Letter

$T\bar{T}$ deformed soft theorem

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In this paper, we derive a $T\bar{T}$ deformed soft graviton theorem in terms of celestial holography. As a concrete example, it illustrates that a two-dimensional irrelevant deformation can be applied to a four-dimensional theory at the level of amplitudes.

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I. INTRODUCTION

Newton's constant has a negative mass dimension; therefore, general relativity is not renormalizable in the usual sense, which 't Hooft conclusively confirms with Veltman in the early 1970s [1]. As an effective theory, general relativity can properly describe gravitational interaction at the low energy scale in the Wilson scheme. Then the search for a consistent ultraviolet (UV) completion for general relativity has been a tremendous physical problem for more than half a century, see, e.g., [2] for a comprehensive introduction.

In general, conformal field theory (CFT) is central to describing the fixed points of the renormalization group flow. A common way of flowing away from fixed points to probe the dynamics at higher energy scales is to consider irrelevant deformations of the theory. In particular, it was recently discovered that the composite operator $T\bar{T}$ could lead to a tractable and even solvable irrelevant deformation in two-dimensional (2D) spacetime [3,4]. The deformed theories in the deep UV are expected to be UV complete. One crucial piece of evidence is that the $T\bar{T}$ deformed massless free scalar field theory relates to the Nambu-Goto action in static gauge. The associated nonlocal property of $T\bar{T}$-deformed theories have already been discovered while studying the effective theory of long relativistic strings [5,6]. Furthermore, the relation between the $T\bar{T}$-deformation theories and string theory have been intensively investigated in [7–12].

Thanks to celestial holography, any quantum scattering amplitude of massless particles in four-dimensional (4D) asymptotically Minkowskian spacetime can be rewritten as a correlation function on the celestial sphere at null infinity [13–15], namely, celestial CFTs. We argue that a 2D $T\bar{T}$ deformation of celestial CFTs can be applied to a 4D gravitational theory, which could shed light on the construction of a UV complete theory for general relativity. We demonstrate our proposal by deriving a $T\bar{T}$ deformation soft graviton theorem in the context of celestial CFTs. Soft theorems and asymptotic symmetries are mathematically equivalent in many theories with massless particles, revealing the symmetry origin of universal factorization properties of scattering amplitudes in the soft limit [16–37]. In particular, the subleading soft graviton theorem [38] implies that the tree-level S matrix for quantum gravity in four-dimensional Minkowski space has Virasoro symmetry [19]. Moreover, a 2D stress tensor was constructed from the subleading soft graviton theorem [39]. It provokes the writing of scattering amplitudes in a basis [13,14] manifesting the conformal symmetries. Then 4D tree-level scattering amplitudes are mapped to 2D correlators of CFTs on the celestial sphere [15], see also [40–42] for recent reviews and references therein. This connection allows one to deform a 4D theory with a 2D operator.

We start from the 2D charge defined by the stress tensor induced by the subleading soft graviton theorem [39]. By introducing a soft graviton propagator, we can obtain the shadow of the subleading soft factor from the 2D charge. Then the subleading soft factor can be recovered by an inverse shadow transformation. The 2D stress tensor can be deformed by the $T\bar{T}$ operator in the standard way, which is

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given in a perturbative expansion of the deformation parameter λ. Accordingly, the deformed 2D charge leads to the shadow of the deformed soft factor. We perform the inverse shadow transformation and give the explicit form of the soft factor up to λ^2 order. The deformed soft theorem should be considered a universal factorization property of UV-complete quantum gravity. If confirmed by the ordinary amplitude calculation in momentum space, it provides remarkable evidence for the ongoing celestial holography program [40–43].

II. SOFT THEOREM IN ASYMPTOTIC FLAT SPACETIME

The undeformed theory lives on the asymptotic flat spacetime (AFS) background with retarded Bondi coordinates $(u, r, z, \bar{z})$. The AFS metric can be expanded near future null infinity $\mathcal{I}^+$ $(r \to \infty)$

$$\begin{aligned} ds^2 = &-\frac{\mathring{R}}{2}du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \\ &+ \frac{2M}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 \\ &+ D^zC_{zz}dudz + D^{\bar{z}}C_{\bar{z}\bar{z}}dud\bar{z} + \cdots, \end{aligned} \tag{1}$$

where the retarded time $u = t - r$ is the coordinate of the null vector on $\mathcal{I}^+$. D_z and $\mathring{R}$ are the covariant derivatives and Ricci scalar of the transverse metric $\gamma_{z\bar{z}}$, respectively. We choose $\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$ for the celestial sphere. Correspondingly, $\mathring{R} = 2$. The asymptotic shear C_{AB} and the Bondi mass aspect M are independent of r. The Bondi news tensor is defined as follows:

$$N_{AB} = \partial_u C_{AB}, \qquad A, B = z \quad \text{or} \quad \bar{z}. \tag{2}$$

One can use $\kappa h_{\mu\nu}$ $(\kappa = \sqrt{32\pi G_N})$ to denote the perturbative part of AFS metric, which can be expanded as outgoing graviton modes

$$\begin{aligned} h^{\text{out}}_{\mu\nu}(x) = \sum_{\alpha=\pm}\int \frac{d^3q}{(2\pi)^3}\frac{1}{2\omega}[&\bar{\varepsilon}^{\alpha}_{\mu\nu}a^{\text{out}}_{\alpha}(\vec{q})e^{iq\cdot x} \\ &+ \varepsilon^{\alpha}_{\mu\nu}a^{\text{out}}_{\alpha}(\vec{q})^{\dagger}e^{-iq\cdot x}], \end{aligned} \tag{3}$$

where we have adopted natural units $8\pi G_N = 1$, and $q = (\omega, \vec{q})$ is the 4-momentum of graviton. The polarization tensor $\varepsilon^{\pm}_{\mu\nu}$ can be factorized as two polarization vectors $\varepsilon^{\pm}_{\mu\nu} = \varepsilon^{\pm}_{\mu}\varepsilon^{\pm}_{\nu}$. The momenta and the polarization vectors can be parametrized as

$$\begin{aligned} &q^{\mu}(\omega, z, \bar{z}) = \omega\left(1, \frac{z+\bar{z}}{1+z\bar{z}}, \frac{-i(z-\bar{z})}{1+z\bar{z}}, \frac{1-z\bar{z}}{1+z\bar{z}}\right), \\ &\varepsilon^{+}_{\mu}(q) = \frac{1}{\sqrt{2}}(-\bar{z}, 1, -i, -\bar{z}), \qquad \varepsilon^{-}_{\mu}(q) = \bar{\varepsilon}^{+}_{\mu}(q). \end{aligned} \tag{4}$$

The canonical quantification gives

$$[a^{\text{out}}_{\alpha}(\vec{q}), a^{\text{out}}_{\beta}(\vec{q'})^{\dagger}] = 2\omega_q\delta_{\alpha\beta}(2\pi)^3\delta^{(3)}(\vec{q} - \vec{q}'). \tag{5}$$

Comparing (3) with (1), one obtains the mode expansion for the shear and news tensors as

$$\begin{aligned} C_{\bar{z}\bar{z}} &= -\frac{i\hat{\varepsilon}_{\bar{z}\bar{z}}}{4\pi^2}\int_0^{\infty} \mathrm{d}\omega[a^{\text{out}}_{-}(q)e^{-i\omega u} - a^{\text{out}}_{+}(q)^{\dagger}e^{i\omega u}], \\ N_{\bar{z}\bar{z}} &= -\frac{\hat{\varepsilon}_{\bar{z}\bar{z}}}{4\pi^2}\int_0^{\infty} \omega\mathrm{d}\omega[a^{\text{out}}_{-}(q)e^{-i\omega u} + a^{\text{out}}_{+}(q)^{\dagger}e^{i\omega u}], \\ \hat{\varepsilon}_{\bar{z}\bar{z}} &= \frac{2}{(1+z\bar{z})^2} = \gamma_{z\bar{z}}, \end{aligned}$$

where we used the stationary-phase approximation [28]. Hence, the nth moment of the news tensor can be written in the mode expansion as

$$\begin{aligned} N^{(n)}_{\bar{z}\bar{z}} &= \int_{-\infty}^{\infty} duu^n N_{\bar{z}\bar{z}}, \\ &= \frac{(-i)^n}{2}\lim_{\omega\to 0}\partial^n_{\omega}\int_{-\infty}^{\infty} du(e^{i\omega u} + (-1)^n e^{-i\omega u})N_{\bar{z}\bar{z}}, \\ &= -\frac{(-i)^n\hat{\varepsilon}^{+}_{\bar{z}\bar{z}}}{4\pi}\lim_{\omega\to 0}\partial^n_{\omega}\{\omega[a^{\text{out}}_{-}(q) + (-1)^n a^{\text{out}}_{+}(q)^{\dagger}]\}. \end{aligned} \tag{6}$$

In the Heisenberg picture, the n-point tree level scattering problem in AFS can be regarded as that the asymptotic states $|\text{in}\rangle = |q_1, s_1; \cdots; q_m, s_m\rangle$ defined on $\mathcal{I}^-$ and $|\text{out}\rangle = |q_{m+1}, s_{m+1}; \cdots; q_n, s_n\rangle$ defined on $\mathcal{I}^+$ are fixed and the $\mathcal{S}$ matrix depends on the time evolution. We use q_k, s_k to denote the 4-momentum and helicity of kth massless hard particle with finite energy ω_k. The expression of $q_k, \varepsilon^{\pm}_{\mu}(q_k)$ can be similarly parametrized as (4). The n-point amplitude of massless hard particles is defined as follows:

$$\mathcal{A}_n = \langle \text{out}|\mathcal{S}|\text{in}\rangle = \langle \mathcal{O}_1 \cdots \mathcal{O}_n\rangle, \tag{7}$$

where $\mathcal{O}_k$ is annihilation or creation operator of kth hard particle [44],

$$\mathcal{O}_k(\omega_k, z_k, \bar{z}_k) = a^{\text{out}}_k(q_k)\theta(\omega_k) + a^{in\dagger}_k(-q_k)\theta(-\omega_k).$$

By introducing the following Mellin transform,

$$\mathcal{O}_{\Delta_k, s_k}(z_k, \bar{z}_k) = \int_0^{\infty} d\omega_k\omega_k^{\Delta_k - 1}\mathcal{O}_k(\omega_k, z_k, \bar{z}_k), \tag{8}$$

one can connect the operators between 2D and 4D. Therefore, the 4D amplitude is equivalent to the 2D correlation function by implementing Mellin transform for all hard particles

$$\langle X_n\rangle = \prod_{k=1}^{n}\left(\int_0^{\infty} d\omega_k \omega_k^{\Delta_k-1}\right)\delta^{(4)}\left(\sum_{k=1}^{n}\epsilon_k q_k\right)\mathcal{A}_n, \quad (9)$$

where $X_n = \prod_{k=1}^{n}\mathcal{O}_{\Delta_k,s_k}(z_k,\bar{z}_k)$, and $\epsilon_k = 1,-1$ for particles in $|\mathrm{out}\rangle$ and $|\mathrm{in}\rangle$ state, respectively. See more details of the Mellin transform in [15].

In terms of the soft theorem, an amplitude containing n hard particles and a soft graviton with energy $\omega \to 0$ can be expanded by the power of soft energy ω

$$\begin{aligned}\mathcal{A}^{\pm}_{n+1}(q)|_{\omega\to 0} &= \lim_{\omega\to 0}\langle \mathrm{out}; q, \pm 2|\mathcal{S}|\mathrm{in}\rangle, \\ &= \lim_{\omega\to 0}\langle \mathrm{out}|a^{\mathrm{out}}_{\pm}(q)\mathcal{S}|\mathrm{in}\rangle = \sum_{n=0}^{\infty} S^{(n)\pm}\omega^{n-1}\mathcal{A}_n.\end{aligned} \quad (10)$$

Since the nth news tensor can be expressed by the generator of nth order soft factor (6), one can read off the results of the insertion of the nth moment into the amplitude as

$$\langle \mathrm{out}|N^{z(n)}_{\bar{z}}\mathcal{S}|\mathrm{in}\rangle = -\frac{(-i)^n n!}{4\pi}S^{(n)-}\langle \mathrm{out}|\mathcal{S}|\mathrm{in}\rangle. \quad (11)$$

The soft factors are universal for the first three orders [38]. In particular, the subleading soft graviton factors can be written in the position space as [39]

$$\begin{aligned} S^{(1)+}_{(\vec{z},\vec{z}_k)} &= \sum_{k=1}^{n}\frac{(\bar{z}-\bar{z}_k)^2}{z-z_k}\left[\frac{2\hat{\bar{h}}_k}{\bar{z}-\bar{z}_k} - \Gamma^{\bar{z}_k}_{\bar{z}_k\bar{z}_k}\hat{\bar{h}}_k - \partial_{\bar{z}_k} - s_k\Omega_{\bar{z}_k}\right], \\ S^{(1)-}_{(\vec{z},\vec{z}_k)} &= \sum_{k=1}^{n}\frac{(z-z_k)^2}{\bar{z}-\bar{z}_k}\left[\frac{2\hat{h}_k}{z-z_k} - \Gamma^{z_k}_{z_k z_k}\hat{h}_k - \partial_{z_k} + s_k\Omega_{z_k}\right], \\ \hat{h}_k &= \frac{1}{2}(s_k - \omega_k\partial_{\omega_k}), \qquad \hat{\bar{h}}_k = \frac{1}{2}(-s_k - \omega_k\partial_{\omega_k}), \end{aligned} \quad (12)$$

where $\vec{z} = (z,\bar{z})$ and $\vec{z}_k = (z_k,\bar{z}_k)$ are the locations of the soft graviton and the hard particles, respectively, Γ^z_{zz} is the Levi-Civita connection of the celestial sphere metric $\gamma_{z\bar{z}}$ and $\Omega_z = \frac{\Gamma^z_{zz}}{2}$ is the spin connection.

III. FROM 4D SUPERROTATION CHARGE TO 2D VIRASORO CHARGE

In 4D AFS, the gravitational scattering has Bondi-Metzner-Sachs invariance [17], which reveals the symmetry origin of the soft graviton theorem. The Bondi-Metzner-Sachs symmetry consists of supertranslations and superrotations related to the leading [18] and subleading [19,21] soft graviton theorem, respectively. The superrotation charge includes two parts, namely the soft part and the hard part [19,21,40,41],

$$\mathcal{Q} = \mathcal{Q}_S + \mathcal{Q}_H, \quad (13)$$

which are given by

$$\begin{aligned}\mathcal{Q}_H &= -2i\int_{\mathcal{I}^+}\gamma_{z\bar{z}}\mathrm{d}^2z\mathrm{d}u(Y^z T^{(4)}_{uz} + uD_z Y^z T^{(4)}_{uu}), \\ \mathcal{Q}_S &= i\int_{S^2}\mathrm{d}^2z Y^z D_z^3 N^{z(1)}_{\bar{z}} = i\int \mathrm{d}^2z Y^z\partial_z^3 N^{z(1)}_{\bar{z}}\end{aligned} \quad (14)$$

for the holomorphic case, where $T^{(4)}_{\mu\nu}$ is the 4D total stress tensor [28] and Y^A is the superrotation parameter. The Ward identity of the superrotation charge yields the insertion of soft charge $\mathcal{Q}_S$ as [19]

$$\begin{aligned}\langle \mathrm{out}|\mathcal{Q}_S\mathcal{S}|\mathrm{in}\rangle = \sum_{k=1}^{n}&(Y^{z_k}(\partial_{z_k} - s_k\Omega_{z_k}) + D_{z_k}Y^{z_k}\hat{h}_k) \\ &\times \langle \mathrm{out}|\mathcal{S}|\mathrm{in}\rangle.\end{aligned} \quad (15)$$

Applying the Mellin transform for hard particles in (15) yields

$$\langle Q_Y X_n\rangle = \sum_{k=1}^{n}[Y^{z_k}(\partial_{z_k} - s_k\Omega_{z_k}) + D_{z_k}Y^{z_k}h_k]\langle X_n\rangle, \quad (16)$$

where $(h_k,\bar{h}_k) = \left(\frac{\Delta_k+s_k}{2},\frac{\Delta_k-s_k}{2}\right)$ are the conformal weights of kth hard particle on the celestial sphere. The subscript Y in the charge indicates that it corresponds to a 2D charge operator. Remarkably, Eq. (16) recovers the Ward identity of 2D Virasoro charge constructed in [39] from 2D stress tensor,

$$Q_Y = \frac{1}{2\pi i}\oint_{\mathcal{C}}\mathrm{d}z T_{zz}Y^z, \quad (17)$$

where the integral contour $\mathcal{C}$ separates the locations of all hard particles $\vec{z}_k$ and soft particles $\vec{z}$. The insertion of the 2D stress tensor in (17) into the correlator yields [39]

$$\begin{aligned}\langle T_{zz}X_n\rangle = \sum_{k=1}^{n}\Bigg[&\frac{h_k}{(z-z_k)^2} + \frac{\Gamma^{z_k}_{z_k z_k}}{z-z_k}h_k \\ &+ \frac{1}{z-z_k}(\partial_{z_k} - s_k\Omega_{z_k})\Bigg]\langle X_n\rangle,\end{aligned} \quad (18)$$

which is precisely the conformal Ward identity of the stress tensor on celestial sphere [45]. While the operator product expansions of the stress tensor are derived by inserting two components of the stress tensor into the amplitude and implementing Mellin transform [46],

$$
\begin{aligned}
&T_{zz}T_{z'z'} \sim \frac{2T_{z'z'}}{(z-z')^2} + \frac{\partial_{z'}T_{z'z'}}{z-z'} + \text{regular},\\
&T_{zz}T_{\bar{z}'\bar{z}'} \sim \text{regular},
\end{aligned}
\tag{19}
$$

which indicates that the central charge of the corresponding CFT on the celestial sphere is vanishing and the stress tensor is traceless.

Following the above procedures, one can easily recover the correspondence between the 4D and 2D charges for the antiholomorphic part.

IV. FROM 2D VIRASORO CHARGE TO SUBLEADING SOFT GRAVITON THEOREM

Superrotations reveal the symmetry origin of the subleading soft graviton theorem [19,21,47]. Hence, the 2D Virasoro charge is related to the subleading soft graviton theorem in the context of celestial holography [40–43]. It is shown that soft theorems can be directly derived in the 2D conformal basis [48–50]. Here, we propose a direct way to reveal the subleading conformally soft graviton theorem [48–50] from 2D Virasoro charge. The 2D stress tensor with dimension $\Delta = 2$ is the shadow transformation of the subleading soft-graviton operator [51] with dimension $\Delta = 0$ [48–50]. One can refer to the shadow transformation in the Supplemental Material [52] and also in [14,62–64]. Thanks to this shadow relation, one can verify that the 2D Virasoro charge associated with a particular choice [28] of the superrotation parameter $Y^z = \frac{1}{w-z}$ is the shadow of the subleading soft-graviton operator. Further, one can apply the same choice for superrotation charge to recover the subleading soft graviton theorem [19,28]. We would refer to the particular choice Y^z [28] as a soft graviton propagator. In such a way, the 2D Virasoro charge reveals the symmetry origin of the conformally soft graviton theorem [48–50] in the 2D context. It can be justified by inserting the Mellin transform of the 2D charge associated with the soft graviton propagator into a 4D amplitude,

$$
\langle \text{out}|Q_Y \mathcal{S}|\text{in}\rangle = \tilde{S}^{(1)-}_{(\vec{w})}\langle \text{out}|\mathcal{S}|\text{in}\rangle,
\tag{20}
$$

where

$$
\begin{aligned}
\tilde{S}^{(1)-}_{(\vec{w})} &= \sum_{k=1}^{n}\left[\frac{\hat{h}_k}{(w-z_k)^2} + \frac{\Gamma^{z_k}_{z_k z_k}\hat{h}_k}{w-z_k} + \frac{\partial_{z_k} - s_k\Omega_{z_k}}{w-z_k}\right],\\
\tilde{S}^{(1)+}_{(\vec{w})} &= \sum_{k=1}^{n}\left[\frac{\hat{\bar{h}}_k}{(\bar{w}-\bar{z}_k)^2} + \frac{\Gamma^{\bar{z}_k}_{\bar{z}_k\bar{z}_k}\hat{\bar{h}}_k}{\bar{w}-\bar{z}_k} + \frac{\partial_{\bar{z}_k} + s_k\Omega_{\bar{z}_k}}{\bar{w}-\bar{z}_k}\right].
\end{aligned}
\tag{21}
$$

The factor $\tilde{S}^{(1)-}_{(\vec{w},\vec{z}_k)}$ is related to soft factor $S^{(1)-}_{(\vec{z},\vec{z}_k)}$ in (12) by the shadow transformation as

$$
\begin{aligned}
\tilde{S}^{(1)-}_{(\vec{w})} &= \frac{3!}{4\pi}\int d^2z \frac{1}{(w-z)^4} S^{(1)-}_{(\vec{z})},\\
S^{(1)-}_{(\vec{z})} &= \frac{1}{2\pi}\int d^2w \frac{(z-w)^2}{(\bar{z}-\bar{w})^2}\tilde{S}^{(1)-}_{(\vec{w})}.
\end{aligned}
\tag{22}
$$

V. TWO-DIMENSIONAL CHARGES FROM $T\bar{T}$ DEFORMED STRESS TENSOR

The $T\bar{T}$ flow effect on action is

$$
\frac{\partial S^{[\lambda]}}{\partial \lambda} = -\int \sqrt{\gamma^{[\lambda]}} d^2x O^{[\lambda]}_{T\bar{T}}, \qquad O^{[\lambda]}_{T\bar{T}} = \frac{1}{2}(T^{AB}_{[\lambda]}T^{[\lambda]}_{AB} - T^2_{[\lambda]}),
\tag{23}
$$

where λ is the coupling constant of $T\bar{T}$ deformation and T denotes the trace of stress tensor. The superscript [0] denotes quantities before $T\bar{T}$ deformation, while [λ] denotes the deformed quantities. The flow equation can be exactly solved with variational principle [65]

$$
\begin{aligned}
\hat{T}^{[\lambda]}_{AB} &= T^{[\lambda]}_{AB} - \gamma^{[\lambda]}_{AB}T^{[\lambda]} = \hat{T}^{[0]}_{AB} - \hat{T}^{[0]}_{AC}\hat{T}^{[0]}_{BD}\gamma^{CD}_{[0]}\lambda,\\
\gamma^{[\lambda]}_{AB} &= \gamma^{[0]}_{AB} - 2\hat{T}^{[0]}_{AB}\lambda + \hat{T}^{[0]}_{AC}\hat{T}^{[0]}_{BD}\gamma^{CD}_{[0]}\lambda^2,\\
T^{[\lambda]} &= \frac{T^{[0]} - 2O^{[0]}_{T\bar{T}}\lambda}{1 + T^{[0]}\lambda - O^{[0]}_{T\bar{T}}\lambda^2}.
\end{aligned}
\tag{24}
$$

One can check that the deformed stress tensor is conserved, namely $\gamma^{AB}_{[\lambda]}D^{[\lambda]}_A T^{[\lambda]}_{BC} = 0$, where $D^{[\lambda]}_A$ is the covariant derivative with respect to the deformed metric $\gamma^{[\lambda]}_{AB}$. On the celestial sphere, the perturbative terms of components of the deformed stress tensor are

$$
\begin{aligned}
T^{[\lambda]}_{zz} &= T^{[0]}_{zz}\left[1 + 4\sum_{n=1}^{\infty}(O^{[0]}_{T\bar{T}}\lambda^2)^n\right],\\
T^{[\lambda]}_{\bar{z}\bar{z}} &= T^{[0]}_{\bar{z}\bar{z}}\left[1 + 4\sum_{n=1}^{\infty}(O^{[0]}_{T\bar{T}}\lambda^2)^n\right],\\
T^{[\lambda]}_{z\bar{z}} &= -\gamma^{[0]}_{z\bar{z}}\left[3 + 4\sum_{n=1}^{\infty}(O^{[0]}_{T\bar{T}}\lambda^2)^n\right]O^{[0]}_{T\bar{T}}\lambda.
\end{aligned}
\tag{25}
$$

For the deformed stress tensor, the corresponding 2D charge is

$$
Q^{[\lambda]} = \frac{1}{2\pi i}\oint_{\mathcal{C}} dx^A T^{[\lambda]}_{AB}Y^B = Q^{[\lambda]}_Y + Q^{[\lambda]}_{\bar{Y}},
\tag{26}
$$

where

$$Q_Y^{[\lambda]} = \frac{1}{2\pi i}\oint_C \mathrm{d}z T_{zz}^{[\lambda]} Y^z + \frac{1}{2\pi i}\oint_C \mathrm{d}\bar{z} T_{z\bar{z}}^{[\lambda]} Y^z,$$
$$Q_{\bar{Y}}^{[\lambda]} = \frac{1}{2\pi i}\oint_C \mathrm{d}\bar{z} T_{\bar{z}\bar{z}}^{[\lambda]} Y^{\bar{z}} + \frac{1}{2\pi i}\oint_C \mathrm{d}z T_{z\bar{z}}^{[\lambda]} Y^{\bar{z}}. \tag{27}$$

Inserting the deformed stress tensor (25), we obtain the minus helicity charge (27) in series expansion of λ as

$$\begin{aligned} Q_Y^{[\lambda]} = &\oint \frac{\mathrm{d}z}{2\pi i} Y^z T_{zz}^{[0]} - 3\lambda \oint \frac{\mathrm{d}\bar{z}}{2\pi i} Y^z \gamma^{z\bar{z}[0]} T_{zz}^{[0]} T_{\bar{z}\bar{z}}^{[0]} \\ &- 4\sum_{s=1}^{\infty} \lambda^{2s+1} \oint \frac{\mathrm{d}\bar{z}}{2\pi i} Y^z (T_{zz}^{[0]})^{s+1} (T_{\bar{z}\bar{z}}^{[0]})^{s+1} (\gamma_{[0]}^{z\bar{z}})^{2s+1} \\ &+ 4\sum_{s=1}^{\infty} \lambda^{2s} \oint \frac{\mathrm{d}z}{2\pi i} Y^z (T_{zz}^{[0]})^{s+1} (T_{\bar{z}\bar{z}}^{[0]})^{s} (\gamma_{[0]}^{z\bar{z}})^{2s}. \end{aligned} \tag{28}$$

To close this section, two remarks about the deformed charges are as follows. First, since the deformed stress tensor has three independent components, one cannot directly apply the connection [39] between the 2D traceless stress tensor and the subleading soft graviton theorem to our case. Alternatively, we construct the deformed charge by contracting the deformed stress tensor with the super-rotation (conformal killing) vectors. Second, since the contour integration associated with the λ odd order terms in (28) is irrelevant to minus helicity soft graviton propagator Y^z, the integration contour cannot attach the soft graviton propagator to any hard particles. It corresponds to disconnected correlators, namely a soft graviton propagator plus the correlation function of hard particles.

VI. $T\bar{T}$ DEFORMED SOFT THEOREM

The deformed charges play essential roles in obtaining the deformed soft graviton theorem. In particular, one can insert the charge (28) into the amplitudes to get the shadow of a deformed subleading minus helicity soft graviton theorem. In the Heisenberg picture, the asymptotic states $\langle\mathrm{out}|$ and $|\mathrm{in}\rangle$ are the same as the ones in undeformed theory, the information of deformation is hidden in $\mathcal{S}$ matrix,

$$\mathcal{A}_n^{[\lambda]} = \langle \mathrm{out}|\mathcal{S}^{[\lambda]}|\mathrm{in}\rangle. \tag{29}$$

As discussed previously, after the insertion of $Q_Y^{[\lambda]}$ into the amplitudes, together with $Y^z = \frac{1}{w-z}$, the shadow of the soft factor will be obtained by shrinking the integral contour away from locations of hard particles z_k,

$$\tilde{S}^{[\lambda](1)-}_{(\vec{w},\vec{z}_k)} \langle \mathrm{out}|\mathcal{S}^{[\lambda]}|\mathrm{in}\rangle = \langle \mathrm{out}|Q_Y^{[\lambda]}\mathcal{S}^{[\lambda]}|\mathrm{in}\rangle, \tag{30}$$

which is equivalent to the insertion of $T_{ww}^{[\lambda]}$ and subtracting all extra δ functions because the integral contour does not pass through the poles of the delta function. The shadow of the deformed soft factor is given by

$$\tilde{S}^{(1)-}_{[\lambda](\vec{w},\vec{z}_k)} = \tilde{S}^{(1)-}_{(\vec{w},\vec{z}_k)} + 4\sum_{s=1}^{\infty} \lambda^{2s} (\gamma_{[0]}^{z\bar{z}})^{2s} [\tilde{S}^{(1)-}_{(\vec{w},\vec{z}_k)}]^{s+1} [\tilde{S}^{(1)+}_{(\vec{w},\vec{z}_k)}]^{s}, \tag{31}$$

where $\tilde{S}^{(1)\pm}_{(\vec{w},\vec{z}_k)}$ is the shadow of undeformed subleading soft factor (21). The $T\bar{T}$ deformation does not change the helicity of T_{zz} and $T_{\bar{z}\bar{z}}$, and they follow the same shadow formula (22). The explicit formula of the soft factor is

$$\begin{aligned} S^{(1)-}_{[\lambda]}(\vec{z},\vec{z}_k) = S^{(1)-}_{(\vec{z},\vec{z}_k)} + \frac{2}{\pi}\sum_{s=1}^{\infty} \lambda^{2s} \int \mathrm{d}^2 w \frac{(z-w)^2}{(\bar{z}-\bar{w})^2} \\ \times (\gamma_{[0]}^{z\bar{z}})^{2s} [\tilde{S}^{(1)-}_{(\vec{w},\vec{z}_k)}]^{s+1} [\tilde{S}^{(1)+}_{(\vec{w},\vec{z}_k)}]^{s}, \end{aligned} \tag{32}$$

where $S^{(1)-}_{(\vec{z},\vec{z}_k)}$ is the undeformed subleading soft factor (21). The leading order of λ expansion restores the undeformed soft factor. The explicit forms at λ^2 order and the strategy of computing surface integral on the celestial sphere in the complex stereographic coordinates are presented in the Supplemental Material [52]. Finally, the soft factor can be translated into momentum space with the relation between the celestial sphere coordinates and null momenta and polarization vectors in (4). Similarly, the plus helicity soft graviton theorem can be obtained from the antiholomorphic charge $Q_{\bar{Y}}$ defined in (27).

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