Supplemental Material for: Cavity Light-Matter Hybridization Driven by Quantum Fluctuations

Giacomo Passetti,[1,*] Christian J. Eckhardt,[1,2,*] Michael A. Sentef,[2] and Dante M. Kennes[1,2]

[1]*Institut für Theorie der Statistischen Physik, RWTH Aachen University and JARA-Fundamentals of Future Information Technology, 52056 Aachen, Germany*

[2]*Max Planck Institute for the Structure and Dynamics of Matter, Center for Free-Electron Laser Science (CFEL), Luruper Chaussee 149, 22761 Hamburg, Germany*

(Dated: May 25, 2023)

I. HIGH-FREQUENCY EXPANSION

In this part we will show how to obtain the approximate form of the wave-function (Eq. (2) of the main part) that is accurate in the high frequency limit and how to compute the photon number and the entanglement entropy with it. We consider the Hamiltonian to second order in the light-matter coupling which gives the correct ground state in the thermodynamic limit $L \to \infty$ and if the light-matter coupling is small which is fulfilled since $gt_{\mathrm{h}} \ll \Omega$ in the here considered limit

$$H^{g^2} = \Omega a^\dagger a + T + H_{\mathrm{int}} + \frac{g}{\sqrt{L}}\left(a^\dagger + a\right) J - \frac{g^2}{2L}\left(a^\dagger + a\right)^2 T. \tag{1}$$

We leave the part of the Hamiltonian containing fermion-fermion interactions general at this point. Later in the paper we set it to

$$H_{\mathrm{int}} = U \sum_i \left(n_i - \frac{1}{2}\right)\left(n_{i+1} - \frac{1}{2}\right) \tag{2}$$

for the concrete example of the XXZ chain. We perform two consecutive transformations of the Hamiltonian according to

$$\begin{aligned}
\tilde{H}^{g^2} &= e^{S^{\mathrm{sq}}[\xi(T)]} e^{S^{\mathrm{d}}[\alpha(J)]} H^{g^2} e^{-S^{\mathrm{d}}[\alpha(J)]} e^{-S^{\mathrm{sq}}[\xi(T)]} \\
S^{\mathrm{d}}[\alpha(J)] &= \alpha(J)\left(a^\dagger - a\right) \; ; \; \alpha(J) = \frac{g}{\Omega}\frac{J}{\sqrt{L}} \\
S^{\mathrm{sq}}[\xi(T)] &= \frac{1}{2}\xi(T)\left(a^2 - \left(a^\dagger\right)^2\right) \; ; \; \xi(T) = \frac{1}{2}\ln\left(1 - 2\frac{g^2}{\Omega}\frac{T}{L}\right)
\end{aligned} \tag{3}$$

which yields for $\tilde{H}$[1]

$$\tilde{H}^{g^2} = \tilde{\Omega}\beta^\dagger\beta + T + H_{\mathrm{int}} \tag{4}$$

where we have dropped terms multiplied with $\frac{1}{\Omega}$ since they will become irrelevant in the high frequency limit. Here β annihilates a squeezed coherent state of the cavity mode. In particular the interacting part of the Hamiltonian is transformed first by the displacement transformation S^{d} as

$$e^{S^{\mathrm{d}}} H_{\mathrm{int}} e^{-S^{\mathrm{d}}} = H_{\mathrm{int}} + \frac{g}{\sqrt{L}\Omega}\left(a^\dagger - a\right)[J, H_{\mathrm{int}}] + \mathcal{O}\left(\frac{1}{\Omega^2}\right) = H_{\mathrm{int}} + \mathcal{O}\left(\frac{1}{\Omega}\right). \tag{5}$$

such that the interaction term H_{int} stays unaffected by the displacement transformation to leading order in the large frequency limit. Essentially the same argument applies to the squeezing transformation S^{sq}.

---

* These authors contributed equally

$\tilde{\Omega}$ is a renormalized frequency that is identical to the original frequency Ω to leading order in $\frac{1}{\Omega}$

$$\tilde{\Omega} = \Omega\sqrt{1 - 2\frac{g^2}{\Omega}\frac{\langle T\rangle}{L}} = \Omega + \mathcal{O}\left(\frac{1}{\Omega}\right) \tag{6}$$

such that we will keep performing computations with the original frequency Ω.

In the new squeezed-coherent basis light and matter degrees of freedom decouple to leading order in $\frac{1}{\Omega}$ such that we can write the ground state wave-function as

$$|\tilde{\Psi}\rangle = |\psi\rangle \otimes |0_\beta\rangle \tag{7}$$

where $|\psi\rangle$ is the fermionic part of the wave-function and $|0_\beta\rangle$ is the squeezed-coherent vacuum. In order to calculate the photon number and light-matter entanglement, we need to obtain the ground state wave-function in the original basis which we do by applying the inverse transformation to Eq. (3). In order to do this we write the fermionic part of the wave-function in the common eigenbasis of T and J ($[T, J] = 0$) ie. in the Bloch basis

$$|\tilde{\Psi}\rangle = \sum_u c_u |j_u, t_u\rangle \otimes |0_\beta\rangle \tag{8}$$

where $c_u \in \mathbb{C}$ are complex numbers fulfilling the normalization condition $\sum_u |c_u|^2 = 1$. t_u and j_u denote the eigenvalue of T and J respectively to the corresponding Bloch state $T|j_u, t_u\rangle = t_u|j_u, t_u\rangle$ and $J|j_u, t_u\rangle = j_u|j_u, t_u\rangle$. Here the index u also counts potential degeneracies. The ground state in the original cavity-basis can thus be computed as

$$\begin{aligned}
|\Psi\rangle &= e^{-S^{\mathrm{d}}[\alpha(J)]} e^{-S^{\mathrm{sq}}[\xi(T)]} |\tilde{\Psi}\rangle \\
&= e^{-S^{\mathrm{d}}[\alpha(J)]} e^{-S^{\mathrm{sq}}[\xi(T)]} \sum_u c_u |j_u, t_u\rangle \otimes |0_\beta\rangle \\
&= \sum_u c_u |j_u, t_u\rangle \otimes e^{-S^{\mathrm{d}}[\alpha(j_u)]} e^{-S^{\mathrm{sq}}[\xi(t_u)]} |0_\beta\rangle \\
&= \sum_u c_u |j_u, t_u\rangle \otimes |\alpha(j_u), \xi(t_u)\rangle .
\end{aligned} \tag{9}$$

Here $|\alpha(j_u), \xi(t_u)\rangle$ is the squeezed coherent state with coherent displacement $\alpha(j_u) = -\frac{g}{\Omega}\frac{j_u}{\sqrt{L}}$ and squeezing $\xi(t_u) = \frac{1}{2}\ln\left(1 - 2\frac{g^2}{\Omega}\frac{t_u}{L}\right)$. Eq. (9) is the approximate form of the ground state wave-function that is accurate in the high frequency limit reported in the main part of the paper in Eq. (2). It has been obtained without explicit knowledge of the interacting part of the Hamiltonian H_{int}. The information about the particular form of the interaction is encoded in the coefficients c_u since these interactions determine the precise nature of the fermionic groundstate. However, the form of the composite fermionic and photonic wave-function consisting of combined current and kinetic energy operator eigenstates $|j_u, t_u\rangle$ and a corresponding squeezed coherent state $|\alpha(j_u), \xi(t_u)\rangle$ generally holds in the high frequency limit for a Hamiltonian as written in Eq. (1).

We continue by computing the photon number N_{phot} with this form of the wave-function

$$
\begin{aligned}
N_{\text{phot}} = \langle a^\dagger a \rangle &= \sum_u |c_u|^2 \langle \alpha(j_u), \xi(t_u) | a^\dagger a | \alpha(j_u), \xi(t_u) \rangle \\
&= \sum_u |c_u|^2 \langle \xi(t_u) | a^\dagger a + \frac{g^2}{\Omega^2} \frac{j_u^2}{L} | \xi(t_u) \rangle \\
&= \sum_u |c_u|^2 \frac{g^2}{\Omega^2} \frac{j_u^2}{L} + \sum_u |c_u|^2 \sinh^2 \left(\frac{1}{2} \ln \left(1 - 2 \frac{g^2}{\Omega} \frac{t_u}{L} \right) \right) \\
&= \frac{g^2}{\Omega^2} \frac{\langle J^2 \rangle}{L} + \frac{g^4}{\Omega^2} \frac{\langle T^2 \rangle}{L^2} + \mathcal{O}(\frac{1}{\Omega^4}).
\end{aligned}
\tag{10}
$$

The current operator has vanishing expectation value in the ground state $\langle J \rangle = 0$ since inversion symmetry is not broken. On the other hand

$$
\frac{\langle T^2 \rangle}{L^2} = \frac{\langle T \rangle^2}{L^2} + \frac{\Delta T^2}{L^2} \to \frac{\langle T \rangle^2}{L^2} \; (L \to \infty). \tag{11}
$$

The fluctuation contribution from the kinetic energy term is suppressed in the thermodynamic limit (the fluctuations of the kinetic energy operator ΔT^2 scale like L) such that we get for the photon number

$$
N_{\text{phot}} = \frac{g^2}{\Omega^2} \frac{\Delta J^2}{L} + \frac{g^4}{\Omega^2} \frac{\langle T \rangle^2}{L^2}. \tag{12}
$$

We will next turn to computing the entanglement entropy between light and matter. For this we compute the reduced density matrix of the photons ρ_{ph} by performing a partial trace over the fermionic degrees of freedom Tr_{f} on the density matrix ρ of the combined system as obtained from the ground state wave-function Eq. (9)

$$
\rho_{\text{ph}} = \text{Tr}_{\text{f}} \rho = \sum_u |c_u|^2 |\alpha(j_u), \xi(t_u)\rangle \langle \alpha(j_u), \xi(t_u)| \tag{13}
$$

which we find to be a statistical ensemble of squeezed coherent states. This is the reduced density matrix of the photons reported in Eq. (4) in the main part. We aim at calculating the entropy

$$
S = -\text{k}_{\text{B}} \text{Tr} \rho_{\text{ph}} \ln \rho_{\text{ph}}. \tag{14}
$$

From now we set $\text{k}_{\text{B}} = 1$. The intuition for the computation is that the squeezing part of the wave-function does not contribute significantly to the entropy since it is centered around the expectation value of the squeezing

$$
\xi_0 = \frac{1}{2} \ln \left(1 - 2 \frac{g^2}{\Omega} \frac{\langle T \rangle}{L} \right) \tag{15}
$$

while fluctuations around that value are suppressed in the thermodynamic limit as previously seen in the case of the photon number. To explicitly see this, we insert a 1 into the Tr Eq. (14) and use the cyclic property of the Tr

$$\begin{aligned} S &= -\mathrm{Tr} e^{S^{\mathrm{sq}}[\xi_0]} e^{-S^{\mathrm{sq}}[\xi_0]} \rho_{\mathrm{ph}} \ln \rho_{\mathrm{ph}} \\ &= -\mathrm{Tr} \tilde{\rho}_{\mathrm{ph}} \ln \tilde{\rho}_{\mathrm{ph}} \end{aligned} \tag{16}$$

where

$$\tilde{\rho}_{\mathrm{ph}} = \sum_u |c_u|^2 |\alpha(j_u), \xi(t_u) - \xi_0\rangle\langle \alpha(j_u), \xi(t_u) - \xi_0|. \tag{17}$$

One can write this using the squeezing transformation and writing the squeezing factor ξ to leading order in $\frac{1}{\Omega}$ as

$$\tilde{\rho}_{\mathrm{ph}} = \sum_u |c_u|^2 e^{\frac{1}{8}\left(a^2 - \left(a^\dagger\right)^2\right) \frac{t_u - \langle T\rangle}{L}} |\alpha(j_u)\rangle\langle\alpha(j_u)| e^{-\frac{1}{8}\left(a^2 - \left(a^\dagger\right)^2\right) \frac{t_u - \langle T\rangle}{L}}. \tag{18}$$

From expanding the exponentials it becomes clear that the leading order contribution to the matrix elements of $\tilde{\rho}_{\mathrm{ph}}$ from the squeezing will be proportional to $\frac{\langle (T - \langle T\rangle)^2\rangle}{L^2} = \frac{\Delta T^2}{L^2} \to 0$ $(L \to \infty)$ and therefore vanish in the thermodynamic limit. We can therefore discard the squeezing for the computation of the entanglement entropy and instead write

$$S = -\mathrm{Tr} \rho'_{\mathrm{ph}} \ln \rho'_{\mathrm{ph}} \; ; \; \rho'_{\mathrm{ph}} = \sum_u |c_u|^2 |\alpha(j_u)\rangle\langle\alpha(j_u)|. \tag{19}$$

This is the form of the density matrix used in the main part Eq. (5) to argue about current fluctuations as a necessary condition for light-matter entanglement. Writing the coherent states in their photon number representation one obtains

$$\begin{aligned} \rho'_{\mathrm{ph}} &= \sum_u |c_u|^2 e^{-|\alpha(j_u)|^2} \sum_{n,m} \frac{\alpha(j_u)^n}{\sqrt{n!}} \frac{(\alpha(j_u)^*)^m}{\sqrt{m!}} |n\rangle\langle m| \\ &\overset{j_u \in \mathbb{R}}{=} \sum_u |c_u|^2 e^{-\frac{g^2}{\Omega^2}\frac{j_u^2}{L}} \sum_{n,m} \frac{(-\frac{g}{\Omega}\frac{j_u}{\sqrt{L}})^n}{\sqrt{n!}} \frac{(-\frac{g}{\Omega}\frac{j_u}{\sqrt{N}})^m}{\sqrt{m!}} |n\rangle\langle m|. \end{aligned} \tag{20}$$

Thus the matrix elements of the reduced density matrix of the photons read, in the photon number basis

$$\langle n|\rho'_{\mathrm{ph}}|m\rangle = \sum_u |c_u|^2 e^{-\frac{g^2}{\Omega^2}\frac{j_u^2}{L}} \sum_{n,m} \frac{(-\frac{g}{\Omega}\frac{j_u}{\sqrt{L}})^{n+m}}{\sqrt{n!}\sqrt{m!}}. \tag{21}$$

We can relate this quantity to moments of the current operator. For this we first note

$$
\begin{aligned}
&\langle\Psi|e^{-\frac{g^2}{\Omega^2}\frac{J^2}{L}}\frac{\left(-\frac{g}{\Omega}\frac{J}{\sqrt{L}}\right)^{(n+m)}}{\sqrt{n!}\sqrt{m!}}|\Psi\rangle \\
&=\sum_{u,v} c_u^* c_v \langle j_u| \otimes \langle\alpha_u|e^{-\frac{g^2}{\Omega^2}\frac{J^2}{L}}\frac{\left(-\frac{g}{\Omega}\frac{J}{\sqrt{L}}\right)^{(n+m)}}{\sqrt{n!}\sqrt{m!}}|j_v\rangle \otimes |\alpha_v\rangle \\
&=\sum_{u} |c_u|^2 e^{-\frac{g^2}{\Omega^2}\frac{j_u^2}{L}}\frac{\left(-\frac{g}{\Omega}\frac{j_u}{\sqrt{L}}\right)^{(n+m)}}{\sqrt{n!}\sqrt{m!}}.
\end{aligned}
\tag{22}
$$

This shows that in the high frequency limit, the entanglement entropy between light and matter is completely determined by the moments of the current operator together with the light-matter coupling g and the frequency Ω. When assuming separability of higher order correlation functions as $\langle J^{2n}\rangle \approx \langle J^2\rangle^n$ (keeping in mind that $\langle J^{2n+1}\rangle = 0$, $n \in \mathbb{N}$ due to symmetry) we can also write

$$
\langle e^{-\frac{g^2}{\Omega^2}\frac{J^2}{L}}\frac{\left(-\frac{g}{\Omega}\frac{J}{\sqrt{L}}\right)^{(n+m)}}{\sqrt{n!}\sqrt{m!}}\rangle \approx e^{-\frac{g^2}{\Omega^2}\frac{\langle J^2\rangle}{L}}\frac{\left(-\frac{g}{\Omega}\right)^{(n+m)}\frac{\langle J^{(n+m)}\rangle}{\sqrt{L}^{(n+m)}}}{\sqrt{n!}\sqrt{m!}}. \tag{23}
$$

Therefore we write the matrix elements of the reduced density matrix as

$$
\langle n|\rho'_{\mathrm{ph}}|m\rangle \approx e^{-\frac{g^2}{\Omega^2}\frac{\langle J^2\rangle}{L}}\frac{\left(-\frac{g}{\Omega}\right)^{(n+m)}\frac{\langle J^{(n+m)}\rangle}{\sqrt{L}^{(n+m)}}}{\sqrt{n!}\sqrt{m!}}. \tag{24}
$$

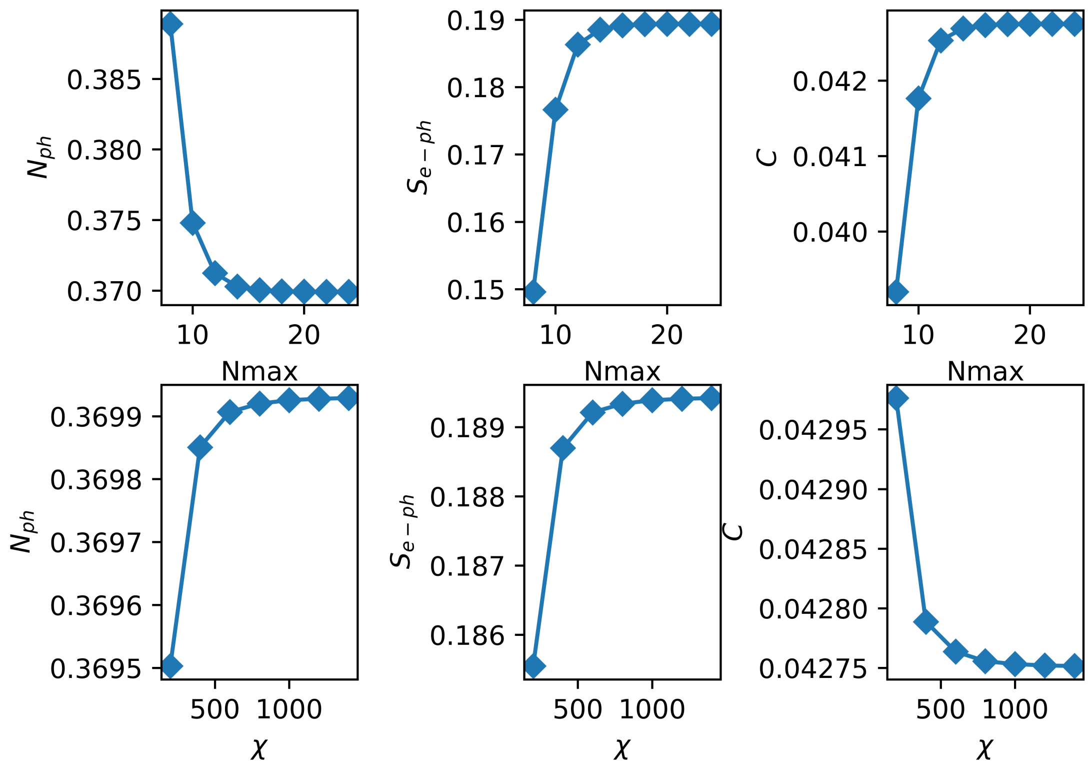
Since we are assuming $\Omega \gg t_{\mathrm{h}}, U$, we can limit ourselves to the lowest order photon number states for calculating the entanglement entropy

$$
\begin{aligned}
\rho'_{\mathrm{ph}} &= e^{-\frac{g^2}{\Omega^2}\frac{\langle J^2\rangle}{L}}\left(|0\rangle\langle 0| + \frac{g^2}{\Omega^2}\frac{\langle J^2\rangle}{L}|1\rangle\langle 1|\right) + \mathcal{O}\left(\frac{g^4 t_{\mathrm{h}}^4}{\Omega^4}\right) \\
&= \frac{1}{1+\frac{g^2}{\Omega^2}\frac{\langle J^2\rangle}{L}}\left(|0\rangle\langle 0| + \frac{g^2}{\Omega^2}\frac{\langle J^2\rangle}{L}|1\rangle\langle 1|\right) + \mathcal{O}\left(\frac{g^4 t_{\mathrm{h}}^4}{\Omega^4}\right).
\end{aligned}
\tag{25}
$$

In principle, one would also need to consider elements of the density matrix of the form $|0\rangle\langle 2|$ but one can show that they don't contribute to the entropy in our level of approximation. The $\mathcal{O}$ notation is to be understood in the operator norm here. The entropy of the density matrix Eq. (25) is

$$
\begin{aligned}
S_{\mathrm{ph}} = &-\frac{1}{1+\frac{g^2}{\Omega^2}\frac{\langle J^2\rangle}{L}}\ln\left(\frac{1}{1+\frac{g^2}{\Omega^2}\frac{\langle J^2\rangle}{L}}\right) \\
&-\frac{\frac{g^2}{\Omega^2}\frac{\langle J^2\rangle}{L}}{1+\frac{g^2}{\Omega^2}\frac{\langle J^2\rangle}{L}}\ln\left(\frac{\frac{g^2}{\Omega^2}\frac{\langle J^2\rangle}{L}}{1+\frac{g^2}{\Omega^2}\frac{\langle J^2\rangle}{L}}\right)
\end{aligned}
\tag{26}
$$

which matches Eq. (7) of the main part.



Supplementary Figure 1. Convergence analysis for the relevant quantities N_{ph}, $S_{\mathrm{e-ph}}$ and C. All the curves have been obtained setting $L = 110$, $U = 2.4t_{\mathrm{h}}$, $g = 4$ and $\Omega = 1t_{\mathrm{h}}$.

II. NUMERICAL CONVERGENCE

Here we report convergence checks for our DMRG results. There are two parameters that set the limit of the accuracy of our numerical approximation, which are photon cutoff N_{max} and the bond dimension χ. The photon cutoff sets the dimension of the local Hilbert space that describes the bosonic degree of freedom, which we considered to be finite. The maximum bond dimension χ defines the maximum dimension for the virtual bonds in the MPS representation of the ground state wavefunction[2]. The plots reported in Fig. 1 have all been obtained simulating a system of size $L = 110$ for $U = 2.4t_{\mathrm{h}}$ and $g = 4$, which relatively to the typical parameters discussed in the paper corresponds to high value of the coupling in the critical region. We show the convergence behaviour for three significant quantities discussed in the main text, nominally N_{ph}, $S_{\mathrm{e-ph}}$ and C. Unless otherwise stated, the DMRG parameters for the results shown in the paper are $\chi = 1000$ and $N_{max} = 16$.

III. CONNECTION TO HIGH U LIMIT VIA PERTURBATION THEORY

In the main part we noted, that in the CDW phase one needs to account for the fact that the light-matter coupling does not only create a light-excitation by creating a photon but also a charge excitation through a fermionic hopping. In this part we will show that this essentially corresponds to a replacement of $\frac{1}{\Omega^2} \to \frac{1}{(\Omega+U)^2}$ in the formulas found in the high frequency limit. To this end, we will consider the large U limit and perform second order perturbation theory in $\frac{t_\mathrm{h}}{U}$ again only keeping terms including the second order light-matter coupling. We thus define

$$\begin{aligned} H_0 &= \Omega a^\dagger a + U \sum_j \left(n_j - \frac{1}{2}\right)\left(n_{j+1} - \frac{1}{2}\right) \\ V &= T + \frac{g}{\sqrt{L}}\left(a^\dagger + a\right) J - \frac{g^2}{2L}\left(a^\dagger + a\right)^2 T. \end{aligned} \tag{27}$$

where the definitions of T (kinetic energy operator) and J (current operator) are in the main part. Denoting by $|\psi_n^{(0)}\rangle$ the eigenstates of the unperturbed system H_0 with eigenenergy E_n one can write the GS of the perturbed system to second order in t_h as

$$\begin{aligned} |\psi_0^{(2)}\rangle = |\psi_0^{(0)}\rangle &+ \sum_{n>0} \frac{\langle\psi_n^{(0)}|V|\psi_0^{(0)}\rangle}{E_0 - E_n} |\psi_n^{(0)}\rangle + \\ &+ \sum_{n,m>0} \frac{\langle\psi_n^{(0)}|V|\psi_m^{(0)}\rangle\langle\psi_m^{(0)}|V|\psi_0^{(0)}\rangle}{(E_n - E_m)(E_0 - E_m)} |\psi_n^{(0)}\rangle \\ &- \sum_{n>0} \frac{\langle\psi_n^{(0)}|V|\psi_0^{(0)}\rangle\langle\psi_0^{(0)}|V|\psi_0^{(0)}\rangle}{(E_n - E_0)^2} |\psi_n^{(0)}\rangle + \\ &- \frac{1}{2}\sum_{n>0} \frac{|\langle\psi_n^{(0)}|V|\psi_0^{(0)}\rangle|^2}{(E_0 - E_n)^2} |\psi_0^{(0)}\rangle. \end{aligned} \tag{28}$$

With this state one can calculate the photon number to second order in t_h as

$$\begin{aligned} N_\mathrm{ph}^{(2)} &= \langle\psi_0^{(2)}|a^\dagger a|\psi_0^{(2)}\rangle = \\ &= \sum_{n,m>0} \frac{\langle\psi_0^{(0)}|V^\dagger|\psi_m^{(0)}\rangle\langle\psi_n^{(0)}|V|\psi_0^{(0)}\rangle}{(E_0 - E_n)(E_0 - E_m)} \langle\psi_m^{(0)}|a^\dagger a|\psi_n^{(0)}\rangle \\ &= \sum_{n>0} \frac{\langle\psi_0^{(0)}|V^\dagger|\psi_n^{(0)}\rangle\langle\psi_n^{(0)}|V|\psi_0^{(0)}\rangle}{(E_0 - E_n)^2} \langle\psi_n^{(0)}|a^\dagger a|\psi_n^{(0)}\rangle \end{aligned} \tag{29}$$

where in the second line we assumed the basis states to be written in the photon number basis and used the fact that the operator $a^\dagger a$ is diagonal in that basis. Noting that only the second and third term in V (Eq. (27)) create a non-zero photon number and $\langle\psi_0^{(0)}|V|\psi_0^{(0)}\rangle = 0$ we can write

Eq. (29) as

$$N_{\rm ph}^{(2)} = \frac{1}{(U+\Omega)^2}\langle\psi_0^{(0)}|\left(\frac{g}{\sqrt{L}}\left(a^\dagger+a\right)J - \frac{g^2}{2L}\left(a^\dagger+a\right)^2 T\right)^2|\psi_0^{(0)}\rangle. \tag{30}$$

The GS of the unperturbed system $|\psi_0^{(0)}\rangle$ can be written as

$$|\psi_0^{(0)}\rangle = |\psi^{\rm CDW}\rangle \otimes |0\rangle \tag{31}$$

where $|0\rangle$ is the photon vacuum state and $|\psi^{\rm CDW}\rangle$ is the state with staggered occupation (or the symmetric superposition of the two possible states in the case of no explicit symmetry breaking). Inserting this into Eq. (30) we obtain

$$\begin{aligned} N_{\rm ph}^{(2)} =& \frac{g^2}{L(U+\Omega)^2}\langle 0|\left(a^\dagger+a\right)^2|0\rangle\langle\psi^{\rm CDW}|J^2|\psi^{\rm CDW}\rangle + \\ & \frac{g^4}{4L^2(U+\Omega)^2}\langle 0|\left(a^\dagger+a\right)^4|0\rangle\langle\psi^{\rm CDW}|T^2|\psi^{\rm CDW}\rangle - \\ & \frac{g^3}{\sqrt{L}^3(U+\Omega)^2}\langle 0|\left(a^\dagger+a\right)^3|0\rangle\langle\psi^{\rm CDW}|J\,T|\psi^{\rm CDW}\rangle. \end{aligned} \tag{32}$$

The last term vanishes due to the uneven number of photon creation and annihilation operators. The other two require the calculation of the $\langle J^2\rangle$ and $\langle T^2\rangle$ in the CDW state. We find

$$\begin{aligned} &\langle\psi^{\rm CDW}|J^2|\psi^{\rm CDW}\rangle = \langle\psi^{\rm CDW}|T^2|\psi^{\rm CDW}\rangle = \\ &-t_{\rm h}^2\sum_{i,j}\delta_{i,j}\langle\psi|c_{i+1}^\dagger c_i c_j^\dagger c_{j+1} + c_i^\dagger c_{i+1} c_{j+1}^\dagger c_j|\psi\rangle = Lt_{\rm h}^2. \end{aligned} \tag{33}$$

Thus the second term in Eq. (32) also vanishes in the thermodynamic limit $L\to\infty$. This is consistent with our observations in the main part since in the CDW state $\langle T\rangle = 0$ and the contribution from fluctuations is suppressed in the large L limit as seen explicitly here. Thus we obtain the same relation of the photon number to the current fluctuations as in the high frequency limit (Eq. (3) of the main part) but with the replacement $\frac{1}{\Omega^2}\to\frac{1}{(\Omega+U)^2}$.

Overall we get for the photon number in the large U limit

$$N_{\rm ph}^{(2)} = \frac{g^2 t_h^2}{(U+\Omega)^2}. \tag{34}$$

This matches precisely the results obtained with DMRG for large interaction values $U \gg t_{\rm h}$ and $U \gg \Omega$.

A. Entanglement entropy accounting for charge excitations in the CDW

We calculate the entanglement entropy to second (leading) order in $\frac{t_\mathrm{h}}{U}$ in perturbation theory. To this end we split up the perturbative wave-function as

$$\begin{aligned}|\phi\rangle = &\left(c_0|\phi_0\rangle + c_1^0|\phi_1^0\rangle + c_2^0|\phi_2^0\rangle\right) \otimes |0\rangle + \left(c_1^1|\phi_1^1\rangle + +c_2^1|\phi_2^1\rangle\right) \otimes |1\rangle + \left(c_1^2|\phi_1^2\rangle c_2^2|\phi_2^2\rangle\right) \otimes |2\rangle \\ &+ c_2^3|\phi_2^3\rangle \otimes |3\rangle + +c_2^4|\phi_2^4\rangle \otimes |4\rangle.\end{aligned} \tag{35}$$

Here $|\phi_0\rangle$ denotes the fermionic part of the unperturbed GS which in the large U limit is the staggered-occupied state. For the coefficients c_m^n and the fermionic wave-functions $|\phi_m^n\rangle$, m always denotes the order in perturbation theory that one gets by repeatedly applying J and T to the GS and n denotes the photon number that only increases when applying terms that contain photon creators or annihilators. In order to obtain the reduced density matrix for the photons we need to calculate several scalar products of the fermionic states. Writing only those that don't vanish or are of higher order in the perturbation we get for the reduced density matrix

$$\begin{aligned}\rho_\mathrm{ph} = \mathrm{Tr}_{\mathrm{e}^-}\rho = &\underbrace{\left(|c_0|^2 + c_0\left(c_2^0\right)^* \langle\phi_2^0|\phi_0\rangle + c_0^* c_2^0\langle\phi_0|\phi_2^0\rangle\right)}_{=:\kappa_0}|0\rangle\langle 0| + \underbrace{|c_1^1|^2}_{=:\kappa_1}|1\rangle\langle 1| + \underbrace{|c_1^2|^2}_{:=\kappa_2}|2\rangle\langle 2| \\ &+ \underbrace{\left(c_0\left(c_2^2\right)^*\langle\phi_2^2|\phi_0\rangle + c_1^0\left(c_1^2\right)^*\langle\phi_1^2|\phi_1^0\rangle\right)}_{=:\kappa_{02}}|0\rangle\langle 2| + \underbrace{\left(c_0^* c_2^2\langle\phi_0|\phi_2^0\rangle + \left(c_1^0\right)^* c_1^2\langle\phi_1^0|\phi_1^2\rangle\right)}_{=:\kappa_{02}^*}|2\rangle\langle 0|.\end{aligned} \tag{36}$$

Written in matrix form this reads

$$\rho_\mathrm{ph} = \begin{pmatrix}\kappa_0 & 0 & \kappa_{02}\\ 0 & \kappa_1 & 0\\ \kappa_{02}^* & 0 & \kappa_2\end{pmatrix} \overset{\text{reorder}}{\rightarrow} \begin{pmatrix}\kappa_0 & \kappa_{02} & 0\\ \kappa_{02}^* & \kappa_2 & 0\\ 0 & 0 & \kappa_1\end{pmatrix} \tag{37}$$

As seen in the previous section, κ_2 involves the expectation value $\frac{1}{L^2}\langle\psi^\mathrm{CDW}|T^2|\psi^\mathrm{CDW}\rangle$ and therefore $\kappa_2 \to 0$ for $L \to \infty$. To find the entropy we diagonalize the remaining matrix which we can now do by blocks – in fact we only need to diagonalize the upper 2×2 block. The eigenvalues read

$$v_\pm = \frac{1}{2}\left(\kappa_0 \pm \sqrt{\kappa_0^2 + 4|\kappa_{02}|^2}\right) = \frac{1}{2}\left(\kappa_0 \pm \kappa_0\sqrt{1 + 4\frac{|\kappa_{02}|^2}{\kappa_0^2}}\right). \tag{38}$$

Since $\kappa_0 \sim 1$ and $\kappa_{02} \sim \left(\frac{t_\mathrm{h}}{U}\right)^2$ (as can be seen from Eq. (36)) we can expand the square root and conclude that only one eigenvalue κ_0 is different from zero within our level of approximation.

We now remember that we have calculated $\kappa_1 = \frac{g^2 t_\mathrm{h}^2}{(U+\Omega)^2}$ before and fix κ_0 by normalization such that we get for the entropy

$$-S = \left(1 - \frac{g^2 t_\mathrm{h}^2}{(U+\Omega)^2}\right)\ln\left(1 - \frac{g^2 t_\mathrm{h}^2}{(U+\Omega)^2}\right) + \frac{g^2 t_\mathrm{h}^2}{(U+\Omega)^2}\ln\left(\frac{g^2 t_\mathrm{h}^2}{(U+\Omega)^2}\right) + \mathcal{O}\left(\frac{t_\mathrm{h}^3}{U^3}\right). \tag{39}$$

This matches to leading order in $\frac{1}{U}$ the result obtained in the high-frequency limit when replacing $\frac{1}{\Omega^2} \to \frac{1}{(\Omega+U)^2}$ and inserting the current fluctuations in the large U according to Eq. (33).

IV. CAVITY-FERMIONS MEAN FIELD DECOUPLING

We start from the full model, defined as

$$\begin{aligned} H = -t_h \sum_i \left(e^{i\frac{g}{\sqrt{L}}(a^\dagger + a)} c_i^\dagger c_{i+1} + h.c. \right) + \\ +U \sum_i \left(n_i - \frac{1}{2} \right) \left(n_{i+1} - \frac{1}{2} \right) + \Omega a^\dagger a \end{aligned} \tag{40}$$

We further define

$$K_L = \sum_i c_i^\dagger c_{i+1} \quad K_R = \sum_i c_{i+1}^\dagger c_i. \tag{41}$$

The single photonic mode can be decoupled from the chain with a mean-field approach, by setting to zero the fluctuations

$$\begin{cases} (K_R - \langle K_R \rangle) \left(e^{ig(a^\dagger + a)} - \langle e^{ig(a^\dagger + a)} \rangle \right) = 0 \\ (K_L - \langle K_L \rangle) \left(e^{-ig(a^\dagger + a)} - \langle e^{-ig(a^\dagger + a)} \rangle \right) = 0. \end{cases} \tag{42}$$

This leads to the decoupled hamiltonian

$$\begin{aligned} H = -t_h \left(\langle e^{ig(a^\dagger + a)} \rangle K_R + \langle e^{-ig(a^\dagger + a)} \rangle K_L \right) + \\ +U \sum_i \left(n_i - \frac{1}{2})(n_{i+1} - \frac{1}{2} \right) + \\ +\Omega a^\dagger a - t_h \left(e^{ig(a^\dagger + a)} \langle K_R \rangle + e^{-ig(a^\dagger + a)} \langle K_L \rangle \right) \\ +t_h \left(\langle K_R \rangle \langle e^{ig(a^\dagger + a)} \rangle + \langle K_L \rangle \langle e^{-ig(a^\dagger + a)} \rangle \right) \end{aligned} \tag{43}$$

Now we can observe that one can always write

$$\langle e^{\pm ig(a^\dagger + a)} \rangle = \gamma e^{\pm i\Gamma} \tag{44}$$

And that in general:

$$\langle K_R \rangle = \kappa e^{i\chi} = \langle K_L \rangle^* . \tag{45}$$

Considering only the XXZ model, for the couple of parameters (t, U) we can get the ground state that will be labeled as $\langle GS_{t,U} \rangle$ satisfying $H^{XXZ}_{t,U} |GS_{t,U}\rangle = E^{GS}_{t,u}$.

To shorten the notation we define $t_{eff} = \gamma t_h e^{i\Gamma}$ and

$$H_f(\gamma, \phi) = -t_{eff} \sum_i \left(c_i^\dagger c_{i+1} + h.c. \right) + \tag{46}$$

$$U \sum_i \left(n_i - \frac{1}{2} \right) \left(n_{i+1} - \frac{1}{2} \right)$$

$$H_b = \Omega a^\dagger a - 2t_h \kappa \cos \left(g(a + a^\dagger) + \chi \right).$$

that allows to write the MF hamiltonian as

$$H_{MF} = H_f + H_b + 2t_h \kappa \gamma \cos(\chi + \phi). \tag{47}$$

The minimization of the energy will be then obtained through a numerical procedure that, starting from the solution for the uncoupled model, iteratively diagonalize the bosonic and the fermionic part until one gets a wavefunction:

$$|\Psi\rangle = |\psi\rangle_f \otimes |\phi\rangle_b, \tag{48}$$

tensor product of a fermionic wavefunction $|\psi\rangle_f$ and a bosonic wavefunction $|\phi\rangle_b$, that satisfy the self-consistency conditions:

$$\langle\Psi|K_R|\Psi\rangle = \kappa \mathrm{e}^{i\chi} \tag{49}$$

$$\langle\Psi|\mathrm{e}^{ig(a+a^\dagger)}|\Psi\rangle = \gamma \mathrm{e}^{i\Gamma} \tag{50}$$

$$H_{MF}(\gamma, \phi, \kappa, \chi) |\Psi\rangle = E(\gamma, \phi, \kappa, \chi) |\Psi\rangle. \tag{51}$$

V. SUPPLEMENTARY REFERENCES

[1] Eckhardt, C. J. *et al.* Quantum floquet engineering with an exactly solvable tight-binding chain in a cavity. *Communications Physics* **5** (2022). URL https://doi.org/10.1038/s42005-022-00880-9.

[2] Schollwöck, U. The density-matrix renormalization group. *Rev. Mod. Phys.* **77**, 259–315 (2005). URL https://link.aps.org/doi/10.1103/RevModPhys.77.259. Publisher: American Physical Society.