking-Ellis classification of $T_{\mu\nu}$ is performed according to the properties of the Lorentz-invariant eigenvalues λ and eigenvectors n^{μ} (or Lorentz-covariant eigenvectors $n^{(\alpha)} = E^{(\alpha)}_{\mu} n^{\mu}$) that are determined by the following eigenvalue equations [32, 33]:

$$T^{(\alpha)(\beta)} n_{(\beta)} = \lambda \eta^{(\alpha)(\beta)} n_{(\beta)} \quad \Leftrightarrow \quad T^{\mu\nu} n_{\nu} = \lambda g^{\mu\nu} n_{\nu}. \tag{2.2}$$

The characteristic equation to determine λ is

$$\det\left(T^{(\alpha)(\beta)} - \lambda \eta^{(\alpha)(\beta)}\right) = 0. \tag{2.3}$$

Since $n_{(\alpha)} n^{(\alpha)} = n_{\mu} n^{\mu}$ holds, a Lorentz-covariant vector $n^{(\alpha)}$ is referred to as timelike, spacelike, and null if $n_{(\alpha)} n^{(\alpha)}$ is negative, positive, and zero, respectively.

In three or higher dimensions ($n \geq 3$), $T_{\mu\nu}$ is classified into four types as summarized in Table 1 [32, 33]. In two dimensions ($n = 2$), $T_{\mu\nu}$ is classified into type I, II, or IV. By local Lorentz transformations, we can write each type of $T^{(\alpha)(\beta)}$ in a canonical form and then equivalent representations of the standard energy conditions are available [33].

Table 1: Eigenvectors of type-I–IV energy-momentum tensors.

Type	Eigenvectors
I	1 timelike, $n - 1$ spacelike
II	1 null (doubly degenerated), $n - 2$ spacelike
III	1 null (triply degenerated), $n - 3$ spacelike
IV	2 complex, $n - 2$ spacelike

Type I

The canonical form of type I is

$$T^{(\alpha)(\beta)} = \mathrm{diag}(\rho, p_1, p_2, \cdots, p_{n-1}) \tag{2.4}$$

for which the characteristic equation (2.3) gives

$$(\lambda + \rho)(\lambda - p_1) \cdots (\lambda - p_{n-1}) = 0, \tag{2.5}$$

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so that the eigenvalues are $\lambda = \{-\rho, p_1, p_2, \cdots, p_{n-1}\}$. The eigenvector of $\lambda = -\rho$ is timelike and other eigenvectors are spacelike. The standard energy conditions are equivalent to the following inequalities:

$$\text{NEC}: \quad \rho + p_i \geq 0 \quad \text{for} \quad i = 1, 2, \cdots, n-1, \tag{2.6}$$

$$\text{WEC}: \quad \rho \geq 0 \text{ in addition to NEC}, \tag{2.7}$$

$$\text{DEC}: \quad \rho - p_i \geq 0 \quad \text{for} \quad i = 1, 2, \cdots, n-1 \text{ in addition to WEC}, \tag{2.8}$$

$$\text{SEC}: \quad (n-3)\rho + \sum_{j=1}^{n-1} p_j \geq 0 \quad \text{in addition to NEC}. \tag{2.9}$$

Type II

The canonical form of type II is

$$T^{(\alpha)(\beta)} = \begin{pmatrix} \rho+\nu & \nu & 0 & 0 & \cdots & 0 \\ \nu & -\rho+\nu & 0 & 0 & \cdots & 0 \\ 0 & 0 & p_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & p_{n-1} \end{pmatrix} \tag{2.10}$$

with $\nu \neq 0$, for which the characteristic equation (2.3) gives

$$(\lambda + \rho)^2 (\lambda - p_2) \cdots (\lambda - p_{n-1}) = 0, \tag{2.11}$$

so that the eigenvalues are $\lambda = \{-\rho, p_2, \cdots, p_{n-1}\}$. The Lorentz-covariant eigenvector $n_{(\alpha)} = \bar{k}_{(\alpha)}$ of the doubly degenerate eigenvalue $\lambda = -\rho$ is null, while the Lorentz-covariant eigenvectors $n_{(\alpha)} = w_{i(\alpha)}$ $(i = 2, 3, \cdots, n-1)$ of the eigenvalues $\lambda = p_i$ are spacelike. $\bar{k}_{(\alpha)}$ and $w_{i(\alpha)}$ are given by

$$\bar{k}_{(\alpha)} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad w_{2(\alpha)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots, \quad w_{n-1(\alpha)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \tag{2.12}$$

with which $T^{(\alpha)(\beta)}$ can be written as

$$T^{(\alpha)(\beta)} = \nu \bar{k}^{(\alpha)} \bar{k}^{(\beta)} - \rho \, \eta_2^{(\alpha)(\beta)} + \sum_{i=2}^{n-1} p_i w_i^{(\alpha)} w_i^{(\beta)}, \tag{2.13}$$

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where $\eta_2^{(\alpha)(\beta)} := \mathrm{diag}(-1, 1, 0, \cdots, 0)$. The standard energy conditions are equivalent to the following inequalities:

$$\text{NEC}: \quad \nu \geq 0 \text{ and } \rho + p_i \geq 0 \text{ for } i = 2, 3, \cdots, n-1, \tag{2.14}$$

$$\text{WEC}: \quad \rho \geq 0 \text{ in addition to NEC}, \tag{2.15}$$

$$\text{DEC}: \quad \rho - p_i \geq 0 \text{ for } i = 2, 3, \cdots, n-1 \text{ in addition to WEC}, \tag{2.16}$$

$$\text{SEC}: \quad (n-4)\rho + \sum_{j=2}^{n-1} p_j \geq 0 \text{ in addition to NEC}. \tag{2.17}$$

Type III

The canonical form of type III is

$$T^{(\alpha)(\beta)} = \begin{pmatrix} \rho + \nu & \nu & \zeta & 0 & 0 & \cdots & 0 \\ \nu & -\rho + \nu & \zeta & 0 & 0 & \cdots & 0 \\ \zeta & \zeta & -\rho & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & p_3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & p_{n-1} \end{pmatrix} \tag{2.18}$$

with $\zeta \neq 0$, for which the characteristic equation (2.3) gives

$$(\lambda + \rho)^3 (\lambda - p_3) \cdots (\lambda - p_{n-1}) = 0, \tag{2.19}$$

so that the eigenvalues are $\lambda = \{-\rho, p_3, \cdots, p_{n-1}\}$. The eigenvector of the triply degenerate eigenvalue $\lambda = -\rho$ is null and other eigenvectors are spacelike. Any type-III energy-momentum tensor violates all the standard energy conditions.

We note that ν in Eq. (2.18) can be set to zero by local Lorentz transformations if and only if ζ is non-zero [39]. Nevertheless, the expression (2.18) with non-vanishing ν admits a limit $\zeta \to 0$ to type II and may be useful to identify a type-III energy-momentum tensor in a given spacetime.

Type IV

The canonical form of type IV is

$$T^{(\alpha)(\beta)} = \begin{pmatrix} \rho & \nu & 0 & 0 & \cdots & 0 \\ \nu & -\rho & 0 & 0 & \cdots & 0 \\ 0 & 0 & p_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & p_{n-1} \end{pmatrix} \tag{2.20}$$

with $\nu \neq 0$, for which the characteristic equation (2.3) gives

$$[(\lambda + \rho)^2 + \nu^2](\lambda - p_2) \cdots (\lambda - p_{n-1}) = 0, \tag{2.21}$$

so that the eigenvalues are $\lambda = \{-\rho + i\nu, -\rho - i\nu, p_2, \cdots, p_{n-1}\}$. The eigenvectors of the complex eigenvalues $\lambda = -\rho \pm i\nu$ are complex and other eigenvectors are spacelike. Any type-IV energy-momentum tensor violates all the standard energy conditions.

We note that a canonical form of $T^{(\alpha)(\beta)}$ in the textbook [32] is different from Eq. (2.20). However, the expression (2.20) may be more useful as pointed out in [39].

2.2 Energy-momentum tensor of massive thin-shells

For junction conditions, we follow the definitions and notations adopted in [28]. We consider an $(n-1)$-dimensional junction hypersurface Σ between two $n(\geq 2)$-dimensional spacetime regions $(\mathcal{M}_+, g_{\mu\nu}^+)$ and $(\mathcal{M}_-, g_{\mu\nu}^-)$. The metric $g_{\mu\nu}^\pm$ is expressed in the coordinates $x_\pm^\mu$ on $(\mathcal{M}_\pm, g_{\mu\nu}^\pm)$. In the following subsections, we identify an n-dimensional spacetime $(\mathcal{M}, g_{\mu\nu})$ with the line element

$$ds_n^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \tag{2.22}$$

as $(\mathcal{M}_+, g_{\mu\nu}^+)$ or $(\mathcal{M}_-, g_{\mu\nu}^-)$ and its boundary as Σ. Suppose that Σ is described by $\Phi(x) =$ constant in the bulk spacetime (2.22). We set the same intrinsic coordinates y^a on both sides of Σ and introduce the standard notation $[X]$ defined by

$$[X] := X^+ - X^-, \tag{2.23}$$

where $X^\pm$ are X's evaluated either on the $+$ or $-$ side of Σ.

2.2.1 Non-null shell

Here we consider the case where Σ is non-null as shown in Fig. 1. A unit normal vector n^μ to Σ is given by

$$n_\mu := \frac{\varepsilon \nabla_\mu \Phi}{(\varepsilon g^{\rho\sigma} \nabla_\rho \Phi \nabla_\sigma \Phi)^{1/2}} \tag{2.24}$$

and satisfies $n^\mu n_\mu = \varepsilon$, where $\varepsilon = 1 \ (-1)$ corresponds to the case where Σ is a timelike (spacelike) hypersurface. We choose n^μ to point from $\mathcal{M}_-$ to $\mathcal{M}_+$.

In the bulk spacetime $(\mathcal{M}, g_{\mu\nu})$, Σ is described by $x^\mu = x^\mu(y)$ and the line element on Σ is given by
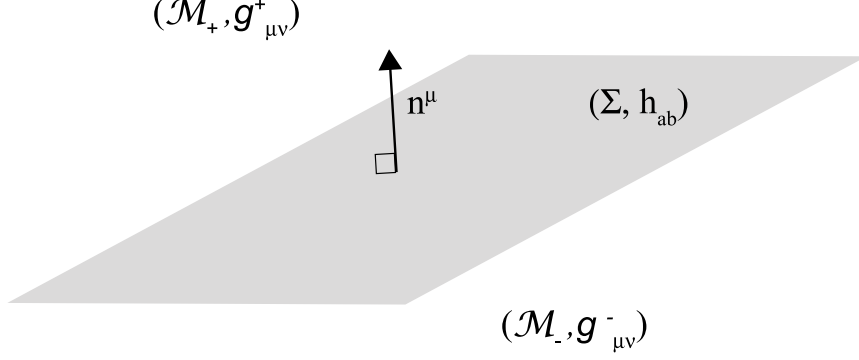
$$ds_\Sigma^2 = h_{ab}(y)dy^a dy^b, \tag{2.25}$$

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Figure 1: A non-null hypersurface Σ partitions a spacetime into two regions $\mathcal{M}_+$ and $\mathcal{M}_-$.

where the induced metric h_{ab} on Σ is defined by

$$h_{ab}(y) := g_{\mu\nu} e_a^\mu e_b^\nu, \qquad e_a^\mu := \frac{\partial x^\mu}{\partial y^a}. \tag{2.26}$$

h_{ab} and its inverse h^{ab} are used to raise or lower Latin indices, respectively. A projection tensor is defined by $h_{\mu\nu} := g_{\mu\nu} - \varepsilon n_\mu n_\nu$ which satisfies $h_{\mu\nu} n^\nu = 0$ and $h_{ab} = h_{\mu\nu} e_a^\mu e_b^\nu$ (and therefore $h_{\mu\nu} = h_{ab} e_\mu^a e_\nu^b$). The extrinsic curvature (or the second fundamental form) $K_{\mu\nu}$ of Σ and its trace are defined by

$$K_{\mu\nu} := h_\mu^{\ \rho} h_\nu^{\ \sigma} \nabla_\rho n_\sigma \left(\equiv \frac{1}{2} \mathcal{L}_n h_{\mu\nu} \right), \tag{2.27}$$

$$K := g^{\mu\nu} K_{\mu\nu} = \nabla_\mu n^\mu, \tag{2.28}$$

where $\mathcal{L}_n$ is the Lie derivative with respect to n^μ. $K_{\mu\nu}$ is symmetric and tangent to Σ, so that $K_{\mu\nu} n^\nu = 0$ holds.

The first junction conditions at Σ are given by

$$[h_{ab}] = 0, \tag{2.29}$$

which means that the induced metric on Σ is the same on both sides of Σ. Under the first junction conditions, one obtains the second junction conditions from the gravitational field equations[1] that determine the induced energy-momentum tensor $t_{\mu\nu}$ on Σ. $t_{\mu\nu}$ is symmetric and tangent to Σ, so that $t_{\mu\nu} n^\nu = 0$ holds. In general relativity, the second junction conditions are obtained from the Einstein equations $G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa_n T_{\mu\nu}$ and referred to as the *Israel junction conditions* [1], which are given by

$$- \varepsilon \left([K_{\mu\nu}] - h_{\mu\nu}[K] \right) = \kappa_n t_{\mu\nu} \quad \Leftrightarrow \quad - \varepsilon \left([K_{ab}] - h_{ab}[K] \right) = \kappa_n t_{ab}, \tag{2.30}$$

---

[1]Independent second junction conditions may also be obtained from the field equations for matter fields in the bulk spacetime $\mathcal{M}$. (See [28] in the case of a scalar field.)

where $K_{ab} := K_{\mu\nu}e_a^\mu e_b^\nu$, $K \equiv K_{ab}h^{ab}$, and $t_{ab} := t_{\mu\nu}e_a^\mu e_b^\nu$. We wil use the following expression

$$
\begin{aligned}
K_{ab} =& (\nabla_\mu n_\nu)e_a^\mu e_b^\nu \\
=& - n_\mu e_{a,b}^\mu - \Gamma_{\mu\nu}^\kappa n_\kappa e_a^\mu e_b^\nu ,
\end{aligned}
\tag{2.31}
$$

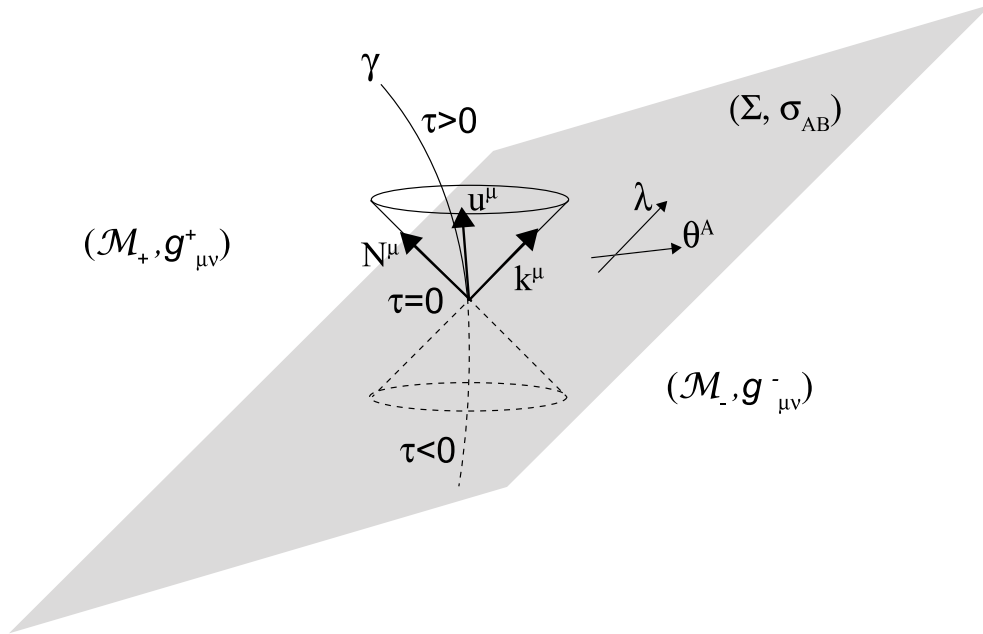where we have used $n_\mu e_a^\mu = 0$ at the last equality.

2.2.2   Null shell

Figure 2: A null hypersurface Σ partitions a spacetime into two regions $\mathcal{M}_+$ and $\mathcal{M}_-$.

In the case where Σ is null, our convention is such that $\mathcal{M}_-$ is in the past of Σ and $\mathcal{M}_+$ is in the future as shown in Fig. 2. Since the unit normal vector (2.24) cannot be used for null Σ, we introduce a null vector k^μ defined by

$$
k^\mu = -\nabla^\mu \Phi,
\tag{2.32}
$$

which is tangent to the generators of Σ. (See Sec. 3.1 in the textbook [4] for the proof.) Here the minus sign is chosen so that k^μ is future-directed when Φ increases toward the future. We install intrinsic coordinates $y^a = (\lambda, \theta^A)$ $(A = 2, 3, \cdots, n-1)$ on Σ, where λ is an arbitrary parameter on the null generators of Σ and the other $n-2$ coordinates θ^A label the generators. λ can be an affine parameter on one side of Σ but it is in general not possible on both sides of Σ.

The tangent vectors $e_a^\mu := \partial x^\mu / \partial y^a$ on each side of Σ are naturally segregated into a null vector k^μ that is tangent to the generators and spacelike vectors e_A^μ that point in the directions transverse to the generators. k^μ and e_A^μ are explicitly written as

$$k^\mu \equiv e_\lambda^\mu = \left(\frac{\partial x^\mu}{\partial \lambda}\right)_{\theta^A}, \qquad e_A^\mu = \left(\frac{\partial x^\mu}{\partial \theta^A}\right)_\lambda, \tag{2.33}$$

which satisfy $k^\mu k_\mu = 0 = k_\mu e_A^\mu$. The line element on Σ is

$$ds_\Sigma^2 = g_{\mu\nu} e_a^\mu e_b^\nu dy^a dy^b = \sigma_{AB} d\theta^A d\theta^B, \tag{2.34}$$

where the induced metric σ_{AB} on Σ is defined by

$$\sigma_{AB} := g_{\mu\nu} e_A^\mu e_B^\nu. \tag{2.35}$$

A basis is completed by adding a transverse null vector N^μ which satisfies

$$N_\mu N^\mu = 0, \qquad N_\mu k^\mu = -1, \qquad N_\mu e_A^\mu = 0. \tag{2.36}$$

The completeness relations of the basis are given as

$$g^{\mu\nu} = -k^\mu N^\nu - N^\mu k^\nu + \sigma^{AB} e_A^\mu e_B^\nu, \tag{2.37}$$

where σ^{AB} is the inverse of σ_{AB}. The first junction conditions at Σ are given by

$$[\sigma_{AB}] = 0, \tag{2.38}$$

which means that the induced metric on Σ is the same on both sides of Σ.

Since k^μ is not normal but tangent to the generators of Σ, we introduce the transverse curvature C_{ab} that properly represents the transverse derivative of the metric:

$$C_{ab} := \frac{1}{2}(\mathcal{L}_N g_{\mu\nu}) e_a^\mu e_b^\nu = (\nabla_\mu N_\nu) e_a^\mu e_b^\nu, \tag{2.39}$$

where we have used that $\nabla_\nu(N_\mu e_a^\mu) = 0$ and an identity $(\nabla_\nu e_a^\mu) e_b^\nu \equiv (\nabla_\nu e_b^\mu) e_a^\nu$ at the last equality. The jump in the transverse curvature $[C_{ab}]$ is directly related to the induced energy-momentum tensor $t_{\mu\nu}$ on Σ.

In the reformulation by Poisson [3], one needs to introduce an arbitrary congruence of timelike geodesics γ that arbitrarily intersect Σ, of which unit tangent vector is u^μ, in order to derive the expression of $t_{\mu\nu}$. Each member of the congruence corresponds to the world line of a geodesic observer that intersects Σ and performs measurements there. The congruence corresponds to a whole family of such observers and gives operational meaning to the distributional character of $t_{\mu\nu}$. We parametrize γ by the proper time τ such that $\tau = 0$ at Σ, $\tau < 0$ in $\mathcal{M}_-$, and $\tau > 0$ in $\mathcal{M}_+$. Then, a displacement along a member of the congruence is described by $dx^\mu = u^\mu d\tau$. Continuity of u^μ across Σ requires

$$[-u_\mu k^\mu] = 0, \qquad [u_\mu e_A^\mu] = 0, \tag{2.40}$$

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while $u_\mu N^\mu$ may be discontinuous across Σ.

Then, under the first junction conditions (2.38), one obtains the second junction conditions from the gravitational field equations, which determine the induced energy-momentum tensor $t_{\mu\nu}$ on Σ. Generally, $t_{\mu\nu}$ is obtained in the following form:

$$t_{\mu\nu} = (-k_\eta u^\eta)^{-1} S_{\mu\nu}, \tag{2.41}$$

where

$$S_{\mu\nu} := \mu k_\mu k_\nu + j_A (k_\mu e^A_\nu + e^A_\mu k_\nu) + p\sigma_{AB} e^A_\mu e^B_\nu. \tag{2.42}$$

Here μ, j^A, and p are respectively interpreted as the shell's surface density, surface current, and isotropic surface pressure. Those quantities multiplied by $(-k_\eta u^\eta)^{-1}$ are the quantities that the geodesic observer corresponding to γ measures. $t_{\mu\nu}$ is symmetric and tangent to Σ, so that $t_{\mu\nu} k^\nu = 0$ holds. In general relativity, the second junction conditions are obtained from the Einstein equations $G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa_n T_{\mu\nu}$ and referred to as the *Barrabès-Israel junction conditions* [2,3], which are given by

$$\kappa_n \mu = -\sigma^{AB}[C_{AB}], \qquad \kappa_n j^A = \sigma^{AB}[C_{\lambda B}], \qquad \kappa_n p = -[C_{\lambda\lambda}]. \tag{2.43}$$

2.3 Main results

Now we are ready to present our main results. If Σ is timelike ($\varepsilon = 1$), h_{ab} has the Lorentzian signature and consequently the induced energy-momentum tensor $t^{\mu\nu}(= t^{ab} e^\mu_a e^\nu_b)$ can be any of the Hawking-Ellis types I through IV. In contrast, on a spacelike Σ ($\varepsilon = -1$), as shown below, $t^{\mu\nu}$ is of type I and simple equivalent representations of the standard energy conditions for $t^{\mu\nu}$ are available.

Proposition 1 *An induced energy-momentum tensor $t_{\mu\nu}$ on a spacelike hypersurface Σ is of the Hawking-Ellis type I and a non-vanishing $t_{\mu\nu}$ violates the DEC. The NEC, WEC, and SEC are all equivalent to that $p_i \geq 0$ are satisfied for all $i(= 1, 2, \cdots, n-1)$, where p_i are eigenvalues of the eigenvalue equations $t_{ab} v^b = \lambda h_{ab} v^b$.*

Proof: For $\varepsilon = -1$, we can choose $E^\mu_{(0)} = n^\mu$ and set $t_{(a)(b)} = t_{\mu\nu} E^\mu_{(a)} E^\nu_{(b)}$ ($a, b = 1, 2, \cdots, n-1$) be diagonal without loss of generality by using degrees of freedom to rotate the spacelike basis vectors $E^\mu_{(a)}$. In this orthonormal frame, $t_{(\alpha)(\beta)}$ is given in the form of the Hawking-Ellis type I as $t_{(\alpha)(\beta)} = \text{diag}(0, p_1, p_2, \cdots, p_{n-1})$ and the proposition follows from Eqs. (2.6)–(2.9) because p_i are eigenvalues of the eigenvalue equations $t_{(a)(b)} v^{(b)} = \lambda \delta_{(a)(b)} v^{(b)}$. By identifications $E^\mu_{(a)} \equiv e^\mu_a / |e^\mu_a|$, the eigenvalue equations are written as $t_{ab} v^b = \lambda h_{ab} v^b$, where $v^b = v^\mu e^b_\mu$ is a vector on Σ. ∎

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Next, we consider the case where Σ is null. The Hawking-Ellis type of $t_{\mu\nu}$ and equivalent representations of the standard energy conditions for $t_{\mu\nu}$ are given as follows.

Proposition 2 *Define J^2 for an induced energy-momentum tensor $t_{\mu\nu}$ in the most general form (2.41) on a null hypersurface Σ by*

$$J^2 := j_A j_B \sigma^{AB}. \tag{2.44}$$

For $t_{\mu\nu}$, the Hawking-Ellis type and equivalent representations of the standard energy conditions are as shown in the following table.

	Hawking-Ellis type	NEC, WEC, SEC	DEC
$J = 0$	II	$\mu \geq 0,\ p \geq 0$	$\mu \geq 0,\ p = 0$
$J \neq 0,\ p = 0$	III	violated	violated
$Jp \neq 0,\ J^2 \neq \mu p$	II	$\mu p > J^2,\ p > 0$	violated
$Jp \neq 0,\ J^2 = \mu p$	I	$p > 0$	violated

Proof. We introduce orthonormal basis vectors $E_{(\alpha)}^{\mu}$ at the location of Σ such that a timelike basis vector $E_{(0)}^{\mu}$ and a spacelike basis vector $E_{(1)}^{\mu}$ are given by

$$\begin{cases} E_{(0)}^{\mu} = (k^{\mu} + N^{\mu})/\sqrt{2} \\ E_{(1)}^{\mu} = (-k^{\mu} + N^{\mu})/\sqrt{2} \end{cases} \Leftrightarrow \begin{cases} k^{\mu} = (E_{(0)}^{\mu} - E_{(1)}^{\mu})/\sqrt{2} \\ N^{\mu} = (E_{(0)}^{\mu} + E_{(1)}^{\mu})/\sqrt{2} \end{cases}. \tag{2.45}$$

If $j_A e_{\mu}^{A}$ is non-vanishing, using degrees of freedom to rotate the spacelike basis vectors $E_{(i)}^{\mu}$ $(i = 2, 3, \cdots, n-1)$, we can set $E_{\mu(2)}$ point the direction of $j_A e_{\mu}^{A}$ such that $j_A e_{\mu}^{A} = -J E_{\mu(2)}$ without loss of generality, where J satisfies Eq. (2.44). If $j_A e_{\mu}^{A}$ is vanishing (and then $J^2 = 0$), we don't specify the direction of $E_{\mu(2)}$. Now Eq. (2.42) is written as

$$S^{\mu\nu} = \frac{1}{2}\mu(E_{(0)}^{\mu} - E_{(1)}^{\mu})(E_{(0)}^{\nu} - E_{(1)}^{\nu}) - \frac{1}{\sqrt{2}}J\left[(E_{(0)}^{\mu} - E_{(1)}^{\mu})E_{(2)}^{\nu} + E_{(2)}^{\mu}(E_{(0)}^{\nu} - E_{(1)}^{\nu})\right]$$
$$+ p\left[g^{\mu\nu} + \frac{1}{2}(E_{(0)}^{\mu} - E_{(1)}^{\mu})(E_{(0)}^{\nu} + E_{(1)}^{\nu}) + \frac{1}{2}(E_{(0)}^{\mu} + E_{(1)}^{\mu})(E_{(0)}^{\nu} - E_{(1)}^{\nu})\right], \tag{2.46}$$

where we have used Eq. (2.37). Then, we obtain orthonormal components of the induced energy-momentum tensor (2.41) as $t^{(\alpha)(\beta)} = (-k_{\eta}u^{\eta})^{-1}S^{(\alpha)(\beta)}$, where

$$S^{(\alpha)(\beta)} := \begin{pmatrix} \mu/2 & \mu/2 & J/\sqrt{2} & 0 & 0 & \cdots & 0 \\ \mu/2 & \mu/2 & J/\sqrt{2} & 0 & 0 & \cdots & 0 \\ J/\sqrt{2} & J/\sqrt{2} & p & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & p & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & p \end{pmatrix}. \tag{2.47}$$

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Since the factor $(-k_\eta u^\eta)^{-1}$ is positive, the Hawking-Ellis types and the energy conditions for $t^{(\alpha)(\beta)}$ and $S^{(\alpha)(\beta)}$ are the same.

For $J = 0$, $S^{(\alpha)(\beta)}$ is in the canonical type II form (2.10) with $\nu = \mu/2$, $\rho = 0$, and $p_2 = p_3 = \cdots = p_{n-1} = p$. Then, the standard energy conditions are equivalent to

$$\text{NEC, WEC, SEC}: \quad \mu \geq 0 \text{ and } p \geq 0, \tag{2.48}$$
$$\text{DEC}: \quad \mu \geq 0 \text{ and } p = 0 \tag{2.49}$$

by Eqs. (2.14)–(2.17). For $J \neq 0$ with $p = 0$, $S^{(\alpha)(\beta)}$ is in the canonical type-III form (2.18) with $\nu = \mu/2$, $\rho = 0$, and $\zeta = J/\sqrt{2}$, and then all the standard energy conditions are violated. Hereafter we assume $Jp \neq 0$.

The characteristic equation of the eigenvalue equations gives $\lambda^2(\lambda - p)^{n-2} = 0$ and hence the eigenvalues are $\lambda = \{0, p\}$. For the eigenvalue $\lambda = p(\neq 0)$, the corresponding $n - 2$ Lorentz-covariant eigenvectors are spacelike and their normalized forms are given by $n_{(\alpha)} = \{w_{i(\alpha)}\}$ $(i = 2, 3, \cdots, n - 1)$, where

$$w_{2(\alpha)} = \begin{pmatrix} J/(\sqrt{2}p) \\ -J/(\sqrt{2}p) \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad w_{3(\alpha)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \quad w_{n-1(\alpha)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}. \tag{2.50}$$

Comparing

$$S^{(\alpha)(\beta)} - \sum_{i=2}^{n-1} p w_i^{(\alpha)} w_i^{(\beta)}$$

$$= \begin{pmatrix} (\mu p - J^2)/(2p) & (\mu p - J^2)/(2p) & 0 & 0 & 0 & \cdots & 0 \\ (\mu p - J^2)/(2p) & (\mu p - J^2)/(2p) & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{2.51}$$

with Eqs. (2.10) and (2.13), we find that $S^{(\alpha)(\beta)}$ is of type II for $\mu p \neq J^2$ and of type I for $\mu p = J^2$. For $\mu p \neq J^2$, the canonical form of $S^{(\alpha)(\beta)}$ is given by Eq. (2.10) with $\nu = (\mu p - J^2)/(2p)$, $\rho = 0$, and $p_2 = p_3 = \cdots = p_{n-1} = p$. Then, by Eqs. (2.14)–(2.17), the NEC, WEC, and SEC are equivalent to $\mu p > J^2$ and $p > 0$, while the DEC is violated. For $\mu p = J^2$, the canonical form of $S^{(\alpha)(\beta)}$ is given by Eq. (2.4) with $\rho = 0$, $p_1 = 0$, and $p_2 = p_3 = \cdots = p_{n-1} = p$. Then, by Eqs. (2.6)–(2.9), the NEC, WEC, and SEC are equivalent to $p > 0$, while the DEC is violated. ∎

3 Applications in general relativity in four dimensions

Propositions 1 and 2 in the previous section are the main results in the present paper. Without imposing a spacetime symmetry, we have shown that the induced energy-momentum tensor $t_{\mu\nu}$ on a shell Σ is of the Hawking-Ellis type I if Σ is spacelike and either of type I, II, or III if Σ is null. Then, we have derived equivalent representations of the standard energy conditions for $t_{\mu\nu}$. In particular, on a spacelike shell or on a null shell with non-vanishing surface current, $t_{\mu\nu}$ inevitably violates the DEC. Those fully general results have been obtained without imposing a spacetime symmetry and can be used in any theory of gravity. In this section, as a demonstration, we apply Propositions 1 and 2 to several physical situations in general relativity in four dimensions ($n = 4$).

3.1 Shells in the Schwarzschild spacetime

3.1.1 Black bounce with a spacelike shell

As the first application, we consider a spacelike massive thin shell constructed by gluing two Schwarzschild bulk spacetimes. We write the bulk metric in the diagonal coordinates $x^\mu = (t, r, \theta, \phi)$ as

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2 \gamma_{AB}\mathrm{d}z^A \mathrm{d}z^B, \tag{3.1}$$

$$f(r) := 1 - \frac{2M}{r}, \qquad \gamma_{AB}\mathrm{d}z^A \mathrm{d}z^B = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2, \tag{3.2}$$

where M is a positive mass parameter and $z^A (A = 2, 3)$ are coordinates on the unit two-sphere S^2. Let r_0 be a constant satisfying $0 < r_0 < 2M$ and consider two spacetimes which are described by the metric (3.1) with the same positive mass M and defined in the domain $r \geq r_0$. We glue them at a spacelike hypersurface Σ described by $r = r_0$. In the resulting spacetime, the big bounce occurs at a spacelike bounce hypersurface Σ inside the event horizon of a black hole as shown in Fig. 3. Such a spacetime is referred to as a *black bounce*. Our model is a thin-shell version of the Simpson-Visser black-bounce model [42], in which the metric is analytic everywhere. We will show that all the standard energy conditions are violated on Σ.

As seen from $\mathcal{M}_+$, the unit normal one-form to Σ is given by

$$n_\mu^+ \mathrm{d}x^\mu = -\frac{1}{\sqrt{-f(r_0)}}\mathrm{d}r. \tag{3.3}$$

Note that the timelike vector $n^\mu = (0, \sqrt{-f(r_0)}, 0, 0)$ points an increasing direction of r, which is consistent with the assumption in Sec. 2.2.1 that n^μ points from $\mathcal{M}_-$ to $\mathcal{M}_+$. The
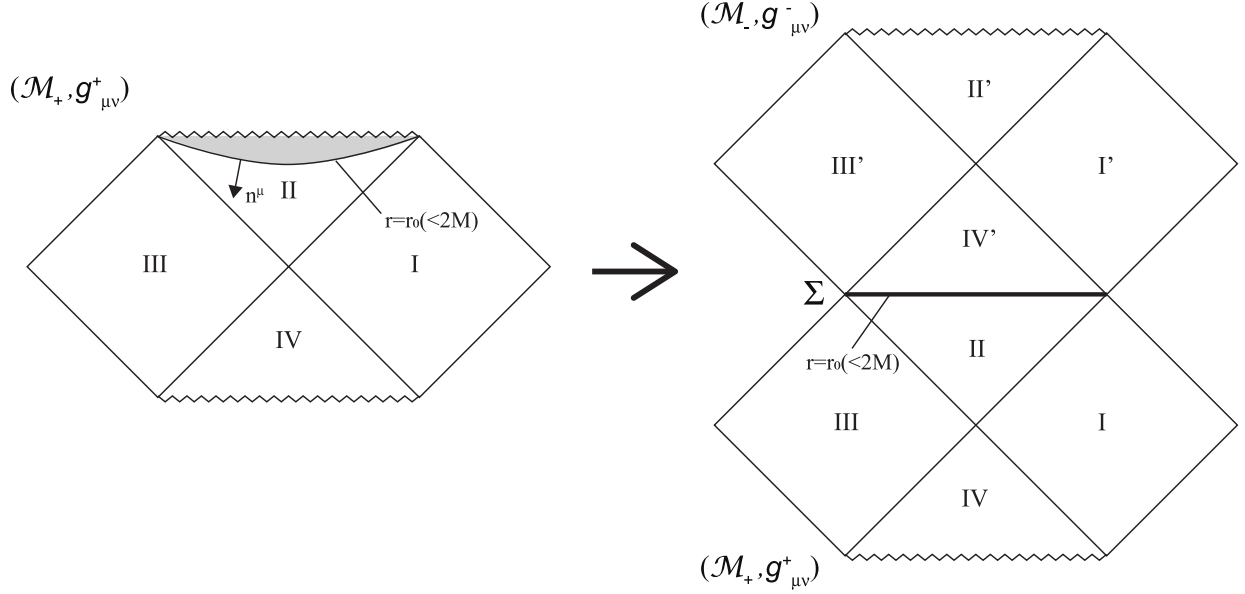
Figure 3: A Penrose diagram of the thin-shell black-bounce spacetime constructed by gluing two identical Schwarzschild spacetimes at a spacelike hypersurface $r = r_0(< 2M)$.

induced metric h_{ab} on the spacelike Σ is given by

$$ds_\Sigma^2 = h_{ab}(y)dy^a dy^b = -f(r_0)dt^2 + r_0^2 \gamma_{AB}dz^A dz^B \,, \tag{3.4}$$

where $a = 1, 2, 3$ and we have identified $y^1 \equiv t$ and $y^A \equiv z^A$. Since $r_0 = 2M$ is the same on both sides of Σ, the first junction conditions $[h_{ab}] = 0$ are satisfied.

Using Eq. (2.31) with Eq. (3.3) and

$$e_t^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial t}, \qquad e_A^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial z^A}, \tag{3.5}$$

we obtain non-zero components of K_{ab} as seen from $\mathcal{M}_+$ as

$$K_{tt}^+ = -\frac{1}{2}\sqrt{-f}f'|_{r=r_0} \,, \quad K_{AB}^+ = r\sqrt{-f}\gamma_{AB}|_{r=r_0} \,, \tag{3.6}$$

which give

$$K^+ = K_{ab}^+ h^{ab} = \left(-\frac{f'}{2\sqrt{-f}} + \frac{2\sqrt{-f}}{r} \right)\Bigg|_{r=r_0} \,, \tag{3.7}$$

where a prime denotes differentiation with respect to r. As seen from $\mathcal{M}_-$, the unit normal is given by

$$n_\mu^- dx^\mu = \frac{1}{\sqrt{-f(r_0)}}dr \tag{3.8}$$

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instead of Eq. (3.3), which points a decreasing direction of r. As a result, K_{ab} as seen from $\mathcal{M}_-$ is given by $K_{ab}^- = -K_{ab}^+$.

Then, the Israel junction conditions (2.30) with $\varepsilon = -1$ give

$$\kappa_4 t_{ab} = 2(K_{ab}^+ - h_{ab} K^+), \tag{3.9}$$

of which non-zero components are given by

$$t_{tt} = -\left.\frac{4(-f)^{3/2}}{\kappa_4 r}\right|_{r=r_0} =: \lambda_1, \qquad t_{AB} = \left.\frac{r(rf' + 2f)}{\kappa_4 \sqrt{-f}}\gamma_{AB}\right|_{r=r_0} =: \lambda_2 \gamma_{AB}. \tag{3.10}$$

Eigenvalues of $t_{ab} v^b = \lambda h_{ab} v^b$ are given by $\lambda = \{p_1, p_2\}$, where

$$p_1 := -\left.\frac{4\sqrt{-f}}{\kappa_4 r}\right|_{r=r_0}, \qquad p_2 := \left.\frac{rf' + 2f}{\kappa_4 r \sqrt{-f}}\right|_{r=r_0}. \tag{3.11}$$

Since p_1 is negative, all the standard energy conditions are violated on Σ by Proposition 1.

3.1.2 Lightlike impulse as a null shell

Next, we study the energy conditions of a lightlike impulse in the Schwarzschild spacetime as a null shell, which has been studied in [2,3]. We write the bulk metric in the single-null coordinates (v, r, z^A) as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -f(r) dv^2 + 2\epsilon dv dr + r^2 \gamma_{AB} dz^A dz^B, \tag{3.12}$$

where $f(r)$ and γ_{AB} are defined by Eq. (3.2) and $\epsilon = \pm 1$. Non-zero components of the inverse metric are given by

$$g^{vv} = 0, \qquad g^{vr} = \epsilon, \qquad g^{rr} = f, \qquad g^{AB} = r^{-2}\gamma^{AB}, \tag{3.13}$$

where γ^{AB} is the inverse of γ_{AB}. For a given ϵ, we consider a spacetime described by the metric (3.12) with $M = M_+(M_-)$ defined in the domain $v \geq (\leq)v_0$ as $\mathcal{M}_+(\mathcal{M}_-)$, where v_0 is a constant. We glue them at a null hypersurface Σ defined by $v = v_0$ and the resulting spacetime describes a lightlike impulse in the Schwarzschild spacetime as shown in Fig. 4. We will derive the equivalent inequalities to the standard energy conditions on Σ.

The null vector $k^\mu = -g^{\mu\nu}\nabla_\nu v$ tangent to the null generators of Σ is given by

$$k^\mu \frac{\partial}{\partial x^\mu} = -\epsilon \frac{\partial}{\partial r}, \tag{3.14}$$

which satisfies $k_\mu k^\mu = 0$ and $k^\nu \nabla_\nu k^\mu = 0$. Hence, k^μ is parametrized by an affine parameter λ. By $k^r = dr/d\lambda = -\epsilon$, we identify $-\epsilon r$ with λ. Since z^A are constant along the

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(a)

$(\mathcal{M}_+, g^+{}_{\mu\nu})$

II

II'

r=2M-

v=v₀

r=2M₊

I'

III

Nᵘ

I

kᵘ

IV

Σ

$(\mathcal{M}_-, g^-{}_{\mu\nu})$

(b)

$(\mathcal{M}_+, g^+{}_{\mu\nu})$

II

Σ

III

Nᵘ

I

kᵘ

r=2M₊

v=v₀

I'

IV

r=2M-

IV'

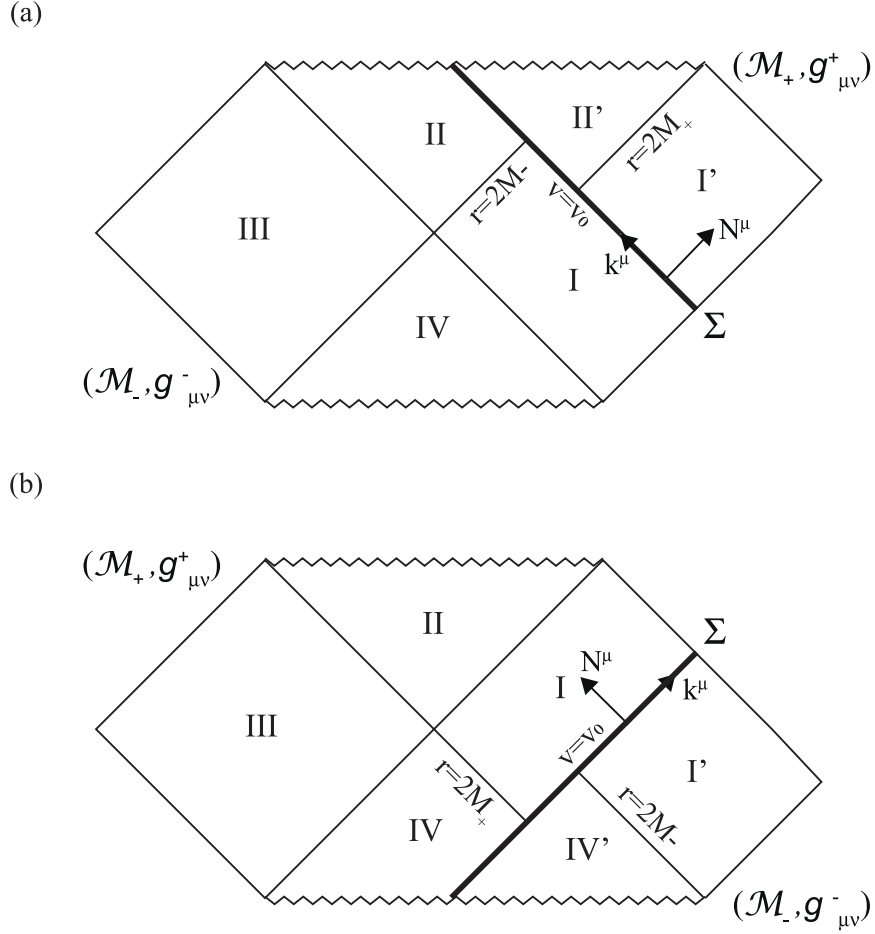$(\mathcal{M}_-, g^-{}_{\mu\nu})$

Figure 4: Penrose diagrams of the Schwarzschild spacetime with a lightlike impulse as a null shell for (a) $\epsilon = 1$ with $M_+ > M_-$ and (b) $\epsilon = -1$ with $M_+ < M_-$. These shells satisfy all the standard energy conditions.

generators, we install coordinates $y^a = (\lambda, \theta^A)$ on Σ such that $\lambda = -\epsilon r$ and $\theta^A = z^A$. Now the parametric equations $x^\mu = x^\mu(\lambda, \theta^A)$ describing Σ are $v = v_0$, $r = -\epsilon\lambda$, and $z^A = \theta^A$, where θ^A label the generators of Σ. $e^\mu_a := \partial x^\mu / \partial y^a$ are given by

$$e^\mu_\lambda \frac{\partial}{\partial x^\mu} = k^\mu \frac{\partial}{\partial x^\mu}, \qquad e^\mu_A \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial z^A}. \tag{3.15}$$

By Eq. (2.34), the induced metric on Σ is given by

$$ds^2_\Sigma = \sigma_{AB} d\theta^A d\theta^B = \lambda^2 \gamma_{AB} dz^A dz^B. \tag{3.16}$$

Since σ_{AB} is independent from $M_\pm$, the first junction conditions $[\sigma_{AB}] = 0$ are satisfied.

The transverse null vector N^μ completing the basis is given by

$$N^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial v} + \frac{1}{2}\epsilon f \frac{\partial}{\partial r}. \tag{3.17}$$

The expression $N_\mu dx^\mu = -(f/2)dv + \epsilon dr$ shows $N_\mu N^\mu = 0$, $N_\mu e^\mu_\lambda = -1$, and $N_\mu e^\mu_A = 0$. Then, by Eq. (2.39), nonvanishing components of the transverse curvature of Σ are computed to give

$$C_{AB} = \frac{1}{2}\epsilon r f \gamma_{AB} \bigg|_{r=-\epsilon\lambda}. \tag{3.18}$$

Now we have $[C_{\lambda\lambda}] = [C_{\lambda A}] = 0$ and

$$[C_{AB}] = \frac{1}{2}\epsilon r(f_+ - f_-)\gamma_{AB} \bigg|_{r=-\epsilon\lambda} = -\epsilon(M_+ - M_-)\gamma_{AB}. \tag{3.19}$$

Then, the Barrabès-Israel junction conditions (2.43) give $p = 0$, $j^A = 0$, and

$$\mu = \frac{2\epsilon(M_+ - M_-)}{\kappa_4 \lambda^2}. \tag{3.20}$$

As $p = 0$ and $J = 0$ are satisfied, $t^{\mu\nu}$ on Σ is of the Hawking-Ellis type II and all the standard energy conditions are satisfied (violated) for $\epsilon(M_+ - M_-) \geq (<)0$ by Proposition 2.

3.1.3 Accretion of a slowly-rotating null shell

As the third application, we consider a situation that a slowly-rotating null shell is collapsing in the Schwarzschild spacetime with its mass $M - m$. This system has been studied in [3]. The past (or interior) spacetime $\mathcal{M}_-$ of the shell is described by the Schwarzschild spacetime metric

$$ds^2_- = -F(r)d\bar{t}^2 + F(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{3.21}$$

$$F(r) := 1 - \frac{2(M - m)}{r} \tag{3.22}$$

in the coordinates $x_-^\mu = (\bar{t}, r, \theta, \varphi)$, where M and m are constant. The future (or exterior) spacetime $\mathcal{M}_+$ is described by the slowly rotating Kerr metric:

$$\mathrm{d}s_+^2 = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) - \frac{4Ma}{r}\sin^2\theta\mathrm{d}t\mathrm{d}\phi,$$

$$f(r) := 1 - \frac{2M}{r} \tag{3.23}$$

in the coordinates $x_+^\mu = (t, r, \theta, \phi)$, where a is a rotation parameter. Non-zero components of the inverse metric on $\mathcal{M}_+$ are

$$g^{tt} = -\frac{r^4}{r^4 f + 4M^2 a^2 \sin^2\theta}, \qquad g^{t\phi} = -\frac{2Mar}{r^4 f + 4M^2 a^2 \sin^2\theta}, \tag{3.24}$$

$$g^{rr} = f, \qquad g^{\theta\theta} = r^{-2}, \qquad g^{\phi\phi} = \frac{r^2 f}{\sin^2\theta(r^4 f + 4M^2 a^2 \sin^2\theta)}. \tag{3.25}$$

The metric (3.23) is a solution to the vacuum Einstein equations in the slow-rotation approximation, in which we consider up to the linear order of a. We will show that, due to the ambiguity to define a null vector, one cannot obtain a definite conclusion on the Hawking-Ellis type of the shell and the energy conditions for the null shell in this approximation.

On $\mathcal{M}_+$, we define $v = v(t, r)$ and $r_* = r_*(r)$ by

$$v := t + r_*, \tag{3.26}$$

$$r_* := r + 2M \ln\left|\frac{r}{2M} - 1\right| \left(= \int f(r)^{-1}\mathrm{d}r\right) \tag{3.27}$$

and let Σ be a hypersurface described by $v = v_0 =$ constant. A vector $k^\mu = -g^{\mu\nu}\nabla_\nu v$ is given by

$$k^\mu \frac{\partial}{\partial x^\mu} = -g^{tt}\frac{\partial}{\partial t} - \frac{\partial}{\partial r} - g^{t\phi}\frac{\partial}{\partial \phi}$$

$$\simeq f^{-1}\frac{\partial}{\partial t} - \frac{\partial}{\partial r} + \frac{2Ma}{r^3 f}\frac{\partial}{\partial \phi}, \tag{3.28}$$

which satisfies

$$k_\mu k^\mu = \frac{4M^2 a^2 \sin^2\theta}{f(r^4 f + 4M^2 a^2 \sin^2\theta)} \simeq 0, \tag{3.29}$$

$$k^\nu \nabla_\nu k^\mu \frac{\partial}{\partial x^\mu} = -\frac{4M^2 a^2 \sin^2\theta[2r^3 f^2 + f'(r^4 f + 2M^2 a^2 \sin^2\theta)]}{f(r^4 f + 4M^2 a^2 \sin^2\theta)^2}\frac{\partial}{\partial r}$$

$$+ \frac{4M^2 a^2 r^2 \sin\theta\cos\theta}{(r^4 f + 4M^2 a^2 \sin^2\theta)^2}\frac{\partial}{\partial \theta} \simeq 0. \tag{3.30}$$

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Hence, k^μ is null and affinely parametrized in the slow-rotation approximation. In this approximation, Σ is a null hypersurface and k^μ is tangent to the null generators of Σ. By $k^r = -1$ and $k^\theta = 0$, the generators are affinely parametrized by $\lambda = -r$ and θ is constant on each generator.

On $\mathcal{M}_+$, we also define $\psi = \psi(r, \phi)$ by

$$\psi := \phi + \frac{a}{r}\left(1 + \frac{r}{2M}\ln f\right). \tag{3.31}$$

Along generators of Σ, we have

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{k^\phi}{k^r} \simeq -\frac{2Ma}{r^3 f}, \tag{3.32}$$

which is integrated to give

$$\phi - \phi_0 \simeq -\frac{a}{r}\left(1 + \frac{r}{2M}\ln f\right), \tag{3.33}$$

where ϕ_0 is an integration constant. Therefore, ψ defined by Eq. (3.31) is constant on the generators of Σ in the slow-rotation approximation. Thus, we install coordinates $y^a = (\lambda, \theta^A)$ on Σ, where $\theta^A = (\theta, \psi)$. Now the parametric equations $x^\mu = x^\mu(\lambda, \theta^A)$ describing Σ as seen from $\mathcal{M}_+$ are

$$t = -r_*(-\lambda) + v_0, \qquad r = -\lambda, \qquad \theta = \theta, \qquad \phi = \psi + \frac{a}{\lambda}\left(1 - \frac{\lambda}{2M}\ln f(-\lambda)\right). \tag{3.34}$$

Then, $e_a^\mu := \partial x^\mu / \partial y^a$ are given by

$$e_\lambda^\mu \frac{\partial}{\partial x^\mu} = k^\mu \frac{\partial}{\partial x^\mu}, \qquad e_\theta^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial \theta}, \qquad e_\psi^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial \phi}, \tag{3.35}$$

which satisfy $k_\mu k^\mu \simeq 0$ and $k_\mu e_A^\mu = 0$. By Eq. (2.34), the induced metric on Σ is given by

$$\mathrm{d}s_\Sigma^2 = \sigma_{AB}\mathrm{d}\theta^A\mathrm{d}\theta^B = \lambda^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\psi^2). \tag{3.36}$$

The transverse vector N^μ completing the basis in the slow-rotation approximation is given by

$$N_\mu \mathrm{d}x^\mu = \frac{1}{2}(-f\mathrm{d}t + \mathrm{d}r), \tag{3.37}$$

which satisfy $N_\mu k^\mu = -1$, $N_\mu e_A^\mu = 0$, and

$$N_\mu N^\mu = \frac{fM^2 a^2 \sin^2\theta}{r^4 f + 4M^2 a^2 \sin^2\theta} \simeq 0. \tag{3.38}$$

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From Eq. (2.39), the transverse curvature of Σ is computed to give

$$C^+_{\lambda\lambda} = \frac{4M^2a^2\sin^2\theta[3r^3f^2 + f'(r^4f + M^2a^2\sin^2\theta)]}{f(r^4f + 4M^2a^2\sin^2\theta)^2}\bigg|_{r=-\lambda} \simeq 0, \tag{3.39}$$

$$C^+_{\lambda\theta} = \frac{2M^2a^2r^4f\sin\theta\cos\theta}{(r^4f + 4M^2a^2\sin^2\theta)^2}\bigg|_{r=-\lambda} \simeq 0, \tag{3.40}$$

$$C^+_{\lambda\psi} = \frac{3Mar^2f\sin^2\theta}{r^4f + 4M^2a^2\sin^2\theta}\bigg|_{r=-\lambda} \simeq \frac{3Ma\sin^2\theta}{r^2}\bigg|_{r=-\lambda}, \tag{3.41}$$

$$C^+_{\theta\theta} = \frac{1}{2}rf\bigg|_{r=-\lambda}, \qquad C^+_{\psi\psi} = \frac{1}{2}rf\sin^2\theta\bigg|_{r=-\lambda}. \tag{3.42}$$

We can obtain the results on $\mathcal{M}_-$ by replacing such that $M \to M - m$, $a \to 0$, and $(t,\phi) \to (\bar{t},\varphi)$. Thus, as seen from $\mathcal{M}_-$, we have $\mathrm{d}s^2_{\Sigma-} = \sigma_{AB}\mathrm{d}\theta^A\mathrm{d}\theta^B$ and

$$C^-_{\lambda\lambda} = 0, \qquad C^-_{\lambda\theta} = 0, \qquad C^-_{\lambda\psi} = 0, \qquad C^-_{AB} = \frac{F}{2r}\sigma_{AB}\bigg|_{r=-\lambda}, \tag{3.43}$$

where σ_{AB} is given by Eq. (3.36), so that the first junction conditions $[\sigma_{AB}] = 0$ are satisfied. Then, using the Barrabès-Israel junction conditions (2.43) and

$$[C_{\lambda\lambda}] = [C_{\lambda\theta}] = 0, \qquad [C_{\lambda\psi}] = \frac{3Ma\sin^2\theta}{r^2}\bigg|_{r=-\lambda}, \qquad [C_{AB}] = -\frac{m}{r^2}\sigma_{AB}\bigg|_{r=-\lambda}, \tag{3.44}$$

we obtain

$$\mu = \frac{2m}{\kappa_4\lambda^2}, \qquad j^\theta = 0, \qquad j^\psi = \frac{3Ma}{\kappa_4\lambda^4}, \qquad p = 0. \tag{3.45}$$

Since we have $p = 0$ and $J \neq 0$, the energy-momentum tensor $t_{\mu\nu}$ on the shell Σ is of the Hawking-Ellis type III and violates all the standard energy conditions by Proposition 2.

However, this conclusion under the slow-rotation approximation cannot be definite as shown below. Now $t_{\mu\nu}$ on the shell Σ is given by

$$t^{\mu\nu} = (-k_\eta u^\eta)^{-1}[\mu k^\mu k^\nu + j^\psi(k^\mu e^\nu_\psi + e^\mu_\psi k^\nu)]. \tag{3.46}$$

This $t^{\mu\nu}$ can be written in the slow-rotation approximation as

$$t^{\mu\nu} = (-k_\eta u^\eta)^{-1}\mu\ell^\mu\ell^\nu, \tag{3.47}$$

$$\ell^\mu := k^\mu + \frac{j^\psi}{\mu}e^\mu_\psi, \tag{3.48}$$

where components of ℓ^μ in the coordinates (t,r,θ,ϕ) are given by

$$\ell^\mu \simeq \left(f^{-1}, -1, 0, \frac{2Ma}{r^3f} + \frac{3Ma}{2mr^2}\right)\bigg|_{r=-\lambda}. \tag{3.49}$$

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Since $\ell_\mu \ell^\mu \simeq 0$ is satisfied, ℓ^μ is null in the slow-rotation approximation. As a consequence, $t^{\mu\nu}$ in the form of Eq. (3.47) is of the Hawking-Ellis type II and all the standard energy conditions are satisfied (violated) for $\mu \geq (<)0$ by Proposition 2 in spite that $t^{\mu\nu}$ in the form of Eq. (3.46) is of type III and all the standard energy conditions are inevitably violated. Hence, due to the ambiguity to define a null vector, one cannot obtain a definite conclusion on the Hawking-Ellis type of $t_{\mu\nu}$ and the energy conditions on Σ. To avoid this problem, higher-order effects of the rotation parameter a must be taken into account. We will see in the next subsection how the result in the full-order analysis is different from the one in the slow-rotation approximation.

3.2   Cylindrically symmetric rotating null shell

As the fourth application, we consider a rotating cylindrically symmetric null shell collapsing in the Minkowski spacetime. The past (or interior) spacetime $\mathcal{M}_-$ of the shell is described by the Minkowski metric

$$ds_-^2 = -dt^2 + d\rho^2 + \rho^2 d\varphi^2 + dz^2 \tag{3.50}$$

in the cylindrical coordinates $x_-^\mu = (t, \rho, \varphi, z)$. The future (or exterior) spacetime $\mathcal{M}_+$ of the shell is described by the following locally flat metric

$$ds_+^2 = -(dT + md\Phi)^2 + \mathcal{C}^2 dr^2 + r^2 d\Phi^2 + dz^2 \tag{3.51}$$

in the coordinates $x_+^\mu = (T, r, \Phi, z)$, where $\mathcal{C}(> 0)$ and m are constants. Non-vanishing components of the inverse metric on $\mathcal{M}_+$ are

$$g^{TT} = -\frac{r^2 - m^2}{r^2}, \quad g^{T\psi} = -\frac{m}{r^2}, \quad g^{rr} = \mathcal{C}^{-2}, \quad g^{\Phi\Phi} = r^{-2}, \quad g^{zz} = 1. \tag{3.52}$$

Non-vanishing components of the Levi-Cività connection on $\mathcal{M}_+$ are

$$\Gamma_{r\Phi}^T = -\frac{m}{r}, \quad \Gamma_{r\Phi}^\Phi = \frac{1}{r}, \quad \Gamma_{\Phi\Phi}^r = -\frac{r}{\mathcal{C}^2}. \tag{3.53}$$

The exterior metric (3.51) describes a spinning cosmic string [43, 44] and admits a closed timelike curve in the region $r < |m|$ where $v^\mu(\partial/\partial x^\mu) = \partial/\partial\Phi$ is timelike.

The dynamics of a massive thin-shell Σ as a matching hypersurface between $\mathcal{M}_-$ and $\mathcal{M}_+$ has been investigated for timelike Σ in [45] and for null Σ in [46]. Here we follow the argument in [46] in more detail.

In order to identify the description of the null shell Σ, we first study the most general affinely parametrized ingoing null geodesic γ described by $x_+^\mu = x_+^\mu(\lambda)$ with an affine

parameter λ in the exterior spacetime $\mathcal{M}_+$, of which tangent vector is given by $k^\mu (= \mathrm{d}x_+^\mu/\mathrm{d}\lambda)$. Since the exterior spacetime (3.51) admits the following Killing vectors

$$\xi_1^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial T}, \qquad \xi_2^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial \Phi}, \qquad \xi_3^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial z}, \tag{3.54}$$

there are three conserved quantities $E := -k_\mu \xi_1^\mu$, $K := k_\mu \xi_2^\mu$, and $V_z := k_\mu \xi_3^\mu$ along γ, which give

$$k^T = \frac{E(r^2 - m^2) - mK}{r^2}, \qquad k^\Phi = \frac{mE + K}{r^2}, \qquad k^z = V_z. \tag{3.55}$$

We assume $E > 0$ so that γ is future directed in a far region $r \to \infty$. Then, the null condition $\mathrm{d}s_+^2 = 0$ along γ with Eq. (3.55) gives the following master equation for $r(\lambda)$:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 = \frac{(E^2 - V_z^2)(r^2 - b^2)}{\mathcal{C}^2 r^2} \quad \to \quad k^r = \frac{\mathrm{d}r}{\mathrm{d}\lambda} = -\frac{\sqrt{(E^2 - V_z^2)(r^2 - b^2)}}{\mathcal{C}r}, \tag{3.56}$$

where the minus sign is taken for ingoing γ and b is defined by

$$b := \frac{K + mE}{\sqrt{E^2 - V_z^2}} \tag{3.57}$$

for $E^2 \neq V_z^2$. The master equation (3.56) requires $E^2 - V_z^2 \geq 0$ and shows that $r = |b|$ is the turning radius. The general solution of the master equation for $E^2 \neq V_z^2$ is given by

$$\lambda - \lambda_0 = -\mathcal{C}\sqrt{\frac{r^2 - b^2}{E^2 - V_z^2}} \quad \to \quad r(\lambda)^2 = \frac{E^2 - V_z^2}{\mathcal{C}^2}(\lambda - \lambda_0)^2 + b^2, \tag{3.58}$$

where λ_0 is an integration constant and $\lambda = \lambda_0$ corresponds to the turning radius $r = b$. The null geodesic γ enters the region with closed timelike curves for $|b| < |m|$, or equivalently $|K + mE| < |m|\sqrt{E^2 - V_z^2}$.

Using Eqs. (3.55) and (3.56), we obtain

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{\mathcal{C}(Er^2 - mb\sqrt{E^2 - V_z^2})}{r\sqrt{(E^2 - V_z^2)(r^2 - b^2)}}, \tag{3.59}$$

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} = -\frac{b\mathcal{C}}{r\sqrt{r^2 - b^2}}, \tag{3.60}$$

$$\frac{\mathrm{d}z}{\mathrm{d}r} = -\frac{V_z \mathcal{C} r}{\sqrt{(E^2 - V_z^2)(r^2 - b^2)}} \tag{3.61}$$

along γ. We define three functions as

$$v(T, r) := T + r_*(r), \qquad \psi(\Phi, r) := \Phi + r_\Phi(r), \qquad Z(z, r) := z + r_z(r), \tag{3.62}$$

where

$$r_*(r) := \int \frac{\mathcal{C}(Er^2 - mb\sqrt{E^2 - V_z^2})}{r\sqrt{(E^2 - V_z^2)(r^2 - b^2)}} dr, \tag{3.63}$$

$$r_\Phi(r) := \int \frac{b\mathcal{C}}{r\sqrt{r^2 - b^2}} dr, \tag{3.64}$$

$$r_z(r) := \int \frac{V_z\mathcal{C}r}{\sqrt{(E^2 - V_z^2)(r^2 - b^2)}} dr, \tag{3.65}$$

and then v, ψ, and Z are constant along γ by Eqs. (3.59)–(3.61). By coordinate transformations $T = v - r_*(r)$, $\Phi = \psi - r_\Phi(r)$, and $z = Z - r_z(r)$, the exterior metric (3.51) becomes

$$ds_+^2 = -dv\left(dv - \frac{2\mathcal{C}Er}{\sqrt{(E^2 - V_z^2)(r^2 - b^2)}} dr + 2md\psi\right)$$
$$- \frac{2\mathcal{C}Kr}{\sqrt{(E^2 - V_z^2)(r^2 - b^2)}} drd\psi - \frac{2V_z\mathcal{C}r}{\sqrt{(E^2 - V_z^2)(r^2 - b^2)}} drdZ$$
$$+ (r^2 - m^2)d\psi^2 + dZ^2 \tag{3.66}$$

in the coordinates (v, r, ψ, Z). Now we consider a hypersurface given by $v = v_0 =$ constant. Its normal vector $l_\mu dx^\mu = -(\nabla_\mu v)dx^\mu = -dv$ satisfies

$$l_\mu l^\mu = g^{vv} = -\frac{V_z^2(r^2 - m^2) + K^2}{(K + mE)^2 - (E^2 - V_z^2)r^2} \tag{3.67}$$

and therefore $v = v_0$ is a null hypersurface for $K = V_z = 0$.

Hence, we identify k^μ with $K = 0$ and $V_z = 0$ (and then $b = m$ and $Z = z$) as the tangent vector to the generators of the null shell Σ described by $v = v_0$. Then, the exterior metric (3.51) and k^μ are given in the coordinates (v, r, ψ, z) as

$$ds_+^2 = -dv\left(dv - \frac{2\mathcal{C}r}{\sqrt{r^2 - m^2}} dr + 2md\psi\right) + (r^2 - m^2)d\psi^2 + dz^2, \tag{3.68}$$

$$k^\mu \frac{\partial}{\partial x^\mu} = -\frac{E\sqrt{r^2 - m^2}}{\mathcal{C}r}\frac{\partial}{\partial r}. \tag{3.69}$$

Since ψ and $z(= Z)$ are constant along the generators, we shall install intrinsic coordinates $y^a = (\lambda, \theta^A)$ on Σ as $\theta^A \equiv (\psi, z)$, where $\lambda = \lambda(r)$ is given by Eq. (3.58) with $V_z = 0$ and $b = m$. Then, $e_a^\mu := dx^\mu/dy^a$ are given by

$$e_\lambda^\mu \frac{\partial}{\partial x^\mu} = k^\mu \frac{\partial}{\partial x^\mu}, \qquad e_\psi^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial \psi}, \qquad e_z^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial z}, \tag{3.70}$$

which satisfy $k_\mu k^\mu = 0$ and $k_\mu e_A^\mu = 0$. The induced metric on Σ is given by

$$ds_{\Sigma+}^2 = \sigma_{AB}^+ d\theta^A d\theta^B = (r^2 - m^2)|_\Sigma d\psi^2 + dz^2 = \frac{E^2}{\mathcal{C}^2}(\lambda - \lambda_0)^2 d\psi^2 + dz^2. \tag{3.71}$$

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The transverse null vector completing the basis is given by

$$N_\mu \mathrm{d}x^\mu = -\frac{r^2}{2E(r^2 - m^2)}\mathrm{d}v + \frac{\mathcal{C}r}{E\sqrt{r^2 - m^2}}\mathrm{d}r, \tag{3.72}$$

which satisfies $N_\mu N^\mu = 0$, $N_\mu k^\mu = -1$, and $N_\mu e^\mu_A = 0$. Then, from Eq. (2.39), non-vanishing components of the transverse curvature of Σ are computed to give

$$\begin{aligned} C^+_{\lambda\psi} =C^+_{\psi\lambda} &= \left.\frac{m}{\mathcal{C}\sqrt{r^2 - m^2}}\right|_\Sigma = -\frac{m}{E(\lambda - \lambda_0)}, \\ C^+_{\psi\psi} &= \left.\frac{r^2}{2E\mathcal{C}\sqrt{r^2 - m^2}}\right|_\Sigma = -\frac{1}{2E^2(\lambda - \lambda_0)}\left\{\frac{E^2}{\mathcal{C}^2}(\lambda - \lambda_0)^2 + m^2\right\}. \end{aligned} \tag{3.73}$$

By setting $m = 0$ and $\mathcal{C} = 1$ and replacing (T, r, Φ) by (t, ρ, φ), we can obtain the results on $\mathcal{M}_-$ described by the flat metric (3.50). Equation (3.58) with $V_z = 0$ and $b = 0$ gives

$$\rho = -\bar{E}(\lambda - \lambda_0), \tag{3.74}$$

where we have used the same λ and λ_0 as those in $\mathcal{M}_+$ without loss of generality by an affine transformation $\lambda \to a\lambda + b$. $\bar{E}$ is a constant related to the conserved quantity along a null generator of Σ associated with the Killing vector $\xi^\mu(\partial/\partial x^\mu) = \partial/\partial t$. As seen from $\mathcal{M}_-$, $\lambda = \lambda_0$ corresponds to the axis of symmetry $\rho = 0$. From Eqs. (3.71), we obtain the induced metric on Σ as

$$\mathrm{d}s^2_{\Sigma-} = \sigma^-_{AB}\mathrm{d}\theta^A \mathrm{d}\theta^B = \rho^2|_\Sigma \mathrm{d}\psi^2 + \mathrm{d}z^2 = \bar{E}^2(\lambda - \lambda_0)^2\mathrm{d}\psi^2 + \mathrm{d}z^2. \tag{3.75}$$

From Eq. (3.73), we obtain the transverse curvature of Σ as

$$C^-_{\lambda\psi} = C^-_{\psi\lambda} = 0, \qquad C^-_{\psi\psi} = \left.\frac{\rho}{2\bar{E}}\right|_\Sigma = -\frac{1}{2}(\lambda - \lambda_0). \tag{3.76}$$

By the first junction conditions $[\sigma_{AB}] = 0$, we obtain

$$\bar{E} = \frac{E}{\mathcal{C}}. \tag{3.77}$$

Then, using the Barrabès-Israel junction conditions (2.43) and

$$[C_{\lambda\lambda}] = 0, \qquad [C_{\lambda\psi}] = [C_{\psi\lambda}] = -\frac{m}{E(\lambda - \lambda_0)}, \tag{3.78}$$

$$[C_{\psi\psi}] = -\frac{1}{2E^2(\lambda - \lambda_0)}\left\{\frac{E^2(1 - \mathcal{C}^2)}{\mathcal{C}^2}(\lambda - \lambda_0)^2 + m^2\right\}, \tag{3.79}$$

we obtain

$$\mu = \frac{E^2(1 - \mathcal{C}^2)(\lambda - \lambda_0)^2 + m^2\mathcal{C}^2}{2\kappa_4 E^4(\lambda - \lambda_0)^3}, \quad j^\psi = -\frac{m\mathcal{C}^2}{\kappa_4 E^3(\lambda - \lambda_0)^3}, \quad j^z = 0, \quad p = 0. \tag{3.80}$$

Using $\lambda - \lambda_0 = -\mathcal{C}\sqrt{r^2 - m^2}/E$ from Eq. (3.58), we can write those quantities in terms of r as

$$\mu = \frac{(\mathcal{C}^2 - 1)r^2 - \mathcal{C}^2 m^2}{2\kappa_4 E\mathcal{C}(r^2 - m^2)^{3/2}}\bigg|_\Sigma, \quad j^\psi = \frac{m}{\kappa_4 \mathcal{C}(r^2 - m^2)^{3/2}}\bigg|_\Sigma, \quad j^z = 0, \quad p = 0. \tag{3.81}$$

This result is consistent with the one in [46], in which $\bar{E} = 1$ is assumed. Since we have $p = 0$ and $J \neq 0$, the induced energy-momentum tensor $t_{\mu\nu}$ on the null shell is of the Hawking-Ellis type III and violates all the standard energy conditions by Proposition 2.

Our result is conclusive because we have not used any approximation. $t_{\mu\nu}$ on the shell is now given by

$$t^{\mu\nu} = (-k_\eta u^\eta)^{-1}[\mu k^\mu k^\nu + j^\psi(k^\mu e_\psi^\nu + e_\psi^\mu k^\nu)]. \tag{3.82}$$

For our interest, let us see what happens in the slow-rotation approximation up to the linear order of m/r. In this approximation, Eq. (3.82) can be written as

$$t^{\mu\nu} \simeq (-k_\eta u^\eta)^{-1}\mu \ell^\mu \ell^\nu, \tag{3.83}$$

$$\ell^\mu := k^\mu + \frac{j^\psi}{\mu}e_\psi^\mu. \tag{3.84}$$

The components of ℓ^μ in the coordinates (v, r, ψ, z) are

$$\ell^\mu \frac{\partial}{\partial x^\mu} = -\frac{E\sqrt{r^2 - m^2}}{\mathcal{C}r}\frac{\partial}{\partial r} + \frac{2mE}{(\mathcal{C}^2 - 1)r^2 - \mathcal{C}^2 m^2}\frac{\partial}{\partial \psi} \tag{3.85}$$

and its squared norm is given by

$$\ell_\mu \ell^\mu = \frac{4m^2 E^2(r^2 - m^2)}{[(\mathcal{C}^2 - 1)r^2 - \mathcal{C}^2 m^2]^2} \simeq 0. \tag{3.86}$$

Since ℓ^μ is null in the slow-rotation approximation, one could misunderstand from Eq. (3.83) that $t^{\mu\nu}$ is of the Hawking-Ellis type II and all the standard energy conditions are satisfied (violated) for $\mu \geq (<)0$. Of course, it is a wrong conclusion caused by the approximation.

3.3 Cosmological phase transition

As the last application, this subsection considers a sudden transition of the universe from the anisotropic Bianchi I expansion to the isotropic flat Friedmann-Lemaître-Robertson-Walker (FLRW) expansion. We consider the past spacetime $\mathcal{M}_-$ described by the following Bianchi-I metric

$$ds_-^2 = -dt^2 + a(t)^2(dx^2 + dy^2) + dz_-^2 \tag{3.87}$$

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in the coordinates $x^\mu_- = (t, x, y, z_-)$ and the future spacetime $\mathcal{M}_+$ described by the flat FLRW metric

$$\mathrm{ds}^2_+ = -\mathrm{d}t^2 + a(t)^2(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2_+) \tag{3.88}$$

in the coordinates $x^\mu_+ = (t, x, y, z_+)$ with the same scale factor $a(t)$. We assume that the scale factor $a(t)$ is the same both in $\mathcal{M}_-$ and $\mathcal{M}_+$ as a solution to the Einstein equations $G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa_4 T_{\mu\nu}$. As the Einstein tensor in $\mathcal{M}_-$ and $\mathcal{M}_+$ are given by

$$G^\mu{}_\nu|_- = \mathrm{diag}\left(-\frac{\dot{a}^2}{a^2}, -\frac{\ddot{a}}{a}, -\frac{\ddot{a}}{a}, -\frac{\dot{a}^2 + 2a\ddot{a}}{a^2}\right), \tag{3.89}$$

$$G^\mu{}_\nu|_+ = \mathrm{diag}\left(-\frac{3\dot{a}^2}{a^2}, -\frac{\dot{a}^2 + 2a\ddot{a}}{a^2}, -\frac{\dot{a}^2 + 2a\ddot{a}}{a^2}, -\frac{\dot{a}^2 + 2a\ddot{a}}{a^2}\right), \tag{3.90}$$

the matter fields in $\mathcal{M}_-$ and $\mathcal{M}_+$ are different. In [3], the author assumed $a(t) = (t/t_0)^{1/2}$ and $\Lambda = 0$, where t_0 is a constant. Then, we have

$$G^\mu{}_\nu|_- = \mathrm{diag}\left(-\frac{1}{4t^2}, \frac{1}{4t^2}, \frac{1}{4t^2}, \frac{1}{4t^2}\right), \tag{3.91}$$

$$G^\mu{}_\nu|_+ = \mathrm{diag}\left(-\frac{3}{4t^2}, \frac{1}{4t^2}, \frac{1}{4t^2}, \frac{1}{4t^2}\right). \tag{3.92}$$

In this case, a matter field in $\mathcal{M}_-$ may be a stiff fluid, namely a perfect fluid obeying an equation of state $p = \rho$, while a matter field in $\mathcal{M}_+$ may be a radiation fluid, namely a perfect fluid obeying an equation of state $p = \rho/3$. Hereafter we keep $a(t)$ arbitrary.

3.3.1 Transition at spacelike Σ

First, we consider the case where the transition occurs at a spacelike hypersurface Σ given by $t = t_0 =$ constant both on $\mathcal{M}_-$ and $\mathcal{M}_+$ and then the first junction conditions require $a(t_0)^2 = 1$. The induced metric on Σ is given by

$$\mathrm{ds}^2_\Sigma = h_{ab}\mathrm{d}y^a\mathrm{d}y^b = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2, \tag{3.93}$$

where we have installed intrinsic coordinates $y^a = (x, y, z)$ on Σ and the parametric equations $x^\mu = x^\mu(y^a)$ describing Σ are given by

$$t = t_0, \quad x = x, \quad y = y, \quad z_- = z \quad \text{on } \mathcal{M}_-, \tag{3.94}$$

$$t = t_0, \quad x = x, \quad y = y, \quad z_+ = z \quad \text{on } \mathcal{M}_+. \tag{3.95}$$

The unit normal vector n^μ to Σ is given by

$$n^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial t} \tag{3.96}$$

both on $\mathcal{M}_-$ and $\mathcal{M}_+$. Note that the timelike vector n^μ points in an increasing direction of t, which is consistent with the assumption in Sec. 2.2.1 that n^μ points from $\mathcal{M}_-$ to $\mathcal{M}_+$.

Using $e_a^\mu = \delta_a^\mu$ in Eq. (2.31), we obtain $K_{ab} = \Gamma_{ab}^0$ and therefore non-zero components of K_{ab} on $\mathcal{M}_-$ and $\mathcal{M}_+$ are given by

$$K_{xx}^- = K_{yy}^- = a\dot{a}|_{t=t_0}, \tag{3.97}$$

$$K_{xx}^+ = K_{yy}^+ = K_{zz}^+ = a\dot{a}|_{t=t_0}, \tag{3.98}$$

which give

$$K^- = K_{ab}^- h^{ab} = \left.\frac{2\dot{a}}{a}\right|_{t=t_0}, \qquad K^+ = K_{ab}^+ h^{ab} = \left.\frac{3\dot{a}}{a}\right|_{t=t_0}. \tag{3.99}$$

Thus, $[K_{ab}]$ admits only a single non-zero component $[K_{zz}] = a\dot{a}|_{t=t_0}$. From the Israel junction conditions (2.30) with $\varepsilon = -1$ and $[K] = \dot{a}/a|_{t=t_0}$, we obtain non-zero components of t_{ab} as

$$\kappa_4 t_{xx} = \kappa_4 t_{yy} = -a\dot{a}|_{t=t_0}. \tag{3.100}$$

Since eigenvalues of $t_{ab}v^b = \lambda h_{ab}v^b$ are given by $\lambda = \{0, -\dot{a}/(\kappa_4 a)|_{t=t_0}\}$, the DEC is violated and the NEC, WEC, and SEC are satisfied (violated) for $\dot{a} \leq (>)0$ by Proposition 1.

3.3.2 Transition at null Σ

Next, we consider the case where the transition occurs at a null hypersurface Σ. The system with $a(t) = (t/t_0)^{1/2}$ and $\Lambda = 0$ has been studied in [3] as a simplified version of the example in [47], however, it contains an error that changes the conclusion.

As seen from $\mathcal{M}_-$, we describe the transition null hypersurface Σ by $t - z_- =$ constant. The null vector $k^\mu = -g^{\mu\nu}\nabla_\nu(t - z_-)$ is computed to give

$$k^\mu\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial t} + \frac{\partial}{\partial z_-}, \tag{3.101}$$

which satisfies $k^\nu\nabla_\nu k^\mu = 0$. Hence, k^μ is tangent to the null generators of Σ which are parametrized by an affine parameter λ. By $k^t = 1$, t is an affine parameter on this side of Σ. Since x and y are constant on the generators, we install intrinsic coordinates $y^a = (\lambda, \theta^A)$ on Σ as $\lambda = t$ and $\theta^A = (x, y)$, and then $e_a^\mu := \partial x^\mu/\partial y^a$ are given by

$$e_\lambda^\mu = k^\mu, \qquad e_A^\mu = \delta_A^\mu, \tag{3.102}$$

which satisfies $k_\mu e_A^\mu = 0$ and $g_{\mu\nu}e_A^\mu e_B^\mu = a^2\delta_{AB}$. From Eq. (2.34), we obtain the induced metric on Σ as

$$ds_{\Sigma-}^2 = \sigma_{AB}^- d\theta^A d\theta^B = a(\lambda)^2(dx^2 + dy^2). \tag{3.103}$$

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The transverse null vector completing the basis is given by

$$N_\mu dx^\mu = -\frac{1}{2}(dt + dz_-), \tag{3.104}$$

which satisfies $N_\mu N^\mu = 0$, $N_\mu k^\mu = -1$, and $N_\mu e_A^\mu = 0$. From Eq. (2.39), non-vanishing components of the transverse curvature of Σ are computed to give

$$C_{AB}^- = \frac{\dot{a}}{2a}\sigma_{AB}^-\Big|_{t=\lambda}, \tag{3.105}$$

where a dot denotes differentiation with respect to t.

As seen from $\mathcal{M}_+$, we describe Σ by $\int a^{-1}dt - z_+ =$constant, which is obtained by integrating $dt = a(t)dz_+$. The null vector $k^\mu = -g^{\mu\nu}\nabla_\nu(\int a^{-1}dt - z_+)$ tangent to the null generators of Σ is given by[2]

$$k^\mu\frac{\partial}{\partial x^\mu} = a^{-1}\frac{\partial}{\partial t} + a^{-2}\frac{\partial}{\partial z_+}, \tag{3.106}$$

which satisfies $k^\nu\nabla_\nu k^\mu = 0$. Hence, the null generators of Σ are parametrized by an affine parameter λ also on this side of Σ. However, t is *not* an affine parameter on this side and $k^t = dt/d\lambda = a(t)^{-1}$ is integrated to give $\lambda = \int a dt$. We write the inverse function of $\lambda = \int a dt$ as $t = t_+(\lambda)$. Since x and y are constant on the generators, we install intrinsic coordinates $y^a = (\lambda, \theta^A)$ on Σ as $\lambda = \int a dt$ and $\theta^A = (x, y)$, and then $e_a^\mu := \partial x^\mu/\partial y^a$ are given by

$$e_\lambda^\mu = k^\mu, \qquad e_A^\mu = \delta_A^\mu, \tag{3.107}$$

which satisfies $k_\mu e_A^\mu = 0$ and $g_{\mu\nu}e_A^\mu e_B^\mu = a^2\delta_{AB}$. From Eq. (2.34), we obtain the induced metric on Σ as

$$ds_{\Sigma+}^2 = \sigma_{AB}^+ d\theta^A d\theta^B = a(t)^2|_{t=t_+(\lambda)}(dx^2 + dy^2). \tag{3.108}$$

Since the coordinate t is the same on $\mathcal{M}_-$ and $\mathcal{M}_+$, the first junction conditions $[\sigma_{AB}] = 0$ are satisfied. The transverse vector completing the basis is

$$N_\mu dx^\mu = -\frac{1}{2}(a dt + a^2 dz_+), \tag{3.109}$$

which satisfies $N_\mu N^\mu = 0$, $N_\mu k^\mu = -1$, and $N_\mu e_A^\mu = 0$. From Eq. (2.39), non-vanishing components of the transverse curvature of Σ are computed to give

$$C_{AB}^+ = \frac{1}{2}\dot{a}\sigma_{AB}^+\Big|_{t=t_+(\lambda)}. \tag{3.110}$$

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[2]In [3], k^μ is erroneously identified as $k^\mu\partial/\partial x^\mu = \partial/\partial t + a^{-1}\partial/\partial z_+$, which affects the following argument.

The jump of the transverse curvature at Σ is obtained in terms of t as

$$[C_{\lambda\lambda}] = [C_{\lambda B}] = 0, \qquad [C_{AB}] = \frac{1}{2}\left(\dot{a}|_{t=t_+(\lambda)} - \frac{\dot{a}}{a}\bigg|_{t=\lambda}\right)\sigma_{AB} = \frac{\dot{a}}{2}\left(1 - \frac{1}{a}\right)\sigma_{AB}, \qquad (3.111)$$

where $\sigma_{AB} \equiv \sigma_{AB}^+(= \sigma_{AB}^-)$. Then, the Barrabès-Israel junction conditions (2.43) give

$$\kappa_4 \mu = -\dot{a}\left(1 - \frac{1}{a}\right), \qquad j^A = 0, \qquad p = 0. \qquad (3.112)$$

As we have $\mu \neq 0$, $p = J = 0$, the induced energy-momentum tensor $t_{\mu\nu}$ on the null shell is of the Hawking-Ellis type II and all the standard energy conditions are satisfied (violated) for $\dot{a}(1 - a^{-1}) \leq (>)0$ by Proposition 2. Thus, in the expanding universe with $\dot{a} > 0$, such a phase transition of the cosmic expansion at null Σ is possible without violating any of the standard energy conditions if it occurs in the early stages of the universe where $a \leq 1$ holds.

Acknowledgments

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