Conformal anomalies in 6D four-derivative theories: A heat-kernel analysis

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We compute the conformal anomalies for some higher-derivative (nonunitary) 6D Weyl invariant theories using the heat-kernel expansion in the background-field method. To this aim we obtain the general expression for the Seeley-DeWitt coefficient b_6 for 4-derivative differential operators with background curved geometry and gauge fields, which was known only in flat space so far. We consider 4-derivative scalars and Abelian vectors as well as 3-derivative fermions, confirming the result of the literature obtained via indirect methods. We generalize the vector case by including the curvature coupling FFWeyl.

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I. INTRODUCTION

The calculation of conformal anomalies for six-dimensional theories has recently been of interest, also in the higher-derivative case (see [1–8] and references therein). In the context of conformal field theory, six-dimensional spacetime plays a very important role, as no interacting unitary supersymmetric conformal field theory can exist in more than six dimensions [9] and there is no known example even in the nonsupersymmetric case. It is however difficult to study unitary theories in six dimensions due to the lack of perturbative renormalizability for standard 2-derivative actions. Higher-derivative theories, despite being nonunitary, can be considered as a formal UV completion of standard 2-derivative theories [10] and can therefore help to shed light on the properties of conformal field theories and of the space of QFTs in higher dimensions, see e.g., [11–14].

The conformal anomaly $\mathcal{A}$ in six dimensions takes the form [15–17]

$$\begin{aligned}\mathcal{A}\cdot(4\pi)^3 &= g^{mn}\langle T_{mn}\rangle\cdot(4\pi)^3\\ &= -a\mathbb{E}_6 + c_1 I_1 + c_2 I_2 + c_3 I_3, \end{aligned}\qquad (1.1)$$

where $\mathbb{E}_6$ is the Euler density in six dimensions and the invariants I_i are built from the Weyl tensor (I_1, $I_2 \sim \mathrm{Weyl}^3$, $I_3 \sim \mathrm{Weyl}\nabla^2\mathrm{Weyl}$—see Appendix A for explicit expressions). Equation (1.1) also appears in the UV divergent part of the effective action, and the anomaly coefficients a, c_i enter in the stress tensor two-, three- and four-point functions. In (1.1) we ignored scheme-dependent total-derivative contributions.

An efficient way of determining the UV divergent part of the effective action, and equivalently the conformal anomaly coefficients, is the heat-kernel method. By providing a representation of the determinant of a differential operator preserving background covariance, the heat kernel is particularly suited to study one-loop effects. In the present case the relevant terms are captured by

$$\Gamma_\infty = -\frac{\log\Lambda}{(4\pi)^3}\int\sqrt{g}b_6,\qquad \mathcal{A} = \frac{1}{(4\pi)^3}b_6,$$
$$b_6 = b_6(\Delta_{\mathrm{b}}) - b_6(\Delta_{\mathrm{f}}) \pm b_6(\Delta_{\mathrm{gh}}),\qquad (1.2)$$

where b_6 is a combination of the heat kernel coefficients $b_6(\Delta)$ of the operators Δ governing the quadratic fluctuations. In writing (1.2) we assume real bosons (b) and Weyl or Majorana fermions (f) in the gamma-matrix representation. The last term schematically represents ghost (gh) contributions. The heat-kernel coefficients for second-order differential operators have been known for a long time and have been widely applied to physics [18–21]. The coefficients for higher powers of the Laplacian and its deformations have also been considered, albeit with less completeness, see e.g., [19,22–25].

In particular, for the scope of this paper we need to consider operators of the form

$$\Delta_4 = \nabla^4 + V^{mn}\nabla_m\nabla_n + 2N^m\nabla_m + U,\qquad (1.3)$$

where $V_{mn} = V_{nm}$, the covariant derivative contains spacetime as well as gauge connections and the coefficient

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functions V, N, U are generally matrix valued. The coefficient $b_6(\Delta_4)$ was recently computed in flat spacetime in [25] (see also [26]) using an argument based on special factorized cases in terms of 2-derivative operators first proposed by [23] in the context of four-dimensional quadratic gravity. Here, we extend the result to include a geometrical background, thereby providing a direct way to compute the conformal anomaly coefficients.

We then use the newly obtained coefficient $b_6(\Delta_4)$ to provide a direct calculation of the anomaly coefficients of some classically Weyl-invariant scalar, spinor, and vector models. Most of these have been recently computed using indirect techniques [3,5,24]; our results provide an independent confirmation based on a conceptually straightforward and well-established procedure. In the case of the vector we furthermore extend the case studied in the literature by including an extra coupling with the background geometry with the structure FFWeyl.

The fields considered in this paper also appear as lower-spin contributions to six-dimensional (2,0) conformal supergravity theory, where the graviton kinetic term is a combination of the I_i's above. This theory, constructed in [27] to a level which is sufficient for the one-loop anomaly calculation, contains however a 6-derivative operator which therefore escapes the scope of the present paper. It would be interesting to compute the conformal anomaly of this theory, for which the a coefficients are known from holographic considerations [3] and suggest that (2,0) conformal supergravity coupled to 26 (2,0) tensor multiplets is anomaly free.

Finally, let us comment on zero modes, which are ignored in this paper. In general, a differential operators like the one in (1.3) admits normalizable zero modes, which have to be treated separately from the rest of the spectrum. For fourth-order operators that factorize into second-order ones exhibiting zero modes, these will be naturally inherited, however there could be additional ones depending on the interplay between the two factors. These aspects fall outside the scope of this paper and deserve further study.

This paper is organized as follows. Section II presents some relevant facts about the heat kernel expansion and discusses the derivation of the heat-kernel coefficient $b_6(\Delta_4)$ using the factorization ansatz. In Sec. III we apply such newly computed coefficient to the calculation of conformal anomalies (1.1) to 4-derivative scalar, 4-derivative gauge vector and 3-derivative spinor. Appendix A summarizes notation. Appendix B presents some more complete facts on the heat-kernel expansion that are useful for this paper and provides additional explicit formulas. Appendix C, for completeness, lists a basis for $b_6(\Delta_4)$ used in one of the decompositions. Appendix D discusses some diagrammatic checks for our result of $b_6(\Delta_4)$.

II. HEAT-KERNEL COEFFICIENT $b_6(\Delta_4)$ ON A GEOMETRIC BACKGROUND

A. Preliminary considerations

Here we recall some basic facts about the heat-kernel expansion relevant for the calculation. Further details are given in Appendix B.

We consider an elliptic differential operator Δ of even order $2n$ defined on a d-dimensional manifold without boundaries with the schematic structure

$$\Delta = (-\nabla^2)^n + \text{lower derivative terms}, \tag{2.1}$$

where $\nabla = \partial + \Gamma + A$ is a covariant derivative with geometric and gauge connection. We denote the associated spacetime and internal curvatures as $[\nabla, \nabla] = R + \mathfrak{F}$. One can express the logarithmically divergent part of $\det \Delta$ as (see e.g., [10,19,26] and references therein)

$$(\log\ \det \Delta)_\infty = -\frac{2\log \Lambda}{(4\pi)^{d/2}} \int \sqrt{g} b_d(\Delta), \tag{2.2}$$

where Λ is the UV cutoff and b_p is the trace of a local covariant quantity built using the differential operator as well as the covariant derivative and it is defined modulo boundary terms. We shall refer to these coefficients as Seeley-DeWitt heat-kernel coefficients. Let us now consider two differential operators Δ and Δ'. Using the factorization property of the determinant for combined operators we obtain the key relation

$$b_d(\Delta\Delta') = b_d(\Delta) + b_d(\Delta'), \tag{2.3}$$

which allows one to relate the heat-kernel coefficients of operators of different order (again modulo total derivatives). We stress here that the relation (2.3) is valid for the logarithmic part only and not for power-law divergences.

In the case of the 2-derivative operator

$$\Delta_2 = -\nabla^2 + X, \tag{2.4}$$

where ∇_m has internal and spacetime connections, in $6d$ one has the expression (B8) [18–20], which can be schematically represented as

$$b_6(\Delta_2) = b_6^{\mathrm{g}}(\Delta_2) + b_6^{\mathrm{gc}}(\Delta_2) + b_6^{\mathrm{m}}(\Delta_2), \tag{2.5}$$

where we distinguished the purely gravitational terms ("g"), those that originate from the generally-covariantized flat-spacetime expression ("gc"), and the terms which mix gravitational and gauge terms ("m"), so that $b_6^{\mathrm{m}}(\Delta_2)$ vanishes in flat spacetime as well as $b_6^{\mathrm{g}}(\Delta_2) = \mathfrak{C} \cdot \mathrm{tr}\mathbb{1}$ ($\mathfrak{C}$ is given in (B7) and $\mathbb{1}$ is the identity in the internal space where tr acts).

B. Derivation of $b_6(\Delta_4)$

We are interested in the coefficient $b_6(\Delta_4)$, where the operator Δ_4 has the structure (1.3). One can equivalently present the operator (1.3) in the "symmetric" form (B10). Following (2.5) we correspondingly decompose

$$b_6(\Delta_4) = b_6^{\mathrm{g}}(\Delta_4) + b_6^{\mathrm{gc}}(\Delta_4) + b_6^{\mathrm{m}}(\Delta_4). \qquad (2.6)$$

The strategy to compute it is the following. First, we make an ansatz based on dimensional and covariance considerations, taking into account algebraic relations between different terms due to Bianchi identities, symmetries of tensors, and boundary terms. Then we consider special cases for Δ_4, where it can be decomposed as the produce of second-order operators, $\Delta_4 = \Delta_2\Delta_2'$. Using (2.3) with the explicit expression for (2.5) in (B5) allows us to gather enough information to reconstruct $b_6(\Delta_4)$.

From these considerations it is immediate to see that

$$b_6^{\mathrm{g}}(\Delta_4) = 2b_6^{\mathrm{g}}(\Delta_2) = 2\mathfrak{G}\cdot \mathrm{tr}\mathbb{1}. \qquad (2.7)$$

Furthermore, this procedure was already applied to (1.3) in [25] (see also [26]) to the flat-spacetime case, therefore $b_6^{\mathrm{gc}}(\Delta_4)$ can be immediately obtained,

$$\begin{aligned} b_6^{\mathrm{gc}}(\Delta_4) = \mathrm{tr}\Big[&-\frac{1}{30}(\nabla_m\mathfrak{F}_{mn})^2 + \frac{1}{45}\mathfrak{F}_{mn}\mathfrak{F}_{nr}\mathfrak{F}_{rm} + \frac{1}{360}V_{mn}V_{nr}V_{rm} + \frac{1}{480}V_{mn}V_{mn}V + \frac{1}{2880}VVV + \frac{1}{30}V_{mn}\nabla_{(n}\nabla_{r)}V_{rm} \\ &+ \frac{1}{120}V_{mn}\nabla^2V_{mn} - \frac{1}{40}V_{mn}\nabla_m\nabla_nV + \frac{1}{240}V\nabla^2V - \frac{1}{12}V_{mn}V_{nr}\mathfrak{F}_{mr} + \frac{1}{6}\mathfrak{F}_{mn}\nabla_{(m}\nabla_{r)}V_{rn} + \frac{1}{24}V\mathfrak{F}_{mn}\mathfrak{F}_{mn} \\ &- \frac{1}{6}V_{mn}\mathfrak{F}_{mr}\mathfrak{F}_{nr} - \frac{1}{3}\mathfrak{F}_{mn}\nabla_mN_n - \frac{1}{6}V_{mn}\nabla_mN_n + \frac{1}{12}V\nabla_mN_m - \frac{1}{6}N_mN_m - \frac{1}{12}UV\Big], \end{aligned} \qquad (2.8)$$

where $V = g^{mn}V_{mn}$ and $\mathfrak{F}$ is the internal curvature. What remains to be determined is therefore only $b_6^{\mathrm{m}}(\Delta_4)$. On dimensional and covariance grounds we make the ansatz

$$\begin{aligned} b_6^{\mathrm{m}}(\Delta_4) = \mathrm{tr}[&c_1R\mathfrak{F}_{mn}\mathfrak{F}_{mn} + c_2RV_{mn}V_{mn} + c_3RVV + c_4RU + c_5R\nabla^2V + c_6R\nabla_m\nabla_nV_{mn} + c_7R\nabla_mN_m + c_8R_{mn}V_{nr}V_{rm} \\ &+ c_9R_{mn}V_{mn}V + c_{10}R_{mn}\nabla^2V_{mn} + c_{11}R_{mn}V_{nr}\mathfrak{F}_{mr} + c_{12}R_{mn}\mathfrak{F}_{mr}\mathfrak{F}_{nr} + c_{13}R_{mnrq}V_{mr}V_{nq} + c_{14}R_{mnrq}\mathfrak{F}_{mn}\mathfrak{F}_{rq} \\ &+ c_{15}R_{mnrq}R_{mnrq}V + c_{16}R_{mrqk}R_{nrqk}V_{mn} + c_{17}R_{mn}R_{mn}V + c_{18}R^2V + c_{19}R_{mn}R_{mr}V_{rn} + c_{20}RR_{mn}V_{mn} \\ &+ c_{21}R_{mnrq}R_{mr}V_{nq}]. \end{aligned} \qquad (2.9)$$

All other combinations vanish or reduce to these by means of the Bianchi identities, integration by parts, and symmetry properties. As we shall explain, we find

$$\begin{aligned} &c_1 = \frac{1}{36}, \quad c_2 = \frac{1}{144}, \quad c_3 = \frac{1}{288}, \quad c_4 = -\frac{1}{6}, \quad c_5 = \frac{1}{60}, \quad c_6 = -\frac{1}{20}, \quad c_7 = \frac{1}{6}, \\ &c_8 = -\frac{1}{36}, \quad c_9 = -\frac{1}{72}, \quad c_{10} = -\frac{1}{60}, \quad c_{11} = -\frac{1}{12}, \quad c_{12} = \frac{1}{45}, \quad c_{13} = \frac{1}{36}, \quad c_{14} = \frac{1}{90}, \\ &c_{15} = \frac{1}{360}, \quad c_{16} = -\frac{1}{90}, \quad c_{17} = -\frac{1}{360}, \quad c_{18} = \frac{1}{144}, \quad c_{19} = \frac{1}{45}, \quad c_{20} = -\frac{1}{36}, \quad c_{21} = -\frac{1}{90}. \end{aligned} \qquad (2.10)$$

The full explicit expression of $b_6(\Delta_6)$ is given in (B9).

To fix the values of the coefficients c_i's we resort to the following two decompositions:

(1) $\Delta_4 = \Delta_2^X\Delta_2^Y$, where the 2-derivative operators have the structure (2.4) and the background gauge connection is non-Abelian.

(2) $\Delta_4 = \Delta_+\Delta_-$, with an Abelian gauge connection and $\Delta_\pm = -(\nabla_m \pm B_m)^2$.

In total we find an overdetermined system of 49 equations with unique solution (2.10). The following two subsections provide details of the derivation.

1. Decomposition 1

The fourth-order operators obtained from the composition

$$\Delta_4 = \Delta_2^X\Delta_2^Y \qquad (2.11)$$

has the structure (1.3) with $V_{mn} = -\delta_{mn}(X+Y)$, $N_m = -\nabla_mY$, $U = XY - \nabla^2Y$, and therefore $V = -6(X+Y)$.

From the general expression (2.9) we get

$$
\begin{aligned}
b_6^{\mathrm{m}}(\Delta_4) = \mathrm{tr}[&c_1\mathfrak{F}_{mn}\mathfrak{F}_{mn}R + c_{12}\mathfrak{F}_{mr}\mathfrak{F}_{mn}R_{nr} + c_{14}\mathfrak{F}_{mn}\mathfrak{F}_{rs}R_{mnrs} - (6c_{18}+c_{20})R^2X - (6c_{18}+c_{20})R^2Y \\
&- (6c_{17}+c_{19}+c_{21})R_{mn}R_{mn}X - (6c_{17}+c_{19}+c_{21})R_{mn}R_{mn}Y + (6c_2+36c_3+c_8+6c_9+c_{13})RX^2 \\
&+ (6c_2+36c_3+c_8+6c_9+c_{13})RY^2 - (6c_{15}+c_{16})R_{mnrs}R_{mnrs}Y - (6c_{15}+c_{16})R_{mnrs}R_{mnrs}X \\
&- (6c_5+c_6+c_{10})R\nabla^2X - (c_4+6c_5+c_6+c_7+c_{10})R\nabla^2Y \\
&+ (12c_2+72c_3+c_4+2c_8+12c_9+2c_{13})RXY],
\end{aligned}
\tag{2.12}
$$

and from the factorization we have

$$
\begin{aligned}
b_6^{\mathrm{m}}(\Delta_2) + b_6^{\mathrm{m}}(\Delta_2') = \mathrm{tr}\Big[&-\frac{1}{36}\mathfrak{F}_{mn}\mathfrak{F}_{mn}R - \frac{1}{45}\mathfrak{F}_{mr}\mathfrak{F}_{mn}R_{nr} - \frac{1}{90}\mathfrak{F}_{mn}\mathfrak{F}_{rs}R_{mnrs} + \frac{1}{72}R^2X + \frac{1}{72}R^2Y - \frac{1}{180}R_{mn}R_{mn}X \\
&- \frac{1}{180}R_{mn}R_{mn}Y - \frac{1}{12}RX^2 - \frac{1}{12}RY^2 + \frac{1}{180}R_{mnrs}R_{mnrs}X + \frac{1}{180}R_{mnrs}R_{mnrs}Y \\
&+ \frac{1}{30}R\nabla^2X + \frac{1}{30}R\nabla^2Y\Big].
\end{aligned}
\tag{2.13}
$$

Equating the two we obtain 14 linear equations.

2. Decomposition 2

We are considering

$$
\begin{aligned}
\Delta_4 &= \Delta_+\Delta_-, \\
-\Delta_\pm &= (\nabla_m \pm B_m)^2 \\
&= \nabla^2 \pm 2B_m\nabla_m \pm (\nabla_m B_m) + B_m B_m,
\end{aligned}
\tag{2.14}
$$

with $\nabla = \partial + \Gamma + A$, A being an Abelian connection. The field strengths therefore read

$$
\begin{aligned}
\mathfrak{F}^\pm_{mn} &\equiv [\nabla^\pm_m, \nabla^\pm_n] \\
&= \mathfrak{F}_{mn} + [B_m, B_n] \pm (\nabla_m B_n - \nabla_n B_m).
\end{aligned}
\tag{2.15}
$$

The coefficients for the operator Δ_4 (2.14) in the notation (1.3) read

$$
\begin{aligned}
V_{mn} &= -4\nabla_{(m}B_{n)} + 2B^2\delta_{mn} - 4B_{(m}B_{n)}, \\
V &= -4\nabla_n B_n + 8B^2,
\end{aligned}
\tag{2.16}
$$

$$
\begin{aligned}
N_m = &-B_nR_{mn} - \nabla^2B_m - \nabla_m\nabla_nB_n + \nabla_mB^2 + B_mB^2 \\
&- B^2B_m - 2B_n\nabla_nB_m - B_m\nabla_nB_n + 2B_n\mathfrak{F}_{nm},
\end{aligned}
\tag{2.17}
$$

$$
\begin{aligned}
U = &-\nabla^2\nabla_nB_n + \nabla^2B^2 - 2B_m\nabla_m\nabla_nB_n + 2B_m\nabla_mB^2 \\
&- (\nabla_nB_n)^2 + B^4 + (\nabla_nB_n)B^2 - B^2\nabla_nB_n \\
&- 2\nabla_mB_n\mathfrak{F}_{mn} - 2B_mB_n\mathfrak{F}_{mn} + 2B_m\nabla_n\mathfrak{F}_{mn}.
\end{aligned}
\tag{2.18}
$$

The only place where the spacetime curvature explicitly appears is B_nR_{mn} in N_m.

In considering the factorization ansatz (2.3), for simplicity we focus on terms of order 1, 2, 4, 5, and 6 in B. It turns out that this provides us with enough information to determine $b_6^{\mathrm{m}}(\Delta_4)$.

To start, we need to determine a basis for the invariants that can appear in the expression of $b_6(\Delta_4)$. In doing so one needs to be careful about the possibility of adding total derivatives, the symmetries of the objects involved and their algebraic relations. We identified a basis consisting of 45 elements listed in Appendix C. In such a basis, we can evaluate (2.9) as

$$
\begin{aligned}
b_6^{\mathrm{m}}(\Delta_4) = &c_{12}\mathfrak{F}_{an}\mathfrak{F}_{am}R_{mn} + 4c_{11}\mathfrak{F}_{an}B_aB_mR_{mn} - 16(c_{13}+2c_9)B^2B_mB_nR_{mn} - 4(2c_{13}+c_{19})B_aB_mR_{an}R_{mn} \\
&+ 2(4c_{17}+c_{19}+c_{21})B^2R_{mn}R_{mn} + c_1\mathfrak{F}_{am}\mathfrak{F}_{am}R + (24c_2+64c_3+c_4+4c_8+16c_9+4c_{13})B^2B_mB_mR \\
&- 8(2c_8-3c_{13})B_aR_{mn}\nabla_a\nabla_nB_m - 2(2c_6+c_7-2c_8+2c_{13}+2c_{20})B_aB_mR_{am}R \\
&- 4(c_{10}-2c_{17}-c_{21})B_aR_{mn}\nabla_aR_{mn} + 2(4c_{18}+c_{20})B^2R^2 + c_{14}\mathfrak{F}_{am}\mathfrak{F}_{nc}R_{amnc} \\
&- 4(2c_8-2c_{13}+c_{21})B_aB_mR_{nc}R_{anmc} + 4(2c_{13}-c_{16})B_aB_mR_{ancr}R_{mncr} + 2(4c_{15}+c_{16})B^2R_{mncr}R_{mncr} \\
&+ 2(4c_{18}+c_{20})B_aR\nabla_aR + 2(4c_{15}+c_{16})B_aR_{mncr}\nabla_aR_{mncr} - 2(c_4+4c_6+2c_7-2c_8-4c_9-2c_{13})B_aR\nabla_a\nabla_mB_m \\
&+ 4(c_8-2c_{10}-2c_{13})R_{an}\nabla_mB_n\nabla_mB_a + 8c_{13}B_aR_{acmn}\nabla_c\nabla_nB_m
\end{aligned}
$$

$$
\begin{aligned}
&+(16c_3-c_4-4c_6-2c_7+8c_9+8c_{13})R\nabla_aB_a\nabla_mB_m+2(c_4+8c_5+2c_6+c_7+c_8+2c_{10})B_aR\nabla^2B_a\\
&-8(2c_9+3c_{13})B_aR_{an}\nabla_n\nabla_mB_m-(c_4+4c_5+4c_6+2c_7+2c_{10})R\nabla^2\nabla_aB_a-8(c_{10}+c_{13})B_aR_{am}\nabla_n\nabla_nB_m\\
&+2(4c_2-2c_6-c_7+2c_8-2c_{13})R\nabla_aB_m\nabla_mB_a+2(4c_2+c_4+8c_5+2c_6+c_7+c_8+2c_{10})R\nabla_mB_a\nabla_mB_a\\
&-(2c_4+2c_7+c_{11})\mathfrak{F}_{am}B_a\nabla_mR-4c_8B_aR_{mn}\nabla_n\nabla_mB_a+2(2c_6+c_7+2c_{10}+c_{19}+2c_{20})B_aR_{am}\nabla_mR\\
&-2c_{11}B_aR_{mn}\nabla_n\mathfrak{F}_{am}+2c_{11}B_aR_{am}\nabla_n\mathfrak{F}_{mn}+4(2c_{10}-2c_{16}-c_{21})B_aR_{amnc}\nabla_cR_{mn}+2c_{11}\mathfrak{F}_{mn}B_a\nabla_nR_{am}\\
&+4(2c_{10}+c_{19}-c_{21})B_aR_{mn}\nabla_nR_{am},
\end{aligned}\tag{2.19}
$$

and from the factorization we have

$$
\begin{aligned}
b_6^{\mathrm{m}}(\Delta_+)+b_6^{\mathrm{m}}(\Delta_-)=&-\frac{1}{45}\mathfrak{F}_{an}\mathfrak{F}_{am}R_{mn}+\frac{1}{3}\mathfrak{F}_{an}B_aB_mR_{mn}+\frac{14}{45}B_aB_mR_{an}R_{mn}-\frac{1}{36}\mathfrak{F}_{am}\mathfrak{F}_{am}R+\frac{11}{45}B_aB_mR_{am}R\\
&-\frac{1}{90}\mathfrak{F}_{am}\mathfrak{F}_{nc}R_{amnc}-\frac{22}{45}B_aB_mR_{nc}R_{anmc}-\frac{4}{15}B_aB_mR_{ancr}R_{mncr}+\frac{2}{45}B_aR\nabla_a\nabla_mB_m\\
&-\frac{8}{9}B_aR_{mn}\nabla_a\nabla_nB_m-\frac{1}{5}R\nabla_aB_a\nabla_mB_m+\frac{1}{18}B_aR\nabla_m\nabla_mB_a+\frac{3}{10}R\nabla_aB_m\nabla_mB_a+\frac{1}{5}R_{an}\nabla_mB_n\nabla_mB_a\\
&-\frac{1}{12}\mathfrak{F}_{am}B_a\nabla_mR-\frac{1}{6}B_aR_{mn}\nabla_n\mathfrak{F}_{am}+\frac{1}{6}B_aR_{am}\nabla_n\mathfrak{F}_{mn}+\frac{1}{6}\mathfrak{F}_{mn}B_a\nabla_nR_{am}+\frac{4}{45}B_aR_{am}\nabla_n\nabla_nB_m\\
&+\frac{4}{9}B_aR_{an}\nabla_n\nabla_mB_m-\frac{1}{9}B_aR_{mn}\nabla_n\nabla_mB_a-\frac{2}{9}B_aR_{acmn}\nabla_c\nabla_nB_m.
\end{aligned}\tag{2.20}
$$

Equating the two expressions we obtain 35 linear equations.

III. APPLICATIONS

A. 4-derivative scalar field

A 4-derivative Weyl-covariant differential operator [3,5] in d-dimensions was constructed by Paneitz (cf. [28]; the $4D$ case was first given in [29,30]),

$$
\begin{aligned}
\Delta_4=&\nabla^4+\nabla_m[(4S_{mn}-(d-2)g_{mn}S)\nabla_n]-(d-4)S_{mn}S_{mn}\\
&+d\frac{d-4}{4}S^2-\frac{d-2}{2}(\nabla^2S),
\end{aligned}\tag{3.1}
$$

where S_{mn} is the Schouten tensor

$$
\begin{aligned}
S_{mn}&=\frac{1}{d-2}\left[R_{mn}-\frac{1}{2(d-1)}Rg_{mn}\right],\\
S=S_{mm}&=\frac{1}{2(d-1)}R.
\end{aligned}\tag{3.2}
$$

Such operator allows one to consider the following Weyl-invariant action in $6D$ for a real scalar, from which we can compute the corresponding effective action and conformal anomaly via (1.2),

$$
S=\frac{1}{2}\int dx^6\sqrt{g}\phi\Delta_4\phi,\qquad b_6=b_6(\Delta_4).\tag{3.3}
$$

The operator (3.1) is written in the symmetric form (B10). Direct application of (B9) gives

$$
(a,c_i)=\frac{1}{7!}\left(\frac{4}{9},\frac{224}{3},8,-10\right),\tag{3.4}
$$

in agreement with the recent independent analysis of [[24], (16)–(19) with $k=2$].

B. 4-derivative gauge vector

We consider the following Weyl-invariant action for an Abelian gauge vector A_m,

$$
\begin{aligned}
S=&\int\sqrt{g}\left[\nabla_rF_{rm}\nabla_nF_{nm}-\left(R_{mn}-\frac{1}{5}g_{mn}R\right)F_{mp}F_{np}\right]\\
&+\xi\int\sqrt{g}F_{mn}F_{rs}W_{mnrs},
\end{aligned}\tag{3.5}
$$

where $F_{mn}=\nabla_mA_n-\nabla_nA_m$ is the field strength. The first integral provides a Weyl-invariant kinetic term for A_m as considered in [5]. The second integral is Weyl invariant by itself (where W is the Weyl tensor) and can therefore be added with an arbitrary numerical coefficient ξ. In terms of the gauge field A_m the action reads

$$
S=\frac{1}{2}\int\sqrt{g}A_m[\Delta_{4A}]_{mn}A_n+\frac{1}{2}\int\sqrt{g}A_m\nabla_m\nabla^2\nabla_nA_n,\tag{3.6}
$$

where the operator Δ_{4A} is a 4-derivative one. It is more convenient to present it in the symmetrized form (B10) with coefficients

$$
\begin{aligned}
[\hat{V}_{mn}]_{ac} = &(1+\xi)g_{ac}R_{mn} - (1-\xi)g_{mn}R_{ac} \\
&- \frac{2+\xi}{5}g_{ac}g_{mn}R + \frac{2+\xi}{5}Rg_{m(a}g_{c)n} \\
&- 2(1+\xi)g^{(m}_{(a}R^{n)}_{c)} + 4\xi R_{a(mn)c},
\end{aligned} \tag{3.7}
$$

$$
[\hat{N}_m]_{ac} = \frac{1-3\xi}{2}\nabla_{(a}R_{c)m} + \frac{1+3\xi}{20}g_{m(a}\nabla_{c)}R \tag{3.8}
$$

$$
\begin{aligned}
[\hat{U}]_{ac} = &\frac{1-\xi}{2}R_{am}R_{cm} - \frac{1+\xi}{2}R_{ms}R_{amcs} + \frac{2+\xi}{10}R_{ac}R \\
&- \nabla^2 R_{ac} + 2\xi R_{amrs}R_{crms}.
\end{aligned} \tag{3.9}
$$

The second term in (3.6) can be gauge fixed away by choosing the covariant gauge $\nabla_m A_m = 0$ and averaging over gauges with the Gaussian weight, $-\nabla^2$.

The effective action for (3.5) thus constructed reads

$$
Z = \left[\frac{\det \Delta_{4A}}{[\det \Delta_{2,0}]^3}\right]^{1/2}, \qquad \Delta_{2,0} = -\nabla^2, \tag{3.10}
$$

where $\Delta_{2,0}$ acts on scalars and comes from ghost and gauge-fixing contributions. The divergent part of the effective action has therefore the structure (1.2) governed by the coefficient

$$
b_6 = b_6(\Delta_{4A}) - 3b_6(-\nabla^2), \tag{3.11}
$$

where the first term can be evaluated with (B9) and the second one with (B8). In computing $b_6(\Delta_4)$ we use that $\mathbb{1}$ is the identity in the space of six-dimensional vectors and that the curvature $\mathfrak{F}$ is given by $[\mathfrak{F}_{mn}]_{ac} = R_{mnac}$.

In the form (1.2) we obtain

$$
\begin{aligned}
&a = \frac{275}{8\cdot 7!}, \quad c_1 = \frac{28}{7!}(97 - 60\xi + 4\xi^2 - 4\xi^3), \\
&c_2 = \frac{1}{7!}(911 - 840\xi + 392\xi^2 - 392\xi^3), \quad c_3 = -\frac{150}{7!}.
\end{aligned} \tag{3.12}
$$

The case $\xi = 0$ was considered in [5] via indirect methods; our result agrees. We notice that ξ does not affect the a coefficient (as expected) and that c_1 and c_2 do not exhibit common zeroes. The fact that ξ does not enter c_3 is probably accidental at one loop.

C. 3-derivative fermion

We consider here a 3-derivative Weyl spinor Ψ with the kinetic operator given in [5] (see also [27])

$$
\begin{aligned}
&S = \int \bar{\Psi}\Delta_3\Psi, \\
&-i\Delta_3 = \nabla\!\!\!/^3 + 2S_{mn}\gamma_m\nabla_n + \gamma^m\nabla_m S,
\end{aligned} \tag{3.13}
$$

where S_{mn} is the Schouten tensor as in (3.2). In (3.13) and in the following we consider Dirac gamma matrix notation with $\{\gamma_m, \gamma_n\} = 2g_{mn}$, γ_m being eight-dimensional.

We have (1.2) with

$$
b_6 = -b_6(\Delta_3) \equiv b_6(\Delta_1) - b_6(\Delta_3\Delta_1), \qquad \Delta_1 = i\nabla\!\!\!/, \tag{3.14}
$$

where we evaluate the heat-kernel coefficient $b_6(\Delta_3)$ considering composition $\Delta_3\Delta_1$ with the first-order Dirac operator Δ_1 (acting on Weyl spinors) and applying (2.3). The 4-derivative operator $\Delta_3\Delta_1$ has the structure (1.3) with[1]

$$
\begin{aligned}
&V_{mn} = 2\gamma_r\gamma_{(n}S_{m)r} - \frac{1}{2}Rg_{mn}, \qquad N_m = \frac{1}{2}\nabla_a S\gamma^a\gamma^m - \frac{1}{4}\nabla_m R, \\
&U = \frac{1}{16}R^2 - \frac{1}{4}\nabla^2 R + \frac{1}{4}S_{ma}R_{anrs}\gamma^m\gamma^n\gamma^r\gamma^s,
\end{aligned} \tag{3.15}
$$

which via (B9) results in

$$
b_6(\Delta_3\Delta_1) = \frac{1}{7!}\left[-\frac{10}{9}\mathbb{E}_6 - \frac{448}{3}I_1 - \frac{172}{3}I_2 + 4I_3\right]. \tag{3.16}
$$

The heat kernel coefficient of the Dirac operator can be computed by squaring it and using (B9), which gives

$$
\begin{aligned}
b_6(\Delta_1) &= \frac{1}{2}b_6\left[(\Delta_1)^2 \equiv -\nabla^2 + \frac{1}{4}R\right] \\
&= \frac{1}{7!}\left[\frac{191}{144}\mathbb{E}_6 + \frac{448}{3}I_1 - 16I_2 - 20I_3\right].
\end{aligned} \tag{3.17}
$$

The expression (3.17) was originally obtained in [17]. In computing (3.16) and (3.17) we have used $[\nabla_m, \nabla_n] = \frac{1}{4}R_{mnrs}\gamma^r\gamma^s$ and $R_{mnrs}\gamma^m\gamma^n\gamma^r\gamma^s = -2R$. Here $\mathbb{1}$ of (B9) is the identity in the eight-dimensional spinor space, where tr is taken.

In conclusion the conformal anomaly coefficients derived from (1.2) with (3.14) are

$$
(a, c_i) = \frac{1}{7!}\left(\frac{39}{16}, \frac{896}{3}, \frac{220}{3}, -24\right). \tag{3.18}
$$

Our result (3.18) coincides with that of [5] obtained via indirect methods,[2] thereby providing an independent direct confirmation.

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[1]The self-adjoint requirements discussed in Appendix B 3 are not to be imposed, as the operator Δ_{3+1} does not come from functional integration.

[2]Our values (3.18) are double those given in (6.3) of [5], since formal Majorana-Weyl spinors are used there.

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APPENDIX A: NOTATION AND CONVENTIONS

We work in 6 euclidean dimensions and indicate spacetime indices with Latin lowercase letters. The metric is g_{mn} and we do not distinguish between upper and lower indices. We write the covariant derivative with Levi-Civita and gauge connection as $\nabla = \partial + \Gamma + A$. We introduce the gauge and Riemann curvatures

$$\mathfrak{F}_{mn} = \partial_m A_n - \partial_n A_m + [A_m, A_n],$$
$$R^m{}_{nac} = \partial_a \Gamma^m_{nc} \pm \cdots, \qquad R_{mn} = R^a{}_{man}, \qquad R = R^m_m, \tag{A1}$$

so that $[\nabla_m, \nabla_n]\phi = \mathfrak{F}_{mn}\phi$ on a spacetime scalar and $[\nabla_m, \nabla_n] V_a = R_{mnac} V^c$ on a spacetime vector without gauge indices. We reserve F_{mn} for the Maxwell field strength and keep $\mathfrak{F}_{mn}$ in the general case. The Weyl tensor reads

$$W_{mnrs} = R_{mnrs} + g^{[r}_{[n} R^{s]}_{m]} + \frac{1}{10} g_{m[r} g_{s]n} R. \tag{A2}$$

We note the following identities that are used in the main text without mentioning,

$$2(\nabla_m F_{mn})^2 = (\nabla_m F_{rs})^2 + 2F_{ma}F_{na}R_{mn} - R_{mnrs}F^{mn}F^{rs} + \text{t.d.},$$
$$2R_{rnac}R_{macn} = -R_{rnac}R_{mnac} \tag{A3}$$

The basis of the anomaly is

$$\begin{aligned}
\mathbb{E}_6 &= -\varepsilon_{mnrspq}\varepsilon_{abcdef}R^{mnab}R^{rscd}R^{pqef}\\
&= -32 R_{mnac}R_{acsr}R_{srmn} + \ldots,\\
I_1 &= W_{amnc}W_{mrsn}W_{racs},\\
I_2 &= W^{acmn}W^{rsac}W^{mnrs},\\
I_3 &= W_{amnr}\left[g_{ac}\nabla^2 + 4R_{ac} - \frac{6}{5}g_{ac}R\right]W_{cmnr}.
\end{aligned} \tag{A4}$$

APPENDIX B: RELEVANT FACTS ABOUT THE HEAT KERNEL

In this appendix we provide further details about the heat-kernel expansion to complete the discussion of Sec. II A.

1. Generalities

A standard representation for the determinant of a differential operator of order ℓ is

$$\log\ \det \Delta_\ell = -\int d^d x \sqrt{g} \int_\varepsilon^\infty \frac{dt}{t} \mathrm{tr}\langle x | e^{-t\Delta_\ell} | x\rangle, \tag{B1}$$

where tr is the trace over internal indices of the operator and $\varepsilon = \Lambda^{-\ell}$ is a UV cutoff. The matrix element in the integrand is the heat kernel. It has an asymptotic expansion for $t \to 0^+$ that allows us to write (see e.g., [19–21,26])

$$\begin{aligned}
\mathrm{tr}\langle x | e^{-t\Delta_\ell} | x\rangle &\equiv \mathrm{tr}K(t; x, x; \Delta_\ell)\\
&\simeq \sum_{p\geq 0} \frac{2}{(4\pi)^{d/2}\ell} t^{(p-d)/\ell} a_p(\Delta_\ell, d, g; x).
\end{aligned} \tag{B2}$$

The Seeley-DeWitt coefficients a_p are local covariant expressions of dimension p constructed out of the background metric and gauge field, exhibiting an explicit nontrivial dependence on the spacetime dimension d when $\ell \neq 2$, see e.g., [35] for explicit examples. However, we are interested in the spacetime integral of the trace of the Seeley-DeWitt coefficients, which in the present context acquire the interpretation of a Lagrangian density. We therefore focus on the simpler invariant quantity

$$b_p(\Delta_\ell, d, g; x) = \mathrm{tr}\, a_p(\Delta_\ell, d, g; x) \quad \text{modulo total derivatives,} \tag{B3}$$

which with abuse of notation we also call heat-kernel coefficients. Some of the arguments of b_p are often omitted. As a consequence, one can express the divergent part of (B1) as

$$\begin{aligned}
(\log\ \det \Delta_\ell)_\infty &= -\frac{2}{(4\pi)^{d/2}} \times \left[\sum_{p=0}^{d-1} \frac{B_p(\Delta_\ell)}{d-p}\Lambda^{d-p} + B_d(\Delta_\ell)\log\frac{\Lambda}{\mu}\right],\\
B_p(\Delta_\ell) &= \int d^d x \sqrt{g}\, b_p(\Delta_\ell),
\end{aligned} \tag{B4}$$

where μ is a renormalization scale and the explicit dependence on ℓ dropped out. In (B4) we find both power-law divergences (which we ignore for the scope of this paper) and the logarithmic term relevant for the conformal anomalies. In dimensional regularization one has only the logarithmic term of (B4) with the formal substitution $\log\frac{\Lambda}{\mu} \to \frac{1}{n-d}$, where n is the original integer number of dimensions.

Let us now consider two differential operators Δ and Δ'. Using the factorization property of the determinant for combined operators we further obtain the key relation

$$b_d(\Delta\Delta') = b_d(\Delta) + b_d(\Delta'). \tag{B5}$$

This factorization ansatz is only true for the coefficient $b_{p=d}$, i.e., with the index p equal to the spacetime dimension d. Indeed, only the logarithmically divergent term of the expansion (B4) is universal, while the power-law divergences are regularization dependent. This implies that the factorization ansatz (B5) does not capture the explicit d-dependence of the b_ps.

2. Explicit formulas

In this section we give some explicit expressions for the heat kernel coefficients of 2- and 4-derivative elliptic operators. We consider here operators of the standard forms

$$\Delta_2 = -\nabla^2 + X,$$
$$\Delta_4 = \nabla^4 + V_{mn}\nabla_m\nabla_n + 2N_m\nabla_m + U, \tag{B6}$$

with $V_{mn} = V_{nm}$; X, V, N, U are covariant coefficient functions, in general matrix valued. In (B6), Δ_2 is the most general second-order operator and Δ_4 is the most general fourth-order operator without 3-derivative term. At this point, no further restriction is imposed on the coefficient functions, as the operators are not necessarily self-adjoint.

Let us define the purely geometrical contribution as

$$\mathfrak{E} = \frac{1}{945}R_{mn}R_{pq}R_{mpnq} + \frac{1}{7560}R_{mn}R_{mpqr}R_{npqr} - \frac{4}{2835}R_{mn}R_{np}R_{pm} + \frac{17}{45360}R_{mn}{}^{pq}R_{pq}{}^{rs}R_{rs}{}^{mn} - \frac{1}{1620}R_m{}^p{}_n{}^qR_p{}^r{}_q{}^sR_r{}^m{}_s{}^n + \frac{1}{840}R_{mn}\nabla^2R^{mn} + \frac{1}{1296}R^3 + \frac{1}{1080}RR_{mpqr}R_{mpqr} - \frac{1}{1080}RR_{mn}R_{mn} + \frac{1}{336}R\nabla^2R. \tag{B7}$$

For second-order differential operators of the form (B6) we have, following [18,19,36],

$$b_6(\Delta_2) = \mathrm{tr}\Big[-\frac{1}{60}(\nabla_m\mathfrak{F}_{mn})^2 + \frac{1}{90}\mathfrak{F}_{mn}\mathfrak{F}_{nr}\mathfrak{F}_{rm} - \frac{1}{12}X\mathfrak{F}_{mn}\mathfrak{F}_{mn} + \frac{1}{12}X\nabla^2X - \frac{1}{6}X^3 + \frac{1}{12}RX^2 - \frac{1}{72}R^2X - \frac{1}{30}X\nabla^2R + \frac{1}{180}XR_{mn}R_{mn} - \frac{1}{180}XR_{mnrs}R_{mnrs} + \frac{1}{72}R\mathfrak{F}_{mn}\mathfrak{F}^{mn} - \frac{1}{90}R_{mn}\mathfrak{F}^{mr}\mathfrak{F}^{rn} + \frac{1}{180}R_{mnpq}\mathfrak{F}_{mn}\mathfrak{F}_{pq} + \mathfrak{E}\cdot\mathbb{1}\Big], \tag{B8}$$

where $\mathbb{1}$ is the identity in the internal space, where tr acts.

For fourth-order differential operators of the form (B6) we have (with $V = V_{mm}$)

$$\begin{aligned} b_6^{d=6}(\Delta_4) = \mathrm{tr}\Big[&-\frac{1}{30}(\nabla_m\mathfrak{F}_{mn})^2 + \frac{1}{45}\mathfrak{F}_{mn}\mathfrak{F}_{np}\mathfrak{F}_{pm} + \frac{1}{360}V_{mn}V_{np}V_{pm} + \frac{1}{480}V_{mn}V_{mn}V + \frac{1}{2880}VVV + \frac{1}{30}V_{mn}\nabla_{(n}\nabla_{p)}V_{pm} \\ &+ \frac{1}{120}V_{mn}\nabla^2V_{mn} - \frac{1}{40}V_{mn}\nabla_m\nabla_nV + \frac{1}{240}V\nabla^2V - \frac{1}{12}V_{mn}V_{np}\mathfrak{F}_{mp} + \frac{1}{6}\mathfrak{F}_{mn}\nabla_{(m}\nabla_{p)}V_{pn} + \frac{1}{24}V\mathfrak{F}_{mn}\mathfrak{F}_{mn} \\ &- \frac{1}{6}V_{mn}\mathfrak{F}_{mp}\mathfrak{F}_{np} - \frac{1}{3}\mathfrak{F}_{mn}\nabla_mN_n - \frac{1}{6}V_{mn}\nabla_mN_n + \frac{1}{12}V\nabla_mN_m - \frac{1}{6}N_mN_m - \frac{1}{12}UV + \frac{1}{36}R\mathfrak{F}_{mn}\mathfrak{F}_{mn} \\ &+ \frac{1}{144}RV_{mn}V_{mn} + \frac{1}{288}RVV - \frac{1}{6}RU + \frac{1}{60}R\nabla^2V - \frac{1}{20}R\nabla_m\nabla_nV_{mn} + \frac{1}{6}R\nabla_mN_m - \frac{1}{36}R_{mn}V_{np}V_{pm} \\ &- \frac{1}{72}R_{mn}V_{mn}V - \frac{1}{60}R_{mn}\nabla^2V_{mn} - \frac{1}{12}R_{mn}V_{np}\mathfrak{F}_{mp} + \frac{1}{45}R_{mn}\mathfrak{F}_{mp}\mathfrak{F}_{np} + \frac{1}{36}R_{mnpq}V_{mp}V_{nq} + \frac{1}{90}R_{mnpq}\mathfrak{F}_{mn}\mathfrak{F}_{pq} \\ &+ \frac{1}{360}R_{mnpq}R_{mnpq}V - \frac{1}{90}R_{mpqk}R_{npqk}V_{mn} - \frac{1}{360}R_{mn}R_{mn}V + \frac{1}{144}R^2V + \frac{1}{45}R_{mn}R_{mp}V_{pn} - \frac{1}{36}RR_{mn}V_{mn} \\ &- \frac{1}{90}R_{mnpq}R_{mp}V_{nq} + 2\mathfrak{E}\cdot\mathbb{1}\Big]. \end{aligned} \tag{B9}$$

The formula (B9) extends the result of [25] with the present paper.

3. A note on self-adjointness

Often we are interested in the operators of the form (B6) arising after path integration. This operation typically projects on self-adjoint part, which imposes restrictions on the coefficient functions. These conditions are not automatically taken into account in expression for b_p such as (B8) and (B9), which apply to generic operators, and have to be imposed by hand when isolating the differential operator in the quadratic part of the action.

For the second-order operator Δ_2 these translate on the requirement that $X = X^\dagger$, where $\dagger$ is the appropriate conjugation on the internal index structure (i.e., it is transposition or Hermitian conjugation for real or complex fields respectively).

The discussion for Δ_4 is slightly more subtle. It is convenient to rewrite the operator in the symmetric form

$$\Delta_4 = \nabla^4 + \nabla_r \hat{V}_{rk} \nabla_k + \hat{N}_k \nabla_k + \nabla_k \hat{N}_k + \hat{U},$$
$$\hat{V}_{rk} = \hat{V}_{kr}, \tag{B10}$$

where the derivatives act on everything to their right. The relation with the nonsymmetric form in (B6) is given by

$$V_{mn} = \hat{V}_{mn}, \qquad N_m = \hat{N}_m + \frac{1}{2}\nabla_m \hat{V}_{mn},$$
$$U = \hat{U} + \nabla_m \hat{N}_m. \tag{B11}$$

The form (B10) is convenient because self-adjointness amounts to the conditions

$$\hat{V}_{mn} = \hat{V}^{\dagger}_{mn}, \qquad \hat{N}_m = -\hat{N}^{\dagger}_m, \qquad \hat{U} = \hat{U}^{\dagger}, \tag{B12}$$

where again † is the appropriate conjugation of the internal indices.

APPENDIX C: BASIS OF THE INVARIANTS FOR THE DECOMPOSITION $\Delta_4 = \Delta_+ \Delta_-$

In this appendix we list the basis of the invariants used to study the decomposition (2.14). We consider terms of $\mathcal{O}(B^n)$, with $n = 1, 2, 4, 5, 6$. In total we find 45 elements; any other combination can be expressed in terms of these by integration by parts, use of Bianchi identities and other symmetry properties.

(i) $\mathcal{O}(B^1)$: 19 elements

$$\nabla_n \mathfrak{F}_{mn} R_{ma} B_a, \quad \mathfrak{F}_{mn} \nabla_n R_{ma} B_a, \quad \nabla_a \mathfrak{F}_{mn} R_{ma} B_n, \quad \nabla_m \mathfrak{F}_{mn} R B_n,$$
$$\mathfrak{F}_{mn} \nabla_m R B_n, \quad \mathfrak{F}_{mn} \nabla_a \mathfrak{F}_{mn} B_a, \quad \mathfrak{F}_{mn} \nabla_a \mathfrak{F}_{ma} B_n,$$
$$R_{mnac} \nabla_c \mathfrak{F}_{mn} B_a, \quad R_{mnac} \mathfrak{F}_{ma} \nabla_n B_c, \quad R_{mnac} \mathfrak{F}_{ma} \nabla_c B_n, \quad R_{mnac} \nabla_r R_{mnac} B_r,$$
$$R_{mn} \nabla_r R_{mn} B_r, \quad R_{mr} \nabla_r R_{mn} B_n, \quad R_{mn} \nabla_n R B_m, \quad R \nabla_m R B_m,$$
$$R_{mnac} \nabla_c R_{ma} B_n, \quad R_{mnac} \nabla_n R_{ma} B_c, \quad \nabla^2 \nabla_m \mathfrak{F}_{mn} B_n, \quad \nabla^2 R \nabla_m B_m.$$

(ii) $\mathcal{O}(B^2)$: 26 elements

$$\mathfrak{F}_{mn} \mathfrak{F}_{mn} B_a B_a, \quad \mathfrak{F}_{ma} \mathfrak{F}_{mc} B_a B_c, \quad R_{ma} \mathfrak{F}_{mc} B_a B_c, \quad R_{mnac} \mathfrak{F}_{ma} B_n B_c,$$
$$R^2 B_a B_a, \quad R R_{mn} B_m B_n, \quad R_{ma} R_{mc} B_a B_c, \quad R_{mnac} R_{mnac} B_r B_r, \quad R_{mnac} R_{rnac} B_m B_r,$$
$$\mathfrak{F}_{mn} B_m \nabla^2 B_n, \quad \mathfrak{F}_{ma} B_m \nabla_a \nabla_n B_n, \quad \mathfrak{F}_{ma} \nabla_a B_m \nabla_n B_n, \quad \mathfrak{F}_{ma} B_n \nabla_n \nabla_a B_m,$$
$$R_{mnac} B_m B_a R_{nc}, \quad B_a \nabla^2 \nabla^2 B_a, \quad B_a \nabla_a \nabla^2 \nabla_m B_m \quad R_{mn} B_m \nabla^2 B_n,$$
$$R_{mn} \nabla_a B_m \nabla_a B_n, \quad R_{mn} B_a \nabla_m \nabla_n B_a, \quad R_{mn} B_m \nabla_n \nabla_a B_a, \quad R_{mn} B_a \nabla_a \nabla_n B_m,$$
$$R B_m \nabla_m \nabla_n B_n, \quad R \nabla_m B_m \nabla_n B_n, \quad R B_a \nabla^2 B_a, \quad R \nabla_a B_c \nabla_a B_c, \quad R_{mnac} B_m \nabla_n \nabla_a B_c.$$

(iii) $\mathcal{O}(B^4)$: 8 elements

$$B_a B_a B_r \nabla^2 B_r, \quad B_a B_a \nabla_m B_r \nabla_m B_r, \quad B_a \nabla_m \nabla_n B_a B_m B_n, \quad B_a B_a B_m \nabla_m \nabla_n B_n,$$
$$B_a B_a \nabla_m B_m \nabla_n B_n, \quad B_a B_a \nabla_n B_m \nabla_m B_n, \quad R_{mn} B_m B_n B_a B_a, \quad R B_m B_m B_a B_a.$$

(iv) $\mathcal{O}(B^5)$: 1 element

$$B_m B_m B_a B_n \nabla_n B_a.$$

(v) $\mathcal{O}(B^6)$: 1 element

$$(B_m B_m)^3.$$

APPENDIX D: DIAGRAMMATIC CHECKS OF $b_6^{\mathbf{m}}(\Delta_4)$

In this appendix we compute diagrammatically some of the terms in (2.9) and (2.10) as an independent consistency check. We consider a free scalar in dimensional regularization ($d = 6 - 2\varepsilon$) with

$$S = \frac{1}{2}\int \sqrt{g}\phi\Delta_{4\phi}\phi,$$
$$\Gamma_\infty = \frac{1}{(4\pi)^3\varepsilon}\int \sqrt{g}b_6(\Delta_{4\phi}), \tag{D1}$$

where $\Delta_{4\phi}$ has the structure (1.3) with spacetime connection only and with V, N, U being spacetime-covariant functions.

To set the perturbative expansion in powers of the external fields we use $h_{mn} = g_{mn} - \delta_{mn}$ and for simplicity we assume $h_{mm} = h_{mn}\delta_{mn} = 0$. We construct the diagrams for the following three correlators: $\langle Uh\rangle$ (giving c_4), $\langle Nh\rangle$ (giving c_7), $\langle Vh\rangle$ (giving c_5, c_6, c_{10}). The results of this calculation are all in agreement with the solution in (2.10).

To diagrammatically compute the divergent part of the effective action from (D1) we need the free propagator

$$\langle\phi(p)\phi(-p)\rangle = \frac{1}{p^4} \tag{D2}$$

and the following vertices,

$$\begin{aligned}
S_h &= \frac{1}{2}\int h_{mn}(-p-q)H_{mn}(p,q)\phi(q)\phi(q), \qquad H_{mn}(p,q) = p_{(m}q_{n)}(p^2+q^2);\\
S_V &= \frac{1}{2}\int V_{mn}(p-q)W_{mn}(p,q)\phi(q)\phi(q), \qquad W_{mn}(p,q) = -\frac{1}{2}(q_m q_n + p_m p_n);\\
S_N &= \frac{1}{2}\int N_m(p-q)M_m(p,q)\phi(q)\phi(q), \qquad M_m(p,q) = i(p_m+q_m);\\
S_U &= \frac{1}{2}\int U(-p-q)\phi(p)\phi(q).
\end{aligned} \tag{D3}$$

To the terms under considerations only a single two-propagator diagram contributes, and the corresponding terms in the effective action are found to be

$$\begin{aligned}
\langle V_{rs}(q)h_{mn}(-q)\rangle &= -\frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^4(q-p)^4}H_{mn}(p,q-p)W_{rs}(p-q,-p),\\
\langle N_r(q)h_{mn}(-q)\rangle &= -\frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^4(q-p)^4}H_{mn}(p,q-p)M_r(p-q,-p),\\
\langle U(q)h_{mn}(-q)\rangle &= -\frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^4(q-p)^4}H_{mn}(p,q-p).
\end{aligned} \tag{D4}$$

The loop integrals can be evaluated using standard two-propagator technology (see e.g., [26]) and the divergent parts read

$$\begin{aligned}
\langle V_{rs}(q)h_{mn}(-q)\rangle_\infty &= \frac{1}{240(4\pi)^3\varepsilon}[6q_rq_sq_mq_n - \delta_{rm}\delta_{sn}q^4 + 2\delta_{rn}q_sq_mq^2 - 2\delta_{rs}q_mq_nq^2],\\
\langle N_r(q)h_{mn}(-q)\rangle_\infty &= \frac{i}{12(4\pi)^3\varepsilon}q_rq_mq_n,\\
\langle U(q)h_{mn}(-q)\rangle_\infty &= -\frac{1}{12(4\pi)^3\varepsilon}q_mq_n,
\end{aligned} \tag{D5}$$

which correctly reproduce the values for c_4, c_5, c_6, c_7, c_{10}.

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